COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Simulation of Bond Portfolio Development with Default Risk

DIPLOMA THESIS

2017 **Peter VODIČKA**

COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Simulation of Bond Portfolio Development with Default Risk

DIPLOMA THESIS

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Study Field: 1114 Applied Mathematics
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Supervisor: doc. Mgr. Igor Melicherčík, PhD.

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Title: Simulation of bond portfolio development with default risk.

Aim: The goal is to study Merton model for bond pricing with risk default. Calibration of the model on real data and consequently simulation of portfolio development.

Supervisor:	doc. Mgr. Igor Melicherčík, PhD.
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ZADANIE ZÁVEREČNEJ PRÁCE

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Simulovanie vývoja portfólia dlhopisov s rizikom defaultu.

Cieľ: Naštudovať Mertonov model oceňovania dlhopisov s rizikom defaultu. Kalibrácia modelu na základe reálnych dát a následne simulovanie vývoja portfólia dlhopisov.

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študent vedúci práce

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# Abstract

VODIČKA, Peter: Simulation Of Bond Portfolio Development With Default Risk [Diploma Thesis], Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics; Supervisor: doc. Mgr. Igor Melicherčík, PhD., Bratislava, 50p.

When investing in a portfolio of corporate bank bonds, we focus on quantifying one of the risks - the default risk. For the purpose of estimating the value and volatility of banks' asset value, we use the theoretical concept of pricing of financial derivatives. The Black and Scholes Model is a pillar of structural models such as Merton and Moody's Kealhofer, McQuown and Vasicek Model. Since one of our goals is to price the bonds, we build the probability of default term structure in a risk-neutral world, which is obtained by simulating bank asset values. These results are then used in the calculation of the fair price of the bonds and subsequently lead to determining the portfolio value. We apply the benefits of portfolio diversification with default risk on both, theoretical and practical levels. An important parts of the thesis is the simulation of correlated asset values and the simulation of bond portfolio development.

Keywords: Merton Model, Default Probability Term Structure, Bond Portfolio Development

Abstrakt

VODIČKA, Peter: Simulovanie vývoja portfólia dlhopisov s rizikom defaultu [Diplomová práca], Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky; školiteľ: doc. Mgr. Igor Melicherčík, PhD., Bratislava, 50s.

Pri investovaní do portfólia korporátnych bankových dlhopisov sa budeme venovať kvantifikovaniu jedného z rizík - rizika defaultu. Na odhadnutie hodnoty a volatility aktív bánk použijeme teoretický koncept oceňovania finančných derivátov. Black a Scholesov model je pilierom štrukturálnych modelov - Mertonov a Moody's Kealhofer, McQuown a Vasicek Model. Keďže jedným z cieľov je oceňovanie dlhopisov, v práci sa venujeme budovaniu časovej štruktúry pravdepodobností defaultov v rizikovo neutrálnom svete. Tú získame pomocou simulácií hodnôt aktív banky. Časové štruktúry pravdepodobností defaultov použijeme na výpočet férovej ceny dlhopisu, hodnote portfólia a medziročného výnosu fondu. Teoreticky odvodíme a prakticky použijeme diverzifikáciu portfólia s rizikom defaultu. Významnou časťou v práci je simulácia korelovaných hodnôt aktív bánk a simulovanie vývoja portfólia berúc v úvahu riziko defaultu.

Kľúčové slová: Mertonov model, časová štruktúra pravdepodobností defaultu, vývoj dlhopisového portfólia

Contents

Introduction

When investing in a bond portfolio, it is extremely necessary to quantify two main risks. The first risk is associated with change in the interest rate (”interest rate risk”). We are well aware that increasing interest rate causes a decline in the firm’s bond value. Today, however, we are witnessing a decline in interest rates all around the globe. The other risk is risk of default. It expresses the state of being uncertain about a firm’s ability to service its obligations and debts.
In the worst case scenario, the consequence of interest rate risk may be decrease in the bond value. Default itself results in a potentially significant loss for investors. The reason is clear. The bankrupt firm is not paying the coupon and may, or may not, pay share of the nominal value. This depends on the seniority of the purchased bond in the portfolio. In addition, the bonds bear quantitative information and also qualitative status, which is expressed by rating. Its range varies according to the credit rating agency. Globally there are three major credit rating agencies: Standard & Poor’s, Fitch Group and Moody’s Corporation. In general, the corporate bonds are more risky than the government bonds. An important rule says that the firm’s credit rating cannot be better than the rating of country itself.
According to the Moody’s Analytics team there is no way to distinguish unambiguously between firms that will default and those that will not. The best we can do is to assess probability of the entity’s default.

Prior to writing the thesis and the programming itself, we have studied various related sources and literature, which are listed at the end of the thesis. In addition to the article on key modeling methodology titled *Modeling Default Risk* [1] and published by Moody’s, we have also worked with the notes and knowledge gained through the Master’s study. First of all, the foundations were laid down in the first year of the studies through the *Methods in Risk Management* [2], the course led by Dr. Pavol Jurča. Pivotal parts were comprised in the Lecture 8 on *Credit Risk*, which set the theoretical basis, and also Lecture 9 on particular credit risk models.
Secondly, courses *Financial Mathematics I. and II.*, both are led by the thesis supervisor, assistant professor Igor Melicherčík and the book [3] represents also a valuable source of financial modeling theory.
Another important course for this thesis was *Credit Risk in Banking* [6], which was included in the last winter semester of exchange programme. Lectures and presentations given by Dr Sebastiano Vitali. During the preparatory phase of this thesis, westudied *Credit Risk Management and Modeling* [7], the author of which is assistant professor of finance Jiří Witzany.
Equally important for this thesis were the theses by EFM programme graduates, namely A. Pišková in 2004: *Modeling the Portfolio of Bonds with Risk Default* [4] and K. Kadlečíková in 2009: *Credit Default Swap Valuation and Comparison of Development during Financial Crisis* [5].

The contents of thesis is divided into four major chapters: *Theoretical Concepts and Introduction to Credit Risk Modeling, Portfolio Risk Default, Asset Value and Bond Price Modeling and Applied Credit Risk Modeling.*

The purpose of the Chapter 1 is to introduce credit risk in terms of firm’s default risk, which is effectively the key concept of this thesis. In this part we present theoretical approach how to measure this particular kind of risk. We define and describe structural models as Merton and Moody’s Kealhofer, McQuown and Vasicek Model. Both models are very similar. The common building pillar is the Black and Scholes Model of pricing option derivatives.

Chapter 2 focuses on transition from the standalone risk to the portfolio risk. The portfolio point of view addresses and solves the problem of default correlations between two and more firms. We show several formulas to portfolio profit and portfolio loss. The last Subchapter 2.3 is devoted to optimization problem to find optimal weights which is equivalent to Markowitz minimum variance portfolio. Portfolio risk default in the form of *Joint Default Frequencies.*

Chapter 3 has two parts. In the first one we extend one dimensional model of the firm’s asset value to multi-dimensional system of stochastic equations. The goal is to correctly model correlated asset values. Second Subchapter 3.2 introduces the bond price modeling.

Chapter 4 is a final part of thesis. This applied part is structurally divided into bond performance over the past years from 2009 up to 2017 and the future years from 2018 up to 2022. We deal

with the real-data-constructed bond portfolio. All senior unsecured bonds have issuers like big financial houses in Europe. A correct bond pricing requires for estimation of default probability term structures in risk-neutral world. In the first half of the Chapter we describe in detail the first algorithm using simulations of bank's asset value to build this term structures. By-products are our own credit rating history and fair prices of bonds.
Furthermore, we describe the second algorithm of portfolio diversification and we point out every step how we estimate inputs required for optimalization. Using all optimum weights and bond prices we can then fully concentrate on portfolio management. We compare four different ways of investing into bonds portfolio. We monitor return of these funds.
The last and most complex part is the simulation of the naively diversified portfolio until the maturity of all the selected bonds. This Subchapter describes in detail the third algorithm of simulation of correlated bank's asset values. This algorithm will alternate with the first algorithm of generating default probability term structures.

Chapter 1

Theoretical Concepts and Introduction to Credit Risk Modeling

The main sources of this Chapter is the Moody's Analytics documentation [1] and book of authors A. Resti and A. Sironi [8].

It is necessary to distinguish two main areas for measuring credit risk:

1. **Credit risk management** - models are used to determine the loss distribution of general portfolio over fixed time period, typically at least one year. This is also the goal of this thesis.

2. **Analysis of credit-risky securities** - dynamic models and continuous time are necessary because the pay-off of the most credit-risky products depends on the exact timing of default.

Credit risk is omnipresent in every portfolio and should be closely linked to the structure of existing credit portfolio models. As already mentioned in the introduction the credit risk is strongly related to default risk. It is the risk that the value of our portfolio changes due to unexpected events. These events are represented by negligible or significant changes in the credit quality of issuers. In this thesis the issuer is considered as a private firm. An issued security is a bond as a sufficient alternative to a loan.
Firm's default is a truly rare event. According to Moody's Analytics, in 2003 typical firm had a default probability of around 2% annually. However, there is considerable variation in default probabilities across firms. For example, the odds of a firm with a AAA rating defaulting are only about 2 in 10,000 per annum. Then a single A-rated firm has odds of around 10 in 10,000 per annum, five times higher than a AAA. At the bottom of the Moody's rating scale, a CCC-rated firm's odds of defaulting are 4 in 100, which is 200 times the odds of a AAA-rated firm.

Depending on the problem formulation, credit risk models can be divided into the following:

1. **Structural or firm-value models**,

2. **Reduced-form models**.

This grouping cut across the dynamic and static models.

1.1 Structural Models of Default

The model of default is known as a structural or firm-value model. There are three main elements according Moody's Analytics documentation [1] that determine the default probability of a firm:

1. **Asset Value** - the market value of the firm's asset value. This is a measure of the present value of the future free cash flows produced by the firm's asset discounted back at the appropriate discount rate. This measures the firm's prospects and incorporates relevant information about the firm's industry and the economy.

2. **Asset Risk** - the uncertainty or risk of the asset value. This is a measure of the firm's business and industry risk. The value of the firm's asset is an estimate and is thus uncertain. As a result, the value of the firm's asset should always be understood in the context of the firm's business or asset risk.

3. **Leverage** - the extent of the firm's contractual liabilities. Whereas the relevant measure of the firm's assets is always their market value, the book value of liabilities relative to the market value of asset is the pertinent measure of the firm's leverage, since that is the amount the firm must repay.

1.1.1 Merton Model

Robert Cox Merton in 1974 based on the book called *The Pricing of Options and Corporate Liabilities*, written by F. Black and M. Scholes, developed a model to provide method how to measure corporate credit risk. His model is the prototype of all structural models because it takes into account also the financial structure of the firm. All assumptions used by professor Merton were simplistic and idealistically described the market.

Let us assume that the firm is publicly traded on a specific frictionless stock market with zero transaction costs, taxes and whose asset value follows stochastic process (V_t). The firm's asset value follows dynamic process on the probability space $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and is modeled by stochastic equation (hereinafter "eq.")

$$\mathrm{d}V_t = \mu_V V_t \,\mathrm{d}t + \sigma_V V_t \,\mathrm{d}W_t, \tag{1.1.1}$$

where $\omega \in \Omega$ are events, $\mathcal{F}$ is σ-algebra, $\mathbb{P}$ present chosen probabilistic measure, $\left(W_t\right)_{t \geq 0}$ is Wiener process in $\mathbb{R}^1$. Also μ_V and σ_V are constants representing immediately expected value of asset and volatility of asset value. Firm's asset value V_t has log-normal distribution with value at certain time t expressed by eq.

$$V_t = V_0 \exp\left\{\left(\mu_V - \frac{1}{2}\sigma_V^2\right)t + \sigma_V W_t\right\}. \tag{1.1.2}$$

All firm's liabilities we substitute by a single debt obligation or zero-coupon bond with face value B and maturity T. The value at time t and $0 \leq t < T$ of equity and debt is denoted by E_t and B_t. The value of the firm's asset V is represented by sum

$$V_t = E_t + B_t. \tag{1.1.3}$$

The firm finances itself by equity (issuing shares) and by debt (issuing bonds). In the next chapters we will show that both are derivative securities of the underlying V_t. Also, it is necessary to mention that Merton's idealistic firm cannot pay out dividends or issue a completely new debt.

The default event occurs if the firm fails to pay a payment to its debt holders. Please note that the debt B is homogeneous in time t. At maturity T there are the following two possible cases to happen:

1. When the inequality

$$V_T > B \tag{1.1.4}$$

holds, the debt holders receive properly B and shareholders receive duly the remaining amount

$$V_T - B. \tag{1.1.5}$$

2. Otherwise, the credit risk occurs as the default event if

$$V_T \leq B. \tag{1.1.6}$$

The firm which issued a bond cannot meet its own financial obligations and the shareholders have to hand over control to the bondholders by the law. In this case

$$\begin{aligned} B &= V_T \\ E_T &= 0. \end{aligned} \tag{1.1.7}$$

The debt B in accordance with eq. (1.1.3) fulfill following chain of relations

$$\begin{aligned} B_T &= V_T - E_T = \min\big(V_T, B\big) = V_T - \big(V_T - B\big)^+ \\ &= B - \big(B - V_T\big)^+. \end{aligned} \tag{1.1.8}$$

The value of the debt at T is equivalent to the nominal value of the debt minus the payoff of the European put option on firm's asset V_T with exercise price equal to B. As mentioned above, if value of the firm's asset V_T is bigger than debt B at maturity T we can exercise this option. Consequently, the real debt is equivalent to the value of the debt minus the value of put option on the underlying asset. The value of the debt for $t = 0$ is the difference

$$\exp\big\{-rT\big\}B - \text{Put} \tag{1.1.9}$$

The firm's debt option pricing formula is

$$B_t = V_t\mathcal{N}\big(-d_{t,1}\big) + \exp\big\{-r(T-t)\big\}B\mathcal{N}\big(d_{t,2}\big), \tag{1.1.10}$$

where $0 \le t \le T$. From eq. (1.1.8) we evaluate the European put option for $t = 0$

$$\begin{aligned} B_0 &= V_0\mathcal{N}\big(-d_{0,1}\big) + \exp\big\{-rT\big\}B - \exp\big\{-rT\big\}B\mathcal{N}\big(-d_{0,2}\big) \\ &= V_0\mathcal{N}\big(-d_{0,1}\big) + \exp\big\{-rT\big\}B\Big(1 - \mathcal{N}\big(-d_{0,2}\big)\Big) \\ &= V_0\mathcal{N}\big(-d_{0,1}\big) + \exp\big\{-rT\big\}B\mathcal{N}\big(d_{0,2}\big). \end{aligned} \tag{1.1.11}$$

Conventionally

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\xi^2}{2}}\,\mathrm{d}\xi \tag{1.1.12}$$

is the standardized normal distribution function $\mathcal{N}(0,1)$ with following parameters

$$\begin{aligned} d_{t,1} &= \frac{\log\Big(\frac{V_t}{B}\Big) + \Big(r + \frac{1}{2}\sigma_V^2\Big)(T-t)}{\sigma_V\sqrt{T-t}}, \\ d_{t,2} &= d_{t,1} - \sigma_V\sqrt{T-t}. \end{aligned} \tag{1.1.13}$$

The present value, abbr. PV of the future pay-off of the non-default zero-coupon bond $C = 0$ with the face value B is discounted value to $t = 0$

$$p(0,T) = B\exp\big\{-rT\big\}. \tag{1.1.14}$$

With the credit risk is also associated the computation of *a credit spread*. The firm pays the credit spread over the default-free interest rate that is proportional to its default probability to compensate lenders for this uncertainty. The formula for deriving the credit spread is

$$\exp\big\{-(r+s)T\big\}B = V_0\mathcal{N}\big(-d_{0,1}\big) + p(0,T)B\mathcal{N}\big(d_{0,2}\big) \tag{1.1.15}$$

and the credit spread is defined as

$$s = -\frac{1}{T}\log\Big(\frac{V_0}{B}\mathcal{N}\big(-d_{0,1}\big) + p(0,T)\mathcal{N}\big(d_{0,2}\big)\Big) - r. \tag{1.1.16}$$

The credit spread is a risk premium add-on to the base interest rate (hereinafter "IR") used when pricing firm's debt issues. It reflects the firm's credit rating or the risk rating at the maturity of the issue. Besides that it bears information about current market spread rates, as well as other components such as security and liquidity. Credit spreads are quoted in basis points (bps).

Using all reference theory it is easy to determine the firm's default probability PD_t. It is precise probability of situation that inequality (1.1.6) holds.

Theorem 1.1.1.
The Probability of Default, abbr. PD can be computed from the formula

$$\begin{aligned} PD_t &= \mathcal{N}\big(-d_{t,2}\big) \\ &= 1-\mathcal{N}\big(d_{t,2}\big). \end{aligned} \tag{1.1.17}$$

Proof.

$$\begin{aligned} PD_t &= \mathbb{P}\Big[V_t < B\Big] \\ &= \mathbb{P}\Bigg[V_t \exp\Big\{\Big(\mu_V - \frac{1}{2}\sigma_V^2\Big)(T-t) + \sigma_V W_{T-t}\Big\} < B\Bigg] \\ &= \mathbb{P}\Bigg[W_{T-t} < \frac{\log\Big(\frac{B}{V_t}\Big) - \Big(\mu_V - \frac{1}{2}\sigma_V^2\Big)(T-t)}{\sigma_V}\Bigg] \\ &= \mathbb{P}\Bigg[W_1 < \frac{-\log\Big(\frac{V_t}{B}\Big) - \Big(\mu_V - \frac{1}{2}\sigma_V^2\Big)(T-t)}{\sigma_V\sqrt{T-t}}\Bigg] \\ &= \mathbb{P}\Big[W_1 < -d_{t,2}\Big] \\ &= \mathcal{N}\big(-d_{t,2}\big), \end{aligned} \tag{1.1.18}$$

where $W_{T-t} \sim \mathcal{N}(0, T-t)$ and $W_1 \sim \mathcal{N}(0,1)$. □

The Distance to Default, abbr. DD is defined by

$$\begin{aligned} DD_t &= d_{t,2} \\ &= \frac{\log\Big(\frac{V_t}{B}\Big) + \Big(\mu_V - \frac{1}{2}\sigma_V^2\Big)(T-t)}{\sigma_V\sqrt{T-t}}. \end{aligned} \tag{1.1.19}$$

Cons of Merton model are as follows [6]:

- simplified debt structure and possibility to exercise only in T;
- Gaussian distribution assumption;
- no negotiation between shareholders and debtholders;
- no arbitrage assumption;
- Black and Scholes Option Pricing Model assumes continuous negotiation of the underlying asset;
- no downgrading risk.

Pros of Merton model are as follows:

- simple to understand;
- easy to implement;
- no need to adjust for liquidity and liquidity risk;
- it works with main variables as leverage and volatility;
- structural approach.

1.1.2 Moody's Kealhofer, McQuown and Vasicek Model

Kealhofer, McQuown and Vasicek model, abbr. KMV model is a later product of already introduced Merton's crucial framework. It was developed under the name of Moody's Analytics, which is a subsidiary of Moody's Corporation, and it was established in 2007.

KMV model can also generate *Probability of Default*, abbr. PD and *Expected Default Frequency*, abbr. EDF. Modeling approach is very similar and essentially consists of these three steps:

1. Estimation of firm's asset value V_t and volatility σ_V from the market value of firm's equity E_t, volatility of equity σ_E and the book value of firm's liabilities B.
2. Calculation of the *Default Point*, abbr. DP and *Distance to Default*, abbr. DD using V_t, σ_V and B.
3. Calculation of PD is determined directly from the DD.

Since the equity value can be seen as the European call option on the asset value of the firm V_t with exercise price equal to B and the firm's capital is traded the same as debt, we can use the option pricing theory again. See Figure (1.1) for graphical interpretation of the European call option on firm's equitiy.

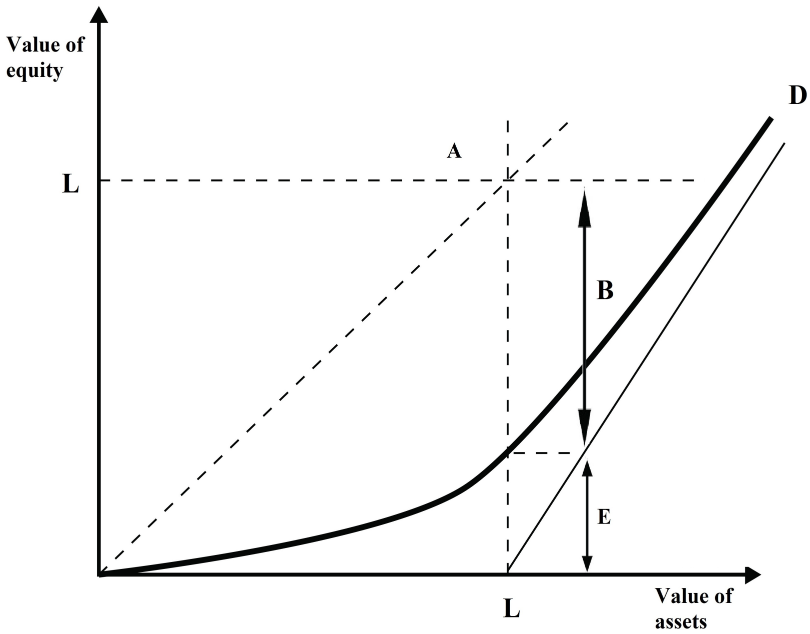

Fig. 1.1: The firm's equity as the European call option.

We use the Black and Scholes partial differential equation's result for the European call option on firm's equity

$$\begin{aligned} E_t &= \text{Call}^{\text{BS}}(V, t, \sigma_V, B, T, \mu_V) \\ &= V_t \mathcal{N}\big(d_{t,1}\big) - \exp\big\{-r(T-t)\big\} B \mathcal{N}\big(d_{t,2}\big), \end{aligned} \tag{1.1.20}$$

where $T - t$ is time remaining to expiration. The KMV uses this option nature of E to derive the underlying V and σ_V implied by the market value, volatility of equity σ_E and the book value of liabilities B. It has to be highlighted that we have available E_t the value of the capitalization at time t, together with its volatility σ_E. It is impossible to observe V_t and its σ_V. We only know that σ_V is somehow related to, but quite different from σ_E. A firm's leverage has the effect of magnifying its underlying σ_V.

We already showed stochastic differential eq. for modeling the process of V. The same approach can be applied for the firm's equity E

$$\frac{\mathrm{d}E}{E} = \mu_E \,\mathrm{d}t + \sigma_E \,\mathrm{d}W_t, \tag{1.1.21}$$

where μ_E is expressing immediately expected drift and σ_E is volatility of firm's equity.

As before, we use the *Itô's lemma* on equity's dynamics to derive

$$\begin{aligned} \mathrm{d}E &= \frac{\partial E}{\partial V}\,\mathrm{d}V + \frac{\partial E}{\partial t}\,\mathrm{d}t + \frac{1}{2}\sigma_E^2 V^2 \frac{\partial^2 E}{\partial V^2}(\mathrm{d}t)^2 \dots \\ &= \left(\frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 E}{\partial V^2} + \mu_V F \frac{\partial E}{\partial V} + \frac{\partial E}{\partial t}\right)\mathrm{d}t + \sigma_V V \frac{\partial E}{\partial V}\,\mathrm{d}W_t. \end{aligned} \tag{1.1.22}$$

By derivation with respect to the first, the second power of firm's asset value and time we gain all option characteristics and two useful formulas.

1. By comparison of volatility coefficients about stochastic term $\mathrm{d}W_t$ from eq. (1.1.21) with (1.1.22) we gain an interesting pattern eq.

$$\sigma_E = \sigma_V \frac{V}{E}\frac{\partial E}{\partial V}. \tag{1.1.23}$$

2. A ratio $\frac{\partial E}{\partial V}$ is analogical to *Asset Delta* and it is called firm's *Equity Delta*

$$\Delta^E = \mathcal{N}\big(d_1\big). \tag{1.1.24}$$

3. Directly via drift comparison yields to formula

$$\mu_E = \frac{1}{2}\sigma_V^2 \frac{V^2}{E}\frac{\partial^2 E}{\partial V^2} + \mu_V \frac{V}{E}\frac{\partial E}{\partial V} + \frac{1}{E}\frac{\partial E}{\partial t}. \tag{1.1.25}$$

4. Equity Gamma Γ^E is defined

$$\Gamma^E = \frac{\partial^2 E}{\partial V^2} = \frac{\mathcal{N}\big(d_1\big)}{V\sigma_V\sqrt{T-t}} \tag{1.1.26}$$

   and it is positive.

5. Equity Theta θ^E

$$\theta^E = \frac{\partial E}{\partial t} = -\frac{V\mathcal{N}\big(d_1\big)\sigma_V}{2\sqrt{T-t}} - \mu_V B\exp\big\{-\mu_V(T-t)\big\}\mathcal{N}\big(d_2\big). \tag{1.1.27}$$

Using option greeks above we have a formula for μ_V derived from eq. (1.1.25)

$$\mu_V = \frac{\mu_E E - \theta^E - \frac{1}{2}\sigma_V^2 V^2 \Gamma^E}{V\Delta^E}. \tag{1.1.28}$$

The estimation of exact values V_t and σ_V is more complex. We solve following non-linear system of eq.

$$\begin{aligned} V_t\mathcal{N}\big(d_1\big) - \exp\big\{-r(T-t)\big\}B\mathcal{N}\big(d_2\big) - E_t = 0 \\ \frac{V_t}{E}\mathcal{N}\big(d_1\big)\sigma_V - \sigma_E = 0 \end{aligned} \tag{1.1.29}$$

to get firm's asset value V_t and its volatility σ_V. It is appropriate to use correct numerical methods for solving this non-linear system. Since we showed (1.1.24) that function $V_t\mathcal{N}\big(d_1\big) - \exp\big\{-\mu_V(T-t)\big\}B\mathcal{N}\big(d_2\big) - E_t$ is an increasing in V_t, we have surely only one solution. The same applies for $V_t\mathcal{N}\big(d_1\big)\sigma_V - E\sigma_E$.

A measure of firm's leverage is stated by *the Leverage Ratio*, abbr. LR and it is defined as

$$LR = \frac{B_0}{V_0}. \tag{1.1.30}$$

Using LR we can rewrite firm's equity as follows

$$E_0 = V_0\Big(\mathcal{N}\big(d_{0,1}\big) - LR\mathcal{N}\big(d_{0,2}\big)\Big), \tag{1.1.31}$$

Up to now we could understand B as a constant value of book liabilities. KMV's authors had opened the opportunity that the credit risk computation distinguishes two types of liabilities with respect to T:

1. short-term liabilities, abbr. STL,
2. long-term liabilities, abbr. LTL.

According the Moody's Analytics documentation [1], we can quote and point out several KMV's credit risk characteristic indicators.

1. If we combine STL and LTL correctly, we get *theDefault Point*, abbr. "DP" sometimes referred to *the Default Threshold*, abbr. DT, which shows an approximate V at which observed firm will default. The value of DP lies between total and current liabilities

   $$DP = STL + \frac{1}{2}LTL. \tag{1.1.32}$$

   The DP estimation is based on an extensive empirical research by Moody's rating agency, which has looked at thousands of defaulting firms, observing each firm's DP in relation to the V_t at the time of default. We already know that the default event occurs when V_t is lower than B. KMV substitutes B for DP. Therefore, the DP will be considered as nominal debt B .

2. The KMV definition of *the Distance to Default*, abbr. DD is intuitive and can be expressed by formula

   $$DD = \frac{\text{market asset value} - DP}{\text{market asset value} \times \text{asset volatility}}. \tag{1.1.33}$$

   The numerator is a relevant net worth of the firm. Apparently, if the firm is about to default, then market net worth reaches 0. The DD equals the number of standard deviations away from the DP.

Now, suppose we have obtained both values V_T and σ_V from non-linear system (1.1.29). *The Probability of Default* is derived from (1.1.18) and uses definition of KMV's DD

$$PD_t = \mathcal{N}\big(-DD_t\big). \tag{1.1.34}$$

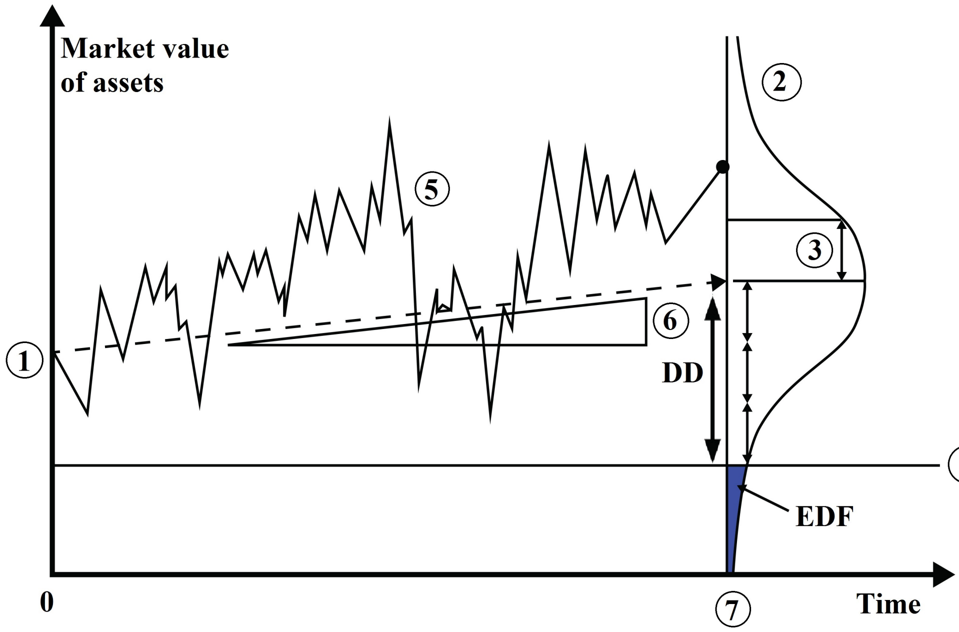

Fig. 1.2: Graphic representation of derived credit risk parameters: ① - the current firm's asset value: V_0; ② - the probability distribution of the future firm's asset value at time H: V_H; ③ - the volatility of the V_H; ④ - the benchmark of the default point: DP; ⑤ - a possible asset value trajectory; ⑥ - the expected growth rate of V_t over H; ⑦ - the length of observed horizon: H

Cons of KMV model are as follows [6]:

- Gaussian distribution assumption on the asset process;
- no arbitrage assumption;
- the firm must be listed in a market;
- market assumed to be efficient.

Pros of KMV model are as follows:

- functions DD and EDF can be updated more often than the classic rating grade;
- in rating grade approach, companies with the same rating share the same PD;
- debt structure is not oversimplified;
- input data are easier to define;

Firm's DD functions computed for different segments could be various. For financial institutions we can modify DD to *the Distance to Capital*, abbr. DC according to paper written by Larsen & Mange (2008). The only difference between the DC and the DD formula (1.1.19) is that B is multiplied by λ where $\lambda = \frac{1}{1-\text{PCAR}}$. $PCAR$ means *The Prudential Capital Assessment Review* expressing *Basel* requirement on the bank capital adequacy.

As a result of the expert meeting on this topic in National Bank of Slovakia, there was presented advice that there is need to distinguish between the regulatory approach and the approach already presented. The regulatory approach for financial institutions is based on substitution *Risk Weighted Assets*, abbr. RWA instead of value of bank's asset V in definition of default (1.1.6). $RWAs$ are multiplied by the risk weight α and compared with the bank's equity.

Example 1.1.2.

Before we preceed to portfolio default theory, let us to perform the standalone risk calculation on the public traded firm Deutsche Bank, abbr. DBK in 2017.

1. We compute daily market equity value E_t for $t = 1, \dots, 756$ where $t = 1$ equals 1 January 2014 and $t = 756$ equals to 31 December 2016 as multiplication

$$\textit{daily stock price} \times \textit{outstanding shares}. \tag{1.1.35}$$

   It is also possible to ascertain this value from the regular financial statements.

2. We determine Deutsche's equity standard deviation σ_E from increments of stock prices at least from the last 3 years and we get $\sigma_E = 66.52\%$.
3. We choose risk-free IR r instead of μ_V (see Chapter 3). E.g. Euro-zone goverment bond $r = -0.79\%$ in 2017 with maturity 1Y.
4. From the balance sheet we know the last book value of bank's liabilities before January 2017. DBK's is $B =$ EUR 1.525 Trillion.
5. We solve non-linear system (1.1.29) and we get asset value $V =$ EUR $1,530.633$ Billion and $\sigma_V = 0.54$ %.
6. Having all parameters we estimate the DD (1.1.19) for next year and we get 2.06 standard deviations.
7. Since $PD^Q_{1Y} = \mathcal{N}(-DD)$ we finally have 4.78% (478 basis points) probability of default for DBK in next 1 year under risk-free measure Q (see Chapter 3).

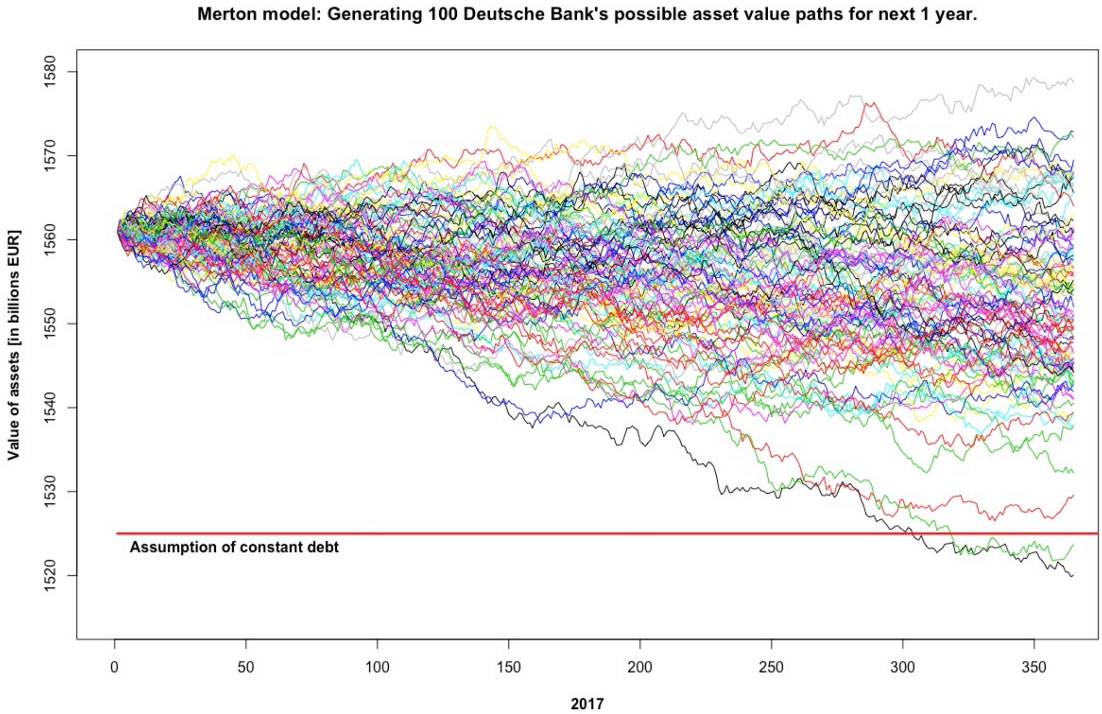

Fig. 1.3: Generating 100 possible Deutsche Bank's asset value trajectories for 1Y according the Merton model using risk-free rate.

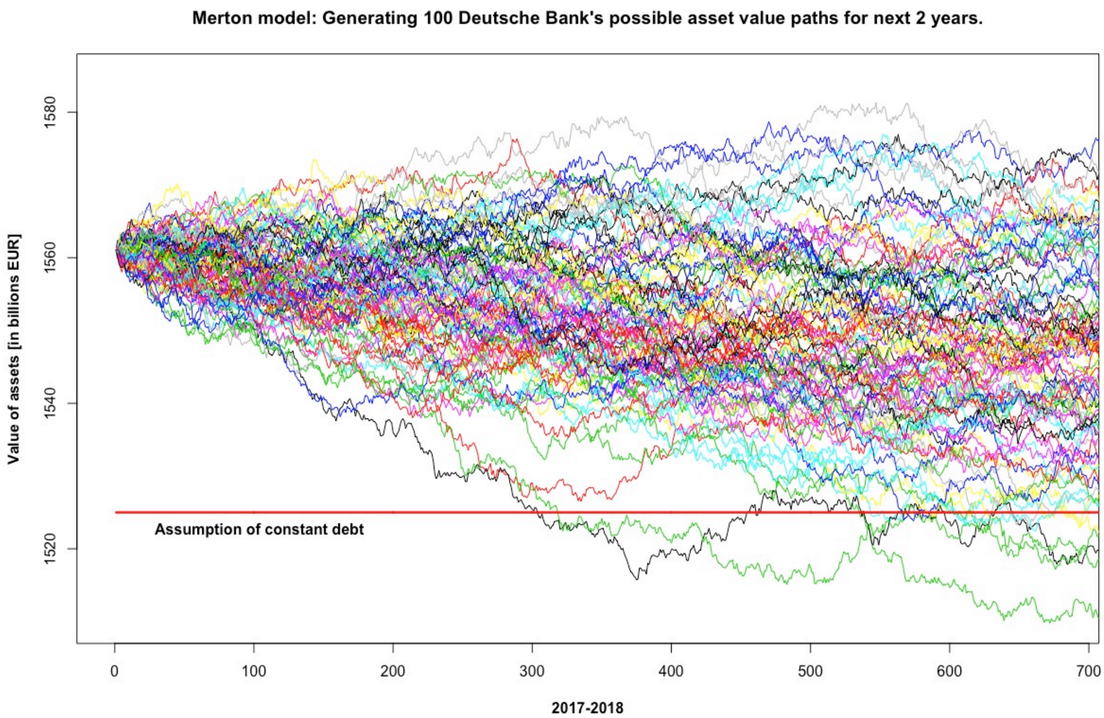

Fig. 1.4: Generating 100 possible Deutsche Bank's asset value trajectories for 2Y according the Merton model using risk-free rate.

Chapter 2

Portfolio Risk Default

In Chapter 2 is primary source of theory [4].

2.1 Portfolio Profit and Loss

If default event occurs it is desirable to quantify the approximate value of loss. Firstly, assume the portfolio which consists of a single bond.

1. *Expected Loss*, abbr. EL or *Credit Loss Rate*, abbr. CLR is defined as multiplication of EDF and *Loss Given Default*, abbr. LGD

$$EL = EDF \times LGD. \tag{2.1.1}$$

   where LGD is easily understood as share of an firm's asset that is lost if a borrower defaults. It can be expressed as

$$LGD = 1 - RR, \tag{2.1.2}$$

   where RR is *the Recovery Rate*.

2. *Unexpected Loss*, abbr. UL is representing volatility of expected loss and can be expressed as

$$UL = LGD \times \sqrt{EDF \times (1 - EDF)}. \tag{2.1.3}$$

Moreover, suppose that we measure expected and unexpected portfolio credit loss, whereas portfolio consists of m bonds. We need to measure diversification and to specify the loss value of the whole portfolio.

1. *The Expected Portfolio Loss*, abbr. EL_π is weighted average of expected loss of all bonds individually, EL_i where $i = 1, \ldots, m$

$$EL_\pi = w_1 EL_1 + \ldots + w_m EL_m. \tag{2.1.4}$$

2. *The Unexpected Portfolio Loss*, abbr. UL_π is more complex than simply weighted average of UL_i where $i = 1, \ldots, m$ because total loss is also dependent on the default correlation ρ_{ij} between every pair of firm's bonds

$$UL_\pi = \sqrt{w_1 w_1 UL_1 UL_1 \rho_{11} + w_1 w_2 UL_1 UL_2 \rho_{12} + \ldots + w_m w_m UL_m UL_m \rho_{mm}}. \tag{2.1.5}$$

Naturally, we need to distinguish the market value of a loan to determine the price for which it is possible to buy or sell. The value is expressed by discounting all cash values in the future to time $t = 0$ with respect to correct discount factor.

If the firm defaults, we lose exactly LGD, otherwise we will receive profit Y.

1. *The Profit* Y from a single bond is defined as

$$Y = r + \textit{Expected Risk Premium} + \textit{Unexpected Risk Premium}, \tag{2.1.6}$$

where r is the risk-free IR.

2. *The Expected Profit* $\bar{Y}$ from a single bond equals

$$\begin{aligned} \bar{Y} &= \mathbb{E}[Y] \\ &= EDF \times \big(r - LGD\big) + (1 - EDF) \times Y. \end{aligned} \tag{2.1.7}$$

3. *The Expected Risk Premium* is defined as

$$\frac{LGD \times EDF}{1 - EDF}. \tag{2.1.8}$$

If we do not know *Unexpected Risk Premium* we rewrite profit eq. with assumption of zero risk

$$\begin{aligned} Y &= r_f + \textit{Expected Risk Premium} \\ \bar{Y} &= EDF \times \big(r - LGD\big) + (1 - EDF) \times \left(r + \frac{LGD \times EDF}{1 - EDF}\right) \end{aligned} \tag{2.1.9}$$

and we get conclusion that expected profit is equal to risk-free IR

$$\bar{Y} = r. \tag{2.1.10}$$

2.2 Default Correlation

For a purpose of this thesis it is important to measure the default correlations between two and more firms in the portfolio.

1. The first step is to find correct probability that two firms will default at the same time. We use definition of default (1.1.6). The probability that two firms i and j will default at the same time can be expressed as joint probability that value of firm's asset i drops below book liabilities B_i at the same time as value of asset of firm j drops below book value liabilities B_j.

   The Joint Default Frequency, abbr. JDF is defined as

$$JDF_{ij} = \mathbb{P}\Big[V_{t,i} \leq B_i \wedge V_{t,j} \leq B_j\Big]. \tag{2.2.1}$$

   Following Black and Scholes Option Pricing Model's assumptions let us take two random variables $\varepsilon_i, \varepsilon_j$ which have standardized normal distribution.

   We compute

$$\begin{aligned} JDF_{ij} &= \mathbb{P}\Big[-DD_i \leq \varepsilon_i, -DD_j \leq \varepsilon_j\Big] \\ JDF_{ij} &= \mathbb{P}\left[-\frac{\log\Big(\frac{V_{t,i}}{B_i}\Big) + \Big(\mu_i - \frac{1}{2}\sigma_{V,i}^2\Big)t}{\sigma_{V,i}\sqrt{t}} \geq \varepsilon_i, -\frac{\log\Big(\frac{V_{t,j}}{B_j}\Big) + \Big(\mu_j - \frac{1}{2}\sigma_{V,j}^2\Big)t}{\sigma_{V,j}\sqrt{t}} \geq \varepsilon_j\right]. \end{aligned} \tag{2.2.2}$$

   More precisely, we have

$$JDF_{ij} = \frac{1}{2\pi\sqrt{1-\rho_{ij}^2}} \int_{-\infty}^{-DD_i} \int_{-\infty}^{-DD_j} \exp\left[-\frac{1}{2(1-\rho_{ij}^2)}\Big(x^2 - 2\rho_{ij}xy + y^2\Big)\right] \mathrm{d}x\,\mathrm{d}y, \tag{2.2.3}$$

   where ρ_{ij} is correlation between these two firm's assets values.

All computations yield to key relationship

$$JDF_{ij} = \mathcal{N}_2\Big(- DD_i, -DD_j, \rho_{ij}\Big), \tag{2.2.4}$$

where $\mathcal{N}_2$ is standard *bivariate* normal cumulative distribution function.

2. The second step is to estimate the final formula of default correlation between two firms. Let us denote X and Y as random variables of firms i and j defaults. X is dummy random variable and it is equal 1 if firm i defaults or 0 if not with following probabilities

$$\begin{aligned} X &= 1 \text{ with probability } EDF_i, \\ Y &= 1 \text{ with probability } EDF_j, \\ X &= 0 \text{ with probability } 1 - EDF_i, \\ Y &= 0 \text{ with probability } 1 - EDF_j. \end{aligned} \tag{2.2.5}$$

Expected value and variance of random variable X (Y similar index j) are

$$\begin{aligned} \mathbb{E}(X) &= EDF_i, \\ \text{var}(X) &= EDF_i \times \big(1 - EDF_i\big). \end{aligned} \tag{2.2.6}$$

More interesting result we get alternative distribution of random variable XY. Value 1 inducts the credit event that both firms default at the same time and 0 if not.

$$\begin{aligned} XY &= 1 \text{ with probability } EDF_{ij}, \\ XY &= 0 \text{ with probability } 1 - EDF_{ij}. \end{aligned} \tag{2.2.7}$$

Expected value and variance of random variable XY are

$$\begin{aligned} \mathbb{E}(XY) &= EDF_{ij}, \\ \text{var}(XY) &= EDF_{ij} \times \big(1 - EDF_{ij}\big). \end{aligned} \tag{2.2.8}$$

We have all components to build default correlation formula for two general random variables

$$\begin{aligned} \rho^D_{XY} &= \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X) \times \text{var}(Y)}} \\ &= \frac{\mathbb{E}(XY) - \mathbb{E}(X) \times \mathbb{E}(Y)}{\sqrt{\text{var}(X) \times \text{var}(Y)}}. \end{aligned} \tag{2.2.9}$$

The Default Correlation Function of firm i and firm j is defined by formula

$$\rho^D_{ij} = \frac{JDF_{ij} - EDF_i \times EDF_j}{\sqrt{EDF_i \times \big(1 - EDF_i\big) \times EDF_j \times \big(1 - EDF_j\big)}}. \tag{2.2.10}$$

2.3 Portfolio Diversification

It is well known that correctly diversified portfolio is bearing less risk. Using *Harry M. Markowitz's portfolio theory* and with precise execution we can reduce risk with a fixed portfolio expected return, abbr. $\mathbb{E}[\boldsymbol{R}]_\pi$. In our case, portfolio risk is hidden in variable called portfolio unexpected loss, abbr. UL_π.

There are often more possible ways how to formulate optimalization problem of diversification. With solution of optimalization we get the efficient frontier. It is a set of optimal portfolios that can provide the best $\mathbb{E}[\boldsymbol{R}]_\pi$ for an exact risk level of UL_π. Or, on the other hand the lowest risk level for a pre-defined $\mathbb{E}[\boldsymbol{R}]_\pi$.

Portfolios that fall outside the efficient frontier are considered as sub-optimal because they either carry too much risk relative to $\mathbb{E}[\boldsymbol{R}]$ or too little yield relatively to the volatility.

The optimalization problem can be formulated as follows

$$\begin{aligned} \min UL_\pi^2 \;\Leftrightarrow\; \min \boldsymbol{w}^T V \boldsymbol{w} \\ \sum_{i=1}^{m} w_i = 1 \\ \sum_{i=1}^{m} w_i \mathbb{E}[R_i] = \mathbb{E}[\boldsymbol{R}]_\pi \end{aligned} \tag{2.3.1}$$

Subject to minimizing function we can solve the problem with quadratic programming methods. As output we want to get optimal weights w_i^*. The matrix V is defined

$$V_{ij} = UL_i UL_j \rho_{ij}^D. \tag{2.3.2}$$

In the later analysis it is important to deal with properties of the matrix V. The quadratic optimalization problem has a solution only if $V \in \mathbb{R}^{m \times m}$ is a regular, positive definite matrix $\boldsymbol{w}^T V \boldsymbol{w} \succ 0$ for $\forall \boldsymbol{w} \in \mathbb{R}^m$.

The following statements are equivalent:

1. The symmetric matrix V is positive definite.
2. All its eigenvalues are positive.
3. It has a unique Cholesky decomposition, i.e. exists nonsingular square matrix L such that $V = L^T L$. (See Subchapter 3.1 *Asset Value Modeling.*)
4. All the leading principal minors of V are positive.

Remark that *a covariance matrix is positive definite.*

Proof.
In the next steps we want to show that V is a covariance matrix. We use relations (2.1.3, 2.2.10 and 2.2.9) sequentially operated.

$$\begin{aligned} V_{ij} &= LGD \times \sqrt{EDF_i \times \big(1 - EDF_i\big)} \times LGD \sqrt{EDF_j \times \big(1 - EDF_j\big)} * \\ &\quad * \frac{JDF_{ij} - EDF_i \times EDF_j}{\sqrt{EDF_i \times \big(1 - EDF_i\big) \times EDF_j \times \big(1 - EDF_j\big)}} \\ &= LGD^2 \times \Big(JDF_{ij} - EDF_i \times EDF_j\Big) \\ &= LGD^2 \times \operatorname{cov}\Big(FIRM_i, FIRM_j\Big), \end{aligned} \tag{2.3.3}$$

whereby LGD is a constant. $FIRM_1$ and $FIRM_j$ are dummies random variables (see Subchapter 2.2 *Default Correlations*) having values 1 if firm i or j defaults or 0 if it does not.

We have shown $V \succ 0$.

□

Based on our experience, the matrix V contains extremely small values. In such a situation the optimization algorithm does not need to converge. In practice it is reasonable to multiply V with a large constant. We say without a proof that optimal weights $\boldsymbol{w}^*$ are invariant to this multiplication.

Chapter 3

Asset Value and Bond Price Modeling

Firstly, we extend necessary theory about the firm's asset value from Subchapter 1.1.1. We begin with one-dimensional problem that upgrades to multi-dimensional system of stochastic equations. Secondly, we introduce useful theory for bond price modeling.

3.1 Asset Value Modeling

Let us start with list of these three assumptions:

1. Firm's asset value $V(t)$ follows GBM (analogous to stock price).
2. $W = \{W_t, t \in [0,T]\}$ is a $\mathcal{F}_t$-Wiener process.
3. $\{\mathcal{F}_t, t \in [0,T]\}$ is a corresponding filtration.

Under below fulfilled conditions of finite integrals

1. $\int_0^t \mu_V(t)\,\mathrm{d}t < \infty$,
2. $\int_0^t \sigma_V(t)\,\mathrm{d}t < \infty$,
3. $\int_0^t \big|V(t)\mu_V(t)\big|\,\mathrm{d}t < \infty$,
4. $\int_0^t V^2(t)\sigma_V^2(t)\,\mathrm{d}t < \infty$,

where $\mu_V(t)$ is drift and $\sigma_V(t)$ is volatility of the firm's asset value $V(t)$. Then we model $V(t)$ as GBM

$$\mathrm{d}V(t) = \mu_V(t)V(t)\,\mathrm{d}t + \sigma_V(t)V(t)\,\mathrm{d}W_t. \tag{3.1.1}$$

If $V(t) \neq 0$ we can form eq. (3.1.1)

$$\frac{\mathrm{d}V(t)}{V(t)} = \mu_V(t)\,\mathrm{d}t + \sigma_V(t)\,\mathrm{d}W_t. \tag{3.1.2}$$

Now, we use Itô's lemma on substituted increasing function $f(t,x) = \log(x)$

$$\begin{aligned}
\mathrm{d}\log\big(V(t)\big) &= \frac{1}{V(t)}\,\mathrm{d}V(t) - \frac{1}{2V^2(t)}\sigma_V^2(t)V^2(t)\,\mathrm{d}t \\
&= \frac{1}{V(t)}\Big(\mu_V(t)V(t)\,\mathrm{d}t + \sigma_V(t)V(t)\,\mathrm{d}W_t\Big) - \frac{1}{2V^2(t)}\sigma_V^2(t)V^2(t)\,\mathrm{d}t \\
&= \left(\mu_V(t) - \frac{1}{2}\sigma_V^2(t)\right)\mathrm{d}t + \sigma_V(t)\,\mathrm{d}W_t
\end{aligned} \tag{3.1.3}$$

and after integration on the interval $[0,t]$ we get

$$\log\big(V(t)\big) = \log\big(V(0)\big) + \int_0^t \left(\mu_V(s) - \frac{1}{2}\sigma_V^2(s)\right)\mathrm{d}s + \int_0^t \sigma_V(s)\,\mathrm{d}W_s. \tag{3.1.4}$$

In consequence, the firm's asset value formula is the solution of (3.1.4)

$$V(t) = V(0)\exp\left\{ \int_0^t \left(\mu_V(s) - \frac{1}{2}\sigma_V^2(s) \right) \mathrm{d}s + \int_0^t \sigma_V(s)\, \mathrm{d}W_s \right\}. \tag{3.1.5}$$

If we assume that μ_V and σ_V of firm's asset are constant in time t, the modeling process can be expressed by simplified eq.

$$V(t) = V(0)\exp\left\{ \left(\mu_V - \frac{1}{2}\sigma_V^2 \right) t + \sigma_V W_t \right\}. \tag{3.1.6}$$

In practice, we model Wiener process using random variable with standard normal distribution $\varepsilon \sim \mathcal{N}(0,1)$

$$V(t) = V(0)\exp\left\{ \left(\mu_V - \frac{1}{2}\sigma_V^2 \right) t + \sigma_V \sqrt{t}\varepsilon \right\}. \tag{3.1.7}$$

We know that $V(t)$ has a log-normal distribution, i.e.

$$\log\big(V(t)\big) \sim \mathcal{N}\left(\log\big(V(0)\big) + \left(\mu_V - \frac{1}{2}\sigma_V^2 \right) t, \sigma_V^2 t \right). \tag{3.1.8}$$

Let us upgrade already introduced **one-asset-dimensional model into a multi-asset-dimensional.**

Henceforward, our portfolio contains m assets. Each value of particular asset moreover depends on correlations with other assets in the portfolio. Let us describe the model how to measure it. It is an essential question we focus on in this part of the thesis. According to eq. (3.1.1) we define the stochastic differential process of firm's asset in a matrix form

$$\begin{pmatrix} \mathrm{d}V_1(t)/V_1(t) \\ \mathrm{d}V_2(t)/V_2(t) \\ \vdots \\ \mathrm{d}V_m(t)/V_m(t) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_1(t) \\ \mathrm{d}W_2(t) \\ \vdots \\ \mathrm{d}W_m(t) \end{pmatrix} \tag{3.1.9}$$

or in an abbreviated form

$$\big(\mathrm{d}\boldsymbol{V}(t)\big)./\big(\boldsymbol{V}(t)\big) = \boldsymbol{\mu}_V\, \mathrm{d}t + \boldsymbol{\sigma}_V\, \mathrm{d}\boldsymbol{W}_t, \tag{3.1.10}$$

where $\boldsymbol{V}(t)$ is a vector of asset values at time t, $\boldsymbol{\mu}_V$ is a vector of drifts, $\boldsymbol{\sigma}$ is volality matrix, $\boldsymbol{W}_t$ is a stochastic vector of m independent $\mathcal{F}_t$-Wiener processes and $t \in [0,T]$. We express $\boldsymbol{\mu}_V$ using expected value as $\mathbb{E}\big[R_{V_i}\big] = \mu_{V_i}, i = 1,\dots,m$ where R_{V_i} is return on asset value. The covariance matrix $\boldsymbol{\sigma}_V\boldsymbol{\sigma}_V^T$ features of m^2 elements

$$\begin{aligned} \boldsymbol{\sigma}_V\boldsymbol{\sigma}_V^T &= \begin{pmatrix} \mathrm{var}\big[R_{V_1}\big] & \mathrm{cov}\big(R_{V_1}, R_{V_2}\big) & \cdots & \mathrm{cov}\big(R_{V_1}, R_{V_m}\big) \\ \mathrm{cov}\big(R_{V_2}, R_{V_1}\big) & \mathrm{var}\big[R_{V_2}\big] & \cdots & \mathrm{cov}\big(R_{V_2}, R_{V_m}\big) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}\big(R_{V_m}, R_{V_1}\big) & \mathrm{cov}\big(R_{V_m}, R_{V_2}\big) & \cdots & \mathrm{var}\big[R_{V_m}\big] \end{pmatrix} \\ &= \mathbf{vol}^T \boldsymbol{\rho} \mathbf{vol} \\ &= \mathbf{vol}^T \boldsymbol{L}\boldsymbol{U} \mathbf{vol} \\ &= \mathbf{vol}^T \boldsymbol{L}\boldsymbol{L}^T \mathbf{vol} \end{aligned} \tag{3.1.11}$$

and $\mathbf{vol}_{m\times 1}$ is a volatility vector. $\boldsymbol{\rho}$ is the correlation matrix between pairwise asset returns R_{V_i} and R_{V_j}. Similarly as the covariance matrix it is a positive definite matrix. It has a unit diagonal. Having matrix with this properties, we can perform Cholesky decomposition to find an unique lower triangular matrix $\boldsymbol{L}$ with strictly positive diagonal elements. See general example of both matrixes.

$$\boldsymbol{\rho} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1m} \\ \rho_{21} & 1 & \cdots & \rho_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m1} & \rho_{m2} & \cdots & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ l_{m1} & l_{m2} & \cdots & l_{mm} \end{pmatrix} \tag{3.1.12}$$

A multivariate GBM is an example of the asymptotic multivariate m-dimensional normal distribution of portfolio asset returns with parameters

$$\boldsymbol{R}_V \sim \mathcal{N}_m\Big(\boldsymbol{\mu}_V \Delta t, \boldsymbol{\sigma}_V \boldsymbol{\sigma}_V^T \Delta t\Big). \tag{3.1.13}$$

Finally, we can determine the formula for the simulation of correlated asset value

$$V_i(t) = V_i(0)\exp\bigg\{\Big(\mu_{V_i} - \frac{1}{2}\text{vol}_i^2\Big)t + \sigma_{i\star}\boldsymbol{W}_t\bigg\}, \tag{3.1.14}$$

where $\text{vol}_i^2 = \sum_j \sigma_{ij}^2$ and labeling $\sigma_{i\star}$ denote whole i row of the matrix σ.

As we see, *covariance* and *correlation* matrixes are important inputs to the introduced theory and which are key components to the simulation procedure. A specific matrix generating process and a simulation algorithm is described in the Chapter 4 *Applied Credit Risk Modeling*.

3.2 Bond Price Modeling

Since we measure credit risk of bond portoflio there is necessity remind bond pricing formulas.

In general the bond is a type of obligation. There are some parameters to determine the bond value. PV is a present value or market value, F is a face value, C_{t_i} a coupon payment in time period t_i, $i = 1, \dots, n$ and n is a number of time periods till maturity and y is a yield to maturity, abbr. YTM. Every bond is characterized by flow of future payments (cash flows) $CF_{t_i} = C_{t_i}$ if $t_i < T$ and $CF_T = C_T + F$. All future payments depend on future IR, future inflation, etc. However, it is not a purpose of this thesis to model also bonds with floating IR and its dependent coupon rates. The bond with coupon payments has present value defined as

$$PV = \sum_{i=1}^{n} \frac{CF_{t_i}}{B_{t_i}}, \tag{3.2.1}$$

where B_{t_i} is continuous discount

$$\begin{aligned} B_{t_i} &= \exp\{r_{t_i} t_i\} \\ &= \exp\bigg\{\int_0^{t_i} f_s \,\mathrm{d}s\bigg\}, \end{aligned} \tag{3.2.2}$$

where f_s is risk-free forward IR. r_{t_i} is IR valid only in $(t_i, t_{i+1}]$ and CF_{t_i} is a cash flow at time t_i. Assume the first case of a single cash flow. Let us start with an example according to the article published in *Journal of Banking and Finance* [9]. Derive value of a zero coupon bond with a promised payment EUR 100 in 1Y with a general RR (2.1.2) if the issuer defaults. A calculation of the single cash flow issue can be performed in 3 steps:

1. The risk-free component $RR \times$EUR 100 $= (1 - LGD) \times$EUR 100 is valued using the default-free discount, i.e.

$$\begin{aligned} PV_{\text{risk-free}} &= PV(\text{risk-free flow}) \\ &= \frac{1 - LGD}{B_{1Y}} \times \text{EUR } 100, \end{aligned} \tag{3.2.3}$$

   where r_{1Y} denotes annual risk-free IR.

2. The risky component is valued using *The Martingale approach*, i.e.

$$\begin{aligned} PV_{\text{risky}} &= PV(\text{risky cash flow}) \\ &= \mathbb{E}_Q(\text{discounted risky cash flow}) \\ &= \frac{\text{EUR } 100 \times \Big(1 - PD^Q\Big) + 0 \times PD^Q}{B_{1Y}} \times LGD \\ &= \frac{1 - PD^Q}{B_{1Y}} \times LGD \times \text{EUR } 100, \end{aligned} \tag{3.2.4}$$

   where the expected value is calculated using the risk neutral probability PD^Q.

3. The PV of zero coupon bond subject to default credit risk follows the sum of default-free component and the risky component, i.e.

$$PV = PV_{\text{risk-free}} + PV_{\text{risky}}. \tag{3.2.5}$$

Graphic illustration of example is in Figure (3.1).

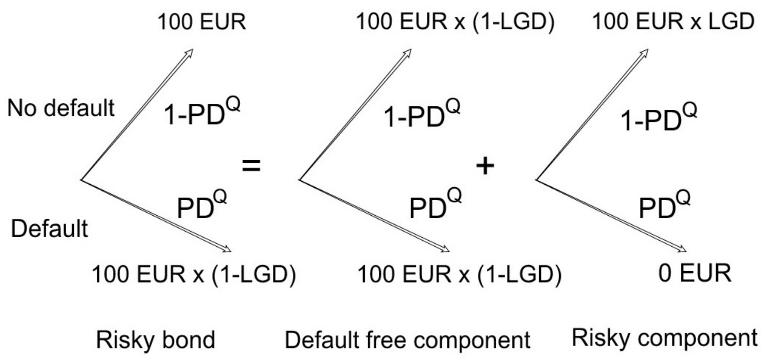

Fig. 3.1: Valuation of a single cash flow bond subject to default risk using the risk neutral probability PD^Q.

The implicit discount rate d which accounts for default risk we get as sum

$$d = r + s, \tag{3.2.6}$$

where s is specified credit spread and r is risk-free IR. By solving the following eq.

$$\frac{1}{1+r+s} = \frac{1-LGD}{1+r} + \frac{LGD \times \big(1 - PD^Q\big)}{1+r} \tag{3.2.7}$$

we get a new explicit formula for bond's credit spread. It can be derived using risk neutral measure Q by formula

$$s = \frac{LGD \times PD^Q \times (1+r)}{1 - LGD \times PD^Q} \tag{3.2.8}$$

We can use the Martingale approach to build generalized pricing model for a bond subject to default risk

$$\begin{aligned} PV &= \mathbb{E}_Q\big[\text{discounted cash flow}\big] \\ &= \sum_{i=1}^{n} \frac{1}{B_{t_i}} \times \mathbb{E}_Q\big[\text{cash flow}_{t_i}\big] \\ &= (1-LGD) \times \sum_{i=1}^{n} \frac{CF_{t_i}}{B_{t_i}} + LGD \times \sum_{i=1}^{n} \frac{CF_{t_i} \times \big(1 - PD_{t_i}^Q\big)}{B_{t_i}}, \end{aligned} \tag{3.2.9}$$

where $CF_{t_1}, CF_{t_2}, \dots, CF_{t_n}$ are specific cash flows and $PD_{t_i}^Q$ denotes the cumulative risk neutral EDF at the specific time horizon. The last unknown parameter is the risk neutral probability PD^Q.

According to [9] we define:

Definition 3.2.1.
The risk neutral PD^Q is the Probability of Default when the value of the firm's assets falls below the Default Point, abbr. DP at time T under the modified risk neutral process for the firm's assets value V_t^Q. It is analogous to (1.1.18)

$$PD_T^Q = \mathcal{N}\Big(-d_2^Q\Big). \tag{3.2.10}$$

Proof.

$$
\begin{aligned}
PD_T^Q &= \mathbb{P}\Big[V_T^Q < B\Big] \\
&= \mathbb{P}\Big[V_T^Q < DP_T\Big] \\
&= \mathbb{P}\bigg[\log\big(V_0\big) + \Big(r_T - \frac{1}{2}\sigma_V^2\Big)T + \sigma_V\sqrt{T}\varepsilon_T < \log\big(DP_T\big)\bigg] \\
&= \mathbb{P}\Bigg[\varepsilon_T < -\frac{\log\Big(\frac{V_0}{DP_T}\Big) + \Big(r_T - \frac{1}{2}\sigma_V^2\Big)T}{\sigma_V\sqrt{T}}\Bigg] \\
&= \mathcal{N}\Big(-d_2^Q\Big),
\end{aligned}
\tag{3.2.11}
$$

where $\sqrt{T}\varepsilon = W_T - W_0$ is normally distributed with zero mean and variance equal to T. □

We already have shown definition (1.1.18)

$$EDF_T = \mathcal{N}\big(-d_2\big), \tag{3.2.12}$$

where d_2 has already been defined (1.1.13). Only difference between d_2^Q and d_2 is usage of risk-free IR r_t instead of μ_V. Since

$$-d_2^Q = -d_2 + \frac{\mu_V - r_t}{\sigma_V}\sqrt{T}, \tag{3.2.13}$$

we come to the following relationship of *The Cumulative Risk Neutral* EDF^Q or PD^Q at horizon T. It is expressed as

$$PD_T^Q = \mathcal{N}\Big[\mathcal{N}^{-1}(EDF) + \frac{\mu_V - r_t}{\sigma_V}\sqrt{T}\Big]. \tag{3.2.14}$$

We distinguish two cases:

1. $\mu_V \geq r_t \;\Rightarrow\; PD_T^Q \geq EDF_T$
2. $\mu_V \leq r_t \;\Rightarrow\; PD_T^Q \leq EDF_T$.

In the first case the risk neutral probability of default after risk price adjustment is higher than actual probability of default. In the second case it is lower. We can get the difference $\mu_V - r_t$ from *The Capital Asset Pricing Model*, abbr. CAPM. It is a well known financial model that describes the relationship between systematic risk and expected return for firm's assets

$$\mu_V - r_t = \beta\pi. \tag{3.2.15}$$

Parameters are:

1. β is *Beta* of particular asset

$$
\begin{aligned}
\beta &= \frac{\operatorname{cov}\big(R_V, R_M\big)}{\operatorname{var}\big(R_M\big)} \\
&= \rho\frac{\sigma_V}{\sigma_M},
\end{aligned}
\tag{3.2.16}
$$

   where R_V and R_M denote return of the firm's asset and the market portfolio respectively. σ_V and σ_M are the volatility of the asset return and the market return, respectively. Paramter ρ is the correlation between returns.

2. π is *The Market Risk Premium* for a unit of *Beta risk*, i.e.

$$\mu_M - r_t \tag{3.2.17}$$

   where μ_M and μ_V denote the expected return on the market portfolio and on the firm's assets value respectively and r_t is the risk-free IR.

We can rewrite eq. (3.2.14)

$$\begin{aligned} PD_T^Q &= \mathcal{N}\left[\mathcal{N}^{-1}\Big(EDF_T\Big) + \frac{\beta\pi}{\sigma_V}\sqrt{T}\right] \\ &= \mathcal{N}\left[\mathcal{N}^{-1}\Big(EDF_T\Big) + \rho\frac{\pi}{\sigma_M}\sqrt{T}\right], \end{aligned} \tag{3.2.18}$$

where

$$\frac{\pi}{\sigma_M} \tag{3.2.19}$$

is called *Market Sharpe Ratio.*

As we will see later using correct IR is key entry in the pricing process. More instructions how to compute and apply risk-free IR r_{t_i} in the next Chapter 4 *Applied Credit Risk Modeling.*

Chapter 4

Applied Credit Risk Modeling

We have decided to use all theoretical tools above to measure credit default risk on real data portfolio.

4.1 Data

Data preparation is splitted into several parts:

1. **Selected Bond Portfolio**

   The selected portfolio is homogeneous. Homogeneous portfolio implies that all selected corporate bonds are from the same business sector. In this thesis we select 10 financial institutions. All bonds have maturity 10 years which are issued by the largest European banks according to market capitalization in the particular countries (information dated December 2016). These banks have their headquarters in Germany, United Kingdom, Italy and France. Selected list of bonds are in the Table (4.1). The Table also provides information on bond's issue volume, annual coupon rate, issue date and maturity. The Table does not provide qualitative characteristics which are ratings published on the certain date by Moody's agency. Rating is a variable that changes over time. The bond's rating may vary from the rating of the issuer itself.

Firm(1)	**Country**	**Issue Volume**(2)	**Annual Coupon Rate**(3)	**Issue Date**	**Maturity**
Deutsche Bank	DE	500.0	1.750	06/08/2012	06/08/2022
Commerzbank	DE	1,250	7.750	03/16/2011	03/16/2021
Royal Bank of Scotland	UK	1,000	5.500	03/23/2010	03/23/2020
Barclays	UK	20.00	5.600	03/17/2011	03/17/2021
HSBC Holdings	UK	1,750	6.000	06/10/2009	06/10/2019
UniCredit Banca	IT	750.0	6.125	04/19/2011	04/19/2021
Banco Intesa Sanpaolo	IT	1,500	6.000	01/27/2011	01/27/2021
BNP Paribas	FR	2,050	3.750	11/25/2010	11/25/2020
Société Générale	FR	5.000	3.830	12/06/2010	12/06/2020
Crédit Agricole	FR	50.00	4.530	04/16/2012	04/16/2022

(1) Specific information from following web pages:

- http://markets.businessinsider.com/bonds
- http://finance.yahoo.com/bonds
- http://www.finanzen.ch/obligationen

(2) In milion.
(3) In percent.

Tab. 4.1: Selected portfolio of banks.

2. **Firm's Stock Information**

   (a) Stock prices in from https://finance.yahoo.com/. Example of possible tickers are DBK.F, CBK.F, RYS1.F, BCY.F, HBC1.F, UCG.MI, IES.F, BNP.F, SGE.F and XCA.F.

   (b) Historical outstanding shares which refer to firm's stock currently held by all its shareholders. It is also necessary to mark all historical stock splits.

   (c) Historical total liabilities. The amount of debt is usually referred to banks on a quarterly basis.

Historical data of total liabilities and oustanding shares are from web page: https://ycharts.com/.

3. **The Risk-Free Interest Rate**

   Risk-free IR r_{t_i} is key to estimate PD^Q. It is necessary to compute risk-free IR term structure annually. It is a zero-coupon default-free IR of corresponding government bond with different maturities. For selected portfolio we have risk-free IR term structures for Euro-zone and United Kingdom.

4. **LGD**

   Moody's Investor Service regularly publishes research articles indicating the number of defaults and corresponding recovery rates RR for financial and non-financial corporate issuers in Europe. Bond's seniority is important when it comes to LGD. In our portfolio we have senior unsecured bonds. Article [10] refers to an average issuer weighted RR for senior unsecured bonds from 1985 to 2016 as 38.39%. In that case LGD is 61.61%. For senior secured bonds LGD value is lower 53.86%. These LGD are estimated over a long period. If we look at other Moody's articles, we see that LGD for senior unsecured bonds only in the last years at around 50%.

   The same value was identified during the expert consultation of the above approach in the National Bank of Slovakia. For a better understanding of the results and comparison we change $LGD \pm 10\%$.

4.2 2009 - 2017

This Subchapter is divided into the following tasks:

1. Key issue is to determine default probability term structures for all bonds during the observed history.
2. An assignment of corresponding credit rating.
3. A computation of fair prices of bond and consecutive comparison with issue prices and historical market prices.
4. Investing in bonds and managing the portfolio using four strategies:
   (a) Strategy 1: Buying all bonds with naive diversification.
   (b) Strategy 2: Buying undervalued bonds with naive diversification.
   (c) Strategy 3: Buying overvalued bonds with naive diversification.
   (d) Strategy 4: Portfolio diversification.

4.2.1 Default Probability Term Structure

As we can see in previous Subchapter 4.2 in 2009 was issued one bond (issuer: HSBC Holding), in 2010 3 more (issuers: Royal Bank of Scotland, BNP Paribas and Société Générale), in 2011 4 more (issuers: Commerzbank, Barclays, UniCredit and Banco Intesa Sanpaolo) and in 2012 last 2 bonds (Deutsche Bank and Credit Agricole). In order to perform valuation of these bonds with respect to default risk, we determine term structures of probability of defaults under risk neutral measure (PD^Q term structures). Recall Example (1.1.2) from Subchapter 1.1.2 and computation of $PD^Q_{1,Y}$ for DBK in the beginning of 2017. Now we want to get values $PD^Q_{2,Y}, \ldots, PD^Q_{10,Y}$.

The Algorithm of Computation the PD^Q Term Structures:

1. We generate matrix of Wiener processes $\boldsymbol{W}_{r\times c}$, where $r = 100\ 000$ simulations and $c = 10$ is number of monitored years. See matrix $\boldsymbol{W}_t$

$$\boldsymbol{W}_t = \begin{pmatrix} W^1_{t_1} & W^1_{t_1} + \left(W^1_{t_2} - W^1_{t_1}\right) & \cdots & W^1_{t_9} + \left(W^1_{t_{10}} - W^1_{t_9}\right) \\ W^2_{t_1} & W^2_{t_1} + \left(W^2_{t_2} - W^2_{t_1}\right) & \cdots & W^2_{t_9} + \left(W^2_{t_{10}} - W^2_{t_9}\right) \\ \vdots & \vdots & \ddots & \vdots \\ W^r_{t_1} & W^r_{t_1} + \left(W^r_{t_2} - W^r_{t_1}\right) & \cdots & W^r_{t_9} + \left(W^r_{t_{10}} - W^r_{t_9}\right) \end{pmatrix}, \tag{4.2.1}$$

where $W^1_{t_1}$ and independent increments $W_{t_i} - W_{t_{i-2}}, i = 2, \dots, 10$ we generate as $\sqrt{\Delta t} \times \mathcal{N}(0, 1)$.

2. We use directly risk-free IR term structures $r_{t_k}, k = 1, \dots, 10$ for Euro-zone and UK from 2009 to 2017.

3. We simulate $V^1_{t_k}, \dots, V^{100\ 000}_{t_k}$ for each year from 2009 to 2017 using

$$V_{t_k} = V_{t_0} \exp\left\{ \left(r_{t_k} - \frac{1}{2}\sigma_V^2 \right) t_k + \sigma_V W_{t_k} \right\}, \tag{4.2.2}$$

where $k = 1, \dots, 10$. V_{t_0} and σ_V are like in the (1.1.2) solution of non-linear system (1.1.29). Inputs are E and σ_E and its estimated from last 3 years. The recommendation is to use the σ_V calculated from the last year.

4. We already know definition of credit default

$$PD^Q_{t_k} = \mathbb{P}\big[V_{t_k} < B\big]. \tag{4.2.3}$$

In this simulation case outputs are made by scoring

$$PD^Q_{t_k} = \frac{\mathbb{1}_{\left\{V^s_{t_k} < B\right\}}}{100\ 000}, \tag{4.2.4}$$

where $s = 1, \dots, 100\ 000$.

Tables of PD^Q term structures are (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8) and the last year 2017 (4.9).

	2009	2010			
	HSBC	**RBS**	**HSBC**	**BNP**	**SG**
1	0.04839	0.00365	0.00725	0.02989	0.05130
2	0.10019	0.01908	0.02225	0.05369	0.08064
3	0.13698	0.03785	0.03588	0.06842	0.09559
4	0.16436	0.05550	0.04652	0.07752	0.10478
5	0.18321	0.07023	0.05431	0.08295	0.10989
6	0.19803	0.08314	0.06003	0.08647	0.11297
7	0.20951	0.09397	0.06438	0.08889	0.11497
8	0.21834	0.10312	0.06770	0.09062	0.11628
9	0.22522	0.11063	0.07061	0.09170	0.11719
10	0.23147	0.11773	0.07273	0.09254	0.11771

Tab. 4.2: PD^Q term structures in 2010 and 2011.

Y	CBK	RBS	BCY	HSBC	UNI	IS	BNP	SG
1	**0.03374**	**0.02910**	**0.02604**	**0.00002**	**0.01204**	**0.01522**	**0.00867**	**0.02279**
2	0.14821	0.09942	0.03932	0.00062	0.02959	0.02979	0.03174	0.02605
3	0.16656	0.13348	0.04575	0.00175	0.03305	0.03893	0.04430	0.02784
4	0.17330	0.13958	0.04851	0.00289	0.03398	0.04437	0.05192	0.02853
5	0.17603	0.14049	0.05006	0.00367	0.03416	0.04799	0.05652	0.02877
6	0.17732	0.14057	0.05093	0.00451	0.03426	0.04994	0.05942	0.02890
7	0.17788	0.14058	0.05130	0.00508	0.03431	0.05126	0.06126	0.02899
8	0.17821	0.14058	0.05157	0.00563	0.03432	0.05214	0.06252	0.02900
9	0.17831	0.14058	0.05170	0.00601	0.03432	0.05268	0.06340	0.02901
10	0.17839	0.14058	0.05179	0.00629	0.03432	0.05310	0.06403	0.02901

Tab. 4.3: PD^Q term structures in 2011.

Y	DBK	CBK	RBS	BCY	HSBC	UNI	IS	BNP	SG	CA
1	**0.02710**	**0.06521**	**0.00547**	**0.00070**	**0.00019**	**0.06552**	**0.08199**	**0.04943**	**0.04167**	**0.03309**
2	0.07428	0.12446	0.03508	0.00485	0.00298	0.08801	0.13761	0.05536	0.07922	0.05401
3	0.11539	0.16422	0.07279	0.01030	0.00793	0.09696	0.15111	0.05852	0.09818	0.06549
4	0.14941	0.19255	0.10170	0.01517	0.01296	0.10105	0.15279	0.05996	0.10925	0.07191
5	0.17573	0.21182	0.12041	0.01981	0.01861	0.10325	0.15296	0.06072	0.11575	0.07566
6	0.19726	0.22728	0.13433	0.02313	0.02297	0.10418	0.15296	0.06111	0.11988	0.07765
7	0.21479	0.23842	0.14234	0.02587	0.02671	0.10472	0.15296	0.06127	0.12262	0.07891
8	0.22856	0.24727	0.14741	0.02817	0.03006	0.10503	0.15296	0.06132	0.12454	0.07969
9	0.24048	0.25391	0.15105	0.02998	0.03289	0.10521	0.15296	0.06137	0.12587	0.08026
10	0.25103	0.26013	0.15352	0.03152	0.03555	0.10528	0.15296	0.06142	0.12692	0.08065

Tab. 4.4: PD^Q term structures in 2012.

Y	DBK	CBK	RBS	BCY	HSBC	UNI	IS	BNP	SG	CA
1	**0.02670**	**0.03011**	**0.01093**	**0.01472**	**0.00160**	**0.05681**	**0.05974**	**0.00658**	**0.02653**	**0.03717**
2	0.08907	0.05421	0.03511	0.03617	0.01054	0.08269	0.14866	0.03403	0.07743	0.04637
3	0.11590	0.07915	0.05763	0.05308	0.02203	0.11319	0.19673	0.04644	0.08800	0.05675
4	0.11991	0.09996	0.07667	0.06523	0.03326	0.13984	0.21245	0.04731	0.08812	0.06497
5	0.12018	0.11538	0.09079	0.07365	0.04317	0.16079	0.21523	0.04734	0.08812	0.07090
6	0.12018	0.12796	0.10274	0.07968	0.05134	0.17809	0.21562	0.04734	0.08812	0.07516
7	0.12018	0.13804	0.11240	0.08432	0.05857	0.19254	0.21564	0.04734	0.08812	0.07831
8	0.12018	0.14631	0.12007	0.08751	0.06440	0.20406	0.21565	0.04734	0.08812	0.08073
9	0.12018	0.15280	0.12599	0.09033	0.06971	0.21389	0.21565	0.04734	0.08812	0.08259
10	0.12018	0.15860	0.13125	0.09236	0.07428	0.22301	0.21565	0.04734	0.08812	0.08413

Tab. 4.5: PD^Q term structures in 2013.

Y	DBK	CBK	RBS	BCY	HSBC	UNI	IS	BNP	SG	CA
1	**0.01581**	**0.02989**	**0.00870**	**0.00141**	**0.00009**	**0.01714**	**0.01263**	**0.00215**	**0.00470**	**0.00033**
2	0.03639	0.04435	0.03165	0.00735	0.00181	0.10315	0.04383	0.01741	0.01491	0.00386
3	0.05785	0.05873	0.05510	0.01413	0.00561	0.18612	0.06004	0.03995	0.02765	0.00988
4	0.07612	0.06979	0.07616	0.02018	0.00991	0.22801	0.06393	0.06422	0.03960	0.01648
5	0.08993	0.07770	0.09244	0.02540	0.01460	0.23848	0.06453	0.08604	0.04959	0.02313
6	0.10159	0.08345	0.10628	0.02915	0.01852	0.24016	0.06456	0.10708	0.05755	0.02866
7	0.11089	0.08783	0.11770	0.03200	0.02212	0.24022	0.06457	0.12535	0.06468	0.03336
8	0.11845	0.09098	0.12742	0.03439	0.02528	0.24022	0.06457	0.14126	0.07022	0.03752
9	0.12441	0.09367	0.13517	0.03616	0.02801	0.24022	0.06457	0.15569	0.07526	0.04108
10	0.12968	0.09569	0.14203	0.03773	0.03041	0.24022	0.06457	0.16911	0.07944	0.04434

Tab. 4.6: PD^Q term structures in 2014.

Y	DBK	CBK	RBS	BCY	HSBC	UNI	IS	BNP	SG	CA
1	**0.01881**	**0.00610**	**0.00543**	**0.00166**	**0.00011**	**0.03962**	**0.01395**	**0.00109**	**0.00103**	**0.01335**
2	0.10392	0.05031	0.07866	0.05648	0.01680	0.04551	0.03500	0.02108	0.02226	0.09153
3	0.18585	0.10676	0.09762	0.06626	0.02228	0.05597	0.06175	0.05700	0.06015	0.17045
4	0.22985	0.14848	0.10177	0.06722	0.02317	0.06622	0.08733	0.08618	0.08781	0.21143
5	0.24283	0.16712	0.10247	0.06728	0.02328	0.07487	0.10844	0.09806	0.09689	0.22123
6	0.24540	0.17386	0.10271	0.06729	0.02332	0.08182	0.12736	0.10175	0.09879	0.22273
7	0.24560	0.17556	0.10278	0.06730	0.02334	0.08794	0.14325	0.10239	0.09897	0.22277
8	0.24561	0.17586	0.10278	0.06731	0.02334	0.09272	0.15643	0.10247	0.09898	0.22277
9	0.24561	0.17587	0.10278	0.06731	0.02334	0.09694	0.16792	0.10247	0.09898	0.22277
10	0.24561	0.17587	0.10278	0.06731	0.02334	0.10049	0.17842	0.10247	0.09898	0.22277

Tab. 4.7: PD^Q term structures in 2015.

Y	DBK	CBK	RBS	BCY	HSBC	UNI	IS	BNP	SG	CA
1	**0.03058**	**0.02117**	**0.02160**	**0.00222**	**0.00100**	**0.02989**	**0.02475**	**0.00118**	**0.00125**	**0.00755**
2	0.04234	0.03373	0.06860	0.01608	0.01045	0.04456	0.04497	0.02662	0.03067	0.08133
3	0.06101	0.05649	0.09679	0.02578	0.01896	0.06620	0.07062	0.07312	0.08274	0.17044
4	0.07897	0.08107	0.10998	0.02993	0.02341	0.08767	0.09442	0.10812	0.11857	0.21579
5	0.09381	0.10287	0.11299	0.03061	0.02447	0.10523	0.11364	0.12271	0.13043	0.22743
6	0.10697	0.12331	0.11430	0.03086	0.02492	0.12067	0.13067	0.12761	0.13332	0.22954
7	0.11811	0.14101	0.11451	0.03090	0.02496	0.13361	0.14459	0.12848	0.13364	0.22965
8	0.12757	0.15622	0.11460	0.03091	0.02496	0.14473	0.15630	0.12861	0.13368	0.22966
9	0.13527	0.16996	0.11461	0.03091	0.02496	0.15416	0.16595	0.12864	0.13368	0.22966
10	0.14234	0.18268	0.11461	0.03091	0.02496	0.16251	0.17461	0.12864	0.13368	0.22966

Tab. 4.8: PD^Q term structures in 2016.

Y	DBK	CBK	RBS	BCY	HSBC	UNI	IS	BNP	SG	CA
1	**0.04776**	**0.00803**	**0.04317**	**0.01724**	**0.00012**	**0.04338**	**0.02862**	**0.00136**	**0.00143**	**0.00244**
2	0.05319	0.05818	0.11536	0.06158	0.00407	0.15644	0.11749	0.03046	0.03456	0.06659
3	0.07299	0.10700	0.16577	0.09477	0.01182	0.22965	0.18736	0.05954	0.06432	0.09886
4	0.10171	0.15100	0.18853	0.10753	0.01721	0.28232	0.24103	0.08655	0.08998	0.12088
5	0.13155	0.18753	0.19953	0.11287	0.02122	0.32173	0.28249	0.10802	0.10984	0.13539
6	0.16189	0.21838	0.20285	0.11380	0.02244	0.35293	0.31683	0.12700	0.12730	0.14617
7	0.19081	0.24439	0.20440	0.11410	0.02296	0.37826	0.34437	0.14309	0.14141	0.15409
8	0.21674	0.26621	0.20495	0.11419	0.02315	0.39913	0.36774	0.15639	0.15294	0.16034
9	0.24195	0.28631	0.20505	0.11420	0.02320	0.41809	0.38884	0.16787	0.16255	0.16476
10	0.26555	0.30394	0.20510	0.11421	0.02321	0.43322	0.40623	0.17845	0.17101	0.16862

Tab. 4.9: PD^Q term structures in 2017.

PD^Q term structure is logically increasing. If default occurs in the first year, in remaining nine years bank stay in default. We also tried to predict bank's debt using risk-free IR as following eq.

$$B_{t_k} = B_{t_0} \exp\{r_{t_k} t_k\} \tag{4.2.5}$$

but results are not satisfactory. In fact, we need to be aware of the 'rolling over debt phenomena'. Financial institution refinances risk constantly. One recommendation and perhaps a better approach is to model bank's liabilities stochastically.

4.2.2 Rating

The purpose of representation of PD^Q with corresponding ratings in the Table (4.12) is not to compare results with Moody's rating agency, evaluate matches and to conclude whether we are successful or not. That is impossible. Our recommendation is to take our rating as a result of the Merton model and algorithm described in the previous Subchapter 4.2.1. Not as reference ratings. In the Table4.10 we refer to article of author Xiaoming Tong: *Modeling Banks' Probability of Default* [13]. Be aware of the fact that ratings are estimated on US bank portfolio.

Moody's Rating	**Aaa**	**Aa1**	**Aa2**	**Aa3**	**A1**	**A2**	**A3**	**Baa1**	**Baa2**	**Baa3**	**Ba1**
Min(1)	0	3	5	6	8	10	16	23	37	61	84
Max(1)	3	5	6	8	10	16	23	37	61	84	110
Moody's Rating	**Ba2**	**Ba3**	**B1**	**B2**	**B3**	**Caa1**	**Caa2**	**Caa3**	**Ca**	**C**	**D**
Min	110	130	160	210	280	340	400	630	1400	3100	6700
Max	130	160	210	280	340	400	630	1400	3100	6700	10000

(1) In basis points (bps).

Tab. 4.10: Bank's 1Y PD values and corresponding ratings.

Moody's Rating (1)	**Credit Quality**
Aaa	Highest quality and ability to repay debt.
Aa	High quality and very strong ability to meet its financial commitments.
A	Upper medium grade of strong ability to repay debt.
Baa	Medium grade of adequate capacity to meet its financial commitments. Not investment grade
Ba	Lower medium grade. Somewhat speculative with risk exposure.
B	Low grade. Speculative with risk exposure.
Caa	Poor quality. An obligor is currently vulnerable.
Ca	Most speculative. An obligor is highly-vulnerable.
C	No interest being paid or bankruptcy petition filed.
D	In default.

(1) Moody's ratings from **Aa** to **Ca** may be modified by addition of **1, 2** or **3** to show relative standing within the rating category.

Tab. 4.11: Bond credit quality rating description by Moody's agency.

	2009	2010	2011	2012	2013	2014	2015	2016	2017
				271	267	158	188	306	478 (1)
DBK				**B2**	**B2**	**Ba3**	**B1**	**B3**	**Caa2** (2)
				A2	**A2**	**A3**	**A3**	**Baa2**	**Baa2** (3)
			337	625	301	299	61	212	80
CBK			**B3**	**Caa3**	**B3**	**B3**	**Baa3**	**B2**	**Baa3**
			A2	**A3**	**Baa1**	**Baa1**	**Baa1**	**Baa1**	**Baa1**
		37	291	55	109	87	54	216	431
RBS		**Baa1**	**B3**	**Baa2**	**Ba1**	**Ba1**	**Baa2**	**B2**	**Caa2**
		A2	**A2**	**A3**	**A3**	**Baa1**	**A3**	**A3**	**Ba1**
			260	7	147	14	17	22	172
BCY			**B2**	**Aa3**	**Ba3**	**A2**	**A3**	**A3**	**B1**
			Aa3	**Aa3**	**A2**	**A2**	**A2**	**A2**	**A2**
	484	73	0	2	16	1	1	10	1
HSBC	**Caa2**	**Baa3**	**Aaa**	**Aaa**	**A2**	**Aaa**	**Aaa**	**A1**	**Aaa**
	Aa3	**Aa2**	**Aa2**	**Aa2**	**Aa3**	**Aa3**	**A1**	**A1**	**A1**
			120	655	568	171	396	299	439
UNI			**Ba2**	**Caa3**	**Caa2**	**B1**	**Caa1**	**B3**	**Caa2**
			A2	**A3**	**A3**	**Baa1**	**Baa1**	**Baa1**	**Baa1**
			152	820	597	126	140	248	286
IS			**Ba3**	**Caa3**	**Caa2**	**Ba2**	**Baa3**	**B2**	**B3**
			A2	**A3**	**Baa2**	**Baa2**	**Baa2**	**Baa1**	**Baa1**
		299	87	494	66	22	11	12	14
BNP		**B3**	**Ba1**	**Caa2**	**Baa3**	**A3**	**A2**	**A2**	**A2**
		Aa2	**Aa2**	**A2**	**A2**	**A1**	**A1**	**A1**	**A1**
		513	228	417	265	47	10	13	14
SG		**Caa2**	**B2**	**Caa2**	**B3**	**Baa2**	**A1**	**A2**	**A2**
		Aa3	**A1**	**A2**	**A2**	**A2**	**A2**	**A2**	**A2**
				331	371	3	136	75	24
CA				**B3**	**Caa1**	**Aaa**	**Ba3**	**A1**	**Baa1**
				A2	**A2**	**A2**	**A2**	**A1**	**A1**

(1) In basis points (bps).
(2) Our rating according Xiaoming Tong's transition table.
(3) Moody's Rating of bank's senior unsecured debt https://www.moodys.com/.

Tab. 4.12: Our rating and Moody's rating.

The second effective way to check the results is the following text. More details can be founded in modeling methodology article [11]. Simplified expression of (3.2.8) is economic intuitive relation of the credit spread

$$\begin{aligned} s &= PD \times LGD + \text{Risk Premium}, \\ s &\approx PD \times LGD. \end{aligned} \tag{4.2.6}$$

So, to extract probability of default from *Credit Default Swap*, abbr. *CDS* spreads, one has to remove the confounding factors as *LGD* and *Risk Premium* that arise from risk aversion.

CDS is basic credit derivative. It is contract where the payoff depends on the creditworthiness of one or more commercial or sovereign entities. When the reference entity is a single firm, bank or country we are talking about *CDS*. When the derivative refers to a set of reference entities - it is *Collaterized Debt Obligation*, abbr. *CDO*. Note that *CDS* provides insurance against the risk of default by particular firm. In particular, the buyer of the insurance obtain the right to sell bonds issued by the firm for their face value when a credit event occurs.

E.g. using *CDS* spreads for Intesa Sanpaolo in 2017 and with assumption $LGD = 50\%$ we get

$$\begin{aligned} s^{\text{IS}}_{\text{min}} &= 124 \text{ bps} \longrightarrow PD^{\text{IS}} \simeq 2.5\%, \\ \overline{s}^{\text{IS}} &= 130 \text{ bps} \longrightarrow PD^{\text{IS}} \simeq 2.60\%, \\ s^{\text{IS}}_{\text{max}} &= 155 \text{ bps} \longrightarrow PD^{\text{IS}} \simeq 3.1\%. \end{aligned} \tag{4.2.7}$$

Our estimated $PD^{\mathbb{Q}}$ for Intesa Sanpaolo in 2017 is 2.86%. E.g. using *CDS* spreads for UniCredit in 2017 we have

$$\begin{aligned} s^{\text{UNI}}_{\text{min}} &= 149 \text{ bps} \longrightarrow PD^{\text{UNI}} \simeq 3\%, \\ \overline{s}^{\text{UNI}} &= 165 \text{ bps} \longrightarrow PD^{\text{UNI}} \simeq 3.3\%, \\ s^{\text{UNI}}_{\text{max}} &= 177 \text{ bps} \longrightarrow PD^{\text{UNI}} \simeq 3.5\%. \end{aligned} \tag{4.2.8}$$

Our estimated PD^Q for UniCredit in 2017 is 4.34%. The historical development of CDS spreads of these two Italian banks is on Figure (4.1).

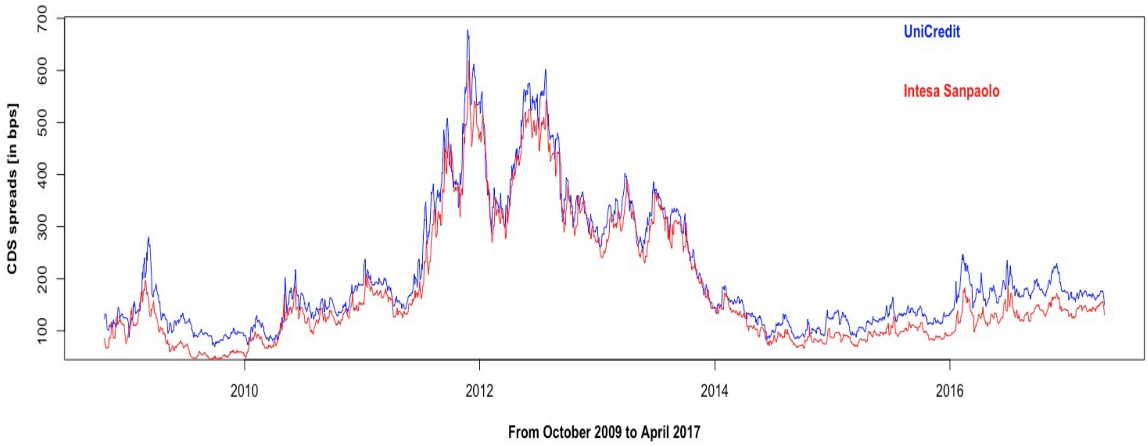

Fig. 4.1: CDS spread curves of UniCredit and Intesa Sanpaolo from October 2009 to April 2017.

Apparently, CDS's spread curves are extremely correlated. At given time frame, highest values were achieved from November to December in 2011 and from June to July in 2012.

$$\begin{aligned} s^{\text{UNI}}_{\max} &= 678 \text{ bps} \longrightarrow PD^{\text{UNI}} \simeq 13.6\%, \\ s^{\text{IS}}_{\max} &= 562 \text{ bps} \longrightarrow PD^{\text{IS}} \simeq 11.2\%. \end{aligned} \tag{4.2.9}$$

4.2.3 Fair Price

Let us move forward to the next task. We will use all listed PD^Q term structures for bond pricing each year beginning issue year.

Let us describe the bond pricing on example of BNP Paribas's bond at issue year 2010. We have introduced the bond pricing formula (3.2.9) which separates the risk-free and risky present value. We use $LGD = 50\%$ and continuous discount (3.2.2). Let us recall risk-free PV formula (3.2.9)

$$\begin{aligned} PV^{\text{BNP}} &= PV^{\text{BNP}}_{\text{risk-free}} + PV^{\text{BNP}}_{\text{risky}} \\ PV^{\text{BNP}}_{\text{risk-free}} &= (1 - LGD) \times \sum_{i=1}^{10} \frac{CF_{t_i}}{B_{t_i}} \\ PV^{\text{BNP}}_{\text{risky}} &= LGD \times \sum_{i=1}^{10} \frac{CF_{t_i} \times \left(1 - PD^Q_{t_i}\right)}{B_{t_i}}. \end{aligned} \tag{4.2.10}$$

PD^Q_t term structure is estimated on BNP Paribas data. Alternatively, we can write

$$\begin{aligned} PV^{\text{BNP}} &= \times \sum_{i=1}^{10} B^{-1}_{t_i} \Big[CF_{t_i} \times (1 - LGD) \times PD^Q_{t_i} + CF_{t_i} \times \left(1 - PD^Q_{t_i}\right) \Big] \\ &= \times \sum_{i=1}^{10} B^{-1}_{t_i} \Big[CF_{t_i} - CF_{t_i} \times LGD \times PD^Q_{t_i} \Big]. \end{aligned} \tag{4.2.11}$$

The last statement has desirable economic interpretation. PV of BNP Paribas is sum of discount cash flows depreciate by possible credit loss in every year.

t_i (2010-19/20)	CF_{t_i} (1)	Discount Factor	$PV^{\text{BNP}(1)}_{\text{risky}}$	$PV^{\text{BNP}}_{\text{risk-free}}$ (1)	$PV^{\text{BNP}(1)}$
1	3.75	0.9917	1.8039	1.8595	3.6634
2	3.75	0.9714	1.7236	1.8214	3.5450
3	3.75	0.9437	1.6484	1.7695	3.4180
4	3.75	0.9106	1.5751	1.7075	3.2826
5	3.75	0.8746	1.5038	1.6399	3.1437
6	3.75	0.8363	1.4324	1.5680	3.0004
7	3.75	0.7982	1.3636	1.4966	2.8602
8	3.75	0.7594	1.2949	1.4239	2.7188
9	3.75	0.7219	1.2295	1.3537	2.5832
10	103.75	0.6859	32.2891	35.5819	67.871
Total					**96.0863**

(1) In EUR.

Tab. 4.13: The pricing of BNP Paribas's bond at the issue date 2010.

It is discussed in detail in the Table (4.13). Important values of real prices of BNP Paribas's bond in 2010 are

$$P^{\text{BNP}}_{\text{min}} = \text{EUR } 96.05,$$
$$\bar{P}^{\text{BNP}} = \text{EUR } 97.11,$$
$$P^{\text{BNP}}_{\text{max}} = \text{EUR } 99.51.$$

In this case estimated fair price is really close to bond market price. Let us show second example of bond and issuer is few times mentioned Deutsche Bank. In the Table (4.14) is performed computation of PV^{DBK} in 2017.

t_i (2017-21/22)	CF_{t_i} (1)	Discount Factor	$PV^{\text{DBK}(1)}_{\text{risky}}$	$PV^{\text{DBK}}_{\text{risk-free}}$ (1)	$PV^{\text{DBK}(1)}$
1	1.75	1.0079	0.8398	0.8819	1.7218
2	1.75	1.0165	0.8422	0.8895	1.7316
3	1.75	1.0234	0.8301	0.8954	1.7255
4	1.75	1.0268	0.8070	0.8984	1.7054
5	101.75	1.0258	45.3235	52.189	97.5125
Total					**104.3969**

(1) In EUR.

Tab. 4.14: The pricing of Deutsche Bank's bond at the beginning of 2017.

Discount factor is greater than one because of the negative IR in the Euro-zone in recent years. 2017's market prices of this bond are

$$P^{\text{DBK}}_{\text{min}} = \text{EUR } 107.59,$$
$$\bar{P}^{\text{DBK}} = \text{EUR } 108.32,$$
$$P^{\text{DBK}}_{\text{max}} = \text{EUR } 108.92.$$

According to our computation, bond present value in 2017 is lower by 3.6%. We say that the bond is overvalued by financial market.

In the Table (4.15) we complete annual PV results and real prices according to Subchapter 3.2. In some years even price of some bonds had high volatility. Our claim is to evaluate bond's overvaluation or undervaluation. Therefore, it is useful to compute percentage difference between our model output and market average price. See the Table (4.16).

We consider the differences in the interval from −5% to +5% as trustworthy. If the difference is bigger than +5% (or lower than −5%) we determine percentage difference between fair price and maximal market price (or minimal market price). These values are in brackets. Our model overestimates more than 55% of PV results. The model adds in bond's present value extremely in 2011 and 2012 and also to Commerzbank from 2011 to 2016 and UniCredit from 2011 to 2013 and from 2015 to 2016. It is worth noting, the model in recent 2 years undervalues PV.

		2009	2010	2011	2012	2013	2014	2015	2016	2017
	Fair				81.04	95.30	93.90	97.18	104.09	104.39
	Max				101.89	103.49	109.47	111.54	111.51	108.92
DBK	**Min**				96.13	96.36	98.54	98.54	108.37	107.59
	Median				99.44	99.79	104.62	108.19	110.40	108.41
	Average				99.34	99.88	104.48	106.82	110.30	108.32
	Fair			127.34	126.72	139.52	135.71	134.37	132.77	125.22
	Max			104.36	106.20	114.60	124.75	125.25	122.75	122.59
CBK	**Min**			63.26	72.15	102.00	114.10	117.43	112.10	119.50
	Median			90.54	89.65	107.78	122.18	121.13	120.07	120.62
	Average			89.12	90.83	108.21	121.18	121.30	119.85	121.01
	Fair		108.67	107.75	121.05	121.13	114.83	116.68	112.97	106.57
	Max		105.28	101.44	120.41	123.41	124.46	124.47	120.28	116.20
RBS	**Min**		95.15	84.15	91.99	112.14	112.30	112.35	115.91	114.71
	Median		103.75	96.69	105.31	118.04	119.30	119.94	118.08	115.62
	Average		102.79	95.23	107.76	118.06	119.43	120.06	118.03	115.57
	Fair			114.24	130.96	124.98	121.14	122.96	121.35	114.35
	Max			101.55	124.85	125.45	127.45	129.10	126.55	122.60
BCY	**Min**			89.55	94.05	115.15	117.8	119.80	120.25	115.80
	Median			96.65	109.95	120.33	123.15	123.25	122.35	122.00
	Average			96.44	110.65	120.65	123.08	123.88	122.74	120.07
	Fair	116.82	113.79	118.13	129.43	125.75	120.4	119.51	115.23	111.50
	Max	110.55	117.8	109.49	121.11	123.4	123.4	122.40	120.21	121.21
HSBC	**Min**	108.71	104.34	95.91	101.50	102.50	101.50	100.50	112.17	112.00
	Median	109.68	111.27	106.05	112.33	115.75	117.54	115.58	116.12	115.38
	Average	109.68	111.77	104.43	112.86	114.83	116.36	114.50	116.37	115.92
	Fair			122.79	122.83	124.05	116.03	130.60	124.60	110.91
	Max			100.90	102.41	106.24	118.01	118.21	117.42	114.78
UNI	**Min**			66.60	73.22	98.64	105.88	108.79	102.83	113.34
	Median			88.05	89.93	103.10	113.08	113.93	113.75	113.97
	Average			88.06	91.03	103.22	112.48	114.03	113.06	114.00
	Fair			112.23	111.22	114.61	122.18	121.63	118.68	109.21
	Max			101.30	113.82	119.24	126.04	127.40	123.58	121.58
IS	**Min**			91.15	100.73	114.10	118.88	121.11	116.14	117.46
	Median			96.95	108.23	115.47	121.17	123.92	120.21	118.21
	Average			96.31	107.26	115.63	122.03	124.09	119.95	118.49
	Fair		96.09	101.85	106.83	114.87	108.54	113.08	109.95	110.54
	Max		99.05	99.85	111.35	113.55	116.66	117.56	115.90	114.01
BNP	**Min**		96.05	87.55	95.89	105.65	105.65	105.65	110.53	112.56
	Median		96.88	96.05	104.45	109.92	110.90	113.19	113.27	113.37
	Average		97.11	96.04	105.07	109.91	111.58	112.81	113.12	113.29
	Fair		95.46	104.06	104.21	113.03	111.50	108.46	109.69	110.52
	Max		101.48	103.53	111.50	111.06	119.53	117.17	118.45	118.45
SG	**Min**		93.15	85.32	94.39	99.93	104.74	107.11	110.26	108.21
	Median		94.23	95.71	104.70	106.45	113.80	115.84	115.43	115.46
	Average		96.67	96.85	105.18	108.54	114.81	116.77	116.71	115.95
	Fair				112.30	120.25	118.62	116.20	112.99	111.88
	Max				109.97	118.64	118.46	119.70	125.93	124.60
CA	**Min**				101.05	100.70	104.86	110.74	112.62	110.08
	Median				106.23	113.15	114.91	114.92	119.93	121.14
	Average				107.66	114.81	115.01	115.54	119.07	120.65

In EUR.

Tab. 4.15: Bond's real prices and fair prices from 2009 to 2017.

	2009	2010	2011	2012	2013	2014	2015	2016	2017
DBK				-18.42 (-15.70)	-4.59	-10.13 (-4.71)	-9.03 (-1.38)	-5.63 (-3.95)	-3.63
CBK			+42.89 (+22.02)	+39.52 (+19.32)	+28.94 (+21.74)	+11.99 (+8.79)	+10.78 (+7.28)	+10.78 (+8.16)	+3.48
RBS		+5.72 (+3.22)	+13.15 (+6.22)	+39.52 (+0.53)	+2.60	-3.85	-2.82	-4.29	-7.79 (-7.10)
BCY			+18.47 (+12.50)	+18.35 (+4.89)	+3.59	-1.58	-0.74	-1.13	-4.77
HSBC	+6.50 (+5.67)	+1.81	+13.12 (+7.89)	+14.68 (+6.87)	+9.50 (+1.90)	+3.48	+4.38	-0.98	-3.81
UNI			+39.45 (+21.70)	+34.93 (+19.94)	+20.18 (+16.76)	+3.16	+14.52 (+10.48)	+10.21 (+6.12)	-2.71
IS			+16.53 (+21.70)	+3.69	-0.88	+0.13	-1.98	-1.06	-7.84 (-7.03)
BNP		-1.05	+6.04 (+2.00)	+1.68	+4.51	-2.73	+0.24	-2.80	-2.42
SG		-1.26	+7.45 (+0.52)	-0.92	+4.14	-2.89	-7.12 (+1.26)	-6.02 (-0.52)	-4.68
CA				+4.31	+4.73	+3.13	+0.57	-5.1 (+0.33)	-7.27 (+1.63)

In percent.

Tab. 4.16: Differences between fair prices and market prices.

4.2.4 Fund Return

In previous tasks we have created the most appropriate data base. Let us use it in terms of portfolio management. We have already introduced four buying strategies.

Important assumptions are:

1. A sufficient number of financial resources at any time.
2. Zero transaction costs.
3. We buy and sell only at the beginning of the year.
4. We can buy any quantities of bonds.

The first assumption is more imaginable if we are pension fund and we receive cash regularly. Amount of free cash at given time can be additive constrain in optimasation problem. Optimal control of portoflio is not the essence of this thesis.

We make *The Return of Fund*, abbr. R^{Fund} which will be tracking only active bonds. Active bonds are bonds which are purchased in the portfolio at that moment.

R^{Fund} is expressed as relative difference

$$R^{\text{Fund}} = \frac{V^{\pi}_{t_i} - V^{\pi}_{t_{i-1}}}{V^{\pi}_{t_{i-1}}} \tag{4.2.12}$$

and $V^{\pi}_{t_i}$ is sum of paid coupons and PV of all bonds at given time t_i in the portfolio. We have

$$V^{\pi}_{t_i} = \sum_{j=1}^{m} CF^{j}_{t_i} + PV^{j}_{t_i}, \tag{4.2.13}$$

where m is number of active bonds. By reason of neither one of the banks has no credit default in 2009-2017, it is easy to calculate the value of the portfolio V_{t_i}. But the future is questionable, see the next Subchapter 4.3. Data are prepared for first three strategies. The last one strategy of portfolio diversification is more complex.

Let us describe **The Algorithm of Finding the Optimal Weights:**

1. A computation of bond expected returns.

   In the formula (2.1.7) we need to choose correct bond profit Y. Historical returns of corporate bonds distinguish according the corresponding rating. For simplification we consider only two ratings categories Aaa and Baa. These categories had corresponding different yields at given year. In the Table (4.17) are estimated expected returns for each bond. These individual profits are compared with risk-free IR. If inequality $\bar{Y}_{t_i} < r_{t_i}$ holds, we prefers to buy rather government bonds. These cases are colored in red in the Table (4.17).

	2009	2010	2011	2012	2013	2014	2015	2016	2017
DBK				2.44	0.55	1.21	0.12	-0.05	-1.65
CBK			1.63	1.79	0.83	0.65	1.07	0.44	0.40
RBS		3.56	1.88	3.6	1.32	1.58	1.11	0.20	-1.38
BCY			2.04	1.89	1.13	1.96	1.01	1.18	-0.06
HSBC	3.14	3.01	3.41	2.03	1.80	2.03	1.08	1.24	0.80
UNI			2.78	0.37	-1.06	1.32	0.14	-0.01	-1.29
IS			2.61	1.77	-0.73	1.55	0.66	0.25	-0.53
BNP		1.82	3.87	1.97	1.55	1.92	1.03	1.23	0.74
SG		-0.83	1.87	1.09	0.51	1.79	1.03	1.23	0.74
CA				1.31	-0.04	2.01	0.41	0.91	0.68
EU		0.83	0.58	0.14	-0.03	0.11	-0.09	-0.41	-0.79
UK	0.73	0.79	0.71	0.4	0.36	0.39	0.31	0.38	0.06

In percent.

Tab. 4.17: Expected profits $\bar{Y}$ and r_{1Y} in Euro-zone and UK.

2. A computation of correlation matrix between bank's asset values $\boldsymbol{\rho}$.

3. A computation of Joint Default Frequency matrix (2.2.4).

We also already know distance to default for every bond every year. See the Table (4.18).

	2009	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**
DBK				2.31917	2.34193	2.54118	2.47161	2.26648	2.06040
CBK			2.22267	1.90334	2.27492	2.27653	2.89155	2.42332	2.79486
RBS		3.06401	2.28827	2.92901	2.68228	2.76604	2.93151	2.41500	2.10887
BCY			2.33631	3.56244	2.56889	3.36013	3.31119	3.22221	2.50663
HSBC	2.05403	2.83119	4.44991	3.91141	3.32229	4.09799	4.04873	3.46087	4.02718
UNI			2.64597	1.90077	1.97439	2.50894	2.14918	2.27653	2.10657
IS			2.55586	1.77889	1.94875	2.62783	2.58972	2.35796	2.29552
BNP		2.27653	2.76729	2.04365	2.86523	3.23214	3.43588	3.41271	3.37086
SG		2.02540	2.39270	2.12558	2.32832	2.98035	3.45232	3.39578	3.35593
CA				2.23141	2.17868	3.76762	2.60665	2.81683	3.19275

In standard deviations.

Tab. 4.18: The DD in 1 year.

Let us show an example of JDF for year 2015 when expected returns of all corporate bonds are bigger than risk-free rates, i.e. $\bar{Y} > r_{1,2015}$. Diagonal of the matrix consists of already known one year PD^{Q} from term structures in 2015. Out of diagonal we have required joint default probabilities.

Probability of defaulting two largest German banks according market capitalization Deutsche Bank and Commerzbank in the end of 2015 is

$$\begin{aligned} JDF_{\text{DBK-CBK}} &= \mathcal{N}_2\Big(-DD_{\text{DBK}}, -DD_{\text{CBK}}, \rho_{\text{DBK-CBK}}\Big) \\ &= \mathcal{N}_2\Big(-2.47161, -2.89155, 0.7562935\Big) \\ &= 0.08\%. \end{aligned} \tag{4.2.14}$$

Correlation ρ between bank's asset values is very high and also joint probability is quite considerable compared with some others bank pairs.

$$JDF^{2015} = \\ \begin{pmatrix} & DBK & CBK & RBS & BCY & HSBC & UNI & IS & BNP & SG & CA \\ DBK & 1.881e{-02} & 8.807e{-04} & 6.338e{-09} & 2.336e{-04} & 2.797e{-10} & 2.844e{-06} & 1.495e{-08} & 2.849e{-04} & 2.362e{-04} & 1.570e{-06} \\ CBK & & 6.099e{-03} & 3.659e{-09} & 5.432e{-05} & 2.207e{-10} & 1.345e{-06} & 5.455e{-06} & 1.950e{-04} & 1.769e{-04} & 2.845e{-05} \\ RBS & & & 5.430e{-03} & 3.432e{-06} & 2.179e{-05} & 1.352e{-03} & 3.677e{-05} & 6.272e{-11} & 5.080e{-14} & 1.704e{-05} \\ BCY & & & & 1.660e{-03} & 4.438e{-08} & 1.585e{-05} & 1.074e{-08} & 2.215e{-05} & 9.218e{-06} & 8.306e{-08} \\ HSBC & & & & & 1.099e{-04} & 2.574e{-05} & 2.263e{-07} & 1.633e{-11} & 2.151e{-14} & 5.496e{-07} \\ UNI & & & & & & 3.962e{-02} & 1.084e{-04} & 1.280e{-07} & 2.633e{-09} & 1.809e{-04} \\ IS & & & & & & & 1.395e{-02} & 2.312e{-08} & 3.739e{-07} & 1.250e{-03} \\ BNP & & & & & & & & 1.090e{-03} & 1.071e{-04} & 1.224e{-06} \\ SG & & & & & & & & & 1.030e{-03} & 6.759e{-06} \\ CA & & & & & & & & & & 1.335e{-02} \end{pmatrix} \tag{4.2.15}$$

One more example from 2017. Expected profits of bonds of Deutsche Bank, Royal Bank of Scotland, Barclays and UniCredit are worse than government bond yields in corresponding countries. Hence, we do not buy this bonds in 2017.

In the next step we compute covariance matrix V_{2017} using following JDF^{2017} matrix

$$JDF^{2017} = \begin{pmatrix} & CBK & HSBC & IS & BNP & SG & CA \\ CBK & 8.030e{-03} & 1.915e{-08} & 1.976e{-03} & 2.705e{-06} & 2.370e{-05} & 4.922e{-05} \\ HSBC & & 1.199e{-04} & 1.324e{-06} & 2.425e{-05} & 1.525e{-05} & 1.659e{-05} \\ IS & & & 2.862e{-02} & 3.069e{-05} & 1.134e{-04} & 1.879e{-04} \\ BNP & & & & 1.359e{-03} & 1.938e{-04} & 2.011e{-04} \\ SG & & & & & 1.430e{-03} & 2.512e{-04} \\ CA & & & & & & 2.439e{-03} \end{pmatrix}. \tag{4.2.16}$$

The most significant joint default probability is JDF_{CBK-IS}. We compute

$$\begin{aligned} JDF_{\text{CBK-IS}} &= \mathcal{N}_2\Big(-DD_{\text{CBK}}, -DD_{\text{IS}}, \rho_{\text{CBK-IS}}\Big) \\ &= \mathcal{N}_2\Big(-2.79486, 2.29552, 0.86912523\Big) \\ &= 0.19\%. \end{aligned} \tag{4.2.17}$$

Again, a high correlation of assets is input to the calculation.

4. A computation of covariance matrix V (2.3.3).
5. Solving optimization problem (2.3.1).

Optimal weights w^* are in the Table (4.19). As reference expected portfolio $\mathbb{E}[\mathbf{R}]_\pi$ at given year we choose average of individual expected asset returns. Surely, from a managerial perspective there are many options how to choose $\mathbb{E}[\mathbf{R}]_\pi$.

	2009	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**
DBK	0	0	0	0	0.165073	0.125504	0.129106	0.112462	0
CBK	0	0	0.112319	0	0.132091	0.118609	0	0.099302	0.160423
RBS	0	0.250828	0.123809	0	0.103663	0.102238	0	0	0
BCY	0	0	0.112501	0.538932	0.114286	0.136835	0.121956	0.09186	0
HSBC	1	0.454289	0.318118	0.361712	0.202641	0.182551	0.276401	0.129045	0.206094
UNI	0	0	0.089863	0.048289	0	0.076823	0.065817	0.109615	0
IS	0	0	0.099859	0	0	0.056535	0.071593	0.132921	0.172354
BNP	0	0.294883	0	0	0.120031	0.068861	0.101461	0.114313	0.157612
SG	0	0	0.143531	0.027478	0.162215	0.072545	0.107371	0.118051	0.142048
CA	0	0	0	0.023589	0	0.059499	0.126295	0.092431	0.161469
$\mathbf{E[R]_\pi}$ (1)	**3.14**	**2.80**	**2.51**	**1.83**	**1.10**	**1.60**	**0.76**	**0.71**	**0.47**

(1) In percent.

Tab. 4.19: Optimal weights w^*.

$\mathbf{E[R]_\pi}$ (1)	**0.12**	**0.14**	**0.40**	**0.66**	**1,00**	**1.03**	**1.07**	**1.08**	**1.10**
DBK	0.676014	0.650778	0.294263	0.17152	0.027393	0.014709	0	0	0
CBK	0	0	0	0	0.006166	0.007868	0.009973	0.012138	0
RBS	0	0	0	0	0.007357	0.010059	0.013929	0.022819	0.863789
BCY	0	0	0.121399	0.145778	0.064335	0.057049	0.044971	0	0
HSBC	0	0	0	0.105933	0.675106	0.722753	0.793516	0.937685	0.136211
UNI	0.318477	0.307315	0.145873	0.087128	0.014617	0.008195	0	0	0
IS	0	0	0.147741	0.093505	0.019139	0.012608	0.002526	0	0
BNP	0	0	0.009501	0.111781	0.075152	0.0716	0.064812	0.010359	0
SG	0	0	0.006029	0.117877	0.080613	0.077002	0.070273	0.016999	0
CA	0.005509	0.041907	0.275194	0.166478	0.030122	0.018157	0	0	0

(1) In percent.

Tab. 4.20: Example of optimal weights depending on portfolio expected return in 2015.

Figure (4.2, 4.3, 4.4 and 4.5) show the development of the fund return for strategy 1, 2, 3 and 4 respectively.

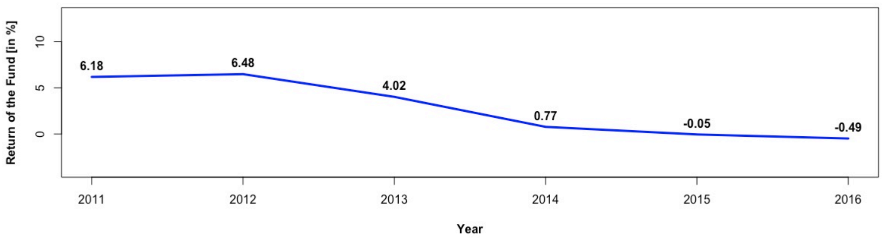

Fig. 4.2: Strategy 1: Buying all bonds with naive diversification.

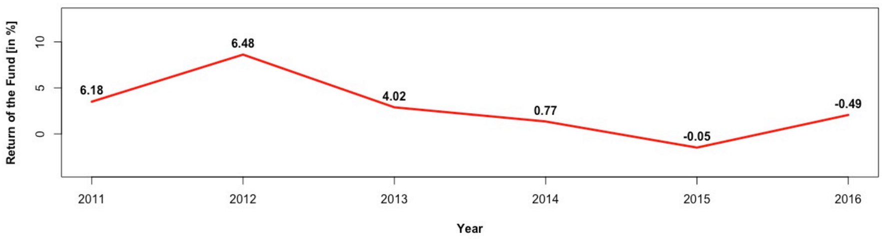

Fig. 4.3: Strategy 2: Buying undervalued bonds with naive diversification.

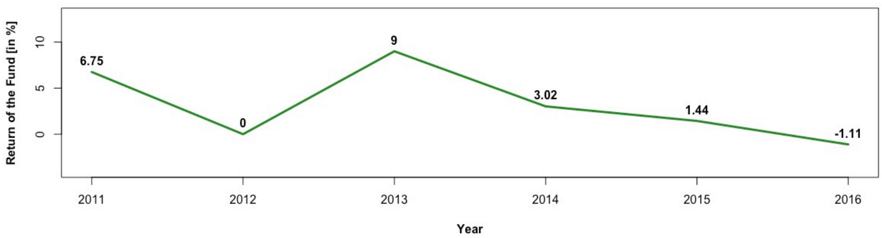

Fig. 4.4: Strategy 3: Buying overvalued bonds with naive diversification.

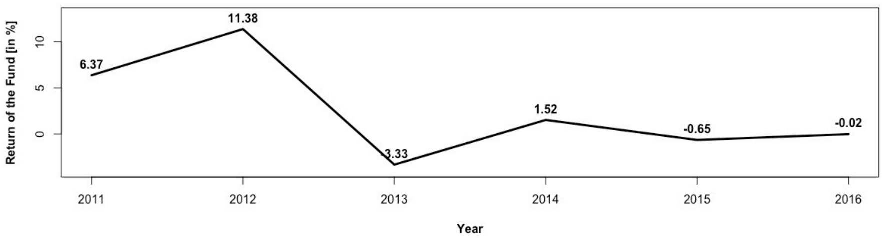

Fig. 4.5: Strategy 4: Portfolio diversification.

4.3 2018 - 2022

In this Subchapter we use theoretical setup from Subchapter 3.1 *Asset Value Modeling*. We model portfolio asset values and we take into account the mutual default correlations between assets. Now, we describe the last algorithm which contains the procedure of computation the PD^Q term structures from 4.2.1 each year. In the next steps, we need to be aware if we operate under risk-neutral measure Q or not. Let us describe the last proceedure in this thesis.

The Algorithm of the Simulated Fund Return.

1. For the purpose of computation in *the real world*, we start with the determination of essential input. It is the vector of annual bank's asset value drifts $\mu_{V^{(i)}}, i = 1, \ldots, 10$. It can be estimated as the trend of linear regression. We have available historical asset values (only in USD)on a quarterly basis, in some cases information available back to the 90s. If we look at historical data, we can see that asset values are declining as well as bank debt, occasionally both V and B increase between some years. It is often possible to observe very large differences in asset values between years, but the value of the bank's debt is also adjusted at a given year. We recommend an interesting article on this issue [12] and at least for the experience to read

specific annual report for particular bank. For each bank we have selected different lengths of time series of annual asset values to estimate the drift due to high changes. Our vector of $\boldsymbol{\mu_V}$ is in the Table (4.21).

Bank	DBK	CBK	RBS	BCY	HSBC	UNI	IS	BNP	SG	CA
$\boldsymbol{\mu_V}$ [(1)]	-0.85	-1.24	-1.14	-1.40	-0.55	-1.02	-0.80	0.40	-0.03	-0.91

[(1)] In percent.

Tab. 4.21: Estimated vector $\boldsymbol{\mu_V}$.

Eventually, in its annual report the bank indicates prediction of increase or decrease in asset value for the next year but it never states a specific percentage. To determine this value as correctly and meaningful as possible from the data available, it requires many years of experience in the field. It would be appropriate to make the model sensitivity of this variable in the potential extension of this thesis.

During portoflio simulation process we do not change $\boldsymbol{\mu_V}$.

2. We compute correlation matrix and perform Cholesky decomposition. Example of used L matrix in 2018

$$L = \begin{pmatrix} 1.00000 \\ 0.74609 & 0.66583 \\ 0.48518 & 0.27789 & 0.82907 \\ 0.54222 & 0.30924 & 0.38116 & 0.68196 \\ 0.52844 & 0.14377 & 0.16989 & 0.22578 & 0.78755 \\ 0.13027 & 0.10896 & 0.05220 & 0.03446 & 0.11931 & 0.97621 \\ 0.60534 & 0.36419 & 0.18418 & 0.17196 & 0.00264 & 0.08069 & 0.65643 \\ 0.65502 & 0.36540 & 0.16802 & 0.12444 & 0.05459 & 0.00610 & 0.16109 & 0.603926 \\ 0.68489 & 0.39034 & 0.19381 & 0.12222 & 0.05151 & 0.04750 & 0.19356 & 0.271431 & 0.45824 \\ 0.56310 & 0.43260 & 0.16368 & 0.08109 & 0.09927 & 0.01061 & 0.10070 & 0.277406 & 0.16520 & 0.58141 \end{pmatrix}. \tag{4.3.1}$$

3. Moreover, we compute covariance matrix and acquire $\boldsymbol{\sigma}$. Example of used volatility matrix $\boldsymbol{\sigma}$ in 2018

$$\boldsymbol{\sigma} = \begin{pmatrix} 0.01318 \\ 0.01216 & 0.01085 \\ 0.02482 & 0.01421 & 0.04241 \\ 0.02051 & 0.01170 & 0.01442 & 0.02580 \\ 0.01454 & 0.00395 & 0.00467 & 0.00621 & 0.02167 \\ 0.00648 & 0.00542 & 0.00260 & 0.00170 & 0.00594 & 0.04863 \\ 0.03008 & 0.01810 & 0.00915 & 0.00854 & 0.00013 & 0.00401 & 0.03262 \\ 0.01469 & 0.00819 & 0.00376 & 0.00279 & 0.00122 & 0.00013 & 0.00361 & 0.01354 \\ 0.01325 & 0.00755 & 0.00375 & 0.00236 & 0.00099 & 0.00091 & 0.00374 & 0.00525 & 0.00886 \\ 0.00738 & 0.00567 & 0.00214 & 0.00106 & 0.00130 & 0.00013 & 0.00132 & 0.00363 & 0.00216 & 0.00762 \end{pmatrix} \tag{4.3.2}$$

and the volatility vector is

$$\mathbf{vol} = \begin{pmatrix} 0.01318 \\ 0.01629 \\ 0.05116 \\ 0.03783 \\ 0.02751 \\ 0.04981 \\ 0.04970 \\ 0.02243 \\ 0.01935 \\ 0.01312 \end{pmatrix} \tag{4.3.3}$$

4. We build $\boldsymbol{W}_t$ matrix of Wiener processes again according to description (4.2.1).

5. As V_0 value, we put the last known asset value from the bank's financial statement.

6. We use formula (3.1.14) to simulate portfolio correlated asset values. We simulate 10 000 asset trajectories for each bank. Hence, some bonds maturity earlier, we simulate less asset values in later years. In the last year, we simulate only Deutsche Bank and Crédit Agricole's asset value. Taking this fact into account, we need to estimate new covariance and correlation matrixes.

7. B is constant and it is the last known value of book liabilities in 1Q 2017 (alternatively in 4Q 2016).

8. By reason of proper bond pricing process, it is necessary to go back into *the risk-neutral world.* PD^Q term structure under risk-neutral measure Q is estimated using simulations as before. Recall Subchapter 4.2.1. This time, PD^Q is computed for each simulated portfolio asset value separately using risk-free IR. The problem is evident. We do not know the future spot risk-free interest rate r_{t_i}. Nevertheless, we have several options how to proceed further:

(a) We replicate the same r_{t_i} term structure.
(b) We use expert r_{t_i} predictions.
(c) We model r_{t_i} using stochastic processes.
(d) In our case, we compute *risk-free forward IR* f_{t_i} from the following eq.

$$\begin{aligned} \exp\Big\{r^{\star}_{t_j} t_j\Big\} &= \exp\Big\{r^{\star}_{t_i} t_i\Big\}\exp\Big\{f_{t_i t_j} \times \big(t_j - t_i\big)\Big\} \\ f_{t_i t_j} &\equiv r\big(t_i\big)_{t_j - t_i}, \end{aligned} \tag{4.3.4}$$

where with $\star$ are marked known spot risk-free IR from the last year from 2017. See spot and forward risk-free IR in Figure (4.6). A simple explanation is that we model the future risk-free IR as today's risk-free forwards.

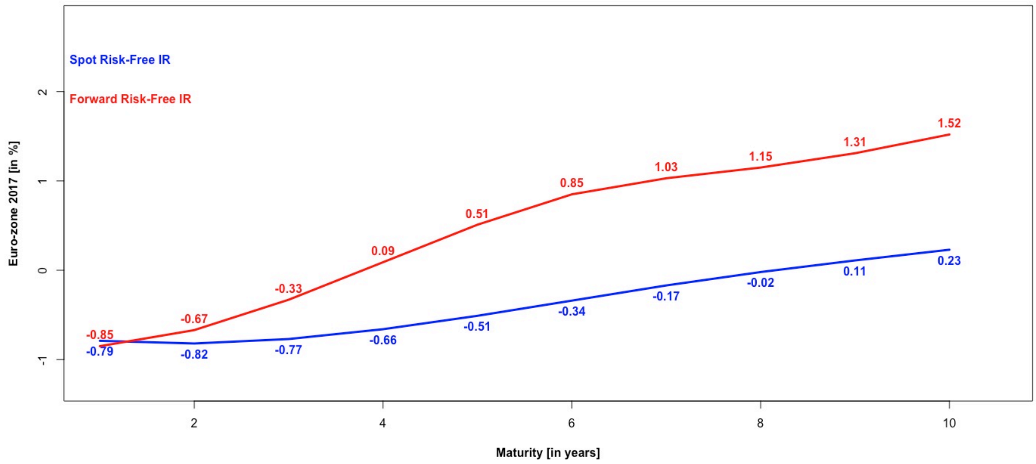

Fig. 4.6: Spot and forward risk-free IR in Euro-zone.

9. We perform the portfolio bond pricing computation using PD^Q_j where $j = 1, \ldots, 10\,000$ and f_{t_i} for each simulation using theory from the Subchapter 3.2. Table (4.22) summerizes $PD^Q_{t_i}$ term structures for each year and each bank estimated from only one simulation j.

2018	**DBK**	**CBK**	**RBS**	**BCY**	**HSBC**	**UNI**	**IS**	**BNP**	**SG**	**CA**
1	0.0393	0.0299	0.0202	0.0205	0.0015	0.0543	0.0368	0.0350	0.0144	0.0255
2	0.0706	0.0511	0.0305	0.0447		0.0976	0.0699	0.0376	0.0869	0.0767
3	0.1195	0.1008		0.0984		0.1214	0.1411			0.1188
4	0.1703									0.1363
2019										
1	0.0789	0.0564	0.0444	0.0452		0.1182	0.0881	0.0253	0.0349	0.0439
2	0.1485	0.1453		0.0932		0.1518	0.2018			0.0629
3	0.2458									
2020										
1	0.1318	0.1177		0.0659		0.1785	0.1410			0.0880
2	0.2253									0.1365
2021										
1	0.5859									0.1469

Tab. 4.22: PD^Q term structures of one simulation in 2018, in 2019, in 2020 and sequentially in 2021.

Y	2017	2018	2019	2020	2021/22
DBK	104.39	97.85	93.46	93.15	72.52
CBK	125.21	118.68	108.86	102.27	
RBS	106.56	108.29	102.94		
BCY	114.34	108.12	105.13	101.90	
HSBC	111.50	105.70			
UNI	110.90	112.58	105.20	97.47	
IS	109.20	116.06	100.50	98.43	
BNP	110.54	106.88	103.31		
SG	110.52	107.12	101.15		
CA	111.87	107.63	110.13	103.07	99.86

(1) In EUR.

Tab. 4.23: Simulated bond fair prices using PD^Q term structures from the Table (4.22.

10. If the bank, based on *the real world* simulation, is not in default, we compute value of the simulated portfolio V^π as before. See definition of V^π (4.2.13) in the Subchapter 4.2.4. It is a sum of coupon cash flows and fair prices. Using PD^Q in the Table 4.22 and fair prices, we can display one possible curve of fund returns in Figure (4.7). Until 2017, we have already computed V^π using real bond prices on the financial market according stock exchanges data.

11. Remark, we only perform Strategy 1: buying all bonds with naive diversification.

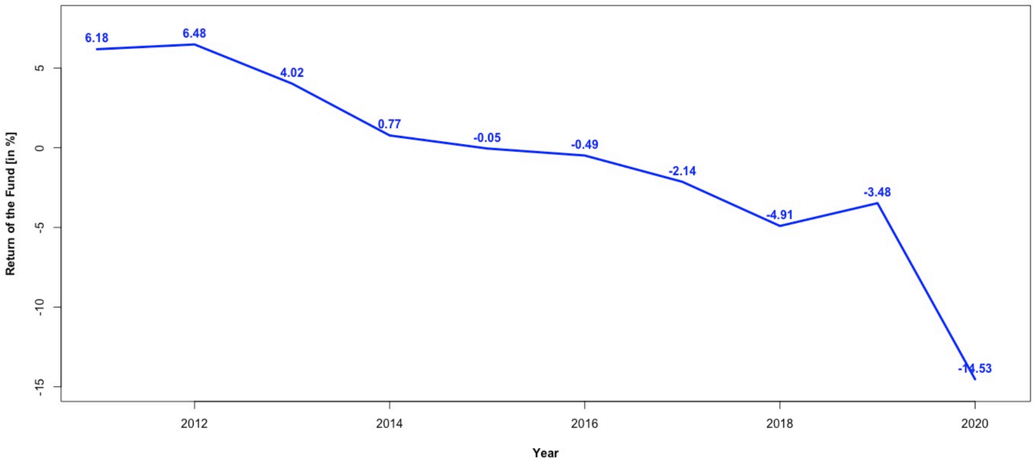

Fig. 4.7: Example of possible fund return.

12. So far, we have not seen actual bank's default of selected banks in the observed history. If the bank's correlated simulated asset value falls below the debt value, the result of that simulation is default in *the real world.* In that case, we compute V^π rather differently. Recall bond pricing examples in the Subchapter 4.2.3. Now, PV_{risky} component is 0. As a bond holder, we get only recovery rate from the value of coupon and from bond fair price at a given year. The default of one or more banks at any year can dramatically change the value of the simulated portfolio V^π. E.g. the another curve of fund returns (in red) is added in Figure (4.9). Red curve line presents again one possible simulated portfolio development. In this case Deutsche Bank defaults in 2019.

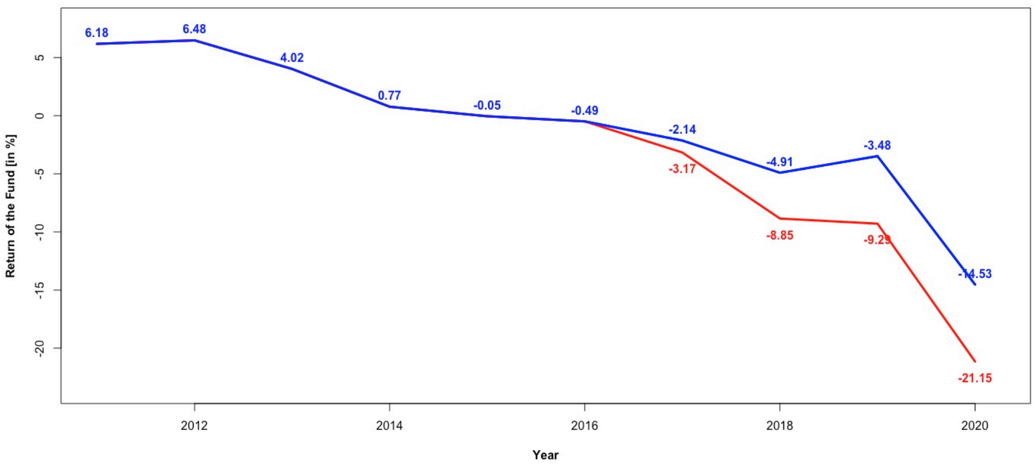

Fig. 4.8: Two simulated fund returns.

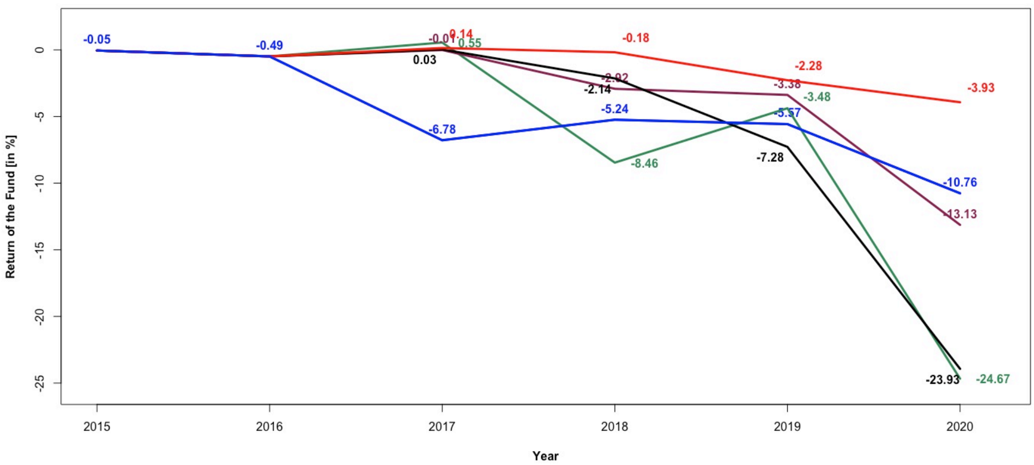

Fig. 4.9: Five different simulated fund returns.

13. The last step in this algorithm is to compute $\bar{R}^{\text{Fund}}$ (4.2.12) which presents the average of simulated fund returns at a given year. As the last result we present the curve of average simulated fund returns as well. Hence, the curve in Figure (4.10) can be misrepresentated (recall *the Jensen's inequality*), we also plot interval lines, which are computed as

$$\bar{R}^{\text{Fund}} \pm \sigma_R, \tag{4.3.5}$$

where σ_R is standard deviation of simulated fund returns.

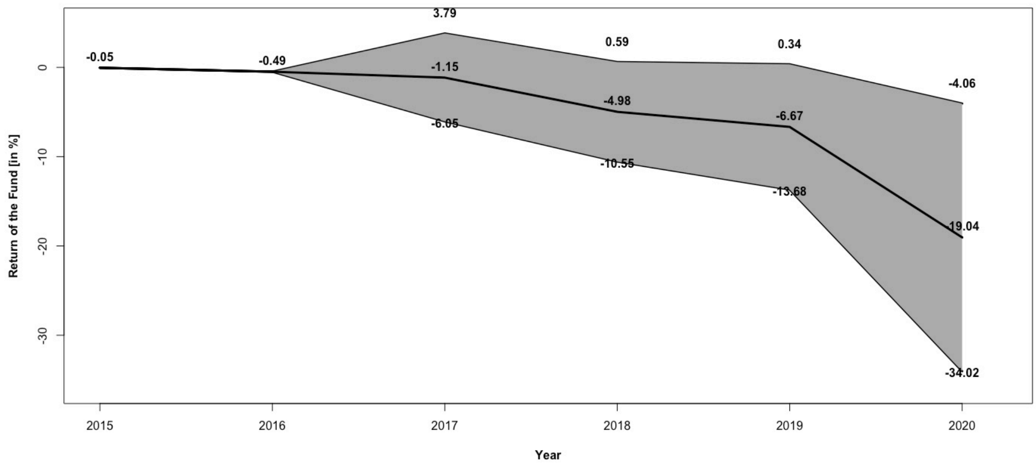

Fig. 4.10: Average of simulated fund returns.

Conclusion

To conclude this thesis, let us summarize the key conclusions and evaluate whether the objectives of this thesis have been met.

Firstly, our goal was to study the Merton Model and in general credit risk structural models. Secondly, we focused on pricing of the bond portfolio and simulation of portfolio development with respect to default risk. It is difficult to assess whether we have been able to sufficiently cover theoretical concepts in the initial theoretical Chapters 1, 2 and 3 as well as whether their description and explanations suffice the needs and knowledge of any reader who comes across this thesis.
It was the last Chapter 4 where all mathematical definitions and theoretical models have been applied on real bond data. That was the best way to better understand the weaker and stronger aspects of these models. However, it needs to be underlined that the whole applied part of the thesis heavily depends on the type of companies selected for the portfolio. Some may consider it inappropriate that all bonds selected for the portfolio are issued by 10 European banks. Nevertheless, we have accepted it as a great challenge and decided to tackle issues associated with it.
The last Chapter 4 contains a large number of results. Behind every outcome, there are many more data and inputs. Hence, data quality and proper parameters defined for all the models are of paramount importance. In the thesis we do not refer to the outcomes as the reference ones, instead we refer to them as our product and as a result of mathematical modeling. If we model banks such as those in our portfolio, asset values are very high, ranging from billions to trillions of euro. Similar affect can be seen also on the liabilities of the bank. These values are publicly shared only on quarterly basis.
The determination of the PD^Q term structures under risk-neutral measure Q is crucial procedure in this thesis. We simulate the uncorrelated bank's assets values from 2009 to 2017 and those of correlated assets between 2018 and 2022 to find out in how many cases we have run below the value of the debt in the corresponding year. It is precisely the PD value determines the bond quality and its credit rating class. We have created our own rating historical series and independently compared it to the Moody's official one. PD^Q term structures are also inputs to fair bond pricing calculation and our price can be compared to the actual market price. Another possible conclusion may be overvaluation or under-valuation of the bond by the financial market.
To the above results we have also added the optimal weights results, taking into account the default risk. Finally, we have created four strategies to monitor development of these four different fund returns on an annual basis. The last part and the most complex one is about simulating portfolio development. The Subchapter 4.3 combines all models mentioned above, except for a model to find optimal weights. To simulate portfolio development, we have decided to apply only naive bond diversification. The observed decline of simulated portfolio value comes by reason of our assumptions of estimated asset drift vector. Such analysis, however, can be subject to futher extension, alternatively follow up on this thesis. In such case it is advisable to discuss asset drifts of the selected banks with an expert on the financial institution.

The conclusion provides room to assess the approach taken and also to offer some recommendations for further actions that may follow up on this topic. To name a few, please see below:

1. In addition to the simulation of the company's asset value, focus on modeling the debt value. As we can see in the last Subchapter 4.3, the fixed debt may, or may not, significantly distort the result. With financial institutions, we also mentioned \rolling of debt phenomenon". In case stochastic models are not used, try to reasonably determine the debt trend.

2. We have worked with a constant value of LGD. Since this is a dynamic value over a 10-year period, it would be advisable to track down different periods when the bank's LGD has risen above 50% in real terms. At the meeting with the expert of the National Bank of Slovakia, a proposal was raised to follow up on a bank index of European financial institutions, to observe its trend and reflect it on LGD. Another approach can be to determine LGD for every bank individually.

3. Simulate development of portfolio, which is non-homogenous and sectors therein are diversificated.

4. Manage portfolio better. In terms of buying and holding to maturity, alternatively work more with the cash flow, or with the duration of the bonds. For instance to invest matured money.

5. Future interest rates could be modeled by stochastic processes which we have studied during EFM programme or simply overlap with equal risk-free interest rates instead of forward rates in future.

6. For the purpose of comparison calculate the PD according to the bank's regulatory requirements.

7. What is very important is to shorten the time interval and compute at shorter intervals than just annually. The theory of continuous modeling is derived in this thesis.

8. Calculate model's sensitivity for parameter modification, e.g. risk-free spot or forward term structure movements.

9. The question that remains open is what importance and impact have the asset correlations on the portfolio modeling similar to the one presented here. The importance of asset correlations varies according similar to the economic cycle. Empirical evidence shows that prior to and during the financial crisis (e.g. in 2007-2008) the behaviour is similar to 'house of cards'.

Last but not least, I would like to say that this topic has clarified many of credit risk issues and has forced me to use the great mathematical apparatus I have acquired during my studies. It has been an excellent experience and I am grateful for it.

Bibliography

[1] Moody's Corporation: *Modeling Default Risk.*
Modeling Methodology, (2003)

[2] Jurča, Pavol: *Quantitative Methods in Risk Management.*
Study Notes on http://www.iam.fmph.uniba.sk/institute/jurca/qrm/qrm.html, Comenius University, Faculty of Mathematics, Physics and Informatics Bratislava, (2016)

[3] Melicherčík, Igor et al.: *Kapitola z Finančnej Matematiky.*
Epos, Bratislava, (2005)

[4] Pišková, Andrea: *Modelovanie Portfólia Dlhopisov s Uvažovaním Rizika Defaultu.*
Diploma Thesis, Comenius University, Faculty of Mathematics, Physics and Informatics, Bratislava, (2004)

[5] Kadlečíková, Katarína: *Ocenenie Credit Default Swapov a Porovnanie Ich Vývoja v Čase Finančnej Krízy.*
Diploma Thesis, Comenius University, Faculty of Mathematics, Physics and Informatics, Bratislava, (2009)

[6] Vitali, Sebastiano: *Credit Risk in Banking.*
Study Notes on http://www.karlin.mff.cuni.cz/ vitali/CreditRisk.html, Charles University, Faculty of Mathematics and Physics, Prague (2017)

[7] Witzany, Jiří: *Credit Risk Management and Modeling.*
Study Notes on http://kbp.vse.cz/wp-content/uploads/2012/12/WitzanyCreditRiskManFG.pdf, University of Economics, Financial Engineering, Prague (2017)

[8] Resti, Andrea et al.: *Risk Management and Shareholders' Value in Banking.*
Wiley Finance, Chapters 11 and 12, (2007)

[9] Crouhy, Michel et al.: *A Comparative Analysis of Current Credit Risk Models.*
Journal of Banking and Finance 24, 59 - 117, (2000)

[10] Moody's Investor Service: *Default and Recovery Rates of European Financial and Non-Financial Corporate Issuers, 1985-2016 Q3.*
Modeling Methodology, (2016)

[11] Moody's Analytics: *Public Firm Expected Default Frequency* $\left(EDF^{TM}\right)$ *Credit Measures Methodology, Performance and Model Extensions.*
Modeling Methodology, (2012)

[12] Halaj, Grzegorz: *Optimal Asset Structure of Bank, Bank Reaction To Stressful Market Conditions*
Macroprudential Research Network, European Central Bank, Working Paper, (2013)

[13] Tong, Xiaoming: *Modeling Banks' Probability of Default*
Applied Economics and Finance, (2015)