COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS



Can realized volatility and asymmetry in realized volatility predict stock market movements?

MASTER THESIS

2019 **Bc. Daniel Križan**

COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Can realized volatility and asymmetry in realized volatility predict stock market movements?

MASTER THESIS

Study programme:	Mathematical Economics, Finance and Modelling
Field of study:	1114 Applied Mathematics
Supervisor:	RNDr. Ing. Peter Molnár PhD.

Bratislava 2019 **Bc. Daniel Križan**





Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname:	Bc. Daniel Križan
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Title: Can realized volatility and asymmetry in realized volatility predict stock market movements?

Annotation: In financial markets, investors are usually rewarded for holding more risky instruments by higher expected return. For example, stocks are more risky than bonds and stocks have higher expected return than bonds. We will investigate, whether similar relationship holds over time. Particularly, we will analyze 20 stock indices around the world and study whether returns of these stock indices during/after periods of high volatility is higher than return during/after periods of low volatility. In order to measure volatility as precisely as possible, we will utilize realized volatility calculated from high frequency data. We will investigate also the predictive power of asymmetry in realized volatility.

Supervisor:	Ing. RNDr. Peter Molnár, PhD.
Department:	FMFI.KAMŠ - Department of Applied Mathematics and Statistics
Head of department:	prof. RNDr. Daniel Ševčovič, DrSc.

Assigned: 31.01.2018

Approved: 31.01.2018 prof. RNDr. Daniel Ševčovič, DrSc.
Guarantor of Study Programme

.....
Student

.....
Supervisor





Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Bc. Daniel Križan
Študijný program: ekonomicko-finančná matematika a modelovanie (Jednoodborové štúdium, magisterský II. st., denná forma)
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Názov: Can realized volatility and asymmetry in realized volatility predict stock market movements?
Môže realizovaná volatilita a asymetria v realizovanej volatilite predpovedať pohyby na akciových trhoch?

Anotácia: Na akciových trhoch sú zvyčajne investori ocenení držaním rizikovejších nástrojov s vyššou očakávanou výnosnosťou. Napríklad, akcie sú rizikovejšie ako dlhopisy a zároveň majú vyššiu očakávanú hodnotu ako dlhopisy. Skúmame či budú vzťahy medzi rôznymi akciami podobné v danom čase. Osobitne budeme analyzovať 20 akciových indexov po celom svete a skúmať či sú výnosy týchto akciových indexov počas(po) periódy s vysokou volatilitou vyššie ako výnosy počas(po) periódy s nízkou volatilitou. Aby boli merania volatility čo najlepšie, tak budeme realizovanú volatilitu počítať z vysokofrekvenčných dát. Taktiež budeme skúmať silu predikcie pomocou asymetrie v realizovanej volatilite.

Vedúci: Ing. RNDr. Peter Molnár, PhD.
Katedra: FMFI.KAMŠ - Katedra aplikovanej matematiky a štatistiky
Vedúci katedry: prof. RNDr. Daniel Ševčovič, DrSc.

Dátum zadania: 31.01.2018

Dátum schválenia: 31.01.2018 prof. RNDr. Daniel Ševčovič, DrSc.
garant študijného programu

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vedúci práce

Abstract

KRIŽAN, Daniel: Can realized volatility and asymmetry in realized volatility predict stock market movements? [Master Thesis], Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics; Supervisor: Ing. RNDr. Peter Molnár, PhD, Bratislava, 2019, 55p.

We examine the relation between returns and realized volatility for five stock indices. We compare the size of returns after periods of high realized volatility and also low realized volatility. Firstly, we demonstrate the leverage effect, and later we try to construct a forecasting model, which includes various measures of realized volatility and logarithmic realized volatility. GARCH models also offer the possibility to include realized volatility, for example in GARCH-X and REAL-GARCH models. These models we compare with EGARCH, GJR-GARCH, and GARCH-M models and review the power of one day ahead prediction. We show the accuracy of prediction using the MAPE and RMSE metrics.

Keywords: returns, leverage effect, realized volatility, forecast, REAL-GARCH, GARCH-X

Abstrakt

KRIŽAN, Daniel: Môže realizovaná volatilita a asymetria v realizovanej volatilite predpovedať pohyby na akciových trhoch? [Diplomová práca], Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky; školiteľ: Ing. RNDr. Peter Molnár, PhD, Bratislava, 2019, 55s.

Cieľom práce je skúmanie vzťahu medzi výnosmi a realizovanou volatilitou pre 5 akciových indexoch. Zaoberáme sa skúmaním výšky výnosov po období s vysokou realizovanou volatilitou a tiež s nízkou realizovanou volatilitou. Lineárnou regresiou najskôr demonštrujeme pákový efekt a neskôr sa pokúsime zostrojiť model pre predikciu výnosov pomocou rôznych meraní realizovanej volatility a jej logaritmovaním. GARCH modely ponúkajú taktiež možnosť zahrnúť realizovanú volatilitu a to napríklad v modeloch GARCH-X a REAL-GARCH. Tieto modely následne porovnávame s EGARCH, GJR-GARCH a GARCH-M v presnosti jednodňovej predikcie a naše výsledky zobrazujeme pomocou metrík MAPE a RMSE.

Kľúčové slová: výnos, pákový efekt, realizovaná volatilita, predikcia, REAL-GARCH, GARCH-X

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Introduction

Nowadays, analysis and modeling of financial high-frequency time series become more critical, especially in asset prices fluctuations, which can be measured by volatility. Volatility is an essential element in trading on financial markets. Participants in the financial market have a different point of view on volatility. Most often is a high level of volatility associated with a loss rather than profit. Investors approach with caution during periods of increased volatility. It turns out that volatility increase as the market declines in prices and is lower when the market grows (French,1987). Since investors perceive the rise in volatility as an increased risk, it leads to a transfer of capital into less risky assets. Volatility for time series is changing over time. One way to compute volatility is through returns from high-frequency data. Nowadays, because of easy access to high-frequency data and good results is this approach trendy mainly for forecasting volatility (Lyócsa, 2016). In our thesis, we look at the relationship between realized volatility and returns, and whether the realized volatility can improve the power of one period ahead forecast. For this purpose, we use for example realized semi-variance, which is strongly related to future returns (Patton, 2015). One of the goals is also to investigate returns of the stock market index after a period of high volatility and after a period of low volatility. We show how linear regression works to estimate returns using different measures of realized volatility. We also look at GARCH models, since the newest approaches can incorporate the realized volatility into these models and we compare them with ordinary GARCH models. The concept of volatility can relate to other features such as leverage effect, volatility clustering or heteroscedasticity. We explain the first and second concept in detail in the first chapter. In the second chapter, we show results from linear regression models, which explain the leverage effect with the utilization of realized volatility. Our results are consistent with results from other authors (Molnar, 2019). Part of the second chapter also includes studies whether returns of our stock indices after periods of high volatility is higher than returns after periods of low volatility. In the last section, we examine one day ahead forecast accuracy among different GARCH models as EGARCH, GARCH-X or REAL-GARCH. Examples of applying external regressors to the GARCH-X model may be various, such as using the intraday range between the highest and the lowest price to proxy volatility (Molnár, 2016). But we focus on using the realized volatility.

1 Theory overview

1.1 Return

Prices of financial assets are the base of the financial system. In the financial sector, we often use the concept of return, which value we can determine from two prices in different moments. Return is defined as an increase or decrease of assets in a specific period.

$$r_t = ln\left(\frac{P_t^c}{P_t^o}\right)$$

Where P_t^c is close price in period t and P_t^o is open price in the same period.

Investors who are willing to take a risk should be rewarded with higher returns. However, there is always a high risk of loss. Among positive features of returns belong ability to make our data stationary, which allows examining more types of models with the assumption of stationarity.

The significance of return go with unit-free measurement, so we can compare time series, which prices are in different currencies. Characteristic of returns follow these facts: (Taylor, 2005) Evidence that returns are not normally distributed. In most cases, the distribution is symmetric but significantly different from the normal distribution. So, we observe positive skewness and high kurtosis which values can indicate a high peak (Mandelbrot, 1963) and fat tails (Fama, 1965).

Secondly, there is no significant correlation between returns and previous values of returns determined in same time series. Finally, Taylor indicates a positive relationship of absolute returns between previous days and the current day, so it shows a correlation between squared returns (Taylor, 2005).

French examines the relation between stock returns and stock market volatility(French,1987). Paper expected excess stock market returns are computed as a difference between the return on the stock market index and a risk-free rate. Paper confirms the hypothesis of a positive relation between expected risk premiums and volatility on S&P500 index analyzing data from the previous century. We examine the relationship between risk presented by volatility and returns. We will ignore the risk-free rate because there is only a small effect on results and negligibly low values of the risk-free rate. Moreover, we don't have risk-free rate data for all our researched indices.

1.2 Realized Variance

We have different ways to determine realized variance (RV). The most basic method is the sum of squared intraday returns. Realized variance can be transformed to realized volatility without measurement error. Andersen and Benzoni confirm that volatility can be estimated more precisely if the frequency of returns increases (Andersen and Benzoni, 2008). We gained realized variance data from Oxford-Man Institute(website) for more than 30 stock market indices(https://realized.oxford-man.ox.ac.uk/), but we decide to use only five of them for the next research. The framework contains a daily open price, daily close price and daily realized measures determined in different ways from intraday data. So we introduce the formula for selected realized variances, which we use in the next part of the thesis:

$$\begin{aligned} medRV_t &= \frac{\pi}{6-4\sqrt{3}+\pi}\left(\frac{m}{m-2}\right)\sum_{i=2}^{m-1} med(|r_{t,i-1}|, |r_{t,i}|\,|r_{t,i+1}|)^2 \\ RV_t &= \sum_{i=1}^{m} r_{t,i}^2 \\ RSV_t &= \sum_{i=1}^{m} r_{t,i}^2 I \end{aligned}$$

Where m is a number of intraday returns and t is a number of daily realized variances determined from intraday returns. I is a naive variable, which is equal to 1 if the intraday return is negative and 0 otherwise. We don't subtract mean in equations of realized variances. In the words of Poon and Granger: 'The statistical properties of a sample mean to make it a very inaccurate estimate of the true mean, especially for small samples, taking deviations around zero instead of the sample mean typically increases volatility forecast accuracy.'(Poon and Granger, 2003)

Volatility estimation is undoubtedly one of the essential parts of the financial sector. Many investors use volatility for measuring risk and making decisions in the creation of their portfolios. For estimate volatility, we can use different approaches. The most straightforward approach is historical volatility. The method consists of determining realized variance from historical data and subsequently compute volatility. It can serve to estimate volatility for future periods and leads to returns prediction.

In the next research, we use realized semi-variance, median realized variance and

realized variance for 5 and 10 minutes intraday data. We transform all to realized volatility by the square root:

$$RVolatility_t = \sqrt{RV_t}$$

Further, we use the RSV shortcut for volatility transformed from realized semi-variance. Shortcut medRV for volatility transformed from median realized variance and RV for volatility transformed from realized variance.

1.3 Volatility clustering

Volatility is not constant over time. We can't observe true volatility, so we are trying to estimate something that we never see in our case by realized variance and modification of realized variance. Changes in volatility can't explain easy, but there is a small part, which can be explained by volatility clustering. Volatility clustering phenomenon was first observed and noted by Mandelbrot. He examined that large changes tend to be followed by large changes and small changes tend to be followed by small changes (Mandelbrot, 1963). It is seemingly attributed to market data. A demonstration of this fact is following: 'While returns themselves are uncorrelated, absolute returns or their squares returns are positive, significant and slowly decaying autocorrelation function for ranging from a few minutes to several weeks' (Granger and Ding, 1993). When volatility is higher, then returns are more spread out. High volatility contains several extreme returns, but there are only a few large or low values of returns in a period of low volatility (Taylor, 2005). Volatility clustering can be partially explained by occurring bad or good news and their impact on returns.

Volatility clustering also relates to the concept of leptokurtic distribution. Compared to a normal distribution, leptokurtic have fat tails. Fat tails indirectly signify that after a period of large changes occur large change period, analogously for small changes (Taylor, 2005). In the next chapters, we examine the distribution of our financial data. The effort to confirm leptokurtic distribution in our data leads to apply Jarque-Bera test to rejection of normality and look at values of kurtosis and skewness.

1.4 Leverage effect

Leverage effect (Black, 1976) demonstrate asymmetric effect caused by positive and negative changes of asset prices on volatility. Specifically, a negative correlation between volatility and return of the asset. It means that negative deflection has a different impact as equivalent positive deflection. The difference is caused by decreasing of the asset price, and it is following by increasing company debt. Leverage effect consequences in times of crisis are volatility increase and more stable level of volatility. Leverage effect can also be reflected in no-tradable days because bad news can accumulate and indicate an increase in volatility in the next tradable days. In the second chapter, we confirm a presence of leverage effect in our data.

2 Realized volatility and returns

2.1 Data

We obtained our data from https://realized.oxford-man.ox.ac.uk. This dataset contains more than 30 indices around the world, but for our needs, we have chosen five (S&P500, AEX, AORD, BSE SENSEX, HSI), which are sufficiently diverse to prevent a repetition of results for groups of indices from similar regions. Our study time intervals are the same for all indices. As we mentioned before, our dataset cover daily open prices, close prices, realized variance and more measurements of modified realized variances from January 2000 to April 2019.

Firstly, we briefly introduce each index. S&P500 is a stock market index, which includes approximately the 500 largest companies by market capitalization listed on the New York Stock Exchange or NASDAQ. Amsterdam Exchange index (AEX) is a stock market index composed of Dutch companies that trade on Euronext Amsterdam. All Ordinaries Index (AORD) is made up of the share prices for 500 of the largest companies listed on the Australian Securities Exchange. S&P Bombay Stock Exchange Sensitive Index (BSE SENSEX) is a market-weighted stock market index of 30 largest companies listed on Bombay Stock Exchange. Hang Seng Index (HSI) is a stock-market index in Hong Kong.

There are many ways how to measure volatility. We can't say, which measure of volatility is best or is the right one. The dataset contains more measures of modified realized variances computed for each index. We observed correlations between daily standard realized variance, median realized variance and daily returns. Later, we consider possibilities for mixing variances to create only one time series of variance, which is enough correlated with returns and transform it into volatility. We don't subtract realized variance average from the realized variance for the reason we mentioned in 1.2.

In Table 1 realized semi-variance shows more related to returns than other measures of variances and it is also more related to returns than standard measures of realized variances as rv5 or rv10 because semi-variance can capture asymmetry in the stock market, unlike other variances.

correlation(RV vs returns)	S&P	AEX	AORD	BSE SENSEX	HSI
Realized Variance (5-min)	-0.056	-0.123	-0.139	-0.187	-0.074
Realized Variance (10-min)	-0.048	-0.112	-0.134	-0.189	-0.069
Median Realized Variance (5-min)	-0.075	-0.094	-0.154	-0.146	0.039
Realized Semi-variance (5-min)	-0.297	-0.304	-0.479	-0.324	-0.346

Table 1: Correlation between different measures of variances and returns of our indices.

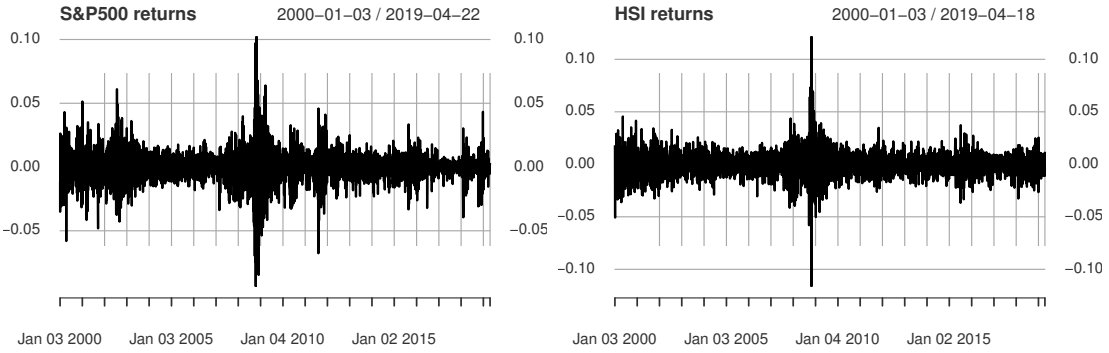

Figure 1: S&P500 and HSI daily returns from January 2000 to April 2019.

We display daily returns in Figure 1 from January 2000 to April 2019 for S&P500 and HSI(Hang Seng Index). We see the most significant fluctuations in 2008 in time of the financial crisis. So, we expect most changes in realized volatility in this period. For other indices(AORD, BSENS, AEX) there is similar behavior of returns, but we don't report these charts into the thesis.

Different measurements of realized volatility can capture different properties of intraday data. So we decided to combine various measures of realized volatility. Thus, daily volatility in both graphs in Figure 2 was computed as the average of 5-min realized volatility, 10-min realized volatility, and 5-min median realized volatility. As we expected the highest increase in volatility is in 2008. The financial crisis was global, so it hit all stock markets over the world by a similar effect. We can see this effect in Figure 2.

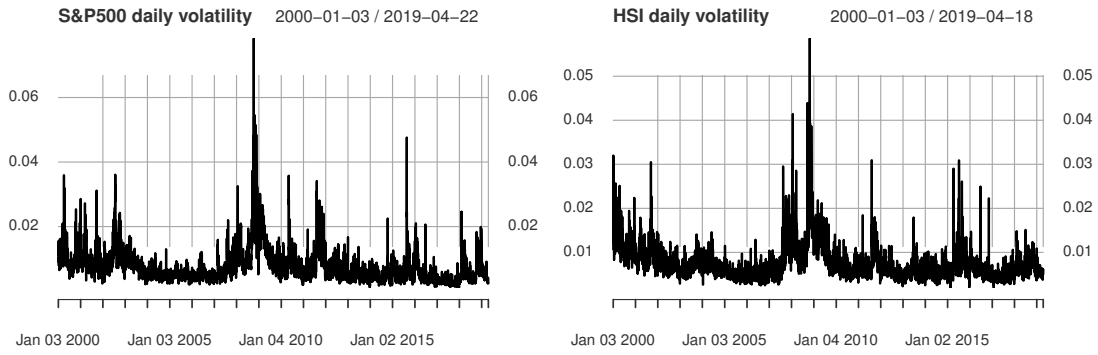

Figure 2: S&P500 and HSI daily mixed volatility(average of 5-min realized volatility, 10-min realized volatility, and 5-min median realized volatility).

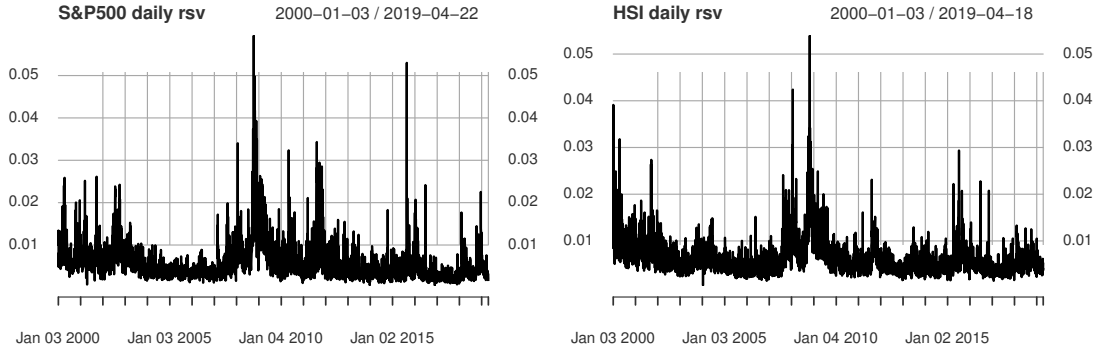

Figure 3: S&P500 and HSI daily realized semi volatility(RSV).

Figure 3 is similar to Figure 2, but it shows that realized semi-variance transformed into volatility is much more correlated with returns according to Table 1 as mixed volatility in Figure 2. In the next subsection, we show the advantages of RSV.

2.2 Examination of leverage effect

Demonstration of leverage effect on daily, weekly and monthly data for our indices is the main objective for this subsection. We expected a negative correlation between volatility and returns in the same period. For verification, we use mixed volatility and realized semi-variance transformed into volatility(RSV).

Firstly we apply a well known linear model to data where we examine the relationship between the observations Y_t and the independent variable X_t:

$$Y_t = \beta_0 + \beta_1 \phi_1(X_t) + \varepsilon_t$$

Where t is a number of daily data, weekly data or monthly data, it depends on the length of examining time scale. β_0, β_1 are regression coefficients and ϕ_1 may be nonlinear function.

In Table 2 we assume linear regressions without ϕ_1. Models use returns as observations Y_1 and independent variable X_1 as RSV. Models in Table 2 always compare volatility and returns at the same time scale. In parentheses, we show p-values of parameters. If the p-value is under 0.05, then the parameter is significant.

vol(t) ret(t)	rsv/nolog	S&P	AEX	AORD	BSENS	HSI
daily	β_0	0.005(0)	0.005(0)	0.006(0)	0.006(0)	0.006(0)
	β_1	-0.927(0)	-0.883(0)	-1.565(0)	-1.038(0)	-1.136(0)
	R^2	0.136	0.11	0.312	0.161	0.155
weekly	β_0	0.011(0)	0.012(0)	0.016(0)	0.013(0)	0.01(0)
	β_1	-0.843(0)	-0.96(0)	-1.708(0)	-1.089(0)	-0.89(0)
	R^2	0.104	0.114	0.259	0.143	0.082
monthly	β_0	0.032(0)	0.042(0)	0.041(0)	0.022(0)	0.017(0)
	β_1	-1.063(0)	-1.63(0)	-2.045(0)	-1.234(0)	-0.849(0)
	R^2	0.221	0.323	0.343	0.164	0.094

Table 2: Parameters and coefficients of determination for linear models between RSV and returns in the same period.

In Table 2 all regression models have significant parameters and show a strong negative correlation between returns and RSV. So we can confirm a leverage effect for all indices. However, a low value of R-squared(coefficient of determination) shows, that we can only partially explain values of daily returns by daily volatility, the same for weeks and months. All Ordinaries Index (AORD) shows the highest values of R^2 for all periods. For daily RSV and daily returns $R^2 = 0.312$, weekly $R^2 = 0.259$ and monthly $R^2 = 0.343$. This pointing at the strongest effect of asymmetry in the AORD index.

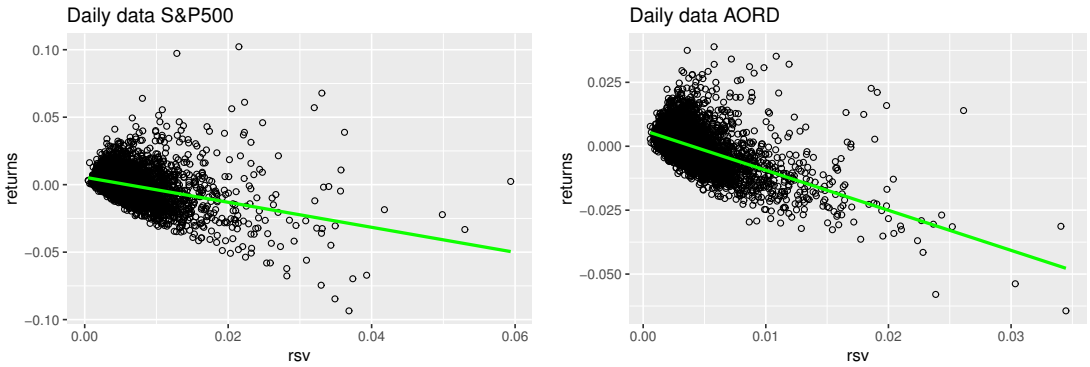

Figure 4: Comparison of linear regressions between daily returns and RSV for S&P500 and AORD indices.

Now, we assume linear regressions with non-linear function ϕ_1. We choose a logarithmic function and look on the impact of log(RSV) on R^2. In other words, we examine if the models can increase the value of our metrics(R^2).

vol(t)-ret(t)	rsv/log	S&P	AEX	AORD	BSENS	HSI
daily	β_0	-0.033(0)	-0.033(0)	-0.043(0)	-0.046(0)	-0.044(0)
	β_1	-0.006(0)	-0.006(0)	-0.007(0)	-0.008(0)	-0.008(0)
	R^2	0.118	0.092	0.28	0.136	0.145
weekly	β_0	-0.056(0)	-0.07(0)	-0.086(0)	-0.088(0)	-0.067(0)
	β_1	-0.012(0)	-0.015(0)	-0.018(0)	-0.019(0)	-0.014(0)
	R^2	0.09	0.098	0.231	0.109	0.08
monthly	β_0	-0.123(0)	-0.2(0)	-0.178(0)	-0.174(0)	-0.124(0)
	β_1	-0.033(0)	-0.053(0)	-0.044(0)	-0.044(0)	-0.032(0)
	R^2	0.18	0.252	0.291	0.119	0.107

Table 3: Results for linear regressions between RSV and returns for our indices.

In Table 3, all linear models have significant parameters. However, lower values of R-squared than in Table 2 indicate, that we can explain returns more properly with linear regressions without logarithmic function.

vol(t)-ret(t)	rv5,rv10,medrv/log	S&P	AEX	AORD	BSENS	HSI
daily	β_0	-0.01(0)	-0.014(0)	-0.01(0)	-0.019(0)	-0.011(0)
	β_1	-0.001(0)	-0.002(0)	-0.002(0)	-0.003(0)	-0.002(0)
	R^2	0.01	0.017	0.014	0.022	0.008
weekly	β_0	-0.034(0)	-0.05(0)	-0.045(0)	-0.059(0)	-0.035(0)
	β_1	-0.008(0)	-0.011(0)	-0.01(0)	-0.013(0)	-0.007(0)
	R^2	0.035	0.053	0.06	0.049	0.02
monthly	β_0	-0.106(0)	-0.176(0)	-0.14(0)	-0.144(0)	-0.098(0)
	β_1	-0.031(0)	-0.05(0)	-0.037(0)	-0.037(0)	-0.026(0)
	R^2	0.15	0.218	0.19	0.086	0.068

Table 4: Results for linear regressions between logarithm of mixed volatility and returns.

In Table 4 we again choose a logarithmic function, but for mixed volatility and look on the significance of parameters and impact on R^2. All parameters are significant, but we can observe significantly lower values for coefficients of determination than in Table 2 and Table 3.

The empirical study shows that volatility transformed from realized semi-variance performs better than the mixed realized volatility measured by three different volatility proxies. In other words, RSV can better capture the leverage effect.

2.3 Volatility and average returns

When volatility increases on high level, investors quickly feel a threat of next crises, and they expect that short-term returns will be reduced. Based on these predictions, they start selling to avoid decreased returns. But Fama points out, that high level of volatility should be associated with price declines and that increase expected returns(Fama,1965). According to Fama, investors can by mistake consider high-level of volatility as a sign of weak short-term profits instead of an indication of previous prices declines. In this subsection, we examine if stock market volatility levels have statistically reliable information about future stock returns. We analyze historical data to see if there have been statistically reliable differences in average returns between more volatile and less volatile periods, specifically if there is any relation between current volatility and subsequent returns. So,

we separately take daily, weekly and monthly returns for all five indices and break them up based on the previous volatility to a group of higher volatility (75th percentile or above) and a group of lower volatility (25th percentile or below). We then calculate the average returns for these groups.

S&P	rv5,rv10,medrv	avg return	avg vol	rsv	avg return	avg vol
daily	Low volatility	-0.00015	0.00331	Low volatility	-0.00025	0.00233
	High volatility	0.00002	0.0145	High volatility	0.00028	0.01151
weekly	Low volatility	0.00013	0.00804	Low volatility	0.0007	0.00606
	High volatility	0.00127	0.03166	High volatility	0.00171	0.02506
monthly	Low volatility	0.0027	0.01862	Low volatility	0.00318	0.01428
	High volatility	0.00341	0.06551	High volatility	0.0049	0.05148
AEX	rv5,rv10,medrv	avg return	avg vol	rsv	avg return	avg vol
daily	Low volatility	-0.00006	0.00396	Low volatility	-0.00015	0.00294
	High volatility	-0.00093	0.01507	High volatility	-0.00068	0.012
weekly	Low volatility	-0.00092	0.00949	Low volatility	-0.00109	0.0073
	High volatility	-0.00373	0.03311	High volatility	-0.00454	0.02628
monthly	Low volatility	-0.00122	0.02128	Low volatility	-0.00117	0.01673
	High volatility	-0.01229	0.06855	High volatility	-0.01145	0.05425
AORD	rv5,rv10,medrv	avg return	avg vol	rsv	avg return	avg vol
daily	Low volatility	0.0004	0.0028	Low volatility	0.00034	0.00179
	High volatility	-0.00021	0.00968	High volatility	-0.00013	0.00757
weekly	Low volatility	0.00046	0.00693	Low volatility	0.00181	0.00477
	High volatility	-0.00074	0.02097	High volatility	-0.00028	0.01639
monthly	Low volatility	0.00457	0.01558	Low volatility	0.00252	0.01149
	High volatility	-0.00571	0.04278	High volatility	-0.00782	0.03301

HSI	rv5,rv10,medrv	avg return	avg vol	rsv	avg return	avg vol
daily	Low volatility	-0.00025	0.00421	Low volatility	-0.00046	0.00318
	High volatility	-0.00064	0.01265	High volatility	-0.00047	0.01085
weekly	Low volatility	-0.0017	0.00983	Low volatility	-0.00177	0.00786
	High volatility	-0.00315	0.02722	High volatility	-0.00281	0.02311
monthly	Low volatility	0.0004	0.02207	Low volatility	-0.00246	0.01834
	High volatility	-0.01774	0.05592	High volatility	-0.01653	0.04727
BSESN	rv5,rv10,medrv	avg return	avg vol	rsv	avg return	avg vol
daily	Low volatility	-0.00107	0.00461	Low volatility	-0.00101	0.00339
	High volatility	-0.00156	0.01673	High volatility	-0.00177	0.01371
weekly	Low volatility	-0.00605	0.01076	Low volatility	-0.00594	0.00852
	High volatility	-0.00714	0.03639	High volatility	-0.00794	0.02964
monthly	Low volatility	-0.02406	0.02405	Low volatility	-0.02166	0.01956
	High volatility	-0.01833	0.07518	High volatility	-0.01892	0.06104

Table 5: Averages of returns after periods with high and low levels of volatility(RSV and mixed volatility) for daily, weekly and monthly data.

Average returns in S&P500 index after a period with higher volatility (75th percentile or above) are slightly higher than average returns when the previous period had lower volatility (25th percentile or below). This result could indicate the truth of Fama hypothesis, but it is not confirmed after examining the other results. Results from AEX, AORD, HSI indices even support investors view of declining returns after a period of high volatility.

Because stock returns have been noisy, these differences in average returns have not been reliably different from zero. In other words, there does not seem to be an economically meaningful difference in average equity returns based on the volatility of the prior period.

So results show the randomness of the relation between recent volatility and future returns. Thus, recent volatility doesn't indicate if future returns will be high and don't mean a low level of future returns. This feature we confirm through regression analysis in the next subsection.

2.4 Linear predictability

We again use linear regression as a statistical process for estimating the strength of the relationship between future returns and volatility. For this purpose, we use mixed volatility and RSV like in the previous subsection.

vol(t-1)-ret(t)	rsv/nolog	S&P	AEX	AORD	BSENS	HSI
daily	β_0	0(0.09)	0(0.33)	0(0.98)	-0.001(0)	0(0)
	β_1	0.087(0.01)	-0.015(0.68)	0(0.95)	0.001(0.97)	0.075(0.07)
	R^2	0.001	0	0	0	0
weekly	β_0	0(0.82)	0.001(0.26)	0.001(0.09)	-0.004(0)	-0.002(0.09)
	β_1	0.003(0.96)	-0.243(0)	-0.198(0.06)	-0.02(0.82)	0.057(0.55)
	R^2	0	0.007	0.003	0	0
monthly	β_0	0.008(0.07)	0.003(0.56)	0.017(0)	-0.019(0.01)	0(0.97)
	β_1	-0.247(0.09)	-0.382(0.04)	-0.849(0)	-0.053(0.78)	-0.254(0.16)
	R^2	0.012	0.017	0.059	0	0.008

Table 6: Relation between RSV and future returns.

In Table 6, we can see some significant parameters for volatility, namely the future daily returns of S&P500, the future weekly returns of AEX and the future monthly returns for AEX. However, looking at the coefficient of determination of these linear models, we conclude that RSV can't even partially model future returns in these cases. The exception is the AORD index for monthly returns, which model has significant parameters and coefficient of determination equal to 0.06. It indicates that the model could partially capture the values of future returns.

In Table 7, we choose a logarithmic function of RSV to linear regressions and look on coefficients of determination. For model with future monthly returns of AEX, we get significant parameters and a determination coefficient equal to 0.04. The model could partially capture the value of future returns. However, the model doesn't overcome the same model without the logarithmic function.

vol(t-1)-ret(t)	rsv/log	S&P	AEX	AORD	BSENS	HSI
daily	β_0	0.001(0.19)	-0.002(0.19)	-0.001(0.28)	-0.003(0.06)	0(0.9)
	β_1	0(0.20)	0(0.29)	0(0.27)	0(0.22)	0(0.72)
	R^2	0	0	0	0	0
weekly	β_0	0.002(0.64)	-0.015(0.02)	-0.007(0.16)	-0.008(0.29)	-0.003(0.62)
	β_1	0(0.68)	-0.003(0.05)	-0.001(0.15)	0(0.68)	0(0.8)
	R^2	0	0.003	0.001	0	0
monthly	β_0	-0.017(0.38)	-0.057(0.02)	-0.066(0)	-0.02(0.47)	-0.051(0.03)
	β_1	-0.005(0.33)	-0.013(0.05)	-0.016(0)	0(0.97)	-0.012(0.06)
	R^2	0.004	0.016	0.04	0	0.014

Table 7: Linear regressions between logarithm of RSV and future returns.

vol(t-1)-ret(t)	rv/log	S&P	AEX	AORD	BSENS	HSI
daily	β_0	0.001(0.45)	-0.002(0.07)	-0.001(0.11)	-0.002(0.15)	-0.001(0.48)
	β_1	0(0.48)	0(0.12)	0(0.11)	0(0.41)	0(0.62)
	R^2	0	0	0	0	0
weekly	β_0	0.001(0.77)	-0.014(0.03)	-0.009(0.12)	-0.01(0.18)	-0.009(0.19)
	β_1	0(0.81)	-0.002(0.06)	-0.002(0.11)	-0.001(0.5)	-0.001(0.28)
	R^2	0	0.003	0.002	0	0.001
monthly	β_0	-0.019(0.29)	-0.054(0.02)	-0.06(0)	-0.02(0.45)	-0.052(0.02)
	β_1	-0.006(0.24)	-0.013(0.05)	-0.016(0)	0(0.97)	-0.013(0.04)
	R^2	0.005	0.016	0.034	0	0.016

Table 8: Linear regressions between logarithm of mixed volatility and future returns.

We use mixed volatility with logarithmic function in Table 8, but don't get better linear regressions than in Table 6 and Table 7. So, we failed in construction elementary linear model to predict future returns only with realized volatility for daily or weekly periods.

Linear and multilinear models cannot capture the volatility clustering phenomenon. The effort to improve led us to the use of GARCH models and mean-reverting stochastic volatility models. The ARCH (Engle, 1982) and GARCH (Bollerslev, 1986) models aim to

describe the phenomenon of volatility clustering and kurtosis effect more accurately. The main idea behind these two models is that volatility is dependent upon past realizations of the asset process and related volatility process. In the next chapter, we describe both models in more details.

3 GARCH modeling

3.1 ARCH, GARCH model

The first model, which provide a possibility for modeling variance is model ARCH (Engle, 1982). Later Bollerslev extends preceding model noted as GARCH (generalized autoregressive conditional heteroskedasticity) (Bollerslev,1986). Both models are non-linear. In the GARCH model, conditional variance depends on previous conditional variances. It turns out that GARCH models can correctly model volatility clustering. GARCH model also has more extended memory as ARCH and highly flexible lag structure. GARCH models, which don't have normal distribution have evidence that is more robust for forecasting than other models (Liu and Morley, 2009). So, we consider these models useful and suitable for our subsequent analysis. GARCH model:

$$\begin{aligned} r_t &= \mu + \epsilon_t \\ \epsilon_t &= \sigma_t z_t \quad z_t \sim N(0,1) \\ \sigma_t^2 &= \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \end{aligned}$$

Where α_i and β_i are model parameters. This model is equivalent to ARCH model if $\forall \beta_i = 0$. There are also boundaries for coefficients to guarantee positive variance:

$$\begin{aligned} \alpha_1, ..., \alpha_{p-1} \geq 0, \alpha_p &> 0 \\ \beta_1, ..., \beta_{q-1} \geq 0, \beta_q &> 0 \\ \omega &> 0 \end{aligned}$$

Another boundary for guarantee stationarity:

$$\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i < 1$$

Models, which don't consider heteroskedasticity and volatility clustering are a less suitable choice for prices of stock indices. GARCH models can ensure these assumptions and give better results as ARCH models (Akgiray, 1989). A disadvantage of ARCH models is a frequent break of conditions of positive parameters. GARCH models can avoid overfitting and have better prediction with fewer parameters (Brooks, 2014).

The appropriate frequency of data to insert into the GARCH model is daily data. We could use weekly or monthly data, but that excludes some useful information. Into GARCH models it's not suitable to insert extreme amount of data, because markets change through time. Also, it is difficult to find a GARCH model that has significant parameters, and residuals meet the condition of non-correlation, so there is no reason to use distant historical data to estimate the current situation.

GARCH models are not especially close to perfect. It turns out if we are predicting with a time horizon of a month or more, then we have a small value from a GARCH model versus a simpler model (Bollerslev,1986). However, for predicting a few days, GARCH is considering as a quite useful model.

3.2 GARCH-M model

Next, we introduce GARCH-in-mean(GARCH-M) proposed by Engle et al., which is the only modification of the mean equation in the GARCH model. This model characterizes the evolution of the mean and the variance of a time series simultaneously (Engle, Lilien and Robins, 1987). GARCH-M model consists of the system:

$$\begin{aligned} r_t &= \mu + \delta g(\sigma_t^2) + \epsilon_t \\ \epsilon_t &= \sigma_t z_t \quad z_t \sim N(0,1) \\ \sigma_t^2 &= \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \end{aligned}$$

Where function $g(\sigma_t^2)$ can be used in 3 ways : $g(\sigma_t^2) = \sqrt{\sigma_t^2}$, $g(\sigma_t^2) = ln(\sigma_t^2)$,$g(\sigma_t^2) = \sigma_t^2$. In next part of thesis we use $g(\sigma_t^2) = \sqrt{\sigma_t^2}$. All boundaries are the same for GARCH and GARCH-M model.

The GARCH-M represents a simple way to model the relation between returns and its conditional variance. Investigation of this relationship between returns and volatility is one of our objectives in the thesis. In our research, we look at the variance coefficient in GARCH-in-mean models. We need a statistically significant and positive variance coefficient in the mean equation to conclude.

The persistence of a GARCH-M model demonstrates how fast large volatility decay after a shock(Lamoureux and Lastrapes,1990). For the GARCH-M(1,1) model the key

statistic is the sum of the two main parameters α_1 and β_1. The sum of α_1 and β_1 should be less than 1. If the sum is greater than 1, then the predictions of volatility are not stationary. The persistence of the model is a key indicator of the predictions. It determines how fast the predictions go to the unconditional volatility. If volatility contains much persistence and model can capture the persistence, then we have an excellent model for predictions.

It is practical to use basic GARCH-M model, but we can't be sure to get a significant variance parameter, so we can't evaluate the relation between conditional volatility and returns. We create GARCH-M models for all our indices and examine the variance parameters in subsection 3.6.

3.3 GJR-GARCH model

Glosten developed the GJR-GARCH model, which can capture positive and negative shocks for the conditional variance asymmetrically (Glosten, 1993). Difference between ordinary GARCH model and GJR-GARCH model is in variance equation:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \beta_t \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_t \epsilon_{t-i}^2 + \gamma_i \epsilon_{t-i}^2 I_{t-i}$$

Where $I_{t-i} = 0$ if $\epsilon_{t-i} \geq 0$ and $I_{t-i} = 1$ if $\epsilon_{t-i} < 0$. γ_i represents the coefficient of leverage effect.

GJR-GARCH model is better than common GARCH because it can capture leverage effect and it leads to better forecasting performance.

3.4 EGARCH model

The exponential generalized autoregressive conditional heteroskedastic model(EGARCH) proposed by Nelson is another type of the GARCH model (Nelson,1991). Volatility can react asymmetrically to good and bad news. Formally, an EGARCH(p,q) variance equation:

$$log(\sigma_t^2) = \omega + \sum_{k=1}^{q} (\alpha_k z_{t-k} + \gamma_k (|z_{t-k}| - E|z_{t-k}|)) + \sum_{k=1}^{p} \beta_k log(\sigma_{t-k}^2))$$

z_t denote a series of *i.i.d.* standardized random variables with mean 0 and variance 1. Coefficient α_k captures the sign effect, and γ_k can capture the size effect of the shocks on volatility. The first is a common effect of heteroskedasticity in squared residuals, while the second is an asymmetrical effect(leverage effect). Since $log(\sigma_t^2)$ can be negative, there are no boundaries for parameters.

3.5 REAL-GARCH and GARCH-X models

The effort to use realized volatility in GARCH models have led to the GARCH-X model. It's just adding the calculated realized volatility to variance equation:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \gamma x_{t-1}$$

The advantage of the GARCH-X model is that we can use different measures of realized volatility, which can specify current conditional volatility.

REAL-GARCH model differs from GARCH-X. However, compared to the GARCH model, both models can quickly respond to fluctuations. The following three equations are a formulation of REAL-GARCH model (Hansen et al.,2012):

$$\begin{aligned} r_t &= \mu_t + \sigma_t z_t \quad z_t \sim i.i.d(0,1) \\ log\sigma_t^2 &= \omega + \sum_{i=1}^{q} \alpha_i log x_{t-i} + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \\ log x_t &= \xi + \delta log\sigma_t^2 + \tau(z_t) + u_t \quad u_t \sim N(0,\lambda) \end{aligned}$$

Where we have define returns (r_t), the log of the conditional variance (σ_t^2) and the log of the realized measure (x_t). The asymmetric reaction to shocks is defined by function : $\tau(z_t) = \eta_1 z_t + \eta_2(z_t^2 - 1)$.

A key feature of REAL-GARCH model is an equation that relates the realized measure to the conditional variance of returns, which makes it possible to determine the accuracy of GARCH models with realized volatility in comparison with conventional GARCH models.

3.6 Application of GARCH models

This subsection examines the prediction power of ARMA-GARCH models and their modifications. Models Real-GARCH or GARCH-X contain realized volatility, so we determine if these models have better power of prediction like ordinary GARCH models based on conditional volatility without parameter including the behavior of realized volatility. We take our best ARMA-GARCH model, based on the AIC and BIC criteria to examine forecast accuracy of one day ahead prediction by rolling approach. We create all following GARCH models with daily returns from 2011-12-16 until 2018-01-01. For examination of prediction accuracy, we use the period from 2018-01-01 until 2019-04-12. For each index, this represents approximately 330 daily data. Finally, results from one day ahead forecast we compare to the real historical returns by metrics MAPE and RMSE.

It is necessary to mention formula for essential metrics that we use for evaluating the prediction strength of our GARCH models: root mean square error(RMSE) and mean absolute percentage error(MAPE). RMSE measures how different a set of values are. If we compare values from RMSE, the lowest value indices the closest predicted and observed sets. RMSE formula:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(\widehat{y_i} - y_i)^2}{n}}$$

where y_i is the observed value for the i-th observation and $\widehat{y_i}$ is the predicted value.

We use the MAPE metric because it has an intuitive interpretation in terms of relative error. Moreover, it does not matter the size of scale the stock indices move on since MAPE captures percentage changes. MAPE formula:

$$MAPE = \frac{100}{n}\sum_{i=1}^{n}\left|\frac{y_i - \widehat{y_i}}{y_i}\right|$$

where y_i is the observed value for the i-th observation and $\widehat{y_i}$ is the predicted value.

Table 9 provides an overview of summary statistic: mean, standard deviation, skewness, kurtosis, and Jarque-Bera test of normality of returns. The value of the skewness coefficient is negative for all indices, which means that the distributions of returns have long left tails. The values of the kurtosis coefficients are high, indicating that every time series of returns are leptokurtic. As expected, the Jarque-Berov test rejects normality for all stock indices at a 5% level of significance.

	S&P	AEX	AORD	BSESN	HSI
mean	0.00029	0.00001	0.00009	-0.00113	-0.00115
standard deviation	0.00741	0.00764	0.00726	0.00787	0.00804
skewness	-0.361	-0.207	-0.416	-0.213	-0.21
kurtosis	6.32	5.148	4.618	5.077	4.992
p-value of JB-test	0	0	0	0	0

Table 9: Summary statistic for our indices.

The behavior of our prices represented by time series is not constant. They show an irregularly increasing and decreasing trend. We can't expect that conditional variances are constant in such a large number of observations. To solve the problem of stationarity, we used the transformation of prices to returns. We still need to confirm that this transformation has made the time series of our indices stationary.

ADF test	S&P	AEX	AORD	BSESN	HSI
p-values	0	0	0	0	0

Table 10: P-values of Augmented Dickey–Fuller test for our indices.

To fit GARCH models on returns, they need to meet an assumption of stationarity. Stationary data have a constant mean, volatility, and covariance over the whole period. For confirming of data stationarity, we use augmented dickey fuller test(ADF) in Table 10.

Firstly, we must find a suitable ARIMA model so, we determine orders of AR and MA. We work with stationary returns, so we expected the order of differencing equal to zero for all models. After designing of ARIMA model, we investigate heteroscedasticity and the presence of autocorrelation in residuals. So, we applied the Q-test (Ljung-Box statistic), the results being centralized below (Table 11). In our calculations, we used 10,15 and 20-period lags for residuals and also for squared residuals.

In the standard procedure, ACF and PACF are used to determine the order of the autoregressive process and order of moving-average process. However, in our case, we firstly use the auto.arima function from the forecast package in R, which returns the best ARIMA model according to the AIC, AICc, and BIC. Then we check the significance of

ARMA	S&P	AEX	AORD	BSESN	HSI
order of AR	2	0	3	0	0
order of MA	0	0	2	0	0
$resQ(10)$	0.301	0.13	0.46	0.44	0.29
$resQ(15)$	0.076	0.2	0.79	0.36	0.31
$resQ(20)$	0.07	0.06	0.83	0.49	0.4
$res^2Q(10)$	0	0	0	0	0
$res^2Q(15)$	0	0	0	0	0
$res^2Q(20)$	0	0	0	0	0

Table 11: ARMA modeling for our indices.

parameters. If parameters are not significant, we choose fewer parameters and control their significance. However, it is necessary to look at the results of Ljung-Box statistics if the condition of nonlinearity in residuals is fulfilled and also if the condition of heteroskedasticity in square residuals is fulfilled. If the conditions are not met, it is necessary to change the parameters for suitability of GARCH models.

Ljung-Box statistic for residuals is higher than 5% for all indices. So, we can consider no serial correlation in our data after application of the ARMA model. We can see that squared residuals of most data series present serial correlation till the 20-lag (the Q-test being significant at 5%), thus confirming the presence of heteroscedasticity.

Since we confirm the presence of heteroskedasticity in returns of our indices, we can create a model, which can capture heteroskedasticity. As we mentioned, most suitable models are GARCH and his modifications like EGARCH, GJR-GARCH or GARCH-M. However, we want to examine also impact of realized volatility to GARCH models, so we also add GARCH-X and REAL-GARCH models.

In the following tables, we estimate the parameters of our GARCH models. We test Ljung-Box statistic for residuals and squared residuals and also look at AIC and BIC criteria to score their quality.

S&P returns	parameters	sGARCH	EGARCH	GJRGARCH	GARCH-M
mean equation	AR(1)	-0.972(0)	-0.924(0)	-0.941(0)	0.111(0.93)
	AR(2)	-0.036(0.15)	-0.036(0.01)	-0.034(0.16)	0.013(0.83)
	MA(1)	0.925(0)	0.877(0)	0.892(0)	-0.162(0.9)
	μ	0(0)	0(0)	0(0.01)	0(0.02)
	δ	N/A	N/A	N/A	0.23(0)
variance equation	ω	0(0)	-0.621(0)	0(0)	0(0.01)
	α_1	0.206(0)	0.215(0)	0.002(0.75)	0.204(0)
	β_1	0.748(0)	0.937(0)	0.797(0)	0.751(0)
	γ_1	N/A	0.189(0)	0.304(0)	N/A
$resQ(10)$		0.47	0.5	0.47	0.37
$resQ(15)$		0.18	0.22	0.2	0.26
$resQ(20)$		0.09	0.1	0.11	0.1
$res^2Q(10)$		0.33	0.79	0.74	0.37
$res^2Q(15)$		0.37	0.79	0.81	0.44
$res^2Q(20)$		0.66	0.94	0.94	0.72
AIC		-7.208	-7.278	-7.251	-7.214
BIC		-7.187	-7.254	-7.227	-7.19

Table 12: ARMA-GARCH models and their parameters for S&P500 index.

Table 12 shows the basic types of GARCH models and their application on the S&P500 index. ARMA model has been added to each GARCH model to ensure conditions for modeling with GARCH. Table 12 also shows the results of the Ljung-Box tests, which are always higher than 0.05 for residuals, and even at square residuals. It indicates no correlation in residuals and rejection of heteroscedasticity according to squared residuals. The best model in Table 12 according to AIC and BIC criteria is EGARCH. It also has all values significant. (P-values are given in parentheses after the parameter value). So, we use the EGARCH model in the final phase of the thesis to one day ahead forecast for the S&P500 index.

S&P returns and vol	parameters	GARCHX	GARCHX	REALGARCH	REALGARCH
mean equation	AR(1)	0.946(0)	0.937(0)	-0.553(0.17)	0.906(0)
	AR(2)	0.021(0.4)	0.027(0.29)	-0.034(0.16)	0.063(0)
	MA(1)	-0.982(0)	-0.982(0)	0.502(0.21)	-0.982(0)
	μ	0(0)	0(0)	0(0.34)	0(0)
variance equation	ω	0(1)	0(0)	-0.068(0.59)	-0.676(0)
	α_1	0.148(0)	0.192(0)	0.677(0)	0.921(0)
	β_1	0.602(0)	0.772(0)	0.622(0)	0.446(0)
realized equation	η_1	N/A	N/A	-0.276(0)	-0.073(0)
	η_2	N/A	N/A	0.021(0)	0.068(0)
	δ	N/A	N/A	0.48(0)	0.517(0)
	λ	N/A	N/A	0.31(0)	0.277(0)
	$x(rsv)$	0.003(0.04)	N/A	-0.759(0)	N/A
	$x(mixv)$	N/A	0(0.99)	N/A	-0.098(0.52)
$resQ(10)$		0.47	0.35	0.6	0.32
$resQ(15)$		0.35	0.25	0.21	0.15
$resQ(20)$		0.04	0.03	0.05	0.05
$res^2Q(10)$		0.23	0.48	0.16	0.3
$res^2Q(15)$		0.16	0.41	0.08	0.23
$res^2Q(20)$		0.4	0.7	0.22	0.5
AIC		-7.202	-7.147	-6.75	-6.96
BIC		-7.177	-7.122	-6.71	-6.92

Table 13: Results for ARMA-GARCH models with realized volatility variable for S&P500 index.

In Table 13, in contrast with the previous table, we see the types of GARCH models, which include realized volatility. In this chapter, we work with two measures of realized volatility, namely mixed volatility, which is equal to the average of RV5, RV10, and medRV. The second volatility is RSV. We introduced both in the second chapter, and we apply them to GARCH-X and REAL-GARCH models. The best model in Table 13 is GARCH-X, which contains the RSV as a variable. I also use this model for one day

ahead rolling prediction for the S&P500 index.

AEX returns	parameters	sGARCH	GJRGARCH	EGARCH	GARCH-M
mean equation	μ	0(0.09)	0(0.92)	0(0.67)	0(0.62)
	δ	N/A	N/A	N/A	0.077(0.32)
variance equation	ω	0(0.31)	0(0.93)	-0.237(0)	0(0.25)
	α_1	0.07(0)	0(0.99)	-0.111(0)	0.068(0)
	β_1	0.915(0)	0.924(0)	0.975(0)	0.917(0)
	γ_1	N/A	0.116(0.61)	0.101(0)	N/A
$resQ(10)$		0.26	0.14	0.12	0.28
$resQ(15)$		0.49	0.29	0.28	0.52
$resQ(20)$		0.2	0.16	0.14	0.2
$res^2Q(10)$		0.96	0.94	0.96	0.96
$res^2Q(15)$		0.97	0.93	0.97	0.97
$res^2Q(20)$		0.99	0.98	0.99	0.99
AIC		-7.052	-7.081	-7.09	-7.052
BIC		-7.041	-7.066	-7.08	-7.037

Table 14: GARCH models and their parameters with p-values for AEX index.

In Table 14, EGARCH is the best model, with the lowest values of the AIC and BIC criteria. GJR-GARCH is slightly worse, but it doesn't have a significant γ_1 parameter, so it is not suitable. EGARCH also fulfills all conditions for residuals, because Ljung-box tests p-values are higher than 0.05. The model has significant parameters so, we use it for forecasting future returns for the AEX index.

AEX returns and vol	parameters	GARCHX	GARCHX	REALGARCH	REALGARCH
mean equation	μ	0(0.32)	0(0.11)	0(0.65)	0(0.8)
variance equation	ω	0(0.78)	0(0.06)	0.129(0.49)	-0.057(0.73)
	α_1	0.069(0.46)	0.069(0)	0.709(0)	0.856(0)
	β_1	0.914(0)	0.914(0)	0.626(0)	0.549(0)
realized equation	η_1	N/A	N/A	-0.176(0)	-0.063(0)
	η_2	N/A	N/A	0.036(0)	0.047(0)
	δ	N/A	N/A	0.474(0)	0.481(0)
	λ	N/A	N/A	0.24(0)	0.229(0)
	$x(rsv)$	0(0.99)	N/A	-0.69(0)	N/A
	$x(mixv)$	N/A	0(0.99)	N/A	-0.365(0.04)
$resQ(10)$		0.26	0.26	0.18	0.2
$resQ(15)$		0.49	0.49	0.33	0.39
$resQ(20)$		0.2	0.2	0.07	0.06
$res^2Q(10)$		0.96	0.96	0.98	0.97
$res^2Q(15)$		0.97	0.97	0.98	0.95
$res^2Q(20)$		0.99	0.99	0.97	0.94
AIC		-7.05	-7.05	-6.99	-7.08
BIC		-7.03	-7.03	-6.96	-7.05

Table 15: Results for ARMA-GARCH models with realized volatility variables for AEX index.

In Table 15, for both GARCH-X models, we have a non-significant parameter for realized volatility(RSV and mixed volatility), which is why we don't consider these models suitable. In the REAL-GARCH model with RSV, we have higher p-values of parameters compared to REAL-GARCH with mixed volatility. However, the mixed-volatility REAL-GARCH model shows lower AIC and BIC values, so we choose it for one day ahead prediction.

AORD returns	parameters	sGARCH	EGARCH	GJRGARCH	GARCH-M
mean equation	AR(1)	0.699(0)	0.704(0)	0.7(0)	1.19(0)
	AR(2)	-0.961(0)	-0.949(0)	-0.957(0)	-0.203(0)
	AR(3)	-0.003(0.82)	0(0.99)	-0.003(0.84)	N/A
	MA(1)	-0.698(0)	-0.696(0)	-0.698(0)	-1.181(0)
	MA(2)	0.977(0)	0.965(0)	0.974(0)	0.186(0)
	μ	0(0.09)	0(0.81)	0(0.62)	N/A
	δ	N/A	N/A	N/A	0.027(0.41)
variance equation	ω	0(0.86)	-0.402(0)	0(0)	0(0.86)
	α_1	0.083(0.44)	-0.1(0)	0.004(0.67)	0.082(0.46)
	β_1	0.888(0)	0.959(0)	0.899(0)	0.889(0)
	γ_1	N/A	0.123(0)	0.118(0)	N/A
$resQ(10)$		0.32	0.3	0.29	0.12
$resQ(15)$		0.52	0.44	0.46	0.34
$resQ(20)$		0.7	0.62	0.61	0.41
$res^2Q(10)$		0.89	0.47	0.59	0.8
$res^2Q(15)$		0.91	0.62	0.69	0.83
$res^2Q(20)$		0.94	0.85	0.88	0.84
AIC		-7.11	-7.13	-7.13	-7.11
BIC		-7.08	-7.10	-7.10	-7.08

Table 16: ARMA-GARCH models and their parameters for AORD index.

In Table 16, based on BIC and AIC values, we chose the EGARCH model again. In this case, it had the same AIC and BIC values as GJR-GARCH, but GJR-GARCH has a non-significant parameter α_1, which we consider as an essential parameter. All Ljung-box tests p-values are higher than 0.05 as we expected, so the EGARCH model is suitable for forecasting.

AORD ret and vol	parameters	GARCHX	GARCHX	REALGARCH	REALGARCH
mean equation	AR(1)	0.708(0)	0.706(0)	1.219(0)	0.96(0)
	AR(2)	-0.966(0)	-0.964(0)	-1.014(0)	-0.769(0)
	AR(3)	0.001(0.93)	0.001(0.89)	0.016(0.04)	-0.007(0.8)
	MA(1)	-0.702(0)	-0.699(0)	-1.205(0)	-0.919(0)
	MA(2)	0.977(0)	0.977(0)	0.993(0)	0.729(0)
	μ	0(0.5)	0(0.31)	0(0.66)	0(0.85)
variance equation	ω	0(1)	0(1)	-0.292(0)	-0.501(0)
	α_1	0.059(0)	0.073(0)	0.33(0)	0.423(0)
	β_1	0.726(0)	0.784(0)	0.781(0)	0.723(0)
realized equation	η_1	N/A	N/A	-0.297(0)	-0.017(0)
	η_2	N/A	N/A	0.031(0)	0.099(0)
	δ	N/A	N/A	0.528(0)	0.553(0)
	λ	N/A	N/A	0.266(0)	0.259(0)
	$x(rsv)$	0.002(0)	N/A	-0.43(0.07)	N/A
	$x(mixv)$	N/A	0.001(0)	N/A	0.177(0.16)
$resQ(10)$		0.34	0.35	0.25	0.45
$resQ(15)$		0.54	0.55	0.39	0.47
$resQ(20)$		0.65	0.69	0.45	0.64
$res^2Q(10)$		0.67	0.9	0.01	0.11
$res^2Q(15)$		0.69	0.91	0.05	0.24
$res^2Q(20)$		0.85	0.97	0.16	0.54
AIC		-7.13	-7.12	-6.95	-6.93
BIC		-7.10	-7.09	-6.90	-6.89

Table 17: Results for ARMA-GARCH models with realized volatility variables for AORD index.

Table 17 shows that both REAL-GARCH models are not suitable, because of too high AIC and BIC values as well as non-significant realized volatility parameters. Model GARCH-X with RSV is the model with the lowest p-values of parameters and also the lowest AIC and BIC values. So we choose this model for forecasting.

BSESN returns	parameters	sGARCH	EGARCH	GJRGARCH	GARCH-M
mean equation	μ	0(0)	-0.001(0)	-0.001(0)	0(0.36)
	δ	N/A	N/A	N/A	-0.012(0.92)
variance equation	ω	0(0.27)	-0.234(0)	0(0.12)	0(0.33)
	α_1	0.047(0)	-0.071(0)	0.009(0.2)	0.047(0)
	β_1	0.934(0)	0.975(0)	0.93(0)	0.934(0)
	γ_1	N/A	0.095(0)	0.073(0)	N/A
$resQ(10)$		0.62	0.54	0.54	0.62
$resQ(15)$		0.4	0.32	0.32	0.4
$resQ(20)$		0.58	0.5	0.48	0.58
$res^2Q(10)$		0.76	0.46	0.55	0.77
$res^2Q(15)$		0.95	0.8	0.84	0.96
$res^2Q(20)$		0.91	0.64	0.74	0.91
AIC		-6.92	-6.94	-6.93	-6.92
BIC		-6.9	-6.92	-6.92	-6.9

Table 18: ARMA(3,2)-GARCH models and their parameters for BSESN index.

Table 18 shows that EGARCH is the best model for BSESN index among the basic types of GARCH models since all parameters have a p-value equal to zero, so parameters are significant. GJR-GARCH model has the same BIC as EGARCH, but GJR-GARCH has non-significant parameters. Moreover, EGARCH has the lowest value of AIC, and all p-values of Ljung-box tests are higher than 0.05, so this model fulfills conditions for residuals and squared residuals.

BSESN ret and vol	parameters	GARCHX	GARCHX	REALGARCH	REALGARCH
mean equation	μ	-0.001(0)	-0.001(0)	-0.001(0)	-0.001(0)
variance equation	ω	0(1)	0(1)	-0.037(0.7)	-0.435(0.02)
	α_1	0.025(0.03)	0.026(0.03)	0.412(0)	0.626(0)
	β_1	0.789(0)	0.799(0)	0.769(0)	0.626(0)
realized equation	η_1	N/A	N/A	-0.202(0)	-0.055(0)
	η_2	N/A	N/A	0.034(0)	0.05(0)
	δ	N/A	N/A	0.454(0)	0.499(0)
	λ	N/A	N/A	0.264(0)	0.231(0)
	$x(rsv)$	0.002(0)	N/A	-0.927(0)	N/A
	$x(mixv)$	N/A	0.001(0)	N/A	-0.252(0.29)
$resQ(10)$		0.6	0.6	0.62	0.62
$resQ(15)$		0.41	0.4	0.41	0.33
$resQ(20)$		0.58	0.58	0.58	0.51
$res^2Q(10)$		0.68	0.7	0.48	0.48
$res^2Q(15)$		0.9	0.92	0.79	0.82
$res^2Q(20)$		0.54	0.54	0.76	0.67
AIC		-6.94	-6.93	-6.77	-7.03
BIC		-6.92	-6.92	-6.75	-7.01

Table 19: Results for ARMA(3,2)-GARCH models with realized volatility variable for BSESN index.

In Table 19 we see that although REAL-GARCH with mixed volatility has the smallest BIC and AIC values, the model is not suitable because of a non-significant realized volatility parameter. REAL-GARCH with RSV looks better in terms of the significance of the parameters, but very poorly in AIC and BIC values. So we chose the GARCH-X model with RSV that meets all our requirements.

HSI returns	parameters	sGARCH	EGARCH	GJRGARCH	GARCH-M
mean equation	μ	0(0)	-0.001(0)	-0.001(0)	0(0.29)
	δ	N/A	N/A	N/A	-0.005(0.96)
variance equation	ω	0(0.39)	-0.227(0)	0(0.13)	0(0.31)
	α_1	0.05(0)	-0.075(0)	0.01(0.18)	0.05(0)
	β_1	0.932(0)	0.976(0)	0.929(0)	0.931(0)
	γ_1	N/A	0.096(0)	0.078(0)	N/A
$resQ(10)$		0.65	0.63	0.63	0.65
$resQ(15)$		0.53	0.47	0.49	0.53
$resQ(20)$		0.62	0.57	0.55	0.63
$res^2Q(10)$		0.73	0.42	0.5	0.72
$res^2Q(15)$		0.93	0.74	0.78	0.93
$res^2Q(20)$		0.9	0.65	0.73	0.9
AIC		-6.88	-6.9	-6.9	-6.88
BIC		-6.87	-6.89	-6.88	-6.87

Table 20: GARCH models and their parameters for HSI index.

Table 20 shows that EGARCH is again better than other models. All parameters are significant and, also AIC value is the lowest.

As we mentioned before the GARCH-M model represents a simple way to examine the relation between returns and its conditional volatility. In our GARCH-in-mean models, we examine the hypothesis that an increase in volatility leads to an increase in returns. For confirming this hypothesis, we need a statistically significant and positive volatility parameter (δ) in the mean equation. For GARCH-M models in tables 14,16,18,20, their volatility parameters are not significant. So we can't find out the relation between conditional volatility and returns. However, the GARCH-M model in Table 12 has a significant parameter of δ. This parameter is positive, so this model indicates a positive relationship between conditional volatility and returns of S&P500 index. But based on one model we can't confirm this hypothesis for all indices.

HSI returns and vol	parameters	GARCHX	GARCHX	REALGARCH	REALGARCH
mean equation	μ	0(0.03)	0(0.05)	0(0.03)	0(0.03)
variance equation	ω	0(1)	0(1)	-0.385(0)	-0.934(0)
	α_1	0.018(0.21)	0.024(0.09)	0.342(0)	0.427(0)
	β_1	0.785(0)	0.794(0)	0.772(0)	0.679(0)
realized equation	η_1	N/A	N/A	-0.176(0)	-0.022(0)
	η_2	N/A	N/A	0.039(0)	0.066(0)
	δ	N/A	N/A	0.533(0)	0.617(0)
	λ	N/A	N/A	0.245(0)	0.22(0)
	$x(rsv)$	0.002(0)	N/A	-0.162(0.51)	N/A
	$x(mixv)$	N/A	0.001(0)	N/A	0.9(0)
$resQ(10)$		0.18	0.17	0.19	0.19
$resQ(15)$		0.38	0.36	0.4	0.4
$resQ(20)$		0.24	0.22	0.29	0.29
$res^2Q(10)$		0.24	0.2	0.35	0.35
$res^2Q(15)$		0.57	0.51	0.6	0.6
$res^2Q(20)$		0.72	0.65	0.78	0.78
AIC		-6.98	-6.97	-6.95	-7.16
BIC		-6.96	-6.96	-6.92	-7.13

Table 21: Results for ARMA-GARCH models with realized volatility variable for HSI index.

In Table 21, the REAL-GARCH model with mixed volatility dominates. It has the smallest values of AIC and BIC criteria compare to other models and all parameters significant, unlike GARCH-X models. REAL-GARCH model also has all p-values of Ljung-box tests higher than 0.05. So we choose this model to predict future returns of the HSI index.

3.7 Distribution of error term

Table 9 shows we can see high values of kurtosis. It means that the distribution of returns in our financial data differs significantly compared to the normal distribution(kurtosis of the normal distribution is equal to 3).

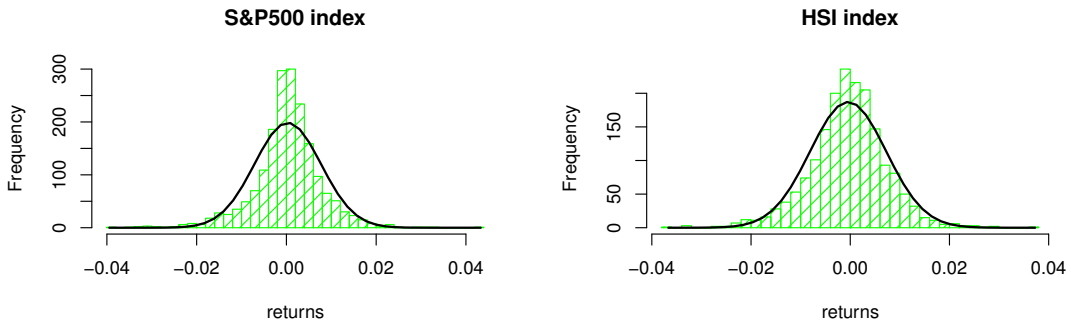

Figure 5: Daily returns distribution for HSI and S&P500 indices compared to the normal distribution.

Models with normally distributed errors are not fully capable of capturing returns distribution in our time series. When we model the returns of our indices, we nevertheless assume that the error term is from a normal distribution. Many statistical packages, such as R software and other statistical modeling tools can provide generalized error distribution or Student's t distribution for residuals instead of normal distribution. These distributions better describe fat tails than other distributions. For our GARCH models, we repeat estimate of the parameters assuming that residuals have Student's t distribution and generalized error distribution. According to AIC and BIC criteria, we can designate the best models. We can see the results in Table 22.

In Table 22, we see for each of our indices two models that we selected in the previous subsection(3.6). Then we examine their AIC and BIC criteria for Student's t distribution(std) and generalized error distribution(ged). In all cases, the normal distribution is worse than ged or std. Models with the lowest AIC and BIC criteria we use in the next subsection(3.8) to one day ahead prediction, and later we determine the accuracy of their forecasting by metrics MAPE and RMSE.

	crit	NORM	GED	STD		crit	NORM	GED	STD
GARCHX	AIC	-7.202	-7.224	-7.251	EGARCH	AIC	-7.278	-7.335	-7.336
S&P	BIC	-7.177	-7.197	-7.223	S&P	BIC	-7.254	-7.308	-7.309
REALGARCH	AIC	-7.212	-7.238	-7.243	EGARCH	AIC	-7.097	-7.126	-7.128
AEX	BIC	-7.185	-7.209	-7.213	AEX	BIC	-7.082	-7.108	-7.111
GARCHX	AIC	-7.137	-7.152	-7.15	EGARCH	AIC	-7.135	-7.145	-7.153
AORD	BIC	-7.107	-7.119	-7.118	AORD	BIC	-7.105	-7.113	-7.12
GARCHX	AIC	-6.943	-6.97	-6.961	EGARCH	AIC	-6.94	-6.968	-6.974
BSESN	BIC	-6.928	-6.952	-6.943	BSESN	BIC	-6.925	-6.95	-6.956
REALGARCH	AIC	-7.165	-7.194	-7.191	EGARCH	AIC	-6.978	-7.008	-7.005
HSI	BIC	-7.137	-7.164	-7.161	HSI	BIC	-6.963	-6.99	-6.986

Table 22: AIC and BIC criteria of normal, Student's t-distribution and generalized error distribution apply to best selected GARCH models.

3.8 Forecasting with GARCH models

In this subsection, we examine the accuracy of one day ahead prediction by rolling approach. As we mentioned before we take daily returns from 2011-12-16 to 2019-04-12. We apply prediction by rolling approach from 2018-01-01 to 2019-04-12. For each index, this represents approximately 330 daily data. For this prediction, we take our best ARMA-GARCH models from Table 22, based on the lowest AIC and BIC.

Later we transform results from one day ahead prediction from returns back to close prices for better interpretation of accuracy metrics. We compare our estimated close prices to the real historical close prices by metrics MAPE and RMSE in Table 24.

Table 24 shows similar values for the accuracy of GARCH models within the same index. Only in the S&P500 index, both metrics confirm the advantage of the model that includes realized volatility (ARMA (2,1) - GARCH-X (RSV) STD) versus the EGARCH model. In the other indices, we can't judge, which of the models is better, because both metrics prefer a different model.

model	index	mape[%]	rmse
ARMA(2,1)- GARCH-X(RSV) STD	S&P	0.6031	23.968
ARMA(2,1)- EGARCH STD	S&P	0.6024	23.933
REAL-GARCH(mixed vol) STD	AEX	0.4477	3.143
EGARCH STD	AEX	0.4479	3.143
ARMA(3,2)- GARCH-X(RSV) GED	AORD	0.4821	39.602
EGARCH STD	AORD	0.4829	39.582
GARCH-X(RSV) GED	BSESN	0.5472	245.534
EGARCH STD	BSESN	0.5474	245.491
REAL-GARCH(mixed vol) GED	HSI	0.6858	252.573
EGARCH GED	HSI	0.6859	252.53

Table 23: Accuracy of one day ahead prediction by rolling approach with the most suitable GARCH models for our indices.

Finally, we see that GARCH models don't have worse one day ahead prediction than GARCH models that include realized volatility. But, GARCH models are suitable only for a short-term forecast. For longer-term forecasting, we would have to look for other models.

Conclusion

We examined the relationship between volatility and returns for indices: S&P500, AEX, AORD, BSESN, HSI from January 2000 to April 2019. Firstly, we created a necessary theoretical background for a better understanding of basic terms. Our data and their descriptions were shown in section (2.1). We chose different measures of realized volatility and included them in our models(RSV and mixed volatility). Also, we showed the correlation between individual measurements of realized volatility and returns. In section (2.2) we demonstrate the leverage effect on daily, weekly and monthly data by linear regression. Realized volatility in linear regression had a significant parameter for all linear models so it can partially explain returns in the same period. Moreover, parameters for all models were negative, which indicate a negative correlation between returns and realized volatility in the same period. We also investigated the level of returns for the stock market indices after periods of high volatility and after periods of low volatility. We get inconclusive results so, there is no rule for returns after level of high or low realized volatility. Later, in section (2.4) we tried to create a predicting model for returns, but none of our models have significant parameters. Generally, it is quite hard to predict returns. If we can do it correctly we can earn a lot of money, so we didn't expect significant results. In the last chapter, we firstly introduced different types of GARCH models. We decided to use them for one day ahead return forecast by rolling approach. Moreover, we found models, which could include realized volatility(GARCH-X and REAL-GARCH). These models we compared with GARCH, EGARCH, GJR-GARCH, and GARCH-M in section 3.6 and took the best models according to AIC and BIC criteria. We were looking for distribution, which could better capture the distribution of errors for GARCH models in section 3.7. Instead of the normal distribution, we tested our best models with generalized error distribution and student's distribution. After that, in 3.8 we review the power of one day ahead forecasting for our best models. We showed the accuracy of forecasting by using the MAPE and RMSE metrics.

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Appendix

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library(datasets);library(quantmod);library(quantmod);library(astsa);
library(stats);library(highfrequency);library(xts);library(forecast);
library(rugarch);library(plm);library(scatterplot3d);library(dplyr);
library(roll);library(ggplot2);library(e1071);library(normtest);
library(LSTS);library(tseries);library(aTSA);library(reshape2);
library(MTS);library(lmtest);library(Metrics);library(MLmetrics)
setwd("~/R/diplomovka")
#data
panel <- read.csv("oxfordman.csv",header=TRUE,sep = ",")
panel <- na.omit(panel)
rv <- panel[,-15:-20]
panel2 <- panel[panel$Symbol == ".SPX",]
panel2 <- panel2[,-2:-14]
panel2<- xts(panel2[, -1], order.by=as.POSIXct(as.character(panel2$date),
    format = "%m/%d/%Y" ))
close_SPX <- panel2[,6]
open_SPX <- panel2[,4]

# returns
return_SPX <- log(close_SPX)-log(open_SPX)
week_return_SPX <- apply.weekly(return_SPX, sum)
week_return_SPX <- week_return_SPX[-1,]
week_return_SPX <- week_return_SPX[-length(week_return_SPX),]
month_return_SPX <- apply.monthly(return_SPX, sum)
month_return_SPX <- month_return_SPX[-1,]
month_return_SPX <- month_return_SPX[-length(month_return_SPX),]


# realized variances dataset
SPX_rv <- rv[rv$Symbol == ".SPX",]
SPX_rv <- SPX_rv[,-2]
SPX_rv <- xts(SPX_rv[, -1], order.by=as.POSIXct(as.character(SPX_rv$date),
    format = "%m/%d/%Y" ))

# correlation between returns and realized semivariance in the same period
vec_cor = NULL
```

```
for (i in 1:12){
  vec_cor[i] <- cor(as.numeric(SPX_rv[,i]),return_SPX)
}
vec_cor

# RSV
daily_rv_SPX <- apply.daily(SPX_rv[,1] , rowMeans)
daily_vol_SPX <- sqrt(daily_rv_SPX)

week_vol_SPX <- apply.weekly(SPX_rv[,1],colSums)
week_vol_SPX <- week_vol_SPX[-1,]
week_vol_SPX <- week_vol_SPX[-length(week_vol_SPX),]
week_vol_SPX<- sqrt(week_vol_SPX)

month_vol_SPX <- apply.monthly(SPX_rv[,1],colSums)
month_vol_SPX <- month_vol_SPX[-1,]
month_vol_SPX <- month_vol_SPX[-length(month_vol_SPX),]
month_vol_SPX <- sqrt(month_vol_SPX)

#mixed volatility(same variable as RSV)
daily_rv_SPX <- apply.daily(SPX_rv[,c(2,7,11)] , rowMeans)
daily_vol_SPX <- sqrt(daily_rv_SPX)

week_vol_SPX <- apply.weekly(SPX_rv[,c(2,7,11)],colSums)
week_vol_SPX <- week_vol_SPX[-1,]
week_vol_SPX <- apply.weekly(week_vol_SPX,rowMeans)
week_vol_SPX <- week_vol_SPX[-length(week_vol_SPX),]
week_vol_SPX<- sqrt(week_vol_SPX)

month_vol_SPX <- apply.monthly(SPX_rv[,c(2,7,11)],colSums)
month_vol_SPX <- month_vol_SPX[-1,]
month_vol_SPX <- apply.weekly(month_vol_SPX,rowMeans)
month_vol_SPX <- month_vol_SPX[-length(month_vol_SPX),]
month_vol_SPX <- sqrt(month_vol_SPX)

#logaritmic function
daily_vol_SPX <- log(daily_vol_SPX)
week_vol_SPX <- log(week_vol_SPX)
```

```
73 month_vol_SPX <- log(month_vol_SPX)
74
75 #graphs for volatility and returns
76 plot(return_SPX, main = 'S&P500 returns')
77 plot(daily_vol_SPX, main = 'S&P500 daily volatility')
78 plot(week_vol_SPX, main = 'S&P500 week-volatility')
79 plot(week_return_SPX, main = 'S&P500 week-return')
80 plot(month_vol_SPX, main = 'S&P500 month-volatility')
81 plot(month_return_SPX, main = 'S&P500 return-month')
82
83 # linear regression for daily returns and realized volatility(leverage
      effect)
84 day_lm <- lm(return_SPX~daily_vol_SPX)
85 summary(day_lm)
86 # plot of daily returns vs daily volatility
87 df <- data.frame(as.numeric(daily_vol_SPX[,1]),as.numeric(return_SPX[,1]))
88 colnames(df) <- c('daily_volatility','daily_return')
89 ggplot(df, aes(x=daily_volatility, y=daily_return))+
90 geom_point(shape=1) + geom_smooth(method=lm , color="green", se=FALSE) +
91 labs(title="Daily data S&P500",y="returns",x='RSV')
92 # linear regression for weekly returns and realized volatility
93 week_lm <- lm(week_return_SPX~week_vol_SPX)
94 summary(week_lm)
95 plot(as.numeric(week_vol_SPX[,1]),as.numeric(week_return_SPX[,1]),
96   xlab = 'weekly volatility', ylab= 'weekly returns',main = 'SPX' )
97 abline(week_lm,col = 'red')
98 # linear regression for monthly returns and realized volatility
99 month_lm <- lm(month_return_SPX~month_vol_SPX)
100 summary(month_lm)
101 plot(as.numeric(month_vol_SPX[,1]),as.numeric(month_return_SPX[,1]),
102   xlab = 'monthly volatility', ylab= 'monthly returns',main = 'SPX')
103 abline(month_lm,col = 'red')
104
105 # predictability of future daily returns by realized volatility
106 daily_volt1_SPX <- daily_vol_SPX[-length(daily_vol_SPX),]
107 returnt_SPX <- return_SPX[-1,]
108 daily_lmt1 <- lm(returnt_SPX~ daily_volt1_SPX )
109 summary(daily_lmt1)
```

```
plot(as.numeric(daily_volt1_SPX[,1]),as.numeric(returnt_SPX[,1]),
  xlab = 'daily volatility', ylab= 'daily returns',main = 'SPX' )
abline(daily_lmt1,col = 'red')

# average of daily returns after day of low volatility and high volatility
volreturn <- cbind(as.numeric(daily_volt1_SPX[,1]),
  as.numeric(returnt_SPX[,1]))
head(volreturn)
volreturn <- volreturn[order(volreturn[,1]),]
a <- round(length(volreturn)/8)
b <- round(length(volreturn)/2)
c <- volreturn[1:a,]
d <- volreturn[(b-a):b,]
x <- volreturn[a : (b-a),]
mean(c[,2])
mean(c[,1])
mean(d[,2])
mean(d[,1])

# predictability of future weekly returns by realized volatility
week_volt1_SPX <- week_vol_SPX[-length(week_vol_SPX),]
week_returnt_SPX <- week_return_SPX[-1,]
week_lmt1 <- lm(week_returnt_SPX~week_volt1_SPX)
summary(week_lmt1)
plot(as.numeric(week_volt1_SPX[,1]),as.numeric(week_returnt_SPX[,1]),
  xlab = 'weekly volatility', ylab= 'weekly returns',main = 'SPX' )
abline(week_lmt1,col = 'red')

# average of weekly returns after week of low volatility and high
    volatility
volreturnweek <- cbind(as.numeric(week_volt1_SPX[,1]),
  as.numeric(week_returnt_SPX[,1]))
head(volreturnweek)
volreturnweek <- volreturnweek[order(volreturnweek[,1]),]
e <- round(length(volreturnweek)/8)
f <- round(length(volreturnweek)/2)
g <- volreturnweek[1:e,]
h <- volreturnweek[(f-e):f,]
```

```
mean(g[,2])
mean(g[,1])
mean(h[,2])
mean(h[,1])

# predictability of future monthly returns by realized volatility
month_volt1_SPX <- month_vol_SPX[-length(month_vol_SPX),]
month_returnt_SPX <- month_return_SPX[-1,]
month_lmt1 <- lm(month_returnt_SPX~month_volt1_SPX)
summary(month_lmt1)
plot(as.numeric(month_volt1_SPX[,1]),as.numeric(month_returnt_SPX[,1]),
  xlab = 'monthly volatility', ylab= 'monthly returns',main = 'SPX' )
abline(month_lmt1,col = 'red')

# average of monthly returns after month of low volatility and high
    volatility
volreturnmonth <- cbind(as.numeric(month_volt1_SPX[,1]),
  as.numeric(month_returnt_SPX[,1]))
head(volreturnmonth)
volreturnmonth <- volreturnmonth[order(volreturnmonth[,1]),]
i <- round(length(volreturnmonth)/8)
j <- round(length(volreturnmonth)/2)
k <- volreturnmonth[1:i,]
l <- volreturnmonth[(j-i):j,]
mean(k[,2])
mean(k[,1])
mean(l[,2])
mean(l[,1])

# table 10
return_SPX[4519]
mean(return_SPX[3000:4844])
sd(return_SPX[3000:4844])
skewness(return_SPX[3000:4844])
kurtosis(return_SPX[3000:4844])
jarque.bera.test(return_SPX)

# table 11
```

```
184 adf.test(as.numeric(return_SPX[3000:4844]))
185
186 #table 12 ARMA models
187
188 auto.arima(return_SPX[3000:4844])
189 mod <- arima(return_SPX[3000:4844],order = c(2,0,1))
190 c <-mod$residuals
191 coeftest(mod)
192 Box.test(c , lag=10, type="Ljung")
193 Box.test(c , lag=15, type="Ljung")
194 Box.test(c , lag=20, type="Ljung")
195 Box.test(c^2 , lag=10, type="Ljung")
196 Box.test(c^2 , lag=15, type="Ljung")
197 Box.test(c^2 , lag=20, type="Ljung")
198
199 #GARCH tables
200
201 #sGARCH
202 garchm1 = ugarchspec(variance.model=list(garchOrder=c(1,1)),
203                      mean.model=list(armaOrder=c(2,1), include.mean=T))
204 fit1_daily<-ugarchfit(spec=garchm1, data=return_SPX[3000:4844])
205 show(fit1_daily)
206 a <- fit1_daily@fit$residuals
207 b <- fit1_daily@fit$sigma
208 stand.rez <- a/b
209
210 # GARCH-M
211
212 garchm2 = ugarchspec(variance.model=list(garchOrder=c(1,1)),
213                      mean.model=list(armaOrder=c(2,1), archm=T, archpow=1,
                         include.mean=T))
214 fit2_daily<-ugarchfit(spec=garchm2, data=return_SPX[3000:4844])
215 show(fit2_daily)
216 a <- fit2_daily@fit$residuals
217 b <- fit2_daily@fit$sigma
218 stand.rez <- a/b
219
220
```

```
# GJR GARCH
garchm3 =ugarchspec(variance.model=list(model = "gjrGARCH",
garchOrder=c(1,1)), mean.model=list(armaOrder=c(2,1), include.mean=T))
fit3_daily<-ugarchfit(spec=garchm3, data=return_SPX[3000:4844])
show(fit3_daily)
c <- fit3_daily@fit$residuals
d <- fit3_daily@fit$sigma
stand.rez <- c/d


# EGARCH

garchm4 = ugarchspec(variance.model=list(model = 'eGARCH',
          garchOrder=c(1,1)),distribution.model = "std",
               mean.model=list(armaOrder=c(2,1), include.mean=T))
fit4_daily<-ugarchfit(spec=garchm4, data=return_SPX[3000:4844])
show(fit4_daily)
c <- fit4_daily@fit$residuals
d <- fit4_daily@fit$sigma
stand.rez <- c/d


# forecast
fit4_daily = ugarchfit(data=return_SPX[3000:4844], spec = garchm4,
       out.sample = 326)

forc1 = ugarchforecast(fit4_daily , n.ahead= 326, n.roll = 326)
b<- sigma(forc1);
c <- fitted(forc1)
odhad_return <- t(as.numeric(c[1,]))
odhad_return <- odhad_return[-length(odhad_return)]
length(odhad_return)
odhad_close <- NULL
for (i in 1:326) {
  odhad_close[i] <- open_SPX[(length(open_SPX)-326)+i]*exp(odhad_return[i])
}
real_close <- (close_SPX)[(length(open_SPX)-325):length(open_SPX)]

```

```
u<-rbind(odhad_close,as.numeric(real_close))
plot(as.numeric(open_SPX)[4519:4844])
lines(u[1,],col = 'red',type = 'p')
rmse(u[2,], u[1,])
MAPE(u[1,],u[2,])

# GARCH-X

garchm5 = ugarchspec(variance.model=list(model = 'sGARCH',
       garchOrder=c(1,1),external.regressors = daily_vol_SPX[2999:4843]),
       mean.model=list(armaOrder=c(2,1), include.mean=T),
       distribution.model = "std")
fit5_daily<-ugarchfit(spec = garchm5, data=return_SPX[3000:4844],
        solver = "hybrid")
show(fit5_daily)
c <- fit5_daily@fit$residuals
d <- fit5_daily@fit$sigma
stand.rez <- c/d


# forecast
fit5_daily = ugarchfit(data=return_SPX[3000:4844], spec = garchm5,
        out.sample = 326, solver = "hybrid")

forc1 = ugarchforecast(fit5_daily , n.ahead= 326, n.roll = 326)
c <- fitted(forc1)
odhad_return <- t(as.numeric(c[1,]))
odhad_return <- odhad_return[-length(odhad_return)]
length(odhad_return)
odhad_close <- NULL
for (i in 1:326) {
  odhad_close[i] <- open_SPX[(length(open_SPX)-326)+i]*exp(odhad_return[i])
}
real_close <- (close_SPX)[(length(open_SPX)-325):length(open_SPX)]

u<-rbind(odhad_close,as.numeric(real_close))
plot(as.numeric(open_SPX)[4519:4844])
lines(u[1,],col = 'red',type = 'p')
```

```
297 rmse(u[2,], u[1,])
298 MAPE(u[1,],u[2,])
299
300 # real GARCH
301
302 garchm6 = ugarchspec(variance.model=list(model = 'realGARCH',
303         garchOrder=c(1,1)), mean.model=list(armaOrder=c(2,1),
304         include.mean=T),distribution.model = "norm")
305 fit6_daily<-ugarchfit(spec=garchm6, data=return_SPX[2930:4730],
306 solver = "hybrid", realizedVol = daily_vol_SPX[2930:4730])
307 show(fit6_daily)
308 c <- fit6_daily@fit$residuals
309 d <- fit6_daily@fit$sigma
310 stand.rez <- c/d
311
312 #Ljung-box tests for residuals and squared residuals
313 Box.test(stand.rez , lag=10, type="Ljung")
314 Box.test(stand.rez , lag=15, type="Ljung")
315 Box.test(stand.rez , lag=20, type="Ljung")
316 Box.test(stand.rez^2 , lag=10, type="Ljung")
317 Box.test(stand.rez^2 , lag=15, type="Ljung")
318 Box.test(stand.rez^2 , lag=20, type="Ljung")
319
320 # leptokurtic distribution(distribution of returns) vs normal distribution
321 h <- hist(return_SPX[2930:4730], breaks = 50, density = 10,
322           col = "green", xlab = "returns", main = "SPX index")
323 xfit <- seq(min(return_SPX[2930:4730]),
324       max(return_SPX[2930:4730]), length = 40)
325 yfit <- dnorm(xfit, mean = mean(return_SPX[2930:4730]),
326     sd = sd(return_SPX[2930:4730]))
327 yfit <- yfit * diff(h$mids[1:2]) * length(return_SPX[2930:4730])
328 lines(xfit, yfit, col = "black", lwd = 2)
```