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Faculty of Mathematics, Physics and Informatics

Probabilistic and analytic methods for pricing American style of Asian options

Dissertation Thesis

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Kľúčové slová: Asian options, lookback options, early exercise boundary, limit at the expiry

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ABSTRACT

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In the thesis, we analyze floating strike American style Asian options with various averaging and lookback options.

As one of the main results, we present a new, unifying method for calculation of the limit of early exercise boundary at expiry. The method can be used for any financial derivative that can be transformed into a so called Doob-Meyer decomposition of Snell envelope of its discounted pay-off function. Results for the limit of early exercise boundary of American style of option strategies calculated by this approach are compared to results calculated by the PSOR method.

The early exercise boundary of analyzed options is estimated by the first order of polynomial expansion. We use the condition of smoothness of solution to derive the expansion close to expiry. The result is consistent with already known values derived for plain vanilla options.

In the thesis, we also present a differential equation for the early exercise boundary of analyzed options. This equation is derived from the modification of Black–Scholes partial differential equation.

Key words: Asian options, lookback options, early exercise boundary, limit at the expiry

ABSTRAKT

V práci analyzujeme tzv. floating strike ázijské opcie amerického typu s rôznym typom priemerovania a tzv. lookback opcie.

Jedným z hlavných výsledkov je jednotná metóda na výpočet hodnoty hranice skorého uplatnenia blízko expirácie. Metóda je použiteľná pre ľubovoľný finančný derivát, ktorý sa dá transformovať na tvar Doob-Meyerovho rozkladu Snellovej obálky jeho diskontovanej pay-off funkcie. Výsledky získané pre hranicu skorého uplatnenia pre americký typ opčných stratégií vypočítané prezentovanou metódou sú konfrontované s výsledkami spočítanými metódou PSOR.

Hranica skorého uplatnenia pre analyzované opcie je aproximovaná polynomickým rozvojom prvého stupňa. Rozvoj v blízkosti expirácie je odvodený na základe podmienky hladkého napojenia. Získané hodnoty sú konzistentné s už známymi hodnotami pre tzv. vanilla opcie.

V práci tiež uvádzame diferenciálnu rovnicu na výpočet voľnej hranice pre analyzované opcie. Táto rovnica je odvodená na základe modifikovanej Black–Scholesovej parciálnej diferenciálnej rovnice.

Kľúčové slová: ázijské opcie, lookback opcie, hranica skorého uplatnenia, limita v expirácii

Preface

"Discas oportet, quamdiu est, quod nescias."

– Proverb

Mathematics and finance have been connected from the very beginning. The evolution in one of them is followed by developments in the other. The chaos and randomness of financial markets caused by (not always rational) behavior of agents involved in the system has to be supported by the order of mathematics. As we have already seen in the past, the market powers can handle with changes in the financial world themselves. Sooner or later, the equilibrium taking into account new situation is established. However, the price has often been too high to be satisfied with the result. Therefore the mathematical background is a necessary complement to market mechanisms and instincts of a good trader.

The most important task for a trader is to manage his portfolio to decrease the risk of loss as much as possible. Solving this problem is usually very difficult and thus every little help comes to hand. Solving partial, generalized problems is very good approach to the hedging.

The way to master risk of each element of the market leads through the understanding of its behavior. This can be achieved by finding the price fair for both the holder and the seller. However, this is not so easy in general. Even some of the most elemental components of the financial world are priced by (more or less precise) approximative methods under restrictive assumptions. The improvements of models and application of the results in the real market can resemble to a never-ending circle of tries and errors, but every little step that pushes our modeled estimates closer to the reality creates a necessary piece of the puzzle...

Contents

Preface **i**

Introduction **1**

Goals of the thesis **7**

1 Preliminaries **9**

1.1 Random variables and stochastic processes 9

1.2 Conditioned expected values 11

1.3 Itô calculus 12

1.4 Martingales 14

1.5 Snell envelope and Doob-Meyer decomposition 15

1.6 Girsanov's theorem 16

2 Financial derivatives **19**

2.1 General properties 19

2.1.1 Futures 21

2.1.2 Swaps 21

2.2 Options 21

2.2.1 European plain vanilla options 22

2.2.2 American plain vanilla options 24

2.3 Forwards 26

3 Exotic options **29**

3.1 Classification of exotic options 29

3.1.1 Time dependence 30

3.1.2 Cash-flows 30

3.1.3 Path dependence 31

3.1.4 Dimensionality . . . 31
3.1.5 Order . . . 32
3.1.6 Embedded decisions . . . 32
3.2 Path-dependent options . . . 32
3.2.1 Asian options . . . 33
3.2.2 Lookback options . . . 34
3.2.3 Barrier options . . . 35
3.3 Other exotic derivatives . . . 35
3.4 Hedging of the exotic portfolio . . . 46

4 A probabilistic model for pricing American style options **47**
4.1 Model . . . 47
4.2 Calculation of the formula . . . 49
4.2.1 Approximation for the general average . . . 54
4.2.2 Geometric average . . . 57
4.2.3 Approximation for the arithmetic average . . . 59
4.2.4 Lookback options . . . 61

5 Limit value of the early exercise boundary at expiry **67**
5.1 Limit value theorem . . . 67
5.2 Calculation of the early exercise boundary position at expiry . . . 69
5.2.1 Plain vanilla options . . . 69
5.2.2 Bullish and bearish spreads . . . 70
5.2.3 Strangles and straddles . . . 73
5.2.4 Condors and butterflies . . . 74
5.2.5 Shout options . . . 77
5.2.6 British vanilla options . . . 78

6 Early exercise boundary of path-dependent options **81**
6.1 Limit of the early exercise boundary at expiry . . . 81
6.2 Expansion of the early exercise boundary close to expiry . . . 87
6.2.1 General average . . . 91
6.2.2 Geometric average . . . 93
6.2.3 Arithmetic average . . . 94
6.2.4 Lookback option . . . 96

7 Partial differential equation for path-dependent options **99**
7.1 Derivation of the Black–Scholes equation for Asian options 99
7.2 Modified Black–Scholes equation 101

8 Transformation method **105**
8.1 American style of Asian options 106
8.2 Fixed domain transformation 108
8.2.1 An equivalent form of the equation for the free boundary . . . 111
8.3 A numerical approximation scheme 112
8.4 Computational examples of the free boundary approximation 115

Conclusions **119**

A American style vanilla option **121**
A.1 Calculation of the formula 121
A.2 Limit of the early exercise boundary at expiry 123
A.3 Expansion of the exercise boundary close to expiry 124

B Greeks **129**
B.1 First order Greeks 131
B.2 Second order Greeks 136
B.3 Third order Greeks 142
B.4 Other Greeks 147

C Proofs **149**
C.1 A probabilistic model for pricing American style options 149
C.2 Limit value of the early exercise boundary at expiry 162
C.3 American style vanilla option 165

D Numerical simulations of the early exercise boundary at expiry **167**
D.1 PSOR method 167
D.2 Comparison of analytic and numerical approach 168

List of symbols **177**

Bibliography **181**

Index **187**

Introduction

"On n'a pas besoin de lumière, quand on est conduit par le ciel."

– Jean Baptiste Poquelin Molière

The valuation is an elemental feature of trading system. Traders need to calculate the value of each traded commodity as exact as possible. The bias in valuation can cause unreasonable loss for the trader or an arbitrage opportunity. The price of financial instruments on a market is a bid-ask equilibrium achieved by the market powers. Even this price is accepted by all traders on the market, it is not necessary the correct value. Calculation of the exact price of financial instruments can prevent the instability in the trading system. Also the financial instruments that are traded over the counter (OTC) cannot rely on the market powers (because the number of traders involved into OTC transactions is much lower than those involved into the exchange trading).

The evolution in trading system impacts the development of financial derivatives. Origins of option derivatives lay in far history[1][2]. The very first record of option transaction appears in the Bible. In the book of Genesis, Jacob agreed to work for Laban for seven years to receive a permission to marry Laban's younger daughter, Rachel. In other words, Jacob paid seven years of labor to receive *the right, but not the obligation* to marry Rachel. However, this contract did not involve any speculative feature.

In the Ancient Greece, Aristotle in his work *Politics* in 332 BC describes a story about the famous philosopher and astronomer Thales of Miletus, who (according to his knowledge of stars) predicted a great harvest. He bought the rights to use all olive presses in Chios and Miletus as the first man when the harvest come. At the

[1]The Options Institute (1999, Chapter 1)

[2] http://www.optiontradingpedia.com/history_of_options_trading.htm

time of harvest, he sold these rights to olive farmers with high profit.

In the 17^{th} century, Europe was influence by the tulip mania. Tulip bulbs imported from Turkey and Netherlands became the symbol of wealth and prices of the most rare of them grew to astronomic heights. The bulbs were sold faster than they could grow, thus call options were introduced into the tulip market. The mania culminated on February 1637, when prices became so high that nobody was able to afford it and the bubble collapsed. The Dutch economy and tulip speculators were crushed by the tiny flower bulbs. In the end of the century, an organized market for call and put options was created in London. The public was still threatened by Dutch experiences and the Parliament was pushed into regulation of the option market. This concluded into a ban of option trading in 1733 known as Barnard's act. This market restriction caused by the unreasonable fear kept financial derivatives officially illegal until 1860. However, the Barnard's act was ineffective, because nor the penalties nor the risk of loss were able to stop the option market.

In 90. of the 18^{th} century, options came also to the United States together with the foundation of exchange in New York. Later, Chicago Board of Trade (CBOT; 1848) and the Chicago Mercantile Exchange (CME; 1919) were founded to keep up with growing market.

First, the simple derivatives (as forwards and plain vanilla options) were used to hedge the risk of portfolio. The breakpoint in valuation methods for the financial derivatives is dated to early 70. of the 20^{th} century. The cornerstone laid by Black and Scholes (1973) and Merton (1973) or its modifications occur in majority of all pricing techniques. However, the idea presented in 1973 was not the first attempt in valuation of plain vanilla options. Almost identical concept was presented 65 years earlier in the unpublished paper *Theorie der Prämiengeschäfte* by Italian mathematician Vincenz Bronzin (cf. Hafner and Zimmermann 2009).

The revolution in pricing of plain vanilla options was accompanied by foundation of the Chicago Board Options Exchange (CBOE; 1973) and the Options Clearing Corporation (OCC; 1973).

The well known Black–Scholes partial differential equation and the theory behind it are considered as an important basis in the financial engineering. However, the theory of valuation has undergone many changes since that time. The progress in valuation of simple financial instruments pushed traders into inventing less predictable derivatives. The growth of their variety traded on markets has increased the need for more general and more accurate valuation.

The most basic classification of financial derivatives is according to their expiration time (one of their main properties). The European style of derivatives can be exercised only at the expiration time T. On the other hand, by buying the American style of derivatives the holder obtains right to exercise it at any moment by the expiration time. The early exercise boundary of financial derivative $x_t^* = x^*(t)$ splits the $t - x$ (time–underlying) space into continuation region $\mathcal{C}$ and stopping region $\mathcal{S}$. The derivative is exercised if spot value of underlying is in the stopping region, i.e. $(t, x_t) \in \mathcal{S}$ and is held otherwise, i.e. $(t, x_t) \in \mathcal{C}$ (cf. Hull 1997, Geske and Johnson 1984, Geske and Roll 1984, Karatzas 1988, Chadam 2008, Kwok 2008, Kuske and Keller 1998, Mallier 2002, Pascucci 2008).

The valuation of (exotic) options is usually done by one of two methods. It is calculated either by a differential equation and in the case of more complicated derivatives the numerical scheme of solving the differential equation is used. The second method is based on the theory of conditioned expected value and martingales. This method is also used as a background theory for the Monte Carlo method. However, if we consider the American style of an option, the valuation becomes more difficult, because we need to calculate the early exercise boundary as well. If there is no way how one could calculate the early exercise function, it can be partially approximated by more simple functions (e.g. Taylor expansion).

In this thesis, we present new method for calculation of the analytic value of limit of early exercise boundary at expiry. However, many authors have considered particular problems and there are also results for some types of derivatives (cf. Albanese and Campolieti 2006, Alobaidi and Mallier 2006, Bokes and Ševčovič 2011, Chiarella and Ziogas 2005, Dai and Kwok 2006, Detemple 2006, Kwok 2008, Ševčovič 2008, Wilmott et al. 1995, Wu et al. 1999, etc.). The presented method is a unified approach to this problem. It can be used to determine the limit of exercise boundary for American style of a general derivative that can be written in form

$$V_{am}(t, x_t) = V_{eu}(t, x_t) + \mathbb{E}_t \left[\int_t^T \mathbf{1}_{\mathcal{S}}(u, x_u) f_b(u, x_u) \, du \right],$$

where V_{eu} is the price of European style of derivative, $\mathbb{E}_t$ is conditioned expected value according to the information at time t, $\mathbf{1}_{\mathcal{S}}$ is the indicator function for stopping region $\mathcal{S}$ and f_b is American style bonus function. Such decomposition (for plain vanilla option) was introduced in Kim (1990). The method presented in this thesis was introduced in Bokes (2011).

Asian options belong to a group of so-called path-dependent options. Their payoff functions depend on the spot value of underlying during the whole or some part(s) of life span of option. Usually, Asian options depend on the (arithmetic or geometric) average of the spot price of the underlying. They can be used as a useful tool for hedging highly volatile assets or goods. Since the price of an underlying varies during the life span of option, the holder of Asian option can be secured from the risk of a sudden price jumps to undesirable region (too high for the call option holder or too low for the put option holder). Among path-dependent options, Asian options play an important role as they are quite common in currency and commodity markets like e.g. oil industry (cf. Wilmott et al. 1995, Hull 1997, Wu et al. 1999, Hansen and Jørgensen 2000, Detemple 2006, Dai and Kwok 2006, Wystup 2006, Kwok 2008, Kim and Oh 2004, Wu and Fu 2003, Linetsky 2004).

In this thesis, we focus on the floating strike Asian call or put options whose strike price depends on the averaged path history of the underlying asset. More precisely, we are interested in pricing American style Asian call and put options having the payoff functions $\Omega(S,A) = (S-A)^+$ and $\Omega(S,A) = (A-S)^+$, respectively. The strike price A is given as an average of the underlying over the time history $[0,T]$. If we consider the general type of average in form

$$(A_t)^p = \frac{1}{t}\int_0^t (S_u)^p \, du,$$

we can transform this expression into maximum value ($A_t \to M_t$) and into minimum value ($A_t \to m_t$) for $p \to \infty$ and $p \to -\infty$, respectively. This generalization allows us to deal also with lookback options, i.e. options similar to Asian type, where the average is replaced by extreme value.

We calculate the price of American type of Asian option with various types of averaging (including lookback options). The calculation is based on the theory of conditioned expected values and has been motivated by Hansen and Jørgensen (2000). As we have already mentioned above, the American type pricing problem is accompanied with the problem of early exercise boundary. We calculate the first two elements of the expansion of the free boundary at the expiry in terms of $\sqrt{T-t}$. To derive the absolute element, we use our new method. The first order element is calculated from the marginal condition of the free boundary guaranteeing the smoothness of an American type option (cf. Bokes 2010, Bokes and Ševčovič 2011).

The calculation of the early exercise boundary itself as an explicit function leads to mathematical problems that we are not able to handle yet. Thank to several transformations of the problem of American type of Asian option, we derive an integral-differential equation for the early exercise boundary. This equation can be solved by a numerical approximation scheme also presented in the thesis. We derive the equation for various types of averaging. However, for the sake of simplicity, the scheme is presented only for the Asian call option with continuous arithmetic averaging.

In the first chapter, we summarize some preliminaries of mathematical theory (e.g. Itô calculus, Girsanov's theorem etc.) often used in the valuation of options (and other derivatives).

The second and third chapter present basic characteristics of financial derivatives with major respect to the options and a subgroup of exotic options called path-dependent options, respectively. We also summarize the classification of the most important exotic options. The main focus of the third chapter is given to Asian, lookback and barrier options.

In the next chapter, we calculate the value of Asian option with various averages and lookback options by the theory of conditioned expected values and martingales.

The fifth and sixth chapter present the analysis of behavior of early exercise boundary. In the former chapter we introduce a new, unified approach of calculating the limit of early exercise boundary at expiry of the general financial derivative. The latter chapter presents the results of new method on the floating strike Asian and lookback options. Moreover, we calculated the first order expansion of the early exercise boundary of floating strike Asian and lookback options.

In the seventh chapter, we summarize the modification of Black–Scholes partial differential equation for pricing path-dependent options.

The eighth chapter presents the transformation of partial differential equations from the previous chapter. Moreover, we present an integral-differential equation and a numerical approximation scheme for the calculation of early exercise boundary.

In the first chapter of appendix of the thesis, we present methods from fourth, fifth and sixth chapter applied on the plain vanilla options.

The second chapter in the appendix consists of a survey of sensitivity indicators - the Greeks with analytic and graphic examples for European plain vanilla options. For the European Asian options with geometric average, approximation of European Asian option with arithmetic average and European lookback options, we provide examples of the sensitivity only by graphic examples.

Next chapter in the appendix contains supporting lemmas and proofs of theorems and lemmas from the thesis.

The concluding chapter in appendix obtains numerical comparison of results from the fifth chapter with values calculated by the projected successive over relaxation (PSOR) method.

Goals of the thesis

In the thesis we study and analyze several questions and problems that are related to the valuation of the American style Asian options. The main goals of the thesis can be summarized as follows:

g.1 **Valuation of the Asian option with non-zero dividend rate.** To extend the model introduced in paper Hansen and Jørgensen (2000) by the dividend rate q, the kernel a and general average. Using the theory of conditioned expectations to calculate the value of the American style Asian floating strike option with the general average. To perform similar calculation also for the value of floating strike lookback options. *[These results are contained in papers Bokes (2010), Bokes and Ševčovič (2011).]*

g.2 **The limit of early exercise boundary.** To create a new method for calculation of the limit of early exercise boundary at expiry. The method is applicable for the general financial derivative that can be written in form of Doob-Meyer decomposition of Snell envelope. *[These results are contained in paper Bokes (2011).]*

g.3 **Approximation of the early exercise boundary of the American style Asian option.** To calculate the function that approximate the early exercise boundary of the floating strike Asian option close to expiration T for general averaging. *[These results are contained in paper Bokes and Ševčovič (2011).]*

g.4 **Equation for the early exercise boundary of the Asian option.** To create an equation and numerical approximation scheme for the calculation of the early exercise boundary of floating strike Asian options with various averages and lookback options. *[These results are contained in paper Bokes and Ševčovič (2011).]*

CHAPTER 1

Preliminaries

In this chapter, we summarize some results from stochastic calculus. We mainly focus on random variables, stochastic processes, Itô calculus and martingales. The following definitions and theorems are in major adapted from Karatzas and Shreve (1988, Chapter 1), Melicherčík and Olšárová (2005, Chapter 1) and Revuz and Yor (2005, first chapters). More detailed information can be found in Revuz and Yor (2005), Kallenberg (1997), Karatzas and Shreve (1988, 1998), Malliaris (1982), Durrett (1996) or Melicherčík and Olšárová (2005).

1.1 Random variables and stochastic processes

DEFINITION 1.1. *Let $(\mathbf{\Omega}, \mathcal{F}, P)$ be a probability space.* **Random variable** *is measurable function X on $\mathbf{\Omega}$.*

The **expected value** *of random variable X is defined as*

$$\mathbb{E}\left[X\right] = \int X \, dP.$$

DEFINITION 1.2. *Let T be a set, $(E, \mathcal{E})$ a measurable space. A* **stochastic process** *indexed by T, taking its values in $(E, \mathcal{E})$, is a family of measurable mappings X_t, $t \in T$, from a probability space $(\mathbf{\Omega}, \mathcal{F}, P)$ into $(E, \mathcal{E})$. The space $(E, \mathcal{E})$ is called the* **state space**.

DEFINITION 1.3. *Let $(\mathbf{\Omega}, \mathcal{F}, P)$ be a probability space and let $\mathcal{X}$ be a nonempty family of nonnegative random variables defined on $(\mathbf{\Omega}, \mathcal{F}, P)$. The* **essential supremum** *of $\mathcal{X}$, denoted by $\operatorname{ess\,sup} \mathcal{X}$, is a random variable X^* satisfying*

(i) $\forall X \in \mathcal{X}$, $X \leq X^*$ *almost surely.*

(ii) If Y is a random variable satisfying $X \leq Y$ almost surely for all $X \in \mathcal{X}$, then $X^* \leq Y$ almost surely.

Definition 1.4. *A continuous-time stochastic process* $\{W_t : 0 \leq t < T\}$ *is called a* **Wiener process** *(or standard Brownian motion) on* $[0,T)$ *(see* Figure *1.1 (left)) if*

(i) $W_0 = 0$ *almost surely.*

(ii) *For any* $0 \leq t \leq t+\Delta < T$ *the increment* $W_{t+\Delta} - W_t$ *has the Gaussian distribution with mean* 0 *and variance* Δ*, i.e.* $W_{t+\Delta} - W_t \sim \mathcal{N}(0, \Delta)$.

(iii) *For any finite set of times* $0 < t_1 < t_2 < \ldots < t_n < T$ *the random variables*

$$W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_T - W_{t_n}$$

are independent.

(iv) *For all* ω *in a set of probability one,* $W_t(\omega)$ *is a continuous function of* t.

Definition 1.5. *Suppose* $\mu \in \mathbb{R}$ *and* $\sigma > 0$. *A continuous-time stochastic process* $\{B_t : 0 \leq t < T\}$ *is called a* **Brownian motion** *with drift* μ *and variance* σ^2 *on* $[0,T)$ *if*

(i) $B_0 = 0$ *almost surely.*

(ii) *For any* $0 \leq t \leq t+\Delta < T$ *the increment* $B_{t+\Delta} - B_t$ *has the Gaussian distribution with mean* $\mu\Delta$ *and variance* $\sigma^2\Delta$*, i.e.* $B_{t+\Delta} - B_t \sim \mathcal{N}(\mu\Delta, \sigma^2\Delta)$.

(iii) *For any finite set of times* $0 < t_1 < t_2 < \ldots < t_n < T$ *the random variables*

$$B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_T - B_{t_n}$$

are independent.

(iv) *For all* ω *in a set of probability one,* $B_t(\omega)$ *is a continuous function of* t.

Remark 1.1. *Relation between Brownian motion and Wiener process is described by the equation*

$$B_t = \mu t + \sigma W_t.$$

Remark 1.2. *According to the fact that the Wiener process is a subset of the Brownian motion (*$\mu = 0$ *and* $\sigma = 1$*), we will often refer to the Wiener process as a (standard) Brownian motion.*

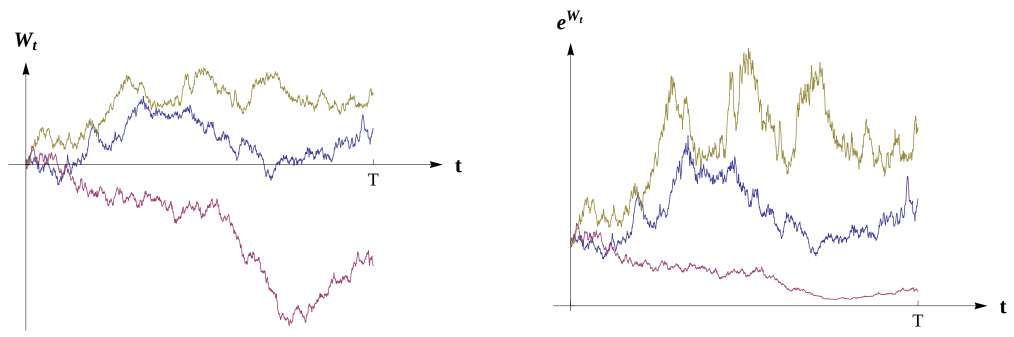

FIGURE 1.1: *Paths of simulated Wiener process (left) and the respective geometric Wiener process (right).*

DEFINITION 1.6. *Let $\{B_t : 0 \leq t < T\}$ be a Brownian motion. The stochastic process $\{G_t : 0 \leq t < T\}$ defined by*

$$G_t = G_0 e^{B_t}, \quad G_0 \in \mathbb{R}$$

is called a **geometric Brownian motion** *with drift μ and variance σ^2 (see* FIGURE *1.1 (right)).*

1.2 Conditioned expected values

DEFINITION 1.7. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X : \Omega \to \mathbb{R}$ a random variable, with property $\mathbb{E}(|X|) < \infty$. Let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra.* **Conditioned expected value** *$\mathbb{E}(X|\mathcal{H})$ is a random variable with following properties*

(i) $\mathbb{E}(X|\mathcal{H})$ *is* $\mathcal{H}$*-measurable.*

(ii) $\int_H \mathbb{E}(X|\mathcal{H})\, dP = \int_H X\, dP$ *for* $\forall H \in \mathcal{H}$.

THEOREM 1.1. *Let X, Y be random variables on probability space $(\Omega, \mathcal{F}, P)$ with property $\mathbb{E}(|X|) < \infty$, $\mathbb{E}(|Y|) < \infty$. Let $a, b \in \mathbb{R}$ and $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra. Then*

(i) $\mathbb{E}(aX + bY|\mathcal{H}) = a\mathbb{E}(X|\mathcal{H}) + b\mathbb{E}(Y|\mathcal{H})$

(ii) $\mathbb{E}(\mathbb{E}(X|\mathcal{H})) = \mathbb{E}(X)$

(iii) If X is $\mathcal{H}$-measurable, then $\mathbb{E}(X|\mathcal{H}) = X$.

(iv) If Y is $\mathcal{H}$-measurable, then $\mathbb{E}(YX|\mathcal{H}) = Y\mathbb{E}(X|\mathcal{H})$.

THEOREM 1.2. *Let X be a random variable on probability space $(\mathbf{\Omega}, \mathcal{F}, P)$ with property*

$$\mathbb{E}(|X|) < \infty.$$

Let $\mathcal{G}$, $\mathcal{H}$ be a σ-algebrae with property

$$\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}.$$

Then

$$\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}).$$

1.3 Itô calculus

Integrals in the stochastic calculus can also have stochastic features. The form of **Itô integral** is introduced by the following expression

$$\int_A f(t,\omega)\, dW_t(\omega),$$

where $A \subset \mathbb{R}^+$.

DEFINITION 1.8. *Let $W_t(\omega)$ be a Wiener process on probability space $(\mathbf{\Omega}, \mathcal{F}, P)$. The symbol $\mathcal{F}_t^W$ states for the smallest σ-algebra on $\mathbf{\Omega}$ generated by sets of type*

$$\{\omega; W_{t_1}(\omega) \in F_1, \ldots, W_{t_k}(\omega) \in F_k\},$$

where $k = 1, 2, \ldots$ and for $\forall j$ $t_j \leq t$ and $F_j \subset \mathbb{R}$ are Borel sets.

DEFINITION 1.9. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be an increasing system of σ-algebrae on $\mathbf{\Omega}$. The stochastic process*

$$g(t,\omega) : [0,\infty) \times \mathbf{\Omega} \to \mathbb{R}$$

*is $\mathcal{F}_t$-**adapted** if for $\forall t \geq 0$ is function*

$$\omega \to g(t,\omega)$$

$\mathcal{F}_t$-measurable.

THEOREM 1.3. *Let $W_t(\omega)$ be a Wiener process on probability space $(\mathbf{\Omega}, \mathcal{F}, P)$. Let the function*

$$f(t,\omega) : [0,\infty) \times \mathbf{\Omega} \to \mathbb{R}$$

has properties

(i) *the function is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ are Borel sets on $\mathbb{R}^+$,*

(ii) *stochastic process $f(t,\omega)$ is $\mathcal{F}_t^W$-adapted,*

(iii) $\mathbb{E}(\int_A f^2(t,\omega)\,dt) < \infty$, *where $A \subset \mathbb{R}$.*

Then for the function f, Itô integral is defined.

THEOREM 1.4. *(Itô isometry)*

Let the function $f(t,\omega)$ satisfy conditions in the THEOREM *1.3, then for $A \subset \mathbb{R}$*

$$\mathbb{E}\left[\left(\int_A f(t,\omega)\,dW_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_A f^2(t,\omega)\,dt\right].$$

LEMMA 1.1. *(Itô lemma)*

Let $X_t(\omega)$ be an Itô process

$$dX_t(\omega) = u(t,\omega)\ dt + v(t,\omega)\ dW_t(\omega).$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$Y_t(\omega) = g(t, X_t(\omega))$$

is also an Itô process

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)\ dt + \frac{\partial g}{\partial x}(t,X_t)\ dX_t + \frac{1}{2}v^2\frac{\partial^2 g}{\partial x^2}(t,X_t)\ dt.$$

LEMMA 1.2. *(Multidimensional Itô lemma)*

Let $X_t(\omega) = (X_t^1(\omega), \dots, X_t^n(\omega))$ be an n-dimensional Itô process

$$dX_t(\omega) = u(t,\omega)\ dt + v(t,\omega)\ dW_t(\omega).$$

Let the covariance of $dW_t = (dW_t^1, \dots dW_t^n)$ be defined by

$$\mathbb{C}ovar\left[dW_t^i, dW_t^j\right] = \begin{cases} dt & \text{for } i = j \\ \rho_{ij}dt & \text{for } i \neq j \end{cases},$$

where $\rho_{ij} \in [-1,1]$ is the correlation coefficient.

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R}^n)$. Then

$$Y_t(\omega) = g(t, X_t(\omega))$$

is also an Itô process

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)\ dt + \sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(t,X_t)\ dX_t^i + \frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}v^iv^j\frac{\partial^2 g}{\partial x_i \partial x_j}(t,X_t)\ dt.$$

1.4 Martingales

Definition 1.10. *Let $\mathbb{P} = (\boldsymbol{\Omega}, \mathcal{F}, P)$ be a probability space. System of σ-algebrae $\{\mathcal{M}_t\}_{t\geq 0}$, $\mathcal{M}_t \subset \mathcal{F}$ with property*

$$0 \leq s < t \to \mathcal{M}_s \subset \mathcal{M}_t$$

is called a **filtration** *on the space $\mathbb{P}$.*

Definition 1.11. *Let $(\boldsymbol{\Omega}, \mathcal{F}, P)$ be a probability space. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfy the* **usual conditions***, if it is right continuous and $\mathcal{F}_0$ contains all P-negligible events in $\mathcal{F}$.*

Definition 1.12. *Let $(\boldsymbol{\Omega}, \mathcal{F}, P)$ be a probability space. An adapted process A is called* **increasing***, if for almost every $\omega \in \boldsymbol{\Omega}$ we have*

(i) $A_0(\omega) = 0$.

(ii) $t \mapsto A_t(\omega)$ *is a nondecreasing, right continuous function*

and $\mathbb{E}[A_t] < \infty$ holds for every $0 \leq t < \infty$. An increasing process is called **integrable** *if $\mathbb{E}[A_\infty] < \infty$, where $A_\infty = \lim_{t\to\infty} A_t$.*

Definition 1.13. *Let us consider a measurable space $(\boldsymbol{\Omega}, \mathcal{F})$ equipped with a filtration $\{\mathcal{F}_t\}$. A random time T is a* **stopping time** *of the filtration, if the event $\{T \leq t\}$ belongs to the σ-field $\mathcal{F}_t$ for every $t \geq 0$.*

A random time T is an **optional time** *of the filtration, if the event $\{T < t\}$ belongs to the σ-field $\mathcal{F}_t$ for every $t \geq 0$.*

Definition 1.14. *Suppose $\{\mathcal{D}_t\}_{t\geq 0}$ is an increasing system of σ-algebrae on $\boldsymbol{\Omega}$. The function $\tau : \boldsymbol{\Omega} \to [0, \infty)$ is* **Markov time** *with respect to $\{\mathcal{D}_t\}$ if*

$$\{\omega \in \boldsymbol{\Omega} : \tau(\omega) \leq t\} \in \mathcal{D}_t \quad \textit{for } \forall t \geq 0.$$

Definition 1.15. *Let us consider the class $\mathscr{T}$ ($\mathscr{T}_a$) of all stopping times T of the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfy $P(T < \infty) = 1$ (respectively, $P(T \leq a) = 1$ for a given finite number $a > 0$). The right continuous process $\{X_t\}$ according to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be of class D, if the family $\{X_T\}_{T\in\mathscr{T}}$ is uniformly integrable; of class DL if the family $\{X_T\}_{T\in\mathscr{T}_a}$ is uniformly integrable, for every $0 < a < \infty$.*

DEFINITION 1.16. *A real-valued process $\{M_t\}_{t\geq 0}$ on a probability space $(\mathbf{\Omega}, \mathcal{F}, P)$, adapted to a filtration $\{\mathcal{M}_t\}_{t\geq 0}$ is a* **submartingale** *(with respect to $\{\mathcal{M}_t\}_{t\geq 0}$ and measure P) if*

(i) $\mathbb{E}(|M_t|) < \infty$ *for* $\forall t \geq 0$,

(ii) $\mathbb{E}(M_s|\mathcal{M}_t) \geq M_t$ *almost surely for* $\forall s \geq t$.

A process M such that $-M$is a submartingale is called a **supermartingale** *and a process which is both a submartingale and supermartingale is a* **martingale**.

DEFINITION 1.17. *An increasing process A is called* **natural**, *if for every bounded, right continuous martingale $\{M_t\}$ according to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ we have*

$$\mathbb{E}\left[\int_{(0,t]} M_s dA_s\right] = \mathbb{E}\left[\int_{(0,t]} M_{s^-} dA_s\right], \quad \textit{for } \forall 0 < t < \infty.$$

THEOREM 1.5. *(Martingale representation theorem)*

Let $M_t(\omega), N_t(\omega) \in L^2(\mathbf{\Omega}, P)$, $0 \leq t \leq T$ be $\mathcal{F}_t^W$ martingales. Suppose the volatility of M_t has property

$$P(m_t \neq 0) = 1.$$

Then there exists exactly one $\mathcal{F}_t^W$-adapted process φ_t that satisfies

$$dN_t = \varphi_t \, dM_t$$

and

$$P\left(\int_0^t (\varphi_\xi m_\xi)^2 \, d\xi < \infty\right) = 1.$$

1.5 Snell envelope and Doob-Meyer decomposition

DEFINITION 1.18. *Let $\{Y_t\}_{0\leq t\leq T}$ be an $\mathcal{F}_t$-adapted, integrable process on probability space $(\mathbf{\Omega}, \mathcal{F}, P)$. Define a process $\{Z_t\}_{0\leq t\leq T}$ by*

$$\begin{aligned} Z_T &= Y_T, \\ Z_t &= \max\left[Y_t, \mathbb{E}_t\left[Z_s\right]\right], \textit{ for } \forall s \geq t. \end{aligned}$$

The process Z is called **Snell envelope** *of the process Y. It is (clearly) an adapted process.*

THEOREM 1.6. *The Snell envelope $\{Z_t\}_{0\le t\le T}$ of the process $\{Y_t\}_{0\le t\le T}$ is a supermartingale. Furthermore, it is the smallest supermartingale which dominates Y in the sense $Z_t \ge Y_t$ for all $0 \le t \le T$.*

THEOREM 1.7. *(Doob-Meyer decomposition)*

Let $(\mathbf{\Omega}, \mathcal{F}, P)$ be a probability space and let filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfy the usual conditions.

If the right-continuous submartingale X according to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ is of class DL, then it admits the decomposition

$$X_t = M_t + A_t, \quad 0 \le t$$

as the summand of a right continuous martingale $M = \{M_t\}$ and an increasing process $A = \{A_t\}$ both according to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$. The latter can be taken to be natural; under this additional condition, the decomposition is unique (except for the set of zero measure).

Further, if X is of class D, then M is a uniformly integrable martingale and A is integrable.

1.6 Girsanov's theorem

DEFINITION 1.19. *We say a measure ν is* **absolutely continuous** *with respect to μ, i.e. $\nu \ll \mu$ if $\mu(A) = 0$ implies that $\nu(A) = 0$.*

THEOREM 1.8. *(Radon-Nikodým theorem)*

Let μ and ν be σ-finite measures on space $(\mathbf{\Omega}, \mathcal{F})$. If $\nu \ll \mu$, there is a function $f \in \mathcal{F}$ so that for all $A \in \mathcal{F}$

$$\int_A f d\mu = \nu(A).$$

Function f is usually denoted $\frac{d\nu}{d\mu}$ and called the **Radon-Nikodým derivative***.*

THEOREM 1.9. *(Girsanov's theorem)*

Let $W_t(\omega)$ for $0 \le t \le T$ be the Brownian motion on the space $(\mathbf{\Omega}, \mathcal{F}, P)$. If $\gamma_t(\omega)$ is $\mathcal{F}_t^W$-adapted process, that

$$\mathbb{E}^P\big(e^{\frac{1}{2}\int_0^T \gamma_t^2(\omega)\,dt}\big) < \infty,$$

then there exists a measure Q on $(\mathbf{\Omega}, \mathcal{F})$ with following properties *(i)* $Q \sim P$, *i.e.* $Q(A) > 0 \Leftrightarrow P(A) > 0$ *for* $\forall A$

(ii) $\ln \frac{dQ}{dP}(\omega) = -\int_0^T \gamma_t(\omega)\, dW_t(\omega) - \frac{1}{2}\int_0^T \gamma_t^2(\omega)\, dt$

(iii) $\widetilde{W}_t(\omega) = W_t(\omega) + \int_0^t \gamma_s(\omega)\, ds$ *is a Brownian motion on* $(\mathbf{\Omega}, \mathcal{F}, Q)$.

The expression $\frac{dQ}{dP}$ *is a Radon-Nikodým derivative.*

CHAPTER 2

Financial derivatives

Evolution in trading during last decades has been pushing people into finding new possibilities how to hedge their assets. One of the ways how to take care of risk in a portfolio is to include derivatives. There are many types of financial derivatives that are being traded either on exchange or over-the-counter (OTC). The most important of them are introduced and classified in this chapter. For more information see Wilmott et al. (1995), Taleb (1996), Hull (1997), Melicherčík et al. (2005), Kwok (2008) or Ševčovič et al. (2011).

2.1 General properties

If we want to talk about derivatives, first we should introduce a general definition.

DEFINITION 2.1. (Financial) derivative *is a financial contract. Value of a derivative depends on the value of an underlying financial instrument with more basic structure. Its value at expiration date is exactly determined by the price process of the underlying up to the time of expiry.*

Purpose of derivative is to pass (part of) the (unwanted) risk to the second party by paying the non-arbitrage *price of the contract. This operation is called* **hedging** *(of the portfolio).*

From here, we will use expressions X and $X_t \equiv X\big|_t$ as a *function* X in general and at the time t, respectively.

Variables S and S_t are value and *spot value* of the underlying asset at time t, respectively (e.g. the *stock* or the *foreign exchange rate*). We assume that the underlying asset of derivative satisfies stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{2.1}$$

where $\mu \in \mathbb{R}$ is a drift, $\sigma \in \mathbb{R}^+$ is volatility and W_t is Wiener process. Although there is no reason to state that the volatility σ does not depend on other variables, in this chapter we assume that it is constant.

The solution of equation (2.1) is geometric Brownian motion

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

There are parameters that are common for all derivatives.

- *Expiration time* is denoted by parameter T.
- Variables t and $\tau = T - t$ denotes *time* and *time to expiry*, respectively.
- Continuous *risk-free interest rate* is denoted by parameter r.
- Continuous *rate of benefit from holding (storing)* the underlying asset is denoted by parameter q ($q > 0$ for money or stock with non-zero dividend rate, $q = 0$ for stock with zero dividend rate, $q < 0$ for commodity). Value $-q$ is called the *cost-of-carry*.
- *Volatility* of the return of underlying asset is denoted by parameter σ.
- *Strike (exercise) price* (the price at which the transaction with underlying is made) is denoted by constant or function X, X_i.

All derivatives are characterized by their *pay-off function* Ω satisfying the property $\Omega_T \equiv \Omega\big|_T = V_T$. Usually, but not necessarily, the pay-off function does not depend on time, i.e. $\frac{\partial \Omega}{\partial t} = 0$.

The party involved into a derivative contract by buying it is in the *long* position. The other party is in the *short* position.

We can classify derivatives in several groups. We present the most common of them: *options*, *forwards*, *futures* and *swaps*.

First, we briefly present latter two types of derivatives. Basic features of options and futures are described in separate SECTIONS 2.2 and 2.3, respectively.

2.1.1 Futures

Futures are very similar to forwards (see SECTION 2.3). The main difference between futures and forwards is the way how to trade them. There is a variety of exchanges trading futures, e.g. *Chicago Board of Trade* (CBOT), *Chicago Mercantile Exchange* (CME), etc. To make possible trading of the futures on an exchange, there is a need of standardized content of contracts specified by the exchange.

2.1.2 Swaps

A swap is an agreement between two parties to exchange cash-flows in the future. The dates of cash-flow transfer and the way of calculation is obtained in the swap contract.

2.2 Options

According to the type of transaction, there are (usually) two different types of options.

A **call option** is financial instrument giving the holder *right, but not the obligation* to buy an underlying asset at or by a (certain specified) date T at a (certain specified) price X.

Buying a call option contract hedges upward movement of the price of underlying asset.

A **put option** is financial instrument giving the holder *right, but not the obligation* to sell an underlying asset at or by a (certain specified) date T at a (certain specified) price X.

Buying a put option contract hedges downward movement of the price of underlying asset.

The most basic classification of option contracts is made by the *expiration time*.

- *European* options can be expired only at the expiration time T defined in the contract.

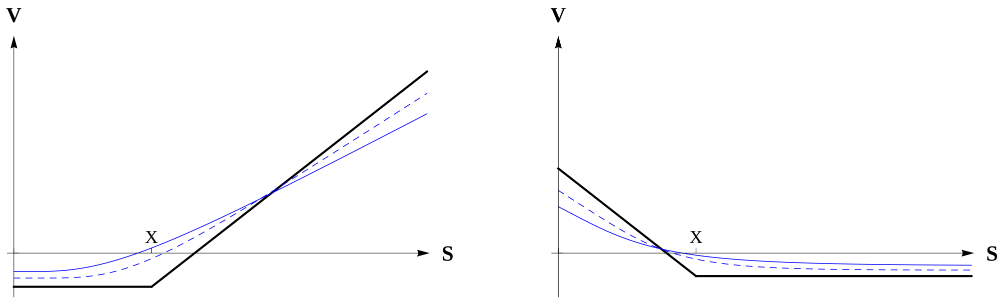

FIGURE 2.1: *Value of the long position of a European plain vanilla call option contract (left) and put option contract (right) at the time $t = 0$ (solid), $0 < t < T$ (dashed) and at the expiration time $t = T$ (bold) (purchase price included).*

- *Bermudan* option can be expired in certain given time moments by the expiration time T defined in the contract. [Bermudan options do not often appear in the classification by expiration time, because they belong to the group of exotic options.]

- *American* options can be expired at every moment by the expiration time T defined in the contract.

The most basic (European and American) options are called *plain vanilla options*. The rest of options is usually marked as *exotic options*, although their classification is not exact. We discuss the exotic options in the following chapter.

2.2.1 European plain vanilla options

The pay-off function of a European plain vanilla option is given by the expression

$$\Omega = (\mathfrak{c}\,(S - X))^+\,, \tag{2.2}$$

where the function $(x)^+ \equiv \max(x, 0)$ and $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively. The strike price X is a constant.

The value of a European plain vanilla option can be calculated as a solution of the Black–Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)\frac{\partial V}{\partial S} - rV = 0. \tag{2.3}$$

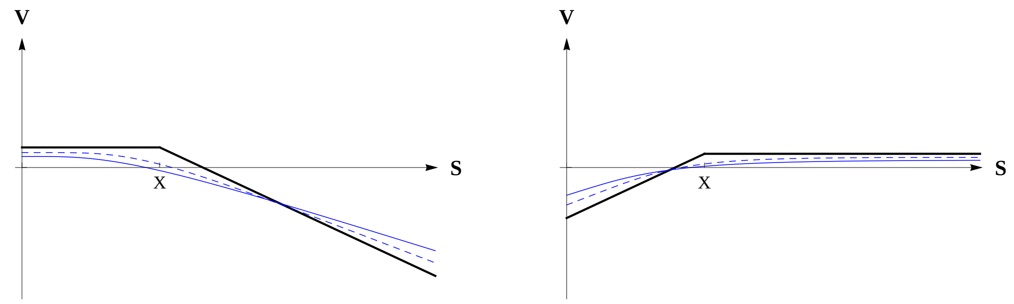

FIGURE 2.2: *Value of the short position of a European plain vanilla call option contract (left) and put option contract (right) at the time $t = 0$ (solid), $0 < t < T$ (dashed) and at the expiration time $t = T$ (bold) (purchase price included).*

TABLE 2.1: *The marginal condition of European plain vanilla option for Black–Scholes partial differential equation.*

call option	put option
$V_t\big\rvert_{S=0} = 0$	$V_t\big\rvert_{S=0} = Xe^{-r(T-t)}$
$\lim_{S\to\infty} \frac{V_t}{S} = e^{-q(T-t)}$	$\lim_{S\to\infty} V_t = 0$

The terminal condition for this problem is the pay-off function (2.2) and the marginal conditions for $\forall t \in [0, T]$ are summarized in TABLE 2.1.

The value of a European vanilla option can be also calculated as conditioned expected value (with risk-neutral measure $\mathcal{Q}$ that exists according to the Girsanov's theorem 1.9) of discounted pay-off function

$$V_t = \mathbb{E}_t{}^{\mathcal{Q}} \left(e^{-r(T-t)} \Omega\big\rvert_{S=S_T} \right) = \mathbb{E}^{\mathcal{Q}} \left(e^{-r(T-t)} \Omega\big\rvert_{S=S_T} \middle| \mathcal{F}_t^W \right),$$

where Ω is defined in (2.2) and $\mathcal{F}_t^W$, defined in DEFINITION 1.8, represents the information in the time t.

The value function at time t is equal to

$$V(t, S) = \mathfrak{c} \left(e^{-q(T-t)} S \boldsymbol{\Phi} \left(\mathfrak{c}\, d_t \right) - e^{-r(T-t)} X \boldsymbol{\Phi} \left(\mathfrak{c} \left(d_t - \sigma\sqrt{T-t} \right) \right) \right), \tag{2.4}$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively. The $\boldsymbol{\Phi}(\,\cdot\,)$ is the cumulative distribution function *CDF* of the normal probability distribution $\mathcal{N}(0, 1)$ and

$$d_t = \frac{\ln S - \ln X + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{2.5}$$

The cash-flow value of an option is calculated as a difference of the value (2.4) and the purchase price (PP) of an option contract discounted to time t.

$$\begin{aligned} V_t^{\text{cash-flow}} &= V_t - e^{rt}PP = V_t - e^{rt}V_0\big|_{S=S_0} \quad \text{for a long position,} \\ V_t^{\text{cash-flow}} &= -V_t + e^{rt}PP = -V_t + e^{rt}V_0\big|_{S=S_0} \quad \text{for a short position.} \end{aligned}$$

2.2.2 American plain vanilla options

American and European style of option contract are different in a possibility to claim the contract earlier than at the expiration date. Such feature gives an advance to the holder of an American style option against the one holding identical, but European style option. Consequently, we have an inequality showing relation between values of these two types of contracts

$$V_t^{eu} \leq V_t^{am} \quad \forall t \in [0, T]. \tag{2.6}$$

Inequality (2.6) turns into equality (in case of vanilla options) for call option with zero rate of benefit from holding the underlying, i.e. $\mathfrak{c} = 1$ and $q = 0$.

The pay-off of an American option is defined by equation

$$\Omega\big|_{S=S_{T^*}} = \left(\mathfrak{c}\left(S_{T^*} - X\right)\right)^+, \tag{2.7}$$

where T^* is the expiration time for an American option, all other variables and parameters have the same meanings and properties as in the pay-off function for a European option (2.2).

The American style option contract can be exercised anytime by the expiration time T, thus the value of this derivative cannot be less than its pay-off function, i.e.

$$V_T^{eu} = \Omega^{eu} = \Omega^{am} = V_T^{am} \leq V_t^{am} \quad \forall t \in [0, T].$$

As in the case of a European option, we can use two methods for pricing the option. However, the closed formula for value of an American plain vanilla option has not been derived yet. The most common approach to pricing is solving the partial differential problem by numerical methods.

DEFINITION 2.2. *The* **early exercise boundary** $S^* = S^*(t)$ *is a function that splits the* (t, S)*-space into two regions. The* stopping region $\mathcal{S} \equiv \{\mathcal{S}_t, t \in [0, T]\}$ *where the option*

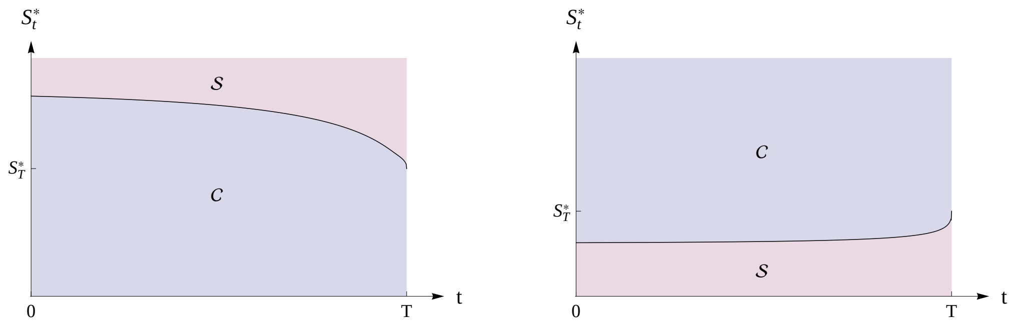

FIGURE 2.3: *The continuous region $\mathcal{C}$, the stopping region $\mathcal{S}$ and the early exercise boundary S^* of an American call option contract (left) and put option contract (right).*

contract is exercised and the continuation region $\mathcal{C} \equiv \{\mathcal{C}_t, t \in [0,T]\}$ *where the option contract is held.*

For the plain vanilla options, the stopping and continuation regions are summarized in TABLE *2.2.*

The **optimal stopping time** $T^* = T^*(S)$ *is the inversion function of the early exercise boundary.*

TABLE 2.2: *The stopping region $\mathcal{S}$ and continuous region $\mathcal{C}$ for plain vanilla option.*

	call option	put option
stopping region $\mathcal{S}$	$\{(S_t^*, \infty), t \in [0,T]\}$	$\{(0, S_t^*), t \in [0,T]\}$
continuation region $\mathcal{C}$	$\{(0, S_t^*), t \in [0,T]\}$	$\{(S_t^*, \infty), t \in [0,T]\}$

To price an American plain vanilla option, we need to solve the Black–Scholes partial differential equation (2.3) on the continuation region $\mathcal{C}$. The terminal condition is defined by (2.7) and the marginal conditions for $\forall t \in [0,T]$ are summarized in TABLE 2.3.

The last marginal condition in TABLE 2.3 guarantees the smoothness of the function V_t on the merger of $\mathcal{C}$ and $\mathcal{S}$ region. The value of V on the $\mathcal{S}$ region is equal to the pay-off function (2.7), i.e.

$$V_t = \Omega_t \quad \text{for } \forall\, S \in \mathcal{S}_t.$$

TABLE 2.3: *The marginal condition of American plain vanilla option for Black–Scholes partial differential equation.*

	call option	put option
marginal condition	$V_t\big\rvert_{S=0} = 0$	$\lim_{S\to\infty} V_t = 0$
continuation condition	$V_t\big\rvert_{S=S_t^*} = \Omega_t\big\rvert_{S=S_t^*}$	$V_t\big\rvert_{S=S_t^*} = \Omega_t\big\rvert_{S=S_t^*}$
smoothness condition	$\frac{\partial V_t}{\partial S}\big\rvert_{S=S_t^*} = 1$	$\frac{\partial V_t}{\partial S}\big\rvert_{S=S_t^*} = -1$

By solving the introduced problem, we calculate the function of an American plain vanilla option contract V_t, but also the early exercise function S^*.

The price of American option can be also defined by the conditioned expected value as

$$\begin{aligned} V_t &= \operatorname*{ess\,sup}_{T^*\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^{\mathcal{Q}}\left(e^{-r(T^*-t)}\Omega\big\rvert_{S=S_{T^*}}\right) \\ &= \operatorname*{ess\,sup}_{T^*\in\mathcal{T}_{[t,T]}} \mathbb{E}^{\mathcal{Q}}\left(e^{-r(T^*-t)}\Omega\big\rvert_{S=S_{T^*}}\Big\rvert\mathcal{F}_t^W\right), \end{aligned}$$

where $\mathcal{Q}$ is the risk-neutral measure, Ω is defined in (2.7), $\mathcal{F}_t^W$ defined in DEFINITION 1.8 represents the information in time t and $\mathcal{T}_I$ is a set of all Markov times with values within the interval I.

Although there is no closed formula for American style option price, there are many approximations or numerical methods for calculation of the price or behavior of early exercise boundary (cf. Hull 1997, Geske and Johnson 1984, Geske and Roll 1984, Karatzas 1988, Chadam 2008, Kwok 2008, Kuske and Keller 1998, Mallier 2002, Pascucci 2008).

2.3 Forwards

A *forward* is an agreement to buy an underlying at a certain future time (certain specified date) T for a (certain specified constant) price X.

Forward contracts are traded in OTC market. Forwards are usually used to hedge foreign exchange rate movements.

The pay-off function of forward is defined by the expression

$$\Omega = S - X.$$

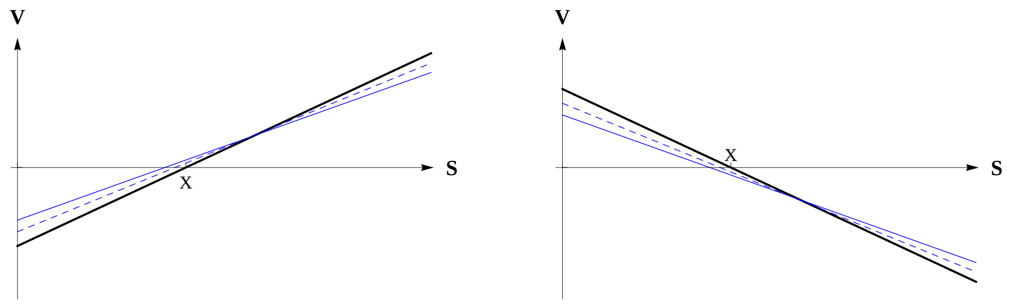

FIGURE 2.4: *Value of the long (left) and the short (right) position of a forward contract (with $r > q$) at the time $t = t_1 < T$ (solid), $t = t_2 \in (t_1, T)$ (dashed) and at the expiration time T (bold).*

To exclude the arbitrage, we have to set the strike price to satisfy the equation

$$V_0 = 0.$$

The non-arbitrage forward strike price is then calculated as

$$X = S_0 e^{(r-q)T}, \tag{2.8}$$

where r is the interest rate and q is the rate of benefit from holding (storing) the underlying asset.

The foreign exchange rate (F) is very common as an underlying asset in the forward contracts. The rate of benefit q is equal to the foreign interest rate r^f and (2.8) becomes

$$X_F = F_0 e^{(r-r^f)T}.$$

The value of a forward contract at time t is calculated as

$$V(t, S) = e^{-q(T-t)}S - e^{-r(T-t)}X.$$

CHAPTER 3

Exotic options

After the important improvements were included into the pricing mechanism of the basic financial instruments (such as plain vanilla options), the market was in need of something more unpredictable. The increasing demand for higher complexity of financial derivatives has brought the exotic features into elements of portfolio. The trend of inventing new, less predictable, financial derivatives persists and already wide family of exotic derivatives is still growing in all dimensions. However, this evolution comes hand in hand with more problems that must be solved to sufficiently secure the portfolio. According to large variability and high unpredictability of exotic options, they can be used to hedge many types of risk that can occur on market.

This chapter deals with elementary classification and main features of the most famous financial instruments selected from the enormous group of exotic options. The main scope lays on the class of path-dependent options, we focus especially on Asian, lookback and barrier options.

Mathematical problems arising from the valuation of financial derivatives presented in this chapter are discussed only marginally. For more detailed analysis of exotic options pricing problems see e.g. Wilmott (2006), Epps (2007), Briys et al. (1998), Zhang (1998), Kwok (2008), Ševčovič et al. (2011), Hull (1997).

3.1 Classification of exotic options

As we have already mentioned, there does not exist a closed definition of the exotic option. By the most common classification, all options except for the plain vanilla options belong into this group. The exotic options varies in many characteristics, thus their exact classification is impossible. In this paper, we use classification according to the main properties of derivatives (this approach is presented in Wilmott (2006)). The derivatives are considered according to following six features: *time dependence*,

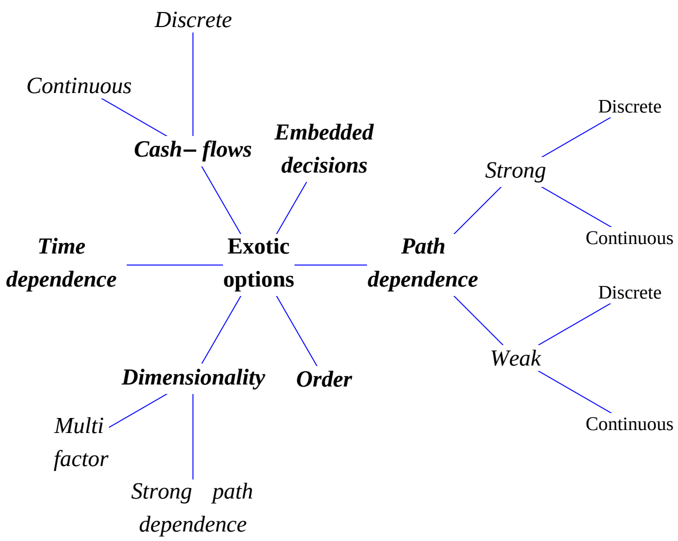

FIGURE 3.1: *The classification scheme of exotic options (based on (Wilmott 2006)).*

cash-flows, *path dependence*, *dimensionality*, *order* and *embedded decision*. The distribution of exotic derivatives is presented in FIGURE 3.1 and in TABLES 3.1-3.3 we classify derivatives presented in this chapter.

3.1.1 Time dependence

The pay-off function or the behavior of a financial derivative contract can depend on the time elapsed from start or remaining to the expiry T.

The example of time dependence feature is a *Bermudan* options. The Bermudan style derivative is permitted to be exercised on certain dates or during certain periods up to the expiration time. According to this feature, this derivative is referred to as time-inhomogeneous.

3.1.2 Cash-flows

Some derivatives can consist of cash-flows that are paid to holder through the life of contract. The cash-flows can be further divided into *discrete* and *continuous*.

According to the non-arbitrage condition, so called *jump condition* have to be involved into the valuation of a derivative with discrete cash-flows, i.e.

$$\lim_{t \to T_0 -} V(t) = \mathbb{E}\left[H_{T_0}\right] + \lim_{t \to T_0 +} V(t),$$

where V is the value of derivative and H_{T_0} is the cash-flow paid to holder of a derivative at time T_0.

For a continuous cash-flow derivative contracts, payments are usually defined as a function of a spot value of underlying asset. Instead of including the jumping condition, cash-flows are implemented into the pricing model as source term.

3.1.3 Path dependence

The pay-off function of path-dependent derivatives usually depends on the spot price at the maturity

$$\Omega = f(t, S_t, S_\cdot), \tag{3.1}$$

where the function of time $S_\cdot = S(\,\cdot\,)$ represents the path of the spot price up to time t. There are two varieties of path dependence: *strong* and *weak*. Both cases can be either continuous or discrete according to the sampling of path of underlying asset.

Strong path dependence

The pay-off function of strongly path-dependent derivative contracts depends on the value and path (behavior) of the underlying asset during its life. The path property is usually captured by an independent variable, e.g. the average in Asian options.

Weak path dependence

The pay-off function of weakly path-dependent derivative contracts depends only on the path (behavior) of the underlying asset during its life.

3.1.4 Dimensionality

The dimensionality of derivative contract refers to the number of variables involved into the model. The plain vanilla option is two dimensional as there are two variables: time t and underlying asset S. There are two types of dimensionality: strong path dependence and multi factor. In the former one, we increase the dimensionality by including a variable capturing path property. The latter case covers derivatives with multiple underlying assets, i.e. multiple sources of randomness.

3.1.5 Order

The pay-off function of the higher order derivative contracts depends on some other financial derivative, e.g. options on options. The plain vanilla options are of the first order.

3.1.6 Embedded decisions

Many derivative contracts have included some property that can be activated by the decision of holder or writer. American style derivative with its early exercise boundary obtain this feature. The implement assumption in the pricing of derivatives with embedded decision is that the holder behaves rationally and wants to increase his benefit as much as possible.

3.2 Path-dependent options

The path-dependent options belong to the group of the most frequently used and analyzed exotic options. Unlike the other types of the exotic options, the pay-off function of path-dependent options depends, in some non-trivial way, on the path history of the spot price of underlying during (the whole or part of) the life of an option.

In a very fundamental sense, also the American style of an option can be considered as a path-dependent option. In this paper we do not use such classification, we consider the American style early exercise rights as a possible feature of any option.

There are three main groups of the path-dependent options: Asian options, lookback options and barrier options. The classification of these path-dependent options is presented in TABLE 3.1.

TABLE 3.1: *Classification of main path-dependent exotic options.*

Derivative	Time dependence	Cash-flows	Path dependence	Dimension	Order	Embedded decisions
Asian option	Yes/No[a]	No	Strong	3[b]	first	No
Lookback option	Yes/No[a]	No	Strong	3[b]	first	No
Barrier option knock-out	No	No	Weak	2	first	No
Barrier option knock-in	No	No	Weak	2	second[c]	No

[a] The derivative is time dependent for discrete sampling of the underlying asset but it is not for continuous sampling.

[b] The dimension can be reduced by suitable substitution (see SECTION 4.2).

[c] The problem can be solved as difference between vanilla and barrier knock-out option.

3.2.1 Asian options

Asian option is a path-dependent derivative. It depends on the average A of the value of underlying asset reached during the (not necessary whole) life of option contract. There are two types of Asian options: *fixed strike* and *floating strike* option.

The pay-off function of the fixed strike (aka average rate) option depends on difference between the average and given constant strike price X, i.e. the underlying asset in the plain vanilla option pay-off is replaced by an average:

$$\Omega^{Asian} = (\mathfrak{c}(A - X))^+ ,$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively.

The pay-off function of the floating strike (or average strike) option depends on difference between the spot value of the underlying asset at the time of expiry and the average at the expiry as well, i.e. the strike price in the plain vanilla option pay-off is replaced by an average:

$$\Omega^{Asian} = (\mathfrak{c}(S - A))^+ ,$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively.

There is a variety of averages that are used in the Asian style contracts, e.g. arithmetic, geometric, weighted arithmetic, etc. In the real market a discrete version of the average is more common than the continuous averaging. The average can be also capped or floored to ensure the average in certain boundaries. The Asian option with such an average is called *capped Asian option*.

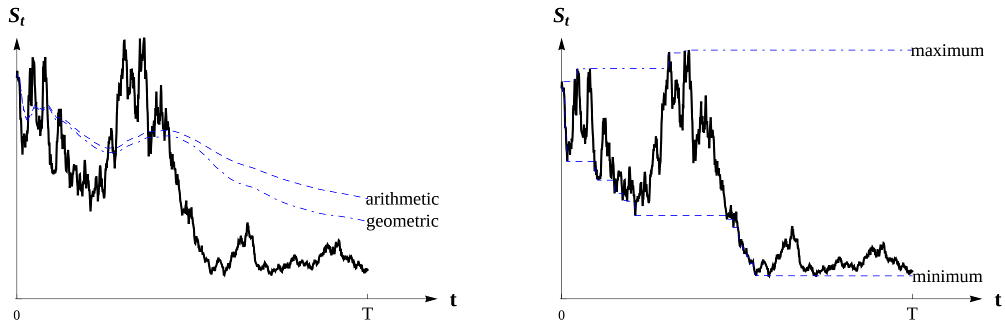

FIGURE 3.2: *The arithmetic and geometric average of an underlying S (left). The maximum and minimum value of an underlying asset S (right).*

According to their pay-off, Asian options can be used to hedge highly volatile underlying assets. The average of an underlying smooths big jumps caused by the volatility of the market (see FIGURE 3.2).

The American style Asian options are called *Hawaiian options*, however, we do not use this name in this thesis.

3.2.2 Lookback options

Lookback option is a path-dependent derivative. The pay-off function of lookback option depends on the extreme value (maximum M or minimum m) of value of the underlying asset reached during (the whole or part of) the life of the option contract.

Similarly to the Asian style options, lookback options can be divided in two main groups according to the pay-off function. The first type is *extreme rate* options

$$\begin{aligned} \Omega^{min} &= (\mathfrak{c}\,(m - X))^+ \,, \\ \Omega^{max} &= (\mathfrak{c}\,(M - X))^+ \,, \end{aligned}$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively. The second type of lookback options is called *extreme strike* options

$$\begin{aligned} \Omega^{min} &= S - m, \\ \Omega^{max} &= M - S. \end{aligned}$$

Notice that it is not reasonable to create an extreme strike lookback put option for minimum value or call option for maximum option.

If the maximum or minimum is guaranteed by certain lower or upper boundary, respectively, the derivative is called *capped Lookback option*. A perpetual lookback option is called *Russian option*.

3.2.3 Barrier options

Barrier option is a path-dependent derivative. In some cases the barrier option contracts are more attractive to the traders, because they are cheaper than regular options.

Barrier option is a financial derivative that change its property when hits specified barrier (lower barrier $B_L(t)$ and/or upper barrier $B_U(t)$).

The basic classification of barrier options is naturally based on the effect that takes place at the time when the underlying hits the barrier.

The most common barrier option ceases to exist when hits the barrier or reversely change from inactive to active. In the first case, the option is active from the beginning. At the time of hitting the barrier T_B, the holder takes the *rebate* (defined by the function $R(t)$) and the option expires. Such options are called *knock-out* barrier options and their pay-off function is

$$\Omega^{\text{out}} = \begin{cases} (\mathfrak{c}\,(S-X))^+, & S_t \in (B_L(t), B_U(t)) \text{ for } \forall t \in [0,T], \\ e^{r(T-T_B)} R(T_B), & \text{otherwise}. \end{cases}$$

In the other case, the option is inactive at the beginning. If the underlying spot value hits the barrier, the option starts to exist. The holder takes the rebate if the barrier is not achieved. Such options are called *knock-in* barrier options and their pay-off is

$$\Omega^{\text{in}} = \begin{cases} R(T), & S_t \in (B_L(t), B_U(t)) \text{ for } \forall t \in [0,T], \\ (\mathfrak{c}\,(S-X))^+, & \text{otherwise}. \end{cases}$$

3.3 Other exotic derivatives

In this section, we present exotic derivatives traded on the financial markets. However, the full list of exotic contracts traded is much wider and is growing each day. The classification of most interesting of them is presented in TABLES 3.2-3.3.

Packages

The package is a set of financial assets (derivatives - vanilla options and forwards, cash and underlying assets) connected together by a contract. The most often used packages are well know strategies as bull spreads, bear spreads, condor spreads, butterfly spreads, straddles, strangles etc.

Binary options

Binary options have discontinuous pay-offs. The two basic types of binary options are *cash-or-nothing* and *asset-or-nothing*. The cash-or-nothing option pays a constant amount of cash at the *in the money* area and nothing at the *out the money* and at the *at the money* area. The asset-or-nothing option pays an asset at the *in the money* area and nothing at the *out the money* and at the *at the money* area.

The pay-off function for cash-or-nothing is

$$\Omega^{CoN} = H\ \mathbf{1}_{ITM}(S),$$

where H is the amount of contracted money and $\mathbf{1}_{ITM}(\,\cdot\,)$ is the indicator function of in-the-money region. The pay-off function for asset-or-nothing is

$$\Omega^{AoN} = S\ \mathbf{1}_{ITM}(S),$$

where $\mathbf{1}_{ITM}(\,\cdot\,)$ is the indicator function of in-the-money region. The in-the-money region for asset-or-nothing option is usually set as interval (X, ∞).

Compound options

Compound options are options on options. There are two strike prices and two expiration dates, one for the ”inner” and one for the ”outer” option. There are four types of the compound options: put on a put, put on a call, call on a put and call on a call.

Let V be value function of an option with expiration time $\mathcal{T}$ and let $T < \mathcal{T}$ be expiration time of a compound option. The pay-off function of compound option is

$$\Omega^{compound} = \left(\mathfrak{c}\left(V(T,S) - X\right)\right)^{+},$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ is for call or put option, respectively.

Chooser options

Chooser option gives an option to its holder to choose between several options at the certain exactly given time, e.g. a call and a put option.

Let C and P be value of a call and a put options with expiration time $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. The chooser option (for these two options) with expiration time $T < \min\left[\mathcal{T}_1, \mathcal{T}_2\right]$ has pay-off function

$$\Omega^{chooser} = \max\left[C(T,S) - X_C, P(T,S) - X_P, 0\right],$$

where X_C and X_P are strike prices of call and put option, respectively.

Extendible options

The extendible option is regular plain vanilla options with feature that at some specified time T, the holder (or the writer) can either exercise the option or extend the option's life and even change the strike price. The pay-off function for the case with holder in charge is

$$\Omega^{extendible} = \max\left[\left(\mathfrak{c}\left(S - X\right)\right)^+, V(T,S)\right],$$

where V is an option (the same call/put type as the extendible one) with expiration time $\mathcal{T} > T$ and $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ is for call or put option, respectively. For the case with writer in charge, the maximum in pay-off function is replaced by minimum.

Range notes

Range note is a derivative that gives fixed income at rate H all the time that underlying asset (usually equity or foreign exchange rate) lies within a given set I. The pay-off function of the range note is

$$\Omega^{range} = H \int_{t_0}^{T} \mathbf{1}_I(S_t)dt,$$

where $\mathbf{1}_I$ is the indicator function of set I and t_0 and T are time of settlement and expiration time, respectively. Range note can be defined on several sets with different rates (negative as well).

For example, a range note called *in-out range accrual note* on a foreign exchange rate pays fixed income for positive part of the ratio of difference of time (or days) that spot value was within the range I and time (or days) it was outside the range. The pay-off function of such range note is given by

$$\Omega^{range} = H\left(\frac{\int_{t_0}^{T}\mathbf{1}_I(S_t)dt}{T-t_0} - \left(1 - \frac{\int_{t_0}^{T}\mathbf{1}_I(S_t)dt}{T-t_0}\right)\right)^{+}.$$

Passport options

Passport option (aka perfect trader) is a call option on the trading account, i.e. the holder receives the positive part of value of his trading account π. To solve this problem, one needs to include a state variable π into the model. The pay-off function of passport option is

$$\Omega^{passport} = \pi^{+}.$$

For suitable selection of the type of variable π, the dimension of this problem can be reduced, e.g. for π given by stochastic differential equation

$$d\pi = r(\pi - kS)\,dt + k\,dS,$$

where r is interest rate and k is amount of options in the portfolio and is called a strategy (the parameter k can vary through time, i.e. it can be replaced by parameter k_t).

Forward-start options

These options start at some time in the future. After this period of time the derivative behaves as a regular option (its type depends on the contract). The start of the option is usually conditioned by the position at the money or the strike is set to the actual value of the underlying asset at forward-start time. The pay-off function of the latter case is

$$\Omega^{fwd-start} = \left(\mathfrak{c}\left(S - S_{fwd}\right)\right)^{+},$$

where S_{fwd} is the value of underlying asset at forward-start time and $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively.

Break/cancelable forward

The break/cancelable forward can be terminated by the holder at certain time specified in contract.

Shout options

Shout option is an European vanilla option, with the property that the holder can "shout" to the writer once (or more times) during the life of the option (this can be done only if $S > X$ for call option and $S < X$ for put option). After the "shout", the option expires and holder gets another pure plain vanilla option with strike price set to the actual value of underlying asset. The pay-off function of this option is

$$\Omega^{shout} = \mathfrak{c}\, e^{r(T-T_{shout})} \left(S_{shout} - X\right) + \left(\mathfrak{c}\left(S - S_{shout}\right)\right)^{+},$$

where T_{shout} is optimal shouting time, S_{shout} is the value of underlying asset and $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively.

Problem of pricing the shout option is similar to pricing American vanilla option. Instead of early exercise boundary we are looking for optimal shouting boundary.

Volatility derivatives

The volatility option is considered with discrete sampling. It depends on the process

$$\sigma_i = \sqrt{\frac{1}{\Delta t}\frac{1}{n-1}\sum_{i=1}^{n}\left(\ln\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2},$$

where $\Delta t = t_i - t_{i-1}$ is difference of time moments in the discretization grid. For the purpose of valuation it is necessary to include two new state variables σ and $S_i^- = S_{t_{i-1}}$, i.e. the past value of underlying asset. For more details see Wilmott (2006).

The pay-off function of volatility option is

$$\Omega^{volatility} = \sigma.$$

The variation of volatility option is variance option, i.e. the option on value σ^2. Another example of derivative where realized variance is involved is called *variance swap* with the pay-off function

$$\Omega^{var-swap} = \sigma^2 - \sigma_{fix}^2,$$

where σ_{fix} is fixed volatility given by the contract. Similar derivatives are also made for correlation, e.g. *correlation swap* (hedging basket options).

Cliquet/Ratchet options

The cliquet or ratchet option is periodic financial derivative resetting the strike price to the value of actual price of the underlying asset. At each reset time the holder receives payment of the difference between old and new strike price or the payment can be also accumulated until the final maturity.

Coupe options

The coupe option is periodic financial derivative resetting the strike price to the worse of the actual (or original) strike price and value of actual price of the underlying asset. At each reset time the holder receives payment of the difference between old strike price and actual price of underlying asset or the payment can be also accumulated until the final maturity. Coupe option is similar to cliquet option, but is cheaper.

Israeli options

The Israeli or game option is a plain vanilla option, with feature that the seller of option can cancel the contract but must pay the early exercise pay-off and the penalty fee.

HYPER option

The HYPER option, i.e. High Yielding Performance Enhancing Reversible option is an American vanilla option that can be exercised over and over. On each exercise, the type of the option changes from call to put and vice versa.

Parisian options

The Parisian option is a barrier option where the barrier feature is activated only after the underlying asset spends some time beyond the barriers. According to this feature, there is need to measure the time spent outside the barriers. The classical Parisian option resets the elapsed time once the underlying returns inside the region. The *Parasian* contract does not reset the elapsed time. Clearly, we need to include state variable keeping the value of elapsed time into the model.

TABLE 3.2: *Classification of selected exotic options (part 1).*

Derivative	Time dependence	Cash-flows	Path dependence	Dimension	Order	Embedded decisions
Package	No	No	No	2	first	No
Binary option	No	No	No	2	first	No
Compound option	No	No	No	2	second	No
Chooser option	No	No	No	2	second	No (or trivial)
Extendible option	No	No	No	2	second	Yes
Range note	No	Yes (continuous)	Weak	2	first	No
In-out range accrual note	No	No	Strong	3	first	No
Passport option	No	No	Weak	3	first	Yes
Forward-start option	No	No	Weak	2	second	No
Break/cancelable forward	Yes	No	No	2	first	Yes
Shout option	No	Yes (discrete)	Strong	3	second	Yes
Volatility option	Yes/No[a]	No	Strong (discrete)	4[c]	first	No
Cliquet/Ratchet option	Yes	Yes/No[b] (discrete)	Strong	3	first	No
Coupe option	Yes	Yes/No[b] (discrete)	Strong	3	first	No
Israeli option	No	No	Weak	2	first	Yes
HYPER option	No	Yes (discrete)	Strong	3	second	Yes

[a] The derivative is time dependent for discrete sampling of the underlying asset but it is not for continuous sampling.

[b] The cash-flow can be accumulated until final maturity or paid at each reset date.

[c] The dimension can be reduced by suitable substitution.

Installment knock-out option

The installment knock-out option is an out barrier option with feature, that the holder can pay the installment to keep the option alive. If he does not want to pay the installment, he loses the contract.

Edokko options

The Edokko (*Edo* is the old name for Tokyo and *ko* means people) or Tokyo option is a knock-out barrier option with the so called *caution* region. After the first hit of the barrier, the option moves from the safety position into the caution status. Hitting the barrier while in caution state, the option vanishes. There is an optional feature of this derivative, that holder can make certain payment given by the contract to reset from the caution back into the safety status. For more details see Fujita and Miura (2002).

Lookback-Asian options

Lookback-Asian option is a member of group of strongly path-dependent options, where more than one path variable is included. Here the average and extreme values are involved in the value of the option in various ways.

Ladders

The ladder option is a lookback option with discretely sampled value of underlying asset. The process of this maximum $\widehat{M}$ and minimum $\widehat{m}$ is given by

$$\widehat{M_t} = \max\{S \in \widehat{\mathbb{D}}; S \leq M_t\}$$

and

$$\widehat{m}_t = \min\{S \in \widehat{\mathbb{D}}; S \geq m_t\},$$

respectively. The set $\widehat{\mathbb{D}}$ is set of discrete values of underlying asset, e.g. even values and $M_t = \max_{\tau \leq t} S_\tau$ and $m_t = \min_{\tau \leq t} S_\tau$. The pay-off function is then given as a function

$$\Omega^{ladder} = f(\widehat{M}) \quad \text{or} \quad \Omega^{ladder} = f(\widehat{m}),$$

for maximum or minimum, respectively.

Basket options

Basket option involves more than one risky underlying. It depends on the value of the basket of assets (underlying assets or indices). The pay-off function of the basket option on underlying assets S^i is given by

$$\Omega^{basket} = \left(\mathfrak{c}\left(\sum_i w_i S^i - X\right)\right)^+,$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively. The constant w_i denotes the share (or weight) of i^{th} underlying asset in basket or index. For the multi-asset options pricing, the correlation matrix should be involved in the calculation.

Rainbow options

Rainbow option involves more than one risky underlying. It is similar to the basket option, but the weights of underlying assets depend on their performance. For the multi-asset options pricing, the correlation matrix should be involved in the calculation. There are many types of the rainbow options, we present several most common of them.

The *exchange option* or *Margrabe option* is a derivative contract, in which the holder exchanges one asset for another. The pay-off function of this option for assets S^1 and S^2 is

$$\Omega^{margrabe} = \left(S^1 - S^2\right)^+ .$$

The *best of* n *assets plus cash option* has pay-off function given by

$$\Omega^{best+cash} = \max\left[H, \max_i S^i\right],$$

where H is the value of cash set according to the contract.

The *better of* n *assets option* is a variation of best of n assets plus cash option with $H = 0$, i.e. the pay-off is positive part of the best performing asset.

The *worse of* n *assets option* the pay-off is positive part of the worst performing asset.

The *maximum of* n *assets option* and *minimum of* n *assets option* are vanilla option with underlying asset replaced by the best or the worst performing of underlying assets, respectively.

Mountain range options

The mountain range options combine characteristics of basket and range options, i.e. the derivative depends on more underlying assets and there is particular period of time when the option is active. For the multi-asset options pricing, the correlation matrix should be involved in the calculation. There are several types of mountain range options.

TABLE 3.3: *Classification of selected exotic options (part 2).*

Derivative	Time dependence	Cash-flows	Path dependence	Dimension	Order	Embedded decisions
Parisian option knock-out	No	No	Strong (continuous)	3	first	No
Parisian option knock-in	No	No	Strong (continuous)	3	second[c]	No
Installment knock-out option	Yes	Yes	Weak	2	first	Yes
Edokko option	No	No	Weak	3	first	Yes
Lookback-Asian option	Yes/No[a]	No	Strong	4[b]	first	No
Ladder	Yes/No[a]	No	Strong	3[b]	first	No
Simple basket of n assets	No	No	No	n+1	first	No
Rainbow option of n assets	No	No	No	n+1	first	No
Margrabe option	No	No	No	3	first	No
Mountain range of n assets	Yes	No	Yes/No	>n+1	first	No
Napolean option	No	No	No	2	first	No
Quanto option	No	No	No	3	first	No

[a] The derivative is time dependent for discrete sampling of the underlying asset but it is not for continuous sampling.

[b] The dimension can be reduced by suitable substitution.

[c] The problem can be solved as difference between vanilla and Parisian knock-out option.

The *Altiplano option* is a multi-asset call option with a feature, that the holder receives a compensatory coupon if the underlying asset never reaches the strike price (or some other given barrier) during given period.

The *Annapurna option* is a multi-asset option, where the holder is rewarded if all underlying assets in the basket never fall below certain barrier during a given period.

The *Atlas option* is a multi-asset call option, where some of the best and some of the worst performing assets are removed from the basket before the execution of the option.

The *Everest option* is a long-term multi asset option. The holder gets the pay-off based on the worst performing asset in the basket. The Everest option usually last for 10 to 15 years and the basket usually contains 10 to 25 assets.

The *Himalayan option* is multi-asset call option, where the best performing asset is thrown out of the basket at specified sampling dates, leaving just one asset in the basket at the end.

Napolean options

The Napolean option is a financial contract typically based on the stock index. The pay-off of this derivative is a fixed coupon and the worst return of the index over specified time periods.

Quanto options

The quanto option is a financial derivative with underlying asset in one currency and pay-off in the another one.

British (style of) options

The British option is a class of early exercise options based on hedging of the true drift μ of underlying asset. The derivative defines a *contract drift* μ_c (constant value specified at the start of the contract). The holder can exercise the option whenever prior the expiration time T and he receives the best prediction of the pay-off according to the contract drift, i.e. the (general) pay-off function of British option is given by

$$\Omega^{GB} = \mathbb{E}_t^{\mathcal{R}} \left[\Omega_T \big|_{S=S_T} | S_t = S \right],$$

where Ω_T is the pay-off function at expiration time T, $\mathbb{E}_t^{\mathcal{R}}[X] = \mathbb{E}^{\mathcal{R}}[X|\mathcal{F}_t]$ is the conditioned expectation with information available at time t (the information set is represented by the filtration $\mathcal{F}_t$ of the σ-algebra $\mathcal{F}$ where the Brownian motion is supported) and $\mathcal{R}$ is a probability measure according to which the underlying is driven by stochastic differential equation

$$dS_t = \mu_c S_t \, dt + \sigma S_t \, dW_t^{\mathcal{R}}.$$

It is clear, that the pay-off function of the British option depends on time.

The most simple example of the British type derivatives is British vanilla option. The pay-off function at the expiry T is the same as for the plain vanilla options, i.e. $\Omega_T^{vanilla} = (\mathfrak{c}\,(S - X))^+$. Consequently, the pay-off function at time t is

$$\Omega^{GB,vanilla} = \mathfrak{c} \left(e^{\mu_c (T-t)} S \boldsymbol{\Phi} \left(\mathfrak{c}\, d_t^{\mu_c} \right) - X \boldsymbol{\Phi} \left(\mathfrak{c} \left(d_t^{\mu_c} - \sigma \sqrt{T-t} \right) \right) \right),$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call or put option, respectively. The function $\Phi(\,\cdot\,)$ is the cumulative distribution function *CDF* of the normal probability distribution $\mathcal{N}(0,1)$ and $d_t^{\mu_c} = \frac{\ln \frac{S}{X} + (\mu_c + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

The British feature is a property that can be add to many (more or less simple) derivative contracts, e.g. vanilla, Asian, lookback, Russian etc. For more detailed information see Peskir and Samee (2008a,b), Glover et al. (2009a,b).

3.4 Hedging of the exotic portfolio

Portfolio consisting of financial instruments (and/or underlying assets) is usually secured using the indicators of sensitivity called *Greeks* (i.e. partial derivatives according to some of the parameters). The more exotic features we add to the portfolio, the more complex Greeks we need to include into its hedging mechanism.

While dealing with a portfolio consisting of simple derivatives as vanilla options, it suffice to consider simpler Greeks (i.e. first order Greeks). On the other hand, if there are exotic features in the portfolio, we need to include more complex sensitivities (e.g. Greeks of higher order, i.e. Greeks on Greeks). We discuss these indicators in detail in the APPENDIX B.

A probabilistic model for pricing American style options

The main purpose of this section is to derive an integral equation for valuation of an American style Asian option paying continuous dividends (we call the underlying S and the rate of benefit q the stock and the dividend rate, respectively). We follow the idea of derivation from Hansen and Jørgensen (2000). Their formula for a floating strike option was derived using the theory of martingales and conditioned expected values. We extend their formula for general financial derivative on underlying(s) driven by a Brownian motion (this includes also Asian options on underlying paying non-zero dividend yield and having a general form of floating strike averaging). In a more detail, we discuss Asian options with geometric, arithmetic and weighted arithmetic averaging operator. This chapter is based on results from the paper Bokes (2010)[1] and the first part of paper Bokes and Ševčovič (2011)[2].

4.1 Model

The pricing model is based on the assumption of stochastic behavior of the underlying asset in time. Throughout the thesis (except for the generalizations of problems) we shall assume that the underlying asset price S_t is driven by a stochastic process satisfying the following stochastic differential equation

$$dS_t = (r-q)S_t \, dt + \sigma S_t \, dW_t^P, \quad 0 \le t \le T. \tag{4.1}$$

It starts almost surely from the initial price $S_0 > 0$. Here the constant parameter $r > 0$ denotes the risk-free interest rate whereas $q \ge 0$ is a continuous dividend rate.

[1] TB: 2010, *Valuation of the American-style of Asian option by a solution to an integral equation, Acta Universitatis Matthiae Belii*

[2] TB and Ševčovič, D.: 2011, *Early exercise boundary for American type of floating strike Asian option and its numerical approximation, Applied Mathematical Finance*

The constant parameter σ stands for the volatility of underlying asset return and W_t^P is the Wiener process with respect to the standard risk-neutral probability measure P. A solution to equation (4.1) corresponds to the geometric Brownian motion

$$S_t = S_0 e^{(r-q-\frac{1}{2}\sigma^2)t+\sigma W_t^P}, \quad 0 \leq t \leq T.$$

The bond (risk-free) market is driven by differential equation

$$dB_t = rB_t \, dt, \tag{4.2}$$

with $B_0 = 1$, i.e. $B_t = e^{rt}$.

As we have already mentioned above we shall derive the value of an American style Asian option with floating strike. If we define the optimal stopping time as T^*, the pay-off of option is set by

$$\Omega\Big|_{(S,A)=(S_{T^*},A_{T^*})} = \left(\mathfrak{c}\left(S_{T^*} - A_{T^*}\right)\right)^+ ,$$

where A_t is a continuous average of the stock value during the interval $[0, t]$ and $\mathfrak{c} = 1$ for a call option and $\mathfrak{c} = -1$ for a put option.

We may consider several different types of averages. We define parametric class of averages with parameter p

$$(A_t)^p = \frac{1}{\int_0^t a(s)\, ds} \int_0^t a(t-u)(S_u)^p \, du. \tag{4.3}$$

The integrable kernel function $a(\,\cdot\,) \geq 0$ is usually defined either as $a(s) = e^{-\lambda s}$ where $\lambda > 0$ is constant (the exponential kernel) or as $a(s) = 1$ (the constant kernel; this is a special case of the exponential kernel with $\lambda \to 0$). By the appropriate choice of the kernel, we can create both continuous and discrete averages. We discuss several possible choices of the parameter p and the kernel $a(\,\cdot\,)$.

By the choice of the constant kernel and value $p = 0$ we obtain continuous geometric average (the expression is calculated as the limit $\lim_{p\to 0} A_t$)

$$\ln A_t = \frac{1}{t} \int_0^t \ln S_u \, du. \tag{4.4}$$

In the case of the constant kernel and value $p = 1$ we obtain continuous arithmetic average

$$A_t = \frac{1}{t} \int_0^t S_u \, du, \tag{4.5}$$

the choice of the general kernel and value $p = 1$ gives us the weighted arithmetic average (in the following sections we usually replace general kernel by the exponential one)

$$A_t = \frac{1}{\int_0^t a(s)\,ds} \int_0^t a(t-u) S_u\,du. \tag{4.6}$$

If we choose the constant kernel and value $|p| \to \infty$, the Asian (floating strike) option transforms into the lookback (floating price) option. For $p \to -\infty$ the model suits for put (minimum) lookback option

$$A_t = \inf_{u\in[0,t]} S_u = m_t \tag{4.7}$$

and for $p \to \infty$ the model suits for call (maximum) lookback option

$$A_t = \sup_{u\in[0,t]} S_u = M_t. \tag{4.8}$$

[We define these averages also in the SECTION 7.2.]

4.2 Calculation of the formula

According to Hansen and Jørgensen (2000), the American style contingent claims can be priced by the conditioned expectations approach. First, we present a theorem that can be used to calculate the value of general American style financial derivative on underlying(s) driven by a Brownian motion.

The value of American style of derivative on underlying x with the pay-off function $\Omega : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is calculated by solving the problem of mathematic programming

$$V(t,x) = \operatorname*{esssup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^{\mathcal{Q}} \left[\frac{\mathcal{N}_t}{\mathcal{N}_s} \Omega\left(s, x_s\right) \middle| x_t = x \right], \tag{4.9}$$

where $\mathcal{N}_t$ is the numeraire, $\mathcal{Q}$ is the risk-neutral measure, $\mathcal{T}_{[t,T]}$ denotes the set of all stopping times in the interval $[t,T]$ and $\mathbb{E}_t^{\mathcal{Q}}[X] = \mathbb{E}^{\mathcal{Q}}[X|\mathcal{F}_t]$ is the conditioned expectation with information available at time t (the information set is represented by the filtration $\mathcal{F}_t$ of the σ-algebra $\mathcal{F}$ where the Brownian motion is supported) and the esssup is the essential supremum (see DEFINITION 1.3).

THEOREM 4.1. *Assume stochastic behavior of the underlying(s) driven by a stochastic differential equation*

$$dx_t^i = \mu^i \, dt + \sigma^i \, dW_t^i,$$

for $i \in \{1, \ldots, n\}$ on their domain $\mathbb{D} \subset \mathbb{R}^n$. The values $\mu^i \in \mathbb{R}$, $\sigma^i \geq 0$ and $dW_t = (dW_t^1, \ldots, dW_t^n)$ are drift, volatility and differential of standard n-dimensional Brownian motion under the joined risk-neutral measure $\mathcal{Q}$, respectively. The covariance matrix Σ of dW_t is defined for $i, j \in \{1, \ldots, n\}$ by

$$\Sigma_{i,j} = \mathbb{C}ovar\left[dW_t^i, dW_t^j\right] = \rho_{ij} dt,$$

where $\rho_{ij} \in [-1, 1]$ is the correlation coefficient and $\rho_{ii} = 1$.

Let function $\mathcal{N} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and $\Omega : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be the numeraire and pay-off function, respectively. Moreover, assume that both functions are differentiable on $x \in \mathbb{D} \subset \mathbb{R}^n$ except for the set of zero measure.

Then the value $V(t, x_t)$ of American style of derivative on underlying asset x_t is given by

$$V(t, x_t) = v(t, x_t) + e(t, x_t), \tag{4.10}$$

where

$$v(t, x_t) \equiv \mathcal{N}(t, x_t) \, \mathbb{E}_t^{\mathcal{Q}} \left[(\mathcal{N}(T, x_T))^{-1} \, \Omega(T, x_T) \right], \tag{4.11}$$

$$e(t, x_t) \equiv \mathcal{N}(t, x_t) \, \mathbb{E}_t^{\mathcal{Q}} \left[- \int_t^T \mathbf{1}_{\mathcal{S}}(u, x_u) f_d(u, x_u) \, du \right], \tag{4.12}$$

and

$$f_d(t, x_t) = \frac{\partial \left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)} \right)}{\partial t} + \sum_{i=1}^n \mu^i \frac{\partial \left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)} \right)}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^n \rho_{ij} \sigma^i \sigma^j \frac{\partial^2 \left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)} \right)}{\partial x^i \partial x^j}. \tag{4.13}$$

REMARK 4.1. *It is worthwhile noting that the expression (4.10) of the value of an American style derivative can be restated as follows:*

$$\widetilde{V}_{am}(t, x) = \widetilde{V}_{eu}(t, x) + \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T \mathbf{1}_{\mathcal{S}}(u, x_u) f_b(u, x_u) \, du \right]$$

where $\widetilde{V}_{eu}$ stands for the price of European style derivative and $\mathbf{1}_{\mathcal{S}}(u, x_u) f_b(u, x_u)$, $u \in [0, T]$, represents a surplus bonus of the American style derivative against the European style derivative.

For the case of Asian options, expression (4.9) becomes

$$V(t,S,A) = \operatorname*{esssup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^P\left[e^{-r(s-t)}\left(\mathfrak{c}(S_s - A_s)\right)^+ \middle| S_t = S, A_t = A\right], \tag{4.14}$$

where $\mathcal{T}_{[t,T]}$ denotes the set of all stopping times in the interval $[t,T]$ and $\mathbb{E}_t^P[X] = \mathbb{E}^P[X|\mathcal{F}_t]$ is the conditioned expectation with information available at time t (the information set is represented by the filtration $\mathcal{F}_t$ of the σ-algebra $\mathcal{F}$ where the Brownian motion is supported) and the esssup is the essential supremum (see DEFINITION 1.3).

We can solve the problem simply using THEOREM 4.1, but we want to decrease the dimension of problem so we change the probability measure by the martingale

$$\eta_t = e^{-(r-q)t}\frac{S_t}{S_0} = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t^P}. \tag{4.15}$$

The new probability measure $\mathcal{Q}$ is defined as $d\mathcal{Q} = \eta_T\, dP$ and according to Girsanov's theorem 1.9, the process

$$W_t^{\mathcal{Q}} = W_t^P - \sigma t$$

is a standard Brownian motion with respect to the measure $\mathcal{Q}$. The value of stock under this measure is defined by

$$S_t = S_0 e^{(r-q+\frac{1}{2}\sigma^2)t + \sigma W_t^{\mathcal{Q}}} \tag{4.16}$$

According to Harrison and Kreps (1979), all asset prices f_t discounted by time value of money is martingale under the measure P, i.e.

$$e^{-rt} f_t = \mathbb{E}_t^P\left[e^{-rT} f_T\right].$$

If we change the measure to $\mathcal{Q}$, we have

$$\begin{aligned} f_t &= \mathbb{E}_t^P\left[e^{-r(T-t)} f_T\right] \\ &= \mathbb{E}_t^{\mathcal{Q}}\left[\frac{\eta_t}{\eta_T} e^{-r(T-t)} f_T\right] = \mathbb{E}_t^{\mathcal{Q}}\left[\frac{S_t}{S_T} e^{-q(T-t)} f_T\right]. \end{aligned} \tag{4.17}$$

The expression (4.17) yields that all assets discounted by the full stock price (dividends included) priced under the new measure are $\mathcal{Q}$-martingales, i.e.

$$\frac{f_t}{e^{qt} S_t} = \mathbb{E}_t^{\mathcal{Q}}\left[\frac{f_T}{e^{qT} S_T}\right].$$

According to this fact, we can reduce the dimension of stochastic variables. We introduce a new variable $x_t = \frac{A_t}{S_t}$. We obtain:

$$
\begin{aligned}
V(t,S,A) &= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^P \left[e^{-r(s-t)} \left(\mathfrak{c}(S_s - A_s)\right)^+ \Big|_{A_t=A}^{S_t=S} \right] \\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^Q \left[\frac{\eta_t}{\eta_T} e^{-r(s-t)} \left(\mathfrak{c}(S_s - A_s)\right)^+ \Big|_{A_t=A}^{S_t=S} \right] \\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^Q \left[e^{-(r-q)t} S_t e^{-r(s-t)} \left(\mathfrak{c}(S_s - A_s)\right)^+ \ \mathbb{E}_s^Q \left[\frac{e^{rT}}{e^{qT} S_T} \right] \Big|_{A_t=A}^{S_t=S} \right] \\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^Q \left[e^{-(r-q)t} S_t e^{-r(s-t)} \left(\mathfrak{c}(S_s - A_s)\right)^+ \ \frac{e^{(r-q)s}}{S_s} \Big|_{A_t=A}^{S_t=S} \right] \\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^Q \left[e^{-q(s-t)} S_t \left(\mathfrak{c} \left(1 - \frac{A_s}{S_s} \right) \right)^+ \Big|_{A_t=A}^{S_t=S} \right] \\
&= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} e^{-q(s-t)} S \ \mathbb{E}_t^Q \left[\left(\mathfrak{c}(1 - x_s)\right)^+ \Big|_{A_t=A}^{S_t=S} \right]. \qquad (4.18)
\end{aligned}
$$

The expression (4.18) can be rewritten in terms of the new variable $x = \frac{A}{S}$ as follows:

$$
\widetilde{V}(t,x) = e^{-qt} \frac{V(t,S,A)}{S} = e^{-qT_t^*} \ \mathbb{E}_t^Q \left[\left(\mathfrak{c}(1 - x_{T_t^*}) \right)^+ \right], \qquad (4.19)
$$

where $T_t^* = \inf\{s \in [t,T] | x_s = x_s^*\}$ and x^* is the exercise boundary.

Definition 4.1. *The stopping region $\mathcal{S}$ and continuation region $\mathcal{C}$ for an American style Asian call and put option (4.19) are defined by*

$$
\begin{aligned}
\mathcal{S}_{call} = \mathcal{C}_{put} &= \{(t,x) | t \in [0,T], 0 \le x < x_t^*\}, \\
\mathcal{C}_{call} = \mathcal{S}_{put} &= \{(t,x) | t \in [0,T], x_t^* < x < \infty\},
\end{aligned}
$$

where $[0,T] \in t \mapsto x_t^* \in \mathbb{R}$ is a continuous function determining the early exercise boundary. By $\mathbf{1}_{\mathcal{S}}(\,\cdot\,)$ we shall denote the indicator function of the set $\mathcal{S}$, i.e. $\mathbf{1}_{\mathcal{S}}(t,y) = 1$ for $(t,y) \in \mathcal{S}$ and $\mathbf{1}_{\mathcal{S}}(t,y) = 0$ otherwise.

In the following theorem, we present a solution to the pricing problem with one stochastic variable x_t formulated in (4.19). It is a generalization of the result by Hansen and Jørgensen (2000) and Wu et al. (1999) for the case of a nontrivial dividend rate $q \ge 0$ and a general form of the averaging of the floating strike price.

THEOREM 4.2. *The value $\widetilde{V}(t,x_t) = e^{-qt}V(t,x_t)$ of the American style floating strike Asian call ($\mathfrak{c}=1$) or put option ($\mathfrak{c}=-1$) on the underlying asset x_t is given by*

$$\widetilde{V}(t,x_t) = \widetilde{v}(t,x_t) + \widetilde{e}(t,x_t), \tag{4.20}$$

where

$$\begin{aligned} \widetilde{v}(t,x_t) &\equiv \mathbb{E}_t^{\mathcal{Q}}\left[e^{-qT}\left(\mathfrak{c}(1-x_T)\right)^+\right], & (4.21)\\ \widetilde{e}(t,x_t) &\equiv \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \mathfrak{c}\, e^{-qu} x_u \mathbf{1}_{\mathcal{S}}(u,x_u)\left(\frac{dA_u}{A_u} - (r - q x_u^{-1})\,du\right)\right], & (4.22)\end{aligned}$$

with the average given by the function A_t, stopping region $\mathcal{S}$ and continuous dividend rate $q \geq 0$.

The value of $\frac{dA_t}{A_t}$ depends on the average used in the valuation. The expression for the general average has form

$$\frac{dA_t}{A_t} = \frac{1}{p\int_0^t a(s)\,ds}\left(\frac{a(0) + \int_0^t a'(t-u)\left(\frac{S_u}{S_t}\right)^p du}{(x_t)^p} - a(t)\right) dt. \tag{4.23}$$

This expression is unusable in its general form, thus we restrict the weighted averaging to the exponential kernel $a(s) = e^{-\lambda s}$, and the expression simplifies into

$$\frac{dA_t}{A_t} = \frac{\lambda}{p(1-e^{-\lambda t})}\left(\frac{1}{(x_t)^p} - 1\right) dt. \tag{4.24}$$

For the geometric averaging we have the expression

$$\frac{dA_t^g}{A_t^g} = -\frac{1}{t}\ln x_t^g\, dt, \tag{4.25}$$

for the arithmetic average it is

$$\frac{dA_t^a}{A_t^a} = \frac{1}{t}\left(\frac{1}{x_t^a} - 1\right) dt, \tag{4.26}$$

and for the weighted arithmetic averaging with exponential kernel the expression becomes

$$\frac{dA_t^{wa}}{A_t^{wa}} = \frac{\lambda}{1-e^{-\lambda t}}\left(\frac{1}{x_t^{wa}} - 1\right) dt. \tag{4.27}$$

Finally, for the lookback options the expression has form

$$\frac{dm_t}{m_t} = \frac{dM_t}{M_t} = 0. \tag{4.28}$$

The expression for the lookback option is calculated from the expression for the general average, where $x_t^{\infty} = \frac{M_t}{S_t} \geq 1$ and $x_t^{-\infty} = \frac{m_t}{S_t} \leq 1$.

We calculate the exact or approximate formula for the American style Asian option with various floating strike averages. The next lemma will be useful in calculations to follow.

LEMMA 4.1. *Let* $z = \ln Z \sim \mathcal{N}(\alpha, \beta^2)$ *and define*

$$\gamma_p \equiv \frac{\ln K - \alpha}{\beta} - p\beta,$$

where $K > 0$ *and* $p \in \mathbb{R}$*. We have*

(i) $\mathbb{E}\left[\mathbf{1}_{\{Z \leq K\}}\right] = \boldsymbol{\Phi}(\gamma_0)$

[this expression is a special case of the expression (iii)],

(ii) $\mathbb{E}\left[\mathbf{1}_{\{Z \geq K\}}\right] = \boldsymbol{\Phi}(-\gamma_0)$

[this expression is a special case of the expression (iv)],

(iii) $\mathbb{E}\left[\mathbf{1}_{\{Z \leq K\}} Z^p\right] = e^{p\alpha + \frac{p^2\beta^2}{2}} \boldsymbol{\Phi}(\gamma_p)$*,*

(iv) $\mathbb{E}\left[\mathbf{1}_{\{Z \geq K\}} Z^p\right] = e^{p\alpha + \frac{p^2\beta^2}{2}} \boldsymbol{\Phi}(-\gamma_p)$*,*

(v) $\mathbb{E}\left[(K - Z)^+\right] = K\boldsymbol{\Phi}(\gamma_0) - e^{\alpha + \frac{\beta^2}{2}} \boldsymbol{\Phi}(\gamma_1)$*,*

(vi) $\mathbb{E}\left[(Z - K)^+\right] = e^{\alpha + \frac{\beta^2}{2}} \boldsymbol{\Phi}(-\gamma_1) - K\boldsymbol{\Phi}(-\gamma_0)$*,*

(vii) $\mathbb{E}\left[\mathbf{1}_{\{Z \leq K\}} Z \ln Z\right] = e^{\alpha + \frac{\beta^2}{2}} \left((\alpha + \beta^2)\boldsymbol{\Phi}(\gamma_1) - \beta \mathit{\Phi}(\gamma_1)\right)$*,*

(viii) $\mathbb{E}\left[\mathbf{1}_{\{Z \geq K\}} Z \ln Z\right] = e^{\alpha + \frac{\beta^2}{2}} \left((\alpha + \beta^2)\boldsymbol{\Phi}(-\gamma_1) + \beta \mathit{\Phi}(\gamma_1)\right)$*,*

where $\boldsymbol{\Phi}(\,\cdot\,)$ *and* $\mathit{\Phi}(\,\cdot\,)$ *are standard normal cumulative distribution and density functions, respectively.*

4.2.1 Approximation for the general average

The probabilistic distribution function of the general average defined by (4.3) cannot be expressed by an explicit expression. Following the idea of Hansen and Jørgensen (2000) we approximate the probabilistic distribution of the variable $x_t = \frac{A_t}{S_t}$ for the general average A_t by the log-normal conditioned distribution, i.e. $\ln x_u|\mathcal{F}_t \sim \mathcal{N}(\alpha_{t,u}, (\beta_{t,u})^2)$ at time t. Moreover, it is not possible to calculate the approximation with the general kernel, thus we use the exponential kernel $a(s) = e^{-\lambda s}$, with $\lambda > 0$. The result for the constant kernel can be calculated as a limit $\lambda \to 0$.

We use following lemma in the derivation of the approximation.

LEMMA 4.2. *Consider that the variable ξ_u has a log-normal conditioned distribution with parameters $\alpha_{t,u}$ and $\beta_{t,u}$, i.e.*

$$\ln \xi_u|\mathcal{F}_t \sim \mathcal{N}(\ln \xi_t + \alpha_{t,u}, (\beta_{t,u})^2).$$

Then the variable $(\xi_u)^p$ where $p \in \mathbb{R}$ has a log-normal conditioned distribution with parameters $p\alpha_{t,u}$ and $p\beta_{t,u}$, i.e.

$$\ln (\xi_u)^p|\mathcal{F}_t \sim \mathcal{N}(\ln (\xi_t)^p + p\alpha_{t,u}, p^2(\beta_{t,u})^2).$$

According to the LEMMA 4.2, we define the parameters of the approximate log-normal distribution by

$$\alpha_{t,u} = \frac{2}{p} \ln \mathbb{E}_t^{\mathbb{Q}}\left[(x_u)^p\right] - \frac{1}{2p} \ln \mathbb{E}_t^{\mathbb{Q}}\left[(x_u)^{2p}\right], \tag{4.29}$$

$$\beta_{t,u} = \sqrt{\frac{1}{p^2} \ln \mathbb{E}_t^{\mathbb{Q}}\left[(x_u)^{2p}\right] - \frac{2}{p^2} \ln \mathbb{E}_t^{\mathbb{Q}}\left[(x_u)^p\right]}. \tag{4.30}$$

LEMMA 4.3. *Consider the variable $(x_u)^p = \left(\frac{A_u}{S_u}\right)^p$, where A_u and S_u are defined as the general average (4.3) with exponential kernel $a(s) = e^{-\lambda s}$, with $\lambda > 0$ and as in (4.16), respectively. First two conditioned moments $\mathbb{E}_t^{\mathbb{Q}}\left[(x_u)^p\right]$ and $\mathbb{E}_t^{\mathbb{Q}}\left[(x_u)^{2p}\right]$ of $(x_u)^p$ entering the expressions for the functions $\alpha_{t,u} = \alpha(t,u,x_t)$ and $\beta_{t,u} = \beta(t,u,x_t)$ can be calculated, for $t \leq u$, as follows:*

$$\mathbb{E}_t^{\mathbb{Q}}\left[(x_u)^p\right] = (x_t)^p \frac{1-e^{-\lambda t}}{1-e^{-\lambda u}}\, e^{-\kappa_{\lambda,p}(u-t)} + \frac{\lambda}{1-e^{-\lambda u}}\, \frac{1-e^{-\kappa_{\lambda,p}(u-t)}}{\kappa_{\lambda,p}}, \tag{4.31}$$

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[(x_u)^{2p}\right] = {} & (x_t)^{2p} \frac{(1-e^{-\lambda t})^2}{(1-e^{-\lambda u})^2}\, e^{-(2\kappa_{\lambda,p}-p^2\sigma^2)(u-t)} \\ & +2(x_t)^p \frac{\lambda(1-e^{-\lambda t})}{(1-e^{-\lambda u})^2}\, e^{-\kappa_{\lambda,p}(u-t)}\, \frac{1-e^{-\kappa_{\lambda,p}(u-t)}}{\kappa_{\lambda,p}} \\ & +\lambda^2 \frac{(\kappa_{\lambda,p}-p^2\sigma^2) - 2(\kappa_{\lambda,p}-p^2\frac{\sigma^2}{2})e^{-\kappa_{\lambda,p}(u-t)} + \kappa_{\lambda,p}\, e^{-2(\kappa_{\lambda,p}-p^2\frac{\sigma^2}{2})(u-t)}}{(1-e^{-\lambda u})^2\, \kappa_{\lambda,p}\, (\kappa_{\lambda,p}-p^2\frac{\sigma^2}{2})\, (\kappa_{\lambda,p}-p^2\sigma^2)},\end{aligned} \tag{4.32}$$

where $\kappa_{\lambda,p} = \lambda + p\left(r - q + (1-p)\frac{\sigma^2}{2}\right)$.

Now, we return to the problem of valuation of an Asian option. First we replace the general form of the average in (4.22) by the expression for the general average with the exponential kernel defined by (4.23). Recall that for the floating strike Asian call or put option, the stopping region $\mathcal{S} = \{(t,x), x \geq 0, \mathfrak{c}\, x_t^* > \mathfrak{c}\, x\}$, where x_t^* is the exercise boundary and $\mathfrak{c} = 1$ for the case of a call option whereas $\mathfrak{c} = -1$ for a put option.

Using LEMMA 4.1 we calculate the value of both (4.21) and (4.22). The European part of the option has value

$$\begin{aligned}\widetilde{v}(t,x) &= \mathbb{E}_t^{\mathcal{Q}}\left[e^{-qT}\left(\mathfrak{c}(1-x_T)\right)^+\right] = e^{-qT}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\mathfrak{c}(1-x_T)\right)^+\right] \\ &= \mathfrak{c}\, e^{-qT}\left(\mathbf{\Phi}\left(-\mathfrak{c}\,\frac{\alpha_{t,T}}{\beta_{t,T}}\right) - e^{\alpha_{t,T}+\frac{(\beta_{t,T})^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_{t,T}}{\beta_{t,T}}+\beta_{t,T}\right)\right)\right) \qquad (4.33)\end{aligned}$$

and the American early exercise bonus premium

$$\begin{aligned}\widetilde{e}(t,x) &= \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \mathfrak{c}\, e^{-qu}x_u\mathbf{1}_{\mathcal{S}}(x_u)\left(\frac{dA_u}{A_u}-(r-\frac{q}{x_u})\,du\right)\right] \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x\}}\left(\frac{x_u\lambda}{p(1-e^{-\lambda u})}((x_u)^{-p}-1)-rx_u+q\right)\right]du \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\left(q\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x\}}\right]-\left(\frac{\lambda}{p(1-e^{-\lambda u})}+r\right)\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x\}}x_u\right]\right. \\ &\quad\left.+\frac{\lambda}{p(1-e^{-\lambda u})}\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x\}}(x_u)^{1-p}\right]\right)du \qquad (4.34) \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\left(q\mathbf{\Phi}(\mathfrak{c}\,\gamma_{0,t,u})-\left(r+\frac{\lambda}{p(1-e^{-\lambda u})}\right)e^{\alpha_{t,u}+\frac{(\beta_{t,u})^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_{1,t,u})\right. \\ &\quad\left.+\frac{\lambda}{p(1-e^{-\lambda u})}e^{(1-p)\alpha_{t,u}+(1-p)^2\frac{(\beta_{t,u})^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_{1-p,t,u})\right)du,\end{aligned}$$

where $\mathbf{\Phi}(\,\cdot\,)$ is the standard normal cumulative distribution function and

$$\gamma_{p,t,u} = \frac{\ln x_u^* - \alpha_{t,u}}{\beta_{t,u}} - p\beta_{t,u}. \qquad (4.35)$$

Returning to the original variables we have the approximate value of American style Asian option with a continuous arithmetic averaging

$$\begin{aligned}V(t,S,A) &= Se^{qt}\widetilde{V}(t,x) \\ &= Se^{qt}(\widetilde{v}(t,x)+\widetilde{e}(t,x)) \\ &= v(t,S,A)+e(t,S,A), \qquad (4.36)\end{aligned}$$

where

$$v(t,S,A) = \mathfrak{c}\, S e^{-q(T-t)} \left(\mathbf{\Phi}\left(-\mathfrak{c}\,\frac{\alpha_{t,T}}{\beta_{t,T}}\right) - e^{\alpha_{t,T}+\frac{(\beta_{t,T})^2}{2}} \mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_{t,T}}{\beta_{t,T}} + \beta_{t,T}\right)\right)\right) \quad (4.37)$$

and

$$\begin{aligned} e(t,S,A) &= \mathfrak{c}\, S \int_t^T e^{-q(u-t)} \left(q\mathbf{\Phi}(\mathfrak{c}\,\gamma_{0,t,u}) - \left(r + \frac{\lambda}{p(1-e^{-\lambda u})}\right) e^{\alpha_{t,u}+\frac{(\beta_{t,u})^2}{2}} \mathbf{\Phi}(\mathfrak{c}\,\gamma_{1,t,u}) \right. \\ &\quad \left. + \frac{\lambda}{p(1-e^{-\lambda u})} e^{(1-p)\alpha_{t,u}+(1-p)^2\frac{(\beta_{t,u})^2}{2}} \mathbf{\Phi}(\mathfrak{c}\,\gamma_{1-p,t,u}) \right) du. \quad (4.38) \end{aligned}$$

4.2.2 Geometric average

In this section, we recall the integral equation for pricing American style of Asian geometrically averaged floating strike options. It was derived for the case $q = 0$ by Hansen and Jørgensen (2000) and for the general case $q \geq 0$ by Wu et al. (1999).

The formula for the geometric continuous average can be derived from the result presented in SECTION 4.2.1 as a limit $p \to 0$ and $\lambda \to 0$. The stochastic variable $x_t^g = \frac{A_t^g}{S_t}$ has log-normal probabilistic distribution. We have identified the distribution, so there is no need of any approximation and the formula can be calculated exactly.

LEMMA 4.4. *(Wu et al. 1999) In the case of geometric averaging, the variable $x_t^g = \frac{A_t^g}{S_t}$ has log-normal (conditioned) distribution $\ln x_u^g | \mathcal{F}_t \sim \mathcal{N}(\alpha_{t,u}, \beta_{t,u}^2)$, where $u \geq t$ and parameters $\alpha_{t,u} = \alpha(t,u,x_t)$ and $\beta_{t,u} = \beta(t,u)$ are defined by*

$$\alpha_{t,u}^g = \frac{t}{u}\ln x_t^g - \frac{u^2 - t^2}{2u}(r - q + \frac{\sigma^2}{2}), \quad (4.39)$$

$$\beta_{t,u}^g = \frac{\sigma}{u\sqrt{3}}\sqrt{u^3 - t^3}. \quad (4.40)$$

REMARK 4.2. *The value of limit of expression (4.29) for $p \to 0$ and $\lambda \to 0$ is equal to (4.39). But if we calculate the value of limit of expression (4.30) we have*

$$\lim_{\lambda\to 0}\lim_{p\to 0} \beta_{t,u} = \frac{\sigma}{u\sqrt{3}}\sqrt{u^3 - t^3 - 3t(u-t)^2} \neq \beta_{t,u}^g.$$

Now, one can apply LEMMA 4.1 in order to calculate the formula for option with the geometric averaging. The stopping region $\mathcal{S} = \{(t,x), x \geq 0, \mathfrak{c}\, x_t^* > \mathfrak{c}\, x\}$, where x_t^* is the exercise boundary and $\mathfrak{c} = 1$ for the case of a call option whereas $\mathfrak{c} = -1$ for a put option. If we insert the expression $\frac{dA_t^g}{A_t^g}$ for the geometric average (4.25) into (4.21) and (4.22) we obtain the formula for the European style option

$$\begin{aligned}\widetilde{v}^g(t,x_t) &= \mathbb{E}_t^{\mathcal{Q}}\left[e^{-qT}\left(\mathfrak{c}(1-x_T^g)\right)^+\right] = e^{-qT}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\mathfrak{c}(1-x_T^g)\right)^+\right] \\ &= \mathfrak{c}\, e^{-qT}\left(\mathbf{\Phi}\left(-\mathfrak{c}\,\frac{\alpha_{t,T}^g}{\beta_{t,T}^g}\right) - e^{\alpha_{t,T}^g+\frac{(\beta_{t,T}^g)^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_{t,T}^g}{\beta_{t,T}^g}+\beta_{t,T}^g\right)\right)\right) \qquad (4.41)\end{aligned}$$

and the value of the American early exercise bonus premium

$$\begin{aligned}\widetilde{e}^g(t,x_t) &= \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \mathfrak{c}\, e^{-qu}x_u^g\mathbf{1}_{\mathcal{S}}(x_u^g)\left(\frac{dA_u^g}{A_u^g}-(r-\frac{q}{x_u^g})\,du\right)\right] \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x^g\}}\left(-\frac{1}{u}x_u^g\ln x_u^g - rx_u^g+q\right)\right]du \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\left(q\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x^g\}}\right] - r\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x^g\}}x_u^g\right]\right. \\ &\qquad \left.-\frac{1}{u}\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x^g\}}x_u^g\ln x_u^g\right]\right)du \qquad (4.42) \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\left(q\mathbf{\Phi}(\mathfrak{c}\,\gamma_{0,t,u}^g)\right. \\ &\qquad \left.+e^{\alpha_{t,u}^g+\frac{(\beta_{t,u}^g)^2}{2}}\left(\mathfrak{c}\frac{\beta_{t,u}^g}{u}\mathit{\Phi}(\gamma_{1,t,u}^g)-\left(r+\frac{\alpha_{t,u}^g+(\beta_{t,u}^g)^2}{u}\right)\mathbf{\Phi}(\mathfrak{c}\,\gamma_{1,t,u}^g)\right)\right)du,\end{aligned}$$

where $\mathbf{\Phi}(\,\cdot\,)$ is the standard normal cumulative distribution function, $\mathit{\Phi}(\,\cdot\,)\equiv\mathbf{\Phi}'(\,\cdot\,)$ is the standard normal probability density function and

$$\gamma_{p,t,u}^g = \frac{\ln x_u^* - \alpha_{t,u}^g}{\beta_{t,u}^g} - p\beta_{t,u}^g. \qquad (4.43)$$

Returning to the original variables we obtain the formula of American style floating strike Asian option with geometrically averaged floating strike:

$$\begin{aligned}V^g(t,S,A) &= Se^{qt}\widetilde{V}^g(t,x_t) \\ &= Se^{qt}(\widetilde{v}^g(t,x_t)+\widetilde{e}^g(t,x_t)) \\ &= v^g(t,S,A)+e^g(t,S,A), \qquad (4.44)\end{aligned}$$

where

$$v^g(t,S,A) = \mathfrak{c}\,Se^{-q(T-t)}\left(\mathbf{\Phi}\left(-\mathfrak{c}\frac{\alpha_{t,T}^g}{\beta_{t,T}^g}\right) - e^{\alpha_{t,T}^g+\frac{(\beta_{t,T}^g)^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_{t,T}^g}{\beta_{t,T}^g}+\beta_{t,T}^g\right)\right)\right) \qquad (4.45)$$

and

$$
\begin{aligned}
e^g(t,S,A) &= \mathfrak{c}\, S \int_t^T e^{-q(u-t)} \left(q\boldsymbol{\Phi}(\mathfrak{c}\,\gamma^g_{0,t,u})\right. \\
&\left. + e^{\alpha^g_{t,u}+\frac{(\beta^g_{t,u})^2}{2}} \left(\mathfrak{c}\frac{\beta^g_{t,u}}{u}\boldsymbol{\Phi}'(\gamma^g_{1,t,u}) - \left(r + \frac{\alpha^g_{t,u}+(\beta^g_{t,u})^2}{u}\right)\boldsymbol{\Phi}(\mathfrak{c}\,\gamma^g_{1,t,u})\right)\right) du.
\end{aligned}
\tag{4.46}
$$

If we formally set value of the continuous dividend rate to zero, i.e. $q = 0$, the result is identical to the expression obtained in the paper Hansen and Jørgensen (2000).

4.2.3 Approximation for the arithmetic average

Unfortunately, in the case of an arithmetically averaged floating strike Asian option the probabilistic distribution function of the arithmetic average cannot be expressed in an explicit way. Following the SECTION 4.2.1 (and Hansen and Jørgensen (2000)) we approximate the probabilistic distribution of the variable $x^a_t = \frac{A^a_t}{S_t}$ for the continuous arithmetic average A^a_t by the log-normal conditioned distribution, i.e. $\ln x^a_u|\mathcal{F}_t \sim \mathcal{N}(\alpha^a_{t,u}, (\beta^a_{t,u})^2)$ at time t, where

$$
\alpha^a_{t,u} = 2\ln \mathbb{E}^{\mathbb{Q}}_t\left[x^a_u\right] - \frac{1}{2}\ln \mathbb{E}^{\mathbb{Q}}_t\left[(x^a_u)^2\right], \tag{4.47}
$$

$$
\beta^a_{t,u} = \sqrt{\ln \mathbb{E}^{\mathbb{Q}}_t\left[(x^a_u)^2\right] - 2\ln \mathbb{E}^{\mathbb{Q}}_t\left[x^a_u\right]}. \tag{4.48}
$$

LEMMA 4.5. *Consider the variable $x_u = \frac{A_u}{S_u}$, where A_u and S_u are defined as the arithmetic average (4.5) and as in (4.16), respectively. First two conditioned moments $\mathbb{E}^{\mathbb{Q}}_t\left[x^a_u\right]$ and $\mathbb{E}^{\mathbb{Q}}_t\left[(x^a_u)^2\right]$ of x_u entering the expressions for the functions $\alpha^a_{t,u} = \alpha^a(t,u,x^a_t)$ and $\beta^a_{t,u} = \beta^a(t,u,x^a_t)$ can be calculated, for $t \leq u$, as follows:*

$$
\mathbb{E}^{\mathbb{Q}}_t\left[x^a_u\right] = x^a_t \frac{t}{u} e^{-(r-q)(u-t)} + \frac{1}{(r-q)u}\left(1 - e^{-(r-q)(u-t)}\right), \tag{4.49}
$$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_t\left[(x^a_u)^2\right] &= (x^a_t)^2 \frac{t^2}{u^2} e^{-2(r-q-\frac{\sigma^2}{2})(u-t)} + x^a_t \frac{2te^{-(r-q)(u-t)}}{u^2(r-q)}\left(1 - e^{-(r-q)(u-t)}\right) \\
&+ \frac{(r-q-\sigma^2) - 2(r-q-\frac{\sigma^2}{2})e^{-(r-q)(u-t)} + (r-q)e^{-2(r-q-\frac{\sigma^2}{2})(u-t)}}{u^2(r-q)(r-q-\frac{\sigma^2}{2})(r-q-\sigma^2)}.
\end{aligned}
\tag{4.50}
$$

REMARK 4.3. *If we formally set the value of the continuous dividend rate $q = 0$ in* LEMMA *4.5 we obtain almost identical expression to that of Hansen and Jørgensen*

(2000) except of the second moment $\mathbb{E}_t^{\mathcal{Q}}\left[(x_u^a)^2\right]$ entering (4.47) and (4.48). The expression

$$\mathbb{E}_t^{\mathcal{Q}}\left[(x_u^a)^2\right]_{HJ} = (x_t^a)^2\frac{t^2}{u^2}e^{-2(r-\frac{\sigma^2}{2})(u-t)} + x_t^a\frac{2te^{-r(u-t)}}{u^2(r-\sigma^2)}\left(1-e^{-(r-\sigma^2)(u-t)}\right) + \frac{(r-\sigma^2)-2(r-\frac{\sigma^2}{2})e^{-r(u-t)}+re^{-2(r-\frac{\sigma^2}{2})(u-t)}}{u^2r(r-\frac{\sigma^2}{2})(r-\sigma^2)}.$$

by Hansen and Jørgensen (2000) differs from our (4.50) in the second summand where the term $r-\sigma^2$ is replaced by r in both the denominator and the exponent. The expression $\mathbb{E}_t^{\mathcal{Q}}\left[(x_u^a)^2\right]_{HJ}$ is not consistent with the derivation presented by Hansen and Jørgensen (2000).

Now, we can return to the problem of valuation of an Asian option. First we replace the general form of the average in (4.22) by the expression for the arithmetic average defined by (4.26). The stopping region $\mathcal{S}$ is the same as for the case of geometric averaging $\mathcal{S} = \{(t,x), x \geq 0, \mathfrak{c}\, x_t^* > \mathfrak{c}\, x\}$.

Using LEMMA 4.1 we calculate the value of both (4.21) and (4.22). The European part of the option has value

$$\begin{aligned} \widetilde{v}^a(t,x) &= \mathbb{E}_t^{\mathcal{Q}}\left[e^{-qT}\left(\mathfrak{c}(1-x_T^a)\right)^+\right] = e^{-qT}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\mathfrak{c}(1-x_T^a)\right)^+\right] \\ &= \mathfrak{c}\, e^{-qT}\left(\mathbf{\Phi}\left(-\mathfrak{c}\frac{\alpha_{t,T}^a}{\beta_{t,T}^a}\right) - e^{\alpha_{t,T}^a+\frac{(\beta_{t,T}^a)^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_{t,T}^a}{\beta_{t,T}^a}+\beta_{t,T}^a\right)\right)\right) \end{aligned} \tag{4.51}$$

and the American early exercise bonus premium

$$\begin{aligned} \widetilde{e}^a(t,x) &= \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \mathfrak{c}\, e^{-qu}x_u^a\mathbf{1}_{\mathcal{S}}(x_u^a)\left(\frac{dA_u^a}{A_u^a}-(r-\frac{q}{x_u^a})\,du\right)\right] \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x^a\}}\left(\frac{1}{u}(1-x_u^a)-rx_u^a+q\right)\right]du \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\left(\left(\frac{1}{u}+q\right)\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x^a\}}\right]-\left(\frac{1}{u}+r\right)\mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\,x^*\geq\mathfrak{c}\,x^a\}}x_u^a\right]\right)du \\ &= \int_t^T \mathfrak{c}\, e^{-qu}\left(\left(q+\frac{1}{u}\right)\mathbf{\Phi}(\mathfrak{c}\,\gamma_{0,t,u}^a)-\left(r+\frac{1}{u}\right)e^{\alpha_{t,u}^a+\frac{(\beta_{t,u}^a)^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_{1,t,u}^a)\right)du, \end{aligned} \tag{4.52}$$

where $\mathbf{\Phi}(\,\cdot\,)$ is the standard normal cumulative distribution function and

$$\gamma_{p,t,u}^a = \frac{\ln x_u^* - \alpha_{t,u}^a}{\beta_{t,u}^a} - p\beta_{t,u}^a. \tag{4.53}$$

Returning to the original variables we have the approximate value of American style Asian option with a continuous arithmetic averaging

$$\begin{aligned} V^a(t,S,A) &= Se^{qt}\widetilde{V}^a(t,x) \\ &= Se^{qt}(\widetilde{v}^a(t,x)+\widetilde{e}^a(t,x)) \\ &= v^a(t,S,A)+e^a(t,S,A), \end{aligned} \tag{4.54}$$

where

$$v^a(t,S,A) = \mathfrak{c}\, Se^{-q(T-t)}\left(\mathbf{\Phi}\left(-\mathfrak{c}\frac{\alpha^a_{t,T}}{\beta^a_{t,T}}\right) - e^{\alpha^a_{t,T}+\frac{(\beta^a_{t,T})^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha^a_{t,T}}{\beta^a_{t,T}}+\beta^a_{t,T}\right)\right)\right) \tag{4.55}$$

and

$$\begin{aligned} e^a(t,S,A) &= \mathfrak{c}\, S\int_t^T e^{-q(u-t)}\left(\left(q+\frac{1}{u}\right)\mathbf{\Phi}\left(\mathfrak{c}\,\gamma^a_{0,t,u}\right)\right. \\ &\qquad \left. -\left(r+\frac{1}{u}\right)e^{\alpha^a_{t,u}+\frac{(\beta^a_{t,u})^2}{2}}\mathbf{\Phi}\left(\mathfrak{c}\,\gamma^a_{1,t,u}\right)\right)du. \end{aligned} \tag{4.56}$$

4.2.4 Lookback options

The general average (4.3) can be transformed into the maximum or the minimum value of variable S_t by appropriate choice of the parameter p ($p\to-\infty$ and $p\to\infty$ for minimum and maximum, respectively). The conditioned distribution of both $\ln x_t^{-\infty}$ and $\ln x_t^{\infty}$ can be derived exactly. Distribution of the similar variables is derived by Kwok (2008). We use the idea of the derivation of distribution.

Neither of the stochastic variables $x_t^{-\infty}=\frac{m_t}{S_t}$ and $x_t^{\infty}=\frac{M_t}{S_t}$ has log-normal probabilistic distribution. We define the distribution by the *CDF* of the variables. As for the geometric average, we have identified the distribution, so there is no need of any approximation and the formula can be calculated exactly.

LEMMA 4.6. *Consider geometric Brownian process S_t defined by (4.16) and define stochastic variables $m_t=\inf_{s\in[0,t]}S_s$ and $M_t=\sup_{s\in[0,t]}S_s$.*

Moreover, we define parameters $\alpha_{t,u}=\alpha(t,u,x_t)$ and $\beta_{t,u}=\beta(t,u)$ by

$$\alpha^{\pm\infty}_{t,u} = \ln x_t^{\pm\infty}-(r-q+\frac{\sigma^2}{2})(u-t), \tag{4.57}$$

$$\beta^{\pm\infty}_{t,u} = \sigma\sqrt{u-t}. \tag{4.58}$$

The conditioned CDF*s of stochastic variables* $y_u = \ln x_u^{-\infty} = \ln \frac{m_u}{S_u} \geq 0$ *and* $Y_u = \ln x_u^{\infty} = \ln \frac{M_u}{S_u} \leq 0$ *where* $u \geq t$ *are given by the expressions*

$$F_{min}(y_u)|\mathcal{F}_t = \mathbf{\Phi}\left(\frac{y_u - \alpha_{t,u}^{-\infty}}{\beta_{t,u}^{-\infty}}\right) + e^{-\varsigma y_u}\mathbf{\Phi}\left(\frac{y_u + \alpha_{t,u}^{-\infty}}{\beta_{t,u}^{-\infty}}\right) \tag{4.59}$$

for $y \leq 0$ *and*

$$F_{max}(Y_u)|\mathcal{F}_t = \mathbf{\Phi}\left(\frac{Y_u - \alpha_{t,u}^{\infty}}{\beta_{t,u}^{\infty}}\right) - e^{-\varsigma Y_u}\mathbf{\Phi}\left(\frac{-Y_u - \alpha_{t,u}^{\infty}}{\beta_{t,u}^{\infty}}\right) \tag{4.60}$$

for $Y \geq 0$*, respectively. The constant* $\varsigma = \frac{r-q+\frac{\sigma^2}{2}}{\frac{\sigma^2}{2}}$ *and* $\mathbf{\Phi}(\,\cdot\,)$ *is the* CDF *of the normal probability distribution* $\mathcal{N}(0,1)$*.*

REMARK 4.4. *The value of limit of expressions (4.29) and (4.30) for* $p \to \pm\infty$ *lead to the log-normal distribution (same for both* ∞ *and* $-\infty$*)*

$$\lim_{p\to\pm\infty} F^p(y)|\mathcal{F}_t = \mathbf{\Phi}\left(\frac{y - \alpha_{t,u}^{\pm\infty}}{\beta_{t,u}^{\pm\infty}}\right) \neq \begin{cases} F_{max}(y)|\mathcal{F}_t \\ F_{min}(y)|\mathcal{F}_t \end{cases}.$$

LEMMA 4.7. *Let* $z = \ln Z$ *be a stochastic variable with the* CDF *defined by (4.59) for minimum and by (4.60) for maximum. We define*

$$\begin{aligned} \gamma_p^+ &\equiv \frac{\ln K + \alpha}{\beta} - p\beta, \\ \gamma_p^- &\equiv \frac{\ln K - \alpha}{\beta} - p\beta, \end{aligned}$$

where $K > 0$ *and* $p \in \mathbb{R}$*. We have*

(i) for minimum value

$$\mathbb{E}_{min}\left[\mathbf{1}_{\{Z\leq K\}}\right] = \mathbf{\Phi}(\gamma_0^-) + e^{-\varsigma \ln K}\mathbf{\Phi}(\gamma_0^+)$$

[this expression is a special case of the expression (iii)],

(ii) for maximum value

$$\mathbb{E}_{max}\left[\mathbf{1}_{\{Z\geq K\}}\right] = \mathbf{\Phi}(-\gamma_0^-) + e^{-\varsigma \ln K}\mathbf{\Phi}(-\gamma_0^+)$$

[this expression is a special case of the expression (iv)],

(iii) for minimum value

$$\begin{aligned}\mathbb{E}_{min}\left[\mathbf{1}_{\{Z\le K\}}Z^p\right] &= e^{p\alpha+\frac{p^2\beta^2}{2}}\mathbf{\Phi}(\gamma_p^-)+\frac{p}{p-\varsigma}e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\mathbf{\Phi}(\gamma_{p-\varsigma}^+)\\ &\quad -\frac{\varsigma}{p-\varsigma}e^{(p-\varsigma)\ln K}\mathbf{\Phi}(\gamma_0^+),\end{aligned}$$

(iv) for maximum value

$$\begin{aligned}\mathbb{E}_{max}\left[\mathbf{1}_{\{Z\ge K\}}Z^p\right] &= e^{p\alpha+\frac{p^2\beta^2}{2}}\mathbf{\Phi}(-\gamma_p^-)+\frac{p}{p-\varsigma}e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\mathbf{\Phi}(-\gamma_{p-\varsigma}^+)\\ &\quad -\frac{\varsigma}{p-\varsigma}e^{(p-\varsigma)\ln K}\mathbf{\Phi}(-\gamma_0^+),\end{aligned}$$

where $\mathbf{\Phi}(\,\cdot\,)$ *is standard normal* CDF.

We can return to the problem of valuation of a lookback option. First we replace the general form of the average in (4.22) by the expression for the extreme value (4.28) (this expression is equal to zero). The stopping region $\mathcal{S}$ is the same as in previous sections and is defined by $\mathcal{S}=\{(t,x),x\ge 0,\mathfrak{c}\,x_t^*>\mathfrak{c}\,x\}$.

We recall that lookback floating strike call option is reasonable only for the minimum value of the underlying (according to the inequality $S_t\ge m_t$ for $\forall t$) and put option is reasonable only for the maximum value of the underlying (according to the inequality $S_t\le M_t$ for $\forall t$). Thus, we associate the value of parameter $\mathfrak{c}=1$ with call option and minimum value and $\mathfrak{c}=-1$ with put option and maximum value.

Using LEMMA 4.7 we calculate the value of both (4.21) and (4.22). The European part of the option has value

$$\begin{aligned}\widetilde{v}^{\pm\infty}(t,x) &= \mathbb{E}_t^{\mathcal{Q}}\left[e^{-qT}\mathfrak{c}\left(1-x_T^{\pm\infty}\right)\right]=e^{-qT}\mathbb{E}_t^{\mathcal{Q}}\left[\mathfrak{c}\left(1-x_T^{\pm\infty}\right)\right]\\ &= \mathfrak{c}\,e^{-qT}\left(1-e^{\alpha_{t,T}^{\pm\infty}+\frac{\left(\beta_{t,T}^{\pm\infty}\right)^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_{t,T}^{\pm\infty}}{\beta_{t,T}^{\pm\infty}}+\beta_{t,T}^{\pm\infty}\right)\right)\right.\\ &\quad -\frac{1}{1-\varsigma}e^{-(1-\varsigma)\alpha_{t,T}^{\pm\infty}+\frac{(1-\varsigma)^2\left(\beta_{t,T}^{\pm\infty}\right)^2}{2}}\mathbf{\Phi}\left(\mathfrak{c}\left(\frac{\alpha_{t,T}^{\pm\infty}}{\beta_{t,T}^{\pm\infty}}-(1-\varsigma)\beta_{t,T}^{\pm\infty}\right)\right)\\ &\quad \left.+\frac{\varsigma}{1-\varsigma}\mathbf{\Phi}\left(\mathfrak{c}\frac{\alpha_{t,T}^{\pm\infty}}{\beta_{t,T}^{\pm\infty}}\right)\right) \qquad (4.61)\end{aligned}$$

and the American early exercise bonus premium

$$\begin{aligned}
\widetilde{e}^{\pm\infty}(t,x) &= \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \mathfrak{c}\, e^{-qu} x_u^{\pm\infty} \mathbf{1}_{\mathcal{S}}(x_u^{\pm\infty})\left(\frac{dA_u^{\pm\infty}}{A_u^{\pm\infty}} - (r - \frac{q}{x_u^{\pm\infty}})\, du\right)\right] \\
&= \int_t^T \mathfrak{c}\, e^{-qu} \mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\, x^* \geq \mathfrak{c}\, x^{\pm\infty}\}}\left(-r x_u^{\pm\infty} + q\right)\right] du \qquad (4.62) \\
&= \int_t^T \mathfrak{c}\, e^{-qu}\left(q \mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\, x^* \geq \mathfrak{c}\, x^{\pm\infty}\}}\right] - r \mathbb{E}_t^{\mathcal{Q}}\left[\mathbf{1}_{\{\mathfrak{c}\, x^* \geq \mathfrak{c}\, x^{\pm\infty}\}} x_u^{\pm\infty}\right]\right) du \\
&= \int_t^T \mathfrak{c}\, e^{-qu}\left(q\left(\mathbf{\Phi}\left(\mathfrak{c}\, \gamma_{0,t,u}^-\right) + e^{-\varsigma \ln x_u^*} \mathbf{\Phi}\left(\mathfrak{c}\, \gamma_{0,t,u}^+\right)\right)\right. \\
&\quad -r\left(e^{\alpha_{t,u}^{\pm\infty} + \frac{\left(\beta_{t,u}^{\pm\infty}\right)^2}{2}} \mathbf{\Phi}\left(\mathfrak{c}\, \gamma_{1,t,u}^-\right)\right. \\
&\quad + \frac{1}{1-\varsigma} e^{-(1-\varsigma)\alpha_{t,u}^{\pm\infty} + \frac{(1-\varsigma)^2\left(\beta_{t,u}^{\pm\infty}\right)^2}{2}} \mathbf{\Phi}\left(\mathfrak{c}\, \gamma_{1-\varsigma,t,u}^+\right) \\
&\quad \left.\left. - \frac{\varsigma}{1-\varsigma} e^{(1-\varsigma)\ln x_u^*} \mathbf{\Phi}\left(\mathfrak{c}\, \gamma_{0,t,u}^+\right)\right)\right) du, \qquad (4.63)
\end{aligned}$$

where $\mathbf{\Phi}(\,\cdot\,)$ is the standard normal cumulative distribution function and

$$\gamma_{p,t,u}^+ = \frac{\ln x_u^* + \alpha_{t,u}^{\pm\infty}}{\beta_{t,u}^{\pm\infty}} - p\beta_{t,u}^{\pm\infty}, \qquad (4.64)$$

$$\gamma_{p,t,u}^- = \frac{\ln x_u^* - \alpha_{t,u}^{\pm\infty}}{\beta_{t,u}^{\pm\infty}} - p\beta_{t,u}^{\pm\infty}. \qquad (4.65)$$

Returning to the original variables and substituting α, β and ς we have the value of American style lookback option

$$\begin{aligned}
V^{\pm\infty}(t,S,A) &= S e^{qt} \widetilde{V}^{\pm\infty}(t,x) \\
&= S e^{qt}(\widetilde{v}^{\pm\infty}(t,x) + \widetilde{e}^{\pm\infty}(t,x)) \\
&= v^{\pm\infty}(t,S,A) + e^{\pm\infty}(t,S,A), \qquad (4.66)
\end{aligned}$$

where

$$\begin{aligned}
v^{\pm\infty}(t,S,A) &= \mathfrak{c}\left(S e^{-q(T-t)} \mathbf{\Phi}\left(\mathfrak{c}\, d_t^{\pm\infty}\right) - A e^{-r(T-t)} \mathbf{\Phi}\left(\mathfrak{c}\left(d_t^{\pm\infty} - \sigma\sqrt{T-t}\right)\right)\right. \\
&\quad + S \frac{\sigma^2}{2(r-q)}\left(\frac{S}{A}\right)^{-\frac{2(r-q)}{\sigma^2}} e^{-r(T-t)} \mathbf{\Phi}\left(\mathfrak{c}\left(-d_t^{\pm\infty} + \frac{2(r-q)}{\sigma}\sqrt{T-t}\right)\right) \\
&\quad \left. - \frac{\sigma^2}{2(r-q)} e^{-q(T-t)} S \mathbf{\Phi}\left(-\mathfrak{c}\, d_t^{\pm\infty}\right)\right) \qquad (4.67)
\end{aligned}$$

and

$$
\begin{aligned}
e^{\pm\infty}(t,S,A) \quad = \quad & \mathfrak{c}\int_t^T \left(q\left(Se^{-q(u-t)}\mathbf{\Phi}\left(\mathfrak{c}\left(d_t^{\pm\infty}+\frac{\ln x_u^*}{\sigma\sqrt{u-t}}\right)\right)\right.\right. \\
& +Se^{-q(u-t)}\left(\frac{1}{x_u^*}\right)^{\frac{2(r-q)}{\sigma^2}+1}\mathbf{\Phi}\left(\mathfrak{c}\left(-d_t^{\pm\infty}+\frac{\ln x_u^*}{\sigma\sqrt{u-t}}\right)\right)\Bigg) \\
& -r\left(Ae^{-r(u-t)}\mathbf{\Phi}\left(\mathfrak{c}\left(d_t^{\pm\infty}+\frac{\ln x_u^*-\sigma^2(u-t)}{\sigma\sqrt{u-t}}\right)\right)\right. \qquad (4.68) \\
& -\frac{\sigma^2e^{-r(u-t)}}{2(r-q)}S\left(\frac{S}{A}\right)^{-\frac{2(r-q)}{\sigma^2}}\mathbf{\Phi}\left(\mathfrak{c}\left(-d_t^{\pm\infty}+\frac{\ln x_u^*+2(r-q)(u-t)}{\sigma\sqrt{u-t}}\right)\right) \\
& \left.\left.+Se^{-q(u-t)}\frac{r-q+\frac{\sigma^2}{2}}{r-q}\left(\frac{1}{x_u^*}\right)^{\frac{2(r-q)}{\sigma^2}}\mathbf{\Phi}\left(\mathfrak{c}\left(-d_t^{\pm\infty}+\frac{\ln x_u^*}{\sigma\sqrt{u-t}}\right)\right)\right)\right)du.
\end{aligned}
$$

The function $d_t^{\pm\infty}$ is defined by

$$
d_t^{\pm\infty}=\frac{\ln\frac{S}{A}+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.
$$

Recall that for $\mathfrak{c}=1$ is $A=m$ and for $\mathfrak{c}=-1$ is $A=M$.

If we formally set value of the continuous dividend rate to zero, i.e. $q=0$, the value of European style lookback option is identical to the expression obtained by Kwok (2008).

CHAPTER 5

Limit value of the early exercise boundary at expiry

In this chapter, we present a new method for the position of early exercise boundary x^* at expiry T for general type of derivative. The result is stated for a wide class of integral equations for pricing American style of derivatives. This problem has been already considered by many authors for American style of certain derivatives (cf. Albanese and Campolieti 2006, Alobaidi and Mallier 2006, Bokes and Ševčovič 2011, Chiarella and Ziogas 2005, Dai and Kwok 2006, Detemple 2006, Kwok 2008, Ševčovič 2008, Wilmott et al. 1995, Wu et al. 1999). Presented method is a unified approach solving the generalized problem of finding the position of early exercise boundary at expiry. Notice that there are no restrictions on the underlying asset (i.e. the underlying asset does not have to be driven by equation (4.1)). This chapter is based on results from the recent preprint by Bokes (2011)[1].

5.1 Limit value theorem

The methodology for calculation of the limit value of early exercise boundary at expiry is summarized into THEOREM 5.1.

Let $\mathbb{D} \subset \mathbb{R}^n$ be a subset of Euclidean space $\mathbb{R}^n$. In what follows, we shall denote by ∂A the boundary of set $A \subset \mathbb{D}$ with respect to the topology of $\mathbb{D}$, i.e. $\partial A = \overline{A} \cap \overline{\mathbb{D}\setminus A}$.

THEOREM 5.1. *Consider an American style of derivative V_{am} on the underlying $x \in \mathbb{D} \subset \mathbb{R}^n$ with the stopping and continuation regions defined by the open sets $\mathcal{S} \subset \mathbb{D}$ and $\mathcal{C} \subset \mathbb{D}$, respectively. Let $\mathcal{X}_t^* = \partial\mathcal{S}(t,\,\cdot\,) \equiv \partial\mathcal{C}(t,\,\cdot\,)$ for $t \in [0,T]$ be a (set of) manifold(s)*

[1] TB: 2010, *A unified approach to determining the early exercise boundary position at expiry for American style of general class of derivatives, arXiv:1012.0348v2*

of the early exercise boundary at time t. Suppose that the value of V_{am} is given by the equation

$$V_{am}(t, x_t) = V_{eu}(t, x_t) + \mathbb{E}_t \left[\int_t^T \mathbf{1}_{\mathcal{S}}(u, x_u) f_b(u, x_u)\, du \right], \tag{5.1}$$

where V_{eu} denotes the price of corresponding European style of derivative and $f_b(t,x)$ is a function representing the early exercise bonus. Furthermore, we suppose that

$$V_{am}(t,x) \geq \Omega(t,x) \text{ and } V_{am}(t,x) \geq V_{eu}(t,x) \quad \text{for any } t \in [0,T], x \in \mathbb{D},$$

where $\Omega(t,x)$ is the pay-off function at time t for both American style and European style of derivative, i.e.

$$V_{am}(T,x) = \Omega(T,x) = V_{eu}(T,x) \quad \text{for any } x \in \mathbb{D}.$$

Then the limit of early exercise boundary at expiry is given by

$$\mathcal{X}_T^* = \partial Z_T^+, \tag{5.2}$$

where $Z_T^+ = \{x_T \in \mathbb{D}; f_b(T, x_T) > 0\}$.

REMARK 5.1. *Price process of the American style derivative discounted by the numeraire is a supermartingale according to the risk neutral measure. It is the Snell envelope of pay-off process discounted by the numeraire and (5.1) discounted by the numeraire is the Doob-Meyer decomposition of this supermartingale. For further details see* SECTION *1.5 or Karatzas and Shreve (1998).*

REMARK 5.2. *Notice that according to the second part of the proof of* THEOREM *5.1 (in* SECTION *C.2), we can determine the function of American style bonus function f_b at expiry by the formula*

$$f_b(T,y) = \lim_{t \to T} \frac{\partial}{\partial t} \left(V_{eu}(t,y) - \Omega(t,y) \right).$$

REMARK 5.3. *According to* THEOREM *4.1, the bonus function in (5.1) at the expiry is given by the expression (if the assumptions of theorem are fulfilled)*

$$f_b(T, x_T) = -\mathcal{N}(T, x_T) f_d(T, x_T), \tag{5.3}$$

where $\mathcal{N}$ is the numeraire and f_d is given by (4.13). The values of f_d on the set of zero measure where the pay-off function Ω and $\mathcal{N}$ are not differentiable can be set to the arithmetic average of limes superior and limes inferior at each point of this set.

REMARK 5.4. *The limit of early exercise boundary analyzed in this thesis is the expansion of order zero. For several financial derivatives of American style, higher order expansion was calculated. Further details on this expansion can be found in Dewynne et al. (1993), Ševčovič (2001), Wilmott et al. (1995) for plain vanilla call option, in Stamicar et al. (1999), Zhu (2006), Zhu and He (2007) for plain vanilla put option and in Bokes and Ševčovič (2011) for Asian options.*

5.2 Calculation of the early exercise boundary position at expiry

In this section, we calculate the limit of early exercise boundary at expiry for several types of American style of financial derivatives and their strategies. The underlying of all derivatives presented in this section is driven by a geometric Brownian motion. THEOREM 5.1 does not have limitation on the distribution of underlying and can be used also in other models for underlying assets (e.g. Lévy processes).

We assume that the underlying asset S is driven by stochastic differential equation (4.1).

Although, trading of American style of option strategies is not common, we use them to demonstrate THEOREM 5.1 on more complex types of derivatives.

5.2.1 Plain vanilla options

The European style of vanilla call/put option gives its holder right to buy/sell the underlying S at maturity time T for the expiration price X. The pay-off functions for call and put options are

$$\Omega^{call}(t, S; X) = (S - X)^+ \text{ and } \Omega^{put}(t, S; X) = (X - S)^+ ,$$

respectively. The value of European style of vanilla option (the well known solution of Black–Scholes partial differential equation extended by Merton) for both call and put option is given by

$$C_{eu}(t, S; X) = e^{-q(T-t)} S \Phi(d_t) - e^{-r(T-t)} X \Phi\left(d_t - \sigma\sqrt{T-t}\right), \tag{5.4}$$

$$P_{eu}(t, S; X) = e^{-r(T-t)} X \Phi\left(-d_t + \sigma\sqrt{T-t}\right) - e^{-q(T-t)} S \Phi(-d_t), \tag{5.5}$$

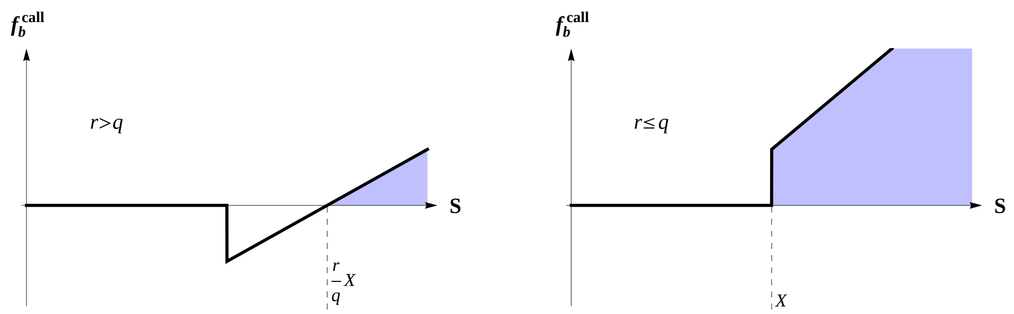

FIGURE 5.1: *The American style bonus function for a call option with $r > q$ (left) and $r \leq q$ (right).*

where $d_t = \frac{\ln\frac{S}{X}+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$.

We know the value of both pay-off function and European style of option, so we can calculate f_b at the expiry according to REMARK 5.2, i.e. for call option and put option we have

$$f_b^{call}(T,S) = \begin{cases} 0 & \text{for } S < X, \\ \frac{X}{2}(q-r) & \text{for } S = X, \\ qS - rX & \text{for } S > X \end{cases} \quad \text{and } f_b^{put}(T,S) = \begin{cases} rX - qS & \text{for } S < X, \\ \frac{X}{2}(r-q) & \text{for } S = X, \\ 0 & \text{for } S > X, \end{cases}$$

respectively.

Finally, we have the boundary of set of positive values of f_b^{call} (see FIGURE 5.1) and f_b^{put} (see FIGURE 5.2)

$$\partial Z_T^{+call} = \max\left[X, \frac{r}{q}X\right] = S_T^{*call} \text{ and } \partial Z_T^{+put} = \min\left[X, \frac{r}{q}X\right] = S_T^{*put},$$

respectively. This result is well known and can be found also in Albanese and Campolieti (2006), Detemple (2006), Kwok (2008), Wilmott et al. (1995) and many other sources.

5.2.2 Bullish and bearish spreads

The most basic strategies consisting of vanilla options of the same type are bullish and bearish spread. Both strategies are difference of two vanilla options.

The European style of bullish spread is difference of two vanilla call options with different strike price, i.e. the pay-off function (for both European style and American

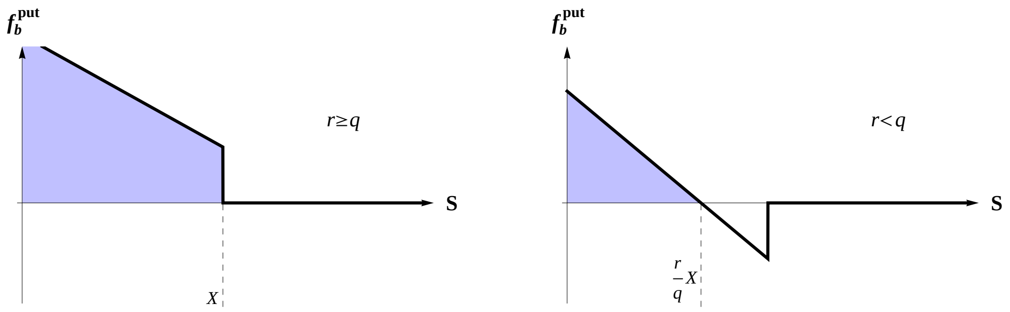

FIGURE 5.2: *The American style bonus function for a put option with $r \geq q$ (left) and $r < q$ (right).*

style) is defined by

$$
\begin{aligned}
\Omega^{bull}(t,S;X_1,X_2) &= \Omega^{call}(t,S;X_1) - \Omega^{call}(t,S;X_2) \\
&= (S-X_1)^+ - (S-X_2)^+ ,
\end{aligned}
$$

for $X_1 < X_2$. The pay-off of European style of bullish spread is a linear combination of vanilla options, so it can be priced by the same linear combination of the value of vanilla options, i.e.

$$
V_{eu}^{bull}(t,S;X_1,X_2) = C_{eu}(t,S;X_1) - C_{eu}(t,S;X_2),
$$

where C_{eu} is defined by (5.4).

According to REMARK 5.2, the American style bonus function at the expiry is

$$
f_b^{bull}(T,S) = \begin{cases} 0 & \text{for } S < X_1, \\ \frac{X_1}{2}(q-r) & \text{for } S = X_1, \\ qS - rX_1 & \text{for } X_1 < S < X_2, \\ \frac{X_2}{2}(q+r) - rX_1 & \text{for } S = X_2, \\ r(X_2 - X_1) & \text{for } X_2 < S. \end{cases}
$$

The boundary of set of positive values of f_b^{bull} (see FIGURE 5.3) is

$$
\partial Z_T^{+bull} = \min\left[\max\left[X_1, \frac{r}{q}X_1\right], X_2\right] = S_T^{*bull}.
$$

The bearish spread is defined as difference of two put options with different strike prices. The pay-off and value of European style of bearish spread is defined for

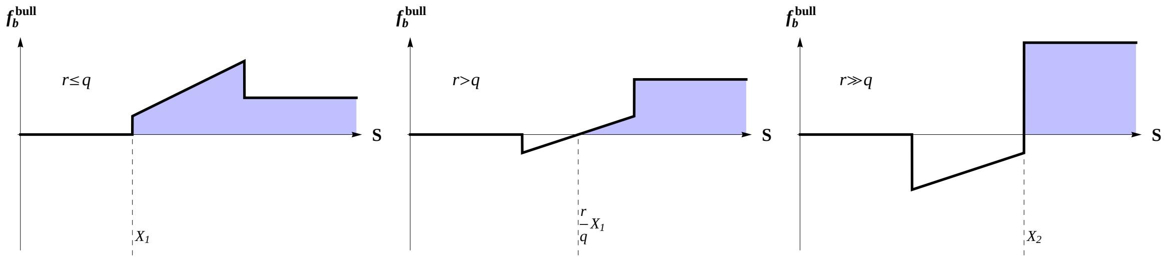

FIGURE 5.3: *The American style bonus function for bullish spread with $r \leq q$ (left), $r > q$ (middle) and $r \gg q$ (right).*

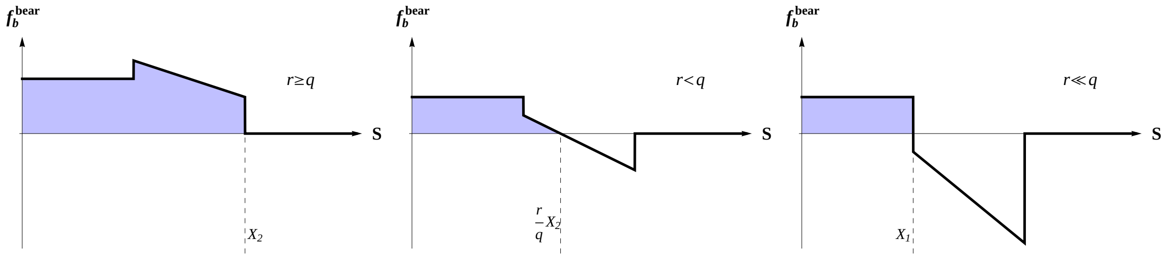

FIGURE 5.4: *The American style bonus function for bearish spread with $r \geq q$ (left), $r < q$ (middle) and $r \ll q$ (right).*

$X_1 < X_2$ by

$$\begin{aligned} \Omega^{bear}(t, S; X_1, X_2) &= \Omega^{put}(t, S; X_2) - \Omega^{put}(t, S; X_1) \\ &= (X_2 - S)^+ - (X_1 - S)^+ \end{aligned}$$

and

$$V_{eu}^{bear}(t, S; X_1, X_2) = P_{eu}(t, S; X_2) - P_{eu}(t, S; X_1),$$

respectively. The function P_{eu} is defined by (5.5).

The bonus function for the American bearish spread at the expiry is

$$f_b^{bear}(T, S) = \begin{cases} r\,(X_2 - X_1) & \text{for } S < X_1, \\ rX_2 - \frac{X_1}{2}\,(r + q) & \text{for } S = X_1, \\ rX_2 - qS & \text{for } X_1 < S < X_2, \\ \frac{X_2}{2}\,(r - q) & \text{for } S = X_2, \\ 0 & \text{for } X_2 < S. \end{cases}$$

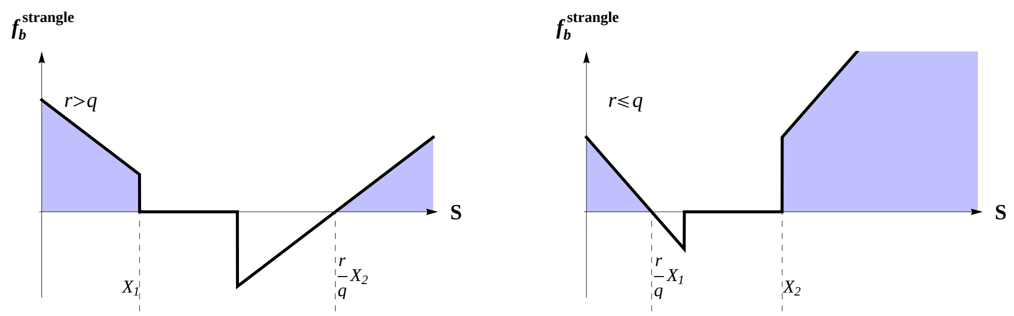

FIGURE 5.5: *The American style bonus function for strangle spread with $r > q$ (left) and $r \leq q$ (right).*

And the boundary of set of positive values of f_b^{bear} (see FIGURE 5.4) is

$$\partial Z_T^{+bear} = \min \left[\max \left[X_1, \frac{r}{q} X_2 \right], X_2 \right] = S_T^{*bear}.$$

5.2.3 Strangles and straddles

The European style of strategies strangle and straddle spread consist also of two vanilla options as bullish and bearish spread. These strategies are created as a sum of one put option and one call option. A straddle spread is restricted case of strangle, so we consider only a strangle spread in the calculation. The pay-off function of the European style of strangle is

$$\begin{aligned} \Omega^{strangle}(t, S; X_1, X_2) &= \Omega^{put}(t, S; X_1) + \Omega^{call}(t, S; X_2) \\ &= (X_1 - S)^+ + (S - X_2)^+ , \end{aligned}$$

usually for $X_1 \leq X_2$, where the case $X_1 = X_2$ is called a straddle spread. This strategy is again a linear combination of vanilla options and so the value of European style of strategy is

$$V_{eu}^{strangle}(t, S; X_1, X_2) = P_{eu}(t, S; X_1) + C_{eu}(t, S; X_2),$$

where the functions C_{eu} and P_{eu} are defined by (5.4) and (5.5), respectively.

The bonus function for the American strangle at the expiry calculated according

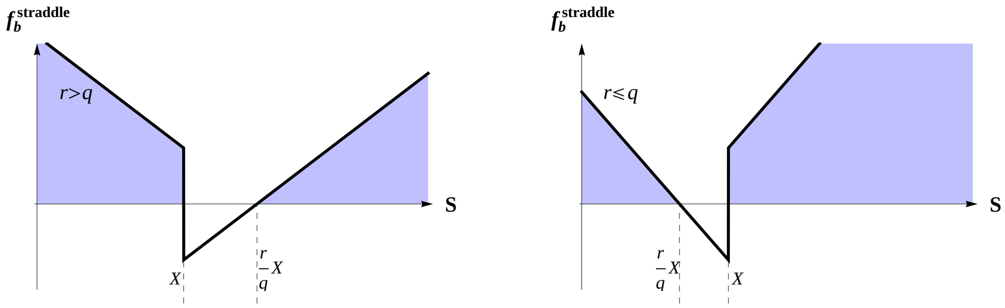

FIGURE 5.6: *The American style bonus function for straddle spread with $r > q$ (left) and $r \leq q$ (right).*

to REMARK 5.2 is

$$f_b^{strangle}(T,S) = \begin{cases} rX_1 - qS & \text{for } S < X_1, \\ \frac{X_1}{2}(r-q) & \text{for } S = X_1, \\ 0 & \text{for } X_1 < S < X_2 \text{ or } X_1 = S = X_2, \\ \frac{X_2}{2}(q-r) & \text{for } S = X_2, \\ qS - rX_2 & \text{for } X_2 < S. \end{cases}$$

For the American style of strangle spread, there are two points in the boundary of set of positive values of $f_b^{strangle}$ (see FIGURE 5.5).

$$\partial Z_T^{+strangle} = \left\{ \min\left[X_1, \frac{r}{q}X_1\right], \max\left[X_2, \frac{r}{q}X_2\right] \right\} = S_T^{*strangle}.$$

For the straddle we have the boundary of set for $X_1 = X_2 = X$ (see FIGURE 5.6)

$$\partial Z_T^{+straddle} = \left\{ X, \frac{r}{q}X \right\} = S_T^{*straddle}.$$

These results are consistent with Alobaidi and Mallier (2006), Chiarella and Ziogas (2005).

5.2.4 Condors and butterflies

The most complex frequently used strategy consisting of vanilla options is European style of condor spread and its restriction butterfly spread. The European style of condor spread is a linear combination of four vanilla call options and its pay-off

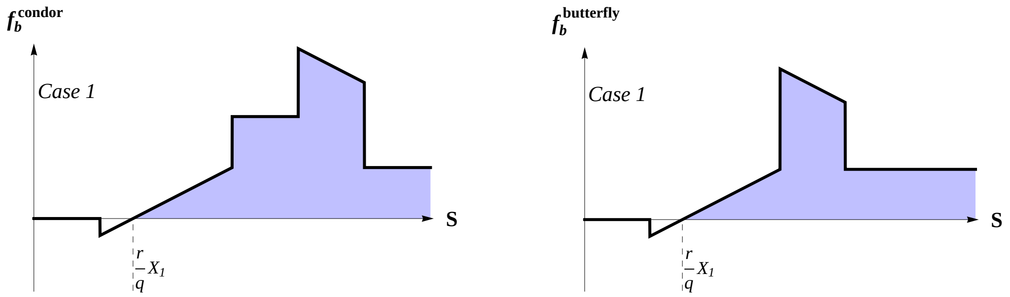

FIGURE 5.7: *The American style bonus function for condor spread (left) and butterfly spread (right) with* $-X_4 + X_3 + X_2 - X_1 > 0$ *and* $r\,(X_3 + X_2 - X_1) \geq qX_4$.

function is

$$\begin{aligned} \Omega^{condor}(t,S;X_1,X_2,X_3,X_4) &= \Omega^{call}(t,S;X_1) - \Omega^{call}(t,S;X_2) \\ &\quad -\Omega^{call}(t,S;X_3) + \Omega^{call}(t,S;X_4) \\ &= (S-X_1)^+ - (S-X_2)^+ - (S-X_3)^+ + (S-X_4)^+ , \end{aligned}$$

for $X_1 < X_2 \leq X_3 < X_4$, where the case $X_2 = X_3$ is called a butterfly spread. The price of a European style of condor is calculated by the formula

$$\begin{aligned} V_{eu}^{condor}(t,S;X_1,X_2,X_3,X_4) &= C_{eu}(t,S;X_1) - C_{eu}(t,S;X_2) \\ &\quad -C_{eu}(t,S;X_3) + C_{eu}(t,S;X_4), \end{aligned}$$

where the function C_{eu} is defined by (5.4).

Once more, we use REMARK 5.2 to calculate the bonus function for American style of condor spread.

$$f_b^{condor}(T,S) = \begin{cases} 0 & \text{for } S < X_1, \\ \frac{X_1}{2}\,(q-r) & \text{for } S = X_1, \\ qS - rX_1 & \text{for } X_1 < S < X_2, \\ \frac{X_2}{2}\,(q+r) - rX_1 & \text{for } S = X_2, \\ r\,(X_2 - X_1) & \text{for } X_2 < S < X_3, \\ \frac{X_3}{2}\,(r-q) + r\,(X_2 - X_1) & \text{for } S = X_3, \\ r\,(X_3 + X_2 - X_1) - qS & \text{for } X_3 < S < X_4, \\ r\,(X_3 + X_2 - X_1) - \frac{X_4}{2}\,(q+r) & \text{for } S = X_4, \\ r\,(-X_4 + X_3 + X_2 - X_1) & \text{for } X_4 < S. \end{cases}$$

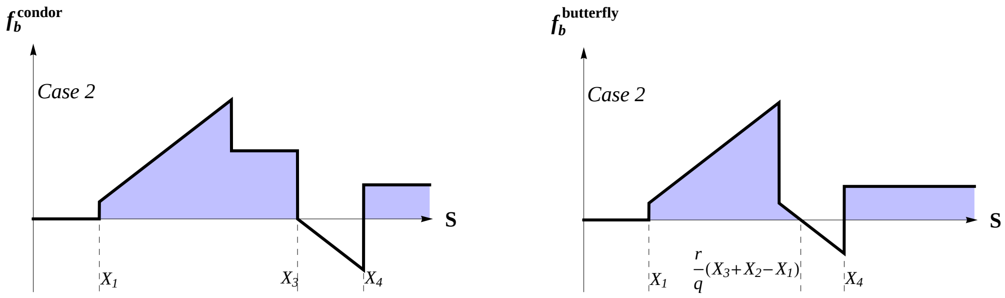

FIGURE 5.8: *The American style bonus function for condor spread (left) and butterfly spread (right) with* $-X_4 + X_3 + X_2 - X_1 > 0$ *and* $r\,(X_3 + X_2 - X_1) < qX_4$.

Notice that for a butterfly spread we have $X_2 = X_3 = X$ and thus the function has form

$$f_b^{butterfly}(T,S) = \begin{cases} 0 & \text{for } S < X_1, \\ \frac{X_1}{2}(q-r) & \text{for } S = X_1, \\ qS - rX_1 & \text{for } X_1 < S < X, \\ r\,(X - X_1) & \text{for } S = X, \\ r\,(2X - X_1) - qS & \text{for } X < S < X_4, \\ r\,(2X - X_1) - \frac{X_4}{2}(q+r) & \text{for } S = X_4, \\ r\,(-X_4 + 2X - X_1) & \text{for } X_4 < S. \end{cases}$$

There are three different cases, when determining the boundary of set of positive values of f_b^{condor}.

In the first case, if we have $-X_4 + X_3 + X_2 - X_1 > 0$ and $r\,(X_3 + X_2 - X_1) \geq qX_4$, then the set of boundary points has only one element (see FIGURE 5.7).

$$\partial Z_T^{+condor} = \min\left[\max\left[X_1, \frac{r}{q}X_1\right], X_2\right] = S_T^{*condor}.$$

In the second case, we have $-X_4 + X_3 + X_2 - X_1 > 0$ and $r\,(X_3 + X_2 - X_1) < qX_4$. The set of boundary points has three elements (see FIGURE 5.8).

$$\begin{aligned} \partial Z_T^{+condor} = & \left\{\min\left[\max\left[X_1, \frac{r}{q}X_1\right], X_2\right],\right. \\ & \left.\max\left[X_3, \frac{r}{q}(X_3 + X_2 - X_1)\right], X_4\right\} = S_T^{*condor}. \end{aligned}$$

In the last case, we have $-X_4 + X_3 + X_2 - X_1 \leq 0$ and the set of boundary points

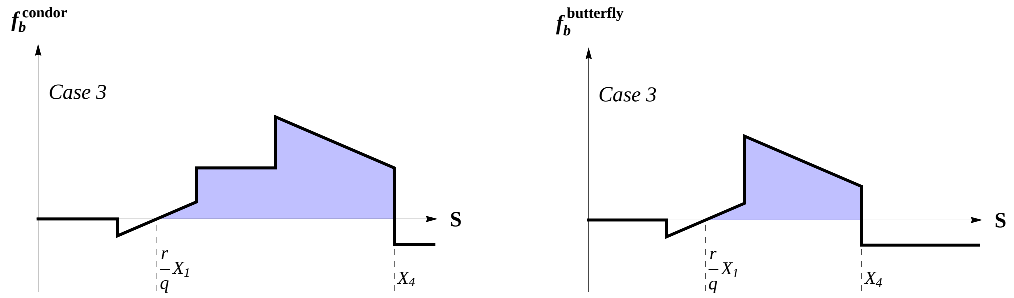

FIGURE 5.9: *The American style bonus function for condor spread (left) and butterfly spread (right) with $-X_4 + X_3 + X_2 - X_1 \leq 0$.*

has two elements (see FIGURE 5.9).

$$\begin{aligned} \partial Z_T^{+condor} \quad &= \quad \left\{ \min \left[\max \left[X_1, \frac{r}{q} X_1 \right], X_2 \right], \right. \\ & \left. \min \left[\max \left[X_3, \frac{r}{q} \left(X_3 + X_2 - X_1 \right) \right], X_4 \right] \right\} = S_T^{*condor}. \end{aligned}$$

5.2.5 Shout options

Shout options are financial derivatives similar to European plain vanilla options. The difference is that the holder of a shout option can once during the life of derivative "shout" to the writer, i.e. the option expires and the strike price is reset to actual spot price of the underlying asset. The shouting action is conditioned by in-the-money position of the option. According to this property, we need to know optimal shouting boundary along with the limit of the boundary at the expiry. The pay-off function of call and put shout option is

$$\Omega^{call,shout}(t, S; X) = \begin{cases} 0 & \text{for } S \leq X, \\ S - X + C_{eu}(t, S; S) & \text{for } S > X \end{cases}$$

and

$$\Omega^{put,shout}(t, S; X) = \begin{cases} X - S + P_{eu}(t, S; S) & \text{for } S < X, \\ 0 & \text{for } S \geq X, \end{cases}$$

respectively. The functions C_{eu} and P_{eu} are defined by (5.4) and (5.5), respectively.

Notice, that the underlying S is under the same measure as for the vanilla option, thus we have the numeraire $\mathcal{N}_t = e^{rt}$. In this case, we use the idea from REMARK 5.3

to determine the bonus function f_b:

$$f_b^{call,shout}(T,S) = \begin{cases} 0 & \text{for } S < X, \\ \infty & \text{for } S \geq X \end{cases} \quad \text{and} \quad f_b^{put,shout}(T,S) = \begin{cases} \infty & \text{for } S \leq X, \\ 0 & \text{for } S > X. \end{cases}$$

The boundary of set of positive values for call is the same as for put shout option:

$$\partial Z_T^{+shout} = X = S_T^{*shout}.$$

This result can be also found in Alobaidi et al. (2011). However, the value of limit was set up only by argumentation and without any mathematical formulation.

5.2.6 British vanilla options

The British vanilla option is financial derivative hedging the real trend of the underlying asset. This feature allows its holder to exercise the option prior to the expiry T and receive the best prediction of the pay-off according to the real trend of underlying restricted to the contract drift μ_c. The pay-off functions of call and put British vanilla option are

$$\Omega^{GB,call} = e^{\mu_c(T-t)} S\mathbf{\Phi}\left(d_t^{\mu_c}\right) - X\mathbf{\Phi}\left(d_t^{\mu_c} - \sigma\sqrt{T-t}\right)$$

and

$$\Omega^{GB,put} = X\mathbf{\Phi}\left(-d_t^{\mu_c} + \sigma\sqrt{T-t}\right) - e^{\mu_c(T-t)} S\mathbf{\Phi}\left(-d_t^{\mu_c}\right),$$

respectively. The function $\Phi(\,\cdot\,)$ is standard normal cumulative distribution function and $d_t^{\mu_c} = \frac{\ln\frac{S}{X} + (\mu_c + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

Notice again, that the underlying S is under the same measure as for the vanilla option, thus we have the numeraire $\mathcal{N}_t = e^{rt}$. We can use the idea from REMARK 5.3 to determine the bonus function f_b:

$$f_b^{GB,call}(T,S) = \begin{cases} 0 & \text{for } S < X, \\ (q+\mu_c)S - rX & \text{for } S \geq X \end{cases}$$

and

$$f_b^{GB,put}(T,S) = \begin{cases} rX - (q+\mu_c)S & \text{for } S \leq X, \\ 0 & \text{for } S > X. \end{cases}$$

Finally, we have the boundary of set of positive values of $f_b^{GB,call}$ and $f_b^{GB,put}$

$$\partial Z_T^{+GB,call} = \max\left[X, \frac{r}{q+\mu_c}X\right] = S_T^{*GB,call}$$

and

$$\partial Z_T^{+GB,call} = \min\left[X, \frac{r}{q+\mu_c}X\right] = S_T^{*GB,call},$$

respectively. This result is consistent with the one presented in Peskir and Samee (2008a,b).

CHAPTER 6

Early exercise boundary of path-dependent options

In this chapter, we analyze the behavior of the early exercise boundary of Asian options near the expiry. First, we discuss the limit at the expiry (the expansion of order zero). In the rest of the chapter, the first order approximation of the early exercise boundary (in terms of $\sqrt{\tau} = \sqrt{T-t}$) is calculated. This chapter is based on results from the second part of paper Bokes and Ševčovič (2011)[1].

6.1 Limit of the early exercise boundary at expiry

In this section, we determine the position of the early exercise boundary x^* at expiry T for floating strike Asian options with continuous geometric and arithmetic averaging and lookback options. The result is calculated according to THEOREM 5.1. To present the idea from REMARK 5.3 we use original variables S_t and A_t, m_t or M_t instead of transformed variable x_t for Asian, lookback call or lookback put options, respectively.

The pay-off functions of floating strike Asian call, Asian put, lookback call and lookback put options are

$$\Omega_{call}^{Asian}(t,S,A) = (S-A)^+ \,, \Omega_{put}^{Asian}(t,S,A) = (A-S)^+ \,,$$

$$\Omega_{call}^{lookback}(t,S,m) = (S-m)^+ \text{ and } \Omega_{put}^{lookback}(t,S,M) = (M-S)^+ \,,$$

respectively. Now, we use (5.3) to determine the bonus function f_b. According to Hansen and Jørgensen (2000) we can use the numeraire $\mathcal{N}_t = e^{rt}$.

[1] TB and Ševčovič, D.: 2011, *Early exercise boundary for American type of floating strike Asian option and its numerical approximation, Applied Mathematical Finance*

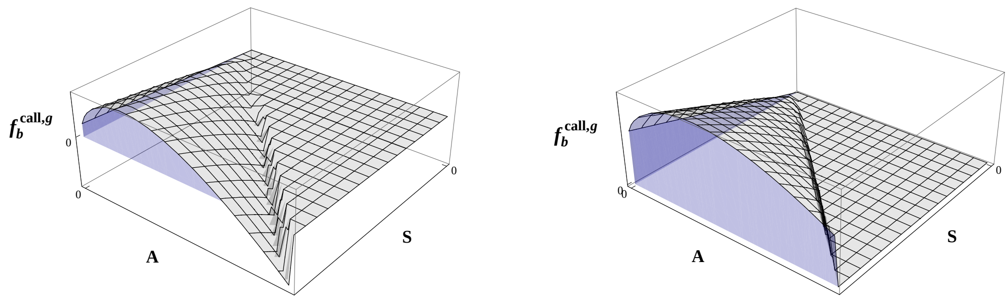

FIGURE 6.1: *The American style bonus function for Asian call option with geometric average and $r > q$ (left) and with $r \leq q$ (right).*

The function f_d for call option and for put option has form

$$f_d^{call}(t, S, A) = \begin{cases} 0 & \text{for } S < A, \\ \frac{1}{2}\left(\limsup_{S \to A} f_d^{call}(t, S, A) + \liminf_{S \to A} f_d^{call}(t, S, A)\right) & \text{for } S = A, \\ e^{-rt}\left(-r\left(S - A\right) + \mu_S - \mu_A\right) & \text{for } A < S \end{cases}$$

and

$$f_d^{put}(t, S, A) = \begin{cases} e^{-rt}\left(-r\left(A - S\right) + \mu_A - \mu_S\right) & \text{for } S < A, \\ \frac{1}{2}\left(\limsup_{S \to A} f_d^{put}(t, S, A) + \liminf_{S \to A} f_d^{put}(t, S, A)\right) & \text{for } S = A, \\ 0 & \text{for } A < S, \end{cases}$$

respectively. The value $\mu_S = (r - q)S$ according to (4.1) and μ_A is drift of the stochastic differential equation

$$dA = \mu_A dt + \sigma_A dW_t^A.$$

According to (4.25) the bonus function at the expiry for call and put Asian options with continuous geometric average has form

$$f_b^{call,g}(T, S, A) = \begin{cases} 0 & \text{for } S < A, \\ \frac{A}{2}\left(q - r\right) & \text{for } S = A, \\ -rA + qS - \frac{1}{T} A \ln \frac{A}{S} & \text{for } A < S \end{cases}$$

and

$$f_b^{put,g}(T, S, A) = \begin{cases} rA - qS + \frac{1}{T} A \ln \frac{A}{S} & \text{for } S < A, \\ \frac{A}{2}\left(r - q\right) & \text{for } S = A, \\ 0 & \text{for } A < S, \end{cases}$$

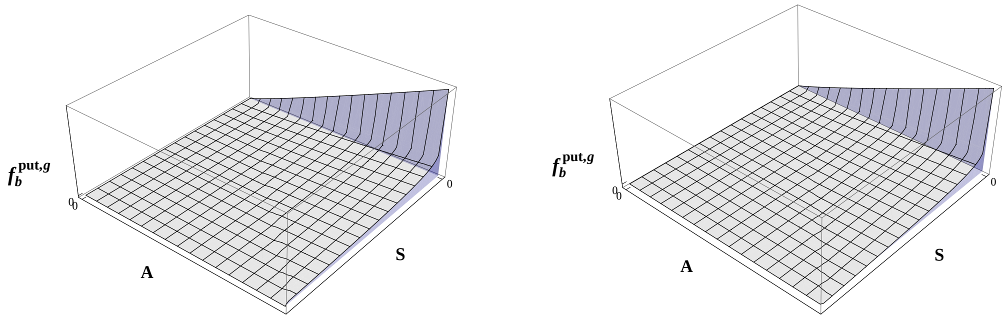

FIGURE 6.2: *The American style bonus function for Asian put option with geometric average and $r \geq q$ (left) and with $r < q$ (right).*

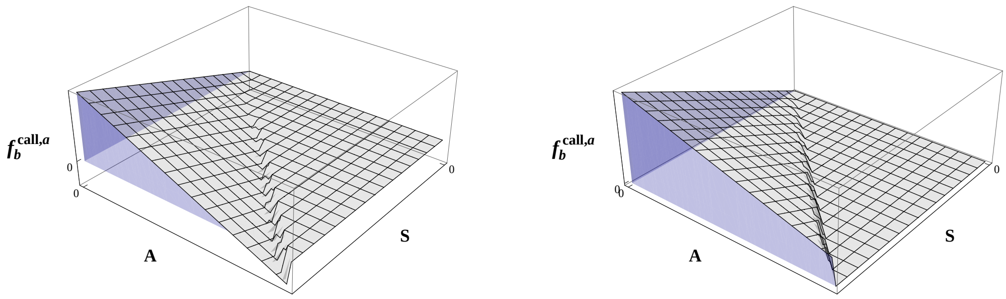

FIGURE 6.3: *The American style bonus function for Asian call option with arithmetic average and $r > q$ (left) and with $r \leq q$ (right).*

respectively. The boundary of set of positive values (see FIGURE 6.1 for call option and FIGURE 6.2 for put option) is given by

$$\partial Z_T^{+call,g} = \left\{ (S, A) \in \mathbb{R}_+^2; \frac{S}{A} = \max \left[1, \widetilde{G} \right] \right\} = \mathcal{X}_T^{*call,g}$$

and

$$\partial Z_T^{+put,g} = \left\{ (S, A) \in \mathbb{R}_+^2; \frac{S}{A} = \min \left[1, \widetilde{G} \right] \right\} = \mathcal{X}_T^{*put,g},$$

where $\widetilde{G}$ is the positive solution of transcendental equation

$$r - qG - \frac{1}{T} \ln G = 0.$$

The solution $\widetilde{G}$ is unique on $\mathbb{R}_+$ for $q \geq 0$ and $T > 0$.

According to (4.26), the bonus function at the expiry for call and put Asian op-

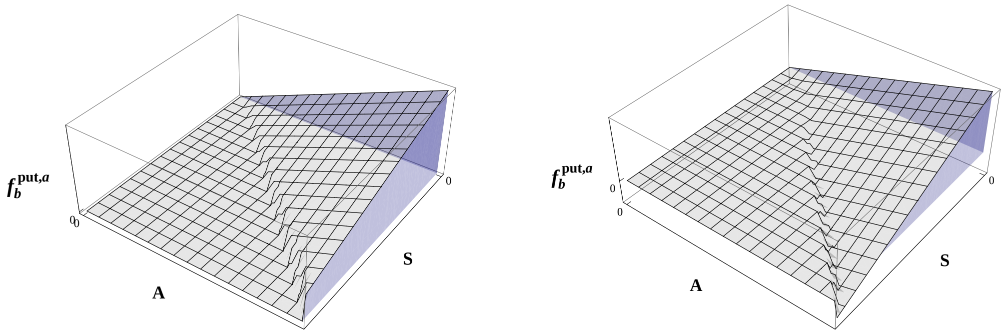

FIGURE 6.4: *The American style bonus function for Asian put option with arithmetic average and $r \geq q$ (left) and with $r < q$ (right).*

tions with continuous arithmetic average has form

$$f_b^{call,a}(T,S,A) = \begin{cases} 0 & \text{for } S < A, \\ \frac{A}{2}(q-r) & \text{for } S = A, \\ \left(q + \frac{1}{T}\right) S - \left(r + \frac{1}{T}\right) A & \text{for } A < S \end{cases}$$

and

$$f_b^{put,a}(T,S,A) = \begin{cases} -\left(q + \frac{1}{T}\right) S + \left(r + \frac{1}{T}\right) A & \text{for } S < A, \\ \frac{A}{2}(r-q) & \text{for } S = A, \\ 0 & \text{for } A < S, \end{cases}$$

respectively. The boundary of set of positive values (see FIGURE 6.3 for call option and FIGURE 6.4 for put option) is given by

$$\partial Z_T^{+call,a} = \left\{ (S,A) \in \mathbb{R}_+^2; \frac{S}{A} = \max\left[1, \frac{r + \frac{1}{T}}{q + \frac{1}{T}}\right] \right\} = \mathcal{X}_T^{*call,a}$$

and

$$\partial Z_T^{+put,a} = \left\{ (S,A) \in \mathbb{R}_+^2; \frac{S}{A} = \min\left[1, \frac{r + \frac{1}{T}}{q + \frac{1}{T}}\right] \right\} = \mathcal{X}_T^{*put,a}.$$

Finally, according to (4.28) the bonus function at the expiry for call and put look-back options has form

$$f_b^{min}(T,S,m) = \begin{cases} 0 & \text{for } S < m, \\ \frac{m}{2}(q-r) & \text{for } S = m, \\ -rm + qS & \text{for } m < S \end{cases}$$

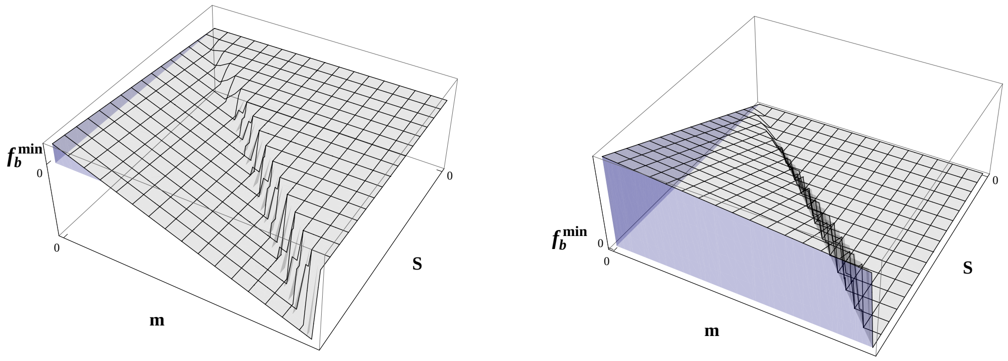

FIGURE 6.5: *The American style bonus function for lookback call (minimum) option with $r > q$ (left) and with $r \leq q$ (right).*

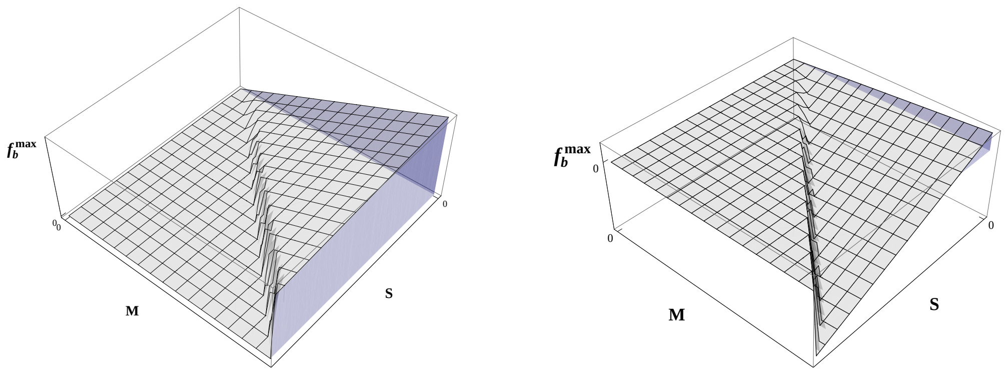

FIGURE 6.6: *The American style bonus function for lookback put (maximum) option with $r \geq q$ (left) and with $r < q$ (right).*

and

$$f_b^{max}(T, S, M) = \begin{cases} rM - qS & \text{for } S < M, \\ \frac{M}{2}(r - q) & \text{for } S = M, \\ 0 & \text{for } M < S, \end{cases}$$

respectively. The boundary of set of positive values (see FIGURE 6.5 for call option and FIGURE 6.6 for put option) is given by

$$\partial Z_T^{+min} = \left\{ (S, m) \in \mathbb{R}_+^2; \frac{S}{m} = \max\left[1, \frac{r}{q}\right] \right\} = \mathcal{X}_T^{*min}$$

and

$$\partial Z_T^{+max} = \left\{ (S, M) \in \mathbb{R}_+^2; \frac{S}{M} = \min\left[1, \frac{r}{q}\right] \right\} = \mathcal{X}_T^{*max}.$$

We use THEOREM 5.1 to obtain the limit of the early exercise boundary at expiry also for American style of Asian options with other types of the strike price averaging method analyzed in this paper. Results are presented in TABLE 6.1 and COROLLARY 6.1.

TABLE 6.1: *The limit of the early exercise boundary position x^*_T at expiry $t = T$ ($\widehat{x}_T$ solves (6.2), $\widehat{x}^w_T$ solves (6.1) and $\widetilde{x}_T$ solves (6.3)).*

x^*_T	put	call
general constant kernel average	$\max\left(\widehat{x}_T, 1\right)$	$\min\left(\widehat{x}_T, 1\right)$
general exponential kernel average	$\max\left(\widehat{x}^w_T, 1\right)$	$\min\left(\widehat{x}^w_T, 1\right)$
geometric average	$\max\left(\widetilde{x}_T, 1\right)$	$\min\left(\widetilde{x}_T, 1\right)$
arithmetic average	$\max\left(\frac{q+\frac{1}{T}}{r+\frac{1}{T}}, 1\right)$	$\min\left(\frac{q+\frac{1}{T}}{r+\frac{1}{T}}, 1\right)$
weighted arithmetic average	$\max\left(\frac{q(1-e^{-\lambda T})+\lambda}{r(1-e^{-\lambda T})+\lambda}, 1\right)$	$\min\left(\frac{q(1-e^{-\lambda T})+\lambda}{r(1-e^{-\lambda T})+\lambda}, 1\right)$
maximum lookback option	$\max\left(\frac{q}{r}, 1\right)$	–
minimum lookback option	–	$\min\left(\frac{q}{r}, 1\right)$

COROLLARY 6.1. *The value of limit of early exercise boundary at expiry x^*_T for the floating strike Asian option is summarized in* TABLE *6.1. In the case of the general average with the exponential kernel, it follows from (4.22) that*

$$\widehat{f}_b(T, x_T) = e^{-qT}\left(\frac{\lambda}{1-e^{-\lambda T}}\frac{x_T}{p}\left(\frac{1}{(x_T)^p} - 1\right) - rx_T + q\right).$$

Then $x^*_T = \widehat{x}^w_T$ is a solution of the transcendent equation

$$\frac{\lambda}{1-e^{-\lambda T}}\left(\frac{1}{(\widehat{x}^w_T)^p} - 1\right) = rp - \frac{qp}{\widehat{x}^w_T} \tag{6.1}$$

if $\widehat{x}^w_T \in ITM$ (in-the-money), otherwise $x^*_T = 1$.

For the constant kernel, we can simply calculate the limit $\lambda \to 0$

$$\frac{1}{(\widehat{x}_T)^p} - 1 = rpT - \frac{qpT}{\widehat{x}_T}. \tag{6.2}$$

Both equations (6.1) and (6.2) have unique solution on $\mathbb{R}^+$.

And in the case of a geometric average, it follows from (4.22) (or from the limit of the general average for $p \to 0$) that $\widetilde{f}_b(T, x_T) = e^{-qT}\left(-\frac{1}{T}x_T \ln x_T - rx_T + q\right)$. Then

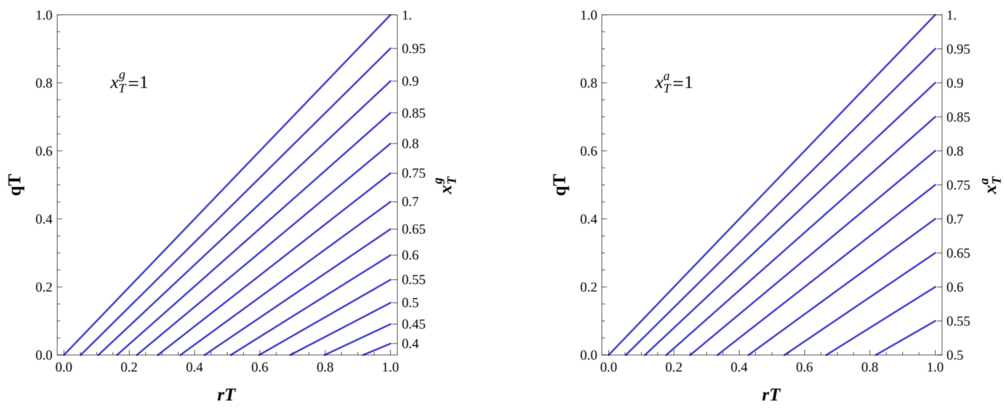

FIGURE 6.7: *Isolines of the limit of early exercise boundary at expiry of call option for the continuous geometric (left) and the continuous arithmetic average (right).*

$x_T^* = \widetilde{x}_T$ *is a solution of the transcendent equation*

$$\ln \widetilde{x}_T = \frac{qT}{\widetilde{x}_T} - rT, \tag{6.3}$$

if $\widehat{x}_T \in ITM$ *(in-the-money), otherwise* $x_T^* = 1$*. As we have already mentioned above, also the equation (6.3) has unique solution on* $\mathbb{R}^+$*.*

The formula for limit of early exercise boundary at expiry (6.3) for geometric averaging is the same as presented by Wu et al. (1999) and Detemple (2006, p. 69). Notice that the same values of limit of early exercise boundary at expiry for the continuous arithmetic average type of an Asian option are derived also in Dai and Kwok (2006) and Ševčovič (2008). The result for geometric, arithmetic and exponentially weighted Asian options can be found in Bokes and Ševčovič (2011). For the case of lookback options, the same result can be found again in Dai and Kwok (2006).

6.2 Expansion of the early exercise boundary close to expiry

Throughout this section, we shall assume the structural assumption on the interest and dividend rates:

$$\mathfrak{c}\, r > \mathfrak{c}\, q, \tag{6.4}$$

where $\mathfrak{c} = 1$ for call option and $\mathfrak{c} = -1$ for put option.

We shall calculate an approximation of the option early exercise boundary function for a call and put option. The approximation is obtained by the first order Taylor series expansion in the $\sqrt{\tau}$ variable, where $\tau = T - t$ is the time to expiry, i.e. we need to calculate the first derivative of x_t^* at expiry T with respect to $\sqrt{\tau}$ variable. Following Kuske and Keller (1998), Dewynne et al. (1993), Ševčovič (2001) we propose an approximation of the early exercise boundary x_t^* in the form

$$\frac{1}{\varrho_{T-t}} = x_t^* = x_T^*(1 + h\sigma\sqrt{T-t}) + O(T-t) \quad \text{as } t \to T,$$

where $h \in \mathbb{R}$ is a constant. To calculate h, we use the condition of smoothness for the value of the option at the early exercise boundary - the smooth pasting principle (cf. Kwok 2008, Dai and Kwok 2006). Since $\widetilde{V}(T,x) = e^{-qT}(\mathfrak{c}(1-x))^+$, we have

$$\begin{aligned} -\mathfrak{c} &= e^{qt}\frac{\partial \widetilde{V}}{\partial x}(t, x_t^*) = e^{qt}\frac{\partial \widetilde{v}}{\partial x}(t, x_t^*) + e^{qt}\frac{\partial \widetilde{e}}{\partial x}(t, x_t^*) \\ &= e^{qt}\frac{\partial \widetilde{v}}{\partial x}(t, x_t^*) + e^{qt}\int_t^T \frac{\partial \widetilde{e}^I}{\partial x}(t, x_t^*, u, x_u^*)\, du \\ &= \widehat{v}_x(t, x_t^*) + \int_t^T \widehat{e}^I_x(t, x_t^*, u, x_u^*)\, du, \end{aligned} \tag{6.5}$$

where e^I denotes integrated function. The first step (common for all averages) of derivation are substitutions $t = T - \tau$ and $u = T - \tau(1-\theta)$ into the previous equation:

$$-\mathfrak{c} = \widehat{v}_x(T-\tau, x^*_{T-\tau}) + \tau\int_0^1 \widehat{e}^I_x\left(T-\tau, x^*_{T-\tau}, T-\tau(1-\theta), x^*_{T-\tau(1-\theta)}\right) d\theta \tag{6.6}$$

This equation should be valid through the time. Thus, we set its derivative with respect to τ equal to zero

$$\begin{aligned} 0 &= \frac{\partial}{\partial \tau}\left(\mathfrak{c} + \widehat{v}_x(T-\tau, x^*_{T-\tau})\right) + \int_0^1 \widehat{e}^I_x\left(T-\tau, x^*_{T-\tau}, T-\tau(1-\theta), x^*_{T-\tau(1-\theta)}\right) d\theta \\ &\quad + \tau\int_0^1 \frac{\partial}{\partial \tau}\widehat{e}^I_x\left(T-\tau, x^*_{T-\tau}, T-\tau(1-\theta), x^*_{T-\tau(1-\theta)}\right) d\theta. \end{aligned}$$

The last element on the right-hand side of previous equation tends to zero with $\tau \to 0$ (for all cases of average presented below). The derivation is straightforward and simple, but very long, space exhausting and similar to the following one, thus we left this proof to the reader.

Next, we calculate the limit for $\tau \to 0$:

$$0 = \lim_{\tau\to 0} \frac{\partial \widehat{v}_x(T-\tau, x^*_{T-\tau})}{\partial \tau} + \lim_{\tau\to 0} \int_0^1 \widehat{e}^I_x \left(T-\tau, x^*_{T-\tau}, T-\tau(1-\theta), x^*_{T-\tau(1-\theta)}\right) d\theta. \quad (6.7)$$

In further derivation, we use following limits (common for all averages) calculated according to TABLE 6.1, LEMMA 4.3 (for general average), LEMMA 4.4 (for geometric average), LEMMA 4.5 (for arithmetic average) and LEMMA 4.6 (for lookback options).

$$\lim_{\tau\to 0} \frac{\ln x^*_T(1+h\sigma\sqrt{\tau(1-\theta)}) - \alpha_{T-\tau,T-\tau(1-\theta)}}{\beta_{T-\tau,T-\tau(1-\theta)}} = -h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}, \quad (6.8)$$

for $\theta \in (0,1)$ and

$$\begin{aligned}
\lim_{\tau\to 0} \mathfrak{c}\, \alpha_{T-\tau,T-\tau(1-\theta)} &= \mathfrak{c}\, \alpha_{T,T} = \mathfrak{c} \ln x^*_T < 0, && (6.9)\\
\lim_{\tau\to 0} \beta_{T-\tau,T-\tau(1-\theta)} &= \beta_{T,T} = 0^+, && (6.10)\\
\lim_{\tau\to 0} \mathbf{\Phi}\left(-\mathfrak{c}\frac{\alpha_{T-\tau,T}}{\beta_{T-\tau,T}}\right) &= \mathbf{\Phi}\left(-\mathfrak{c}\frac{\ln x^*_T}{0^+}\right) = 1, && (6.11)\\
\forall n \in \mathbb{N}\cup\{0\} : \quad \lim_{\tau\to 0} \frac{\mathbf{\Phi}'\left(\frac{\alpha_{T-\tau,T}}{\beta_{T-\tau,T}}\right)}{(\beta_{T-\tau,T})^n} &= 0, && (6.12)\\
\lim_{\tau\to 0} \partial_x \alpha_{T-\tau,T-\tau(1-\theta)} &= \frac{1}{x^*_T}, && (6.13)\\
\lim_{\tau\to 0} \beta_{T-\tau,T-\tau(1-\theta)}\, \partial_\tau \left(\beta_{T-\tau,T-\tau(1-\theta)}\big|_{x=\frac{1}{\varrho\tau}}\right) &= \frac{\theta\sigma^2}{2}, && (6.14)
\end{aligned}$$

for $\theta \in (0,1]$.

Since we have assumed (6.4), we have $0 < x^*_T \neq 1$ (see TABLE 6.1). Notice that both α and β have polynomial order in τ and the derivative of normal cumulative distribution function (i.e. the probability density function) has exponential order in τ variable. In all derivations we have used several properties of the derivative of normal cumulative distribution function $\mathbf{\Phi}(x)$, e.g. $\mathbf{\Phi}'(x) = \mathbf{\Phi}'(-x)$, $\mathbf{\Phi}''(x) = -x\mathbf{\Phi}'(x)$ and $\mathbf{\Phi}'(\frac{a}{b}+c) = e^{-\frac{ac}{b}-\frac{c^2}{2}}\mathbf{\Phi}'(\frac{a}{b})$.

The following lemma will be useful in derivation of asymptotic behavior of the early exercise close to expiry. Its proof is straightforward and follows from increasing behavior of the right-hand side of equation (6.15) as a function of the h variable (see FIGURE 6.8).

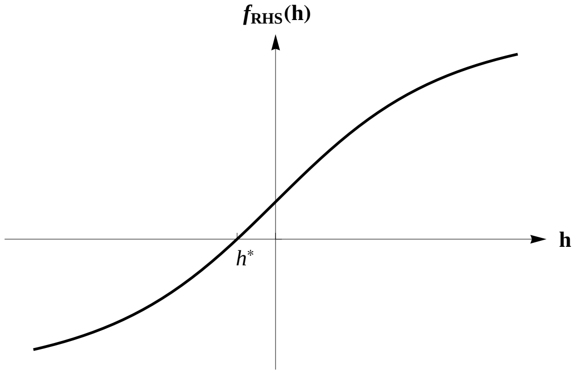

FIGURE 6.8: *The right-hand side of equation (6.15) as a function of the h variable with the unique root h^*.*

LEMMA 6.1. *The implicit equation*

$$0 = 1 - \int_0^1 \mathbf{\Phi}\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right) d\theta + h \int_0^1 \frac{\sqrt{1-\theta}}{\sqrt{\theta}} \mathbf{\Phi}'\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right) d\theta \quad (6.15)$$

has the unique solution h^* having its approximate value $h^* \doteq -0.638833$.

Notice that the first order asymptotic expansion as $t \to T$ of the early exercise boundary $S_t^* \approx S_T^*(1 + 0.638833\,\sigma\sqrt{T-t})$ for the American call option derived by Dewynne et al. (1993), Ševčovič (2001) and in SECTION A.3 contains the same constant $-h^* \doteq 0.638833$ where h^* is a solution of (6.15).

REMARK 6.1. *For the early exercise boundary function $x_t^* = x_t^*(T, r, q, \sigma^2)$ as a function of the model parameters $T, r, \sigma^2 > 0, q \geq 0$, we have the following scaling property:*

$$x_t^*(T, r, q, \sigma^2) = x_{\frac{t}{T}}^*(1, rT, qT, \sigma^2 T).$$

REMARK 6.2. *According to SECTIONS 6.2.1-6.2.4, the early exercise boundary has very similar behavior close to the expiry for all analyzed options. The only difference is in the limit value that multiplies the expression. Moreover, the behavior of free boundary is similar to the plain vanilla option.*

Notice that for $T \to \infty$ and t close to expiry, the problem of Asian and lookback options reduces to the plain vanilla option problem.

6.2.1 General average

We recall that $\alpha_{t,u} = \alpha(t,u,x)$, $\beta_{t,u} = \beta(t,u,x)$ and that we use the approximation $\frac{1}{\varrho_{T-u}} = x^*_u = x^*_T(1 + h\sigma\sqrt{T-u})$. In this section, we use the following notation (to simplify the derivation)

$$
\begin{aligned}
r - q + \frac{\sigma^2}{2} &= \Lambda, \\
x^*_T &= P, \\
\alpha_{T-\tau,T} &= \alpha_T, \\
\beta_{T-\tau,T} &= \beta_T, \\
\alpha_{T-\tau,T-\tau(1-\theta)} &= \alpha_\theta, \\
\beta_{T-\tau,T-\tau(1-\theta)} &= \beta_\theta, \\
\gamma_{p,T-\tau,T-\tau(1-\theta)} &= \gamma_p.
\end{aligned}
$$

We use the value of the Asian option with general average (4.33) and (4.34) and we calculate the derivative of European part of the expression.

$$
\begin{aligned}
\widehat{v}_x(t,x) =& e^{qt}\frac{\partial}{\partial x}\widetilde{v}(t,x) \\
=& \mathfrak{c}\, e^{-q(T-t)}\frac{\partial}{\partial x}\left(\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}\right)\right) - e^{\alpha_T + \frac{\beta_T^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T} + \beta_T\right)\right)\right) \\
=& e^{-q(T-t)}\left(\partial_x\beta_T\mathbf{\Phi}'\left(\frac{\alpha_T}{\beta_T}\right) - \mathfrak{c}\left(\partial_x\alpha_T + \beta_T\partial_x\beta_T\right)e^{\alpha_T + \frac{\beta_T^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T} + \beta_T\right)\right)\right).
\end{aligned}
$$

We calculate additional limits for the general average with exponential kernel

$$\lim_{\tau\to 0}\partial_\tau\left(\alpha_{T-\tau,T-\tau(1-\theta)}\Big|_{x=\frac{1}{\varrho_\tau}}\right) - \frac{h\sigma}{2\sqrt{\tau}} = \theta\left(-\Lambda - \frac{\lambda}{1-e^{-\lambda T}}\frac{1-P^{-p}}{p}\right), \tag{6.16}$$

$$\lim_{\tau\to 0}\partial_\tau\left(\partial_x\alpha_{T-\tau,T-\tau(1-\theta)}\Big|_{x=\frac{1}{\varrho_\tau}}\right) + \frac{h\sigma}{2P\sqrt{\tau}} = -\frac{\theta\lambda}{1-e^{-\lambda T}}P^{-(p+1)}, \tag{6.17}$$

$$\lim_{\tau\to 0}\partial_x\beta_{T-\tau,T-\tau(1-\theta)} = 0, \tag{6.18}$$

$$\lim_{\tau\to 0}\partial_\tau\left(\partial_x\beta_{T-\tau,T-\tau(1-\theta)}\Big|_{x=\frac{1}{\varrho_\tau}}\right) = 0, \tag{6.19}$$

$$\lim_{\tau\to 0}\partial_\tau\left(\beta_{T-\tau,T-\tau(1-\theta)}\Big|_{x=\frac{1}{\varrho_\tau}}\right)\partial_x\beta_{T-\tau,T-\tau(1-\theta)} = 0, \tag{6.20}$$

$$\lim_{\tau\to 0}\frac{\partial_x\beta_{T-\tau,T-\tau(1-\theta)}}{\beta^2_{T-\tau,T-\tau(1-\theta)}} = 0, \tag{6.21}$$

for $\theta \in (0,1]$. Now, we calculate the first part of the limit (6.7). According to limits (6.8) - (6.14), the elements with derivative of *CDF* tends to zero in the limit and limit of the *CDF* tends to 1, thus if we use the equation (6.1), we have

$$\begin{aligned}\lim_{\tau\to 0}\partial_\tau \widehat{v}_x(T-\tau,x) &= \mathfrak{c}\, qP\frac{1}{P} - \mathfrak{c}\, P\left[-\frac{\lambda}{1-e^{-\lambda T}}P^{-(p+1)} - \lim_{\tau\to 0}\frac{h\sigma}{2P\sqrt{\tau}}\right.\\ &\quad \left. +\frac{1}{P}\left(-\Lambda - \frac{\lambda}{1-e^{-\lambda T}}\frac{1-P^{-p}}{p} + \lim_{\tau\to 0}\frac{h\sigma}{2\sqrt{\tau}} + \frac{\sigma^2}{2}\right)\right]\\ &= \mathfrak{c}\left(\frac{q}{P} + \frac{\lambda}{1-e^{-\lambda T}}P^{-p}\right).\end{aligned}$$

Next, we calculate the derivative of integral function of American style option bonus:

$$\begin{aligned}\widehat{e}^I_x(t,x,u,x^*_u) &= \mathfrak{c}\, e^{-q\tau\theta}\frac{\partial}{\partial x}\left(q\mathbf{\Phi}(\mathfrak{c}\,\gamma_0) - \left(r + \frac{\lambda}{p(1-e^{-\lambda(T-\tau(1-\theta))})}\right)e^{\alpha_\theta + \frac{\beta_\theta^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_1)\right.\\ &\quad \left. + \frac{\lambda}{p(1-e^{-\lambda(T-\tau(1-\theta))})}e^{(1-p)\alpha_\theta + (1-p)^2\frac{\beta_\theta^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_{1-p})\right)\\ &= e^{-q\tau\theta}\left(q\partial_x\gamma_0\mathbf{\Phi}'(\gamma_0)\right.\\ &\quad -\mathfrak{c}\left(r + \frac{\lambda}{p(1-e^{-\lambda(T-\tau(1-\theta))})}\right)(\partial_x\alpha_\theta + \beta_\theta\partial_x\beta_\theta)e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_1)\\ &\quad - \left(r + \frac{\lambda}{p(1-e^{-\lambda(T-\tau(1-\theta))})}\right)\partial_x\gamma_1 e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}'(\gamma_1)\\ &\quad + \mathfrak{c}\lambda\frac{(1-p)\partial_x\alpha_\theta + (1-p)^2\beta_\theta\partial_x\beta_\theta}{p(1-e^{-\lambda(T-\tau(1-\theta))})}e^{(1-p)\alpha_\theta+(1-p)^2\frac{\beta_\theta^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_{1-p})\\ &\quad \left. +\lambda\frac{\partial_x\gamma_{1-p}}{p(1-e^{-\lambda(T-\tau(1-\theta))})}e^{(1-p)\alpha_\theta+(1-p)^2\frac{\beta_\theta^2}{2}}\mathbf{\Phi}'(\gamma_{1-p})\right).\end{aligned}$$

Since $\partial_x\gamma_p = -\frac{\partial_x\alpha_\theta}{\beta_\theta} - \frac{(\ln x^*_u - \alpha_\theta)\partial_x\beta_\theta}{\beta_\theta^2} - p\partial_x\beta_\theta$, we use the equation (6.1) and limits (6.8) - (6.14) and (6.16) - (6.21) to calculate the limit

$$\begin{aligned}\lim_{\tau\to 0}\widehat{e}^I_x(t,x,u,x^*_u) &= \mathfrak{c}\left(\frac{q}{P} + \frac{\lambda}{1-e^{-\lambda T}}P^{-p}\right)\\ &\quad \times\left(-\mathbf{\Phi}\left(-\mathfrak{c}\, h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right) + \mathfrak{c}\, h\frac{\sqrt{1-\theta}}{\sqrt{\theta}}\mathbf{\Phi}'\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right)\right).\end{aligned} \tag{6.22}$$

Integrating (6.22) with respect to $\theta \in [0,1]$, putting both partial limits into (6.7), dividing by the nonzero constant $\mathfrak{c}\left(\frac{q}{P} + \frac{\lambda}{1-e^{-\lambda T}}P^{-p}\right)$ and by LEMMA 6.1, we finally obtain

$$x^*_t = P(1 + h^*\sigma\sqrt{T-t}) + O(T-t) \quad \text{as } t\to T,$$

where $h^* \doteq -0.638833\mathfrak{c}$.

6.2.2 Geometric average

We recall that $\alpha^g_{t,u} = \alpha(t,u,x)$, $\beta^g_{t,u} = \beta(t,u)$ and that we use the approximation $\frac{1}{\varrho^g_{T-u}} = x^{g*}_u = x^{g*}_T(1 + h\sigma\sqrt{T-u})$. In this section, we use the following notation (to simplify the derivation)

$$
\begin{aligned}
r - q + \frac{\sigma^2}{2} &= \Lambda, \\
x^{g*}_T &= G, \\
\alpha^g_{T-\tau,T} &= \alpha_T, \\
\beta^g_{T-\tau,T} &= \beta_T, \\
\alpha^g_{T-\tau,T-\tau(1-\theta)} &= \alpha_\theta, \\
\beta^g_{T-\tau,T-\tau(1-\theta)} &= \beta_\theta, \\
\gamma^g_{p,T-\tau,T-\tau(1-\theta)} &= \gamma_p.
\end{aligned}
$$

We use the value of the Asian option with geometric average (4.41) and (4.42) and we calculate the derivative of European part of the expression.

$$
\begin{aligned}
\widehat{v}^g_x(t,x) &= e^{qt}\frac{\partial}{\partial x}\widetilde{v}^g(t,x) \\
&= \mathfrak{c}\, e^{-q(T-t)}\frac{\partial}{\partial x}\left(\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}\right)\right) - e^{\alpha_T + \frac{\beta_T^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T} + \beta_T\right)\right)\right) \\
&= -\mathfrak{c}\, e^{-q(T-t)}\partial_x\alpha_T e^{\alpha_T + \frac{\beta_T^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T} + \beta_T\right)\right).
\end{aligned}
$$

We calculate additional limits for the general average with exponential kernel

$$
\lim_{\tau\to 0}\partial_\tau\left(\alpha^g_{T-\tau,T-\tau(1-\theta)}\big|_{x=\frac{1}{\varrho_\tau}}\right) - \frac{h\sigma}{2\sqrt{\tau}} = \theta\left(-\Lambda - \frac{\ln G}{T}\right), \tag{6.23}
$$

$$
\lim_{\tau\to 0}\partial_\tau\left(\partial_x\alpha^g_{T-\tau,T-\tau(1-\theta)}\big|_{x=\frac{1}{\varrho_\tau}}\right) + \frac{h\sigma}{2G\sqrt{\tau}} = -\frac{\theta}{GT}, \tag{6.24}
$$

for $\theta \in (0,1]$. Now, we calculate the first part of the limit (6.7). According to limits (6.8) - (6.14), the elements with derivative of *CDF* tends to zero in the limit and limit of the *CDF* tends to 1, thus if we use the equation (6.3), we have

$$
\begin{aligned}
\lim_{\tau\to 0}\partial_\tau\widehat{v}^g_x(T-\tau,x) &= \mathfrak{c}\, qG\frac{1}{G} - \mathfrak{c}\, G\left[-\frac{1}{GT} - \lim_{\tau\to 0}\frac{h\sigma}{2G\sqrt{\tau}}\right. \\
&\quad \left. + \frac{1}{G}\left(-\Lambda - \frac{\ln G}{T} + \lim_{\tau\to 0}\frac{h\sigma}{2\sqrt{\tau}} + \frac{\sigma^2}{2}\right)\right] \\
&= \mathfrak{c}\left(\frac{q}{G} + \frac{1}{T}\right).
\end{aligned}
$$

Next, we calculate the derivative of integral function of American style option bonus:

$$
\begin{aligned}
\widehat{e}^{Ig}_{x}(t,x,u,x^*_u) =& \mathfrak{c}\, e^{-q\tau\theta}\frac{\partial}{\partial x}\left(q\mathbf{\Phi}(\mathfrak{c}\,\gamma_0)+e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\left(\frac{\mathfrak{c}\,\beta_\theta}{T-\tau(1-\theta)}\mathbf{\Phi}'(\gamma_1)\right.\right.\\
&\left.\left.-\left(r+\frac{\alpha_\theta+\beta_\theta^2}{T-\tau(1-\theta)}\right)\mathbf{\Phi}(\mathfrak{c}\,\gamma_1)\right)\right)\\
=& e^{-q\tau\theta}\left(q\partial_x\gamma_0\mathbf{\Phi}'(\gamma_0)\right.\\
&+\partial_x\alpha_\theta e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\left(\frac{\beta_\theta}{T-\tau(1-\theta)}\mathbf{\Phi}'(\gamma_1)-\mathfrak{c}\left(r+\frac{\alpha_\theta+\beta_\theta^2}{T-\tau(1-\theta)}\right)\mathbf{\Phi}(\mathfrak{c}\,\gamma_1)\right)\\
&+e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\left(\frac{\beta_\theta\partial_x\gamma_1}{T-\tau(1-\theta)}\mathbf{\Phi}''(\gamma_1)-\left(\frac{\mathfrak{c}\,\partial_x\alpha_\theta}{T-\tau(1-\theta)}\right)\mathbf{\Phi}(\mathfrak{c}\,\gamma_1)\right.\\
&\left.\left.-\partial_x\gamma_1\left(r+\frac{\alpha_\theta+\beta_\theta^2}{T-\tau(1-\theta)}\right)\mathbf{\Phi}'(\gamma_1)\right)\right).
\end{aligned}
$$

Since $\partial_x\gamma_p = -\frac{\partial_x\alpha_\theta}{\beta_\theta}$, we use the equation (6.3) and limits (6.8) - (6.14) and (6.23) - (6.24) to calculate the limit

$$
\begin{aligned}
\lim_{\tau\to 0}\widehat{e}^{Ig}_{x}(t,x,u,x^*_u) =& \mathfrak{c}\left(\frac{q}{G}+\frac{1}{T}\right)\\
&\times\left(-\mathbf{\Phi}\left(-\mathfrak{c}\,h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right)+\mathfrak{c}\,h\frac{\sqrt{1-\theta}}{\sqrt{\theta}}\mathbf{\Phi}'\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right)\right).
\end{aligned}
\tag{6.25}
$$

Integrating (6.22) with respect to $\theta\in[0,1]$, putting both partial limits into (6.7), dividing by the nonzero constant $\mathfrak{c}\left(\frac{q}{G}+\frac{1}{T}\right)$ and by LEMMA 6.1, we finally obtain

$$x^{g*}_t = G(1+h^*\sigma\sqrt{T-t})+O(T-t)\quad\text{as } t\to T,$$

where $h^*\doteq -0.638833\mathfrak{c}$.

6.2.3 Arithmetic average

We recall that $\alpha^a_{t,u}=\alpha(t,u,x)$, $\beta^a_{t,u}=\beta(t,u,x)$ and that we use the approximation $\frac{1}{\varrho^a_{T-u}} = x^{a*}_u = x^{a*}_T(1+h\sigma\sqrt{T-u})$. In this section, we use the following notation (to simplify the derivation)

$$
\begin{aligned}
r-q+\frac{\sigma^2}{2} &= \Lambda,\\
x^{a*}_T &= A,\\
\alpha^a_{T-\tau,T} &= \alpha_T,
\end{aligned}
$$

$$
\begin{aligned}
\beta^a_{T-\tau,T} &= \beta_T, \\
\alpha^a_{T-\tau,T-\tau(1-\theta)} &= \alpha_\theta, \\
\beta^a_{T-\tau,T-\tau(1-\theta)} &= \beta_\theta, \\
\gamma^a_{p,T-\tau,T-\tau(1-\theta)} &= \gamma_p.
\end{aligned}
$$

We use the value of the Asian option with arithmetic average (4.51) and (4.52) and we calculate the derivative of European part of the expression.

$$
\begin{aligned}
\widehat{v}^a_x(t,x) =& e^{qt}\frac{\partial}{\partial x}\widetilde{v}^a(t,x) \\
=& \mathfrak{c}\, e^{-q(T-t)}\frac{\partial}{\partial x}\left(\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}\right)\right) - e^{\alpha_T+\frac{\beta_T^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}+\beta_T\right)\right)\right) \\
=& e^{-q(T-t)}\left(\partial_x\beta_T\mathbf{\Phi}'\left(\frac{\alpha_T}{\beta_T}\right) - \mathfrak{c}\left(\partial_x\alpha_T + \beta_T\partial_x\beta_T\right)e^{\alpha_T+\frac{\beta_T^2}{2}}\mathbf{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}+\beta_T\right)\right)\right).
\end{aligned}
$$

We calculate additional limits for the general average with exponential kernel

$$
\begin{aligned}
\lim_{\tau\to 0}\partial_\tau\left(\alpha_{T-\tau,T-\tau(1-\theta)}\big|_{x=\frac{1}{\varrho\tau}}\right) - \frac{h\sigma}{2\sqrt{\tau}} &= \theta\left(-\Lambda - \frac{1}{T} + \frac{1}{AT}\right), && (6.26) \\
\lim_{\tau\to 0}\partial_\tau\left(\partial_x\alpha_{T-\tau,T-\tau(1-\theta)}\big|_{x=\frac{1}{\varrho\tau}}\right) + \frac{h\sigma}{2A\sqrt{\tau}} &= -\frac{\theta}{A^2T}, && (6.27) \\
\lim_{\tau\to 0}\partial_x\beta_{T-\tau,T-\tau(1-\theta)} &= 0, && (6.28) \\
\lim_{\tau\to 0}\partial_\tau\left(\partial_x\beta_{T-\tau,T-\tau(1-\theta)}\big|_{x=\frac{1}{\varrho\tau}}\right) &= 0, && (6.29) \\
\lim_{\tau\to 0}\partial_\tau\left(\beta_{T-\tau,T-\tau(1-\theta)}\big|_{x=\frac{1}{\varrho\tau}}\right)\partial_x\beta_{T-\tau,T-\tau(1-\theta)} &= 0, && (6.30) \\
\lim_{\tau\to 0}\frac{\partial_x\beta_{T-\tau,T-\tau(1-\theta)}}{\beta^2_{T-\tau,T-\tau(1-\theta)}} &= 0, && (6.31)
\end{aligned}
$$

for $\theta \in (0,1]$. Now, we calculate the first part of the limit (6.7). According to limits (6.8) - (6.14), the elements with derivative of *CDF* tends to zero in the limit and limit of the *CDF* tends to 1, thus if we use the value of $A = \frac{1+qT}{1+rT}$, we have

$$
\begin{aligned}
\lim_{\tau\to 0}\partial_\tau\widehat{v}^a_x(T-\tau,x) &= \mathfrak{c}\,qA\frac{1}{A} - \mathfrak{c}\,A\left[-\frac{1}{A^2T} - \lim_{\tau\to 0}\frac{h\sigma}{2A\sqrt{\tau}}\right. \\
&\quad \left. +\frac{1}{A}\left(-\Lambda - \frac{1}{T} + \frac{1}{AT} + \lim_{\tau\to 0}\frac{h\sigma}{2\sqrt{\tau}} + \frac{\sigma^2}{2}\right)\right] \\
&= \mathfrak{c}\left(r + \frac{1}{T}\right).
\end{aligned}
$$

Next, we calculate the derivative of integral function of American style option bonus:

$$\begin{aligned}\widehat{e}^{Ia}_x(t,x,u,x^*_u) &= \mathfrak{c}\, e^{-q\tau\theta}\frac{\partial}{\partial x}\left(\left(q+\frac{1}{T-\tau(1-\theta)}\right)\mathbf{\Phi}(\mathfrak{c}\,\gamma_0)\right.\\ &\left.-\left(r+\frac{1}{T-\tau(1-\theta)}\right)e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_1)\right)\\ &= e^{-q\tau\theta}\left(\partial_x\gamma_0\left(r+\frac{1}{T-\tau(1-\theta)}\right)\mathbf{\Phi}'(\gamma_0)\right.\\ &-\mathfrak{c}(\partial_x\alpha_\theta+\beta_\theta\partial_x\beta_\theta)\left(r+\frac{1}{T-\tau(1-\theta)}\right)e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}(\mathfrak{c}\,\gamma_1)\\ &\left.-\partial_x\gamma_1\left(r+\frac{1}{T-\tau(1-\theta)}\right)e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}'(\gamma_1)\right).\end{aligned}$$

Since $\partial_x\gamma_p = -\frac{\partial_x\alpha_\theta}{\beta_\theta} - \frac{(\ln x^*_u-\alpha_\theta)\partial_x\beta_\theta}{\beta_\theta^2} - p\partial_x\beta_\theta$, we use the value of $A=\frac{1+qT}{1+rT}$ and limits (6.8) - (6.14) and (6.26) - (6.31) to calculate the limit

$$\begin{aligned}\lim_{\tau\to 0}\widehat{e}^{Ia}_x(t,x,u,x^*_u) &= \mathfrak{c}\left(r+\frac{1}{T}\right)\\ &\times\left(-\mathbf{\Phi}\left(-\mathfrak{c}\,h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right)+\mathfrak{c}\,h\frac{\sqrt{1-\theta}}{\sqrt{\theta}}\mathbf{\Phi}'\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right)\right).\end{aligned} \tag{6.32}$$

Integrating (6.32) with respect to $\theta\in[0,1]$, putting both partial limits into (6.7), dividing by the nonzero constant $\mathfrak{c}\left(r+\frac{1}{T}\right)$ and by LEMMA 6.1, we finally obtain

$$x^{a*}_t = A(1+h^*\sigma\sqrt{T-t})+O(T-t)\quad \text{as } t\to T, \tag{6.33}$$

where $h^*\doteq -0.638833\mathfrak{c}$.

6.2.4 Lookback option

We recall that $\alpha^{\pm\infty}_{t,u}=\alpha(t,u,x)$, $\beta^{\pm\infty}_{t,u}=\beta(t,u)$ and that we use the approximation $\frac{1}{\varrho^{\pm\infty}_{T-u}} = x^{\pm\infty*}_u = x^{\pm\infty*}_T(1+h\sigma\sqrt{T-u})$. In this section, we use the following notation (to simplify the derivation)

$$\begin{aligned} r-q+\frac{\sigma^2}{2} &= \Lambda,\\ x^{\pm\infty*}_T &= L,\\ \alpha^{\pm\infty}_{T-\tau,T} &= \alpha_T,\\ \beta^{\pm\infty}_{T-\tau,T} &= \beta_T,\end{aligned}$$

$$
\begin{aligned}
\alpha^{\pm\infty}_{T-\tau,T-\tau(1-\theta)} &= \alpha_\theta, \\
\beta^{\pm\infty}_{T-\tau,T-\tau(1-\theta)} &= \beta_\theta, \\
\gamma^{+}_{p,T-\tau,T-\tau(1-\theta)} &= \gamma^{+}_{p}, \\
\gamma^{-}_{p,T-\tau,T-\tau(1-\theta)} &= \gamma^{-}_{p}.
\end{aligned}
$$

We use the value of the Asian option with arithmetic average (4.61) and (4.62) and we calculate the derivative of European part of the expression.

$$
\begin{aligned}
\widehat{v}^{\pm\infty}_x(t,x) &= e^{qt}\frac{\partial}{\partial x}\widetilde{v}^{\pm\infty}(t,x) \\
&= \mathfrak{c}\, e^{-q(T-t)}\frac{\partial}{\partial x}\left(1 - e^{\alpha_T+\frac{\beta_T^2}{2}}\boldsymbol{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}+\beta_T\right)\right)\right. \\
&\quad \left. -\frac{1}{1-\varsigma}e^{-(1-\varsigma)\alpha_T+\frac{(1-\varsigma)^2\beta_T^2}{2}}\boldsymbol{\Phi}\left(\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}-(1-\varsigma)\beta_T\right)\right)+\frac{\varsigma}{1-\varsigma}\boldsymbol{\Phi}\left(\mathfrak{c}\frac{\alpha_T}{\beta_T}\right)\right) \\
&= \mathfrak{c}\, e^{-q(T-t)}\partial_x\alpha_T\left(-e^{\alpha_T+\frac{\beta_T^2}{2}}\boldsymbol{\Phi}\left(-\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}+\beta_T\right)\right)\right. \\
&\quad \left. +e^{-(1-\varsigma)\alpha_T+\frac{(1-\varsigma)^2\beta_T^2}{2}}\boldsymbol{\Phi}\left(\mathfrak{c}\left(\frac{\alpha_T}{\beta_T}-(1-\varsigma)\beta_T\right)\right)\right),
\end{aligned}
$$

where $\varsigma = \frac{2\Lambda}{\sigma^2}$. We calculate additional limits for the general average with exponential kernel

$$
\lim_{\tau\to 0}\partial_\tau\left(\alpha_{T-\tau,T-\tau(1-\theta)}\Big|_{x=\frac{1}{\varrho\tau}}\right) - \frac{h\sigma}{2\sqrt{\tau}} = -\theta\Lambda, \tag{6.34}
$$

$$
\lim_{\tau\to 0}\partial_\tau\left(\partial_x\alpha_{T-\tau,T-\tau(1-\theta)}\Big|_{x=\frac{1}{\varrho\tau}}\right) + \frac{h\sigma}{2A\sqrt{\tau}} = 0, \tag{6.35}
$$

$$
\lim_{\tau\to 0}\boldsymbol{\Phi}\left(\mathfrak{c}\frac{\alpha_{T-\tau,T}}{\beta_{T-\tau,T}}\right) = \boldsymbol{\Phi}\left(\mathfrak{c}\frac{\ln x_T^*}{0^+}\right) = 0, \tag{6.36}
$$

for $\theta \in (0,1]$. Now, we calculate the first part of the limit (6.7). According to limits (6.8) - (6.14), the elements with derivative of *CDF* tends to zero in the limit and limit of the *CDF* tends to 1 or to zero, thus we have

$$
\begin{aligned}
\lim_{\tau\to 0}\partial_\tau\widehat{v}^{\pm\infty}_x(T-\tau,x) &= \mathfrak{c}\, qL\frac{1}{L} - \mathfrak{c}\, L\left[-\lim_{\tau\to 0}\frac{h\sigma}{2L\sqrt{\tau}}+\frac{1}{L}\left(-\Lambda+\lim_{\tau\to 0}\frac{h\sigma}{2\sqrt{\tau}}+\frac{\sigma^2}{2}\right)\right] \\
&= \mathfrak{c}\, r.
\end{aligned}
$$

Next, we calculate the derivative of integral function of American style option bonus:

$$\begin{aligned}
\widehat{e}_x^{I\pm\infty}(t,x,u,x_u^*) &= \mathfrak{c}\, e^{-q\tau\theta}\frac{\partial}{\partial x}\left(q\left(\mathbf{\Phi}\left(\mathfrak{c}\,\gamma_0^-\right)+e^{-\varsigma\ln x_u^*}\mathbf{\Phi}\left(\mathfrak{c}\,\gamma_0^+\right)\right)\right.\\
&\quad -r\left(e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}\left(\mathfrak{c}\,\gamma_1^-\right)+\frac{1}{1-\varsigma}e^{-(1-\varsigma)\alpha_\theta+\frac{(1-\varsigma)^2\beta_\theta^2}{2}}\mathbf{\Phi}\left(\mathfrak{c}\,\gamma_{1-\varsigma}^+\right)\right.\\
&\quad \left.\left.-\frac{\varsigma}{1-\varsigma}e^{(1-\varsigma)\ln x_u^*}\mathbf{\Phi}\left(\mathfrak{c}\,\gamma_0^+\right)\right)\right)\\
&= e^{-q\tau\theta}\left(q\left(\partial_x\gamma_0^-\mathbf{\Phi}'\left(\gamma_0^-\right)+\partial_x\gamma_0^+e^{-\varsigma\ln x_u^*}\mathbf{\Phi}'\left(\gamma_0^+\right)\right)\right.\\
&\quad -r\left(\mathfrak{c}\,\partial_x\alpha_\theta e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}\left(\mathfrak{c}\,\gamma_1^-\right)+\partial_x\gamma_1^-e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}'\left(\gamma_1^-\right)\right.\\
&\quad -\mathfrak{c}\,\partial_x\alpha_\theta e^{-(1-\varsigma)\alpha_\theta+\frac{(1-\varsigma)^2\beta_\theta^2}{2}}\mathbf{\Phi}\left(\mathfrak{c}\,\gamma_{1-\varsigma}^+\right)\\
&\quad +\frac{\partial_x\gamma_{1-\varsigma}^+}{1-\varsigma}e^{-(1-\varsigma)\alpha_\theta+\frac{(1-\varsigma)^2\beta_\theta^2}{2}}\mathbf{\Phi}'\left(\gamma_{1-\varsigma}^+\right)\\
&\quad \left.\left.-\frac{\varsigma}{1-\varsigma}\partial_x\gamma_0^+e^{(1-\varsigma)\ln x_u^*}\mathbf{\Phi}'\left(\gamma_0^+\right)\right)\right).
\end{aligned}$$

where $\varsigma=\frac{2\Lambda}{\sigma^2}$. Since $\partial_x\gamma_p^\pm=\pm\frac{\partial_x\alpha_\theta}{\beta_\theta}$, we use the value of $L=\frac{q}{r}$ and limits (6.8) - (6.14) and (6.34) - (6.36) to calculate the limit

$$\lim_{\tau\to 0}\widehat{e}_x^{I\pm\infty}(t,x,u,x_u^*)=\mathfrak{c}\,r\left(-\mathbf{\Phi}\left(-\mathfrak{c}\,h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right)+\mathfrak{c}\,h\frac{\sqrt{1-\theta}}{\sqrt{\theta}}\mathbf{\Phi}'\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right)\right).$$

Integrating (6.37) with respect to $\theta\in[0,1]$, putting both partial limits into (6.7), dividing by the nonzero constant $\mathfrak{c}\,r$ and by LEMMA 6.1, we finally obtain

$$x_t^{\pm\infty *}=L(1+h^*\sigma\sqrt{T-t})+O(T-t)\quad\text{as } t\to T,$$

where $h^*\doteq -0.638833\mathfrak{c}$.

CHAPTER 7

Partial differential equation for path-dependent options

In this chapter, we derive the modified Black–Scholes partial differential equation for financial derivatives dependent on the average of the value of underlying, i.e. $V_t = V(t, S, A)$. The variable representing the general average is defined by

$$A_t = A(t, \mathcal{I}_t), \tag{7.1}$$

where $\mathcal{I}_t = \int_0^t \mathcal{U}(t, u, S_u)\, du$. Solving this equation with appropriate zero and margin conditions, we can price all derivatives with the pay-off function satisfying condition

$$\Omega\Big|_{(S,A)=(S_T,A_T)} = f(S_T, A_T), \tag{7.2}$$

i.e. derivatives with pay-off depending only on value and average of the underlying at the time of expiry.

7.1 Derivation of the Black–Scholes equation for Asian options

Suppose we have a portfolio consisting of underlying S (e.g. a stock), derivative V (e.g. an option) and money B (a risk-less assets, e.g. a bond). We suppose this portfolio to be *self-financing*, i.e. transactions in one group of assets is done only by the resources from the other two groups of assets. Let Q_t^S and Q_t^V denote amount of underlying and derivative in time t respectively, then for $\forall t \in [0, T]$

$$S dQ_t^S + V dQ_t^V + \delta B_t = 0. \tag{7.3}$$

We also suppose the zero investment growth in portfolio, i.e. for $\forall t \in [0, T]$

$$SQ_t^S + VQ_t^V + B_t = 0. \tag{7.4}$$

Change of money amount is influenced by several different factors. Money increase by the return from holding the money rB_tdt, holding the underlying bring us some return (positive or negative) qSQ_t^Sdt and change in amount of money done by transactions δB_t. Together the total change of amount of money is

$$dB_t = rB_tdt + \delta B_t + qSQ_t^S dt. \tag{7.5}$$

As long as the (7.4) holds for $\forall t$, we can differentiate it. Using (7.3) and including (7.5), the equation yields

$$Q_t^S dS + Q_t^V dV + rB_tdt + qSQ_t^S dt = 0. \tag{7.6}$$

Using (7.4) again, dividing equation by Q_t^V and introducing a substitution $\Delta = -\frac{Q_t^S}{Q_t^V}$, we have

$$dV - rVdt + \Delta\left((r-q)Sdt - dS\right) = 0. \tag{7.7}$$

We assume that the underlying follows the stochastic differential equation

$$dS = (r-q)Sdt + \sigma S dW_t \tag{7.8}$$

and the derivative is defined as a function $V_t = V(t,S,A)$, i.e. it follows a stochastic differential equation as well. Applying Itô lemma one has

$$dV = \left(\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \frac{\partial V}{\partial A}dA + \sigma S \frac{\partial V}{\partial S}dW_t. \tag{7.9}$$

According to the (7.1), the value $dA_t = \frac{\partial A_t}{\partial t}dt + \frac{\partial A_t}{\partial \mathcal{I}_t}(\mathcal{U}(t,t,S) + \int_0^t \frac{\partial \mathcal{U}}{\partial t}(t,u,S_u)\,du)\,dt$

Including (7.8) and (7.9) into (7.7) we acquire

$$\begin{aligned}\Bigg(\frac{\partial V}{\partial t} &+ (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} \\ &+ \left(\frac{\partial A}{\partial t} + \frac{\partial A}{\partial \mathcal{I}}\left(\mathcal{U}(t,t,S) + \int_0^t \frac{\partial \mathcal{U}}{\partial t}(t,u,S_u)\,du\right)\right)\frac{\partial V}{\partial A} - rV\Bigg)dt \\ &+ \left(\frac{\partial V}{\partial S} - \Delta\right)\sigma S dW_t = 0.\end{aligned} \tag{7.10}$$

We want to eliminate the stochastic feature in the equation so we set

$$\Delta = \frac{\partial V}{\partial S}. \tag{7.11}$$

Substituting for Δ, the equation yields Black–Scholes partial differential equation

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} \\ &+ \left(\frac{\partial A}{\partial t} + \frac{\partial A}{\partial \mathcal{I}}\left(\mathcal{U}(t,t,S) + \int_0^t \frac{\partial \mathcal{U}}{\partial t}(t,u,S_u)\,du\right)\right)\frac{\partial V}{\partial A} - rV = 0. \quad (7.12)\end{aligned}$$

7.2 Modified Black–Scholes equation

Now, we summarize the modified Black–Scholes equation for floating strike Asian options with various averaging and lookback options.

If we consider the general average

$$A_t = \left(\frac{1}{\int_0^t a(s)\,ds}\int_0^t a(t-u)(S_u)^p\,du\right)^{\frac{1}{p}}. \quad (7.13)$$

the functions are defined as $A(u,\mathcal{I}) = \left(\frac{1}{\int_0^u a(s)\,ds}\mathcal{I}\right)^{\frac{1}{p}}$ and $\mathcal{U}(t,u,S) = a(t-u)S^p$. Then we can rewrite the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + \frac{a(0)S^p + \int_0^t a'(t-u)(S_u)^p\,du - a(t)A^p}{pA^{p-1}\int_0^t a(s)\,ds}\frac{\partial V}{\partial A} - rV = 0. \quad (7.14)$$

For the continuous geometric average

$$A_t^g = e^{\frac{1}{t}\int_0^t \ln S_u\,du}, \quad (7.15)$$

we define the functions as $A(u,\mathcal{I}) = e^{\frac{1}{u}\mathcal{I}}$ and $\mathcal{U}(t,u,S) = \ln S$.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + \frac{\ln S - \ln A^g}{t}A^g\frac{\partial V}{\partial A} - rV = 0. \quad (7.16)$$

For the continuous arithmetic average

$$A_t^a = \frac{1}{t}\int_0^t S_u\,du, \quad (7.17)$$

the functions are defined as $A(u,\mathcal{I}) = \frac{1}{u}\mathcal{I}$ and $\mathcal{U}(t,u,S) = S$.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + \frac{S - A^a}{t}\frac{\partial V}{\partial A} - rV = 0. \quad (7.18)$$

The weighted continuous arithmetic average

$$A_t^{wa} = \frac{1}{\int_0^t a(s)\,ds} \int_0^t a(t-u) S_u \, du, \tag{7.19}$$

have the functions defined as $A(u, \mathcal{I}) = \frac{1}{\int_0^u a(s)\,ds} \mathcal{I}$ and $\mathcal{U}(t, u, S) = a(t-u)S$.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + \frac{a(0)S + \int_0^t a'(t-u)S_u du - a(t)A^{wa}}{\int_0^t a(s)\,ds} \frac{\partial V}{\partial A} - rV = 0. \tag{7.20}$$

If we define $a(s) = e^{-\lambda s}$ for $\lambda > 0$, then $a'(s) = -\lambda a(s)$ and the equation (7.20) simplifies into

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + \lambda \frac{S - A^{wa}}{1 - e^{-\lambda t}} \frac{\partial V}{\partial A} - rV = 0. \tag{7.21}$$

For the lookback options

$$m_t = A_t^{-\infty} = \inf_{u \in [0,t]} S_u \tag{7.22}$$

and

$$M_t = A_t^{\infty} = \sup_{u \in [0,t]} S_u. \tag{7.23}$$

we cannot define functions $A(u, \mathcal{I})$ and $\mathcal{U}(t, u, S)$. Instead of defining these functions, we calculate the limit of the (7.14) for $|p| \to \infty$ (the expression is the same for both $p \to \infty$ and $p \to -\infty$)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0. \tag{7.24}$$

This equation is the same as the basic Black–Scholes partial differential equation for vanilla options (2.3). The only difference is in the terminal and marginal conditions.

The solution of the partial differential equation for American style floating strike option is calculated only on the continuation region

$$\mathcal{C}_{call} = (0, S_t^*) \qquad \text{and} \qquad \mathcal{C}_{put} = (S_t^*, \infty)$$

for call and put option, respectively. The value $S_t^* = S^*(t, A)$ is position of early exercise boundary at time $t \in [0, T]$ for value of average $A > 0$ (or extreme value). The value of option on the stopping region $\mathcal{S}$ is set to the value of the pay-off function Ω for both call and put option.

The terminal condition for all floating strike Asian options is given by their pay-off function

$$V_T = \Omega = (S - A)^+ \quad \text{and} \quad V_T = \Omega = (A - S)^+ .$$

for call and put option, respectively. The average A is replaced by minimum m for floating strike lookback call option and by maximum M for put option.

The American style marginal conditions and condition of smoothness are for call option given by

$$\begin{aligned} &V_t = 0 \quad \text{at } S = 0, \\ &V_t = \Omega, \quad \frac{\partial V_t}{\partial S} = 1 \quad \text{at } S = S_t^* \end{aligned}$$

and for put option are given by

$$\begin{aligned} &V_t = 0 \quad \text{at } S = \infty, \\ &V_t = \Omega, \quad \frac{\partial V_t}{\partial S} = -1 \quad \text{at } S = S_t^*, \end{aligned}$$

where $t \in [0, T]$.

For the lookback options, there is one more condition that guarantees the extreme property of m and M. The time-underlying space is restricted by inequalities

$$S \geq m \quad \text{and} \quad S \leq M$$

for call (minimum) and put (maximum) lookback option, respectively.

According to Goldman et al. (1979), the value of a lookback option at $S = m$ or $S = M$ is unaffected by marginal changes of the current extreme value. Thus, we replace the marginal condition at $S = 0$ and at $S = \infty$ by the extreme condition for $t \in [0, T]$

$$\frac{\partial V_t}{\partial m} = 0 \quad \text{at } S = m \quad \text{and} \quad \frac{\partial V_t}{\partial M} = 0 \quad \text{at } S = M \tag{7.25}$$

for call and put option, respectively.

CHAPTER 8

Transformation method

The purpose of this chapter is to propose an efficient numerical algorithm for determining the early exercise boundary position x_t^* for American style of Asian and lookback options. Construction of the algorithm is based on a solution to a nonlocal parabolic partial differential equation (PDE). The governing PDE is constructed for a transformed variable representing the so-called δ-synthesized portfolio. Furthermore, we employ a front fixing method (also referenced to as Landau's fixed domain transformation) developed by Wu and Kwok (1997), Stamicar et al. (1999), Ševčovič (2001) for plain vanilla options as well as for a class of nonlinear Black–Scholes equations Ševčovič (2007, 2008), Ankudinova and Ehrhardt (2008b,a). In this chapter, we consider lookback options as a version of Asian options, thus the meaning of Asian options includes also lookback options. We again use the constant $\mathfrak{c} = 1$ and $\mathfrak{c} = -1$ for call and put option, respectively. At the end of this chapter, we present numerical results and comparisons for Asian option with arithmetic average achieved by these methods to the recent method developed by Dai and Kwok (2006). This chapter is based on results from the last part of paper Bokes and Ševčovič (2011)[1].

First, we recall the partial differential equation for pricing Asian options derived in CHAPTER 7 (cf. Kwok 2008). We assume that the asset price dynamics is driven by the stochastic differential equation (4.1)

$$dS_t = (r-q)S_t \, dt + \sigma S_t \, dW_t, \quad 0 \le t \le T,$$

with a drift $r - q$, volatility σ and the standard Wiener process W_t. If we apply Itô formula to the function $V = V(t, S, A)$ we obtain

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial A} dA. \tag{8.1}$$

[1] TB and Ševčovič, D.: 2011, *Early exercise boundary for American type of floating strike Asian option and its numerical approximation, Applied Mathematical Finance*

Recall that for arithmetic, geometric, weighted arithmetic continuous averaging or extreme values we have $\frac{dA}{A} = f(\frac{A}{S}, t)dt$, where the function $f = f(x,t)$ is defined as follows (see (4.25)-(4.28)):

$$f(x,t) = \begin{cases} \frac{1}{t}(\frac{1}{x} - 1) & \text{arithmetic averaging,} \\ -\frac{1}{t}\ln x & \text{geometric averaging,} \\ \frac{\lambda}{1-e^{-\lambda t}}(\frac{1}{x} - 1) & \text{exponentially weighted arithmetic averaging,} \\ 0 & \text{maximum or minimum value.} \end{cases} \tag{8.2}$$

Inserting the expression $dA = A\ f(\frac{A}{S}, t)dt$ into (8.1) and following standard arguments from the Black–Scholes theory we obtain the governing equation for pricing Asian option with averaging given by (8.2) in the form:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + S(r-q)\frac{\partial V}{\partial S} + A\ f\left(\frac{A}{S}, t\right)\frac{\partial V}{\partial A} - rV = 0, \tag{8.3}$$

where $0 < t < T$ and $S, A > 0$ (see e.g. Dai and Kwok 2006, Kwok 2008). The above equation is subject to the terminal pay-off condition

$$V(T,S,A) = \Omega(S,A) = (\mathfrak{c}(S-A))^+, \quad S, A > 0.$$

It is well known (see e.g. Kwok 2008, Dai and Kwok 2006) that for Asian options with floating strike we can perform dimension reduction by introducing the following similarity variable:

$$x = \frac{A}{S}, \qquad W(x,\tau) = \frac{V(t,S,A)}{A}$$

where $\tau = T - t$. It is straightforward to verify that $V(t,S,A) = A\ W\left(\frac{A}{S}, T-t\right)$ is a solution of (8.3) iff $W = W(x,\tau)$ is a solution to the following parabolic PDE:

$$\frac{\partial W}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial}{\partial x}\left(x^2\frac{\partial W}{\partial x}\right) + (r-q)x\frac{\partial W}{\partial x} - f(x, T-\tau)\left(W + x\frac{\partial W}{\partial x}\right) + rW = 0, \tag{8.4}$$

where $x > 0$ and $0 < \tau < T$. The initial condition for W immediately follows from the terminal pay-off diagram, i.e.

$$W(x,0) = \left(\mathfrak{c}\left(\frac{1}{x} - 1\right)\right)^+.$$

8.1 American style of Asian options

Following Dai and Kwok (2006), we have the exercise region for American style of Asian options given by

$$\mathcal{E} = \{(t,S,A) \in [0,T]\times[0,\infty)\times[0,\infty),\ V(t,S,A) = \Omega(S,A)\}.$$

This region can be described by the early exercise boundary function $S^* = S^*(t,A)$ such that $\mathcal{E} = \{(t,S,A) \in [0,T]\times[0,\infty)\times[0,\infty),\ \mathfrak{c}\, S \geq \mathfrak{c}\, S^*(t,A)\}$. For American style of an Asian option we have to impose a homogeneous Dirichlet boundary condition $V(t,0,A) = 0$ and $V(t,\infty,A) = 0$ for call and put option, respectively. According to Dai and Kwok (2006) the C^1 continuity condition at the point $(t,S^*(t,A),A)$ of a contact of a solution V with its pay-off diagram implies the following boundary condition at the free boundary position $S^*(t,A)$:

$$\frac{\partial V}{\partial S}(t,S^*(t,A),A) = \mathfrak{c}, \quad V(t,S^*(t,A),A) = \mathfrak{c}\,(S^*(t,A) - A)\,, \tag{8.5}$$

for any $A > 0$ and $0 < t < T$. It is important to emphasize that the early exercise boundary function S^* can be also reduced to a function of one variable by introducing a new state function x_t^* as follows:

$$S^*(t,A) = \frac{A}{x_t^*}.$$

The function $t \mapsto x_t^*$ is a free boundary function for the transformed state variable $x = \frac{A}{S}$. For American style of Asian options the spatial domain for the reduced equation (8.4) is given by

$$\frac{\mathfrak{c}}{\varrho(\tau)} < \mathfrak{c}\, x, \quad (\tau,x) \in (0,T)\times(0,\infty), \quad \text{where } \varrho(\tau) = \frac{1}{x^*_{T-\tau}}\,.$$

Taking into account boundary conditions (8.5) for the option price V we end up with corresponding boundary conditions for the function W:

$$W(x,\tau) = \mathfrak{c}\left(\frac{1}{x} - 1\right),\ \frac{\partial W}{\partial x}(x,\tau) = -\frac{\mathfrak{c}}{x^2} \ \text{ at } \ x = \frac{1}{\varrho(\tau)}, \quad \begin{array}{ll} W(\infty,\tau) = 0 & \text{(call option)} \\ W(0,\tau) = 0 & \text{(put option)} \end{array} \tag{8.6}$$

for any $0 < \tau < T$ and the initial condition

$$W(x,0) = \left(\mathfrak{c}\left(\frac{1}{x} - 1\right)\right)^+ \quad \text{for any } x > 0. \tag{8.7}$$

According to the time-underlying space restriction $S \geq m$ for the lookback call and $S \leq M$ for put option, we need to replace marginal condition at $S = 0$ for call and at $S = \infty$ for put option by extreme condition

$$\frac{\partial V}{\partial m}(t,S,m) = 0 \quad \text{at } S = m \quad \text{and} \quad \frac{\partial V}{\partial M}(t,S,M) = 0 \quad \text{at } S = M \tag{8.8}$$

for call (minimum) and put (maximum) option, respectively (see Goldman et al. 1979). In terms of transformed function W the restriction yields $\mathfrak{c}\, x \le \mathfrak{c}$ and condition (8.8) becomes

$$W(x,\tau) + x\frac{\partial W}{\partial x}(x,\tau) = 0 \quad \text{at } x = 1. \tag{8.9}$$

8.2 Fixed domain transformation

In order to apply the Landau front fixing domain transformation for the early exercise boundary problem (8.4), (8.6), (8.7) we introduce a new state variable ξ and an auxiliary function $\Pi = \Pi(\xi,\tau)$ representing a synthetic portfolio. They are defined as follows:

$$\xi = \ln\left(\varrho(\tau)x\right), \qquad \Pi(\xi,\tau) = W(x,\tau) + x\frac{\partial W}{\partial x}(x,\tau).$$

Clearly, $\mathfrak{c}\, x > \frac{\mathfrak{c}}{\varrho(\tau)}$ iff $\mathfrak{c}\, \xi > 0$ for $\tau \in (0,T)$. The value $\xi = \infty$ and $\xi = -\infty$ of the transformed variable corresponds to the value $x = \infty$ and $x = 0$, respectively, i.e. when expressed in the original variable $S = 0$ and $S = \infty$, respectively. On the other hand, the value $\xi = 0$ corresponds to the free boundary position $x = x_t^*$, i.e. $S = S^*(t, A)$. For the lookback options, we have restrict the space to $0 < \mathfrak{c}\, \xi \le \mathfrak{c} \ln \varrho(\tau)$.

After straightforward calculations we conclude that the function $\Pi = \Pi(\xi,\tau)$ is a solution to the following parabolic PDE:

$$\frac{\partial \Pi}{\partial \tau} + a(\xi,\tau)\frac{\partial \Pi}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} + b(\xi,\tau)\Pi = 0,$$

where the term $a(\xi,\tau)$ depends on the free boundary position ϱ. The terms a, b are given by

$$a(\xi,\tau) = \frac{\dot\varrho(\tau)}{\varrho(\tau)} + r - q - \frac{\sigma^2}{2} - f\left(\frac{e^\xi}{\varrho(\tau)}, T-\tau\right), \tag{8.10}$$

$$b(\xi,\tau) = r - \frac{\partial}{\partial x}\left(x f(x, T-\tau)\right)\Big|_{x=\frac{e^\xi}{\varrho(\tau)}}, \tag{8.11}$$

where $\dot\varrho(\tau) = \frac{\partial \varrho}{\partial \tau}(\tau)$. Notice that for our cases, we have according to (8.2)

$$b(\xi,\tau) = \begin{cases} r + \frac{1}{T-\tau} & \text{arithmetic averaging,} \\ r + \frac{1}{T-\tau}\left(\xi + 1 - \ln\varrho(\tau)\right) & \text{geometric averaging,} \\ r + \frac{\lambda}{1-e^{-\lambda(T-\tau)}} & \text{exponentially weighted arithmetic averaging,} \\ r & \text{maximum or minimum value.} \end{cases}$$

The initial condition for the solution Π can be determined from (8.7) as

$$\Pi(\xi,0)=\begin{cases}-\mathfrak{c} & \text{for } \mathfrak{c}\,\xi<\mathfrak{c}\ln\varrho(0),\\ 0 & \text{for } \mathfrak{c}\,\xi>\mathfrak{c}\ln\varrho(0).\end{cases}$$

Since $\partial_x W(x,\tau)=-\frac{\mathfrak{c}}{x^2}$ and $W(x,\tau)=\mathfrak{c}\left(\frac{1}{x}-1\right)$ for $x=\frac{1}{\varrho(\tau)}$ and $W(\infty,\tau)=0$ (call option) or $W(0,\tau)=0$ (put option) we conclude the Dirichlet boundary conditions for the transformed function $\Pi(\xi,\tau)$

$$\Pi(0,\tau)=-\mathfrak{c},\qquad \Pi(\mathfrak{c}\,\infty,\tau)=0.$$

According to (8.9), the latter condition is for lookback options replaced by

$$\Pi(\mathfrak{c}\ln\varrho(\tau),\tau)=0.$$

It remains to determine an algebraic constraint between the free boundary function $\varrho(\tau)$ and the solution Π. Similarly as in the case of a linear or nonlinear Black–Scholes equation (cf. Ševčovič 2007) we obtain, by differentiation the condition $W\left(\frac{1}{\varrho(\tau)},\tau\right)=\mathfrak{c}\,(\varrho(\tau)-1)$ with respect to τ, the following identity:

$$\mathfrak{c}\frac{d\varrho}{d\tau}(\tau)=\frac{\partial W}{\partial x}\left(\frac{1}{\varrho(\tau)},\tau\right)\left(-\frac{1}{\varrho(\tau)^2}\right)\frac{d\varrho}{d\tau}(\tau)+\frac{\partial W}{\partial\tau}\left(\frac{1}{\varrho(\tau)},\tau\right).$$

Since $\partial_x W\left(\frac{1}{\varrho(\tau)},\tau\right)=-\mathfrak{c}\,\varrho(\tau)^2$ we have $\frac{\partial W}{\partial\tau}(x,\tau)=0$ at $x=\frac{1}{\varrho(\tau)}$. Assuming continuity of the function $\Pi(\xi,\tau)$ and its derivative $\Pi_\xi(\xi,\tau)$ up to the boundary $\xi=0$ we obtain

$$x^2\frac{\partial^2 W}{\partial x^2}(x,\tau)\to\frac{\partial\Pi}{\partial\xi}(0,\tau)+2\mathfrak{c}\,\varrho(\tau),\quad x\frac{\partial W}{\partial x}(x,\tau)\to-\mathfrak{c}\,\varrho(\tau)\quad\text{as } x\to\frac{1}{\varrho(\tau)}.$$

Passing to the limit $x\to\frac{1}{\varrho(\tau)}$ in (8.4) we end up with the algebraic equation

$$q\varrho(\tau)-r+f\left(\frac{1}{\varrho(\tau)},T-\tau\right)=\mathfrak{c}\frac{\sigma^2}{2}\frac{\partial\Pi}{\partial\xi}(0,\tau)\tag{8.12}$$

for the free boundary position $\varrho(\tau)$ where $\tau\in(0,T]$. Notice that according to (8.2) we have for $0<\tau<T$ and

- for arithmetic averaging

$$\varrho(\tau)=\frac{1+r(T-\tau)+\mathfrak{c}\frac{\sigma^2}{2}(T-\tau)\frac{\partial\Pi}{\partial\xi}(0,\tau)}{1+q(T-\tau)},$$

- for geometric averaging

$$\ln \varrho(\tau) + q(T-\tau)\varrho(\tau) = r(T-\tau) + \mathfrak{c}\frac{\sigma^2}{2}(T-\tau)\frac{\partial \Pi}{\partial \xi}(0,\tau),$$

- for exponentially weighted arithmetic averaging

$$\varrho(\tau) = \frac{\lambda + r\left(1-e^{\lambda(T-\tau)}\right) + \mathfrak{c}\frac{\sigma^2}{2}\left(1-e^{\lambda(T-\tau)}\right)\frac{\partial\Pi}{\partial\xi}(0,\tau)}{\lambda + q\left(1-e^{\lambda(T-\tau)}\right)},$$

- for maximum or minimum value

$$\varrho(\tau) = \frac{r + \mathfrak{c}\frac{\sigma^2}{2}\frac{\partial\Pi}{\partial\xi}(0,\tau)}{q},$$

i.e. we can derive an explicit expression for the free boundary position $\varrho(\tau)$ in all cases except for geometric averaging as a function of the derivative $\partial_\xi \Pi(0,\tau)$ evaluated at $\xi = 0$. The value $\varrho(0)$ can be deduced from THEOREM 5.1 and values of $x_T^* = \frac{1}{\varrho(0)}$ for Asian options are summarized in TABLE 6.1.

In summary, we derived the following nonlocal parabolic equation for the synthesized portfolio $\Pi(\xi,\tau)$:

$$\frac{\partial \Pi}{\partial \tau} + a(\xi,\tau)\frac{\partial \Pi}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} + b(\xi,\tau)\Pi = 0,$$

for Asian options: $0 < \tau < T,\ \mathfrak{c}\,\xi > 0,$

for lookback options: $0 < \tau < T,\ 0 < \mathfrak{c}\,\xi \le \mathfrak{c}\ln\varrho(\tau),$

with an algebraic constraint

$$q\varrho(\tau) - r + f\left(\frac{1}{\varrho(\tau)}, T-\tau\right) = \mathfrak{c}\frac{\sigma^2}{2}\frac{\partial \Pi}{\partial \xi}(0,\tau), \quad 0 < \tau < T,$$

subject to the boundary and initial conditions (8.13)

for Asian options: $\Pi(0,\tau) = -\mathfrak{c}, \qquad \Pi(\mathfrak{c}\,\infty,\tau) = 0,$

for lookback options: $\Pi(0,\tau) = -\mathfrak{c}, \qquad \Pi(\mathfrak{c}\ln\varrho(\tau),\tau) = 0,$

$$\Pi(\xi,0) = \begin{cases} -\mathfrak{c} & \text{for } \mathfrak{c}\,\xi < \mathfrak{c}\ln\varrho(0), \\ 0 & \text{for } \mathfrak{c}\,\xi > \mathfrak{c}\ln\varrho(0), \end{cases}$$

where $a(\xi,\tau)$ and $b(\xi,\tau)$ are given by (8.10) and (8.11),

and the starting point $\varrho(0) = \frac{1}{x_T^*}$ is given by THEOREM 5.1.

8.2.1 An equivalent form of the equation for the free boundary

Although equation (8.12) provides an algebraic formula for the free boundary position $\varrho(\tau)$ in terms of the derivative $\partial_\xi \Pi(0,\tau)$ such an expression is not quite suitable for construction of a robust numerical approximation scheme. The reason is that any small inaccuracy in approximation of the value $\partial_\xi \Pi(0,\tau)$ is transferred in to the entire computational domain $0 < \mathfrak{c}\,\xi < \mathfrak{c}\,\infty$ making thus a numerical scheme very sensitive to the value of the derivative of a solution evaluated in one point $\xi = 0$. For the lookback options, the domain is $0 < \mathfrak{c}\,\xi < \mathfrak{c}\ln\varrho(\tau)$. Nevertheless, we use the domain of Asian options in the general equations (to acquire the expression for lookback options, it suffice simply replace ∞ by $\ln\varrho(\tau)$ in the upper boundary of integrals). Although there are various ways to approximate the derivative $\partial_\xi \Pi(0,\tau)$ (see e.g. Kandilarov and Valkov 2011), in our case the following method is more suitable.

In what follows, we present an equation for the free boundary position $\varrho(\tau)$ which is more robust from the numerical approximation point of view.

Integrating the governing equation (8.13) with respect to ξ from 0 to $\mathfrak{c}\,\infty$ yields

$$\frac{d}{d\tau}\int_0^{\mathfrak{c}\,\infty} \Pi\, d\xi + \int_0^{\mathfrak{c}\,\infty} a(\xi,\tau)\frac{\partial \Pi}{\partial \xi}\, d\xi - \frac{\sigma^2}{2}\int_0^{\mathfrak{c}\,\infty} \frac{\partial^2 \Pi}{\partial \xi^2}\, d\xi + \int_0^{\mathfrak{c}\,\infty} b(\xi,\tau)\Pi\, d\xi = 0,$$

where functions a and b are defined by (8.10) and (8.11), respectively. We recall the boundary conditions $\Pi(\mathfrak{c}\,\infty,\tau) = 0$ and $\Pi(\mathfrak{c}\ln\varrho(\tau),\tau) = 0$ for Asian and lookback options, respectively and $\Pi(0,\tau) = -\mathfrak{c}$ for both types of options. Consequently, we have boundary $\partial_\xi \Pi(\mathfrak{c}\,\infty,\tau) = 0$ for Asian options. According to Goldman et al. (1979), the value of lookback option at extreme does not depend on the current value of extreme and thus we have $\partial_\xi \Pi(\mathfrak{c}\ln\varrho(\tau),\tau) = 0$ for lookback option. By applying condition (8.12), we obtain the following differential equation:

$$\begin{aligned}\frac{d}{d\tau}\left(\ln\varrho(\tau) + \mathfrak{c}\int_0^{\mathfrak{c}\,\infty} \Pi(\xi,\tau)\, d\xi\right) \quad &+ \quad q\varrho(\tau) - q - \frac{\sigma^2}{2} \\ &+ \quad \mathfrak{c}\int_0^{\mathfrak{c}\,\infty}\left[r - f\left(\frac{e^\xi}{\varrho(\tau)}, T-\tau\right)\right]\Pi(\xi,\tau)\, d\xi = 0.\end{aligned}$$

In our cases, we obtain according to (8.2)

- for arithmetic averaging

$$\begin{aligned}\frac{d}{d\tau}\left(\ln\varrho(\tau) + \mathfrak{c}\int_0^{\mathfrak{c}\,\infty} \Pi(\xi,\tau)\, d\xi\right) \quad &+ \quad q\varrho(\tau) - q - \frac{\sigma^2}{2} \\ &+ \quad \mathfrak{c}\int_0^{\mathfrak{c}\,\infty}\left[r - \frac{\varrho(\tau)e^{-\xi} - 1}{T-\tau}\right]\Pi(\xi,\tau)\, d\xi = 0,\end{aligned} \tag{8.14}$$

- for geometric averaging

$$\frac{d}{d\tau}\left(\ln\varrho(\tau) + \mathfrak{c}\int_0^{\mathfrak{c}\,\infty}\Pi(\xi,\tau)\,d\xi\right) + q\varrho(\tau) - q - \frac{\sigma^2}{2} + \mathfrak{c}\int_0^{\mathfrak{c}\,\infty}\left[r + \frac{\xi - \ln\varrho(\tau)}{T-\tau}\right]\Pi(\xi,\tau)\,d\xi = 0, \tag{8.15}$$

- for exponentially weighted arithmetic averaging

$$\frac{d}{d\tau}\left(\ln\varrho(\tau) + \mathfrak{c}\int_0^{\mathfrak{c}\,\infty}\Pi(\xi,\tau)\,d\xi\right) + q\varrho(\tau) - q - \frac{\sigma^2}{2} + \mathfrak{c}\int_0^{\mathfrak{c}\,\infty}\left[r - \lambda\frac{\varrho(\tau)e^{-\xi} - 1}{1 - e^{\lambda(T-\tau)}}\right]\Pi(\xi,\tau)\,d\xi = 0, \tag{8.16}$$

- for maximum or minimum value

$$\frac{d}{d\tau}\left(\ln\varrho(\tau) + \mathfrak{c}\int_0^{\mathfrak{c}\ln\varrho(\tau)}\Pi(\xi,\tau)\,d\xi\right) + q\varrho(\tau) - q - \frac{\sigma^2}{2} + \mathfrak{c}\,r\int_0^{\mathfrak{c}\ln\varrho(\tau)}\Pi(\xi,\tau)\,d\xi = 0. \tag{8.17}$$

8.3 A numerical approximation scheme

Our numerical approximation scheme is based on a solution to the transformed system (8.13). For the sake of simplicity, the scheme will be derived for the case of arithmetically averaged Asian call option. Derivation of the scheme for geometric, weighted arithmetic averaging or extreme value call option and for all analyzed put options is similar and therefore omitted.

We restrict the spatial domain $\xi \in (0,\infty)$ to a finite interval of values $\xi \in (0, L)$ where $L > 0$ is sufficiently large. For practical purposes, it is sufficient to take $L \approx 2$. Let denote the time step by $k = \frac{T}{m} > 0$ and the spatial step by $h = \frac{L}{n} > 0$. Here $m, n \in \mathbb{N}$ denote the number of time and space discretization steps, respectively. We denote by $\Pi^j = \Pi^j(\xi)$ the time discretization of $\Pi(\xi,\tau_j)$ and $\varrho^j \approx \varrho(\tau_j)$ where $\tau_j = jk$. By Π_i^j we shall denote the full space–time approximation for the value $\Pi(\xi_i,\tau_j)$. Then for the Euler backward in time finite difference approximation of equation (8.13) we have

$$\frac{\Pi^j - \Pi^{j-1}}{k} + c^j\frac{\partial\Pi^j}{\partial\xi} - \left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi} - 1}{T-\tau_j}\right)\frac{\partial\Pi^j}{\partial\xi} - \frac{\sigma^2}{2}\frac{\partial^2\Pi^j}{\partial^2\xi} + \left(r + \frac{1}{T-\tau_j}\right)\Pi^j = 0$$

where c^j is an approximation of the value $c(\tau_j)$ where the $c(\tau) = \frac{\dot{\varrho}(\tau)}{\varrho(\tau)} + r - q$. The solution $\Pi^j = \Pi^j(x)$ is subject to Dirichlet boundary conditions at $\xi = 0$ and $\xi = L$. We set $\Pi^0(\xi) = \Pi(\xi, 0)$ (see (8.13)). In what follows, we make use of the time step operator splitting method. We split the above problem into a convection part and a diffusive part by introducing an auxiliary intermediate step $\Pi^{j-\frac{1}{2}}$:

(*Convective part*)

$$\frac{\Pi^{j-\frac{1}{2}} - \Pi^{j-1}}{k} + c^j \frac{\partial \Pi^{j-\frac{1}{2}}}{\partial \xi} = 0\,, \tag{8.18}$$

(*Diffusive part*)

$$\frac{\Pi^j - \Pi^{j-\frac{1}{2}}}{k} - \left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi} - 1}{T - \tau_j}\right)\frac{\partial \Pi^j}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi^j}{\partial^2 \xi} + \left(r + \frac{1}{T - \tau_j}\right)\Pi^j = 0. \tag{8.19}$$

Similarly as in Ševčovič (2007) we shall approximate the convective part by the explicit solution to the transport equation $\partial_\tau \tilde{\Pi} + c(\tau)\partial_\xi \tilde{\Pi} = 0$ for $\xi > 0$ and $\tau \in (\tau_{j-1}, \tau_j]$ subject to the boundary condition $\tilde{\Pi}(0,\tau) = -1$ and the initial condition $\tilde{\Pi}(\xi, \tau_{j-1}) = \Pi^{j-1}(\xi)$. It is known that the free boundary function $\varrho(\tau)$ need not be monotonically increasing (see e.g. Dai and Kwok 2006, Ševčovič 2008, Hansen and Jørgensen 2000). Therefore depending whether the value of $c(\tau)$ is positive or negative the boundary condition $\tilde{\Pi}(0,\tau) = -1$ at $\xi = 0$ is either in–flowing ($c(\tau) > 0$) or out–flowing ($c(\tau) < 0$). Hence, the boundary condition $\tilde{\Pi}(0,\tau) = -1$ can be prescribed only if $c(\tau_j) \geq 0$. Let us denote by $C(\tau)$ the primitive function to $c(\tau)$, i.e. $C(\tau) = \ln \varrho(\tau) + (r-q)\tau$. Solving the transport equation $\partial_\tau \tilde{\Pi} + c(\tau)\partial_\xi \tilde{\Pi} = 0$ for $\tau \in [\tau_{j-1}, \tau_j]$ subject to the initial condition $\Pi(\xi, \tau_{j-1}) = \Pi^{j-1}(\xi)$ we obtain:

$$\tilde{\Pi}(\xi,\tau) = \Pi^{j-1}(\xi - C(\tau) + C(\tau_{j-1}))$$

if $\xi - C(\tau) + C(\tau_{j-1}) > 0$ and

$$\tilde{\Pi}(\xi,\tau) = -1$$

otherwise. Hence the full time-space approximation of the half-step solution $\Pi_i^{j-\frac{1}{2}}$ can be obtained from the formula

$$\Pi_i^{j-\frac{1}{2}} = \begin{cases} \Pi^{j-1}(\eta_i), & \text{if } \eta_i = \xi_i - \ln \varrho^j + \ln \varrho^{j-1} - (r-q)k > 0\,, \\ -1, & \text{otherwise.} \end{cases} \tag{8.20}$$

In order to compute the value $\Pi^{j-1}(\eta_i)$ we make use of a linear interpolation between discrete values $\Pi_i^{j-1}, i = 0, 1, \ldots, n$.

Using central finite differences for approximation of the derivative $\partial_\xi \Pi^j$ we can approximate the diffusive part of a solution of (8.19) as follows:

$$\begin{aligned}\frac{\Pi_i^j - \Pi_i^{j-\frac{1}{2}}}{k} &+ \left(r + \frac{1}{T-\tau_j}\right)\Pi_i^j \\ &- \left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi_i} - 1}{T-\tau_j}\right)\frac{\Pi_{i+1}^j - \Pi_{i-1}^j}{2h} - \frac{\sigma^2}{2}\frac{\Pi_{i+1}^j - 2\Pi_i^j + \Pi_{i-1}^j}{h^2} = 0\,.\end{aligned}$$

Therefore the vector of discrete values $\Pi^j = \{\Pi_i^j, i = 1, 2, \dots, n\}$ at the time level $j \in \{1, 2, \dots, m\}$ is a solution of a tridiagonal system of linear equations

$$\alpha_i^j \Pi_{i-1}^j + \beta_i^j \Pi_i^j + \gamma_i^j \Pi_{i+1}^j = \Pi_i^{j-\frac{1}{2}}, \quad \text{for } i = 1, 2, \dots, n, \quad \text{where} \tag{8.21}$$

$$\alpha_i^j(\varrho^j) = -\frac{k}{2h^2}\sigma^2 + \frac{k}{2h}\left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi_i} - 1}{T-\tau_j}\right), \gamma_i^j(\varrho^j) = -\frac{k}{2h^2}\sigma^2 - \frac{k}{2h}\left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi_i} - 1}{T-\tau_j}\right),$$

$$\beta_i^j(\varrho^j) = 1 + \left(r + \frac{1}{T-\tau_j}\right)k - (\alpha_i^j + \gamma_i^j)\,. \tag{8.22}$$

The initial and boundary conditions at $\tau = 0$ and $x = 0, L$, can be approximated as follows:

$$\Pi_i^0 = \begin{cases} -1 & \text{for } \xi_i < \ln\frac{1+rT}{1+qT}, \\ 0 & \text{for } \xi_i \ge \ln\frac{1+rT}{1+qT}, \end{cases}$$

for $i = 0, 1, \dots, n$, and $\Pi_0^j = -1,\ \Pi_n^j = 0$ for $j = 1, \dots, m$.

Finally, we employ the differential equation (8.14) to determine the free boundary position ϱ. Taking the Euler finite difference approximation of $\frac{d}{d\tau}\left(\ln\varrho + \int_0^\infty \Pi\, d\xi\right)$ we obtain

(*Algebraic part*)

$$\ln\varrho^j = \ln\varrho^{j-1} + I_0(\Pi^{j-1}) - I_0(\Pi^j) + k\left(q + \frac{\sigma^2}{2} - q\varrho^{j-1} - I_1(\varrho^{j-1}, \Pi^j)\right) \tag{8.23}$$

where $I_0(\Pi)$ stands for numerical trapezoidal quadrature of the integral $\int_0^\infty \Pi(\xi)\,d\xi$ and $I_1(\varrho^{j-1}, \Pi)$ is a trapezoidal quadrature of the integral

$$\int_0^\infty \left(r - \frac{\varrho^{j-1}e^{-\xi} - 1}{T-\tau_j}\right)\Pi(\xi)\,d\xi.$$

We formally rewrite discrete equations (8.23), (8.20) and (8.21) in the operator form:

$$\varrho^j = \mathcal{F}(\Pi^j), \qquad \Pi^{j-\frac{1}{2}} = \mathcal{T}(\varrho^j), \qquad \mathcal{A}(\varrho^j)\Pi^j = \Pi^{j-\frac{1}{2}}, \tag{8.24}$$

where $\ln \mathcal{F}(\Pi^j)$ is the right-hand side of equation (8.23), $\mathcal{T}(\varrho^j)$ is the transport equation solver given by the right-hand side of (8.20) and $\mathcal{A} = \mathcal{A}(\varrho^j)$ is a tridiagonal matrix with coefficients given by (8.22). The system (8.24) can be approximately solved by means of successive iterations procedure. We define, for $j \geq 1$, $\Pi^{j,0} = \Pi^{j-1}$, $\varrho^{j,0} = \varrho^{j-1}$. Then the $(p+1)$-th approximation of Π^j and ϱ^j is obtained as a solution to the system:

$$\varrho^{j,p+1} = \mathcal{F}(\Pi^{j,p}), \quad \Pi^{j-\frac{1}{2},p+1} = \mathcal{T}(\varrho^{j,p+1}), \quad \mathcal{A}(\varrho^{j,p+1})\Pi^{j,p+1} = \Pi^{j-\frac{1}{2},p+1}. \tag{8.25}$$

Supposing the sequence of approximate discretized solutions $\{(\Pi^{j,p}, \varrho^{j,p})\}_{p=1}^{\infty}$ converges to the limiting value $(\Pi^{j,\infty}, \varrho^{j,\infty})$ as $p \to \infty$ then this limit is a solution to a nonlinear system of equations (8.24) at the time level j and we can proceed by computing the approximate solution in the next time level $j+1$.

8.4 Computational examples of the free boundary approximation

Finally we present several computational examples of application of the numerical approximation scheme (8.25) for the solution $\Pi(\xi,\tau)$ and the free boundary position $\varrho(\tau)$ of (8.13). We consider American style of Asian arithmetically averaged floating strike call options.

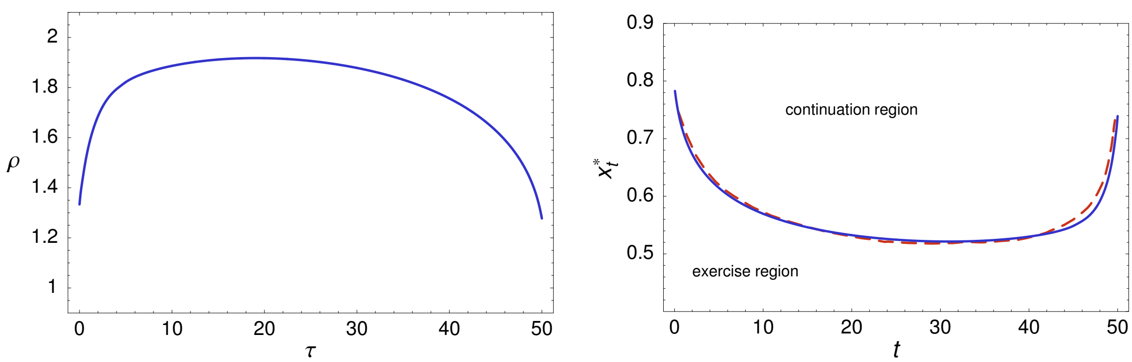

FIGURE 8.1: *The function $\varrho(\tau)$ (left). A comparison of the free boundary position $x_t^* = 1/\varrho(T-t)$ (right) obtained by our method (solid curve) and that of the projected successive over relaxation algorithm by Dai and Kwok (dashed curve).*

In FIGURE 8.1 we show behavior of the early exercise boundary function $\varrho(\tau)$ and the function $x_t^* = \frac{1}{\varrho(T-t)}$. In this numerical experiment we chose $r = 0.06$, $q = 0.04$, $\sigma = 0.2$ and very long expiration time $T = 50$ years. These parameters correspond to the example presented by Dai and Kwok (2006). As far as other numerical parameters are concerned, we chose the mesh of $n = 200$ spatial grid points and we have chosen the number of time steps $m = 10^5$ in order to achieve very fine time stepping corresponding to 260 minutes between consecutive time steps when expressed in the original time scale of the problem.

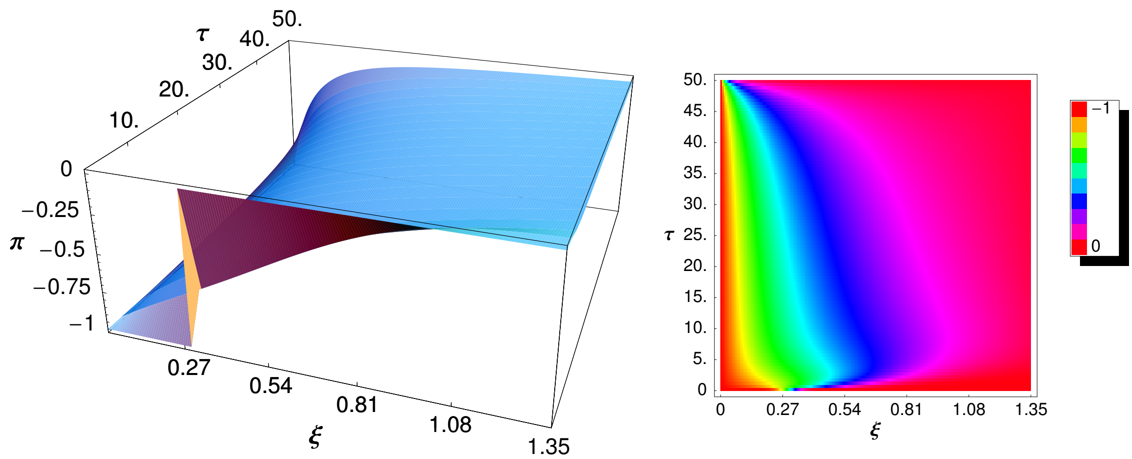

FIGURE 8.2: *A 3D plot (left) and contour plot (right) of the function* $\Pi(\xi, \tau)$.

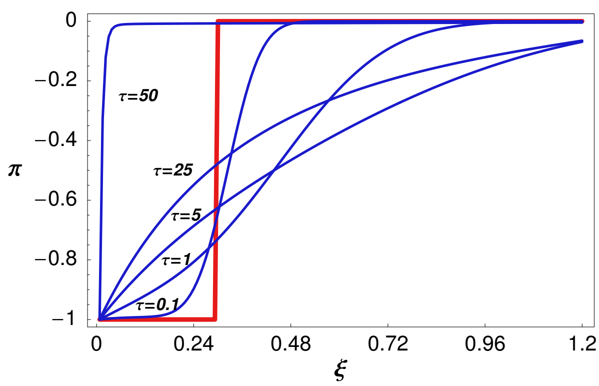

FIGURE 8.3: *Profiles of the function* $\Pi(\xi, \tau)$ *for various times* $\tau \in [0, T]$.

In FIGURE 8.2, we can see the behavior of transformed function Π in both 3D as well as contour plot perspectives. In FIGURE 8.3, we also plot the initial condition

$\Pi(\xi, 0)$ and five time steps of the function $\xi \mapsto \Pi(\xi, \tau_j)$ for $\tau_j = 0.1, 1, 5, 25, 50$.

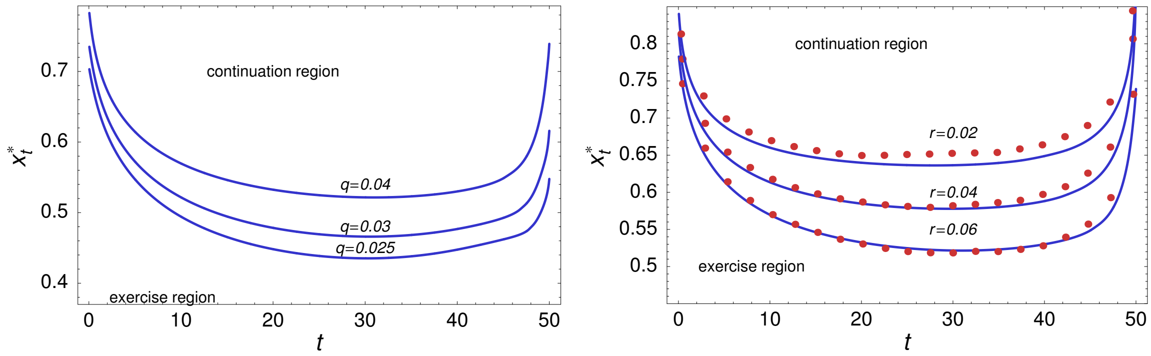

FIGURE 8.4: *A comparison of the free boundary position x_t^* for various dividend yield rates $q = 0.04, 0.03, 0.025$ and fixed interest rate $r = 0.06$ (left). Comparison of x_t^* for various interest rates $r = 0.06, 0.04, 0.02$ and fixed dividend yield $q = 0.04$. Dots represents the solution obtained by Dai and Kwok (right).*

TABLE 8.1: *Comparison of PSOR and our transformation method for $T = 50$, $\sigma = 0.2$, $q = 0.04$.*

	$r = 0.06$	$r = 0.04$	$r = 0.02$
$\Vert x_t^{*,trans} - x_t^{*,psor}\Vert_\infty$	0.09769	0.03535	0.05359
$\Vert x_t^{*,trans} - x_t^{*,psor}\Vert_1$	0.00503	0.00745	0.01437
$\min x_t^{*,trans}$	0.52150	0.57780	0.63619

A comparison of early exercise boundary profiles with respect to varying interest rates r and dividend yields q is shown in FIGURE 8.4. A comparison of the free boundary position $x_t^* = \frac{1}{\varrho(T-t)}$ obtained by our method (solid curve) and that of the projected successive over relaxation algorithm by Dai and Kwok (2006) (dotted curve) for different values of the interest rate r is shown in FIGURE 8.4 (right). The algorithm of Dai and Kwok (2006) is based on a numerical solution to the variational inequality for the function $W = W(x, \tau)$ which is a solution to (8.4) in the continuation region and it is smoothly pasted to its pay-off diagram (8.6). It is clear that our method and that of Dai and Kwok (2006) give almost the same results. A quantitative comparison of both methods is given in TABLE 8.1 for model parameters $T = 50$, $\sigma = 0.2$, $q = 0.04$ and various interest rates $r = 0.02, 0.04, 0.06$. We evaluated discrete $L^\infty(0,T)$ and $L^1(0,T)$ norms of the difference $x_t^{*,trans} - x_t^{*,psor}$ between the numerical solution $x_t^{*,trans}, t \in [0,T]$, obtained by our method and that of Dai and Kwok (2006) denoted by $x_t^{*,psor}$. We also show the minimal value $\min_{t\in[0,T]} x_t^{*,trans}$ of the early exercise boundary.

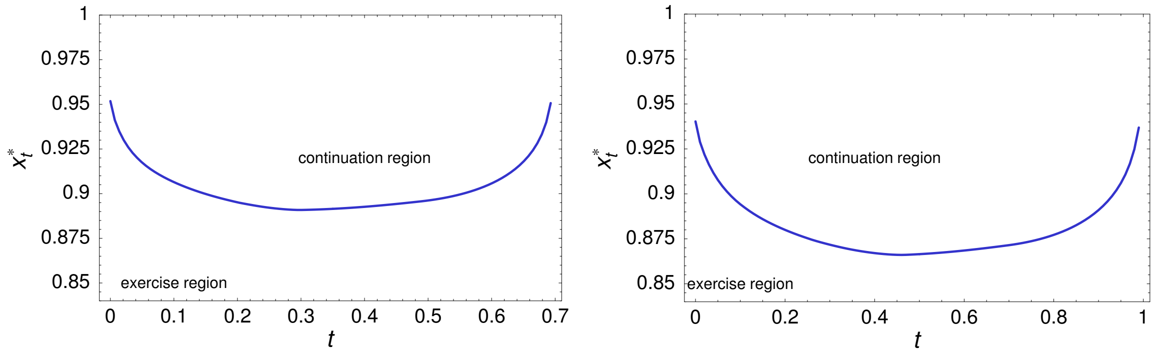

FIGURE 8.5: *The free boundary position for expiration times $T = 0.7$ (left) and $T = 1$ (right).*

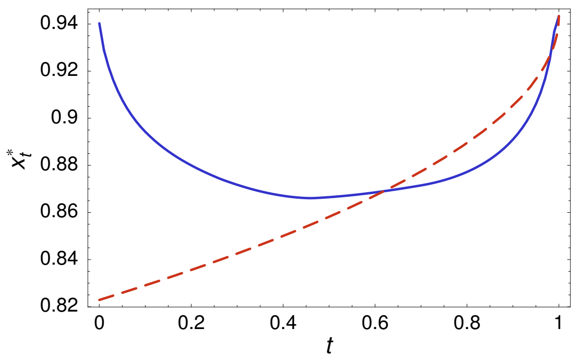

FIGURE 8.6: *A comparison of the free boundary position with its analytic approximation (dashed line).*

Finally, in FIGURE 8.5 we present numerical experiments for shorter expiration times $T = 0.7$ and $T = 1$ (one year) with zero dividend rate $q = 0$ and $r = 0.06$, $\sigma = 0.2$. We also present a comparison of the free boundary position $x_t^* = \frac{1}{\varrho(T-t)}$ and the analytic approximation (6.33) for parameters: $r = 0.06$, $q = 0$, $\sigma = 0.2$ and $T = 1$. It is clear that the analytic approximation (6.33) is capable of capturing the behavior of x_t^* only for times t close to the expiry T. Moreover, the analytic approximation is a monotone function whereas the true early exercise boundary x_t^* is a decreasing function for small values of t and then it becomes increasing (see e.g. FIGURE 8.6).

Conclusions

> *"A man may imagine things that are false, but he can only understand things that are true, for if the things be false, the apprehension of them is not understanding."*
>
> – Sir Isaac Newton

In this thesis, we analyze floating strike American style Asian and lookback options (members of the class of path-dependent options).

Chapter 3 summarizes wide range of exotic options traded on the market. We used a classification based on Wilmott (2006). The main scope of the chapter is path-dependent options.

In Chapter 4, we calculated the value of American style Asian option with general type of average according to the theory of conditioned expected values and martingales (following the idea of Hansen and Jørgensen (2000)). The calculation was also done for the continuous geometric average and we have approximated the value of Asian option with continuous arithmetic averaging. Moreover, we calculated the value of floating strike American style lookback options. The extreme value can be obtain as a limit of the general average.

We created a new method for the calculation of the limit of the early exercise boundary at the expiry. This problem was analyzed by many authors for selected financial derivatives. Our methodology, presented in Chapter 5, is a unifying approach to calculate the limit for general financial derivative that can be written as a Doob-Meyer decomposition of the Snell envelope of its discounted pay-off function. In the Appendix D, we have verified this method by comparing results with numerical results calculated by the PSOR method. The comparison was performed on American style of option strategies.

In CHAPTER 6, we have analyzed the behavior of early exercise boundary close to expiry of Asian and lookback options. According to the methodology from the previous chapter, we calculated the limit value of the free boundary at expiry. The calculation of the first order expansion is based on the marginal condition of smoothness of an American style derivative value at the early exercise boundary.

We present an integral-differential equation for calculation of the early exercise boundary of Asian and lookback options. The equation was derived in CHAPTER 8 by several transformation of the modified Black–Scholes partial differential equation for path-dependent derivatives. The discretization of presented equation leads to numerical approximation scheme.

We have covered selected problems of the topic presented in this thesis. However, there are still open issues up to further research.

APPENDIX A

American style vanilla option

In this chapter, we use methods and theory presented in the main part of the thesis on plain vanilla options. First, we use the model for the derivation of the value presented in SECTION 4.2. Then we summarize the limit of early exercise boundary at expiry already calculated in SECTION 5.2.1. And finally, we calculate the approximation of early exercise boundary at the expiry following derivation from SECTION 6.2.

Most of the results presented in this chapter can be found e.g. in Kwok (2008).

A.1 Calculation of the formula

For the value of an American style vanilla, we have no need to change the probability measure. We use the time value of money as the numeraire.

We set a problem similar to the (4.14)

$$V(t,S) = \underset{s\in\mathcal{T}_{[t,T]}}{ess\sup}\, \mathbb{E}_t^Q \left[e^{-r(s-t)} \left(\mathfrak{c}(S_s - X)\right)^+ \middle| S_t = S \right], \tag{A.1}$$

where $\mathcal{T}_{[t,T]}$ denotes the set of all stopping times in the interval $[t,T]$, $\mathbb{E}_t^Q[X] = \mathbb{E}^Q[X|\mathcal{F}_t]$ is the conditioned expectation with information of time t and X is the strike price of the option ($\mathfrak{c} = 1$ for a call option and $\mathfrak{c} = -1$ for a put option).

We follow the derivation of the formula from SECTION 4.2

$$\begin{aligned} V(t,S) &= \underset{s\in\mathcal{T}_{[t,T]}}{ess\sup}\, \mathbb{E}_t^Q \left[e^{-r(s-t)} \left(\mathfrak{c}(S_s - X)\right)^+ \middle| S_t = S \right] \\ &= \underset{s\in\mathcal{T}_{[t,T]}}{ess\sup}\, e^{-r(s-t)} \mathbb{E}_t^Q \left[\left(\mathfrak{c}(S_s - X)\right)^+ \middle| S_t = S \right]. \end{aligned}$$

We can rewrite last equation as

$$\widetilde{V}(t,S) = e^{-rt} V(t,S) = e^{-rT_t^*}\, \mathbb{E}_t^Q \left[\left(\mathfrak{c}(S_{T_t^*} - X)\right)^+ \right],$$

where $T_t^* = \inf\{s \in [t,T] | S_s = S_s^*\}$ and S^* is the exercise boundary.

Remark A.1. *According to* Table *2.2, the stopping region $\mathcal{S}$ and continuation region $\mathcal{C}$ for call option and put option are defined by*

$$\begin{aligned}\mathcal{C}_{call} = \mathcal{S}_{put} &= \{(t,S) \in [0,T] \times [0,S^*)\},\\ \mathcal{S}_{call} = \mathcal{C}_{put} &= \{(t,S) \in [0,T] \times (S^*,\infty)\},\end{aligned}$$

respectively.

Next, we present an application of Theorem 4.1 for the value of American plain vanilla option.

Theorem A.1. *The value of the American style vanilla option $\widetilde{V}(t,S_t) = e^{-rt}V(t,S_t)$ on stock underlying with dividend rate is given by*

$$\widetilde{V}(t,S_t) = \widetilde{v}(t,S_t) + \widetilde{e}(t,S_t),$$

where

$$\widetilde{v}(t,S_t) \equiv \mathbb{E}_t^Q\left[e^{-rT}\left(\mathfrak{c}(S_T - X)\right)^+\right]$$

and

$$\widetilde{e}(t,S_t) \equiv \mathbb{E}_t^Q\left[\int_t^T \mathfrak{c}\, e^{-ru}\mathbf{1}_{\mathcal{S}}(u,S_u)(qS_u - rX)\,du\right],$$

with stopping region $\mathcal{S}$. Here the function $\mathbf{1}_{\mathcal{S}}(\,\cdot\,)$ is the indicator function of the set $\mathcal{S}$, $\mathfrak{c} = 1$ for call option and $\mathfrak{c} = -1$ for put option.

We use Lemma 4.1 to calculate the expected values. According to the model we know that stock price S has lognormal distribution, i.e. $\ln S_u|\mathcal{F}_t \sim \mathcal{N}(\alpha_{t,u}, \beta_{t,u}^2)$, where

$$\begin{aligned}\alpha_{t,u} &= \ln S_t + (r - q - \frac{1}{2}\sigma^2)(u-t),\\ \beta_{t,u} &= \sigma\sqrt{u-t}.\end{aligned}$$

First part of the formula is well-known Black–Scholes formula for European style vanilla option (2.4):

$$\begin{aligned}v(t,S) &= e^{rt}\widetilde{v}(t,S)\\ &= \mathfrak{c}\left(Se^{-q(T-t)}\mathbf{\Phi}\left(\mathfrak{c}\left(\frac{\alpha_{t,T} - \ln X}{\beta_{t,T}} + \beta_{t,T}\right)\right) - Xe^{-r(T-t)}\mathbf{\Phi}\left(\mathfrak{c}\frac{\alpha_{t,T} - \ln X}{\beta_{t,T}}\right)\right),\end{aligned}$$

where $\mathbf{\Phi}(\,\cdot\,)$ is the *CDF* of the normal probability distribution $\mathcal{N}(0,1)$. We used this α and β notation to make the formula consistent with the rest of this paper.

The stopping region of the American style vanilla option is defined as $\mathfrak{c}\, S \geq \mathfrak{c}\, S^*$, where S^* is the early exercise boundary. We also need to define

$$\gamma_{p,t,u} = \frac{\ln S_u^* - \alpha_{t,u}}{\beta_{t,u}} - p\beta_{t,u},$$

where S_t^* is the value of early exercise boundary at time t. The American style bonus of the formula has form

$$\begin{aligned} e(t,S) &= e^{rt}\widetilde{e}(t,S) \\ &= \int_t^T \mathfrak{c}\, e^{-r(u-t)} \left(q e^{\alpha_{t,u} + \frac{\beta_{t,u}^2}{2}} \mathbf{\Phi}(-\mathfrak{c}\,\gamma_{1,t,u}) - rX\mathbf{\Phi}(-\mathfrak{c}\,\gamma_{0,t,u}) \right) du \\ &= \mathfrak{c} \int_t^T qSe^{-q(u-t)} \mathbf{\Phi}\left(\mathfrak{c} \left(\frac{\alpha_{t,u} - \ln S_u^*}{\beta_{t,u}} + \beta_{t,u} \right) \right) \\ &\qquad\qquad -rXe^{-r(u-t)}\mathbf{\Phi}\left(\mathfrak{c} \frac{\alpha_{t,u} - \ln S_u^*}{\beta_{t,u}} \right) du. \end{aligned}$$

This result can be found e.g. in Kim (1990) or Kwok (2008).

A.2 Limit of the early exercise boundary at expiry

The limit of the early exercise boundary at the expiry for the plain vanilla options is calculated according to THEOREM 5.1. The calculation is presented as an example in SECTION 5.2.1.

We recall that if we know explicit formula for value of European style of derivative, we can simply calculate the American style bonus function f_b at the expiry according to REMARK 5.2:

$$f_b^{vanilla}(T,S) = \begin{cases} 0 & \text{for } \mathfrak{c}\, S < \mathfrak{c}\, X, \\ \mathfrak{c}\frac{X}{2}\,(q-r) & \text{for } S = X, \\ \mathfrak{c}\,(qS - rX) & \text{for } \mathfrak{c}\, S > \mathfrak{c}\, X, \end{cases}$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call option or put option, respectively.

The boundary of set of positive values of $f_b^{vanilla}$ is given by following expression

$$\partial Z_T^{+vanilla} = \mathfrak{c}\,\max\left[\mathfrak{c}\, X, \mathfrak{c}\frac{r}{q}X\right] = S_T^{*vanilla}.$$

The value of limit of the early exercise boundary at expiry S^*_T is summarized in TABLE A.1.

TABLE A.1: *Starting point of the early exercise boundary S^*_T.*

S^*_T	put	call
vanilla option	$\min\left(\frac{rX}{q}, X\right)$	$\max\left(\frac{rX}{q}, X\right)$

This result is well known and can be found also in Kwok (2008), Albanese and Campolieti (2006), Detemple (2006), Wilmott et al. (1995) and many other sources.

A.3 Expansion of the exercise boundary close to expiry

Throughout this section, we shall assume the structural assumption on the interest and dividend rates (6.4), i.e.

$$\mathfrak{c}\, r > \mathfrak{c}\, q,$$

where $\mathfrak{c} = 1$ or $\mathfrak{c} = -1$ for call option or put option, respectively.

We follow the derivation presented in SECTION 6.2. The form of approximation function for early exercise boundary of a plain vanilla option has form

$$\varrho_{T-t} = S^*_t = \frac{rX}{q}(1 + h\sigma\sqrt{T-t}) + O(T-t) \quad \text{as } t \to T,$$

where $h \in \mathbb{R}$ is a constant. To calculate h, we use the condition of smoothness of the value of the option across the early exercise boundary - smooth pasting principle (cf. Kwok 2008, Dai and Kwok 2006).

$$\begin{aligned} \mathfrak{c} &= \frac{\partial V}{\partial S}(t, S^*_t) = \frac{\partial v}{\partial S}(t, S^*_t) + \frac{\partial e}{\partial S}(t, S^*_t) = \frac{\partial v}{\partial S}(t, S^*_t) + \int_t^T \frac{\partial e^I}{\partial S}(t, S^*_t, u, S^*_u)\, du \\ &= \widehat{v}_S(t, S^*_t) + \int_t^T \widehat{e}^I_S(t, S^*_t, u, S^*_u)\, du, \end{aligned}$$

where e^I denotes integrated function and we use the expression for V presented in the end of SECTION A.1. The first step of derivation are substitutions $t = T - \tau$ and

$u = T - \tau(1-\theta)$ into the previous equation:

$$\mathfrak{c} = \widehat{v}_S(T-\tau, S^*_{T-\tau}) + \tau \int_0^1 \widehat{e}^I_S\left(T-\tau, S^*_{T-\tau}, T-\tau(1-\theta), S^*_{T-\tau(1-\theta)}\right) d\theta \tag{A.2}$$

This equation should be valid through the time. Thus, we set its derivative with respect to τ equal to zero

$$\begin{aligned} 0 &= \frac{\partial}{\partial \tau}\left(-\mathfrak{c} + \widehat{v}_S(T-\tau, S^*_{T-\tau})\right) + \int_0^1 \widehat{e}^I_S\left(T-\tau, S^*_{T-\tau}, T-\tau(1-\theta), S^*_{T-\tau(1-\theta)}\right) d\theta \\ &\quad + \tau \int_0^1 \frac{\partial}{\partial \tau}\widehat{e}^I_S\left(T-\tau, S^*_{T-\tau}, T-\tau(1-\theta), S^*_{T-\tau(1-\theta)}\right) d\theta. \end{aligned}$$

The last element on the right-hand side of previous equation tends to zero with $\tau \to 0$. The derivation is straightforward and simple, but very long, space exhausting and similar to the following one, thus we left this proof to the reader.

Next, we calculate the limit for $\tau \to 0$:

$$0 = \lim_{\tau\to 0} \frac{\partial \widehat{v}_S(T-\tau, S^*_{T-\tau})}{\partial \tau} + \lim_{\tau \to 0} \int_0^1 \widehat{e}^I_S\left(T-\tau, S^*_{T-\tau}, T-\tau(1-\theta), S^*_{T-\tau(1-\theta)}\right) d\theta. \tag{A.3}$$

We recall that $\alpha_{t,u} = \alpha(t,u,S)$, $\beta_{t,u} = \beta(t,u)$. In this section, we use the following notation (to simplify the derivation)

$$\begin{aligned} r - q + \frac{\sigma^2}{2} &= \Lambda, \\ \alpha_{T-\tau,T} &= \alpha_T, \\ \beta_{T-\tau,T} &= \beta_T, \\ \alpha_{T-\tau,T-\tau(1-\theta)} &= \alpha_\theta, \\ \beta_{T-\tau,T-\tau(1-\theta)} &= \beta_\theta, \\ \gamma_{p,T-\tau,T-\tau(1-\theta)} &= \gamma_p. \end{aligned}$$

In further derivation, we use following limits calculated according to SECTION A.1 and SECTION A.2

$$\lim_{\tau\to 0} \frac{\alpha_{T-\tau,T-\tau(1-\theta)} - \ln \frac{rX}{q}(1 + h\sigma\sqrt{\tau(1-\theta)})}{\beta_{T-\tau,T-\tau(1-\theta)}} = h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}},$$

for $\theta \in (0,1)$ and

$$\lim_{\tau\to 0} \mathfrak{c}\left(\alpha_{T-\tau,T-\tau(1-\theta)} - \ln X\right) = \mathfrak{c}\left(\alpha_{T,T} - \ln X\right) = \mathfrak{c} \ln\frac{r}{q} > 0,$$

$$
\begin{aligned}
\lim_{\tau\to 0}\beta_{T-\tau,T-\tau(1-\theta)} &= \beta_{T,T} = 0^+,\\
\lim_{\tau\to 0}\mathbf{\Phi}\left(\mathfrak{c}\frac{\alpha_{T-\tau,T}-\ln X}{\beta_{T-\tau,T}}\right) &= \mathbf{\Phi}\left(\mathfrak{c}\frac{\ln\frac{r}{q}}{0^+}\right) = 1,\\
\forall n\in\mathbb{N}\cup\{0\}: \quad \lim_{\tau\to 0}\frac{\mathbf{\Phi}'\left(\frac{\alpha_{T-\tau,T}-\ln X}{\beta_{T-\tau,T}}\right)}{(\beta_{T-\tau,T})^n} &= 0,\\
\lim_{\tau\to 0}\partial_S\alpha_{T-\tau,T-\tau(1-\theta)} &= \frac{1}{S_T^*} = \frac{q}{rX},\\
\lim_{\tau\to 0}\beta_{T-\tau,T-\tau(1-\theta)}\,\partial_\tau\left(\beta_{T-\tau,T-\tau(1-\theta)}\big|_{S=\varrho_\tau}\right) &= \frac{\theta\sigma^2}{2},
\end{aligned}
$$

for $\theta\in(0,1]$.

Since we have assumed (6.4), we have $0<\frac{r}{q}\neq 1$. Notice that both α and β have polynomial order in τ and the derivative of the normal cumulative distribution function (i.e. the probability density function) has exponential order in τ variable. In all derivations we have used several properties of the derivative of normal cumulative distribution function $\mathbf{\Phi}(x)$, e.g. $\mathbf{\Phi}'(x)=\mathbf{\Phi}'(-x)$, $\mathbf{\Phi}''(x)=-x\mathbf{\Phi}'(x)$ and $\mathbf{\Phi}'(\frac{a}{b}+c)=e^{-\frac{ac}{b}-\frac{c^2}{2}}\mathbf{\Phi}'(\frac{a}{b})$.

We calculate the derivative of the European part of the expression (i.e. Greek delta).

$$
\widehat{v}_S(t,S)=\frac{\partial}{\partial S}\widetilde{v}(t,S)=\mathfrak{c}\,e^{-q(T-t)}\mathbf{\Phi}\left(\mathfrak{c}\left(\frac{\alpha_{t,T}-\ln X}{\beta_{t,T}}+\beta_{t,T}\right)\right).
$$

Now, we calculate the first part of the limit (A.3). According to presented limits, the elements with derivative of *CDF* tends to zero in the limit and limit of the *CDF* tends to 1, thus we have

$$
\lim_{\tau\to 0}\partial_\tau\widehat{v}_S(T-\tau,S)=-\mathfrak{c}\,q.
$$

Next, we calculate the derivative of the integral function of American style option bonus:

$$
\begin{aligned}
\widehat{e}_S^I(t,S,u,S_u^*) &= \mathfrak{c}\,e^{-r\tau\theta}\frac{\partial}{\partial S}\left(qe^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}(-\mathfrak{c}\,\gamma_1)-rX\mathbf{\Phi}(-\mathfrak{c}\,\gamma_0)\right)\\
&= e^{-r\tau\theta}\Bigg(\mathfrak{c}\,q\partial_S\alpha_\theta e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}(-\mathfrak{c}\,\gamma_1)\\
&\qquad -q\partial_S\gamma_1 e^{\alpha_\theta+\frac{\beta_\theta^2}{2}}\mathbf{\Phi}'(-\gamma_1)+r\partial_S\gamma_0 X\mathbf{\Phi}'(-\gamma_0)\Bigg).
\end{aligned}
$$

Since $\partial_S \gamma_p = -\frac{\partial_S \alpha_\theta}{\beta_\theta}$, the limit yields

$$\lim_{\tau\to 0} \widehat{e}_S^I(t, S, u, S_u^*) = \mathfrak{c}\, q \left(\boldsymbol{\Phi} \left(\mathfrak{c}\, h \frac{1-\sqrt{1-\theta}}{\sqrt{\theta}} \right) + \mathfrak{c}\, h \frac{\sqrt{1-\theta}}{\sqrt{\theta}} \boldsymbol{\Phi}' \left(h \frac{1-\sqrt{1-\theta}}{\sqrt{\theta}} \right) \right). \tag{A.4}$$

Integrating (A.4) with respect to $\theta \in [0,1]$, putting both partial limits into (A.3), dividing by the nonzero constant $-\mathfrak{c}\, q$ and by LEMMA 6.1, we finally obtain

$$S_t^* = \frac{rX}{q}(1 + h^* \sigma \sqrt{T-t}) + O(T-t) \quad \text{as } t \to T,$$

where $h^* \doteq 0.638833\mathfrak{c}$. This result (for plain vanilla call option) is fully consistent with that of Dewynne et al. (1993), Ševčovič (2001).

APPENDIX B

Greeks

"Timeo Danaos et dona ferentes!"

– PUBLIUS VERGILIUS MARO

Each portfolio has to be managed to reduce the risk the most efficiently possible. The movements of financial derivatives value can be followed and estimated by their sensitivities called Greeks (see e.g. Haug 2006). *Greeks* (or *Greek letters*) are used to hedge the risk caused by derivative contracts in portfolio. The more complex derivatives are included into portfolio, the more advanced indicators have to be used to secure the undesired losses. However, relying only on the Greeks can be unwise.

For the purpose of this section, we define following variables

$$d_1 = d_t \quad \text{and} \quad d_2 = d_t - \sigma\sqrt{T-t}, \tag{B.1}$$

where d_t is defined in (2.5). We recall that $\mathfrak{c} = 1$ for call and $\mathfrak{c} = -1$ for put option.

Greeks of the first, the second and the third order are summarized in table called the Mamma $\mathcal{M}$ (see TABLE B.1). This table shows the relationship of the more common Greeks to the four primary inputs of Black–Scholes model (the spot price of the underlying asset S, the volatility σ, the time to expiry $\tau = T - t$ and the risk-free interest rate r).

TABLE B.1: *The Mamma $\mathcal{M}$ table presenting relationship between Greeks.*

		$\frac{\partial}{\partial S}$	$\frac{\partial}{\partial \sigma}$	$\frac{\partial}{\partial \tau}$	$\frac{\partial}{\partial r}$
Value	V	Delta Δ	Vega $\mathcal{V}$	Theta $-\Theta$	Rho $\mathcal{P}$
Delta	Δ	Gamma Γ	Vanna	Charm	
Vega	$\mathcal{V}$	Vanna	Vomma	DvegaDtime	
Gamma	Γ	Speed	Zomma	Color	
Vomma			Ultima	Totto	

The most basic sensitivities used in the hedging are first order Greeks delta Δ, theta Θ, rho $\mathcal{P}$ and vega $\mathcal{V}$ and second order Greek gamma Γ. We can use these basic Greeks to rewrite the Black–Scholes partial differential equation (2.3) as

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + (r-q)S\Delta - rV = 0.$$

In following sections, we briefly present Greeks and illustrate their behavior on the European style plain vanilla option, Asian option with geometric average, approximation of Asian option with arithmetic average and lookback option. In FIGURE B.1 we present the value of an option and in FIGURES B.2-B.17 the dependence of Greeks on the value of underlying asset S for five time moments $0 = t_1 < t_2 < t_3 < t_4 < T$ (by dotted line, dot-dashed line, dashed line, solid line and thick solid line, respectively). We have fixed the value of average and extremes in figures for the Asian options and lookback option, respectively.

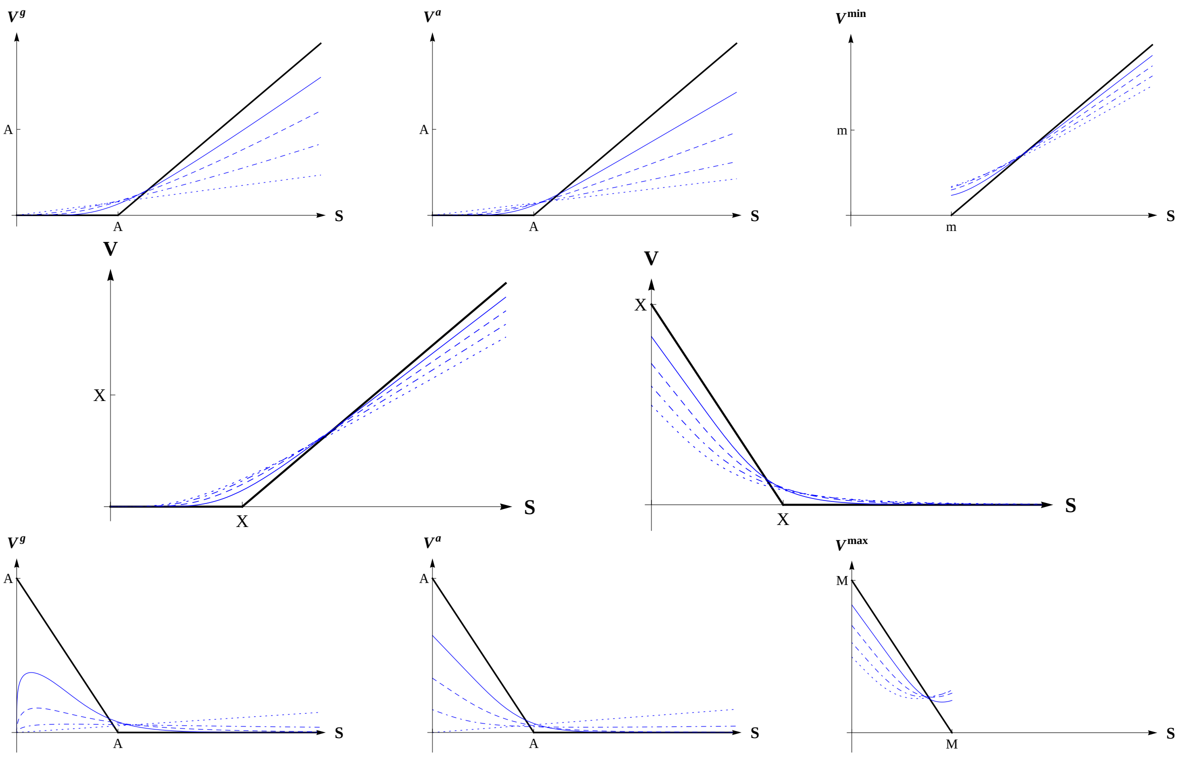

FIGURE B.1: *The value V of a European style call (center left) and put (center right) vanilla option contract. Call (above) and put (below) value of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

B.1 First order Greeks

Delta Δ

The Greek *delta* (see FIGURE B.2) is calculated as the first partial derivative of the value V according to the value of underlying asset S

$$\Delta = \frac{\partial V}{\partial S},$$

i.e. the slope of the option value function V.

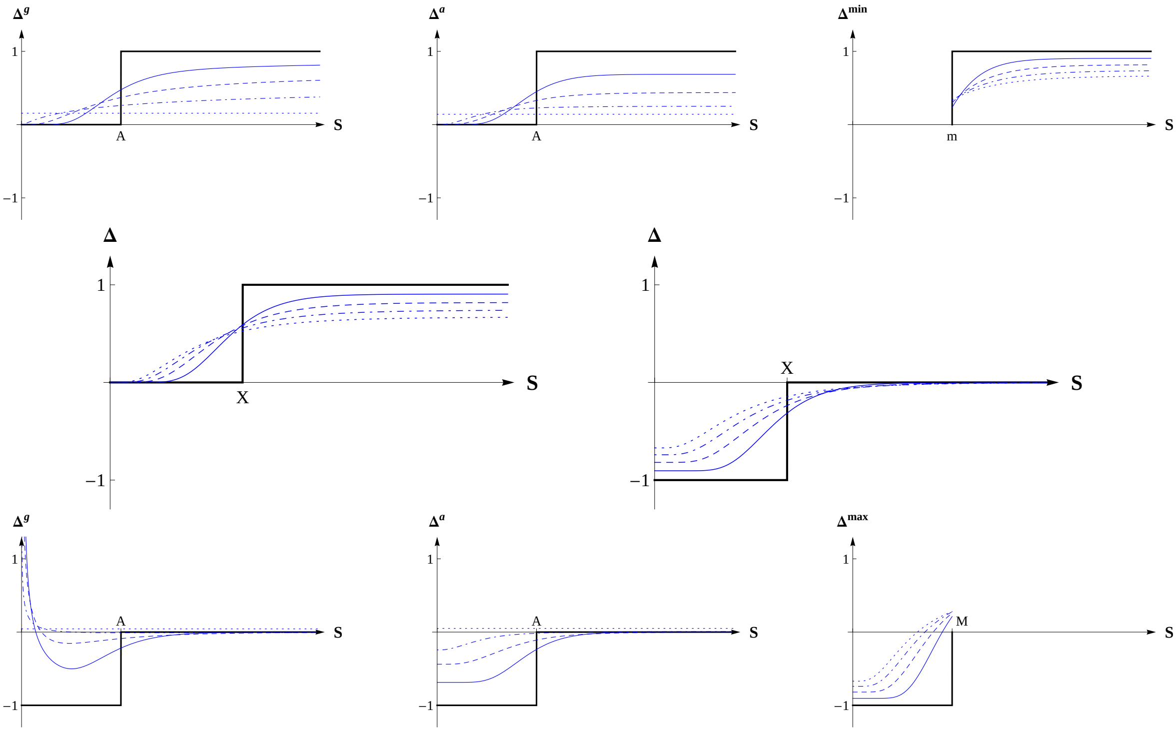

FIGURE B.2: *The Greek delta Δ of a European style call (center left) and put (center right) vanilla option contract. Call (above) and put (below) Greek delta Δ of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

To manage the portfolio by *delta hedging* we have to keep the equation

$$\Delta = -\frac{Q_t^S}{Q_t^V}, \tag{B.2}$$

where Q_t^S is the number of underlying assets at the time t and Q_t^V number of financial derivatives at the time t. Hedging of the portfolio by this scheme can be done either initially by Δ_0 (*static hedging*) or at certain time moments by Δ (*dynamic hedging*). The expression (B.2) is the same as the one used in derivation of Black–Scholes partial differential equation (in SECTION 7.1).

The value of Δ for European vanilla option is

$$\Delta^{EU} = \mathfrak{c}\, e^{-q(T-t)} \boldsymbol{\Phi}\left(\mathfrak{c}\, d_1\right),$$

where d_1 is defined in (B.1) and $\boldsymbol{\Phi}(\,\cdot\,)$ is the *CDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Theta Θ

The Greek *theta* (see FIGURE B.3) is the first partial derivative of the value V with respect to the time t

$$\Theta = \frac{\partial V}{\partial t}.$$

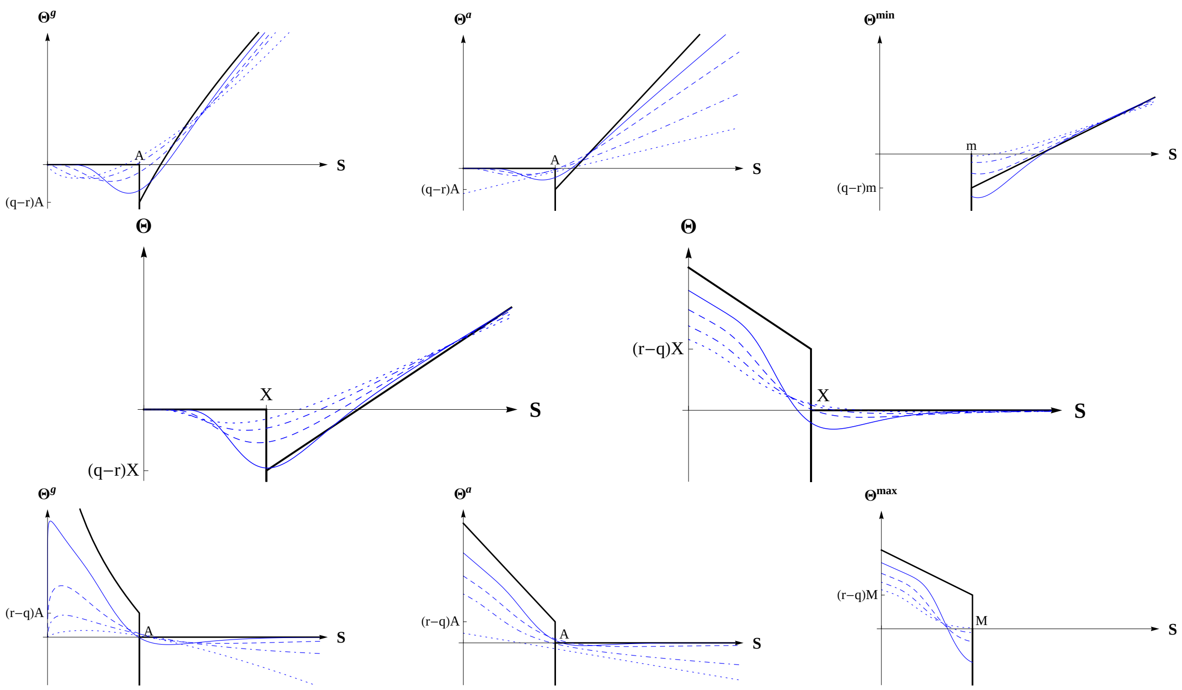

FIGURE B.3: *The Greek theta Θ of a European style call (center left) and put (center right) vanilla option contract. Call (above) and put (below) Greek theta Θ of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

As there is no uncertainty in the time flow, this factor is used in major as a descriptive statistics.

The value of Θ for European vanilla option is

$$\Theta^{EU} = \mathfrak{c}\left(qe^{-q(T-t)}S\mathbf{\Phi}\left(\mathfrak{c}\,d_1\right) - rXe^{-r(T-t)}\mathbf{\Phi}\left(\mathfrak{c}\,d_2\right)\right) - \frac{\sigma e^{-q(T-t)}S\,\mathit{\Phi}\left(d_1\right)}{2\sqrt{T-t}},$$

where d_1 and d_2 are defined in (B.1), $\mathbf{\Phi}(\,\cdot\,)$ and $\mathit{\Phi}(\,\cdot\,)$ are the *CDF* and *PDF* of the normal probability distribution $\mathcal{N}(0,1)$, respectively.

Vega $\mathcal{V}$

The Greek *vega* (see FIGURE B.4) is the first partial derivative of the value V with respect to the volatility σ

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}.$$

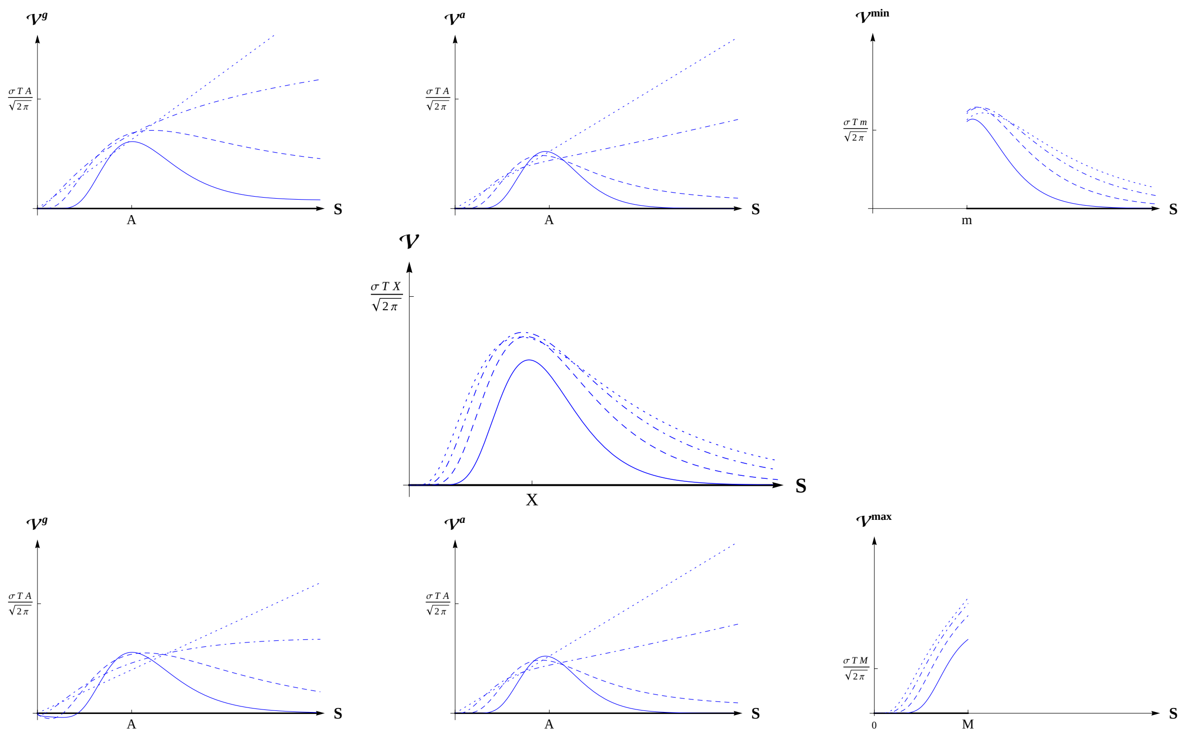

FIGURE B.4: *The Greek vega $\mathcal{V}$ of a European style vanilla option contract (center). Call (above) and put (below) Greek vega $\mathcal{V}$ of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of $\mathcal{V}$ for European vanilla option is

$$\mathcal{V}^{EU} = e^{-q(T-t)} S \,\Phi\left(d_1\right) \sqrt{T-t},$$

where d_1 is defined in (B.1) and $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Rho $\mathcal{P}$

The Greek *rho* (see FIGURE B.5) is the first partial derivative of the value V with respect to the interest rate r

$$\mathcal{P} = \frac{\partial V}{\partial r}.$$

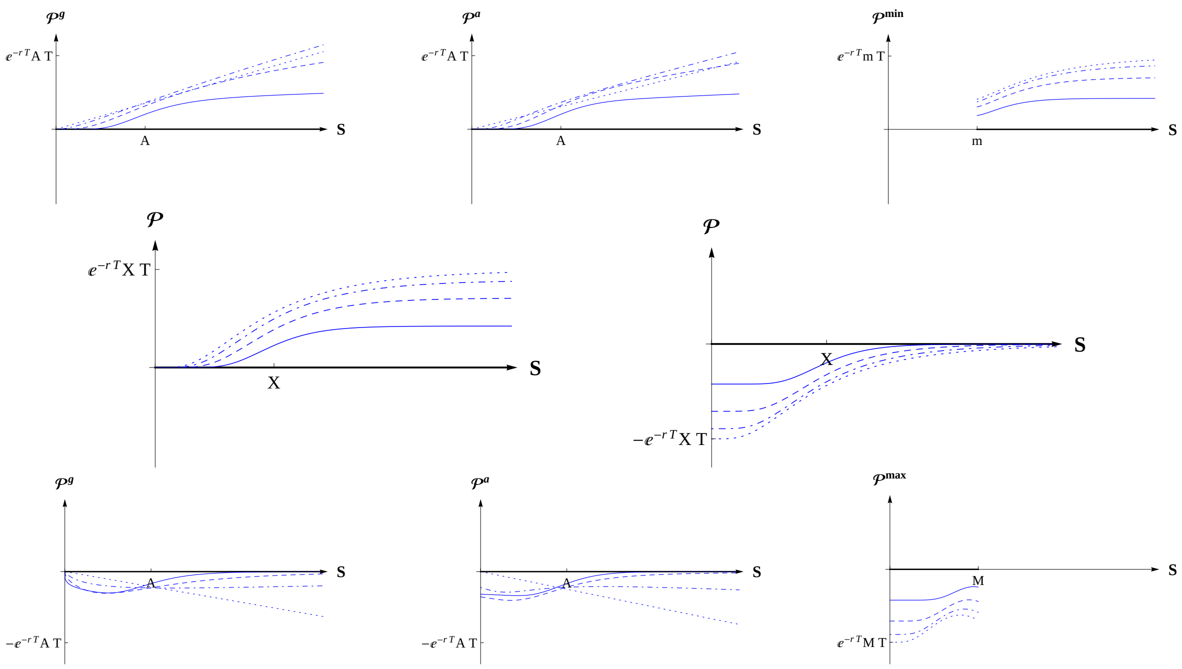

FIGURE B.5: *The Greek rho $\mathcal{P}$ of a European style call (center left) and put (center right) vanilla option contract. Call (above) and put (below) Greek rho $\mathcal{P}$ of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of $\mathcal{P}$ for European vanilla option is

$$\mathcal{P}^{EU} = \mathfrak{c}(T-t)e^{-r(T-t)} X \boldsymbol{\Phi}\left(\mathfrak{c}\, d_2\right),$$

where d_2 is defined in (B.1) and $\boldsymbol{\Phi}(\,\cdot\,)$ is the *CDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Dual delta Δ_{dual}

The Greek *dual delta* (see FIGURE B.6) is the first derivative of the value V with respect to the strike price of derivative X

$$\Delta_{dual} = \frac{\partial V}{\partial X}.$$

[Alternatively for Asian and lookback options, strike price X can be replaced by average and extreme value, respectively.]

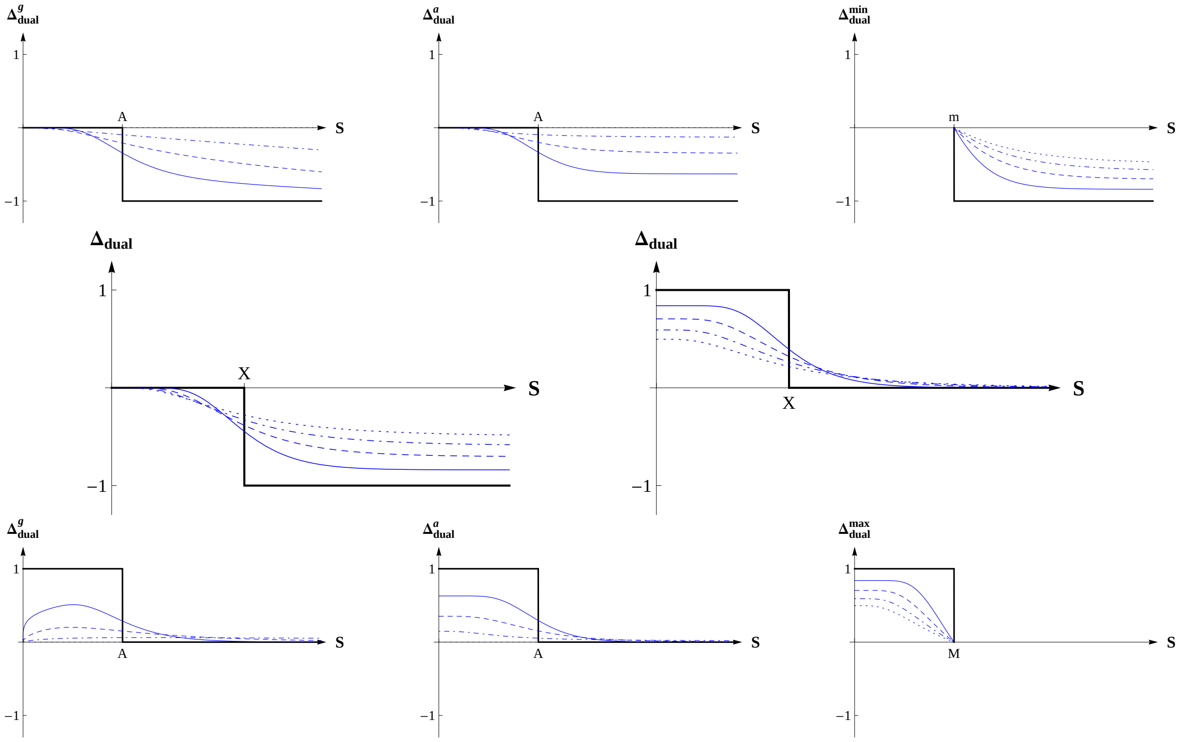

FIGURE B.6: *The Greek dual delta Δ_{dual} of a European style call (center left) and put (center right) vanilla option contract. Call (above) and put (below) Greek dual delta Δ_{dual} of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of Δ_{dual} for European vanilla option is

$$\Delta_{dual}^{EU} = \mathfrak{c}\, e^{-r(T-t)} \mathbf{\Phi}\left(\mathfrak{c}\, d_2\right),$$

where d_2 is defined in (B.1) and $\mathbf{\Phi}(\,\cdot\,)$ is the *CDF* of the normal probability distribution $\mathcal{N}(0,1)$.

B.2 Second order Greeks

Gamma Γ

The Greek *gamma* (see FIGURE B.7) is defined as the second partial derivative of the value V with respect to the value of underlying asset S

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}.$$

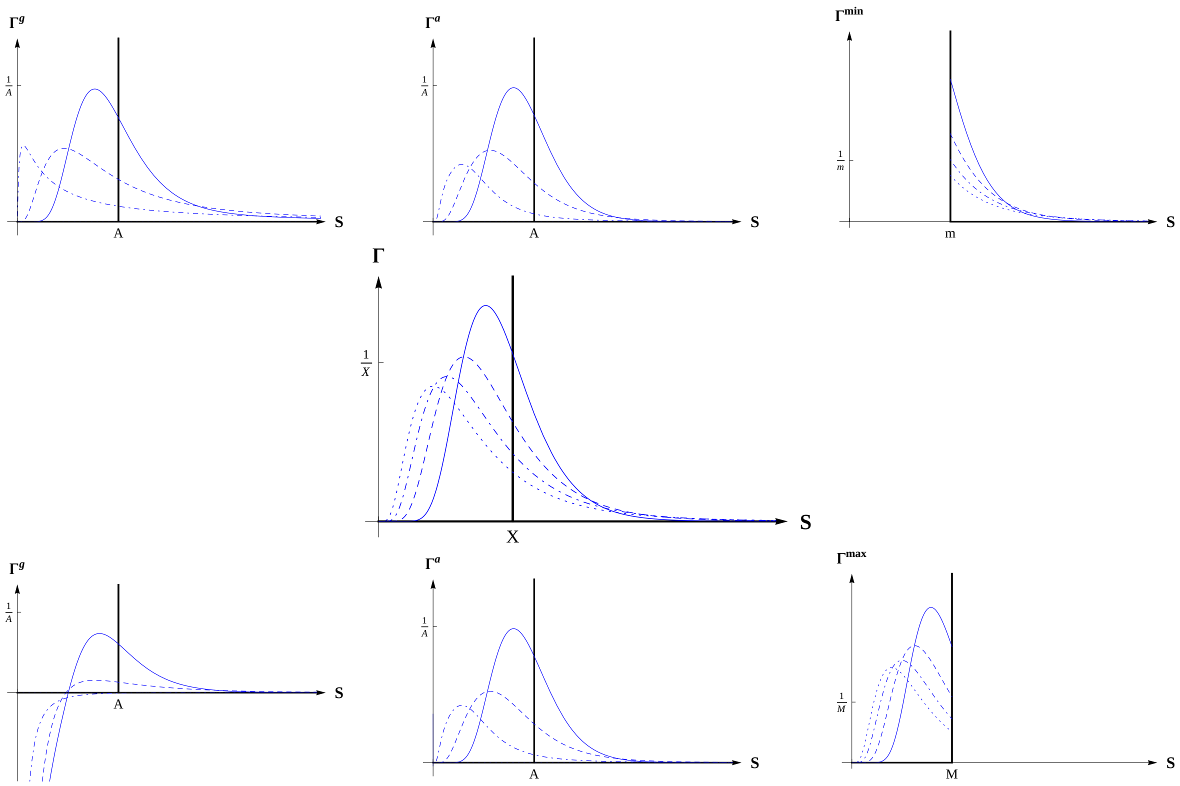

FIGURE B.7: *The Greek gamma Γ of a European style vanilla option contract (center). Call (above) and put (below) Greek gamma Γ of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

Gamma is the rate of change of Δ. These two factors are used together to manage the portfolio. If the absolute value of gamma is low, the delta changes only a little and there is no need to perform delta hedging so frequently as for the high absolute value of gamma.

The value of Γ for European vanilla option is

$$\Gamma^{EU} = e^{-q(T-t)} \frac{\Phi\left(d_1\right)}{S\sigma\sqrt{T-t}},$$

where d_1 is defined in (B.1) and $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Charm

The Greek *charm* (see FIGURE B.8) is the first derivative of Δ with respect to the variable $\tau = T - t$

$$Charm = \frac{\partial \Delta}{\partial \tau} = -\frac{\partial \Theta}{\partial S} = \frac{\partial^2 V}{\partial S \partial \tau}.$$

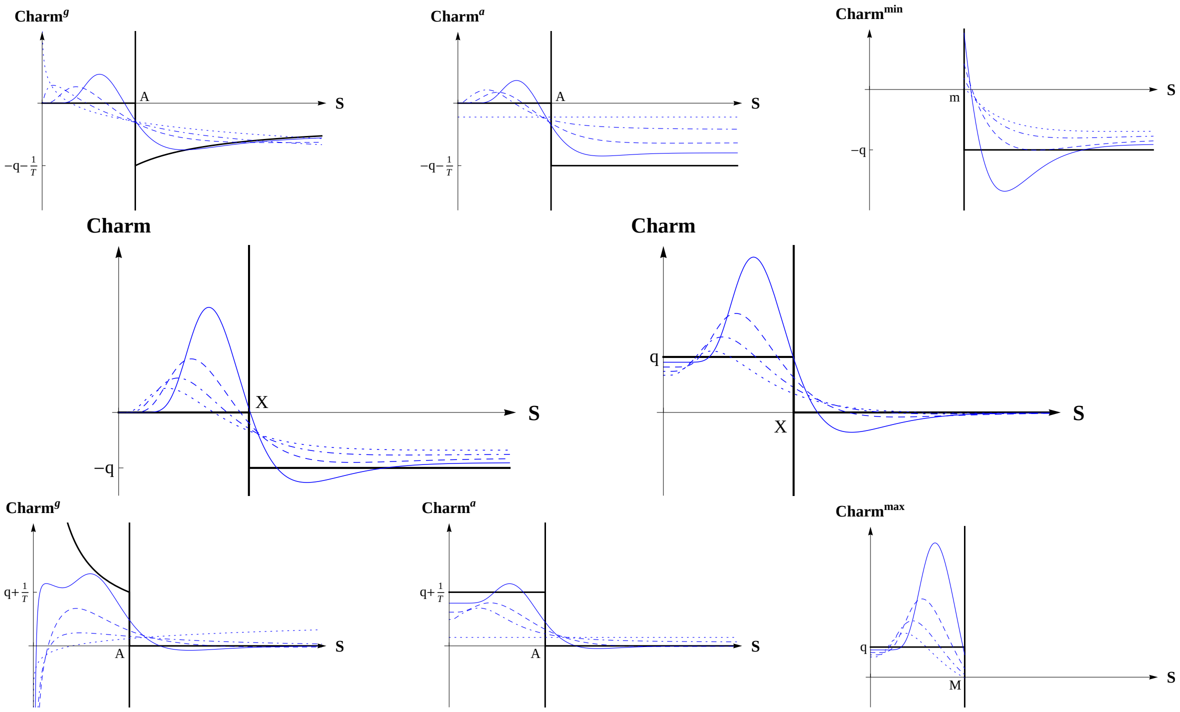

FIGURE B.8: *The Greek charm of a European style call (center left) and put (center right) vanilla option contract. Call (above) and put (below) Greek charm of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *charm* for European vanilla option is

$$Charm^{EU} = -\mathfrak{c}\, q e^{-q(T-t)} \boldsymbol{\Phi}\left(\mathfrak{c}\, d_1\right) + e^{-q(T-t)} \Phi\left(d_1\right) \left(\frac{r-q}{\sigma\sqrt{T-t}} - \frac{d_2}{2(T-t)} \right),$$

where d_1 and d_2 are defined in (B.1), $\mathbf{\Phi}(\,\cdot\,)$ and $\Phi(\,\cdot\,)$ are the *CDF* and the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$, respectively.

Vanna

The Greek *vanna* (see FIGURE B.9) is the first derivative of Δ with respect to the volatility σ

$$Vanna = \frac{\partial \Delta}{\partial \sigma} = \frac{\partial \mathcal{V}}{\partial S} = \frac{\partial^2 V}{\partial S \partial \sigma}.$$

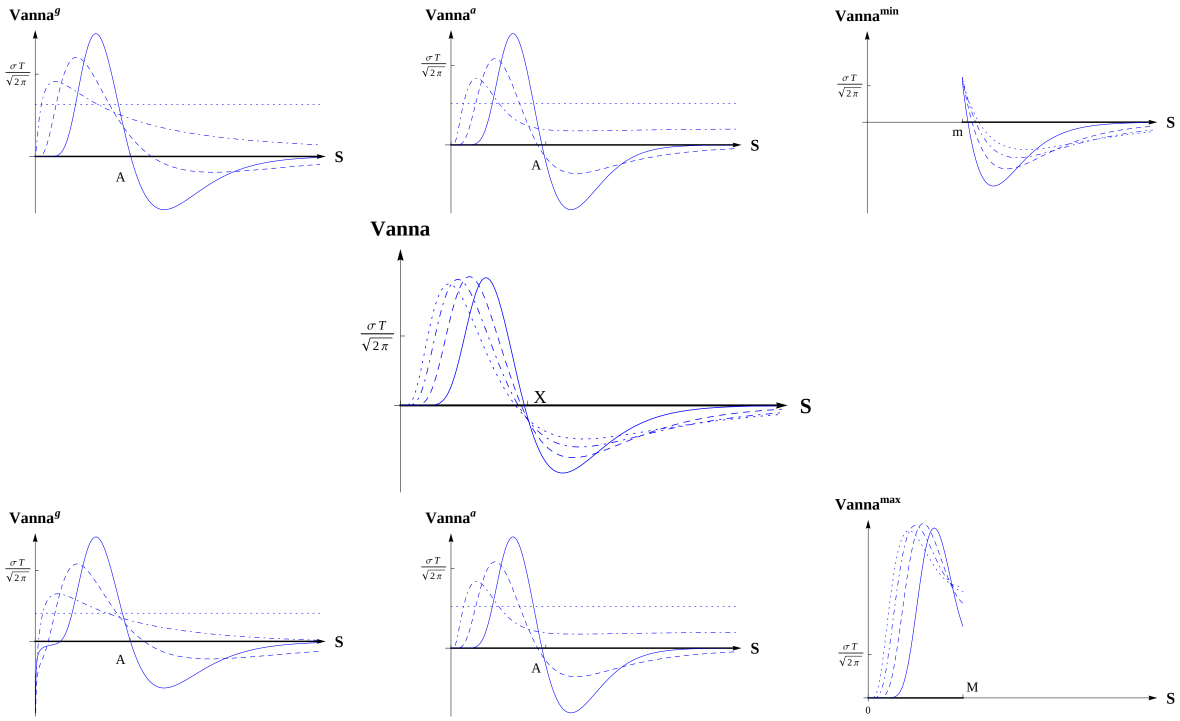

FIGURE B.9: *The Greek vanna of a European style vanilla option contract (center). Call (above) and put (below) Greek vanna of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *vanna* for European vanilla option is

$$Vanna^{EU} = -e^{-q(T-t)}\,\Phi\left(d_1\right)\frac{d_2}{\sigma} = \frac{\mathcal{V}}{S}\left(1 - \frac{d_1}{\sigma\sqrt{T-t}}\right),$$

where d_1 and d_2 are defined in (B.1), $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Vomma

The Greek *vomma* (see FIGURE B.10) is defined as the second partial derivative of the value V with respect to the volatility σ

$$Vomma = \frac{\partial \mathcal{V}}{\partial \sigma} = \frac{\partial^2 V}{\partial \sigma^2}.$$

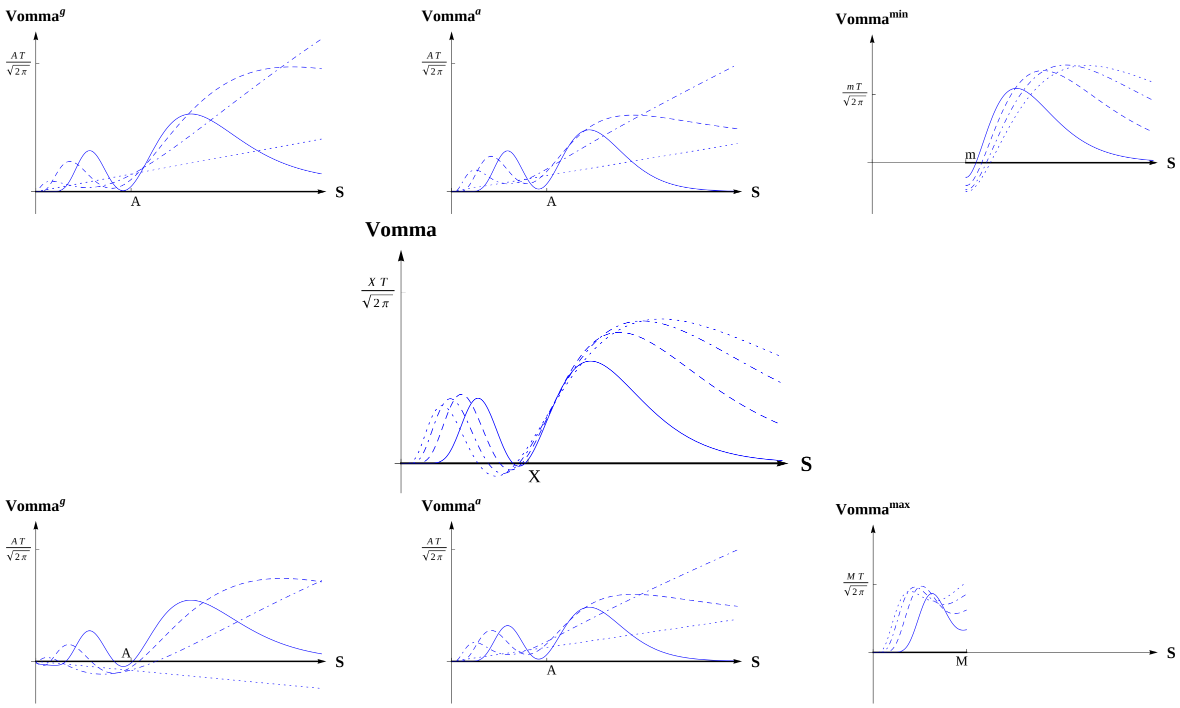

FIGURE B.10: *The Greek vomma of a European style vanilla option contract (center). Call (above) and put (below) Greek vomma of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *vomma* for European vanilla option is

$$Vomma^{EU} = e^{-q(T-t)} S \Phi\left(d_1\right) \frac{d_1 d_2}{\sigma} \sqrt{T-t} = \mathcal{V} \frac{d_1 d_2}{\sigma},$$

where d_1 and d_2 are defined in (B.1), $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

DvegaDtime $\frac{\partial \mathcal{V}}{\partial \tau}$

The Greek *DvegaDtime* (see FIGURE B.11) is the first derivative of $\mathcal{V}$ with respect to the variable $\tau = T - t$

$$DvegaDtime = \frac{\partial \mathcal{V}}{\partial \tau} = -\frac{\partial \Theta}{\partial \sigma} = \frac{\partial^2 V}{\partial \sigma \partial \tau}.$$

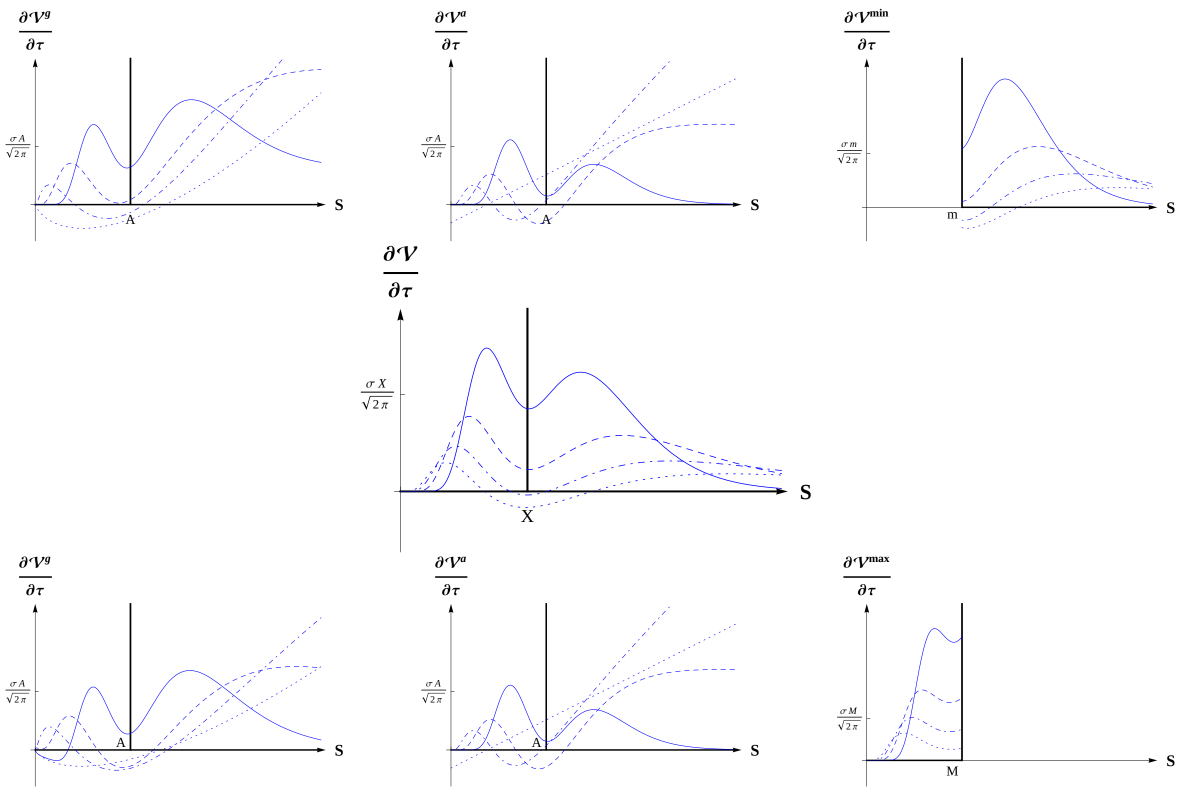

FIGURE B.11: *The Greek DvegaDtime of a European style vanilla option contract (center). Call (above) and put (below) Greek DvegaDtime of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *DvegaDtime* for European vanilla option is

$$DvegaDtime^{EU} = e^{-q(T-t)} S\,\Phi(d_1)\sqrt{T-t}\left(q + \frac{(r-q)d_1}{\sigma\sqrt{T-t}} - \frac{1+d_1 d_2}{2(T-t)}\right),$$

where d_1 and d_2 are defined in (B.1), $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Dual gamma Γ_{dual}

The Greek *dual gamma* (see FIGURE B.12) is defined as the second partial derivative of the value V with respect to the strike price X

$$\Gamma_{dual} = \frac{\partial \Delta_{dual}}{\partial X} = \frac{\partial^2 V}{\partial X^2}.$$

Dual gamma is the rate of change of Δ_{dual}. [Alternatively for Asian and lookback options, strike price X can be replaced by average and extreme value, respectively.]

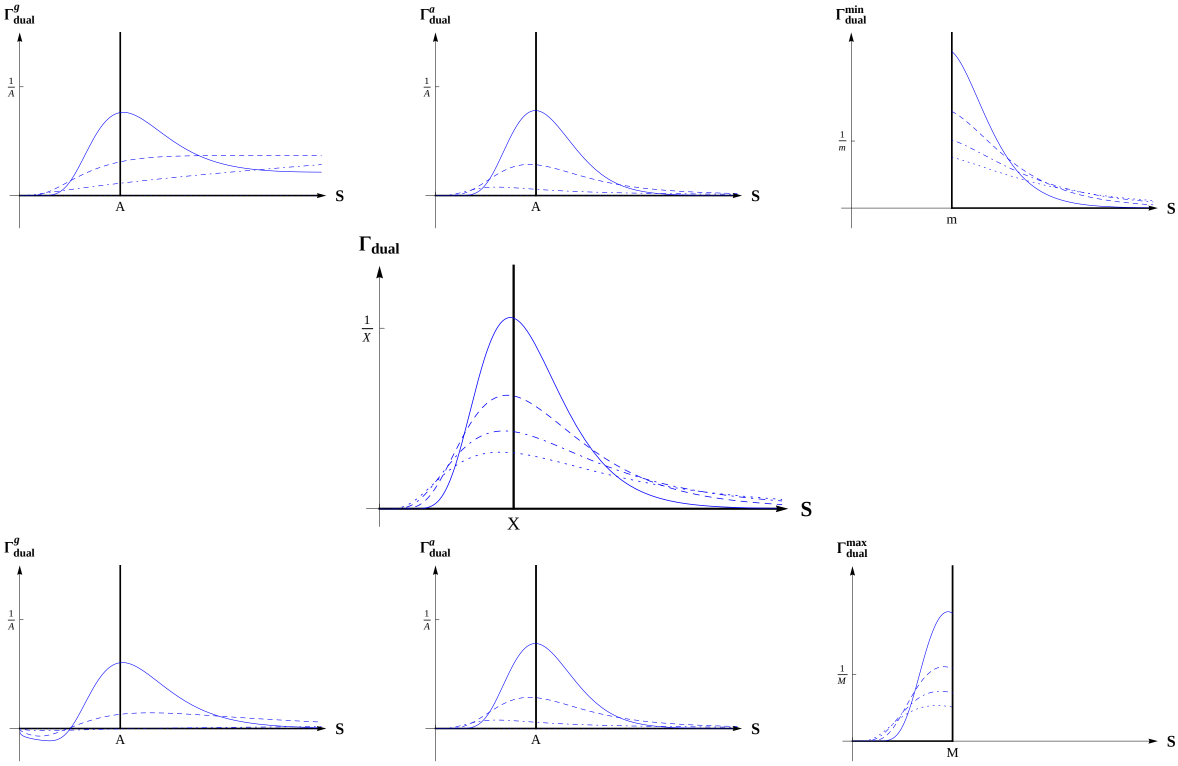

FIGURE B.12: *The Greek dual gamma Γ_{dual} of a European style vanilla option contract (center). Call (above) and put (below) Greek dual gamma Γ_{dual} of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of Γ_{dual} for European vanilla option is

$$\Gamma^{EU}_{dual} = e^{-r(T-t)} \frac{\Phi\left(d_2\right)}{X\sigma\sqrt{T-t}},$$

where d_2 is defined in (B.1) and $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

B.3 Third order Greeks

Speed

The Greek *speed* (see FIGURE B.13) is defined as the third partial derivative of the value V with respect to the value of underlying asset S

$$Speed = \frac{\partial \Gamma}{\partial S} = \frac{\partial^2 \Delta}{\partial S^2} = \frac{\partial^3 V}{\partial S^3}.$$

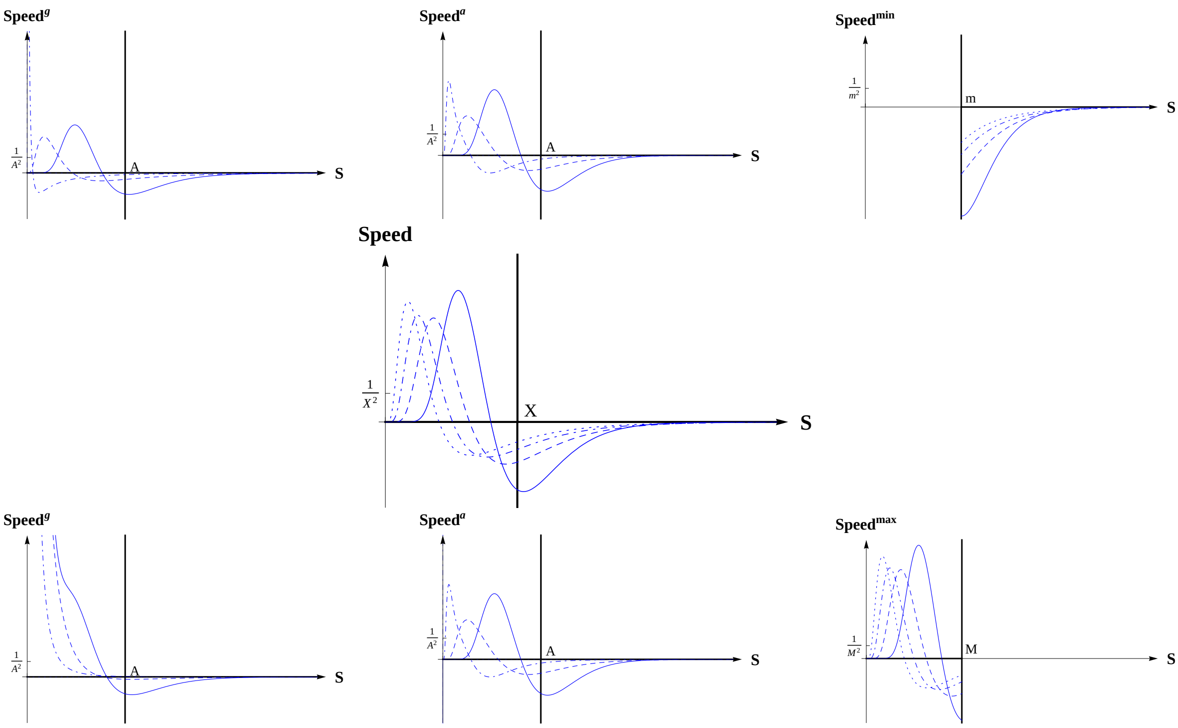

FIGURE B.13: *The Greek speed of a European style vanilla option contract (center). Call (above) and put (below) Greek speed of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *speed* for European vanilla option is

$$Speed^{EU} = -e^{-q(T-t)} \frac{\Phi(d_1)}{S^2 \sigma \sqrt{T-t}} \left(\frac{d_1}{\sigma\sqrt{T-t}} + 1 \right) = -\frac{\Gamma}{S} \left(\frac{d_1}{\sigma\sqrt{T-t}} + 1 \right),$$

where d_1 is defined in (B.1), $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Color

The Greek *color* (see FIGURE B.14) is the first derivative of Γ with respect to the variable $\tau = T - t$

$$Color = \frac{\partial \Gamma}{\partial \tau} = \frac{\partial^2 \Delta}{\partial S \partial \tau} = -\frac{\partial^2 \Theta}{\partial S^2} = \frac{\partial^3 V}{\partial S^2 \partial \tau}.$$

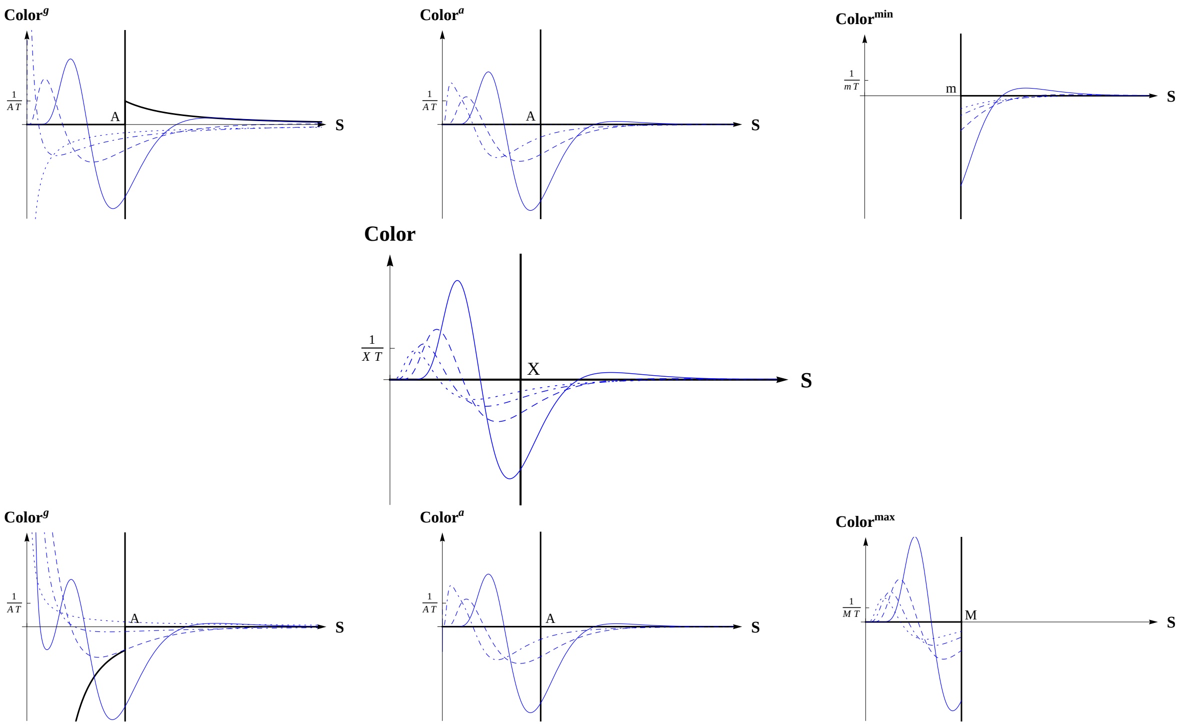

FIGURE B.14: *The Greek color of a European style vanilla option contract (center). Call (above) and put (below) Greek color of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *color* for European vanilla option is

$$Color^{EU} = -e^{-q(T-t)} \frac{\Phi\,(d_1)}{2S\sigma(T-t)^{\frac{3}{2}}} \left(1 + 2q(T-t) - d_1 d_2 + d_1 \frac{2(r-q)\sqrt{T-t}}{\sigma}\right),$$

where d_1 and d_2 are defined in (B.1), $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Zomma

The Greek *zomma* (see FIGURE B.15) is the first derivative of Γ with respect to the volatility σ

$$Zomma = \frac{\partial \Gamma}{\partial \sigma} = \frac{\partial Vanna}{\partial S} = \frac{\partial^2 \Delta}{\partial S \partial \sigma} = \frac{\partial^2 \mathcal{V}}{\partial S^2} = \frac{\partial^3 V}{\partial S^2 \partial \sigma}.$$

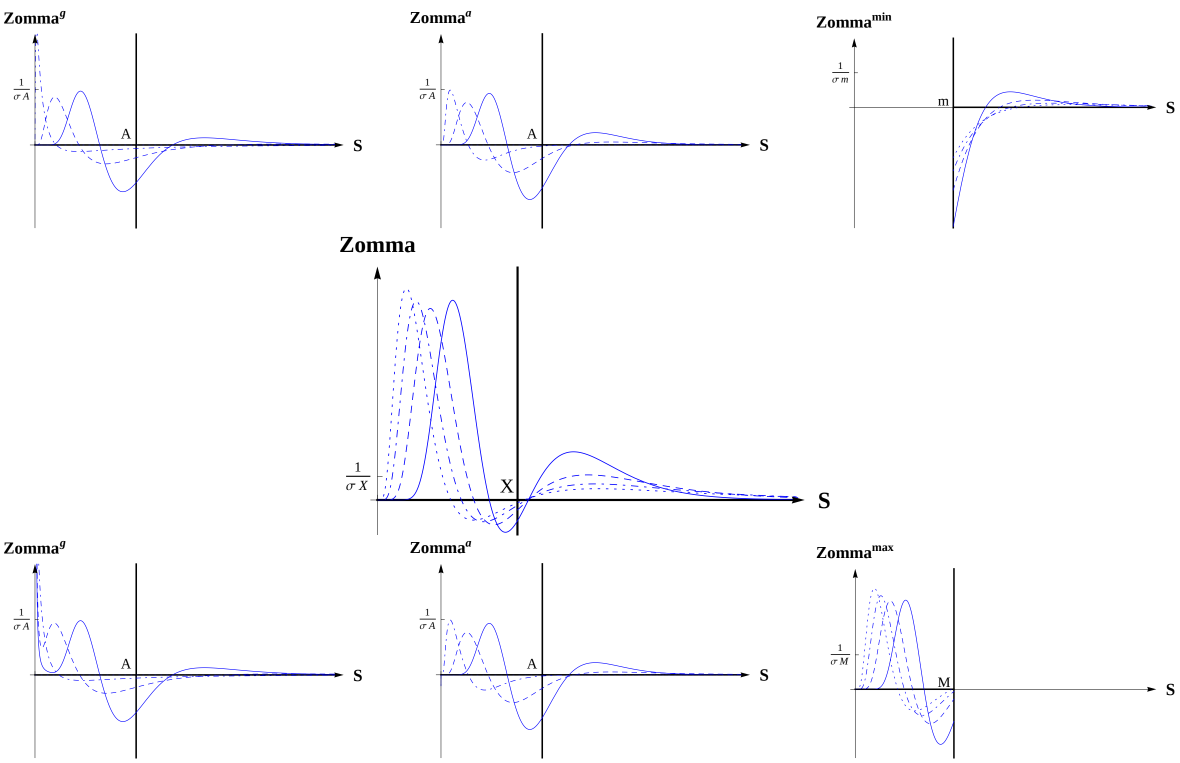

FIGURE B.15: *The Greek zomma of a European style vanilla option contract (center). Call (above) and put (below) Greek zomma of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *zomma* for European vanilla option is

$$Zomma^{EU} = e^{-q(T-t)} \Phi(d_1) \frac{d_1 d_2 - 1}{S\sigma^2 \sqrt{T-t}} = \Gamma \frac{d_1 d_2 - 1}{\sigma},$$

where d_1 and d_2 are defined in (B.1), $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Ultima

The Greek *ultima* (see FIGURE B.16) is defined as the third partial derivative of the value V with respect to the volatility σ

$$Ultima = \frac{\partial Vomma}{\partial \sigma} = \frac{\partial^2 \mathcal{V}}{\partial \sigma^2} = \frac{\partial^3 V}{\partial \sigma^3}.$$

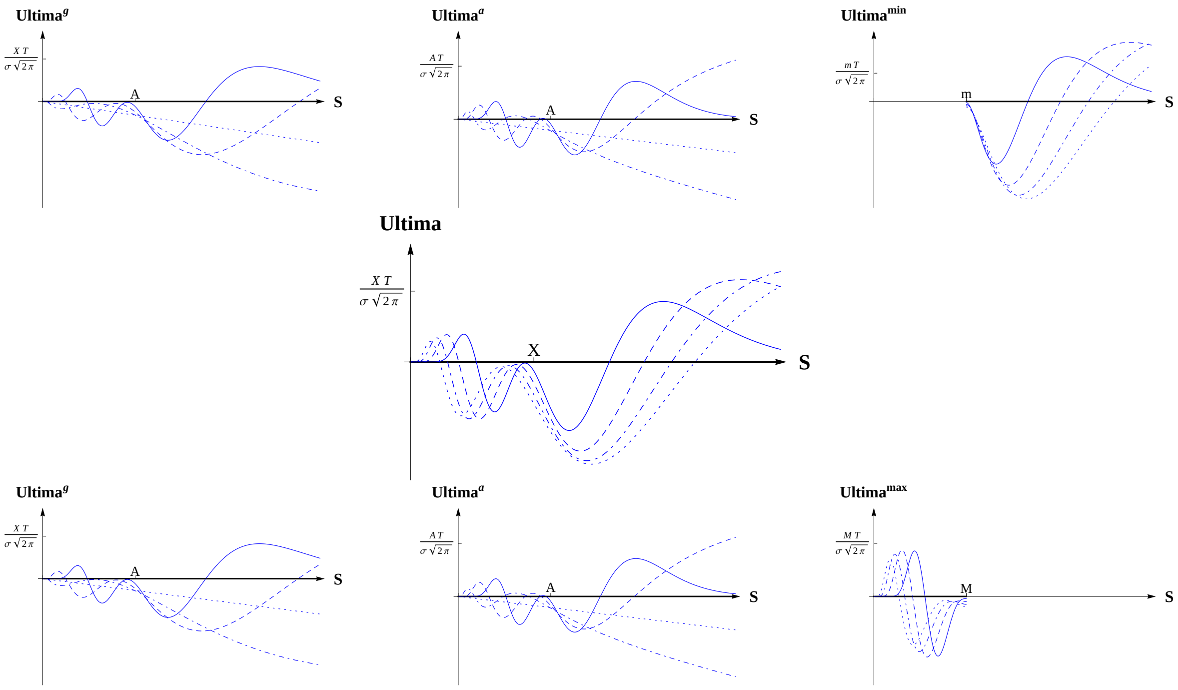

FIGURE B.16: *The Greek ultima of a European style vanilla option contract (center). Call (above) and put (below) Greek ultima of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *ultima* for European vanilla option is

$$\begin{aligned} Ultima^{EU} &= e^{-q(T-t)}\,\Phi\left(d_1\right)\frac{S\sqrt{T-t}}{\sigma^2}\left(d_1^2 d_2^2 - d_1 d_2 - d_1^2 - d_2^2\right) \\ &= \frac{\mathcal{V}}{\sigma^2}\left(d_1^2 d_2^2 - d_1 d_2 - d_1^2 - d_2^2\right), \end{aligned}$$

where d_t is defined in (B.1), $\Phi(\,\cdot\,)$ is the *PDF* of the normal probability distribution $\mathcal{N}(0,1)$.

Totto

The Greek *totto* (see FIGURE B.17) is the first derivative of $vomma$ with respect to the variable $\tau = T - t$

$$Totto = \frac{\partial Vomma}{\partial \tau} = -\frac{\partial^2 \Theta}{\partial \sigma^2} = \frac{\partial^2 \mathcal{V}}{\partial \sigma \partial \tau} = \frac{\partial^3 V}{\partial \sigma^2 \partial \tau}.$$

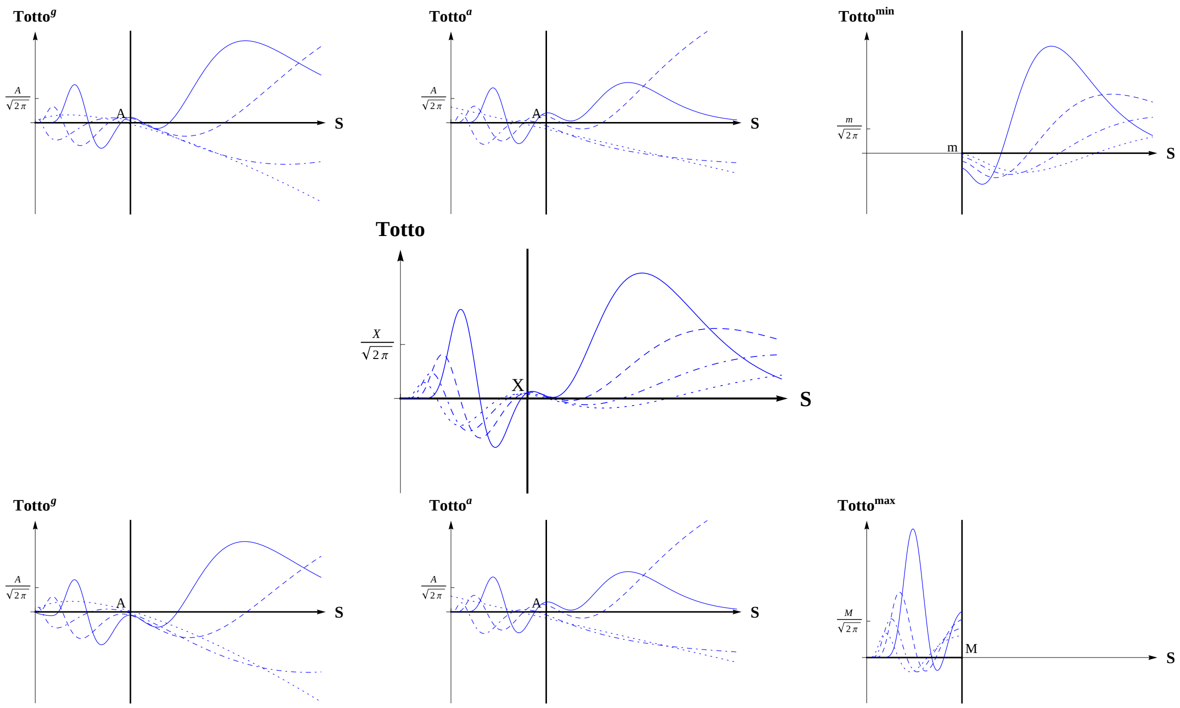

FIGURE B.17: *The Greek totto of a European style vanilla option contract (center). Call (above) and put (below) Greek totto of Asian option contract with geometric averaging (left) and arithmetic averaging approximation (middle) and lookback option (right).*

The value of *totto* for European vanilla option is

$$Totto^{EU} = \frac{d_1 d_2}{\sigma} \frac{\partial \mathcal{V}}{\partial \tau} + \frac{\mathcal{V}}{\sigma} \left(d_2 \frac{\partial d_1}{\partial \tau} + d_1 \frac{\partial d_2}{\partial \tau} \right),$$

where d_1 and d_2 are defined in (B.1), $\Phi(\ \cdot\ )$ is the *PDF* of the normal probability distribution $\mathcal{N}(0, 1)$.

B.4 Other Greeks

Lambda Λ

The Greek *lambda* is elasticity of value of financial derivative according to the price of underlying asset S

$$\Lambda = \frac{S}{V}\frac{\partial V}{\partial S} = \frac{S}{V}\Delta.$$

Option duration

Option duration, also called *omega* is the optimal time to exercise the American or Bermudan style financial derivative. It is also the expected time of hitting the barrier for knock-out barrier options.

Proofs

"Dubito ergo cogito, cogito ergo sum."

– RENÉ DU PERRON DESCARTES

In this chapter, we present proofs for theorems and lemmas presented in the main part of thesis.

C.1 A probabilistic model for pricing American style options

PROOF of THEOREM 4.1 We follow the proof of result for Asian options presented by Hansen and Jørgensen (2000) and we include necessary generalizing modifications.

In the proof, we use the variable $\widetilde{V} = \mathcal{N}^{-1}V = \mathcal{N}^{-1}v + \mathcal{N}^{-1}e = \widetilde{v} + \widetilde{e}$ instead of $V = v + e$.

First, we suppose that the value is in the continuation region, i.e. $(t, x) \in \mathcal{C}$. The derivative is held and so we can apply Itô lemma to calculate the differential

$$\begin{aligned}
d\widetilde{V}\big|_{\mathcal{C}} &= \frac{\partial \widetilde{V}}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial \widetilde{V}}{\partial x^i}dx^i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 \widetilde{V}}{\partial x^i \partial x^j}dx^i dx^j \\
&= \left(\frac{\partial \widetilde{V}}{\partial t} + \sum_{i=1}^{n}\mu^i\frac{\partial \widetilde{V}}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}\sigma^i\sigma^j\frac{\partial^2 \widetilde{V}}{\partial x^i \partial x^j}\right)dt + \sum_{i=1}^{n}\sigma^i\frac{\partial \widetilde{V}}{\partial x^i}dW^i \\
&= \sum_{i=1}^{n}\sigma^i\frac{\partial \widetilde{V}}{\partial x^i}dW^i,
\end{aligned}$$

where the last equality holds true, because $\widetilde{V}\big|_{\mathcal{C}}$ is $\mathcal{Q}$-martingale.

Now we suppose that the value is in the stopping region, i.e. $(t, x) \in \mathcal{S}$. The value of derivative is defined by

$$\widetilde{V}(t, x_t)\big|_{\mathcal{S}} = (\mathcal{N}(t, x_t))^{-1}\,\Omega(t, x_t).$$

Hence the differential $d\widetilde{V}\big|_{\mathcal{S}} = d\left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)}\right)$ has the form

$$\begin{aligned} d\widetilde{V}\big|_{\mathcal{S}} &= \frac{\partial\left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)}\right)}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial\left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)}\right)}{\partial x^i}dx^i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)}\right)}{\partial x^i\partial x^j}dx^i dx^j \\ &= \underbrace{\left(\frac{\partial\left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)}\right)}{\partial t} + \sum_{i=1}^{n}\mu^i\frac{\partial\left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)}\right)}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}\sigma^i\sigma^j\frac{\partial^2\left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)}\right)}{\partial x^i\partial x^j}\right)dt}_{f_d(t,x_t)} \\ &\quad + \sum_{i=1}^{n}\sigma^i\frac{\partial\left(\frac{\Omega(t,x_t)}{\mathcal{N}(t,x_t)}\right)}{\partial x^i}dW^i. \end{aligned}$$

For both regions the following stochastic equation has to be satisfied:

$$d\widetilde{V}(t,x_t) = \mathbf{1}_{\mathcal{S}}(t,x_t)\ f_d(t,x_t)\ dt + dM_t^{\mathcal{Q}}, \tag{C.1}$$

where $M_t^{\mathcal{Q}}$ is a $\mathcal{Q}$-martingale. Integrating (C.1) from t to T and taking expectation we have

$$\mathbb{E}_t^{\mathcal{Q}}\left[\widetilde{V}(T,x_T)\right] - \widetilde{V}(t,x_t) = \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \mathbf{1}_{\mathcal{S}}(u,x_u)f_d(u,x_u)\,du\right] + \underbrace{\mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T dM_u^{\mathcal{Q}}\right]}_{=0},$$

after rearranging elements, we have

$$\begin{aligned} \widetilde{V}(t,x_t) &= \mathbb{E}_t^{\mathcal{Q}}\left[\widetilde{V}(T,x_T)\right] - \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \mathbf{1}_{\mathcal{S}}(u,x_u)f_d(u,x_u)\,du\right] \\ &= \underbrace{\mathbb{E}_t^{\mathcal{Q}}\left[(\mathcal{N}(T,x_T))^{-1}\,\Omega(T,x_T)\right]}_{=\widetilde{v}(t,x_t)} + \underbrace{\mathbb{E}_t^{\mathcal{Q}}\left[-\int_t^T \mathbf{1}_{\mathcal{S}}(u,x_u)f_d(u,x_u)\,du\right]}_{=\widetilde{e}(t,x_t)}. \end{aligned}$$

which completes the proof of THEOREM 4.1. □

In the proof of THEOREM 4.2 we shall use the following lemma.

LEMMA C.1. *The auxiliary variable $x_t = \frac{A_t}{S_t}$ satisfies the following stochastic differential equation:*

$$dx_t = x_t\frac{dA_t}{A_t} - (r-q)x_t\,dt - \sigma x_t\,dW_t^{\mathcal{Q}}.$$

PROOF of LEMMA C.1 We express the differential $dx_t = d\left(\frac{A_t}{S_t}\right)$ as

$$dx_t = \frac{1}{S_t} dA_t - \frac{A_t}{S_t^2} dS_t + \frac{A_t}{S_t^3}(dS_t)^2 = x_t \frac{dA_t}{A_t} - (r-q)x_t\, dt - \sigma x_t\, dW_t^{\mathcal{Q}},$$

and the proof of lemma follows. □

Notice that, when comparing to the original expression due to Hansen and Jørgensen (2000) with a zero dividend rate $q = 0$, the only difference is that the parameter r is replaced by the term $r - q$.

PROOF of THEOREM 4.2 In the proof, we use THEOREM 4.1 and LEMMA C.1. According to (4.19) we have the numeraire and pay-off function

$$\mathcal{N}(t, x_t) = e^{qt}$$

and

$$\Omega(t, x_t) = (\mathfrak{c}(1 - x_t))^+ ,$$

respectively. Moreover, we have the stochastic differential equation

$$dx_t = x_t \frac{dA_t}{A_t} - (r-q)x_t\, dt - \sigma x_t\, dW_t^{\mathcal{Q}} = \widetilde{\mu}\, dt - \widetilde{\sigma}\, dW_t^{\mathcal{Q}}.$$

Then the function (4.13) becomes on the stopping region $\mathcal{S}$

$$\begin{aligned} f_d(t, x_t)\, dt &= \mathfrak{c}(1 - x_t) \frac{\partial\left(e^{-qt}\right)}{\partial t}\, dt + \mathfrak{c}\, e^{-qt} \widetilde{\mu} \frac{\partial\left((1 - x_t)\right)}{\partial x}\, dt + \frac{1}{2} \mathfrak{c}\, e^{-qt} \widetilde{\sigma}^2 \frac{\partial^2\left((1 - x_t)\right)}{\partial x^2}\, dt \\ &= -\mathfrak{c}\, q e^{-qt}(1 - x_t)\, dt - \mathfrak{c}\, e^{-qt} \widetilde{\mu}\, dt \\ &= -\mathfrak{c}\, e^{-qt} \left(x_t \frac{dA_t}{A_t} + (q - r x_t)\ dt \right). \end{aligned}$$

And the proof of theorem follows. □

PROOF of LEMMA 4.1 Consider a stochastic variable $z = \ln Z \sim \mathcal{N}(\alpha, \beta^2)$. Recall that

$$\gamma_p \equiv \frac{\ln K - \alpha}{\beta} - p\beta,$$

where $K > 0$ and $p \in \mathbb{R}$. Functions $\mathbf{\Phi}(\,\cdot\,)$ and $\mathit{\Phi}(\,\cdot\,)$ denote standard normal cumulative distribution and density functions, respectively. The PDF of normal distribution $\mathcal{N}(\alpha, \beta^2)$ has form

$$\mathit{\Phi}(z) = \frac{1}{\sqrt{2\pi}\beta} e^{-\frac{(z-\alpha)^2}{2\beta^2}}.$$

We begin with item (iii):

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{Z\le K\}}Z^p\right] &= \int_{-\infty}^{\ln K} e^{pz}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz \\
&= \int_{-\infty}^{\ln K} \frac{e^{p\alpha+\frac{p^2\beta^2}{2}}}{\sqrt{2\pi}\beta}e^{-\frac{(z-(\alpha+p\beta^2))^2}{2\beta^2}}\,dz \\
&= e^{p\alpha+\frac{p^2\beta^2}{2}}\int_{-\infty}^{\frac{\ln K-(\alpha+p\beta^2)}{\beta}} \frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}\,d\xi \\
&= e^{p\alpha+\frac{p^2\beta^2}{2}}\mathbf{\Phi}\left(\frac{\ln K-\alpha}{\beta}-p\beta\right) \\
&= e^{p\alpha+\frac{p^2\beta^2}{2}}\mathbf{\Phi}(\gamma_p).
\end{aligned}$$

If we set $p=0$, we have item (i). Item (iv) is calculated by

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{Z\ge K\}}Z^p\right] &= \int_{\ln K}^{\infty} e^{pz}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz \\
&= e^{p\alpha+\frac{p^2\beta^2}{2}}\left(1-\mathbf{\Phi}(\gamma_p^-)\right) \\
&= e^{p\alpha+\frac{p^2\beta^2}{2}}\mathbf{\Phi}(-\gamma_p).
\end{aligned}$$

Again, if we set $p=0$, we have item (ii).

We continue with item (v)

$$\begin{aligned}
\mathbb{E}\left[(K-Z)^+\right] &= \int_{-\infty}^{\ln K} (K-e^z)\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz \\
&= K\int_{-\infty}^{\ln K}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz - \int_{-\infty}^{\ln K}e^z\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz \\
&= K\mathbb{E}\left[\mathbf{1}_{\{Z\le K\}}\right]-\mathbb{E}\left[\mathbf{1}_{\{Z\le K\}}Z\right] \\
&= K\mathbf{\Phi}(\gamma_0)-e^{\alpha+\frac{\beta^2}{2}}\mathbf{\Phi}(\gamma_1)
\end{aligned}$$

and item (vi)

$$\begin{aligned}
\mathbb{E}\left[(Z-K)^+\right] &= \int_{\ln K}^{\infty} (e^z-K)\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz \\
&= \int_{\ln K}^{\infty}e^z\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz - K\int_{\ln K}^{\infty}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz \\
&= \mathbb{E}\left[\mathbf{1}_{\{Z\ge K\}}Z\right]-K\mathbb{E}\left[\mathbf{1}_{\{Z\ge K\}}\right] \\
&= e^{\alpha+\frac{\beta^2}{2}}\mathbf{\Phi}(-\gamma_1)-K\mathbf{\Phi}(-\gamma_0).
\end{aligned}$$

Finally, we calculate item (vii)

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{Z\le K\}} Z \ln Z\right] &= \int_{-\infty}^{\ln K} e^z z \frac{1}{\sqrt{2\pi}\beta} e^{-\frac{(z-\alpha)^2}{2\beta^2}}\, dz \\
&= \int_{-\infty}^{\ln K} z \frac{e^{\alpha+\frac{\beta^2}{2}}}{\sqrt{2\pi}\beta} e^{-\frac{(z-(\alpha+\beta^2))^2}{2\beta^2}}\, dz \\
&= e^{\alpha+\frac{\beta^2}{2}} \int_{-\infty}^{\frac{\ln K-(\alpha+\beta^2)}{\beta}} \frac{\beta\xi+(\alpha+\beta^2)}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\, d\xi \\
&= e^{\alpha+\frac{\beta^2}{2}} \left((\alpha+\beta^2) \int_{-\infty}^{\gamma_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\, d\xi + \beta \int_{-\infty}^{\gamma_1} \frac{\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\, d\xi \right) \\
&= e^{\alpha+\frac{\beta^2}{2}} \left((\alpha+\beta^2)\mathbf{\Phi}(\gamma_1) - \beta \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \right]_{-\infty}^{\gamma_1} \right) \\
&= e^{\alpha+\frac{\beta^2}{2}} \left((\alpha+\beta^2)\mathbf{\Phi}(\gamma_1) - \beta \mathit{\Phi}(\gamma_1) \right)
\end{aligned}$$

and item (viii)

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{Z\ge K\}} Z \ln Z\right] &= \int_{\ln K}^{\infty} e^z z \frac{1}{\sqrt{2\pi}\beta} e^{-\frac{(z-\alpha)^2}{2\beta^2}}\, dz \\
&= \int_{\ln K}^{\infty} z \frac{e^{\alpha+\frac{\beta^2}{2}}}{\sqrt{2\pi}\beta} e^{-\frac{(z-(\alpha+\beta^2))^2}{2\beta^2}}\, dz \\
&= e^{\alpha+\frac{\beta^2}{2}} \int_{\frac{\ln K-(\alpha+\beta^2)}{\beta}}^{\infty} \frac{\beta\xi+(\alpha+\beta^2)}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\, d\xi \\
&= e^{\alpha+\frac{\beta^2}{2}} \left((\alpha+\beta^2) \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\, d\xi + \beta \int_{\gamma_1}^{\infty} \frac{\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\, d\xi \right) \\
&= e^{\alpha+\frac{\beta^2}{2}} \left((\alpha+\beta^2)\mathbf{\Phi}(-\gamma_1) - \beta \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \right]_{\gamma_1}^{\infty} \right) \\
&= e^{\alpha+\frac{\beta^2}{2}} \left((\alpha+\beta^2)\mathbf{\Phi}(-\gamma_1) + \beta \mathit{\Phi}(\gamma_1) \right).
\end{aligned}$$

And the proof of lemma follows. □

PROOF of LEMMA 4.2 Consider stochastic variable $X = \ln Y \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\ln Y^p = p \ln Y = pX \sim \mathcal{N}(p\mu, p^2\sigma^2).$$

Let $Y = \frac{\xi_u}{\xi_t}$ then the proof of lemma follows. □

PROOF of LEMMA 4.3 For both moments $\mathbb{E}_t^{\mathcal{Q}}[(x_u)^p]$ and $\mathbb{E}_t^{\mathcal{Q}}[(x_u)^{2p}]$ we follow the idea of the derivation from Hansen and Jørgensen (2000).

In the derivation of the first moment we split the expression into sum

$$\begin{aligned}
\mathbb{E}_t^{\mathcal{Q}}\left[(x_u)^p\right] &= \mathbb{E}_t^{\mathcal{Q}}\left[\frac{\lambda}{1-e^{-\lambda u}}\int_0^u e^{-\lambda(u-v)}\left(\frac{S_v}{S_u}\right)^p dv\right] \\
&= \frac{\lambda}{1-e^{-\lambda u}}\int_0^t e^{-\lambda(t-v)}\left(\frac{S_v}{S_t}\right)^p dv\; e^{-\lambda(u-t)}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_t}{S_u}\right)^p\right] \\
&\quad +\frac{\lambda}{1-e^{-\lambda u}}\int_t^u e^{-\lambda(u-v)}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_v}{S_u}\right)^p\right] dv.
\end{aligned}$$

According to the definition of S_t, we have for all $v, u \geq t$

$$\left(\frac{S_v}{S_u}\right)^p = e^{p(r-q+\frac{\sigma^2}{2})(v-u)+p\sigma(W_v^{\mathcal{Q}}-W_u^{\mathcal{Q}})}.$$

Taking the conditioned expectation

$$\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_v}{S_u}\right)^p\right] = e^{p(r-q+\frac{\sigma^2}{2})(v-u)+\frac{p^2\sigma^2}{2}|v-u|}.$$

In all expression, we need to calculate, the condition $v \leq u$ is satisfied, so we can simplify previous expression to

$$\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_v}{S_u}\right)^p\right] = e^{p\left(r-q+(1-p)\frac{\sigma^2}{2}\right)(v-u)}.$$

Together we have

$$\begin{aligned}
\mathbb{E}_t^{\mathcal{Q}}\left[(x_u)^p\right] &= \frac{1-e^{-\lambda t}}{1-e^{-\lambda u}}(x_t)^p\, e^{-\kappa_{\lambda,p}(u-t)} + \frac{\lambda}{1-e^{-\lambda u}}\int_t^u e^{-\kappa_{\lambda,p}(u-v)}\, dv, \\
\mathbb{E}_t^{\mathcal{Q}}\left[(x_u)^p\right] &= (x_t)^p\frac{1-e^{-\lambda t}}{1-e^{-\lambda u}}\, e^{-\kappa_{\lambda,p}(u-t)} + \frac{\lambda}{1-e^{-\lambda u}}\,\frac{1-e^{-\kappa_{\lambda,p}(u-t)}}{\kappa_{\lambda,p}}.
\end{aligned}$$

where $\kappa_{\lambda,p} = \lambda + p\left(r-q+(1-p)\frac{\sigma^2}{2}\right)$. This proves the first part of the lemma. Now we continue with the evaluation of the second conditioned moment.

$$\begin{aligned}
\mathbb{E}_t^{\mathcal{Q}}\left[(x_u)^{2p}\right] &= \mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{\lambda}{1-e^{-\lambda u}}\int_0^u e^{-\lambda(u-v)}\left(\frac{S_v}{S_u}\right)^p dv\right)^2\right] \\
&= (x_t)^{2p}\frac{(1-e^{-\lambda t})^2}{(1-e^{-\lambda u})^2}\, e^{-2\lambda(u-t)}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_t}{S_u}\frac{S_t}{S_u}\right)^p\right] \\
&\quad +2(x_t)^p\frac{\lambda(1-e^{-\lambda t})}{(1-e^{-\lambda u})^2}\, e^{-\lambda(u-t)}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_t}{S_u}\right)^p\right]\int_t^u e^{-\lambda(u-v)}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_v}{S_u}\right)^p\right] dv \\
&\quad +\frac{\lambda^2}{(1-e^{-\lambda u})^2}\int_t^u\int_t^u e^{-\lambda(2u-z-v)}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_z}{S_u}\frac{S_v}{S_u}\right)^p\right] dv\, dz.
\end{aligned}$$

We now calculate conditioned expectation of expression

$$\left(\frac{S_z}{S_u}\frac{S_v}{S_u}\right)^p = e^{p(r-q+\frac{\sigma^2}{2})(z+v-2u)+p\sigma(W_z^{\mathcal{Q}}+W_v^{\mathcal{Q}}-2W_u^{\mathcal{Q}})}.$$

If we assume that $u \geq z, v$ and let $m = \min\{z, v\}$ and $M = \max\{z, v\}$, we have

$$W_z^{\mathcal{Q}} + W_v^{\mathcal{Q}} - 2W_u^{\mathcal{Q}} = -(2(W_u^{\mathcal{Q}} - W_M^{\mathcal{Q}}) + (W_M^{\mathcal{Q}} - W_m^{\mathcal{Q}})).$$

Then the value of expectation is

$$\mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{S_z}{S_u}\frac{S_v}{S_u}\right)^p\right] = e^{p\left(r-q+(1-p)\frac{\sigma^2}{2}\right)(z+v-2u)+p^2\sigma^2(u-M)}.$$

Now we have calculated all expressions we need to evaluate the second conditioned expectation from the lemma. If we put all together and perform necessary calculation we have

$$\begin{aligned}
\mathbb{E}_t^{\mathcal{Q}}\left[(x_u)^{2p}\right] \;=\; & (x_t)^{2p}\frac{(1-e^{-\lambda t})^2}{(1-e^{-\lambda u})^2}\, e^{-(2\kappa_{\lambda,p}-p^2\sigma^2)(u-t)} \\
& + 2(x_t)^p\frac{\lambda(1-e^{-\lambda t})}{(1-e^{-\lambda u})^2}\, e^{-\kappa_{\lambda,p}(u-t)}\, \frac{1-e^{-\kappa_{\lambda,p}(u-t)}}{\kappa_{\lambda,p}} \qquad \text{(C.2)} \\
& + \lambda^2\frac{(\kappa_{\lambda,p}-p^2\sigma^2) - 2(\kappa_{\lambda,p}-p^2\frac{\sigma^2}{2})e^{-\kappa_{\lambda,p}(u-t)} + \kappa_{\lambda,p}\, e^{-2(\kappa_{\lambda,p}-p^2\frac{\sigma^2}{2})(u-t)}}{(1-e^{-\lambda u})^2\,\kappa_{\lambda,p}\,(\kappa_{\lambda,p}-p^2\frac{\sigma^2}{2})\,(\kappa_{\lambda,p}-p^2\sigma^2)},
\end{aligned}$$

where $\kappa_{\lambda,p} = \lambda + p\left(r - q + (1-p)\frac{\sigma^2}{2}\right)$. And the proof of lemma follows. □

REMARK C.1. *If we let $\lambda \to 0$, $p \to 1$ and set $q = 0$, we obtain the case analyzed in Hansen and Jørgensen (2000) (HJ). However, the expression (C.2) (with the mentioned set up of parameters) does not equal the one derived by HJ (see* REMARK *4.3). The latter expression is not consistent with the derivation presented in the HJ paper. We believe that this problem is caused by a typo and subsequent copy-pasting. We have decided that it is important to highlight this difference.*

PROOF of LEMMA 4.4 The proof of this lemma can be found also in Hansen and Jørgensen (2000). The only difference is that the constant parameter r is replaced by the constant difference $r - q$. Nevertheless, we provide the proof for the reader.

First we recall the definition of the x_t^g

$$\ln x_t^g = \frac{1}{t}\int_0^t \ln S_u\, du - \ln S_t$$

and definition of the S_t

$$S_T = S_t e^{(r-q+\frac{\sigma^2}{2})(T-t)+\sigma \int_t^T dW_u},$$

where $T \geq t$.

We need to calculate the expansion of the $\ln A_T = \frac{1}{T}\int_0^T \ln S_u \, du$ at time t

$$\begin{aligned}
\ln A_T &= \frac{t}{T}\ln A_t + \frac{1}{T}\int_t^T \left(\ln S_t + (r-q+\frac{\sigma^2}{2})(u-t) + \sigma \int_t^u dW_s\right) du \\
&= \frac{t}{T}\ln A_t + \frac{T-t}{T}\ln S_t + (r-q+\frac{\sigma^2}{2})\frac{(T-t)^2}{2T} + \frac{\sigma}{T}\int_t^T \int_s^T du\, dW_s \\
&= \frac{t}{T}\ln A_t + \frac{T-t}{T}\ln S_t + (r-q+\frac{\sigma^2}{2})\frac{(T-t)^2}{2T} + \frac{\sigma}{T}\int_t^T (T-s)\, dW_s
\end{aligned}$$

Now we can expand the expression for the $\ln x_T^g$

$$\begin{aligned}
\ln x_T^g &= \frac{1}{T}\int_0^T \ln S_u \, du - \ln S_T \\
&= \frac{1}{T}\int_0^t \ln S_u \, du + \frac{T-t}{T}\ln S_t + \frac{r-q+\frac{\sigma^2}{2}}{2T}(T-t)^2 \\
&\quad + \frac{\sigma}{T}\int_t^T (T-u)\, dW_u - \ln S_t - (r-q+\frac{\sigma^2}{2})(T-t) - \sigma\int_t^T dW_u \\
&= \frac{t}{T}\ln x_t^g - \frac{r-q+\frac{\sigma^2}{2}}{2T}(T^2-t^2) - \frac{\sigma}{T}\int_t^T u\, dW_u
\end{aligned}$$

The first two elements of $\ln x_T^g$ are deterministic (at time t), so we calculate first two moments only for the Itô integral. Expected value is zero and the variance is calculated by the Itô isometry

$$\mathbb{V}ar_t\left[-\frac{\sigma}{T}\int_t^T u\, dW_u\right] = \frac{\sigma^2}{T^2}\int_t^T u^2\, du = \frac{\sigma^2(T^3-t^3)}{3T^2}$$

This proves the lemma. □

PROOF of LEMMA 4.5 According to the LEMMA 4.3, we set value $p = 1$ and calculate the limit $\lambda \to 0$ in expressions for both moments. And the proof of lemma follows. □

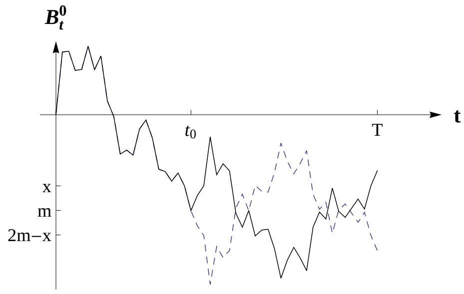

FIGURE C.1: *Reflection principle. The dashed line is reflection $\widetilde{B}_t^0$ of the solid line process B_t^0.*

PROOF of LEMMA 4.6 In the proof of lemma, we use the idea from Kwok (2008).

The derivation of distribution of extreme values is based on the reflection principle. We derive the *CDF* of the stochastic variable $y_t = m_t - B_t$, where $m_t = \inf_{s\in[0,t]} B_s$ and $B_t = \mu t + \sigma W_t \sim \mathcal{N}(\mu t, \sigma^2 t)$ is Brownian motion. First, we set zero drift rate in the Brownian notion, i.e. $\mu = 0$ and $B_t \to B_t^0$.

We need to calculate probability $P(y_T < y, B_T^0 > x)$, where $y \leq 0$ and $y \leq x$.

We follow the derivation of the probability $P(m_T < m, B_T^0 > x)$, where $m \leq 0$ and $m \leq x$ from Kwok (2008). According to the desired probability we expect that the value falls below m and so we can assume that there exists time $0 < t_0 < T$, such that $B_{t_0}^0 = m$ for the first time. Suppose we define a random process

$$\widetilde{B}_t^0 = \begin{cases} B_t^0 & \text{for } t < t_0 \\ 2m - B_t^0 & \text{for } t_0 \leq t \leq T \end{cases} .$$

The process $\widetilde{B}_t^0$ is the mirror reflection of B_t^0 at level m within the time interval $[t_0, T]$ (see FIGURE C.1).

There are two important properties of the relation of the original process and its reflection. First, the following equality holds for $u \geq 0$

$$\widetilde{B}_{t_0+u}^0 - \widetilde{B}_{t_0}^0 = -(B_{t_0+u}^0 - B_{t_0}^0) \sim \mathcal{N}(0, \sigma^2 u).$$

The second property is the following equivalency

$$P(B_t^0 > x) \Leftrightarrow P(\widetilde{B}_t^0 < 2m - x).$$

According to these two properties, we have two equivalent paths of the stochastic process B_t^0. In both cases the process starts in point 0 and then decreases to value m.

Then both of processes move in different direction path of the length $x-m$. Thus we have

$$\begin{aligned} P(m_T < m, B_T^0 > x) &= P(\widetilde{B}_T^0 < 2m-x) = P(B_T^0 < 2m-x) \\ &= \mathbf{\Phi}\left(\frac{2m-x}{\sigma\sqrt{T}}\right),\ m \leq \min(x,0). \end{aligned}$$

Now, we calculate the probability $P(y_T < y, B_T^0 > x)$. We define random process

$$\begin{aligned} \widehat{B}_{T-t} &= B_t - B_T \text{ for } t \in [0,T] \\ &= \mu t + \sigma W_t - \mu T - \sigma W_T \\ &= -\mu(T-t) - \sigma(W_T - W_t). \end{aligned}$$

It is clear that the distribution of the process is

$$\widehat{B}_t \sim \mathcal{N}(-\mu t, \sigma^2 t),$$

i.e. $\widehat{B}_t$ is equivalent to B_t with $\widehat{\mu} = -\mu$ (when we turn the flow of time, the stochastic increments do not change, but the drift decreases). This implies that

$$P(B_T^0 > x) = P(B_0^0 - B_T^0 < -x) = P(\widehat{B}_T^0 < -x) = P(\widehat{B}_T^0 > x).$$

Moreover, $y_t = \inf_{s\in[0,t]} \widehat{B}_s$ and thus

$$P(y_T < y, \widehat{B}_T^0 > x) = \mathbf{\Phi}\left(\frac{2y-x}{\sigma\sqrt{T}}\right),\ y \leq \min(x,0).$$

Next, we apply Girsanov's theorem 1.9 to include the drift. We need to change the probability measure to keep the zero-drift of the Brownian motion with respect to the new probability $\widetilde{P}$. According to the change of the wiener process

$$W_t^{\widetilde{P}} = -\frac{\mu t}{\sigma} + W_t^P,$$

the Radon-Nikodým derivative has form

$$\ln\frac{dP}{d\widetilde{P}} = -\frac{\mu}{\sigma}W_t^{\widetilde{P}} - \frac{\mu^2 T}{2\sigma^2}.$$

We calculate the probability

$$P(y_T < y, \widehat{B}_T > x) \quad = \quad \mathbb{E}^P\left[\mathbf{1}_{\{y_T<y\}}\mathbf{1}_{\{\widehat{B}_T>x\}}\right]$$

$$\begin{aligned}
&= \mathbb{E}^{\widetilde{P}}\left[\mathbf{1}_{\{y_T<y\}}\mathbf{1}_{\{\widehat{B}_T>x\}}e^{-\frac{\mu}{\sigma^2}\widehat{B}_T-\frac{\mu^2T}{2\sigma^2}}\right] \\
\text{(reflection principle)} &= \mathbb{E}^{\widetilde{P}}\left[\mathbf{1}_{\{2y-\widehat{B}_T>x\}}e^{-\frac{\mu}{\sigma^2}(2y-\widehat{B}_T)-\frac{\mu^2T}{2\sigma^2}}\right] \\
&= e^{-\frac{2\mu y}{\sigma^2}}\,\mathbb{E}^{\widetilde{P}}\left[\mathbf{1}_{\{\widehat{B}_T<2y-x\}}e^{\frac{\mu}{\sigma^2}\widehat{B}_T-\frac{\mu^2T}{2\sigma^2}}\right] \\
&= e^{-\frac{2\mu y}{\sigma^2}}\int_{-\infty}^{2y-x}\frac{1}{\sqrt{2\pi\sigma^2T}}e^{-\frac{\xi^2}{2\sigma^2T}}e^{\frac{\mu}{\sigma^2}\xi-\frac{\mu^2T}{2\sigma^2}}\,d\xi \\
&= e^{-\frac{2\mu y}{\sigma^2}}\int_{-\infty}^{2y-x}\frac{1}{\sqrt{2\pi\sigma^2T}}e^{-\frac{(\xi-\mu T)^2}{2\sigma^2T}}\,d\xi \\
&= e^{-\frac{2\mu y}{\sigma^2}}\,\mathbf{\Phi}\left(\frac{2y-x-\mu T}{\sigma\sqrt{T}}\right),\ y\le\min(x,0).
\end{aligned}$$

We derive the decumulative distribution function

$$\begin{aligned}
P(y_T>y,\widehat{B}_T>x) &= P(\widehat{B}_T>x)-P(y_T<y,\widehat{B}_T>x) \\
&= \mathbf{\Phi}\left(\frac{-x-\mu T}{\sigma\sqrt{T}}\right)-e^{-\frac{2\mu y}{\sigma^2}}\,\mathbf{\Phi}\left(\frac{2y-x-\mu T}{\sigma\sqrt{T}}\right),\ y\le\min(x,0).
\end{aligned}$$

If we set $y=x$, the inequality $\widehat{B}_T>y$ is implicitly satisfied by $y_T>y$ and we can calculate the *CDF* of the minimum

$$\begin{aligned}
F_{min}(y) &= 1-\left(\mathbf{\Phi}\left(\frac{-y-\mu T}{\sigma\sqrt{T}}\right)-e^{-\frac{2\mu y}{\sigma^2}}\,\mathbf{\Phi}\left(\frac{y-\mu T}{\sigma\sqrt{T}}\right)\right) \\
&= \mathbf{\Phi}\left(\frac{y+\mu T}{\sigma\sqrt{T}}\right)+e^{-\frac{2\mu y}{\sigma^2}}\,\mathbf{\Phi}\left(\frac{y-\mu T}{\sigma\sqrt{T}}\right),\ y\le 0.
\end{aligned}$$

The last step of the derivation is to calculate the conditioned distribution.

Suppose that $y_t=y_0$, i.e. the minimum of the process is equal y_0 for time interval $[0,t]$. The time point when the minimum was achieved does not change the conditioned distribution, so we can assume that the minimum is at the time t. We shift the Brownian motion by the value y_0 and we have desired conditioned distribution.

$$\begin{aligned}
P(y_T>y,\widehat{B}_T>x\big|y_t=y_0) &= P(y_{T-t}>y,\widehat{B}_{T-t}+y_0>x) \\
&= P(y_{T-t}>y,\widehat{B}_{T-t}>x-y_0) \\
&= \mathbf{\Phi}\left(\frac{-x+y_0-\mu\tau}{\sigma\sqrt{\tau}}\right)-e^{-\frac{2\mu y}{\sigma^2}}\,\mathbf{\Phi}\left(\frac{2y-x+y_0-\mu\tau}{\sigma\sqrt{\tau}}\right),
\end{aligned}$$

for $y\le\min(x,0)$ and $\tau=T-t$. We finish the derivation by the same steps as for the unconditioned distribution. The conditioned *CDF* of the minimum is then

$$F_{min}(y)|\mathcal{F}_t=\mathbf{\Phi}\left(\frac{y-y_0+\mu(T-t)}{\sigma\sqrt{T-t}}\right)+e^{-\frac{2\mu y}{\sigma^2}}\,\mathbf{\Phi}\left(\frac{y+y_0-\mu(T-t)}{\sigma\sqrt{T-t}}\right),\ y\le 0.$$

Finally, we set $B_t = \ln\frac{S_t}{S_0} = (r - q + \frac{\sigma}{2})t + \sigma W_t$ and the proof for distribution of minimum value follows.

The distribution of maximum value can be derived from the distribution of the minimum according to the equality

$$\max_{t\le u\le T}(\mu u + \sigma W_u) = -\min_{t\le u\le T}(-\mu u - \sigma W_u) = -\min_{t\le u\le T}(-\mu u + \sigma W_u).$$

We need to swap $-Y$, $-Y_0$, $-x$ and $-\mu$ for y, y_0, x and μ, respectively. Finally, we have

$$\begin{aligned} P(Y_T < Y, -\widehat{B}_T < x \big| Y_t = Y_0) =& P(Y_{T-t} < Y, -\widehat{B}_{T-t} + y_0 < x) \\ =& P(Y_{T-t} < Y, -\widehat{B}_{T-t} < x - y_0) \\ =& \mathbf{\Phi}\left(\frac{x - Y_0 + \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{-\frac{2\mu Y}{\sigma^2}}\mathbf{\Phi}\left(\frac{-2Y + x - Y_0 + \mu\tau}{\sigma\sqrt{\tau}}\right), \end{aligned}$$

for $Y \ge \max(x, 0)$ and $\tau = T - t$. By setting $x = Y$, we achieve *CDF* for the maximum value

$$F_{max}(Y)|\mathcal{F}_t = \mathbf{\Phi}\left(\frac{Y - Y_0 + \mu(T-t)}{\sigma\sqrt{T-t}}\right) - e^{-\frac{2\mu Y}{\sigma^2}}\mathbf{\Phi}\left(\frac{-Y - Y_0 + \mu(T-t)}{\sigma\sqrt{T-t}}\right),\ Y \ge 0.$$

And the proof of lemma follows. $\square$

PROOF of LEMMA 4.7 Consider a stochastic variable z with *CDF*

$$F(z, \mathfrak{c}) = \mathbf{\Phi}\left(\frac{z-\alpha}{\beta}\right) + \mathfrak{c}\, e^{-\varsigma z}\mathbf{\Phi}\left(\mathfrak{c}\frac{z+\alpha}{\beta}\right)$$

for $\mathfrak{c}\, z \le 0$, where $\mathfrak{c} = 1$ for minimum and $\mathfrak{c} = -1$ for maximum. We recall that $Z = e^z$ and

$$\begin{aligned} \gamma_p^+ &\equiv \frac{\ln K + \alpha}{\beta} - p\beta, \\ \gamma_p^- &\equiv \frac{\ln K - \alpha}{\beta} - p\beta, \end{aligned}$$

where $K > 0$ and $p \in \mathbb{R}$.

First, we calculate the expected value for the minimum.

$$\mathbb{E}\left[\mathbf{1}_{\{Z\le K\}} Z^p\right] = \int_{-\infty}^{\ln K} e^{pz}\frac{\partial F(z,1)}{\partial z}\, dz$$

$$= \int_{-\infty}^{\ln K} e^{pz}\left(\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}} + \frac{e^{-\varsigma z}}{\sqrt{2\pi}\beta}e^{-\frac{(z+\alpha)^2}{2\beta^2}} - \varsigma e^{-\varsigma z}\mathbf{\Phi}\left(\frac{z+\alpha}{\beta}\right)\right)dz$$

We calculate each part of the integral separately.

$$\begin{aligned}
\int_{-\infty}^{\ln K} e^{pz}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz &= \int_{-\infty}^{\ln K} \frac{e^{p\alpha+\frac{p^2\beta^2}{2}}}{\sqrt{2\pi}\beta}e^{-\frac{(z-(\alpha+p\beta^2))^2}{2\beta^2}}\,dz \\
&= e^{p\alpha+\frac{p^2\beta^2}{2}}\int_{-\infty}^{\frac{\ln K-(\alpha+p\beta^2)}{\beta}} \frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}\,d\xi \\
&= e^{p\alpha+\frac{p^2\beta^2}{2}}\mathbf{\Phi}\left(\frac{\ln K-\alpha}{\beta}-p\beta\right) \\
&= e^{p\alpha+\frac{p^2\beta^2}{2}}\mathbf{\Phi}(\gamma_p^-)
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\ln K} e^{(p-\varsigma)z}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z+\alpha)^2}{2\beta^2}}\,dz &= \int_{-\infty}^{\ln K} \frac{e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}}{\sqrt{2\pi}\beta}e^{-\frac{(z+\alpha-(p-\varsigma)\beta^2)^2}{2\beta^2}}\,dz \\
&= e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\int_{-\infty}^{\frac{\ln K+\alpha-(p-\varsigma)\beta^2}{\beta}} \frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}\,dz \\
&= e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\mathbf{\Phi}\left(\frac{\ln K+\alpha}{\beta}-(p-\varsigma)\beta\right) \\
&= e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\mathbf{\Phi}(\gamma_{p-\varsigma}^+)
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\ln K} -\varsigma e^{(p-\varsigma)z}\mathbf{\Phi}\left(\frac{z+\alpha}{\beta}\right)dz &= -\left[\frac{\varsigma}{p-\varsigma}e^{(p-\varsigma)z}\mathbf{\Phi}\left(\frac{z+\alpha}{\beta}\right)\right]_{-\infty}^{\ln K} \\
&\qquad +\frac{\varsigma}{p-\varsigma}\int_{-\infty}^{\ln K}\frac{e^{(p-\varsigma)z}}{\sqrt{2\pi}\beta}e^{-\frac{(z+\alpha)^2}{2\beta^2}}\,dz \\
&= -\frac{\varsigma}{p-\varsigma}e^{(p-\varsigma)\ln K}\mathbf{\Phi}\left(\frac{\ln K+\alpha}{\beta}\right) \\
&\quad +\frac{\varsigma}{p-\varsigma}e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\mathbf{\Phi}\left(\frac{\ln K+\alpha}{\beta}-(p-\varsigma)\beta\right) \\
&= -\frac{\varsigma}{p-\varsigma}e^{(p-\varsigma)\ln K}\mathbf{\Phi}(\gamma_0^+) \\
&\qquad +\frac{\varsigma}{p-\varsigma}e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\mathbf{\Phi}(\gamma_{p-\varsigma}^+)
\end{aligned}$$

Now, we sum all elements and we have the expected value for minimum distribution (iii). By setting the parameter $p=0$ we obtain also the expression (i).

The derivation of the expected value for the maximum is similar.

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{Z\geq K\}}Z^p\right] &= \int_{\ln K}^{\infty} e^{pz}\frac{\partial F(z,-1)}{\partial z}\,dz\\ &= \int_{\ln K}^{\infty} e^{pz}\left(\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}+\frac{e^{-\varsigma z}}{\sqrt{2\pi}\beta}e^{-\frac{(z+\alpha)^2}{2\beta^2}}+\varsigma e^{-\varsigma z}\mathbf{\Phi}\left(-\frac{z+\alpha}{\beta}\right)\right)dz\end{aligned}$$

As for the minimum, we calculate each part of the integral separately.

$$\begin{aligned}\int_{\ln K}^{\infty} e^{pz}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z-\alpha)^2}{2\beta^2}}\,dz &= e^{p\alpha+\frac{p^2\beta^2}{2}}\left(1-\mathbf{\Phi}(\gamma_p^-)\right)\\ &= e^{p\alpha+\frac{p^2\beta^2}{2}}\mathbf{\Phi}(-\gamma_p^-)\end{aligned}$$

$$\begin{aligned}\int_{\ln K}^{\infty} e^{(p-\varsigma)z}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{(z+\alpha)^2}{2\beta^2}}\,dz &= e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\left(1-\mathbf{\Phi}(\gamma_{p-\varsigma}^+)\right)\\ &= e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\mathbf{\Phi}(-\gamma_{p-\varsigma}^+)\end{aligned}$$

$$\begin{aligned}\int_{\ln K}^{\infty} \varsigma e^{(p-\varsigma)z}\mathbf{\Phi}\left(-\frac{z+\alpha}{\beta}\right)dz &= \left[\frac{\varsigma}{p-\varsigma}e^{(p-\varsigma)z}\mathbf{\Phi}\left(-\frac{z+\alpha}{\beta}\right)\right]_{\ln K}^{\infty}\\ &\quad +\frac{\varsigma}{p-\varsigma}\int_{\ln K}^{\infty}\frac{e^{(p-\varsigma)z}}{\sqrt{2\pi}\beta}e^{-\frac{(z+\alpha)^2}{2\beta^2}}\,dz\\ &= -\frac{\varsigma}{p-\varsigma}e^{(p-\varsigma)\ln K}\mathbf{\Phi}(-\gamma_0^+)\\ &\quad +\frac{\varsigma}{p-\varsigma}e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\left(1-\mathbf{\Phi}(\gamma_{p-\varsigma}^+)\right)\\ &= -\frac{\varsigma}{p-\varsigma}e^{(p-\varsigma)\ln K}\mathbf{\Phi}(-\gamma_0^+)\\ &\quad +\frac{\varsigma}{p-\varsigma}e^{-(p-\varsigma)\alpha+\frac{(p-\varsigma)^2\beta^2}{2}}\mathbf{\Phi}(-\gamma_{p-\varsigma}^+)\end{aligned}$$

Now, we sum all elements and we have the expected value for maximum distribution (iv). By setting the parameter $p=0$ we obtain also the expression (ii).

And the proof of lemma follows. □

C.2 Limit value of the early exercise boundary at expiry

LEMMA C.2. *Consider a mutually disjoint decomposition $\mathbb{D}=A\cup\partial A\cup B$ of a topological space $\mathbb{D}$, where $\partial A\equiv\partial B$. Moreover, consider a set Z so that $A\subset Z\subset\overline{A}$, where*

$\overline{A} \equiv A \cup \partial A$ *is the closure of set* A*, then*

$$\partial A = \partial Z.$$

PROOF of LEMMA C.2 Let $a \in \partial A$. For each $\varepsilon > 0$, there exists a neighborhood $O_\varepsilon(a)$ so that $\widetilde{a} \in A \cap O_\varepsilon(a)$ and $\widetilde{b} \in B \cap O_\varepsilon(a)$. This implies, that $a \in \partial Z$, i.e. $\partial A \subset \partial Z$, because $\widetilde{a} \in Z$, but $\widetilde{b} \notin Z \subset \overline{A}$.

Since $\partial A = \overline{A} \cap \overline{\mathbb{D} \backslash A}$ and $\overline{Z} = \overline{A}$, we have $\partial Z = \overline{Z} \cap \overline{\mathbb{D} \backslash Z} \subset \partial A$ and the proof of lemma follows. □

PROOF of THEOREM 5.1 Part 1). First, we show that

$$\mathcal{S}(T, \ \cdot\ ) \subset \{x_T \in \mathbb{D}; f_b(T, x_T) > 0\}.$$

We have

$$\frac{1}{T-t}\mathbb{E}_t\left[\int_t^T \mathbf{1}_{\mathcal{S}}(u, x_u) f_b(u, x_u)\, du\right] = \frac{1}{T-t}\left(V_{am}(t, x_t) - V_{eu}(t, x_t)\right) \geq 0,$$

for any $t \in [0, T)$. In the limit $t \to T$, we can omit the conditioned expected value operator $\mathbb{E}_t$ and we obtain

$$\mathbf{1}_{\mathcal{S}}(T, x_T) f_b(T, x_T) \geq 0.$$

Let $(T, y_T) \in S$, then we obtain

$$f_b(T, y_T) \geq 0.$$

Now suppose that there exists $(T, \widetilde{y}_T) \in \mathcal{S}$ such that $f_b(T, \widetilde{y}_T) = 0$. Notice that in the stopping region $\mathcal{S}$ we have the identity $V_{am}(t, x) = \Omega(t, x)$ for any $(t, x) \in \mathcal{S}$ and, consequently, $\frac{\partial}{\partial t}\left(V_{am}(t, x) - \Omega(t, x)\right) = 0$. Then we have

$$\begin{aligned} 0 = \frac{\partial}{\partial t}\left(V_{am}(t, \widetilde{y}_t) - \Omega(t, \widetilde{y}_t)\right)\Big|_{t=T} &= \frac{\partial V_{eu}}{\partial t}(t, \widetilde{y}_t)\Big|_{t=T} - f_b(T, \widetilde{y}_T) - \frac{\partial \Omega}{\partial t}(t, \widetilde{y}_t)\Big|_{t=T} \\ &= \frac{\partial}{\partial t}\left(V_{eu}(t, \widetilde{y}_t) - \Omega(t, \widetilde{y}_t)\right)\Big|_{t=T}. \end{aligned}$$

In the stopping region, exercising the derivative (American style) gives holder higher pay-off than keeping it (European style), i.e.

$$V_{eu}(t, y) < \Omega(t, y) \quad \text{for } (t, y) \in \mathcal{S},$$

for t sufficiently close to expiry T. The value of difference between European style of derivative and pay-off is increasing (from negative values to zero at maturity). The derivative of this difference is positive in the stopping region, i.e.

$$\frac{\partial}{\partial t}\left(V_{eu}(t,\widetilde{y}_t) - \Omega(t,\widetilde{y}_t)\right)\Big|_{t=T} > 0.$$

This is a contradiction and the proof of first part follows.

Part 2). Now, we show that

$$\{x_T \in \mathbb{D}; f_b(T,x_T) > 0\} \subset \overline{\mathcal{S}}(T,\ \cdot\ ) = \mathcal{S}(T,\ \cdot\ ) \cup \mathcal{X}_T^*.$$

The function f_b can be determined on stopping region by the following property

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left(V_{am}(t,\widetilde{y}_t) - \Omega(t,\widetilde{y}_t)\right)\Big|_{t=T} \\ &= \frac{\partial V_{eu}}{\partial t}(t,\widetilde{y}_t)\Big|_{t=T} - \mathbf{1}_{\mathcal{S}}(T,\widetilde{y}_T) f_b(T,\widetilde{y}_T) - \frac{\partial \Omega}{\partial t}(t,\widetilde{y}_t)\Big|_{t=T}. \end{aligned}$$

To span the function on the whole domain $\mathbb{D}$, we omit the function $\mathbf{1}_{\mathcal{S}}(u,x_u)$ and we have

$$f_b(T,\widetilde{y}_T) = \frac{\partial}{\partial t}\left(V_{eu}(t,\widetilde{y}_t) - \Omega(t,\widetilde{y}_t)\right)\Big|_{t=T}.$$

Notice that the function f_b nullifies movements away from the pay-off function Ω.

On the continuous region $\mathcal{C}$, the holder of a financial derivative does not want to exercise it, because keeping this derivative yields better pay-off, i.e.

$$V_{eu}(t,y) > \Omega(t,y) \quad \text{for } (t,y) \in \mathcal{C},$$

for t sufficiently close to expiry T. The value of difference between European style of derivative and pay-off is decreasing (from positive values to zero at maturity). The derivative of this difference is negative and so is the value of function f_b in the continuation region, i.e.

$$f_b(T,\widetilde{y}_T) = \frac{\partial}{\partial t}\left(V_{eu}(t,\widetilde{y}_t) - \Omega(t,\widetilde{y}_t)\right)\Big|_{t=T} < 0.$$

The function f_b can have positive values only on $\overline{\mathcal{S}}(T,\ \cdot\ )$.

Using LEMMA C.2 we have (5.2) and the proof follows. □

C.3 American style vanilla option

PROOF of THEOREM A.1 In the proof, we use THEOREM 4.1. According to (A.1) we have the numeraire and pay-off function

$$\mathcal{N}(t, x_t) = e^{rt}$$

and

$$\Omega(t, x_t) = (\mathfrak{c}(S_t - X))^+ ,$$

respectively. Moreover, we have the stochastic differential equation

$$dS_t = (r-q)S_t\, dt + \sigma S_t\, dW_t^{\mathcal{Q}} = \widetilde{\mu}\, dt - \widetilde{\sigma}\, dW_t^{\mathcal{Q}}.$$

On the stopping region $\mathcal{S}$, function (4.13) becomes

$$\begin{aligned} f_d(t, x_t) &= \mathfrak{c}(S_t - X)\frac{\partial\left(e^{-rt}\right)}{\partial t} + \mathfrak{c}\, e^{-rt}\widetilde{\mu}\frac{\partial\left((S_t - X)\right)}{\partial S} + \frac{1}{2}\mathfrak{c}\, e^{-rt}\widetilde{\sigma}^2\frac{\partial^2\left((S_t - X)\right)}{\partial S^2} \\ &= -\mathfrak{c}\, re^{-rt}(S_t - X)\, dt + \mathfrak{c}\, e^{-rt}\widetilde{\mu} \\ &= -\mathfrak{c}\, e^{-rt}\left(-rX + qS_t\right). \end{aligned}$$

And the proof of theorem follows. □

APPENDIX D

Numerical simulations of the early exercise boundary at expiry

In this chapter, we compare analytic values of the limit of early exercise boundary (EEB) according to the method from CHAPTER 5 with values calculated by the PSOR (projected successive over relaxation) method introduced in Elliot and Ockendom (1982). For further details on the PSOR method see Kwok (2008) or Wilmott et al. (1995). Values for plain vanilla options, Asian options and lookback options are derived in other sources (e.g. Detemple 2006, Kwok 2008, Ševčovič 2008), thus we present results only for the strategies considered in SECTION 5.2 (bullish, bearish, strangle, straddle and condor spread).

D.1 PSOR method

The PSOR method approach to solving an American plain vanilla options type problem of valuation is based on transformation of the problem by introducing new variables $y = \ln S/X$ and $\tau = T - t$ (for problem with more strike prices X_i, we use their arithmetic average instead of X). The value function $V(t, S)$ is transformed into

$$u(\tau, y) = e^{\alpha y + \beta\tau} V(T - \tau, Xe^y),$$

where $\alpha = \frac{r-q}{\sigma^2} - \frac{1}{2}$ and $\beta = \frac{r-q}{2} + \frac{(r-q)^2}{2\sigma^2} + \frac{\sigma^2}{8}$. Applying this transformation, the Black–Scholes equation changes into the simplest form of a parabolic partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial y^2}.$$

Next, the time–space mesh is created within the range $\tau \in [0, T]$ and $y \in [-L, L]$, where T is the expiration time and L is a sufficiently big constant to cover desired region of the transformed space variable. The precision of result depends on the selection of constants m and n, that set up time and space steps $\Delta t = \frac{T}{m}$ and $\Delta y = \frac{L}{n}$, respectively. According to the definition of steps, there are $m+1$ time steps and $2n+1$ space steps.

In each time step a linear problem is solved using an iterative successive over relaxation (SOR) method with parameter ω (and the tolerance ε^{PSOR}). The calculated iterative results are projected to the transformed pay-off diagram. This is done by taking the maximum of value of transformed pay-off and computed iteration of a solution obtained by the SOR successive iteration (keeping the solution greater than or equal to the pay-off).

The value of early exercise boundary is calculated as a boundary of set where the solution is equal to pay-off function (with the tolerance $\varepsilon^{boundary}$), i.e.

$$S_t^* = \partial \left\{ S > 0 | V(t,S) = \Omega(t,S) \right\}.$$

D.2 Comparison of analytic and numerical approach

In TABLES D.1-D.5, we present values of the limit of early exercise boundary at expiry for bullish, bearish, strangle, straddle and condor spread calculated by the PSOR method, respectively. The discretization parameters L, n, m and both tolerances ε^{PSOR} and $\varepsilon^{boundary}$ varies according to increase the precision of the result. In the calculation, the SOR parameter is set to $\omega = 1.4$ and we use the same value for both tolerances in all cases, i.e. $\varepsilon^{PSOR} = \varepsilon^{boundary} = \varepsilon$. The financial parameters of a contract, namely interest rate r, dividend rate q and strike price(s) X_i, varies to fulfill conditions of examined cases. The volatility $\sigma = 30\%$ of the return of underlying asset is held constant.

Our targeted value is at the expiry, i.e. $\tau = T - T = 0$. However, we are not able to calculate the value of early exercise boundary at this time by the PSOR method. Thus we calculate the value at the origin (birth) of a contract with extremely low expiration time, i.e. the expiration has to be chosen close to zero. In the calculation, we use values from the set $T \in \{10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}\}$.

Each numerical value tends to analytic value calculated by the formulae presented in SECTION 5.2. The numerical results are improving with increasing density of the time–space mesh and decreasing tolerance (as expected). Approaching the highest precision by the set up of parameters, the relative error is decreasing to the order 10^{-4} or lower.

In FIGURES D.1-D.5, we present pay-off functions of bullish, bearish, strangle, straddle and condor spread, respectively. Moreover, an illustrative position of the limit value(s) of the early exercise boundary is depicted for each set up of the financial parameters of a contract.

TABLE D.1: *Comparison of analytic and numerical values of the limit of early exercise boundary for bullish spread.*

r	q	X_1	X_2	T	L	n	m	ε	S^*_{theor}	S^*_{calc}	error
2%	3%	1	2	10^{-5}	1	1000	100	10^{-10}	1	1.00548	0.548%
2%	3%	1	2	10^{-6}	1	1000	100	10^{-10}	1	1.00347	0.347%
2%	3%	1	2	10^{-7}	1	1000	100	10^{-10}	1	1.00247	0.247%
2%	3%	1	2	10^{-8}	1	5000	150	10^{-12}	1	1.00047	0.047%
2%	3%	1	2	10^{-8}	1	10000	200	10^{-14}	1	1.00037	0.037%
2%	3%	1	2	10^{-8}	1	20000	250	10^{-14}	1	1.00027	0.027%
3%	2%	1	2	10^{-5}	1	1000	100	10^{-10}	1.5	1.5	0.000%
3%	2%	1	2	10^{-6}	1	1000	100	10^{-10}	1.5	1.49551	−0.300%
3%	2%	1	2	10^{-7}	1	1000	100	10^{-10}	1.5	1.44986	−3.343%
3%	2%	1	2	10^{-8}	1	5000	150	10^{-12}	1.5	1.49521	−0.319%
3%	2%	1	2	10^{-8}	1	10000	200	10^{-14}	1.5	1.5	0.000%
3%	2%	1	2	10^{-8}	1	20000	250	10^{-14}	1.5	1.49985	−0.010%
5%	2%	1	2	10^{-5}	1	1000	100	10^{-10}	2	1.99864	−0.068%
5%	2%	1	2	10^{-6}	1	1000	100	10^{-10}	2	1.99864	−0.068%
5%	2%	1	2	10^{-7}	1	1000	100	10^{-10}	2	1.99864	−0.068%
5%	2%	1	2	10^{-8}	1	5000	150	10^{-12}	2	1.99984	−0.008%
5%	2%	1	2	10^{-8}	1	10000	200	10^{-14}	2	1.99984	−0.008%
5%	2%	1	2	10^{-8}	1	20000	250	10^{-14}	2	1.99994	−0.003%

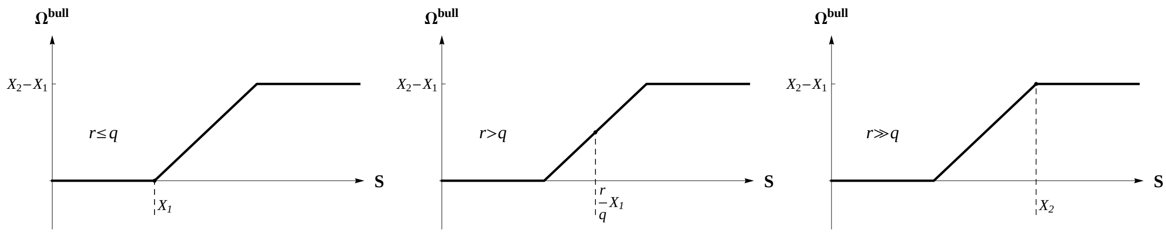

FIGURE D.1: *Limit values of the early exercise boundary at expiry for bullish spread with $r \leq q$ (left), $r > q$ (middle) and $r \gg q$ (right).*

TABLE D.2: *Comparison of analytic and numerical values of the limit of early exercise boundary for bearish spread.*

r	q	X_1	X_2	T	L	n	m	ε	S^*_{theor}	S^*_{calc}	error
2%	3%	1	2	10^{-5}	1	1000	100	10^{-10}	1.33333	1.33304	−0.022%
2%	3%	1	2	10^{-6}	1	1000	100	10^{-10}	1.33333	1.33571	0.178%
2%	3%	1	2	10^{-7}	1	1000	100	10^{-10}	1.33333	1.36542	2.407%
2%	3%	1	2	10^{-8}	1	5000	150	10^{-12}	1.33333	1.33651	0.239%
2%	3%	1	2	10^{-8}	1	10000	200	10^{-14}	1.33333	1.33331	−0.002%
2%	3%	1	2	10^{-8}	1	20000	250	10^{-14}	1.33333	1.33331	−0.002%
3%	2%	1	2	10^{-5}	1	1000	100	10^{-10}	2	1.98867	−0.567%
3%	2%	1	2	10^{-6}	1	1000	100	10^{-10}	2	1.99265	−0.368%
3%	2%	1	2	10^{-7}	1	1000	100	10^{-10}	2	1.99464	−0.268%
3%	2%	1	2	10^{-8}	1	5000	150	10^{-12}	2	1.99904	−0.048%
3%	2%	1	2	10^{-8}	1	10000	200	10^{-14}	2	1.99924	−0.038%
3%	2%	1	2	10^{-8}	1	20000	250	10^{-14}	2	1.99944	−0.028%
2%	5%	1	2	10^{-5}	1	1000	100	10^{-10}	1	1.00147	0.147%
2%	5%	1	2	10^{-6}	1	1000	100	10^{-10}	1	1.00147	0.147%
2%	5%	1	2	10^{-7}	1	1000	100	10^{-10}	1	1.00047	0.047%
2%	5%	1	2	10^{-8}	1	5000	150	10^{-12}	1	1.00027	0.027%
2%	5%	1	2	10^{-8}	1	10000	200	10^{-14}	1	1.00017	0.017%
2%	5%	1	2	10^{-8}	1	20000	250	10^{-14}	1	1.00007	0.007%

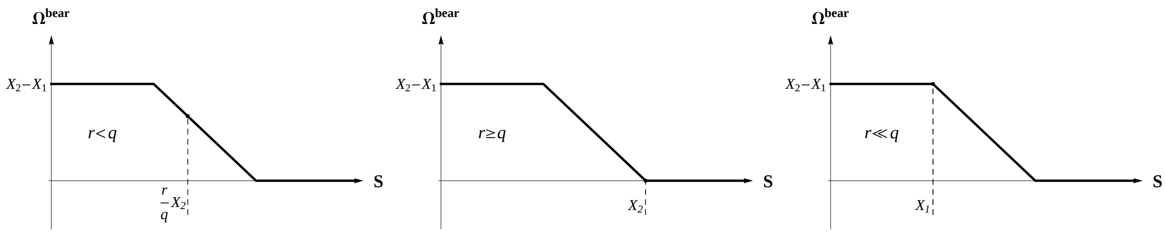

FIGURE D.2: *Limit values of the early exercise boundary at expiry for bearish spread with $r < q$ (left), $r \geq q$ (middle) and $r \ll q$ (right).*

TABLE D.3: *Comparison of analytic and numerical values of the limit of early exercise boundary for strangle spread.*

r	q	X_1	X_2	T	L	n	m	ε	S^*_{theor}	S^*_{calc}	error
3%	2%	1	2	10^{-5}	1	1000	100	10^{-10}	1	0.99448	−0.552%
									3	3.00256	0.085%
3%	2%	1	2	10^{-6}	1	1000	100	10^{-10}	1	0.99647	−0.353%
									3	2.99656	−0.115%
3%	2%	1	2	10^{-7}	1	1000	100	10^{-10}	1	0.99747	−0.253%
									3	2.95195	−1.602%
3%	2%	1	2	10^{-8}	1	5000	150	10^{-12}	1	0.99947	−0.053%
									3	2.99536	−0.155%
3%	2%	1	2	10^{-8}	1	10000	200	10^{-14}	1	0.99967	−0.033%
									3	2.99986	−0.005%
3%	2%	1	2	10^{-8}	1	20000	250	10^{-14}	1	0.99977	−0.023%
									3	2.99986	−0.005%
2%	3%	1	2	10^{-5}	1	1000	100	10^{-10}	0.66667	0.66662	−0.007%
									2	2.01066	0.533%
2%	3%	1	2	10^{-6}	1	1000	100	10^{-10}	0.66667	0.66996	0.494%
									2	2.00665	0.332%
2%	3%	1	2	10^{-7}	1	1000	100	10^{-10}	0.66667	0.6994	4.910%
									2	2.00464	0.232%
2%	3%	1	2	10^{-8}	1	5000	150	10^{-12}	0.66667	0.66996	0.494%
									2	2.00104	0.052%
2%	3%	1	2	10^{-8}	1	10000	200	10^{-14}	0.66667	0.66669	0.003%
									2	2.00064	0.032%
2%	3%	1	2	10^{-8}	1	20000	250	10^{-14}	0.66667	0.66669	0.003%
									2	2.00054	0.027%

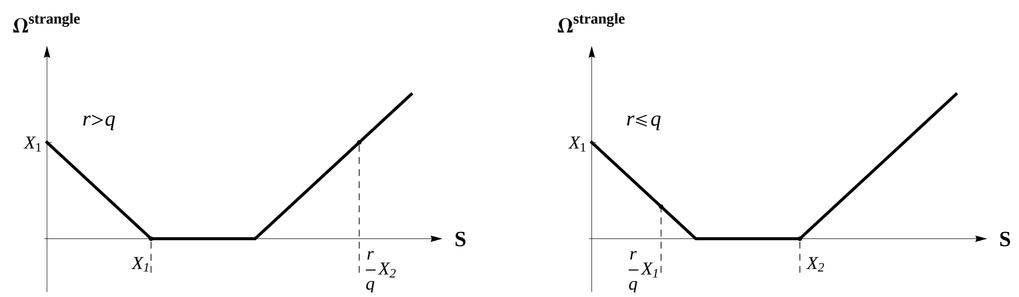

FIGURE D.3: *Limit values of the early exercise boundary at expiry for strangle spread with $r > q$ (left) and $r \leq q$ (right).*

TABLE D.4: *Comparison of analytic and numerical values of the limit of early exercise boundary for straddle spread.*

r	q	X	T	L	n	m	ε	S^*_{theor}	S^*_{calc}	error
3%	2%	1.5	10^{-5}	1	1000	100	10^{-10}	1.5	1.49252	−0.499%
								2.25	2.2512	0.054%
3%	2%	1.5	10^{-6}	1	1000	100	10^{-10}	1.5	1.49551	−0.300%
								2.25	2.24671	−0.146%
3%	2%	1.5	10^{-7}	1	1000	100	10^{-10}	1.5	1.497	−0.200%
								2.25	2.20002	−2.221%
3%	2%	1.5	10^{-8}	1	5000	150	10^{-12}	1.5	1.4991	−0.060%
								2.25	2.24536	−0.206%
3%	2%	1.5	10^{-8}	1	10000	200	10^{-14}	1.5	1.4994	−0.040%
								2.25	2.25008	0.003%
3%	2%	1.5	10^{-8}	1	20000	250	10^{-14}	1.5	1.49963	−0.025%
								2.25	2.24997	−0.002%
2%	3%	1.5	10^{-5}	1	1000	100	10^{-10}	1	0.99947	−0.053%
								1.5	1.50752	0.501%
2%	3%	1.5	10^{-6}	1	1000	100	10^{-10}	1	1.00247	0.247%
								1.5	1.50451	0.300%
2%	3%	1.5	10^{-7}	1	1000	100	10^{-10}	1	1.033	3.300%
								1.5	1.503	0.200%
2%	3%	1.5	10^{-8}	1	5000	150	10^{-12}	1	1.00327	0.327%
								1.5	1.5009	0.060%
2%	3%	1.5	10^{-8}	1	10000	200	10^{-14}	1	0.99997	−0.003%
								1.5	1.5006	0.040%
2%	3%	1.5	10^{-8}	1	20000	250	10^{-14}	1	1.00007	0.007%
								1.5	1.50038	0.025%

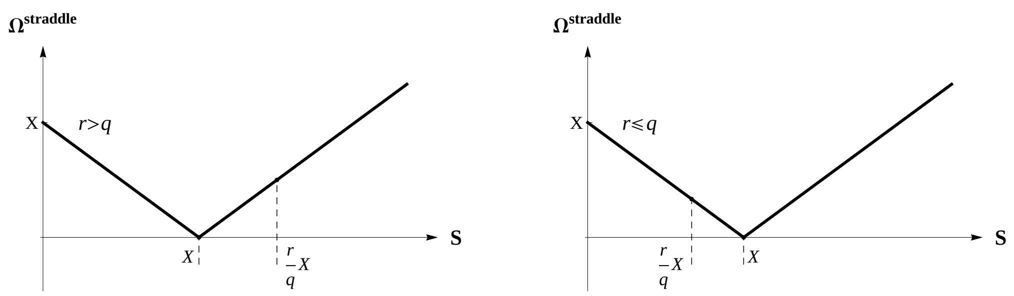

FIGURE D.4: *Limit values of the early exercise boundary at expiry for straddle spread with $r > q$ (left) and $r \leq q$ (right).*

TABLE D.5: *Comparison of analytic and numerical values of the limit of early exercise boundary for condor spread.*

r	q	X_1	X_2	X_3	X_4	T	L	n	m	ε	S^*_{theor}	S^*_{calc}	error
3%	2%	1	3	4	5	10^{-5}	1.5	1000	100	10^{-10}	1.5	1.50104	0.069%
3%	2%	1	3	4	5	10^{-6}	1.5	1000	100	10^{-10}	1.5	1.49654	−0.231%
3%	2%	1	3	4	5	10^{-7}	1.5	1000	100	10^{-10}	1.5	1.45013	−3.324%
3%	2%	1	3	4	5	10^{-8}	1.5	5000	150	10^{-12}	1.5	1.49519	−0.320%
3%	2%	1	3	4	5	10^{-8}	1.5	10000	200	10^{-14}	1.5	1.50013	0.009%
3%	2%	1	3	4	5	10^{-8}	1.5	20000	250	10^{-14}	1.5	1.50002	0.001%
2%	3%	1	3	4	5	10^{-5}	1.5	1000	100	10^{-10}	1	1.00567	0.567%
											4	4.00945	0.236%
											5	5.03622	0.724%
2%	3%	1	3	4	5	10^{-6}	1.5	1000	100	10^{-10}	1	1.00416	0.416%
											4	4.00945	0.236%
											5	5.02113	0.423%
2%	3%	1	3	4	5	10^{-7}	1.5	1000	100	10^{-10}	1	1.00266	0.266%
											4	4.02754	0.688%
											5	5.02113	0.423%
2%	3%	1	3	4	5	10^{-8}	1.5	5000	150	10^{-12}	1	1.00086	0.086%
											4	4.00224	0.056%
											5	5.00309	0.062%
2%	3%	1	3	4	5	10^{-8}	1.5	10000	200	10^{-14}	1	1.00056	0.056%
											4	4.00104	0.026%
											5	5.00234	0.047%
2%	3%	1	3	4	5	10^{-8}	1.5	20000	250	10^{-14}	1	1.00033	0.033%
											4	4.00074	0.019%
											5	5.00159	0.032%
3%	2%	1	2	3	4.5	10^{-5}	1.5	1000	100	10^{-10}	1.5	1.50017	0.012%
											4.5	4.47086	−0.648%
3%	2%	1	2	3	4.5	10^{-6}	1.5	1000	100	10^{-10}	1.5	1.49568	−0.288%
											4.5	4.48429	−0.349%
3%	2%	1	2	3	4.5	10^{-7}	1.5	1000	100	10^{-10}	1.5	0.99609	−33.594%
											4.5	4.48429	−0.349%
3%	2%	1	2	3	4.5	10^{-8}	1.5	5000	150	10^{-13}	1.5	1.49972	−0.018%
											4.5	4.49642	−0.080%
3%	2%	1	2	3	4.5	10^{-8}	1.5	10000	200	10^{-14}	1.5	1.50017	0.012%
											4.5	4.49777	−0.050%
3%	2%	1	2	3	4.5	10^{-8}	1.5	20000	250	10^{-14}	1.5	1.50006	0.004%
											4.5	4.49844	−0.035%

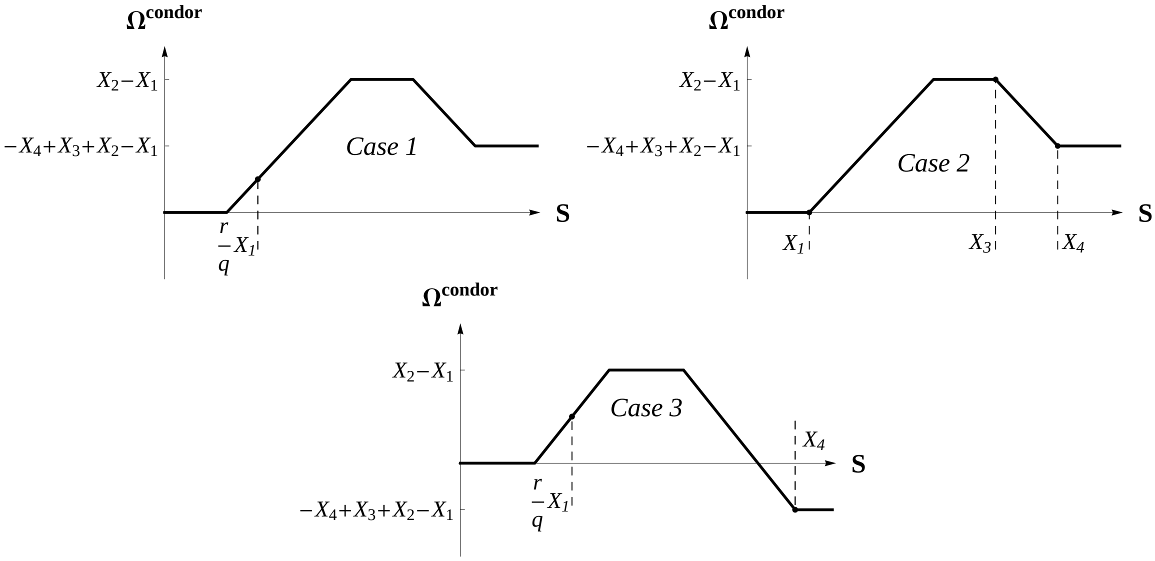

FIGURE D.5: *Limit values of the early exercise boundary at expiry for condor spread with parameters satisfying* $-X_4 + X_3 + X_2 - X_1 > 0$ *and* $r\,(X_3 + X_2 - X_1) \geq qX_4$ *(above left),* $-X_4 + X_3 + X_2 - X_1 > 0$ *and* $r\,(X_3 + X_2 - X_1) < qX_4$ *(above right) and* $-X_4 + X_3 + X_2 - X_1 \leq 0$ *(below).*

List of symbols

We list only the most often used and most important symbols. Other symbols are defined and/or explained in the CHAPTER 1 or at the place of their first usage.

$(\,\cdot\,)^+$ – non-negative part of the value/ function.

∂A – boundary of the set A.

${}^{\infty}, {}^{-\infty}, {}^{\pm\infty}$ – index for a maximum, a minimum and an extreme value, respectively.

$\mathbf{1}_I(\,\cdot\,)$ – indicator function, equals 1 on set I and 0 otherwise.

A – limit at the expiry for Asian option with arithmetic averaging (SECTION 6.2).

A, A_t – average process, value of average process at time t, respectively.

^{a} – index for an arithmetic average.

$a(\,\cdot\,)$ – weight kernel function of weighted average.

$a(\,\cdot\,,\,\cdot\,),\ b(\,\cdot\,,\,\cdot\,)$ – support functions defining the type of averaging (CHAPTER 8).

α, β – coefficients of a log-normal distribution.

B_t – value of a bond (risk-free asset) (CHAPTER 4 and 7).

B_t – Brownian motion (CHAPTER 1).

C_{eu} – value of a European plain vanilla call option.

$\mathcal{C}, \mathcal{C}_t$ – continuation region, continuation region at time t, respectively.

$\mathfrak{c}$ – equals 1 for a call option and -1 for a put option.

$\mathbb{D}$ – domain of a function.

$\mathbb{E}$ – expected value.

$\mathbb{E}_t,\ \mathbb{E}(\,\cdot\,|\mathcal{F}_t)$ – conditioned expected value according to the information at time t.

f_b – American style bonus function.

G – limit at the expiry for Asian option with geometric averaging (CHAPTER 6).

^{g} – index for a geometric average.

H – (amount of) cash.

^{I} – index for an integrated function (SECTION 6.2 and A.3).

L – limit at the expiry for lookback option (SECTION 6.2).

Λ – support parameter; $\Lambda = r - q + \frac{\sigma^2}{2}$ (SECTION 6.2 and A.3).

λ – parameter of weighting exponential kernel function.

M, M_t – maximum process, value of a maximum process at time t, respectively.

m, m_t – minimum process, value of a minimum process at time t, respectively.

μ – drift of a stochastic process.

$\mathcal{N}_t$ – numeraire.

Ω – pay-off function (begins in CHAPTER 2).

P – limit at the expiry for Asian option with general averaging (SECTION 6.2).

P_{eu} – value of a European plain vanilla put option.

$P, Q, \mathcal{Q}, \mathcal{R}$ – probability measures.

p – parameter of the class of general averages.

$\mathbf{\Phi}(\,\cdot\,)$ – standard normal cumulative distribution function (CDF).

$\mathit{\Phi}(\,\cdot\,)$ – standard normal probability density function (PDF).

PP – purchase price of a financial derivative.

q – rate of benefit, dividend rate, $-q$ is cost-of-carry.

r – risk-free interest rate.

ρ_{ij} – correlation coefficient of i^{th} and j^{th} element.

ϱ, ϱ_τ – early exercise boundary and value of the early exercise boundary as a function of time to expiry τ, respectively.

$\mathcal{S}, \mathcal{S}_t$ – stopping region, stopping region at time t, respectively.

S, S_t – underlying asset, spot value of an underlying asset at time t, respectively.

S^*, S_t^* – early exercise boundary, value of the early exercise boundary at time t, respectively.

Σ – covariance matrix.

ς – support parameter; $\varsigma = \frac{r-q+\frac{\sigma^2}{2}}{\frac{\sigma^2}{2}}$.

σ – volatility of a stochastic process.

σ^2 – variance of a stochastic process.

T – expiration time.

T^* – optimal stopping time.

t – time.

τ – time to expiry; $\tau = T - t$.

$\widetilde{V}, \widetilde{V}_t$ – transformed value of a derivative, transformed value of a derivative at time t, respectively.

V, V_t – value of a derivative, value of a derivative at time t, respectively.

v, e – value of a European derivative and an American bonus, respectively.

W_t – Wiener process.

^{wa} – index for a weighted arithmetic average.

$\mathcal{X}_T^*$ – set of limits of early exercise boundary at the expiry.

X, X_i – strike price (CHAPTER 2, 3, 5, APPENDIX A, B, D and SECTION C.3).

x^*, x_t^* – early exercise boundary and value of the early exercise boundary of transformed space variable $x = \frac{A}{S}$, respectively.

x – transformed space variable $x = \frac{A}{S}$.

X, Y, Z – random variables, (super/sub)martingales or stochastic processes.

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Index

$\mathcal{F}_t$-adapted process, 12

adapted process
  increasing, 14
  integrable, 14
  natural, 15
average, 33
  arithmetic, 48, 59, 82, 92, 99
  differential, 53, 104
  geometric, 48, 57, 81, 91, 99
  maximum, 49, 61, 82, 94, 100
  minimum, 49, 61, 82, 94, 100
  parametric class, 48, 54, 88, 99
  weighted arithmetic, 49, 100

Black–Scholes
  formula, 23
  partial differential equation, 22
    Greeks, 128
    path-dependent, 99, 104
    transformation, 165
  transformation, 104
    fixed domain, 106
bonus function, 68, 69, 79
Brownian motion, 10, 50
  geometric, 10, 20, 48
  standard, 9, 51

condition
  extreme, 101, 105
  marginal, 23, 25, 101, 107
  smoothness, 25, 86, 101
  terminal, 23, 25, 101, 107
continuation region, 25, 52, 120

dividend rate, 47, 52
Doob-Meyer decomposition, 16, 68
drift, 20, 80, 103

early exercise boundary, 24, 52, 67, 85
  comparison, 113
  equation, 109
  expression, 107
  numerical scheme, 110
essential supremum, 9, 26, 49
expansion at expiry, 88, 91, 92, 94, 122
expected value, 9
  conditioned, 11, 49
expiration time, 20, 24, 67

filtration, 14, 49
  usual conditions, 14
financial derivative, 19
forward, 26
futures, 21

Girsanov's theorem, 16, 23, 51, 156
Greeks, 46

first order, 129
second order, 134
third order, 140

hedging, 19, 46, 127

Itô
integral, 12
isometry, 13
lemma, 13, 103
multidimensional, 13

limit at expiry, 67, 69, 79, 84, 121, 161
numerical approach, 166

Markov time, 14, 26
martingale, 15
representation theorem, 15
measure
absolutely continuous, 16
risk-neutral, 23, 26, 48, 49

non-arbitrage price, 19, 27
numeraire, 49, 79

option
American plain vanilla, 24, 122
price, 121
Asian, 33, 48, 81, 82, 88, 91, 92, 99, 100
price, 56, 58, 60
call, 21
classification, 21, 29
European plain vanilla, 22
exotic, 22, 29
lookback, 34, 94, 100
price, 64
path-dependent, 32
put, 21
strategies, 69, 166
optional time, 14

path dependence, 31
pay-off function, 20, 48, 69, 79
PSOR, 165
purchase price, 24

Radon-Nikodým
derivative, 16, 156
theorem, 16
random variable, 9
rate of benefit, 20
reflection principle, 155
risk-free interest rate, 20, 47

Snell envelope, 15, 68
state space, 9
stochastic process, 9
stopping region, 24, 52, 120
stopping time, 14, 48, 49
optimal, 25
strike price, 20
submartingale, 15
supermartingale, 15
swap, 21

time, 20
time to expiry, 20

volatility, 20, 48, 103

Wiener process, 9, 20, 103