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VALUATION OF THE AMERICAN-STYLE OF ASIAN OPTION BY A SOLUTION TO AN INTEGRAL EQUATION

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Abstract. We extend the model for valuation of American-style of Asian options introduced by Hansen, Jørgensen (2000) in [3] by including a nontrivial dividend rate q. We use the theory of conditioned expectations to calculate the formula of the American-style Asian floating strike option with a general average of the underlying asset. We determine an integral equation formula for the value of this type of an option with continuous geometric average and approximate formula for the continuous arithmetic average.

1. Introduction

Evolution of trading systems influences the development of the market of financial derivatives. First, the simple derivatives (as forwards and vanilla options) were used to hedge the risk of a portfolio. Progress in valuation of these simple financial instruments pushed traders into inventing less predictable and more complex derivatives. Using financial derivatives with more complicated pay-offs brings into attention also new mathematical problems.

Asian options belong to a group of path-dependent options, i.e. part of exotic options. Here the pay-off depends on the spot value of the underlying during the whole or some part(s) of the life span of the option. Asian options depend on the (arithmetic or geometric) average of the spot price of the underlying.

Asian options can be used as a tool for hedging the high volatility of the price of assets or goods. The price of an underlying varies during the life span of the option, the holder of the Asian option can be secured for the case when the price jumps to the unpleasant region (too high for call holder or too low for put holder) his loss will be reduced.

Asian options can be divided into two subgroups when considering the type of their pay-off function. The average strike Asian option and the fixed strike Asian

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option with the pay-off function for the call option

$$V_T(S, A) = (S - A)^+ \tag{1}$$

and

$$V_T(S, A) = (A - X)^+, \tag{2}$$

respectively.

2. A probabilistic model for pricing of American-Style of Asian options

In this section we provide a formula for the valuation of the early exercise boundary of an American-style Asian option paying nontrivial dividends. We follow the derivation introduced by Hansen, Jørgensen in [3]. Their formula for a floating strike option was derived using the theory of martingales and conditioned expected values. We extend the formula to Asian options on underlying paying non-zero dividend rate.

This model is based on the stochastic behavior of the underlying in time. It is assumed that it is driven by stochastic process satisfying the following stochastic differential equation

$$dS_t = (r - q)S_t \; dt + \sigma S_t \; dW_t^Q \quad \text{on the time interval } [0, T], \tag{3}$$

starting almost surely from the initial price $S_0 > 0$, where the constant parameter $r > 0$ denotes the risk-free interest rate, $q \geq 0$ is a dividend rate, σ is the volatility of stock returns and W_t^Q is a standard Brownian motion with respect to the standard risk-neutral probability measure Q. A solution of equation (3) corresponds to the geometric Brownian motion

$$S_t = S_0 e^{(r-q-\frac{1}{2}\sigma^2)t+\sigma W_t^Q}, \tag{4}$$

for $0 \leq t \leq T$.

The bond (risk-free) market is driven by the differential equation

$$dB_t = rB_t \; dt, \tag{5}$$

with $B_0 = 1$, i.e. $B_t = e^{rt}$.

As we have already mentioned above we shall derive the value of an American-style Asian option with floating strike. If we define the optimal stopping time as T^*, the pay-off of the option is set by

$$V_{T^*} = \Big(\rho(S_{T^*} - A_{T^*})\Big)^+, \tag{6}$$

where V_t is the value of the option at time t, A_t is a continuous average of the stock value during the interval $[0, t]$ and $\rho = 1$ for a call option and $\rho = -1$ for a put option. We may consider either the continuous arithmetic average

$$A_t = \frac{1}{t}\int_0^t S_u \, du, \tag{7}$$

or the continuous geometric average

$$\ln A_t = \frac{1}{t}\int_0^t \ln S_u \, du \tag{8}$$

or the weighted arithmetic average

$$A_t = \frac{1}{t}\int_0^t a(t-u) S_u \, du, \tag{9}$$

where the kernel function $a(.) \geq 0$ with the property $\int_0^\infty a(\zeta)\, d\zeta < \infty$ is usually defined as $a(s) = e^{-\lambda s}$ for $\lambda > 0$.

3. Valuation

We recall that derivation of the more simple type option was introduced in [3]. According to Hansen and Jørgensen, American-style contingent claims can be priced by the conditioned expectations approach. The option prices are evaluated by considering all possible stopping times in the interval $[t, T]$

$$V(t, S, A) = \operatorname*{ess\,sup}_{s \in \mathcal{T}_{[t,T]}} \mathbb{E}_t^Q \Big[e^{-r(s-t)} \Big(\rho(S_s - A_s) \Big)^+ \Big| S_t = S, A_t = A \Big], \tag{10}$$

where $\mathcal{T}_{[t,T]}$ denotes the set of all stopping times in the interval $[t, T]$ and $E_t^Q[X] = E^Q[X|\mathcal{F}_t]$ is the conditioned expectation with information of time t (the information is represented by the filtration $\mathcal{F}_t$ of the σ-algebra $\mathcal{F}$, where the Brownian motion is supported).

To simplify the formula we change the probability measure by the martingale

$$\eta_t = e^{-(r-q)t}\frac{S_t}{S_0} = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t^Q} \tag{11}$$

the new probability measure $\mathcal{Q}$ is defined by

$$d\mathcal{Q} = \eta_T \, dQ. \tag{12}$$

According to Girsanov's theorem, the process

$$W_t^{\mathcal{Q}} = W_t^Q - \sigma t \tag{13}$$

is a standard Brownian motion with respect to the measure $\mathcal{Q}$. The value of the stock under this measure is defined by

$$S_t = S_0 e^{(r-q+\frac{1}{2}\sigma^2)t + \sigma W_t^{\mathcal{Q}}}. \tag{14}$$

All assets priced under this measure are $\mathcal{Q}$-martingales when discounted by the stock price. According to this fact, we can reduce the dimension of stochastic variables. We introduce a variable $\xi_t = \frac{A_t}{S_t}$ and so we can derive

$$\begin{aligned}
V(t,S,A) &= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^{Q}\Big[e^{-r(s-t)}\Big(\rho(S_s-A_s)\Big)^+\Big|S_t=S,A_t=A\Big]\\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^{\mathcal{Q}}\Big[\frac{\eta_t}{\eta_T}e^{-r(s-t)}\Big(\rho(S_s-A_s)\Big)^+\Big|S_t=S,A_t=A\Big]\\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^{\mathcal{Q}}\Big[\frac{e^{r(t-s)}}{e^{(r-q)t}}S_t\Big(\rho(S_s-A_s)\Big)^+\ \mathbb{E}_s^{\mathcal{Q}}\big[\frac{e^{(r-q)T}}{S_T}\big]\Big|S_t=S,A_t=A\Big]\\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^{\mathcal{Q}}\Big[e^{qt-rs}S_t\Big(\rho(S_s-A_s)\Big)^+\ \frac{e^{(r-q)s}}{S_s}\Big|S_t=S,A_t=A\Big]\\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^{\mathcal{Q}}\Big[e^{-q(s-t)}S_t\Big(\rho\Big(1-\frac{A_s}{S_s}\Big)\Big)^+\Big|S_t=S,A_t=A\Big]\\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} e^{-q(s-t)}S\ \mathbb{E}_t^{\mathcal{Q}}\Big[\Big(\rho(1-\xi_s)\Big)^+\Big|S_t=S,A_t=A\Big].
\end{aligned}$$

The last expression can be rewritten in terms of the new variable $\xi = \frac{A}{S}$ as follows:

$$\widetilde{V}(t,\xi) = e^{-qt}\frac{V(t,S,A)}{S} = e^{-qT_t^*}\ \mathbb{E}_t^{\mathcal{Q}}\Big[\Big(\rho(1-\xi_{T_t^*})\Big)^+\Big], \tag{15}$$

where $T_t^* = \inf\{s\in[t,T]|\xi_s=\xi_s^*\}$ and the function $t\mapsto\xi_t^*$ describes the early exercise boundary.

The stopping region $\mathcal{S}$ and continuation region $\mathcal{C}$ for the call and put options are defined by

$$\mathcal{S}_{call} = \mathcal{C}_{put} = \{0\le t\le T, 0\le\xi<\xi_t^*\}, \tag{16}$$

$$\mathcal{C}_{call} = \mathcal{S}_{put} = \{0\le t\le T, \xi_t^*<\xi<\infty\}. \tag{17}$$

Now we solve the problem (with one stochastic variable) formulated in (15). In what follows, we generalize the result by Hansen, Jørgensen (2000) from [3] for the case of a nontrivial dividend rate $q\ge0$.

Theorem 3.1. *The value of the floating strike Asian option on stock underlying with dividend rate $q\ge0$ is given by*

$$\widetilde{V}(t,\xi_t) = \widetilde{v}(t,\xi_t)+\widetilde{e}(t,\xi_t), \tag{18}$$

where

$$\widetilde{v}(t,\xi_t) \equiv \mathbb{E}_t^{\mathcal{Q}}\Big[e^{-qT}\Big(\rho(1-\xi_T)\Big)^+\Big] \tag{19}$$

and

$$\widetilde{e}(t,\xi_t) \equiv \mathbb{E}_t^{\mathcal{Q}}\Big[\int_t^T \rho e^{-qu}\xi_u \mathbf{1}_{\mathcal{S}}(u,\xi_u)\Big(\frac{dA_u}{A_u} - (r - q\xi_u^{-1})du\Big)\Big], \tag{20}$$

with average given by the function A_t and stopping region $\mathcal{S}$. Here the function $1_{\mathcal{S}}(\cdot)$ is the indicator function of the set $\mathcal{S}$, ρ sets the call option by the value 1 and the put option by the value -1.

In the proof of THEOREM 3.1 we will use the following lemma.

Lemma 3.2. *The auxiliary variable $\xi_t = \frac{A_t}{S_t}$ satisfies the following stochastic differential equation:*

$$d\xi_t = \xi_t \frac{dA_t}{A_t} - (r-q)\xi_t\ dt - \sigma\xi_t\ dW_t^{\mathcal{Q}}. \tag{21}$$

Proof of LEMMA 3.2. We express the differential $d\xi_t = d\Big(\frac{A_t}{S_t}\Big)$ as

$$\begin{aligned} d\xi_t &= \frac{1}{S_t}dA_t - \frac{A_t}{S_t^2}dS_t + \frac{A_t}{S_t^3}(dS_t)^2 \\ &= \xi_t\frac{dA_t}{A_t} - (r-q)\xi_t\ dt - \sigma\xi_t\ dW_t^{\mathcal{Q}}, \end{aligned}$$

and the proof of lemma follows. □

Notice that, when comparing to the original expression with a zero dividend rate, $q = 0$, the only difference is that the parameter r is replaced by $r - q$. The value of $\frac{dA_t}{A_t}$ depends on the method of averaging of the underlying used in the valuation. The expression for the arithmetic averaging has form

$$\frac{dA_t^a}{A_t^a} = \frac{1}{t}\Big(\frac{1}{\xi_t^a} - 1\Big)\ dt. \tag{22}$$

As far as, the geometric average is concerned, we have

$$\frac{dA_t^g}{A_t^g} = -\frac{1}{t}\ln\xi_t^g\ dt \tag{23}$$

and for the weighted arithmetic averaging

$$\frac{dA_t^{wa}}{A_t^{wa}} = \frac{1}{t}\Big(\frac{a(0) + \int_0^t a'(t-u)\frac{S_u}{S_t}\ du}{\xi_t^{wa}} - 1\Big)\ dt, \tag{24}$$

where a' is the derivative of the function a. The last equation is unusable in its general form. Neverthless, if we set $a(s) = e^{-\lambda s}$, it becomes

$$\frac{dA_t^{wa}}{A_t^{wa}} = \frac{1}{t}\Big(\frac{1}{\xi_t^{wa}} - (1+\lambda t)\Big)\ dt. \tag{25}$$

Proof of THEOREM 3.1. We follow the proof of the original theorem including necessary modifications related to the form of averaging and the fact that $q \geq 0$.

First, we suppose that (t, ξ) belongs to the continuation region $\mathcal{C}$. The option is held and so we use Itô's lemma to calculate the differential

$$\begin{aligned} d\widetilde{V} &= \frac{\partial \widetilde{V}}{\partial \xi} d\xi + \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial \xi^2}(d\xi)^2 + \frac{\partial \widetilde{V}}{\partial t} dt \\ &= \xi \frac{\partial \widetilde{V}}{\partial \xi}\frac{dA}{A} + \Big[-(r-q)\xi \frac{\partial \widetilde{V}}{\partial \xi} + \frac{1}{2}\sigma^2\xi^2 \frac{\partial^2 \widetilde{V}}{\partial \xi^2} + \frac{\partial \widetilde{V}}{\partial t}\Big] dt - \sigma \xi \frac{\partial \widetilde{V}}{\partial \xi} dW^{\mathcal{Q}} \\ &= -\sigma \xi \frac{\partial \widetilde{V}}{\partial \xi} dW^{\mathcal{Q}}, \end{aligned}$$

where the last equality holds true, because $\widetilde{V}$ is $\mathcal{Q}$-martingale.

Now we suppose that (t, ξ) belongs to the stopping region $\mathcal{S}$. The value of the option is defined by

$$\widetilde{V}(t, \xi_t) = \rho e^{-qt}(1 - \xi_t).$$

So the differential $d\widetilde{V}$ has form

$$\begin{aligned} d\widetilde{V} &= -\rho q e^{-qt}(1-\xi)dt - \rho e^{-qt} d\xi \\ &= -\rho e^{-qt}\xi \frac{dA}{A} + \rho e^{-qt}(r\xi - q)dt + \rho e^{-qt}\sigma \xi dW^{\mathcal{Q}}. \end{aligned}$$

For both regions we have an equation

$$d\widetilde{V}(t, \xi_t) = -\rho e^{-qt}\mathbf{1}_{\mathcal{S}}(t, \xi_t)\Big(\xi_t \frac{dA_t}{A_t} - (r\xi_t - q)dt\Big) + dM_t^{\mathcal{Q}}, \tag{26}$$

where $M_t^{\mathcal{Q}}$ is a $\mathcal{Q}$-martingale. Integrating (26) from t to T and taking expectation we have

$$\begin{aligned} \mathbb{E}_t^{\mathcal{Q}}\Big[\widetilde{V}(T, \xi_T)\Big] - \widetilde{V}(t, \xi_t) &= -\mathbb{E}_t^{\mathcal{Q}}\Big[\int_t^T \rho e^{-qu}\xi_u \mathbf{1}_{\mathcal{S}}(u, \xi_u)\Big(\frac{dA_u}{A_u} - (r - \frac{q}{\xi_u})du\Big)\Big] \\ &\quad + \underbrace{\mathbb{E}_t^{\mathcal{Q}}\Big[\int_t^T dM_u^{\mathcal{Q}}\Big]}_{=0}, \end{aligned}$$

$$\begin{aligned} \widetilde{V}(t, \xi_t) &= \underbrace{\mathbb{E}_t^{\mathcal{Q}}\Big[e^{-qT}\Big(\rho(1-\xi_T)\Big)^+\Big]}_{=\widetilde{v}(t,\xi_t)} \\ &\quad + \underbrace{\mathbb{E}_t^{\mathcal{Q}}\Big[\int_t^T \rho e^{-qu}\xi_u 1_{\mathcal{S}}(\xi_u)\Big(\frac{dA_u}{A_u} - (r - \frac{q}{\xi_u})du\Big)\Big]}_{=\widetilde{e}(t,\xi_t)}. \end{aligned}$$

this completes the proof of THEOREM 3.1. $\square$

Conclusions

In this paper we extended the Hansen and Jørgensen's formula for valuation of the floating strike American-style Asian option by assuming a non-zero dividend rate q. The theory of the martingales and conditioned expected values was used in the calculation of an integral equation for the position of the early exercise boundary. We also present the formula for the weighted arithmetic average with time dependent weights. The presented formula can be used in the comparison of the value of the early exercise boundary to the projected SOR method for Asian option due Kwok, Dai in [1] as well as integral transformation method described in [7].

The numerical experiments and asymptotic analysis of the early exercise boundary will be the subject of the forthcoming paper being prepared.

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