COMENIUS UNIVERSITY BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS,
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Qualitative Properties of Positive Solutions of Parabolic Equations: Symmetry, A priori Estimates, and Blow-up Rates

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by Juraj Földes

ABSTRACT

In this work, we study qualitative properties of non-negative solutions of quasilinear parabolic equations on bounded and unbounded domains. In the first part, using the method of moving hyperplanes, we establish new nonlinear Liouville theorems for parabolic problems on half spaces. Based on the Liouville theorems and scaling techniques, we derive estimates for the blow-up rates of positive solutions of indefinite parabolic problems in bounded and unbounded domains. As a consequence, we obtain new results on the complete blow-up of these solutions and results for the a priori estimates for positive solutions of indefinite elliptic problems.

In the next part, we employ the method of moving hyperplanes, maximum principle, and Harnack inequality to study symmetry properties of positive solution of asymptotically symmetric quasilinear parabolic problems in the whole space. As the result, we prove that if the problem converges exponentially to a symmetric one, then the solution converges to the space of symmetric functions. We also show that this result does not hold true, if the convergence is not exponential.

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ABSTRAKT

V predkladanej práci študujeme kvalitatívne vlastnosti nezáporných riešení kvázilineárnych parabolických rovníc na ohraničených ako aj neohraničených oblastiach. V prvej časti dokážeme nové Liouvillove vety pre nelineárne parabolické rovnice na polpriestoroch s použitím metódy pohyblivých nadrovín. Na základe týchto viet a s využitím škálovania odvodíme odhady pre kladné riešenia divergujúce v konečnom čase, ktoré riešia indefinitné parabolické problémy na ohraničených alebo neohraničených oblastiach. Ako dôsledok dostaneme nové výsledky pre kompletný "blow-up" týchto riešení, ako aj apriórne odhady pre kladné riešenia indefinitných eliptických problémov.

V nasledujúcej časti použijeme metódu pohyblivých nadrovín, princíp maxima a Harnackovu nerovnosť na štúdium symetrií kladných riešení asymptoticky symetrických, kvázilineárnych parabolických problémov definovaných na celom priestore. Ako hlavný výsledok dokážeme, že riešenie konverguje k priestoru symetrický funkcií, pokiaľ samotný problém konverguje exponenciálne k symetrickému problému. Taktiež ukážeme, že toto tvrdenie vo všeobecnosti neplatí, pokiaľ nevyžadujeme exponenciálnu konvergenciu.

Preface

The basic idea of the qualitative theory of differential equations is to investigate properties of solutions of the problem directly from equations, that is, without knowing explicit form of the solutions. Nowadays, one can solve many problems numerically, however many interesting and important problems stays beyond the reach of current computational capacities. Qualitative theory provides theoretical framework for various numerical algorithms, it can also simplify the problem, for example by the use of symmetries, or it can support the validity of the answer.

Also, it is often more revealing to know properties of the solution than the explicit formula. For example, it is rather difficult to see periodicity of trigonometric functions merely by observing their expansions into infinite series. Moreover, knowing certain properties, such as boundedness or decay at infinity, allows one to use theorems and techniques that might not be available for general functions.

This work attempts to refine the results on asymptotic properties of solution to parabolic partial differential equations. More specifically, in the first part we establish blow-up rates for solutions that cease to exist in a finite time. Such problems were studied extensively in the recent years under various assumptions on equations and domains. To the author's best knowledge, this work contains so far unknown results for solutions of nonlinear indefinite problems for which the nonlinearity can change the sign on the boundary of the domain. The proof uses new Liouville type theorems for nonlinear problems on half spaces that can be of independent interest in other parts of mathematics. Since solutions of elliptic problems can be viewed as equilibria of the parabolic ones, our results apply to them and we obtain a priori estimates for solutions of indefinite elliptic problems.

In the second part, we investigate the question, whether the symmetry of the problem implies the symmetry of solutions. An affirmative answer for this question was established for many elliptic and parabolic problems on bounded and unbounded domains. The standing assumption was that the equation as well as the domain was symmetric. Here, we consider evolutionary equations on the whole space that are asymptotically symmetric rather than symmetric, that is, the equation converges to a symmetric one. We prove that if the convergence is exponential, then the solution approaches the space of radially symmetric functions with the common origin. However, if the convergence is not exponential, then the limiting profiles do not necessarily share the same origin.

The rest of the work is organized as follows. Chapter 1 discusses estimates for the blow-up rates. In Section 1.1, we formulate the problem, describe known results, and state our theorems and their possible extensions. Section 1.2 contains parabolic Liouville theorems and in Section 1.4, we formulate doubling lemma and we prove our main results.

Chapter 2 has similar structure. First, in Section 2.1 we formulate the symmetry problem, and we briefly discuss its history. In Section 2.2, we state our main symmetry results. Section 2.3 contains general linear estimates of parabolic problems and in Section 2.4 we prove the symmetry results.

Most results in this work are included in [22] for a priori estimates and in [24] for asymptotically symmetric problems.

Contents

Chapter 1

Liouville theorems and a priori estimates

1.1 Introduction

In this chapter we consider the problem

$$\begin{aligned} u_t &= \Delta u + a(x)|u|^{p-1}u, && (x,t) \in \Omega \times (0,T)\,, \\ u &= 0, && (x,t) \in \partial\Omega \times (0,T)\,, \end{aligned} \tag{1.1}$$

which, if needed, is completed with an initial condition

$$u(\cdot,0) = u_0(\cdot) \in L^\infty(\Omega)\,. \tag{1.2}$$

We assume that Ω is a smooth domain in $\mathbb{R}^N$ and $p > 1$. Furthermore, we suppose that $a : \bar{\Omega} \to \mathbb{R}$ belongs to $C^2(\bar{\Omega})$ and

$$\text{if} \quad \lim_{k\to\infty} a(x_k) = 0, \quad \text{then} \quad \limsup_{k\to\infty} |\nabla a(x_k)| > 0\,. \tag{1.3}$$

Here, $C^k(D)$ denotes the space of k-times differentiable, bounded functions on $D \subset \mathbb{R}^N$, with bounded, continuous derivatives up to k^{th} order.

If Ω is bounded and if we denote

$$\Gamma := \{x \in \bar{\Omega} : a(x) = 0\}, \tag{1.4}$$

$$\Omega^+ := \{x \in \Omega : a(x) > 0\}, \tag{1.5}$$

$$\Omega^- := \{x \in \Omega : a(x) < 0\}\,, \tag{1.6}$$

then (1.3) is equivalent to

$$|\nabla a(x)| \neq 0 \qquad (x \in \Gamma)\,, \tag{1.7}$$

that is, a has nondegenerate zeros in $\bar{\Omega}$. Since u_0 and a are bounded, standard results [44] yield the unique, strong solution of the problem (1.1), (1.2), with the maximal existence time $T_{\max} \in (0, \infty]$. Moreover, by regularity results, if $T_{\max} < \infty$, then $\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to T_{\max}$. We do not indicate the dependence of $T_{\max}$ on u_0 if no confusion seems possible. Here and in the rest of the work we assume $T \in (0, T_{\max}]$.

As a main result of this chapter, we derive an upper bound for the blow-up rate of nonnegative solutions of (1.1). The blow-up rates and related a priori estimates were studied under various assumptions on a, Ω and u in [1, 21, 26, 34, 30, 31, 32, 45, 55, 57, 58, 67, 65, 66], see also references therein. We just briefly describe the results directly connected to our results. First, Friedman and McLeod [26] studied blowing up solutions ($T_{\max} < \infty$) of the problem

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u, && (x,t) \in \Omega \times (0,T)\,, \\ u &= 0, && (x,t) \in \partial\Omega \times (0,T)\,, \end{aligned} \tag{1.8}$$

with $T = T_{\max}$, and the initial condition (1.2). They proved

$$|u(x,t)| \leq C(1 + (T_{\max} - t)^{-\frac{1}{p-1}}) \qquad (x \in \Omega)\,, \tag{1.9}$$

where Ω is a bounded convex domain, $p > 1$, and u is positive, increasing (in time) solution of (1.8). These results were generalized by Giga and Kohn [30] and later by Giga et al. [31, 32]. With the help of localized energy estimates and iterative arguments, they proved that (1.9) holds true if Ω is a bounded convex domain or $\Omega = \mathbb{R}^N$, u is, not necessarily positive, solution of (1.8), (1.2), and $1 < p < p_S$, where

$$p_S = p_S(N) := \begin{cases} \infty & N \leq 2\,, \\ \frac{N+2}{N-2} & N \geq 3\,. \end{cases}$$

In [20] Fila and Souplet employed scaling and Fujita type results to remove the assumption on convexity of Ω and they established (1.9) for all positive solutions of (1.8), (1.2), and $1 < p \leq 1 + \frac{2}{N+1}$.

Finally, Poláčik et al. [55] investigated positive solutions of (1.8) with sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ and $1 < p < p_B$, where

$$p_B = p_B(N) := \begin{cases} \infty & N \leq 1\,, \\ \frac{N(N+2)}{(N-1)^2} & N \geq 2\,. \end{cases} \tag{1.10}$$

Using scaling, doubling lemma and Liouville theorems they obtained

$$u(x,t) \leq C(1 + t^{-\frac{1}{(p-1)}} + (T-t)^{-\frac{1}{(p-1)}}) \qquad ((x,t) \in \Omega \times (0,T))\,, \tag{1.11}$$

where C is a universal constant depending only on p, N and Ω. We remark that the estimates for the initial blow-up rate were previously established by Bidaut-Véron [13] (see also [5]) for $1 < p < p_B$ and $\Omega = \mathbb{R}^N$. Some estimates on the initial blow-up rates for bounded Ω were proved by Quittner et al. [59].

The first a priori estimates for positive solutions of (1.1), (1.2) with sign-changing a were derived in the form (see [57] and references therein)

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^\infty(\Omega)}, \delta, N, p, \Omega, a) \qquad (t \in [0, T_{\max} - \delta], \delta > 0, T_{\max} < \infty)\,. \tag{1.12}$$

Later, Xing [67] obtained an upper estimate for the blow-up rate, of positive solutions of (1.1), (1.2)

$$u(x,t) \leq C(1 + (T_{\max} - t)^{-\frac{3}{2(p-1)}}) \qquad ((x,t) \in \Omega \times (0, T_{\max}), T_{\max} < \infty)$$

when Ω is bounded, $1 < p < p_B$ and $\Gamma \subset \Omega$, that is, when a does not vanish on $\partial\Omega$. Here C depends on $\|u_0\|_{L^\infty(\Omega)}$, N, p, Ω, a.

The next theorem refines the results in [67] in various directions. It includes unbounded domains and it allows for a very general behavior of a on $\partial\Omega$. In addition it also yields an estimate for the initial blow-up rate. Denote $\nu_\Omega(x)$ the unit outward normal vector to $\partial\Omega$ at x.

Theorem 1.1.1. *Let Ω be a uniformly regular domain of class C^2 in $\mathbb{R}^N$ (cf. [4]) and let $1 < p < p_B$. Suppose that $a \in C^2(\bar{\Omega})$ satisfies* (1.3) *and*

$$\left| \frac{\nabla a(x_0)}{|\nabla a(x_0)|} - \nu_\Omega(x_0) \right| \geq \tilde{c} > 0 \qquad (x_0 \in \Gamma \cap \partial\Omega)\,. \tag{1.13}$$

Then every nonnegative solution u of (1.1) *satisfies*

$$u(x,t) \leq C(1 + t^{-\frac{3}{2(p-1)}} + (T-t)^{-\frac{3}{2(p-1)}}) \qquad ((x,t) \in \Omega \times (0,T))\,, \tag{1.14}$$

where C depends on N, p, Ω and a.

Remark 1.1.2. (a) The nonlinearity $|u|^{p-1}u$ in (1.1) can be replaced by $f(u)$ with

$$\lim_{v \to \infty} \frac{f(v)}{v^p} = \ell > 0\,.$$

Then (1.14) holds with C depending on N, f, Ω and a. Also, we can add lower order terms to the right hand side, that is, we can add a function $g : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{u \to \infty} \sup_{(x,t) \in \Omega \times (0,T)} \frac{g(x,t,u)}{u^p} = 0\,.$$

Then (1.14) holds with C depending on N, p, Ω, a and g.

(b) For the blowing-up solutions ($T_{\max} < \infty$) of (1.8) one has (cf. [58, Proposition 23.1]) $\sup_{x\in\mathbb{R}^N} u(x,t) \geq C(T_{\max}-t)^{-\frac{1}{p-1}}$. This shows the optimality of the final blow up estimate in (1.11) for $a \equiv 1$. However, it is not known whether or not the weaker estimate (1.14) is optimal for sign changing a. Below, we show that under additional assumptions the stronger estimate (1.11) holds true even if a changes sign.

(c) If a also depends on t and $p > \frac{N+2}{N}$, the initial blow-up estimate in (1.14) does not hold even if $0 \leq a \leq 1$ (see e.g. [63, 64]). If Ω is bounded, then (1.13) is equivalent to $\frac{\nabla a(x_0)}{|\nabla a(x_0)|} \neq \nu_\Omega(x_0)$ for any $x_0 \in \Gamma \cap \partial\Omega$. It is not known if this assumption is technical or not.

(d) Universal estimates of the form (1.11) or (1.14) are not true for $p \geq p_S$, $N \geq 3$, $\Omega = \mathbb{R}^N$, due to the existence of arbitrarily large stationary radial solutions of (1.1). We require $p < p_B < p_S$ mainly because the Liouville theorem for the problem

$$u_t = \Delta u + u^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,, \tag{1.15}$$

with $p_B \leq p < p_S$ is not known. If one proves such a Liouville theorem for some $p \in [p_B, p_S)$, then the conclusion of Theorem 1.1.1 would hold for the same p as well.

(e) If we restrict ourselves to the class of radial solutions (of course now Ω and a are radially symmetric), then similarly as in [55], one can prove Theorem 1.1.1 for each $1 < p < p_S$. This is possible, since the Liouville theorem is known for nonnegative radial solution of (1.15) for any $1 < p < p_S$ (see [53]).

(f) If a nonnegative solution u of (1.1) is global ($T_{\max} = \infty$), then after letting $T \to \infty$ in (1.14), we obtain

$$u(x,t) \leq C(1 + t^{-\frac{3}{2(p-1)}}) \qquad ((x,t) \in \Omega \times (0,\infty))\,. \tag{1.16}$$

In particular u is bounded on $\Omega \times (1,\infty)$. For previous results, see [13, 55].

Remark 1.1.3. Observe that (1.14) is equivalent to

$$M(x,t) \le C(1 + d^{-1}(t)) \qquad ((x,t) \in \Omega \times (0,T))\,, \tag{1.17}$$

where

$$M := u^{\frac{(p-1)}{3}} \quad \text{and} \quad d(t) := \min\{t, T-t\}^{\frac{1}{2}}\,.$$

Also, for each $x \in \Omega$, one has $d(t) = d_P[(x,t), \Theta]$, where $\Theta := \Omega \times \{0,T\}$ and d_P denotes the parabolic distance:

$$d_P[(x,t),(y,s)] = |x-y| + |t-s|^{\frac{1}{2}} \qquad ((x,t),(y,s) \in \Omega \times (0,T))\,. \tag{1.18}$$

In this notation we obtain yet another form of (1.14)

$$u(x,t) \le C(1 + d_P^{-3/(p-1)}[(x,t), \Theta]) \qquad ((x,t) \in \Omega \times (0,T))\,.$$

If u is a stationary solution of (1.1), that is, if u solves

$$\begin{aligned} 0 &= \Delta u + a(x)|u|^{p-1}u, \qquad && x \in \Omega\,, \\ u &= 0, && x \in \partial\Omega\,, \end{aligned} \tag{1.19}$$

we obtain the following corollary.

Corollary 1.1.4. *Let $\Omega \subset \mathbb{R}^N$ be a uniformly regular domain of class C^2 (cf. [4]), $1 < p < p_S$, and $a \in C^2(\bar{\Omega})$ that satisfies* (1.3) *and* (1.13)*. If u is a nonnegative solution of* (1.19)*, then $u \le C(p, N, \Omega, a)$.*

This corollary extends the results of Du and Li [17] (see also references therein), as it allows a to vanish on $\partial\Omega$. If $1 < p < p_B(N)$, then since $T_{\max} = \infty$, Corollary 1.1.4 follows from (1.16). If we merely assume $1 < p < p_S(N)$, then one has to reprove Theorem 1.1.1 for solutions of (1.19). The only difference is the application of elliptic Liouville theorems [29], instead of parabolic ones, whenever $p < p_B$ is required.

The next propositions shows that final blow-up rates in Theorem 1.1.1 (and main results in [67]) can be improved if $a > 0$ and Ω is a convex bounded set. Notice that a is allowed to vanish on $\partial\Omega$. In this case, the universal bounds (1.12) were already obtained in [57].

Proposition 1.1.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth, convex set and let $1 < p < p_B$. Assume $a \in C^2(\bar{\Omega})$ satisfies* (1.7) *and $a(x) > 0$ for $x \in \Omega$. Then a nonnegative solution u of* (1.1), (1.2) *satisfies*

$$u(x,t) \le C(1 + (T-t)^{-\frac{1}{p-1}}) \qquad ((x,t) \in \Omega \times (0,T)), \tag{1.20}$$

where C depends on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.

If a is changes sign in Ω, we formulate a sufficient conditions for (1.20) only in the one-dimensional case. However, one can generalize the following propositions to higher dimensional case if Ω is convex and certain monotonicity of a and u_0 near $\partial\Omega$ is assumed.

Proposition 1.1.6. *Let $N = 1$ and $\Omega = (0,1)$. Suppose that $a \in C^2([0,1])$ and has exactly one nondegenerate zero $\mu \in [0,1]$, that is, $a(\mu) = 0$ and $a'(\mu) \neq 0$. If*

$$\operatorname{sign}[a(x)](u_0(2\mu - x) - u_0(x)) \le 0 \qquad (x \in (\max\{0, 2\mu - 1\}, \mu)),$$

then a nonnegative classical solution u of (1.1), (1.2) *satisfies* (1.20) *with C depending on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.*

Proposition 1.1.7. *Let $N = 1$ and $\Omega = (0,1)$. Suppose that $a \in C^2([0,1])$ and has exactly two nondegenerate zero $\mu_1 < \mu_2$ in $[0,1]$, that is, $a(\mu_i) = 0$ and $a'(\mu_i) \neq 0$ for $i = 1,2$. If $\max\{\mu_1, 1-\mu_2\} < \mu_2 - \mu_1$ and*

$$\begin{aligned} a(x) < 0, \quad & u_0(2\mu_1 - x) \ge u_0(x) \qquad (x \in (0,\mu_1)), \\ & u_0(2\mu_2 - x) \ge u_0(x) \qquad (x \in (\mu_2, 1)), \end{aligned}$$

then a nonnegative classical solution u of (1.1), (1.2) *satisfies* (1.20) *with C depending on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.*

Let us briefly explain the connection between the blow-up rate and Liouville theorems. We need the following notation:

$$\mathbb{R}^N_c := \{x = (x_1, x') \in \mathbb{R}^N : x_1 > c\} \qquad (c \in \mathbb{R})\,.$$

Assume that (1.14) fails, that is, we assume that there exists $(x_k, t_k) \in \Omega \times (0, T)$ such that

$$u(x_k, t_k) \geq 2kC(1 + t_k^{-\frac{3}{2(p-1)}} + (T - t_k)^{-\frac{3}{2(p-1)}}) \qquad (k \in \mathbb{N})\,.$$

After an application of doubling lemma (see Lemma 1.4.1 below) and appropriate scaling, we obtain a bounded nonnegative function v with $v(0,0) = 1$ that solves

$$v_t = \Delta v + v^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,, \tag{1.21}$$

or

$$\begin{aligned} v_t &= \Delta v + v^p, && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}\,, \\ v &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}\,, \end{aligned} \tag{1.22}$$

for some $c^* \in \mathbb{R}$, provided $a > 0$ in $\bar{\Omega}$. If a satisfies (1.7) and $\Gamma \subset \Omega$, then v can also solve

$$v_t = \Delta v + x_1 v^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,, \tag{1.23}$$

or

$$v_t = \Delta v - v^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,. \tag{1.24}$$

Finally, if we do not assume $\Gamma \subset \Omega$, then v could be a solution of

$$\begin{aligned} v_t &= \Delta v + (x \cdot b) v^p, && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}\,, \\ v &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}\,, \end{aligned} \tag{1.25}$$

where b is a unit vector and $c^* \in \mathbb{R}$, or

$$\begin{aligned} v_t &= \Delta v - v^p, && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}\,, \\ v &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}\,. \end{aligned} \tag{1.26}$$

However, if v satisfies (1.21) and $1 < p < p_B(N)$, we obtain a contradiction to the Liouville theorem proved by Bidaut-Véron [13], which extends classical elliptic results of Gidas and Spruck [29].

If v satisfies (1.22) and $1 < p < p_B(N-1)$, we obtain a contradiction to the Liouville theorem proved by Poláčik et al. [55]. Observe that $p_B(N-1) > p_S(N)$. A contradiction for (1.23) follows from [52] for any $p > 1$. Liouville theorems for (1.24) and (1.26) were proved in [67], but here we provide a simpler proof (see Lemma 1.2.1 and Lemma 1.2.2 below). Nonexistence result for (1.25), based on [52], is new and its proof is given in Section 1.2.

One can also employ Liouville theorems and universal estimates in the investigation of the complete blow-up and the continuity of blow-up time. Let us recall these notions and explain the results.

Let u be a nonnegative solution of (1.1), (1.2) with $T_{\max} < \infty$. Let u_k, $(k \in \mathbb{N})$ be the solution of the approximation problem

$$\begin{aligned} (u_k)_t - \Delta u_k &= f_k(x, u_k), && (x,t) \in \Omega \times (0,\infty)\,, \\ u_k &= 0, && (x,t) \in \partial\Omega \times (0,\infty)\,, \\ u_k(x,0) &= u_0(x) \geq 0, && x \in \Omega\,, \end{aligned}$$

where

$$f_k(x,v) := \begin{cases} a(x)\min\{v^p, k\} & \text{if } a(x) \geq 0, v \in \mathbb{R}\,, \\ a(x)v^p & \text{if } a(x) < 0, v \in \mathbb{R}\,. \end{cases}$$

Since f_k is bounded from above, nonnegative solution u_k exists globally (for all positive times). Since $f_k \leq f_{k+1}$, the maximum principle implies $u_{k+1}(x,t) \geq$

$u_k(x,t)$ for any $(x,t) \in \Omega \times (0,\infty)$. Thus

$$\bar{u}(x,t) := \lim_{k\to\infty} u_k(x,t) \in [0,\infty] \qquad ((x,t) \in \Omega \times [0,\infty))$$

is well defined. Moreover, $\bar{u}(x,t) = u(x,t)$ for any $(x,t) \in \bar{\Omega} \times [0, T_{\max})$. We say that u *blows-up completely* in $D \subset \Omega$ at T, if $\bar{u}(x,t) = \infty$ for any $x \in D$ and $t > T$.

Theorem 1.1.8. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $1 < p < p_B$. Suppose that $a \in C^2(\bar{\Omega})$ satisfies* (1.7) *and* (1.13)*. If $T_{max} < \infty$ for a nonnegative solution u of* (1.1), (1.2)*, then u blows-up completely in Ω^+ at T_{max}. In addition, the function*

$$T : \{u_0 \in L^\infty(\Omega) : u_0 \geq 0\} \to (0,\infty], \qquad T : u_0 \mapsto T_{max}(u_0)$$

is continuous.

If $a \equiv 1$, Baras and Cohen [9] proved complete blow-up of nonnegative solutions of (1.8), (1.2) in Ω at $T_{\max} < \infty$ for each $1 < p < p_S$ (see also [58]). However, for $p > p_S$, $N \leq 10$, and Ω being a ball, there exist radial solutions of (1.8) that do not blow-up completely in Ω at $T_{\max}$. For further discussion see [58] and references therein.

If a changes sign, then one cannot expect the complete blow-up in the whole Ω, since $\bar{u}$ stays bounded in Ω^- for any $t > 0$ (see [43]). Quittner and Simondon [57] proved the complete blow-up of u in Ω^+ at $T_{\max} < \infty$ for $1 < p \leq 1 + 3/(N+1)$ and $\Gamma \subset \Omega$. Later Poláčik and Quittner [52] replaced the former assumption by $1 < p < p_B$ and proved Theorem 1.1.8 under an additional assumption $\Gamma \subset \Omega$.

The rest of the chapter is organized as follows. In Section 1.2 we state and prove parabolic Liouville theorems. In Section 1.4 we formulate doubling lemma and we prove our main results.

1.2 Liouville theorems

Since some results in this section can be of independent interest, we formulate them in more general setting, than required for the proofs of the a priori estimates. Let us define

$$\mathbb{R}^N_\lambda := \{x = (x_1, x') \in \mathbb{R}^N : x_1 > \lambda\} \qquad (\lambda \in \mathbb{R}), \tag{1.27}$$

$$H_\lambda := \partial\mathbb{R}^N_\lambda = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = \lambda\} \qquad (\lambda \in \mathbb{R}). \tag{1.28}$$

The following two lemmas were proved in [67] for increasing function f. Here, we propose a simpler proofs that remove this unnecessary assumption. The elliptic counterparts can be found in [19, 61, 62], see also references therein.

Lemma 1.2.1. *Let f be a continuous function with $f(v) > 0$ for any $v > 0$. If $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonnegative bounded solution of*

$$u_t - \Delta u = -f(u), \qquad (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

then $u \equiv 0$.

Proof. We proceed by a contradiction, that is, we assume $u \not\equiv 0$. Fix $(x^*, t^*) \in \mathbb{R}^N \times \mathbb{R}$ such that

$$u(x^*, t^*) \geq C^* := \frac{1}{2} \sup_{(x,t)\in\mathbb{R}^N\times\mathbb{R}} u(x, t) > 0.$$

For each $\varepsilon > 0$ denote

$$v_\varepsilon(x, t) := u(x, t) - \varepsilon|x - x^*|^2 - \varepsilon(\sqrt{(t - t^*)^2 + 1} - 1) \qquad ((x, t) \in \mathbb{R}^N \times \mathbb{R}).$$

Since $v_\varepsilon(x, t) \to -\infty$ whenever $|t| \to \infty$ or $|x| \to \infty$, there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^N \times \mathbb{R}$ with

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{(x,t)\in\mathbb{R}^N\times\mathbb{R}} v_\varepsilon(x, t).$$

Then for each $\varepsilon > 0$

$$2C^* \geq u(x_\varepsilon, t_\varepsilon) \geq v_\varepsilon(x_\varepsilon, t_\varepsilon) \geq v_\varepsilon(x^*, t^*) = u(x^*, t^*) \geq C^* > 0\,,$$

and

$$(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0\,, \qquad \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \leq 0\,.$$

Consequently,

$$\begin{aligned} 0 &\leq (v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \\ &= u_t(x_\varepsilon, t_\varepsilon) - \Delta u(x_\varepsilon, t_\varepsilon) - \varepsilon \frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon N \\ &= -f(u(x_\varepsilon, t_\varepsilon)) - \varepsilon \frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon N \\ &\leq - \inf_{2C^* \geq v \geq C^*} f(v) + \varepsilon + 2\varepsilon N \qquad (\varepsilon > 0)\,. \end{aligned}$$

Since the first term on the right hand side is negative and independent of ε, we obtain a contradiction for sufficiently small $\varepsilon > 0$. □

Lemma 1.2.2. *Suppose $f \in C^1$ satisfies $f(0) = 0$ and $f(v) > 0$ for any $v > 0$. Let h be a continuous function with $h(x_1) < 0$ for each $x_1 > 0$, and*

$$\limsup_{x_1 \to \infty} h(x_1) < 0\,.$$

If u is a nonnegative bounded solution of the problem

$$\begin{aligned} u_t - \Delta u &= h(x_1) f(u), \qquad && (x, t) \in \mathbb{R}^N_0 \times \mathbb{R}\,, \\ u &= 0, && (x, t) \in H_0 \times \mathbb{R}\,, \end{aligned}$$

then $u \equiv 0$.

Proof. The proof is similar to the proof of Lemma 1.2.1. We again proceed by a contradiction, that is, we assume $u \not\equiv 0$. Fix $(x^*, t^*) \in \mathbb{R}_0^N \times \mathbb{R}$ such that

$$u(x^*, t^*) \geq C^* := \frac{1}{2} \sup_{(x,t)\in\mathbb{R}_0^N \times \mathbb{R}} u(x,t) > 0\,.$$

It is easy to see that there exists a function $\phi \in C^2(\mathbb{R}^N \times \mathbb{R})$ with

$$\begin{gathered}\phi(x,t) \geq 0, \quad |\nabla\phi(x,t)| \leq 1, \quad |\phi_t - \Delta\phi| \leq 1 \qquad ((x,t) \in \mathbb{R}^N \times \mathbb{R})\,, \\ \phi(0,0) = 0, \quad \phi(x,t) \to \infty \quad \text{if } |x| \to \infty \quad \text{or } t \to \pm\infty\,.\end{gathered}$$

For each $\varepsilon \in (0,1)$ denote

$$v_\varepsilon(x,t) := u(x,t) - \varepsilon\phi(x - x^*, t - t^*) \qquad ((x,t) \in \mathbb{R}_0^N \times \mathbb{R})\,.$$

Since u is bounded, $v_\varepsilon(x,t) \to -\infty$ whenever $|t| \to \infty$ or $|x| \to \infty$. Moreover, $v_\varepsilon(x,t) \leq 0 < v_\varepsilon(x^*,t^*)$ for any $(x,t) \in H_0 \times \mathbb{R}$, and therefore there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}_0^N \times \mathbb{R}$ such that

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{(x,t)\in\mathbb{R}_0^N \times \mathbb{R}} v_\varepsilon(x,t)\,.$$

Consequently,

$$2C^* \geq u(x_\varepsilon, t_\varepsilon) \geq v_\varepsilon(x_\varepsilon, t_\varepsilon) \geq v_\varepsilon(x^*, t^*) = u(x^*, t^*) \geq C^* > 0\,,$$

and

$$(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0, \qquad (\Delta v_\varepsilon)(x_\varepsilon, t_\varepsilon) \leq 0\,.$$

Observe that u satisfies

$$u_t = \Delta u + h(x_1)\frac{f(u)}{u}u = \Delta u + c(x,t)u\,.$$

Since $f \in C^1$, $f(0) = 0$, and u is bounded, c is a bounded function in $\{(x,t) \in \mathbb{R}_0^N \times \mathbb{R} : x_1 < 2\}$. Hence, standard parabolic regularity (see for example [42, Theorem 1.15]) implies

$$|\nabla u(x,t)| \le C \qquad ((x,t) \in \bar{\mathbb{R}}^N_0 \times \mathbb{R}, x_1 < 1)\,,$$

and consequently

$$|\nabla v_\varepsilon(x,t)| \le C + 1 \qquad ((x,t) \in \bar{\mathbb{R}}^N_0 \times \mathbb{R}, x_1 < 1)\,,$$

where C is independent of $\varepsilon \in (0,1)$. Furthermore, $v_\varepsilon(x_\varepsilon, t_\varepsilon) \ge C^* > 0$ and $v_\varepsilon(x,t) \le 0$ for all $(x,t) \in H_0 \times \mathbb{R}$ yield $\mathrm{dist}(x_\varepsilon, H_0) = (x_\varepsilon)_1 \ge c_0$, where c_0 is a constant independent of ε. Finally,

$$\begin{aligned}
0 &\le (v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \\
&= u_t(x_\varepsilon, t_\varepsilon) - \Delta u(x_\varepsilon, t_\varepsilon) - \varepsilon[\phi_t(x_\varepsilon, t_\varepsilon) - \Delta\phi(x_\varepsilon, t_\varepsilon)] \\
&\le h((x_\varepsilon)_1) f(u(x_\varepsilon, t_\varepsilon)) + \varepsilon \\
&\le \sup_{y \ge c_0} h(y) \inf_{2C^* \ge v \ge C^*} f(v) + \varepsilon\,.
\end{aligned}$$

Since the first term on the right hand side is negative and independent of ε, we obtain a contradiction for sufficiently small $\varepsilon > 0$. □

Next, consider the problem

$$\begin{aligned}
u_t - \Delta u = h(x \cdot v) f(u), &\qquad (x,t) \in \Omega \times \mathbb{R}\,, \\
u = 0, &\qquad (x,t) \in \partial\Omega \times \mathbb{R}\,,
\end{aligned} \tag{1.29}$$

where

(v1) $v = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^N$ is a unit vector with $v_1 > 0$ and $v_i = 0$ for $i \ge 3$.

About Ω, we assume that

(d1) Ω is a subset of $\mathbb{R}^N$, convex and unbounded in x_1, that is, $x + \xi e_1 \in \Omega$ for any $x \in \Omega$ and $\xi > 0$.

(d2) there is a constant d^* such that $x_2 v_2 \le d^*$ for any $x = (x_1, x_2, \cdots, x_N) \in \Omega$.

Next, the function $h : \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis.

(h1) h is continuous, nondecreasing, and strictly increasing on $(0, \infty)$.

(h2) $h(0) = 0$ and $\lim_{y\to\infty} h(y) = \infty$.

About f we assume

(f1) $f \in C^1([0,\infty))$, with $f(0) = f'(0) = 0$, and $f(v) > 0$, $f'(v) \ge 0$ for each $v > 0$.

The following theorem is a generalization of elliptic [17] and parabolic [52] results proved for $v = e_1$ and $\Omega = \mathbb{R}^N$. The general framework of the proof is similar to one used in [17, 52].

Theorem 1.2.3. *If (v1), (d1), (d2), (h1), (h2), and (f1) hold true, then the only nonnegative, bounded solution u of* (1.29) *is $u \equiv 0$.*

As a corollary we obtain Liouville theorem for indefinite problems on half spaces.

Corollary 1.2.4. *Given unit vectors $b, v \in \mathbb{R}^N$ and a constant c^*, let $\Omega := \{x \in \mathbb{R}^N : x \cdot b > c^*\}$. Consider functions h and f that satisfy (h1), (h2) and (f1) respectively. Let u be a nonnegative, bounded solution of* (1.29). *If $v \ne -b$, then $u \equiv 0$.*

Remark 1.2.5. The statement of Corollary 1.2.4 still holds true if $v = -b$, $c^* \ge 0$ and h in addition to (h1), (h2) satisfies $h(y) < 0$ for $y < 0$. This follows after suitable rotation and translation, from Lemma 1.2.2. However, if $v = -b$ and $c^* < 0$, then there are nontrivial, nonnegative solutions of (1.29) (see Proposition 1.3.1 below).

Proof of Corollary 1.2.4. We rotate the coordinates such that $b = e_2$, $v_1 \geq 0$, and $v_i = 0$ for $i \geq 3$. Then $\Omega = \{x \in \mathbb{R}^N : x_2 > c^*\}$ and (d1) holds true. Notice that (1.29), (h1), (h2), and (f1) are invariant under rotations.

If $v_1 > 0$ and $v_2 \leq 0$, then (v1) and (d2) are satisfied with $d^* = c^* v_2$, and the corollary follows from Theorem 1.2.3.

If $v_2 > 0$, consider another rotation that maps v to e_1 and fixes the space spanned by $\{e_3, \cdots, e_N\}$. Then (v1) and (d2) are clearly satisfied with $d^* = 0$. Also, Ω is transformed to $\Omega' := \{x \in \mathbb{R}^N : x \cdot b' > c^*\}$, where $b' = (v_2, v_1, 0, \cdots, 0)$. In particular $b'_1 > 0$ and (d1) holds. Now, the corollary follows from Theorem 1.2.3.

If $v_1 = 0$ and $v_2 \leq 0$, then $v = -e_2 = -b$, a contradiction to our assumptions. $\square$

Before we proceed, define $Lu := u_t - \Delta u$ and $M := \sup_\Omega u$. Furthermore, given $\lambda \in \mathbb{R}$ set

$$\begin{aligned}
\Sigma_\lambda &:= \{x \in \Omega : x_1 < \lambda\}, \\
x^\lambda &:= (2\lambda - x_1, x_2, \cdots, x_N) \qquad (x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N), \\
w_\lambda(x,t) &:= u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \bar{\Sigma}_\lambda \times \mathbb{R}), \\
\lambda(t) &:= \sup\{\mu : w_\lambda(x,t) \geq 0 \text{ for all } x \in \Sigma_\lambda \text{ and } \lambda < \mu\}, \\
\lambda^* &:= \inf\{\lambda(t) : t \in \mathbb{R}\}.
\end{aligned} \tag{1.30}$$

Observe that (d1) implies $x^\lambda \in \Omega$ for any $x \in \bar{\Sigma}_\lambda$, and therefore w_λ is well defined. Moreover, since u is nonnegative in Ω and vanishes on $\partial\Omega$,

$$w_\lambda(x,t) = u(x^\lambda, t) - u(x,t) = u(x^\lambda, t) \geq 0 \qquad ((x,t) \in (\partial\Omega \cap \bar{\Sigma}_\lambda) \times \mathbb{R}).$$

Clearly $w_\lambda(x,t) = 0$ if $(x,t) \in (\Omega \cap \partial\Sigma_\lambda) \times \mathbb{R}$, and therefore

$$w_\lambda(x,t) \geq 0 \qquad ((x,t) \in \partial\Sigma_\lambda \times \mathbb{R}). \tag{1.31}$$

We divide the proof of Theorem 1.2.3 into several lemmas, in which we implicitly suppose the assumptions of the theorem.

First, notice that $v_1 > 0$ implies

$$x^\lambda \cdot v - x \cdot v = 2(\lambda - x_1)v_1 \geq 0 \qquad (x \in \Sigma_\lambda)\,, \tag{1.32}$$

and consequently by (h1)

$$h(x \cdot v) \leq h(x^\lambda \cdot v) \qquad (x \in \Sigma_\lambda)\,. \tag{1.33}$$

Lemma 1.2.6. *If there are $\lambda \in \mathbb{R}$, $\tilde{x} \in \Sigma_\lambda$ and $\tilde{t} \in \mathbb{R}$ with $h(\tilde{x} \cdot v) \leq 0$ and $w_\lambda(\tilde{x}, \tilde{t}) \leq 0$, then $Lw_\lambda(\tilde{x}, \tilde{t}) \geq 0$. Moreover, if $\tilde{x}_1 \leq \frac{-d^*}{v_1}$, then $w_\lambda(\tilde{x}, \tilde{t}) \leq 0$ implies $Lw_\lambda(\tilde{x}, \tilde{t}) \geq 0$.*

Proof. The positivity and monotonicity of f, and (1.33) yield

$$\begin{aligned} Lw_\lambda(\tilde{x}, \tilde{t}) &= h(\tilde{x}^\lambda \cdot v) f(u(\tilde{x}^\lambda, \tilde{t})) - h(\tilde{x} \cdot v) f(u(\tilde{x}, \tilde{t})) \\ &\geq h(\tilde{x} \cdot v)[f(u(\tilde{x}^\lambda, \tilde{t})) - f(u(\tilde{x}, \tilde{t}))] \geq 0\,, \end{aligned}$$

and the first statement follows. Next, assume $\tilde{x}_1 \leq -\frac{d^*}{v_1}$. Then $v_1 > 0$ and (d2) imply

$$\tilde{x} \cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leq \tilde{x}_1 v_1 + d^* \leq 0\,,$$

and by (h1) and (h2) one has $h(\tilde{x} \cdot v) \leq 0$. Now, the second statement follows from the first one. □

Lemma 1.2.7. $\lambda(t) \geq \frac{-d^*}{v_1}$ *for all* $t \in \mathbb{R}$.

Proof. We proceed by a contradiction, that is, we assume the existence of $\lambda < \frac{-d^*}{v_1}$ and $(\tilde{x}, \tilde{t}) \in \Sigma_\lambda \times \mathbb{R}$ with $w_\lambda(\tilde{x}, \tilde{t}) < 0$. Then, $Lw_\lambda(\tilde{x}, \tilde{t}) \geq 0$ by the second statement of Lemma 1.2.6. One can easily verify that for any sufficiently smooth function $g : (-\infty, \lambda] \to (0, \infty)$

$$g(x_1) L\bar{w}_\lambda(x,t) = Lw_\lambda(x,t) + 2(\partial_{x_1}\bar{w}_\lambda(x,t))g'(x_1) + \bar{w}_\lambda(x,t)g''(x_1) \qquad ((x,t) \in \Sigma_\lambda \times (0,\infty)), \quad (1.34)$$

where $\bar{w}_\lambda(x,t) := w_\lambda(x,t)/g(x_1)$. If we set

$$g(y) := \ln(\lambda + 1 - y) + 1 \qquad (y \in (-\infty, \lambda]),$$

then $g > 0$ and for already fixed $\tilde{x}$ and $\tilde{t}$ we have

$$L\bar{w}_\lambda(\tilde{x},\tilde{t}) \geq 2(\partial_{x_1}\bar{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)} + \bar{w}_\lambda(\tilde{x},\tilde{t})\frac{g''(\tilde{x}_1)}{g(\tilde{x}_1)}\,. \qquad (1.35)$$

Consider the solution of the problem

$$\begin{aligned} z_t - z_{yy} &= F(y, z, z_y), && (y,t) \in \mathbb{R} \times (0,\infty), \\ z(y,0) &= -M, && y \in \mathbb{R}, \end{aligned} \qquad (1.36)$$

where

$$F(y,z,z_y) = \begin{cases} 2z_y g'/g & y < \lambda - 1, \\ 2z_y g'/g - az & y \in [\lambda-1, \lambda], \\ 0 & y > \lambda, \end{cases}$$

and $a := -g''(\lambda-1)/g(\lambda-1) > 0$. Then, the maximum principle implies $z(y,t) < 0$ for all $(y,t) \in \mathbb{R} \times (0,\infty)$, and since $F(y, -M, 0) \geq 0$, z is increasing in t. Also, for any $T \geq 0$ the function $Z : (x,t) \mapsto z(x_1, t+T)$ satisfies

$$L[Z] \leq 2\frac{g'(x_1)}{g(x_1)}\partial_{x_1}Z + \frac{g''(x_1)}{g(x_1)}Z \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty), x_1 < \lambda)\,.$$

Then, the maximum principle on the set where $\bar{w}_\lambda \leq 0$ yields $\bar{w}_\lambda(\tilde{x},\tilde{t}) \geq Z(\tilde{x},\tilde{t}) = z(\tilde{x}_1, \tilde{t} + T)$ for any $T > 0$.

Since z is increasing in t, $\tilde{z}(y) := \lim_{t\to\infty} z(y,t)$ is well defined for each $y \in \mathbb{R}$ and

$$-\tilde{z}_{yy} = F(y, \tilde{z}, \tilde{z}_y), \qquad y \in \mathbb{R}\,.$$

An analysis of this problem (for details see [52, Claim 2]) implies $\tilde{z} \equiv 0$. Thus, $\bar{w}_\lambda(\tilde{x}, \tilde{t}) \geq z(\tilde{x}_1, \tilde{t} + T) \to 0$ as $T \to \infty$, a contradiction. □

Lemma 1.2.8. *The mapping $t \mapsto \lambda(t)$ is nondecreasing. If $\lambda(t_1) = \infty$, this means that $\lambda(t_2) = \infty$ for all $t_1 \leq t_2$.*

Proof. Fix $t_0 \in \mathbb{R}$ and $\lambda < \lambda(t_0)$. Then

$$w_\lambda(x, t_0) \geq 0 \qquad (x \in \Sigma_\lambda)\,,$$

and by (1.31)

$$w_\lambda(x,t) \geq 0 \qquad ((x,t) \in \partial\Sigma_\lambda \times [t_0, \infty))\,.$$

Next, (1.33) and the mean value theorem imply

$$\begin{aligned} Lw_\lambda(x,t) &= h(x^\lambda \cdot v) f(u(x^\lambda, t)) - h(x \cdot v) f(u(x,t)) \\ &\geq h(x \cdot v)[f(u(x^\lambda, t)) - f(u(x,t))] \\ &= h(x \cdot v) f'(\theta(x,t)) w_\lambda(x,t), \qquad (x,t) \in \Sigma_\lambda \times [t_0, \infty)\,, \end{aligned}$$

where $\theta(x,t)$ is a number between $u(x,t)$, and $u(x^\lambda, t)$. In particular $\theta : (x,t) \mapsto [0, \infty)$ is a bounded function. Since by (d2)

$$x \cdot v = x_1 v_1 + x_2 v_2 \leq x_1 v_1 + d^* \leq \lambda + d^* \qquad (x \in \Sigma_\lambda),$$

one has $h(x \cdot v) \leq h(\lambda + d^*)$ for each $x \in \Sigma_\lambda$. Now, the maximum principle, with the coefficient $c(x,t) := h(x \cdot v) f'(\theta(x,t))$ being possibly unbounded from below (see [14, 37]), gives $w_\lambda(x,t) \geq 0$ for all $(x,t) \in \Sigma_\lambda \times [t_0, \infty)$. Since $\lambda < \lambda(t_0)$ was arbitrary, $\lambda(t) \geq \lambda(t_0)$ for each $t \geq t_0$. □

Lemma 1.2.9. *$\lambda^* = \infty$, or equivalently u is nondecreasing in x_1.*

Proof. We proceed by a contradiction, that is, we suppose $\lambda^* < \infty$. Lemma 1.2.7 guarantees $\lambda^* \geq \frac{-d^*}{v_1}$. By the definition of λ^* and Lemma 1.2.8, there exit $\lambda_k \searrow \lambda^*$ and $t_k \searrow -\infty$ with

$$\inf_{x\in\Sigma_{\lambda_k}} w_{\lambda_k}(x,t_k) < 0\,.$$

Since u is bounded, there is $M > 0$ with $u \leq M$. Consequently, by (f1), there exists C_f such that $f' \leq C_f$ on $[0, M]$. Set $b_2 := h(\lambda^* v_1 + d^* + 1)C_f > 0$ and choose $1 > \delta > 0$ with

$$2\delta^{-2} \geq 3^3(2b_2 + 1)\,. \tag{1.37}$$

Since $f'(0) = 0$, we can fix $\eta > 0$ with

$$f'(z) \leq \frac{\delta}{h(\lambda^* + d^* + 1)(\lambda^* + 1 + \frac{d^*}{v_1})^3} \qquad (z \in [0, \eta]). \tag{1.38}$$

Let ε with $0 < \varepsilon < \delta$ be sufficiently small (as specified below), and fix k such that $\lambda_k < \lambda^* + \varepsilon$. To simplify the notation set $\lambda := \lambda_k$ and denote

$$g(y) := 2 - \frac{\delta}{\delta + \lambda - y} \qquad (y \in (-\infty, \lambda]),$$
$$\bar{w}_\lambda(x,t) := \frac{w_\lambda(x,t)}{g(x_1)} \qquad ((x,t) \in \Sigma_\lambda \times \mathbb{R})\,.$$

Observe that $g''(y) \leq 0$ and $g(y) > 0$ for any $y \leq \lambda$. For already fixed λ, define

$$S := \{(x,t) \in \Sigma_\lambda \times \mathbb{R} : w_\lambda(x,t) \leq 0\}\,.$$

Case 1. If $(\tilde{x}, \tilde{t}) \in S$ with $\tilde{x}_1 < \lambda^* - \delta$ and $Lw_\lambda(\tilde{x}, \tilde{t}) \geq 0$, then (1.34) and the concavity of g yield

$$L\bar{w}_\lambda(\tilde{x}, \tilde{t}) \geq 2(\partial_{x_1}\bar{w}_\lambda(\tilde{x}, \tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}\,.$$

Case 2. If $(\tilde{x}, \tilde{t}) \in S$ with $\tilde{x}_1 < \lambda^* - \delta$ and $Lw_\lambda(\tilde{x}, \tilde{t}) < 0$, then Lemma 1.2.6 yields

$h(\tilde{x}\cdot v)>0$. Consequently (h1) and (d2) yield

$$0\le \tilde{x}\cdot v=\tilde{x}_1v_1+\tilde{x}_2v_2\le \tilde{x}_1v_1+d^*\le \lambda^*+d^*+1\,. \tag{1.39}$$

Also, Lemma 1.2.6 implies $\tilde{x}_1>\frac{-d^*}{v_1}$, and therefore

$$\tilde{x}^\lambda\cdot v=(2\lambda-\tilde{x}_1)v_1+\tilde{x}_2v_2\le 2\lambda v_1+2d^*\le 2\lambda^*+2d^*+1\,. \tag{1.40}$$

Now, (1.33) implies $h(\tilde{x}^\lambda\cdot v)\ge h(\tilde{x}\cdot v)>0$ and (h1), (1.39), (1.40) yield

$$h(-1)\le h(x\cdot v)\le h(2(\lambda^*+d^*)+2)\qquad ((x,t)\in\mathbb{R}^{N+1}, d_P[(x,t),S^*]<1)\,,$$

where d_P was defined in (1.18) and S^* is the convex hull of S and the set $\{(x^\lambda,t):(x,t)\in S\}$. Next, boundedness of u and standard local parabolic estimates give

$$|\nabla u(x,t)|\le C_\lambda\qquad ((x,t)\in S^*)\,.$$

Furthermore,

$$u(\tilde{x}^{\lambda^*},\tilde{t})\ge u(\tilde{x},\tilde{t})\ge u(\tilde{x}^\lambda,\tilde{t}) \tag{1.41}$$

and

$$|\tilde{x}^{\lambda^*}-\tilde{x}^\lambda|=|\tilde{x}_1^{\lambda^*}-\tilde{x}_1^\lambda|=2(\lambda-\lambda^*)\le 2\varepsilon\,.$$

Also, by (f1) and $h(\tilde{x}\cdot v)\ge 0$

$$\begin{aligned}0>Lw_\lambda(\tilde{x},\tilde{t})&=h(\tilde{x}^\lambda\cdot v)f(u(\tilde{x}^\lambda,\tilde{t}))-h(\tilde{x}\cdot v)f(u(\tilde{x},\tilde{t}))\\&\ge h(\tilde{x}^\lambda\cdot v)f(u(\tilde{x}^\lambda,\tilde{t}))-h(\tilde{x}\cdot v)f(u(\tilde{x}^{\lambda^*},\tilde{t}))\\&=h(\tilde{x}^\lambda\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t}))-f(u(\tilde{x}^{\lambda^*},\tilde{t}))]+[h(\tilde{x}^\lambda\cdot v)-h(\tilde{x}\cdot v)]f(u(\tilde{x}^{\lambda^*},\tilde{t}))\,.\end{aligned} \tag{1.42}$$

Let us estimate each term separately. Since the segment connecting $\tilde{x}$ and $\tilde{x}^{\lambda^*}$

belongs to S^*, one has by (1.40), (1.41), and the definition of C_f and C_λ

$$\begin{aligned} h(\tilde{x}^\lambda \cdot v)&[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x}^{\lambda^*},\tilde{t}))] \\ &\geq h(2(\lambda^*+d^*)+1)C_f(u(\tilde{x}^\lambda,\tilde{t}) - u(\tilde{x}^{\lambda^*},\tilde{t})) \\ &\geq -2h(2(\lambda^*+d^*)+1)C_f C_\lambda \varepsilon \,. \end{aligned} \tag{1.43}$$

To estimate the second term, notice that $\tilde{x}_1 \leq \lambda^* - \delta$ implies

$$\tilde{x}^\lambda \cdot v - \tilde{x} \cdot v = 2(\lambda - \tilde{x}_1)v_1 \geq 2(\lambda - \lambda^* + \delta)v_1 \geq 2\delta v_1 \,.$$

Thus by the monotonicity of h and (1.39) we have

$$h(\tilde{x}^\lambda \cdot v) - h(\tilde{x} \cdot v) \geq \inf_{y \in [0,\lambda^*+d^*+1]} (h(y+2\delta v_1) - h(y)) > 0 \,. \tag{1.44}$$

A substitution of (1.43) and (1.44) into (1.42) yields

$$0 > -2h(2(\lambda^*+d^*)+1)C_f C_\lambda \varepsilon + [\inf_{y \in [0,\lambda^*+d^*+1]} (h(y+2\delta v_1) - h(y))] f(u(\tilde{x}^{\lambda^*},\tilde{t})) \,,$$

or equivalently

$$f(u(\tilde{x}^{\lambda^*},\tilde{t})) < \frac{2h(2(\lambda^*+d^*)+1)C_f C_\lambda}{\inf_{y \in [0,\lambda^*+d^*+1]} (h(y+2\delta v_1) - h(y))} \varepsilon \,.$$

Hence, by (f1) it follows that for sufficiently small $\varepsilon > 0$ one has $u(\tilde{x}^{\lambda^*},\tilde{t}) \leq \eta$, and for such ε, (1.38) holds true for any $z \in [0, u(\tilde{x}^{\lambda^*},\tilde{t})]$. Then (1.38), (1.39) and (1.41) imply

$$\begin{aligned} Lw_\lambda(\tilde{x},\tilde{t}) &\geq h(\tilde{x} \cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x},\tilde{t}))] \\ &\geq h(\lambda^*+d^*+1) \frac{\delta}{h(\lambda^*+d^*+1)(\lambda^*+1+\frac{d^*}{v_1})^3} w_\lambda(\tilde{x},\tilde{t}) \\ &= \frac{\delta}{(\lambda^*+1+\frac{d^*}{v_1})^3} w_\lambda(\tilde{x},\tilde{t}) \,. \end{aligned}$$

Easy calculations show that

$$\frac{\delta}{(\lambda^*+1+\frac{d^*}{v_1})^3} \le \frac{\delta}{(\delta+\lambda-y)^3} = -\frac{g''(y)}{2} \le -\frac{g''(y)}{g(y)} \qquad \left(y \in \left[\frac{-d^*}{v_1}, \lambda^*\right]\right),$$

and since $\tilde{x}_1 \ge \frac{-d^*}{v_1}$,

$$Lw_\lambda(\tilde{x},\tilde{t}) \ge \frac{\delta}{(\lambda^*+1+\frac{d^*}{v_1})^3} w_\lambda(\tilde{x},\tilde{t}) \ge -\frac{g''(\tilde{x}_1)}{g(\tilde{x}_1)} w_\lambda(\tilde{x},\tilde{t}) = -g''(\tilde{x}_1)\bar{w}_\lambda(\tilde{x},\tilde{t}).$$

Consequently, (1.34) implies

$$L\bar{w}_\lambda(\tilde{x},\tilde{t}) \ge 2(\partial_{x_1}\bar{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}.$$

Case 3. Consider $(\tilde{x},\tilde{t}) \in S$ with $\tilde{x}_1 \in [\lambda^*-\delta, \lambda]$. Then by (d2)

$$\tilde{x}\cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \le \lambda v_1 + d^* \le \lambda^* v_1 + d^* + 1,$$

and therefore for already fixed b_2 and C_f we have

$$\begin{aligned} Lw_\lambda(\tilde{x},\tilde{t}) &\ge h(\tilde{x}\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x},\tilde{t}))] \ge h(\lambda^* v_1 + d^* + 1) C_f w_\lambda(\tilde{x},\tilde{t}) \\ &= b_2 w_\lambda(\tilde{x},\tilde{t}). \end{aligned}$$

Moreover, (1.37) implies

$$-g''(y) = \frac{2\delta}{(\delta+\lambda-y)^3} \ge 2b_2 + 1 \ge g(y)b_2 + 1 \qquad (y \in [\lambda^*-\delta, \lambda]).$$

After a substitution into the previous estimate and then into (1.34), we obtain

$$L\bar{w}_\lambda(\tilde{x},\tilde{t}) \ge 2(\partial_{x_1}\bar{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)} - \frac{\bar{w}_\lambda(\tilde{x},\tilde{t})}{g(\tilde{x}_1)}.$$

The rest of the proof uses comparison principle similarly as in Lemma 1.2.7, for more details see [52, Proof of Claim 4]. $\square$

Proof of Theorem 1.2.3. We proceed by contradiction, that is, we assume $M := \|u\|_{L^\infty(\Omega\times\mathbb{R})} > 0$. Then by the continuity of u, there are $t_0 \in \mathbb{R}$ and a smooth bounded domain $K_0 \subset \Omega$ with $|K_0| \leq 1$ (here $|K_0|$ denotes the Lebesgue measure of K_0) such that $u(x,t_0) > 0$ for all $x \in K_0$. Define

$$K_\sigma := \{x + \sigma e_1 : x \in K_0\} \qquad (\sigma \geq 0)\,.$$

Since Ω is convex and unbounded in x_1, one has $K_\sigma \subset \Omega$ for all $\sigma \geq 0$. Let $\mu > 0$ be the first eigenvalue of the problem

$$\begin{aligned} -\Delta\phi_0 &= \mu\phi_0, & x &\in K_0\,,\\ \phi_0 &= 0, & x &\in \partial K_0\,, \end{aligned}$$

where the eigenfunction ϕ_0 is normalized such that $\max_{K_0}\phi_0 = 1$. Set

$$\phi_\sigma(x) := \phi_0(x_1 - \sigma, x') \qquad (x = (x_1, x') \in K_\sigma)$$

and

$$\psi_\sigma(t) := \int_{K_\sigma} u(x,t)\phi_\sigma(x)\,dx \qquad (t \in \mathbb{R})\,.$$

Since by Lemma 1.2.9 u is nondecreasing in x_1 and $u > 0$ in $K_0 \times \{t_0\}$,

$$\psi_\sigma(t_0) \geq \psi_0(t_0) =: c_0 > 0\,.$$

Denote

$$K^*_\sigma(t) := \{x \in K_\sigma : u(x,t)\phi_\sigma(x) \geq c_0/2\} \qquad (t \geq t_0)\,.$$

If $\psi_\sigma(t^*) \geq c_0$ for some $t^* \geq t_0$, then (using $|K_\sigma| \leq 1$)

$$c_0 \leq \int_{K_\sigma} u(x,t^*)\phi_\sigma(x)\,dx \leq |K^*_\sigma(t^*)| \cdot M + \frac{c_0}{2}|K_\sigma| \leq |K^*_\sigma(t^*)| \cdot M + \frac{c_0}{2}\,.$$

Consequently, $|K_\sigma^*(t^*)| \geq \xi := c_0/(2M) > 0$. Next,

$$\int_{K_\sigma^*(t^*)} u(x,t^*)\phi_\sigma(x)\,dx \geq \xi\frac{c_0}{2} \geq \xi \int_{K_\sigma \setminus K_\sigma^*(t^*)} u(x,t^*)\phi_\sigma(x)\,dx$$
$$= \xi \int_{K_\sigma} u(x,t^*)\phi_\sigma(x)\,dx - \xi \int_{K_\sigma^*(t^*)} u(x,t^*)\phi_\sigma(x)\,dx\,.$$

It follows that

$$\int_{K_\sigma^*(t^*)} u(x,t^*)\phi_\sigma(x)\,dx \geq \frac{\xi}{1+\xi}\int_{K_\sigma} u(x,t^*)\phi_\sigma(x)\,dx = \frac{c_0}{2M+c_0}\psi_\sigma(t^*)\,.$$

Since K is bounded, we can choose R such that K is a subset of the ball of radius R centered at the origin. Then for sufficiently large $\sigma \geq 0$

$$x \cdot v = x_1 v_1 + x_2 v_2 \geq -|x_1 - \sigma|v_1 + v_1\sigma - R|v_2| \geq R(-v_1 - |v_2|) + v_1\sigma$$
$$\geq \frac{1}{2}v_1\sigma \qquad (x \in K_\sigma)\,.$$

Hence, for sufficiently large $\sigma \geq 0$ and (h2) one has

$$\begin{aligned}
\frac{d}{dt}\psi_\sigma(t^*) &= \int_{K_\sigma} \Delta u(x,t^*)\phi_\sigma(x)\,dx + \int_{K_\sigma} h(x\cdot v) f(u(x,t^*))\phi_\sigma(x)\,dx \\
&\geq \int_{K_\sigma} u(x,t^*)\Delta\phi_\sigma(x)\,dx + h\left(\frac{1}{2}v_1\sigma\right)\int_{K_\sigma} f(u(x,t^*))\phi_\sigma(x)\,dx \\
&\geq \int_{K_\sigma} u(x,t^*)\Delta\phi_\sigma(x)\,dx + h\left(\frac{1}{2}v_1\sigma\right)\int_{K_\sigma^*(t^*)} \frac{f(u(x,t^*))}{M} u(x,t^*)\phi_\sigma(x)\,dx \\
&\geq -\mu\psi_\sigma(t^*) + h\left(\frac{1}{2}v_1\sigma\right) f\left(\frac{c_0}{2}\right)\frac{1}{M}\int_{K_\sigma^*(t^*)} u(x,t^*)\phi_\sigma(x)\,dx \\
&\geq \psi_\sigma(t^*)\left[-\mu + h\left(\frac{1}{2}v_1\sigma\right) f\left(\frac{c_0}{2}\right)\frac{1}{M}\frac{c_0}{2M+c_0}\right] \\
&\geq \psi_\sigma(t^*)\,.
\end{aligned}$$

Thus, if $\psi_\sigma(t^*) \geq c_0$, then $\psi'_\sigma(t^*) \geq 0$, and consequently $\psi'_\sigma(t) \geq \psi_\sigma(t) \geq c_0$ for each $t \geq t^*$. Since $\psi_\sigma(t_0) \geq c_0$, one has $\psi'_\sigma(t) \geq c_0 > 0$ for each $t > t_0$. Therefore

$\psi_\sigma(t) \to \infty$ as $t \to \infty$, a contradiction to the boundedness of u. □

1.3 Counterexample

In this section, we show that the statement of the Corollary 1.2.4 does not holt true in general, if $v = -b$ and $c^* < 0$. We construct a function that violates Liouville theorem in one dimensional case only. An counterexample for a higher dimensional elliptic or parabolic problems can be obtained by the extension of this function by a constant.

Proposition 1.3.1. *For each $a > 0$ and $p > 1$, there exists the unique bounded nonnegative, nontrivial solution u of the problem*

$$\begin{aligned} u'' &= x|u|^{p-1}u, \qquad x \in (-a, \infty)\,, \\ u(-a) &= 0\,. \end{aligned} \tag{1.45}$$

Moreover, $u'(x) < 0$ for $x \geq 0$ and $\lim_{x\to\infty} u(x) = 0$.

Proof. Let $u_k : (\tau_k, T_{\max}) \to \mathbb{R}$ be the solution of the initial value problem

$$\begin{aligned} u_k'' &= x|u_k|^{p-1}u_k\,, \qquad x \in (\tau_k, T_{\max})\,, \\ u_k(0) &= 1\,, \qquad u_k'(0) = k\,, \end{aligned} \tag{1.46}$$

where $(\tau_k, T_{\max})$ is the maximal existence interval of u_k. By a standard theory $-\infty \leq \tau_k < 0 < T_{\max} \leq \infty$.

Claim 1. Given k. If $u_k'(x_0) \geq 0$ and $u_k(x_0) > 0$ for some $x_0 > 0$, then $u_k'(x) > 0$ for each $x > x_0$ and $\lim_{x\to T_{\max}} u_k(x) = \infty$.

Proof of Claim 1. First, $u_k''(x_0) = x_0 u_k^p(x_0) > 0$ implies

$$u_k'(x) > u_k'(x_1) > u_k'(x_0) \geq 0 \qquad (x > x_1 > x_0)$$

with x and x_1 sufficiently close to x_0. If $u_k'(x) > u_k'(x_1) > 0$ for each $x > x_1$, the claim follows. Otherwise, there exists $x_2 > x_1$ with $u_k'(x_2) = u_k'(x_1)$. Then, $u_k'(x) > 0$ on $[x_1, x_2]$, and therefore $u_k(x) \geq u_k(x_1)$ for each $x \in [x_1, x_2]$. Consequently $u_k''(x) = xu_k^p(x) > x_1 u_k^p(x_1) > 0$, that is, u_k is strictly convex on $[x_1, x_2]$, a contradiction to $u_k'(x_2) = u_k'(x_1)$. □

Claim 2. If $u_k(x_0) \leq 0$ for some $x_0 > 0$, then $u_k(x) < 0$ for each $x > x_0$ and $\lim_{x \to T_{\max}} u_k(x) = -\infty$.

Proof of Claim 2. We can assume $u_k(x_0) < 0$. Otherwise, $u_k(x_0) = 0$, $u_k'(x_0) = 0$, and we obtain a contradiction to the uniqueness of the solution to the initial value problem.

We proceed by contradiction, that is, we suppose that there is $x_1 > x_0$ such that $u_k(x_1) \geq 0 > u_k(x_0)$. Then, u_k has a local negative minimum x_2 on the interval $[0, x_1]$. But, $u_k''(x_2) = x_2 u_k(x_2) < 0$ a contradiction.

Similarly as in Claim 1, we use concavity of u_k on the interval (x_0, ∞) to prove $\lim_{x \to T_{\max}} u_k(x) = -\infty$. □

Denote

$$\mathcal{K}_0 := \{k : u_k(x) \leq 0 \text{ for some } x \geq 0\},$$
$$\mathcal{K}_2 := \{k : u_k(x) \geq 2 \text{ for some } x \geq 0\}.$$

Claim 3. The sets $\mathcal{K}_0$, $\mathcal{K}_2$ are nonempty, open, and disjoint.

Proof of Claim 3. First, we show that $k = -2 \in \mathcal{K}_0$. This follows if $u_k'(x) < -1$ for each $x \in (0, 1)$. Otherwise, there is $x_0 \in (0, 1)$ with $u_k'(x_0) = -1$ and $u_k'(x) < -1$ for all $x \in (0, x_0)$. Then, $u_k(x) < 1$ for all $x \in (0, x_0)$, and

$$u_k'(x_0) = u_k'(0) + \int_0^{x_0} u_k''(x)\, dx = -2 + \int_0^{x_0} x u_k^p(x)\, dx < -2 + x_0 < -1,$$

a contradiction.

If $k_0 \in \mathcal{K}_0$, then, by Claim 2, there exists x_1 such that $u_{k_0}(x_1) \leq -1$. The continuous dependence of solutions on initial data implies $u_k(x_1) \leq -\frac{1}{2}$ for any k sufficiently close to k_0. Thus, $\mathcal{K}_0$ is open.

By Claim 1, $(0,\infty) \subset \mathcal{K}_2$, and for any $k_0 \in \mathcal{K}_2$ there is x_0 such that $u_{k_0}(x_0) > 3$. Then, the continuous dependence of solutions on the initial data yields $u_k(x_0) > 3$ for any k sufficiently close to k_0. Thus, $\mathcal{K}_2$ is open and nonempty.

Fix $k \in \mathcal{K}_0 \cup \mathcal{K}_2$ and fix $x_0 > 0$ with $0 < u_k(x) < 2$ for any $x \in [0, x_0)$ and $u_k(x_0) = 0$ or $u_k(x_0) = 2$. If $u_k(x_0) = 0$ then, by Claim 2, $u_k(x) < 0$ for each $x > x_0$, and therefore $k \notin \mathcal{K}_2$. If $u_k(x_0) = 2$, then there is $x_1 \in [0, x_0)$ with $u'(x_0) \geq 0$. Claim 1 yields $u(x) \geq u(x_1) = 2 > 0$ for each $x \geq x_1$, and therefore $k \notin \mathcal{K}_0$. This shows $\mathcal{K}_0 \cap \mathcal{K}_2 = \emptyset$. □

Denote

$$M := \mathbb{R} \setminus (\mathcal{K}_0 \cap \mathcal{K}_2)\,.$$

By Claim 3, $M \neq \emptyset$, and since u_k, $k \in M$ is bounded, $T_{\max} = \infty$. Then, by Claim 1, $u_k' < 0$ in $[0,\infty)$ for each $k \in M$. Also, $\lim_{x\to\infty} u_k(x) = 0$ for any $k \in M$. Otherwise, $u_k''(x) = xu^p(x) \geq L > 0$ for a sufficiently large x, a contradiction to $u_k' < 0$.

Claim 4. $M = \{k^*\}$.

Proof of Claim 4. Suppose that there are $k_1, k_2 \in M$ with $k_1 > k_2$. Then, for x_0 sufficiently close to 0 one has

$$\varepsilon := u_{k_1}(x_0) - u_{k_2}(x_0) > 0 \quad \text{and} \quad u_{k_1}'(x) > u_{k_2}'(x) \qquad (x \in [0, x_0])\,.$$

Since $\lim_{x\to\infty} u_{k_1}(x) = \lim_{x\to\infty} u_{k_2}(x) = 0$, there exists $x_1 > x_0$ with $u_{k_1}'(x_1) =$

$u'_{k_2}(x_1)$ and $u_{k_1}(x) > u_{k_2}(x)$ for $x \in (x_0, x_1)$. However,

$$\begin{aligned} u'_{k_1}(x_1) &= u'_{k_1}(x_0) + \int_{x_0}^{x_1} u''_{k_1}(x)\,dx = u'_{k_1}(x_0) + \int_{x_0}^{x_1} x u^p_{k_1}(x)\,dx \\ &> u'_{k_2}(x_0) + \int_{x_0}^{x_1} x u^p_{k_2}(x)\,dx = u'_{k_1}(x_0) + \int_{x_0}^{x_1} u''_{k_2}(x)\,dx = u'_{k_2}(x_1)\,, \end{aligned}$$

a contradiction. □

Claim 5. There exists $x^* < 0$ such that $u_{k^*}(x^*) = 0$.

Proof of Claim 5. For a contradiction assume $u_{k^*}(x) > 0$ for each $x \in (\tau_{k^*}, 0)$. Then, $u''_{k^*}(x) = x u_{k^*} < 0$ for all $x \in (\tau_{k^*}, 0)$. Hence, u'_{k^*} decreases in $(\tau_{k^*}, 0)$, and in particular $u'_{k^*}(0) \le u'_{k^*}(x)$ for each $x \in (\tau_{k^*}, 0)$. Thus, $0 \le u_{k^*}(x) \le 1 + u'_{k^*}(0)x$ for each $x \in (\tau_{k^*}, 0)$, and consequently $\tau_{k^*} = -\infty$.

Next, we show that $u'_{k^*}(x_0) > 0$ for some $x_0 < 0$. If not, then u_{k^*} decreases on $(-\infty, 0)$ and $u_{k^*}(x) \ge u_{k^*}(0) = 1$ for all $x < 0$. However,

$$\begin{aligned} u'_{k^*}(x) &= u'_{k^*}(-1) - \int_{x}^{-1} u''_{k^*}(s)\,ds = u'_{k^*}(-1) + \int_{x}^{-1} (-s) u^p_{k^*}(s)\,ds \\ &\ge u'_{k^*}(-1) + (-1 - x) \qquad (x \le -1)\,, \end{aligned}$$

a contradiction to $u'_{k^*}(x) \le 0$ for large negative x. We already proved that u'_{k^*} decreases for $x < 0$. Hence $u'_{k^*}(x) \ge u'_{k^*}(x_0) > 0$ for each $x < x_0$. Thus, $u_{k^*}(x^*) = 0$ for some $x^* < 0$, a contradiction. □

Now, we finish the proof of the proposition. Notice that u_{k^*} is a bounded non-negative solution of (1.45) with a replaced by $-x^*$. Then

$$x \mapsto \left(\frac{a}{x^*}\right)^{3/(p-1)} u_{k^*}\left(\frac{a}{x^*}x\right)$$

satisfies (1.45) and the existence part follows.

Let u, v be two nonnegative bounded solutions of (1.45) with $u(0) \le v(0)$. Then, $v_\lambda(0) = u(0)$ for some $\lambda \in (0, 1]$, where $v_\lambda(x) := \lambda^{3/(p-1)} v(\lambda x)$. If $u'(0) \ne$

$v'_\lambda(0)$, we obtain a contradiction to Claim 4. If $u'(0) = v'_\lambda(0)$, then $v_\lambda \equiv u$, by the uniqueness of solutions of initial value problems. If $\lambda = 1$, then $u \equiv v$ and the uniqueness follows. Otherwise, $\lambda \in (0, 1)$ and $0 = u(a) = v_\lambda(a) = \lambda^{3/(p-1)} v(\lambda a) > 0$, a contradiction. $\square$

1.4 Proofs of a priori estimates

In this section, we use the notation introduced in the previous sections. Especially, recall the definitions of $\mathbb{R}^N_\lambda$ (see (1.27)), H_λ (see (1.28)), x^λ (see (1.30)), and d_p (see (1.18)).

Our main technical tools are the following doubling lemmas.

Lemma 1.4.1. *Let (X, d) be a compact metric space and let $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Theta := \Sigma \setminus D$. Also, let $M : D \to (0, \infty)$ be a bounded function on compact subsets of D, and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y)d(y, \Theta) > 2k\,,$$

then there exists $x \in D$ such that

$$M(x)d(x, \Theta) > 2k, \quad M(x) \geq M(y)\,,$$

and

$$M(z) \leq 2M(x) \qquad (z \in D \cap B^*(x, kM^{-1}(x)))\,, \tag{1.47}$$

where $B^*(y, R) := \{x \in X : d^*(x, y) \leq R\}$ and $d^*(x, y) = |d(x, \Theta) - d(y, \Theta)|$.

Lemma 1.4.2. *The statement of Lemma 1.4.1 holds true if (X, d) is a complete metric space and $B^*(x, kM^{-1}(x))$ in (1.47) is replaced by $B(x, kM^{-1}(x))$, where $B(x, R) := \{x \in X : d(x, y) \leq R\}$.*

Lemma 1.4.2 was proved in [54, Lemma 5.1]. The proof of Lemma 1.4.1 is analogous to the proof of [54, Lemma 5.1]. One only replaces every d by d^* and uses compactness of X, when passing to the limit.

Proof of Theorem 1.1.1. This proof was partly inspired by the proofs of the corresponding results in [17, 55, 67]. We use the equivalent formulation introduced in Remark 1.1.3. If (1.17) fails, then there exist $(T_k)_{k\in\mathbb{N}} \subset (0,\infty)$, a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative solutions of (1.1) with T replaced by T_k, and $(y_k, s_k)_{k\in\mathbb{N}} \subset \Omega \times (0, T_k)$ such that

$$M_k(y_k, s_k) := u_k^{\frac{p-1}{3}}(y_k, s_k) > 2k(1 + d_k^{-1}(s_k)) \qquad (k \in \mathbb{N}),$$

where $d_k(s) := \min\{s, T_k - s\}^{\frac{1}{2}}$. Now, for each $k \in \mathbb{N}$, Lemma 1.4.2 with $X_k = \Sigma_k = \bar{\Omega} \times [0, T_k]$, $d = d_P$, $D_k = \bar{\Omega} \times (0, T_k)$ and $\Theta_k = \Omega \times \{0, T_k\}$ implies the existence of $(x_k, t_k) \in \bar{\Omega} \times (0, T_k)$ with

$$\begin{aligned} M_k(x_k, t_k) &\geq M_k(y_k, s_k) > 2kd_k^{-1}(t_k) \\ M_k(x_k, t_k) &\geq M_k(y_k, s_k) > 2k \\ 2M_k(x_k, t_k) &\geq M_k(x, t) \qquad ((x, t) \in G_k), \end{aligned} \tag{1.48}$$

where

$$G_k := \{(x, t) \in \Omega \times (0, T_k) : d_P((x, t), (x_k, t_k)) < k\lambda_k\},$$

and

$$\lambda_k := M_k^{-1}(x_k, t_k) \to 0 \qquad \text{as} \quad k \to \infty.$$

Here we used that $d_P((x, t), \Theta_k) = d_k(t)$ for each $(x, t) \in \Sigma_k$. By (1.48)

$$|t - t_k| < k^2\lambda_k^2 < \frac{d_k^2(t_k)}{4} = \frac{1}{4}\min\{t_k, T_k - t_k\} \qquad ((x, t) \in G_k),$$

and therefore

$$\left\{x \in \Omega : |x - x_k| < \frac{k\lambda_k}{2}\right\} \times \left(t_k - \frac{k^2\lambda_k^2}{4}, t_k + \frac{k^2\lambda_k^2}{4}\right) \subset G_k.$$

Since the function a is bounded, we can, after passing to a subsequence, assume that $\mathcal{A} := \lim_{k\to\infty} a(x_k)$ exists.

Case (1). First assume $\mathcal{A} \neq 0$. We define a sequence $(v_k)_{k\in\mathbb{N}}$, of rescaled copies of u as

$$v_k(x,t) := \lambda_k^{\frac{3}{(p-1)}} u(x_k + \lambda_k^{\frac{3}{2}} x, t_k + \lambda_k^3 t) \qquad ((x,t) \in D_k)\,,$$

where

$$D_k := \left\{ x \in \lambda_k^{-\frac{3}{2}}(\Omega - x_k) : |x| < \frac{k}{2\lambda_k^{\frac{1}{2}}} \right\} \times \left(-\frac{k^2}{4\lambda_k}, \frac{k^2}{4\lambda_k} \right). \tag{1.49}$$

Then $v_k(0,0) = 1$ and, by (1.48), $0 \leq v_k(x,t) \leq 2$ for each $(x,t) \in D_k$. Moreover, v_k satisfies

$$(v_k)_t = \Delta v_k + a(x_k + \lambda_k^{\frac{3}{2}} x) v_k^p, \qquad (x,t) \in D_k\,, \tag{1.50}$$

$$v_k = 0, \qquad (x,t) \in \left\{ y \in \lambda_k^{-\frac{3}{2}}(\partial\Omega - x_k) : |y| < \frac{k}{2\lambda_k^{\frac{1}{2}}} \right\} \times \left(-\frac{k^2}{4\lambda_k}, \frac{k^2}{4\lambda_k} \right). \tag{1.51}$$

By passing to a suitable subsequence we may assume either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k^{\frac{3}{2}}} \to \infty \quad \text{or} \quad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k^{\frac{3}{2}}} \to c^* \geq 0\,.$$

If (i) holds, then (1.50), L^p estimates, and the Schauder's estimates yield a subsequence of $(v_k)_{k\in\mathbb{N}}$ converging in $C^{2+\sigma,1+\sigma/2}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$, $\sigma \in (0,1)$ to a function v_∞ satisfying

$$(v_\infty)_t = \Delta v_\infty + \mathcal{A} v_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,.$$

Moreover, $v_\infty(0,0) = 1$ and $v_\infty \leq 2$. However, if $\mathcal{A} > 0$ and $p < p_B(N)$ (for the definition of $p_B(N)$ see (1.10)) this contradicts [13, Remark 2.6]. If $\mathcal{A} < 0$ and $p > 1$ we have a contradiction to Lemma 1.2.1

If (ii) holds, then after an application of a suitable orthogonal change of coordinates, the L^p estimates and the Schauder's estimates, yield a subsequence of

$(v_k)_{k\in\mathbb{N}}$ converging in $C^{2+\sigma,1+\sigma/2}_{\mathrm{loc}}(\mathbb{R}^N_{c^*}\times\mathbb{R})$ to a function v_∞ satisfying

$$
\begin{aligned}
(v_\infty)_t &= \Delta v_\infty + \mathcal{A} v_\infty^p, && (x,t)\in\mathbb{R}^N_{c^*}\times\mathbb{R}\,,\\
v_\infty &= 0, && (x,t)\in\partial\mathbb{R}^N_{c^*}\times\mathbb{R}\,,
\end{aligned}
$$

with $v_\infty(0,0)=1$ and $v_\infty\le 2$. However, if $\mathcal{A}>0$ and $p<p_S(N)\le p_B(N-1)$ this contradicts [55, Theorem 2.1]. If $\mathcal{A}<0$ and $p>1$ we have a contradiction to Lemma 1.2.2.

Case (2). Assume $\mathcal{A}=0$. Since a is bounded in $C^2(\bar\Omega)$, we can assume, after passing to a subsequence, that there exists a vector $\mathcal{B}:=\lim_{k\to\infty}\nabla a(x_k)\in\mathbb{R}^N$. Then (1.3) implies $\mathcal{B}\neq 0$.

If $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence, we can, after appropriate restriction, assume the existence of $x_\infty:=\lim_{k\to\infty}x_k$. Then $\mathcal{A}=a(x_\infty)=0$. Set $\tilde z_k:=x_\infty$ and $V_k:=\mathcal{V}:=\Omega$ for each $k\in\mathbb{N}$

If $(x_k)_{k\in\mathbb{N}}$ does not have a convergent subsequence, we can assume $|x_k-x_l|\ge 3$ for each $k\neq l$. Let V_k be the connected component of $B_1(x_k)\cap\Omega$ containing x_k, where $B_1(y)$ is the unit ball centered at y. By [33, Lemma 6.37], there exists an extension of $a\in C^2(\bar V_k)$ to $C^2(\bar B_1(x_k))$, which we denote again by a. Since $V_k\cap V_l=\emptyset$ for $k\neq l$, the function a is well defined on $\mathcal{V}:=\cup_{k\in\mathbb{N}}\bar B_1(x_k)$.

Denote $\tilde\Gamma:=\{x\in\bar{\mathcal{V}}: a(x)=0\}$. Since $a\in C^2(\mathcal{V})$, $\mathcal{A}=0$, and $\mathcal{B}\neq 0$, there is $(\tilde z_k)_{k\in\mathbb{N}}\subset\tilde\Gamma$ with $|x_k-\tilde z_k|\to 0$ as $k\to\infty$. Define δ_k and $(z_k)_{k\in\mathbb{N}}\subset\tilde\Gamma$ such that

$$
\delta_k := |z_k - x_k| = \mathrm{dist}(x_k,\tilde\Gamma)\le |x_k-\tilde z_k|\to 0\,.
$$

Then $a\in C^2(\mathcal{V})$ yields $\lim_{k\to\infty}\nabla a(z_k)=\lim_{k\to\infty}\nabla a(x_k)\neq 0$. Thus we may assume $|\nabla a(z_k)|\neq 0$, and therefore

$$
\delta_k = \frac{|\nabla a(z_k)(x_k-z_k)|}{|\nabla a(z_k)|}\qquad (k\in\mathbb{N})\,.
$$

Using that $z_k \in \tilde{\Gamma}$, that is, $a(z_k) = 0$, we obtain

$$a(x_k + \lambda_k x) = \nabla a(z_k)(x_k + \lambda_k x - z_k) + O(|\delta_k|^2 + \lambda_k^2 |x|^2)\,. \tag{1.52}$$

We define a sequence $(w_k)_{k\in\mathbb{N}}$, of rescaled copies of u as

$$w_k(x,t) := \lambda_k^{\frac{3}{(p-1)}} u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \qquad ((x,t) \in \tilde{D}_k)\,,$$

where

$$\tilde{D}_k := \left\{ x \in \lambda_k^{-1}(V_k - x_k) : |x| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{4}, \frac{k^2}{4} \right)\,.$$

Then, $w_k(0,0) = 1$ and $0 \le w_k(x,t) \le 2$ for each $(x,t) \in \tilde{D}_k$, and w_k satisfies

$$(w_k)_t = \Delta w_k + \frac{1}{\lambda_k} a(x_k + \lambda_k x) w_k^p, \qquad (x,t) \in \tilde{D}_k\,, \tag{1.53}$$

$$w_k = 0, \qquad (x,t) \in \left\{ y \in \lambda_k^{-1}(\partial\Omega - x_k) : |y| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{4}, \frac{k^2}{4} \right)\,. \tag{1.54}$$

Hence, by (1.52)

$$(w_k)_t = \Delta w_k + \frac{1}{\lambda_k} \left[\nabla a(z_k)(x_k + \lambda_k x - z_k) + O(|\delta_k|^2 + \lambda_k^2 |x|^2) \right] w_k^p, \qquad (x,t) \in \tilde{D}_k\,. \tag{1.55}$$

Case (2a). Assume that there is a suitable subsequence of $(x_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{\nabla a(z_k)(x_k - z_k)}{\lambda_k} = \pm|\mathcal{B}| \lim_{k\to\infty} \frac{\delta_k}{\lambda_k} =: d^* \in \mathbb{R}\,.$$

By passing to a yet another subsequence we may assume that either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k} \to \infty \quad \text{or} \quad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k} \to c^* \ge 0\,.$$

If (i) holds, then (1.55), L^p estimates, and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\rm loc}(\mathbb{R}^N\times\mathbb{R})$ to a function $w_\infty\in C(\mathbb{R}^N\times\mathbb{R})$ that is a weak solution of the problem

$$(w_\infty)_t=\Delta w_\infty+(d^*+\mathcal{B}\cdot x)w_\infty^p, \qquad (x,t)\in\mathbb{R}^N\times\mathbb{R}\,,$$

satisfying $w_\infty(0,0)=1$, $w_\infty\le 2$. Standard regularity theory implies that w_∞ is in fact a classical solution. After a suitable orthogonal transformation and translation, we obtain a nontrivial nonnegative bounded solution of the problem

$$(w_\infty)_t=\Delta w_\infty\pm|\mathcal{B}|x_n w_\infty^p, \qquad (x,t)\in\mathbb{R}^N\times\mathbb{R}\,,$$

a contradiction to [52, Theorem 1.1] for any $p>1$.

If (ii) holds, then $\operatorname{dist}(x_k,\partial\Omega)\to 0$ as $k\to\infty$. After a suitable rotation we have $\nu_\Omega(x_k)\to -e_1$ as $k\to\infty$. Then (1.55), L^p estimates, and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\rm loc}(\mathbb{R}^N_{c^*}\times\mathbb{R})$ to a function $w_\infty\in C(\mathbb{R}^N_{c^*}\times\mathbb{R})$ that is a weak solution of the problem

$$\begin{aligned} (w_\infty)_t&=\Delta w_\infty+(d^*+\mathcal{B}\cdot x)w_\infty^p, && (x,t)\in\mathbb{R}^N_{c^*}\times\mathbb{R}\,,\\ w_\infty&=0, && (x,t)\in\partial\mathbb{R}^N_{c^*}\times\mathbb{R}\,, \end{aligned}$$

with $w_\infty(0,0)=1$ and $w_\infty\le 2$. Standard regularity theory yields that w_∞ is in fact a classical solution. Also $a\in C^2(\bar\Omega)$, $\operatorname{dist}(x_k,\partial\Omega)\to 0$ and (1.13) imply

$$0<\frac{\tilde c}{2}\le\liminf_{k\to\infty}\left|\frac{\nabla a(x_k)}{|\nabla a(x_k)|}+e_1\right|=\left|\frac{\mathcal{B}}{|\mathcal{B}|}+e_1\right|\,.$$

Thus, $\mathcal{B}$ is not a multiple of $-e_1$. Now, after a suitable translation, we obtain a contradiction, to Corollary 1.2.4 for any $p>1$.

Case (2b). After passing to a subsequence, we may assume that

$$\lim_{k\to\infty}\frac{\nabla a(z_k)(x_k-z_k)}{\lambda_k}=\pm|\mathcal{B}|\lim_{k\to\infty}\frac{\delta_k}{\lambda_k}=\pm\infty\,.$$

Setting

$$y = \frac{x}{\alpha_k}\,, \quad s = \frac{t}{\alpha_k^2}\,,$$

where

$$\alpha_k := \left(\frac{\lambda_k}{\delta_k|\nabla a(z_k)|}\right)^{\frac{1}{2}} = \left(\frac{\lambda_k}{|\nabla a(z_k)(x_k - z_k)|}\right)^{\frac{1}{2}} \to 0$$

we transform (1.55) to

$$\begin{aligned}(w_k)_s &= \Delta_y w_k + \frac{\alpha_k^2}{\lambda_k} a(x_k + \lambda_k \alpha_k y) w_k^p \\ &= \Delta_y w_k + \frac{\nabla a(z_k)(x_k - z_k + \lambda_k x) + O(\delta_k^2 + \lambda_k^2 |x|^2)}{|\nabla a(z_k)(x_k - z_k)|} w_k^p \\ &= \Delta_y w_k + [\pm 1 + \alpha_k^3 \nabla a(z_k) y + O(\delta_k + \alpha_k^4 \lambda_k |y|^2)] w_k^p\,, \qquad (y,s) \in \hat{D}_k\,,\end{aligned}$$

where

$$\hat{D}_k := \left\{ y \in (\lambda_k \alpha_k)^{-1}(\Omega - x_k) : |y| < \frac{k}{2\alpha_k} \right\} \times \left(-\frac{k^2}{4\alpha_k^2}, \frac{k^2}{4\alpha_k^2} \right)\,.$$

Moreover, by (1.54)

$$w_k = 0 \qquad \left((y,s) \in \left\{ y \in (\lambda_k \alpha_k)^{-1}(\partial\Omega - x_k) : |y| < \frac{k}{2\alpha_k} \right\} \times \left(-\frac{k^2}{4\alpha_k^2}, \frac{k^2}{4\alpha_k^2} \right) \right)\,.$$

By passing to a yet another subsequence, we may assume either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k \alpha_k} \to \infty \quad \text{or} \quad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k \alpha_k} \to c^* \geq 0\,.$$

If (i) holds, the L^p estimates and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N \times \mathbb{R})$ that is a weak solution of the problem

$$(w_\infty)_t = \Delta w_\infty \pm w_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,,$$

and $w_\infty(0,0) = 1$, $w_\infty \le 2$. Standard regularity theory implies that w_∞ is a classical solution. However, this contradicts [13] (with "+" sign) for any $1 < p < p_B(N)$ and Lemma 1.2.1 (with "-" sign) for any $p > 1$.

If (ii) holds, then after a suitable orthogonal change of coordinates and a translation, the L^p estimates and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\rm loc}(\mathbb{R}^N_{c^*} \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N_{c^*} \times \mathbb{R})$ that is a weak solution of the problem

$$\begin{aligned} (w_\infty)_t &= \Delta w_\infty \pm w_\infty^p, && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}\,, \\ w_\infty &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}\,, \end{aligned}$$

and $w_\infty(0,0) = 1$, $w_\infty \le 2$. Standard regularity theory implies that w_∞ is a classical solution. However this contradicts [55, Theorem 2.1] (with "+" sign) for any $1 < p < p_S(N) \le p_B(N-1)$ and Lemma 1.2.2 (with "-" sign) for any $p > 1$. □

Let us formulate a sufficient condition that guarantees (1.20).

Lemma 1.4.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $1 < p < p_B(N)$, and assume that $a \in C^2(\bar\Omega)$. For a nonnegative classical solution u of* (1.1), (1.2) *define $x^* : (0,T) \to \Omega$ such that*

$$u(x^*(t),t) = \sup_{x\in\Omega} u(x,t) \qquad (t \in (0,T))\,.$$

If there exist $\varepsilon^* > 0$ and $t_0 \in [0,T]$ such that $\operatorname{dist}(x^*(t),\Gamma) \ge \varepsilon^*$ for each $t \in [t_0,T]$, then (1.20) *holds with C depending on N, p, Ω, a, $\|u_0\|_{L^\infty(\Omega)}$, ε^* and t_0.*

Proof. As in the proof of Theorem 1.1.1, we use the equivalent formulation introduced in Remark 1.1.3. Assume that (1.20) fails. Then there exist $(T_k)_{k\in\mathbb{N}} \subset (0,\infty)$, a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative solutions of (1.1), and a sequence $(y_k,s_k)_{k\in\mathbb{N}} \subset \Omega \times (0,T_k)$ such that

$$\tilde M_k(y_k,s_k) > 2k(1 + d_k^{-1}(s_k))\,,$$

where

$$\tilde{M}_k := u_k^{\frac{p-1}{2}}, \qquad d_k(t) = \min\{t, T_k - t\}^{\frac{1}{2}}.$$

Now, Lemma 1.4.1 with compact $X_k = \Sigma_k = \bar{\Omega} \times [0, T_k]$, $D_k = \bar{\Omega} \times (0, T_k)$ and $\Theta_k = \bar{\Omega} \times \{0, T_k\}$ implies the existence of a sequence $(x'_k, t_k) \in \Omega \times (0, T_k)$ with

$$\begin{aligned} \tilde{M}_k(x'_k, t_k) &\geq \tilde{M}_k(y_k, s_k) > 2k d_k^{-1}(t_k) \\ \tilde{M}_k(x'_k, t_k) &\geq \tilde{M}_k(y_k, s_k) > 2k \\ 2\tilde{M}_k(x'_k, t_k) &\geq \tilde{M}_k(x, t) \qquad ((x, t) \in G'_k), \end{aligned} \tag{1.56}$$

where

$$G'_k := \{(x, t) \in \Omega \times (0, T) : d^*_k((x, t), (x'_k, t_k)) < k\lambda'_k\},$$
$$d^*_k((x, t), (y, s)) := |d_k(t) - d_k(s)| \qquad ((x, t), (y, s) \in X_k),$$

and

$$\lambda'_k := \tilde{M}^{-1}(x'_k, t_k) \to 0 \quad \text{as} \quad k \to \infty.$$

Observe that d^*_k does not depend on x, and therefore (1.56) remains true if we replace x'_k by $x_k := x^*(t_k)$ and G'_k by

$$G_k := \{(x, t) \in \Omega \times (0, T) : d^*_k((x, t), (x_k, t_k)) < k\lambda_k\} \subset G'_k,$$

where

$$\lambda_k := \tilde{M}^{-1}(x_k, t_k) \to 0.$$

By our assumptions $\lim_{k\to\infty} a(x_k) \neq 0$. The rest of the proof is now the same as Case (1) in the proof of Theorem 1.1.1 (see also [55, Theorem 4.1]) with v_k replaced by

$$v_k(x, t) := \lambda^{\frac{2}{p-1}} u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \qquad ((x, t) \in D_k),$$

and D_k by

$$D_k := \left\{(x,t) \in \lambda_k^{-1}(\Omega - x_k) : |x| < \frac{k}{2}\right\} \times \left(-\frac{k^2}{2}, \frac{k^2}{2}\right). \qquad \square$$

Proof of Proposition 1.1.5. In the proof we implicitly assume that all constants depend on N, p, Ω, a, $\|u_0\|_{L^\infty(\Omega)}$ and T. Fix any $\xi \in \partial\Omega$ with $a(\xi) = 0$. Since Ω is convex, we can, after a suitable rotation, assume

$$\xi_1 = \sup_{x\in\Omega} x_1, \quad \text{and therofore} \quad \nu_\Omega(\xi) = e_1 .$$

Since ξ is a local minimizer of a in $\bar{\Omega}$, all tangential derivatives of a vanish at ξ. Then (1.7) implies $\partial_{x_1} a(\xi) < 0$. Denote

$$\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\} .$$

Assume $u \not\equiv 0$, otherwise the statement is trivial. Observe, that u satisfies

$$u_t = \Delta u + \alpha(x,t)u, \qquad (x,t) \in \Omega \times (0,T) ,$$

where $\alpha(x,t) = a(x)u^{p-1}$. By Theorem 1.1.1, α is bounded on $\Omega \times (0, T/2)$ and the bound depends only on the implicitly assumed constants. Next, Hopf boundary lemma (see [42, Lemma 2.6]) implies $\partial_{e_1} u(\xi, \frac{T}{2}) < 0$. By the convexity of Ω, we can choose $\lambda < \xi_1$, sufficiently close to ξ_1 such that

$$w_\lambda(x,t) := u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \Omega_\lambda \times (0,T))$$

is well defined. Since $\partial_{x_1} u(\xi, \frac{T}{2}) < 0$ and $\partial_{x_1} a(\xi) < 0$, we can increase $\lambda < \xi_1$ such that

$$w_\lambda(x, \frac{T}{2}) > 0, \quad \text{and} \quad a(x^\lambda) > a(x) \qquad (x \in \Omega_\lambda) .$$

Observe that $\xi_1 - \lambda \geq c_1 > 0$, where c_1 is independent of ξ. Since $a(x^\lambda) > a(x)$

for $x \in \Omega_\lambda$, w_λ satisfies

$$(w_\lambda)_t \geq \Delta w_\lambda + \alpha^*(x,t) w_\lambda, \qquad (x,t) \in \Omega_\lambda \times (0,T)\,,$$

where

$$\alpha^*(x,t) := a(x)\frac{u^p(x^\lambda,t) - u^p(x,t)}{u(x^\lambda,t) - u(x,t)} \qquad ((x,t) \in \Omega_\lambda \times (0,T))$$

is bounded on compact subintervals of $(0,T)$. Similarly as in (1.31)

$$w_\lambda(x,t) \geq 0 \qquad ((x,t) \in \partial\Omega_\lambda \times (0,T))\,.$$

Now, the maximum principle implies $w_\lambda > 0$ in $\Omega_\lambda \times \left(\frac{T}{2}, T\right)$. Therefore $|x^*(t) - \xi| \geq c_0$ for each $t \in \left(\frac{T}{2}, T\right)$. Since c_0 is independent of ξ and $\Gamma \subset \partial\Omega$, one has

$$\operatorname{dist}(x^*(t), \Gamma) \geq \operatorname{dist}(x^*(t), \partial\Omega) \geq c_0 > 0 \qquad \left(t \in \left(\frac{T}{2}, T\right)\right)\,,$$

and the statement of the proposition follows from Lemma 1.4.3. □

Lemma 1.4.4. *Let* $N = 1$, $\Omega = (0,1)$ *and fix* $\mu \in [0, \frac{1}{2})$. *Assume* $a \in C^2([0,1])$ *has exactly one nondegenerate zero* $\mu \in [0, 2\mu]$. *Also assume* $a(x) < 0$ *for* $x \in [0,\mu)$ *and*

$$u_0(x) \leq u_0(x^\mu) \qquad (x \in (0,\mu))\,. \tag{1.57}$$

If $u \not\equiv 0$ *is a nonnegative solution of the problem* (1.1), (1.2), *then* $|x^*(t) - \mu| \geq c_0 > 0$ *and* c_0 *depends on* N, p, a, $\|u_0\|_{L^\infty((0,1))}$, T.

Proof. For each $\lambda \in (0, \frac{1}{2})$, define $w_\lambda : (0,\lambda) \times (0,\infty) \to \mathbb{R}$ as $w_\lambda(x,t) := u(x^\lambda, t) - u(x,t)$. Since $a(x^\mu) \geq 0 \geq a(x)$ for each $x \in [0,\mu]$,

$$a(x^\mu)u^p(x^\mu,t) - a(x)u^p(x,t) \geq 0 \qquad ((x,t) \in [0,\mu] \times (0,T))\,.$$

Thus,

$$(w_\mu)_t - (w_\mu)_{xx} \geq 0 \qquad ((x,t) \in (0,\mu) \times (0,T)) .$$

By (1.57)

$$w_\mu(x,0) = u_0(x^\mu) - u_0(x) \geq 0 \qquad (x \in (0,\mu)) .$$

Since $u \not\equiv 0$, the maximum principle implies $u > 0$ in $(0,1)\times(0,T)$. Then similarly as in (1.31)

$$w_\mu(0,t) > 0 \quad \text{and} \quad w_\mu(\mu,t) = 0 \qquad (t \in (0,T)) .$$

Then, the maximum principle $w_\mu > 0$ in $(0,\mu) \times (0,T)$ and $\partial_x w_\mu(\mu,t) < 0$ for $t \in (0,T)$. Hence, for sufficiently small $\varepsilon_0 > 0$ we obtain

$$w_\lambda(x,T/2) \geq 0 \qquad (x \in (0,\lambda), \lambda \in [\mu, \mu+\varepsilon_0)) .$$

As above one can show

$$w_\lambda(0,t) > 0 \quad \text{and} \quad w_\lambda(\lambda,t) = 0 \qquad (t \in (T/2,T)) .$$

Since $a'(\mu) > 0$, we can decrease $\varepsilon_0 > 0$ to obtain $a(x^\lambda) \geq a(x)$ for each $x \in (0,\lambda)$ and each $\lambda \in [\mu, \mu+\varepsilon_0)$. Then

$$(w_\lambda)_t - \Delta w_\lambda \geq a(x)[u^p(x^\lambda,t) - u^p(x,t)] = c(x,t) w_\lambda \qquad ((x,t) \in (0,\lambda) \times (t_0,T)) ,$$

where $c(x,t)$ is a continuous function on $[0,\lambda]\times[t_0,T)$ (possibly unbounded as $t \to T$) The maximum principle implies $w_\lambda(x,t) > 0$ for each $(x,t) \in (0,\lambda)\times(t_0,T)$. In particular $x^*(t) \geq \lambda > \mu$, and therefore $|x^*(t) - \mu| \geq c_0 > 0$ for each $t \in (t_0,T)$. □

Proof of Proposition 1.1.7. Lemma 1.4.4 with $\mu = \mu_1$ implies $|x^*(t) - \mu_1| > \varepsilon^* > 0$. If we replace x by $1-x$ and use Lemma 1.4.4 with $\mu = 1 - \mu_2$ again, we obtain $|x^*(t) - \mu_2| > \varepsilon^* > 0$. Now, the proposition follows from Lemma 1.4.3. □

Proof of Proposition 1.1.6. Without loss of generality assume $a(0) \leq 0$, otherwise replace x by $1-x$. If $\mu < \frac{1}{2}$, then the proposition follows from Lemma 1.4.4 and Lemma 1.4.3. Assume $\mu \in [\frac{1}{2}, 1]$. Similarly as in the proof of Lemma 1.4.4, we can show that $w_\mu(x,t) := u(x^\mu, t) - u(x,t)$ is well defined on $[\mu, 1]$ and satisfies

$$w_\mu(x,t) < 0 \qquad ((x,t) \in (\mu,1) \times (0,T)) \quad \text{and} \quad w'_\mu(\mu,t) < 0 \qquad (t \in (0,T))\,.$$

Hence, for $\lambda > \mu$ sufficiently close to μ we have $w_\lambda(x, \frac{T}{2}) < 0$ for any $x \in (\lambda, 1)$. Similarly as in Lemma 1.4.4 (using the maximum principle) we prove $w_\lambda(x,t) < 0$ for any $(x,t) \in (\lambda, 1) \times (\frac{T}{2}, T)$. Consequently, $|x^*(t) - \mu| > \lambda - \mu > 0$ for all $t \in (\frac{T}{2}, T)$ and the proposition follows from Lemma 1.4.3. □

Chapter 2

Asmyptotically symmetric equations

2.1 Introduction

In this chapter we study quasilinear parabolic equation

$$\partial_t u = A_{ij}(t, u, \nabla u)u_{x_i x_j} + F(t, u, \nabla u) + G(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (2.1)$$

where ∇g denotes the gradient of a function g. The functions A and F satisfy certain regularity, ellipticity, and symmetry assumptions as specified in the next section. The function G that decays to 0 as t approaches infinity, is considered to be a perturbation of the problem. In (2.1) and also in the rest of the chapter we use summation convention, that is, when an index appears twice in a single term, we are summing over all of its possible values, usually from 1 to N.

Our goal is to show that every positive, classical, global, bounded solution u of (2.1) is asymptotically symmetric. Before we make these statements precise, let us give a brief account of older results.

The first results on reflectional symmetry were established by Gidas, Ni and Nirenberg [27] for positive solutions of elliptic equations on bounded domains.

Specifically, if Ω is a bounded, smooth domain, convex in x_1, and symmetric with respect to the hyperplane

$$H_0 := \{x \in \mathbb{R}^N : x_1 = 0\},$$

and f is a Lipschitz function, then a positive classical solution u of

$$\Delta u + f(u) = 0, \qquad x \in \Omega, \tag{2.2}$$

$$u = 0, \qquad x \in \partial\Omega, \tag{2.3}$$

is even in x_1 and nonincreasing in the set

$$\Omega_0 := \{x \in \Omega : x_1 > 0\}.$$

The used techniques included the maximum principle and the method of moving hyperplanes introduced by Alexandrov [3] and developed by Serrin [60], who used it for overdetermined elliptic problems. Later, the results of Gidas et al. were generalized by Li [39] to fully nonlinear problems and Berestycki and Nirenberg [12] extended them to nonsmooth domains Ω. We refer the reader to the surveys [10, 46, 48] for more results, references, and generalizations.

In another paper, Gidas, Ni and Nirenberg [28] considered (2.2) with $\Omega = \mathbb{R}^N$ and a smooth nonlinearity f satisfying $f(0) = 0$, and certain hypothesis near 0. They proved that each positive solution which decays to 0 at a suitable rate, is radially symmetric. Later, Li [40] showed that any decay of solution as $|x| \to \infty$ is sufficient for symmetry, provided $f(0) = 0$ and $f'(0) < 0$. The later condition was weakened by Li and Ni [41], who assumed that $f'(z) \leq 0$ for any z sufficiently close to 0. All these papers also treat fully nonlinear problems. The described results were extended in various directions such as cooperative systems of equations, more general unbounded domains, or more general equations. We again refer the reader to [10, 46, 48] for more references.

The situation is more complicated for parabolic problems, as one cannot expect the solution to be symmetric, unless the initial data are symmetric. However, one can prove that the solution approaches the space of symmetric functions as time approaches infinity. To make this concept precise, for any open $\Omega \subset \mathbb{R}^N$ we define ω-limit set of u to be

$$\omega(u) := \{z : z = \lim_{n\to\infty} u(\cdot, t_n) \text{ for some } t_n \to \infty\},$$

where the convergence is in the space $C_0(\bar{\Omega})$, the space of continuous functions on $\bar{\Omega}$ that vanish on $\partial\Omega$ and decay to zero at infinity (if Ω is unbounded). The space $C_0(\bar{\Omega})$ is equipped with the supremum norm. If Ω is a bounded domain, symmetric with respect to H_0, we say that u is *asymptotically symmetric* if z is even in x_1 and decreasing in x_1 in Ω_0 for each $z \in \omega(u)$.

The first results on asymptotic symmetry appeared in [35], where Hess and Poláčik proved asymptotic symmetry for positive classical solutions of the problem

$$\begin{aligned} u_t &= \Delta u + f(t,u), && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

Here, Ω is a smooth bounded domain convex in x_1, symmetric with respect to H_0 and f is Hölder in t and Lipschitz in u. In an independent work Babin [6, 7] showed asymptotic symmetry for autonomous fully nonlinear problem and later, Babin and Sell [8] allowed nonlinearity to depend on t. However, these results require additional compactness and positivity assumptions compared to [35].

These drawbacks were removed in [51], where Poláčik proved the asymptotic symmetry for positive, classical solutions of a general fully nonlinear parabolic problem on bounded domains. The results required certain strong positivity assumptions that were further discussed in [23].

Unlike for elliptic equations, symmetric properties of solutions on $\mathbb{R}^N$ are much less understood. The difficulties arise from the fact that the center of symmetry is not a priori fixed. Even if one is able to prove the symmetry of every function $z \in \omega(u)$ with respect to some hyperplane, it is not immediate to show that all functions in $\omega(u)$ are symmetric with respect to the same hyperplane. Having this in mind, we say that u defined on $\Omega = \mathbb{R}^N$ is *asymptotically symmetric*, if there is $\lambda_0 \in \mathbb{R}$ such that all functions $z \in \omega(u)$ are symmetric with respect to the same hyperplane

$$H_{\lambda_0} := \{x \in \mathbb{R}^N : x_1 = \lambda_0\},$$

and decreasing in the halfspace

$$\mathbb{R}^N_{\lambda_0} := \{x \in \mathbb{R}^N : x_1 > \lambda_0\}.$$

In [49], Poláčik proved that a nonnegative solution u of (2.1) is asymptotically symmetric, provided $G \equiv 0$ and assumptions (N1)–(N4), (2.15), (2.16) from the next section are satisfied. In [50], Poláčik discussed entire solutions, that is, solutions defined for all times (positive and negative), and he showed that each nonnegative entire solution is symmetric at each time.

We were not able to locate any symmetry results in the literature if $G \not\equiv 0$. However, these can be obtained if the problem (2.1) is asymptotically autonomous, that is, if F and A_{ij} are independent of t, and u converges to a solution of the elliptic problem

$$0 = A_{ij}(u, \nabla u)u_{x_i x_j} + F(u, \nabla u), \qquad x \in \mathbb{R}^N. \tag{2.4}$$

Then by the symmetry results for elliptic problems [28], this equilibrium is symmetric, and therefore the solution of the parabolic problem is asymptotically symmetric.

The convergence to a nonnegative equilibrium was obtained for asymptotically autonomous problems, that is, for the problems that are approaching an autonomous one as $t \to \infty$. First, let us explain the existing results on the following model problem. Let u be a classical, global, nonnegative solution of the problem

$$\begin{aligned} u_t &= \Delta u + F(u) + G(x,t), && (x,t) \in \Omega \times (0,\infty)\,, \\ u(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,\infty)\,. \end{aligned} \tag{2.5}$$

Huang and Takáč in [36] (see also [15]) proved that the solution u of (2.5) converges to a solution v of the problem

$$\begin{aligned} 0 &= \Delta v + F(v), && x \in \Omega\,, \\ v &= 0, && x \in \partial\Omega\,, \end{aligned} \tag{2.6}$$

provided Ω is a smooth bounded domain, F satisfies certain analyticity assumptions and

$$\sup_{t\in(0,\infty)} t^{1+\delta} \int_t^\infty \|G(\cdot,s)\|_{L^2(\Omega)} ds < \infty\,. \tag{2.7}$$

Huang and Takáč also treated more general gradient-like problems with self-adjoint differential operators.

Later, Chill and Jendoubi [16] considered the problem (2.5) with $\Omega = \mathbb{R}^N$ and

$$F(u) = \sum_{p\in P} c_p |u|^{p-1} u\,,$$

where P is a finite subset of $(1, \frac{N+2}{N-2})$ and $c_q > 0$ for $q = \max_{p\in P} p$. Moreover, they assumed that there exists a compact set $K \subset \mathbb{R}^N$ with $\operatorname{supp} G(\cdot,t) \subset K$ for each $t > 0$. As a result, they proved that (2.7) implies the convergence of positive solutions u, with bounded $H^1(\mathbb{R}^N)$ norm, to a solution of (2.6).

In this work we generalize symmetry results from [49] to nonnegative solutions of the problem (2.1) with $G \not\equiv 0$. Under the assumptions (N1)–(N4) listed in the next section, we prove that each positive solution of (2.1) is asymptotically symmetric, provided there exists $\mu > 0$ with

$$\|G\|_{X_{(t,\infty)}} \le Ce^{-\mu t} \qquad (t \ge 0)\,, \tag{2.8}$$

where

$$X_{(s,t)} := L^\infty(\mathbb{R}^N \times (s,t)) \oplus L^{N+1}(\mathbb{R}^N \times (s,t)) \qquad (t, s \in (0,\infty], s < t) \tag{2.9}$$

is the space of functions f that can be written in the form $f = g + h$ with $g \in L^\infty(\mathbb{R}^N \times (s,t))$ and $h \in L^{N+1}(\mathbb{R}^N \times (s,t))$, equipped with the norm

$$\|f\|_{X_{(s,t)}} = \inf_{g+h=f} \left(\|g\|_{L^\infty(\mathbb{R}^N \times (s,t))} + \|h\|_{L^{N+1}(\mathbb{R}^N \times (s,t))} \right) . \tag{2.10}$$

Notice that G is not assumed to be globally integrable in x. This generalization proves to be useful for perturbations that depends on the solution or derivatives of solution, since these are only assumed to be bounded. Indeed, if instead of $G : (x,t) \to \mathbb{R}$ we consider a function $\tilde{G} : (x,t,u,p,q) \in \Omega \times [0,\infty) \times \mathbb{R}^{1+N+N^2} \to \mathbb{R}$, then our results apply, if

$$\tilde{G} : (x,t) \mapsto \tilde{G}(x,t,u(x,t),Du(x,t),D^2u(x,t)) \qquad ((x,t) \in \mathbb{R}^N \times [0,\infty))$$

satisfies (2.8). An example of such function $\tilde{G}$ is

$$\tilde{G} : (x,t,u,Du,D^2u) \mapsto e^{-t} g(u,Du,D^2u) \, , \tag{2.11}$$

where g is continuous. Notice that problem (2.1) with G replaced by $\tilde{G}$ is fully nonlinear. Therefore, our symmetry results cover certain fully nonlinear problems that converge exponentially to quasilinear ones as $t \to \infty$. However, it is not known if the symmetry results hold for general fully nonlinear equations.

If we apply our results on reflectional symmetry in various directions, the standard arguments show that all functions in the ω-limit set are radially symmetric with respect to the same origin. In a future work we show how to apply our symmetry results in the study of the asymptotic behavior of solution of asymptotically autonomous problems, that is, when F and (A_{ij}) are independent of t.

The asymptotic symmetry of positive solutions does not hold true if we merely assume that G converges to 0 as $t \to \infty$. A counterexample is given in Example 2.2.3 below, with $\|G\|_{X(t,\infty)} \approx \frac{1}{t}$. However, it is not know if the exponential decay (as stated in (2.8)) is necessary. Especially, we leave as an open problem, whether the integrability of $t \mapsto \|G\|_{X(t,\infty)}$ is sufficient for asymptotic symmetry of solutions.

To prove the symmetry results, we extend linear estimates for parabolic equations such as Alexandrov-Krylov estimate and the Harnack inequality to more general inhomogeneities (right hand sides) on unbounded domains. Since these results might be of independent interest, especially for applications to unbounded domains, we devote them a separate section. Once the linear estimates are established, we follow the framework from [49] to prove the symmetry results. The application of methods from [49] is not completely straightforward and a special care should be taken when treating perturbations on unbounded sets, since various constants might depend on the diameter of the set or the length of the time interval. In that case, we restrict our arguments to bounded time intervals and use iterative methods.

The rest of the chapter is organized as follows. In the next section we state our main results. Section 2.3 contains general linear estimates of parabolic problems, and in Section 2.4, we prove the symmetry results.

2.2 Main results

Consider parabolic problem (2.1). We assume that the real valued functions $(A_{ij})_{1\le i,j\le N}, F : (t,u,p) \mapsto \mathbb{R}$ are defined on $[0,\infty) \times [0,\infty) \times \mathbb{R}^N$ and satisfy the following conditions.

(N1) *Regularity.* The functions $(A_{ij})_{1\le i,j\le N}, F$ are continuous on $[0,\infty)\times[0,\infty)\times \mathbb{R}^N$ and continuously differentiable with respect to u and $p = (p_1, \cdots, p_N)$ uniformly in $t \in [0,\infty)$. This means, that if h stands for any of $\partial_u A_{ij}$, $\partial_u F$,

$\partial_{p_k} A_{ij}$ or $\partial_{p_k} F$ for some $1 \le i, j, k \le N$, then for each $M > 0$ one has

$$\limsup_{\substack{0 \le u, v, |p|, |q| \le M, t \ge 0 \\ |u-v|+|p-q| \to 0}} |h(t,u,p) - h(t,v,q)| = 0\,. \tag{2.12}$$

(N2) *Ellipticity.* There is a positive constant α_0 such that for each $\xi \in \mathbb{R}^N$

$$A_{ij}(t,u,p)\xi_i\xi_j \ge \alpha_0|\xi|^2 \qquad ((t,u,p) \in [0,\infty) \times [0,\infty) \times \mathbb{R}^N)\,.$$

(N3) *Symmetry.* For each $(t,u,p) \in [0,\infty) \times [0,\infty) \times \mathbb{R}^N$ and $1 \le i, j \le N$ one has

$$\begin{aligned} A_{ij}(t,u,p) &= A_{ij}(t,u,-p_1,p_2,\cdots,p_N)\,, \\ F(t,u,p) &= F(t,u,-p_1,p_2,\cdots,p_N)\,, \\ A_{1j} = A_{j1} &\equiv 0 \qquad \text{if } j \ne 1\,. \end{aligned}$$

(N4) *Stability of 0.* $F(t,0,0) = 0$ and there is a constant $\gamma > 0$ such that

$$\partial_u F(t,0,0) < -2\gamma \qquad (t \ge 0)\,.$$

Remark 2.2.1. The assumption (N4) and uniform continuity of $\partial_u F$ in t imply the existence of $\varepsilon_\gamma^* > 0$ with

$$\partial_u F(t,u,p) < -\gamma \qquad ((t,u,p) \in [0,\infty) \times [0,\varepsilon_\gamma^*] \times B_{\varepsilon_\gamma^*})\,,$$

where B_r is an open ball centered at the origin with the radius r.

The assumptions on G are as follows (recall that $X_{(s,t)}$ was defined in (2.9)).

(G1) $G \in X_{(t,t+1)}$ for each $t \in [0,\infty)$ and

$$\lim_{t\to\infty} \|G\|_{X_{(t,t+1)}} = 0 \tag{2.13}$$

Some results require exponential decay of G.

(G2) For each $t \in (0,\infty)$ one has $G \in X_{(t,\infty)}$. Moreover, there exist $\mu > 0$ and $C_\mu > 0$ such that

$$\|G\|_{X_{(t,\infty)}} \leq \frac{C_\mu}{2} e^{-\mu t} \qquad (t > 0) \tag{2.14}$$

One can easily verify that, with possibly changed μ, (G2) is equivalent to the following statement. For each $\varepsilon > 0$, there exists $t_\varepsilon > 0$ with $\|\tilde{G}\|_{X_{(t_\varepsilon,\infty)}} \leq \varepsilon$, where $\tilde{G}(x,t) = e^{\mu(t-t_\varepsilon)}G(x,t)$. Notice that if we replace $X_{(t,\infty)}$ by $X_{(t,t+1)}$ in (G2), we obtain an equivalent assumption. As explained in the introduction, the space X allows us to treat, possibly unbounded, perturbations depending on u, Du or D^2u.

We assume that u is a classical, nonnegative, global solution of (2.1), that is, $u \in C^{2,1}(\mathbb{R}^N \times (0,\infty))$ and u satisfies (2.1) everywhere. Moreover, we assume

$$S := \sup_{\substack{(x,t)\in\mathbb{R}^N\times[0,\infty) \\ 1\leq i,j\leq N}} \{|u(x,t)|, |u_{x_i}(x,t)|, |u_{x_ix_j}(x,t)|\} < \infty\,, \tag{2.15}$$

and

$$\limsup_{|x|\to\infty, t\in[0,\infty)} \{|u(x,t)|, |u_{x_i}(x,t)|, |u_{x_ix_j}(x,t)|\} = 0 \qquad (1 \leq i,j \leq N)\,. \tag{2.16}$$

Observe that (N1) combined with (2.15) yields the existence of $\beta_0 > 0$ such that

$$\sup_{t\geq 0} |h(t,v,p) - h(t,w,q)| \leq \beta_0 |(v,p) - (w,q)| \qquad (v,w \in [0,S], p,q \in \mathbb{R}^N, |p|,|q| \leq S)\,, \tag{2.17}$$

where h stands for F or A_{ij}, and S was defined in (2.15). Although we suppose (N2) with fixed constant $\alpha_0 > 0$, we really need it to be true on the range of (u, Du, D^2u) for each considered solution u. Since u is bounded and has bounded derivatives, (N2) needs to hold true only for $u, |p| \leq S$.

By (2.16), there is ρ_γ^* such that $|u|, |\nabla u| < \varepsilon_\gamma^*$ in $(\mathbb{R}^N \setminus B_{\rho_\gamma^*}) \times [0,\infty)$, and therefore by Remark 2.2.1

$$\partial_u F(t, u(x,t), \nabla u(x,t)) < -\gamma \qquad ((x,t) \in (\mathbb{R}^N \setminus B_{\rho_\gamma^*}) \times [0,\infty)) . \tag{2.18}$$

Uniformity of the limit (2.16) in t is not technical. When omitted the symmetry results may fail even for $G \equiv 0$. For more details see [50] and references therein.

It is not sufficient to merely assume $\partial_u F(t,0,0) < 0$ in (N4). Indeed, for appropriate $p > 1$ Poláčik and Yanagida [56] constructed a positive solution of the problem

$$u_t = \Delta u + u^p, \qquad (x,t) \in \mathbb{R}^N \times (0,\infty)$$

satisfying (2.15) and (2.16) that is not asymptotically symmetric. If we set $F(t,u,q) = u^p - e^{-t}u$ and $G(x,t) = e^{-t}u(x,t)$, then $\partial_u F(t,0,0) < 0$ and G satisfies (G2). However, u is not asymptotically symmetric.

The assumptions (2.15) and (2.16) guarantee that u is globally defined and $\{u(\cdot,t) : t \geq 0\}$ is relatively compact in $E := C_0^1(\mathbb{R}^N)$, which stands for the space of $C^1(\mathbb{R}^N)$ functions, bounded together with their first order derivatives, equipped with the standard C^1 norm. Define the ω-limit set of u

$$\omega(u) = \{z : z = \lim_{n\to\infty} u(\cdot, t_n) \text{ for some } t_n \to \infty\} , \tag{2.19}$$

where the convergence is in the topology of the space $C_0^1(\mathbb{R}^N)$. Then $\omega(u)$ is nonempty, compact set in E, and it attracts the solution in the following sense

$$\lim_{t\to\infty} \operatorname{dist}_E (u(\cdot,t), \omega(u)) = 0 . \tag{2.20}$$

We are ready to formulate our first symmetry result.

Theorem 2.2.2. *Assume (N1)–(N4), (G1), and let u be a global solution of* (2.1) *satisfying* (2.15) *and* (2.16). *Then either u converges to* 0 *in $L^\infty(\mathbb{R}^N)$ or there* *exist* $\lambda \in \mathbb{R}$ *and* $\phi \in \omega(u)$ *such that for each* $x \in \mathbb{R}^N_\lambda$ *one has*

$$\begin{aligned} \phi(2\lambda - x_1, x') &= \phi(x) \qquad &((x_1, x') = x \in \mathbb{R}^N)\,, \\ \partial_{x_1}\phi(x) &< 0 \qquad &(x \in \mathbb{R}^N_\lambda)\,. \end{aligned} \tag{2.21}$$

If we in addition assume (G2), then either $\omega(u) = \{0\}$ *or there is* $\lambda \in \mathbb{R}$ *such that* (2.21) *holds for all* $\phi \in \omega(u)$.

The following example shows, that the last statement of Theorem 2.2.2 does not hold if we merely assume (G1). In particular it is not true that all functions in the ω-limit set are symmetric with respect to the same hyperplane.

Example 2.2.3. Let v be a positive function satisfying (2.15), (2.16) and

$$0 = \Delta v + g(v), \qquad x \in \mathbb{R}^N\,, \tag{2.22}$$

for appropriate function g with $g'(0) < 0$. Such a function v exists for example for $g(u) = \lambda u + u^p$ (see e.g. [11] and references therein) with $\lambda < 0$, $1 < p < p_S$, where $p_S := \frac{N+2}{N-2}$ for $N \geq 3$ and $p_S = \infty$ for $N \leq 2$ is the critical Sobolev exponent. By [28], v is radially symmetric and radially decreasing with center at a point $x_0 \in \mathbb{R}^N$. Let $\eta : [0, \infty) \to \mathbb{R}$ be a bounded differentiable function and define $u : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}$ by $u(x,t) := v(x_1 + \eta(t), x')$ for any $(x,t) = ((x_1, x'), t) \in \mathbb{R}^N \times [0, \infty)$. Then u satisfies (2.15), (2.16), and

$$u_t = \Delta u + g(u) + G(x,t), \qquad (x,t) \in \mathbb{R}^N \times [0, \infty)\,,$$

where

$$G(x,t) := v_{x_1}(x_1 + \eta(t), x')\eta'(t) \qquad ((x,t) = ((x_1, x'), t) \in \mathbb{R}^N \times [0, \infty))\,.$$

It is easy to see that we can choose η with the following properties. There are sequences $(s_k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}}$ with $s_k, t_k \to \infty$ as $k \to \infty$ such that $\eta(t_k) = 1$,

$\eta(s_k) = 0$, and there is $C > 0$ with $|\eta'(t)| \le \frac{C}{t}$ for all $t > 0$. Since v_{x_1} is bounded,

$$\lim_{t\to\infty} \|G\|_{X_{(t,t+1)}} \le \lim_{t\to\infty} \|G\|_{L^\infty(\mathbb{R}^N\times(t,t+1))} \le \lim_{t\to\infty} \|v_{x_1}\|_{L^\infty(\mathbb{R}^N\times(0,\infty))} \frac{C}{t} = 0\,,$$

and in particular G satisfies (G1). However, $v(x_1 + s, x') \in \omega(u)$ for any $s \in [0, 1]$, and therefore the functions in $\omega(u)$ are not symmetric with respect to the same hyperplane.

Finally, we state the corollary of Theorem 2.2.2 on asymptotic radial symmetry. We omit the proof since it uses the same arguments as in the case $G \equiv 0$ (cf. [49]). The formulation of results on rotational symmetry, if the problem is rotationally symmetric, is left to the reader.

Corollary 2.2.4. *In addition to (N1) – (N4) and (G2), assume $A_{ij} \equiv 0$ if $i \ne j$ and*

$$A_{ii}(t, u, p) = A_{ii}(t, u, q), \quad F(t, u, p) = F(t, u, q) \quad \textit{whenever} \quad |p| = |q|\,.$$

Let u be a global solution of (2.1) *satisfying* (2.15) *and* (2.16)*. Then either u converges to* 0 *in $L^\infty(\mathbb{R}^N)$ or there exists $\xi \in \mathbb{R}$ such that for each $\phi \in \omega(u)$ there is $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ with*

$$\begin{aligned} \phi(x - \xi) &= \tilde{\phi}(|x|) \qquad && (x \in \mathbb{R}^N)\,, \\ \partial_r \tilde{\phi}(r) &< 0 && (r = |x| > 0)\,. \end{aligned}$$

2.3 Linear equations

This section is devoted to linear parabolic estimates as a preparation for the method of moving hyperplanes. The results that were already published are stated without proofs. However, at some places we have to extend existing results and for those we include proofs as well.

Recall the following standard notation. For an open set $Q \subset \mathbb{R}^N$ we denote by $\partial_P Q$ the parabolic boundary of Q (for precise definition see e.g. [37, 42]). We also define a time cut of Q to be

$$Q_M := \{(x,s) \in \bar{Q} : s \in M\} \qquad (M \subset \mathbb{R})\,. \tag{2.23}$$

If $M = \{t\}$, we often write Q_t instead of $Q_{\{t\}}$.

For bounded sets U, U_1 in $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, the notation $U_1 \subset\subset U$ means $\bar{U}_1 \subset U$, $\operatorname{diam} U$ stands for the diameter of U, and $|U|$ for its Lebesgue measure (if it is measurable). For any $\lambda \in (-\infty, \infty)$ we define an open half space: $\mathbb{R}^N_\lambda := \{x \in \mathbb{R}^N : x_1 > \lambda\}$, and for $\lambda = -\infty$ we set $\mathbb{R}^N_\lambda = \mathbb{R}^N$. The open ball in $\mathbb{R}^N$ centered at x with radius r is denoted by $B(x,r)$ and if the ball is centered at the origin, that is, if $x = 0$, we also write $B_r := B(0,r)$. For any $\lambda \in \mathbb{R}$ and $R > 0$ we set $B^\lambda_R := B_R \cap \mathbb{R}^N_\lambda$. Symbols f^+ and f^- denote the positive and negative parts of a function f: $f^\pm := (|f| \pm f)/2 \geq 0$.

We consider time dependent elliptic operators L of the form

$$L(x,t) = a_{km}(x,t)\frac{\partial^2}{\partial x_k \partial x_m} + b_k(x,t)\frac{\partial}{\partial x_k}. \tag{2.24}$$

To simplify the notation we shall use the following definition.

Definition 2.3.1. Given an open set $Q \in \mathbb{R}^N \times (0,\infty)$ and positive numbers α_0, β_0, we say that an operator L of the form (2.24) belongs to $E(\alpha_0, \beta_0, Q)$ if its coefficients a_{km}, b_k are measurable functions defined on Q and they satisfy

$$\begin{aligned} |a_{km}|, |b_k| &\leq \beta_0 && (k,m = 1,\dots,N)\,, \\ a_{km}(x,t)\xi_k\xi_m &\geq \alpha_0|\xi|^2 && ((x,t) \in Q,\ \xi \in \mathbb{R}^N). \end{aligned}$$

2.3.1 Nonlinear to linear

In this subsection we assume (N1) – (N4) and (G1). At some places, where explicitly stated, we also assume (G2). Fix a positive global solution u of (2.1) satisfying (2.15) and (2.16). We show, how symmetries of the problem give rise to linear equations from the nonlinear ones.

We say that a pair of functions $(\tilde{u}, \tilde{G})$ is *admissible*, if $\tilde{u}$ satisfies (2.15), (2.16), $\tilde{G}$ satisfies (G1), and $\tilde{u}$ is a positive solution of (2.1) with G replaced by $\tilde{G}$. In particular (u, G) is an admissible pair.

Let $(\tilde{u}, \tilde{G})$ be an admissible pair different to (u, G). If we denote $w := u - \tilde{u}$, then

$$\begin{aligned} & w_t = L(x,t)w + c(x,t)w + f(x,t), \qquad (x,t) \in \mathbb{R}^N \times (0,\infty)\,, \\ & \lim_{|x|\to\infty} \sup_{t\in(0,\infty)} |w(x,t)| = 0\,, \end{aligned} \tag{2.25}$$

where L has the form (2.24) with

$$\begin{aligned} a_{ij}(x,t) &= A_{ij}(t, u(x,t), \nabla u(x,t)), \\ b_i(x,t) &= \int_0^1 F_{p_i}(t, u(x,t), \nabla\tilde{u}(x,t) + s(\nabla u(x,t) - \nabla\tilde{u}(x,t)))\, ds \\ &\quad + \tilde{u}_{x_k x_\ell}(x,t) \int_0^1 A_{k\ell,p_i}(t, u(x,t), \nabla\tilde{u}(x,t) + s(\nabla u(x,t) - \nabla\tilde{u}(x,t)))\, ds, \\ c(x,t) &= \int_0^1 F_u(t, \tilde{u}(x,t) + s(u(x,t) - \tilde{u}(x,t)), \nabla\tilde{u}(x,t))\, ds \\ &\quad + \tilde{u}_{x_k x_\ell}(x,t) \int_0^1 A_{k\ell,u}(t, \tilde{u}(x,t) + s(u(x,t) - \tilde{u}(x,t)), \nabla\tilde{u}(x,t))\, ds. \end{aligned} \tag{2.26}$$

Then

$$L \in E(\alpha_0, \beta_0, \mathbb{R}^N \times (0,\infty))\,, \qquad \|c\|_{L^\infty(\mathbb{R}^N\times(0,\infty))} \le \beta_0\,, \tag{2.27}$$

and, by (N1) and Remark 2.2.1

$$c(x,t) < -\gamma\,, \tag{2.28}$$

whenever $u(x,t)$, $\tilde{u}(x,t)$, $|\nabla\tilde{u}(x,t)|$ and $|D^2\tilde{u}(x,t)|$ are smaller than ε^*_γ, where ε^*_γ was defined in Remark 2.2.1. Observe, that we do not impose any smallness assumptions on $|\nabla u(x,t)|$ or $|D^2 u(x,t)|$.

Moreover,

$$f := G - \tilde{G} \in X_{(t,t+1)}, \qquad \lim_{t\to\infty} \|f\|_{X_{(t,t+1)}} = 0\,. \tag{2.29}$$

If we suppose that (G2) holds for G and $\tilde{G}$, then

$$\|f\|_{X_{(t,\infty)}} \leq C_\mu e^{-\mu t} \qquad (t > 0)\,. \tag{2.30}$$

Uniform continuity of derivatives of $(A_{ij})_{1\leq i,j\leq N}$ and F in conjunction with (2.15) yields that (a_{ij}), (b_i), and c are continuous in x and t.

Example 2.3.2. By (N4), $\tilde{u} \equiv 0$ and $\tilde{G} \equiv 0$ is an admissible pair. Thus $w = u - 0 = u$ solves the equation (2.25) such that (2.27) and (2.29) hold true with $f = G$. Moreover,

$$c(x,t) < -\gamma \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty) : u(x,t) \leq \varepsilon_\gamma^*)\,, \tag{2.31}$$

and by (2.18),

$$c(x,t) < -\gamma \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty), |x| \geq \rho_\gamma^*)\,. \tag{2.32}$$

Example 2.3.3. For any $x_0 \in \mathbb{R}^N$ define

$$\tilde{u}(x,t) := u(x + x_0, t) \text{ and } \tilde{G}(x,t) := G(x + x_0, t) \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty))\,.$$

Since $(A_{ij})_{1\leq i,j\leq N}$ and F are independent of x, the pair $(\tilde{u}, \tilde{G})$ is admissible. Therefore $w(x,t) := u(x + x_0, t) - u(x,t)$ satisfies (2.25), such that (2.27) and (2.29) hold true. Moreover, by (2.28) and (2.18)

$$c(x,t) < -\gamma \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty), |x| \geq \rho_\gamma^* + |x_0|)\,.$$

The next example is crucial for the method of moving hyperplanes. To simplify the notation denote $x^\lambda := (2\lambda - x_1, x')$, the reflection of $x = (x_1, x') \in \mathbb{R}^N$ with respect to the hyperplane H_λ. We indicate explicitly the dependence of functions and operators on λ.

Example 2.3.4. By (N3),

$$\tilde{u}(x,t) := u(x^\lambda, t) \text{ and } \tilde{G}(x,t) := G(x^\lambda, t) \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty))$$

form an admissible pair. Thus, $w^\lambda := \tilde{u} - u$ satisfies (2.25) such that (2.27) and (2.29) hold true. Moreover, $|x| > 2|\lambda| + \rho_\gamma^* \geq \rho_\gamma^*$ implies $|x^\lambda| > \rho_\gamma^*$, and therefore (2.28) and (2.18) yield

$$c^\lambda(x,t) < -\gamma \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty), |x| \geq \rho_\gamma^* + 2|\lambda|)\,. \tag{2.33}$$

By (N1), (2.17) (and (G2), if assumed), the constants α_0, β_0, (and also C_μ, μ) are independent of λ. Notice that $w^\lambda(x,t) = 0$ for any $(x,t) \in H_\lambda \times [0,\infty)$. Hence, w^λ satisfies

$$\begin{aligned} &w_t^\lambda = L^\lambda(x,t)w^\lambda + c^\lambda(x,t)w^\lambda + f^\lambda(x,t), && (x,t) \in \mathbb{R}_\lambda^N \times (0,\infty)\,, \\ &w^\lambda = 0, && (x,t) \in H_\lambda \times (0,\infty)\,, \\ &\limsup_{|x|\to\infty, t>0} w^\lambda(x,t) = 0\,. \end{aligned} \tag{2.34}$$

Also, if G satisfies (G2), then $\tilde{G}$ satisfies (G2) as well. Consequently (2.30) holds with f replaced by f^λ. Notice that (a_{ij}) in (2.26) are independent of λ.

2.3.2 Estimates of solutions

The results in this subsection might be of independent interest, therefore we state them in more general setting, than required for the proofs of our symmetry results.

Let Q be a domain in $\mathbb{R}^{N+1}$ (bounded or unbounded), and let α_0, β_0 be positive constants. Consider a general linear parabolic equation

$$v_t = L(x,t)v + c(x,t)v + f(x,t), \qquad (x,t) \in Q\,. \tag{2.35}$$

For any $s < t$ denote $X_{(s,t)}(Q)$ the space of functions $f : Q \to \mathbb{R}$ such that their extension by 0 to $\mathbb{R}^{N+1}$ belongs to $X_{(s,t)}$ (cf. (2.10)). We denote $C_{\rm loc}(\bar{Q})$ the space of continuous functions equipped with the topology induced by the locally uniform convergence.

First, we formulate Alexandrov – Krylov estimate, proved by Alexandrov [2] in the elliptic case, and later extended by Krylov [37] to the parabolic setting. In the literature, one can find many generalizations of these results. Here, we extend Cabré's result [14] to functions f belonging to $X_{(s,t)}(Q)$. If $f \equiv 0$, we refer to the next theorem as the maximum or comparison principle.

Theorem 2.3.5. *Given $\tau < T$, fix an open set $Q \subset \mathbb{R}^N \times (\tau, T)$. If $v \in C_{loc}(\bar{Q}) \cap W^{2,1}_{N+1,loc}(Q)$ is a bounded supersolution of* (2.35) *(it satisfies* (2.35) *with " $=$ " replaced by " $\geq$ ") with $L \in E(\alpha_0, \beta_0, Q)$, a measurable function $c \leq 0$, and $f \in X_{(\tau,T)}(Q)$, then*

$$\sup_Q v^- \leq \sup_{\partial_P Q} v^- + C\|f^-\|_{X_{(\tau,T)}(Q)}\,, \tag{2.36}$$

where C depends on $N, \alpha_0, \beta_0, T - \tau$.

Proof. Fix arbitrary $\varepsilon > 0$ and choose f_1, f_2 such that $f_1^- + f_2^- = f^-$ and

$$\|f^-\|_{X_{(\tau,T)}(Q)} + \varepsilon \geq \|f_1^-\|_{L^{N+1}(Q_{(\tau,T)})} + \|f_2^-\|_{L^\infty(Q_{(\tau,T)})}\,.$$

Since $c \leq 0$, the bounded function $w : Q \to \mathbb{R}$

$$w(x,t) := v(x,t) + \sup_{\partial_P Q} v^- + (t-\tau)\|f_2^-\|_{L^\infty(Q_{(\tau,T)})} \qquad ((x,t) \in Q)\,,$$

satisfies

$$\begin{aligned} &w_t \geq L(x,t)w + c(x,t)w - f_1^-(x,t), && (x,t) \in Q\,, \\ &w \geq 0, && (x,t) \in \partial_P Q\,. \end{aligned} \tag{2.37}$$

Consequently, by [14, Corollary 1.16]

$$\sup_{Q} w^- \leq C\|f_1^-\|_{L^{N+1}(Q)} , \tag{2.38}$$

where C depends on $N, \alpha_0, \beta_0, T-\tau$. Then,

$$\begin{aligned}
\sup_{Q} v^- &\leq \sup_{Q} w^- + \sup_{\partial_P Q} v^- + (t-\tau)\|f_2^-\|_{L^\infty(Q_{(\tau,T)})} \\
&\leq \sup_{\partial_P Q} v^- + C\left(\|f_2^-\|_{L^\infty(Q)} + \|f_1^-\|_{L^{N+1}(Q)}\right) \\
&\leq \sup_{\partial_P Q} v^- + C\left(\|f^-\|_{X_{(\tau,T)}(Q)} + \varepsilon\right) .
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, (2.36) follows. □

Corollary 2.3.6. *If the assumption $c \leq 0$ of the previous theorem is replaced by $c \leq k$ for some $k \in \mathbb{R}$, and all other assumptions are retained, then*

a) if $k \geq 0$

$$\sup_{Q_{[\tau,T]}} v^- \leq e^{k(T-\tau)}\left(\sup_{\partial_P(Q_{[\tau,T]})} v^- + C\|f^-\|_{X_{(\tau,T)}(Q)}\right) ,$$

where C depends on $N, \alpha_0, \beta_0, T-\tau$.

b) if $k < 0$

$$\begin{aligned}
\sup_{Q_T} v^- \leq \max\{e^{k(T-\tau)}\|v^-\|_{L^\infty(Q_\tau)}, &\sup_{(\partial_P Q_{[\tau,T]})\setminus Q_\tau} v^-\} \\
&+ \frac{C}{1-e^k} \sup_{t\in[\tau,T-1]} \|f^-\|_{X_{(t,t+1)}(Q)} ,
\end{aligned}$$

where C depends on N, α_0, β_0. Notice that C is independent of $T-\tau$.

Proof of Corollary 2.3.6. The function $\tilde{v} := e^{-kt}v$ is a a supersolution of (2.35) with c and f replaced by $c-k$ and $\tilde{f}$ respectively, where $\tilde{f}(x,t) = e^{-kt}f(x,t)$.

Since $c-k\le 0$, Theorem 2.3.5 implies

$$e^{-kt_2}\sup_{Q_{t_2}} v^- = \sup_{Q_{t_2}} \tilde v^- \le \sup_{Q_{[t_1,t_2]}} \tilde v^-$$
$$\le \max\{\sup_{Q_{t_1}} \tilde v^-, \sup_{(\partial_P Q_{[t_1,t_2]})\setminus Q_{t_1}} \tilde v^-\} + C\|\tilde f^-\|_{X_{(t_1,t_2)}(Q)}$$
$$(\tau \le t_1 < t_2 \le T)\,, \quad (2.39)$$

where C depends on $N,\alpha_0,\beta_0,t_2-t_1$.

If $k\ge 0$, we set $t_1=\tau$ and elementary manipulations imply

$$\sup_{Q_{t_2}} v^- \le e^{k(t_2-\tau)}\left(\sup_{\partial_P(Q_{[\tau,t_2]})} v^- + C\|f^-\|_{X_{(\tau,t_2)}(Q)}\right).$$

Part a) follows, if we take supremum with respect to $t_2\in[\tau,T]$.

Denote $\Gamma := \sup_{t\in[\tau,T-1]}\|f^-\|_{X_{(t,t+1)}(Q)}$. If $k<0$, then (2.39) with $t_2=t_1+1$ yields

$$\sup_{Q_{t_1+1}} v^- \le \max\{e^k \sup_{Q_{t_1}} v^-, \sup_{(\partial_P Q_{(t_1,t_1+1)})\setminus Q_{t_1}} v^-\} + C\Gamma \qquad (t_1\in[\tau,T-1])\,,$$

where C depends on N,α_0,β_0. Iterating the previous expression for $t_1=\tau+j$, with $j\in\mathbb{N}$, $j\le T-\tau-1$, we obtain

$$\sup_{Q_{\tau+j}} v^- \le \max\{e^{kj}\sup_{Q_\tau} v^-, \sup_{(\partial_P Q_{(\tau,\tau+j)})\setminus Q_\tau} v^-\} + C\Gamma\sum_{i=0}^{j-1} e^{ki}\,. \quad (2.40)$$

Choose $j_0\in\mathbb{N}\cup\{0\}$ such that $\tau+j_0\le T<\tau+j_0+1$. Then (2.39) with $t_1=\tau+j_0$, $t_2=T$ and (2.40) imply

$$\sup_{Q_T} v^- \le \max\{e^{k(T-(\tau+j_0))}\sup_{Q_{\tau+j_0}} v^-, \sup_{(\partial_P Q_{(\tau+j_0,T)})\setminus Q_{\tau+j_0}} v^-\} + C\Gamma$$
$$\le \max\{e^{k(T-\tau)}\sup_{Q_\tau} v^-, \sup_{(\partial_P Q_{(\tau,T)})\setminus Q_\tau} v^-\} + C\Gamma\sum_{i=0}^{\infty} e^{ki}\,,$$

where C depends on N, α_0, β_0 and the part b) follows. □

If $Q = \mathbb{R}^N_\lambda \times (\tau, T)$, $\lambda \in \mathbb{R}$, $\tau < T$, we can change variables such that c becomes negative in the neighborhood of H_λ and it does not change too much away from H_λ. Such results are usually obtained with an application of an appropriate supersolution. The observation that such procedure is possible for thin domains and domains of small measure was proved in [12]. In the next lemma we summarize properties of the supersolution constructed in [49, Lemma 2.5].

Lemma 2.3.7. *Given* $\Theta, \varepsilon > 0$*, there exist a function* $g : [0, \infty) \to \mathbb{R}$ *and a constant* $\delta = \delta(N, \alpha_0, \beta_0, \Theta, \varepsilon) > 0$ *with the following properties:*

$$\begin{gathered} g \in C^1([0,\infty)) \cap C^2([0,\delta)) \cap C^2((\delta,\infty))\,, \\ \frac{1}{2} \leq g \leq 2\,, \\ g''(\xi) + \Theta(|g'(\xi)| + g(\xi)) \leq 0 \qquad (\xi \in (0,\delta))\,, \\ g''(\xi) + \Theta|g'(\xi)| - \varepsilon g(\xi) \leq 0 \qquad (\xi \in (\delta,\infty))\,. \end{gathered}$$

Following [49, Remark 2.6], we obtain the following result.

Remark 2.3.8. Set $Q := \mathbb{R}^N_\lambda \times (\tau, T)$ for some $\lambda \in \mathbb{R}$ and $0 \leq \tau < T \leq \infty$. Let $v \in C_{\rm loc}(\bar{Q}) \cap W^{2,1}_{N+1,loc}(Q)$ be a solution of (2.35) with $L \in E(\alpha_0, \beta_0, Q)$, $\|c\|_{L^\infty(Q)} \leq \beta_0$, and $f \in L^{N+1}(Q)$ satisfying

$$v = 0 \quad ((x,t) \in H_\lambda \times (\tau, T)) \quad \text{and} \quad \lim_{M\to\infty} \sup_{(x,t)\in Q, |x|\geq M} |v(x,t)| = 0\,.$$

For any $\gamma > 0$ set $\Theta = \frac{2\beta_0}{\gamma} + 1$, $\varepsilon = \frac{\gamma}{2}$ and let $\delta = \delta(N, \alpha_0, \beta_0, \gamma) > 0$ and g be as in Lemma 2.3.7. Then

$$w : (x,t) \mapsto \frac{v(x,t)}{g(x_1 - \lambda)} \qquad ((x,t) \in Q)$$

is a solution of

$$\begin{aligned} & w_t = \hat{L}(x,t)w + \hat{c}(x,t)w + \hat{f}(x,t), && (x,t) \in \mathbb{R}^N_\lambda \times (\tau, T)\,, \\ & w = 0, && (x,t) \in H_\lambda \times (\tau, T)\,, \\ & \lim_{M\to\infty} \sup_{(x,t)\in Q, |x|\geq M} |v(x,t)| = 0 \end{aligned} \tag{2.41}$$

with $\hat{L}(x,t) \in E(\alpha_0, 5\beta_0, Q)$, $\|\hat{c}\|_{L^\infty(Q)} \leq 5\beta_0$ and

$$\|\hat{f}\|_{X_{(\tau,T)}(Q)} \leq 2\|f\|_{X_{(\tau,T)}(Q)}\,.$$

Moreover,

$$\hat{c}(x,t) \leq \begin{cases} -\frac{\gamma}{2} & (x,t) \in Q, x_1 \in [\lambda, \lambda+\delta)\,, \\ c(x,t) + \frac{\gamma}{2} & (x,t) \in Q, x_1 \in [\lambda, \infty)\,. \end{cases} \tag{2.42}$$

We conclude this section with a version of Krylov-Safonov Harnack inequality [38] (see also [42]) for sign changing solutions of nonhomogeneous problems. The statement is based on [51, Lemma 3.5], however it was modified to obtain the dependence of κ and κ_1 on diam D instead of diam U.

Lemma 2.3.9. *Given numbers $d > 0$, $\theta > 0$, $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4$, and τ with $\tau_1 - 2\theta \leq \tau \leq \tau_1 - \theta$, consider bounded domains $D, U \subset \mathbb{R}^N$ with*

$$D \subset\subset U, \qquad \textit{dist}\ (\bar{D}, \partial U) \geq d\,,$$

and denote $Q = U \times (\tau, \tau_4)$. Then there exist constants $\kappa, \kappa_1 > 0$ determined only by N, α_0, β_0, d, diam D, θ, $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, and $\tau_4 - \tau_3$ with the following property. If $v \in C_{loc}(\bar{Q}) \cap W^{2,1}_{N+1,loc}(Q)$ is a solution of (2.35), with $L \in E(\alpha_0, \beta_0, Q))$, $\|c\|_{L^\infty(Q)} \leq \beta_0$, and $f \in X_{(\tau,\tau_4)}(Q)$, then

$$\inf_{\bar{D}\times(\tau_3,\tau_4)} v \geq \kappa \|v^+\|_{L^\infty(D\times(\tau_1,\tau_2))} - \kappa_1 \|f\|_{X_{(\tau,\tau_4)}(Q)} - \sup_{\partial_P Q} e^{m(\tau_4-\tau)} v^-\,,$$

where $m = \sup_Q c$.

Sketch of the proof. Since the proof closely follows [51, Proof of Lemma 3.5], we only outline differences (our statement includes a minor correction to [51, Lemma 3.5], as given in the addendum, see [47]). Instead of [51, Lemma 3.6] we employ the original Krylov-Safonov Harnack inequality for nonnegative solutions, [37, 38] where κ depends on N, diam D, α_0, β_0, θ, $\tau_2 - \tau_1$, $\tau_3 - \tau_2$ and $\tau_4 - \tau_3$, but not on diam U. Moreover, we use Theorem 2.3.5 instead of used Alexandrov-Krylov estimate to make κ_1 independent of diam U and to replace L^{N+1} norm of f by X norm. The rest of the proof remains unchanged. □

In the next corollary we formulate Harnack inequality for half spaces and the whole space. Based on the proof, one can easily formulate the results for other unbounded domains. If $\lambda = -\infty$ in the next corollary, we set $\mathbb{R}^N_\lambda := \mathbb{R}^N$ and $H_\lambda := \emptyset$.

Corollary 2.3.10. *Given numbers $d > 0$, $\lambda \in \mathbb{R} \cup \{-\infty\}$, $\theta > 0$, $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4$, and $\tau_1 - 2\theta \leq \tau \leq \tau_1 - \theta$, denote $Q := \mathbb{R}^N_\lambda \times (\tau, \tau_4)$. Fix a bounded domain $D \subset\subset \mathbb{R}^N_\lambda$ with dist$(\bar{D}, H_\lambda) \geq d$. If $v \in C_{loc}(\bar{Q}) \cap W^{2,1}_{N+1,loc}(Q)$ satisfies* (2.35) *with $L \in E(\alpha_0, \beta_0, Q)$, $\|c\|_{L^\infty(Q)} \leq \beta_0$, $f \in X_{(\tau,\tau_4)}(Q)$, and*

$$\lim_{M\to\infty} \sup_{(x,t)\in Q, |x|\geq M} |v(x,t)| = 0 \,, \tag{2.43}$$

then there exist constants κ, κ_1 and p depending on N, α_0, β_0, d, diam (D), θ, $\tau_2 - \tau_1$, $\tau_3 - \tau_2$ and $\tau_4 - \tau_3$ such that

$$\inf_{\bar{D}\times(\tau_3,\tau_4)} v \geq \kappa \|v^+\|_{L^\infty(D\times(\tau_1,\tau_2))} - \sup_{\partial_P Q} e^{\beta_0(\tau_4-\tau)} v^- - \kappa_1 \|f\|_{X_{(\tau,\tau_4)}(Q)} \,.$$

Proof. Choose large enough R such that $D \subset\subset B^\lambda_R$ and dist $(\partial B^\lambda_R, D) \geq \frac{d}{2}$. Then

Lemma 2.3.9 applied with $U = B_R^\lambda$ implies

$$\inf_{\bar{D}\times(\tau_3,\tau_4)} v \geq \kappa \|v^+\|_{L^\infty(D\times(\tau_1,\tau_2))} - \sup_{\partial_P Q} e^{\beta_0(\tau_4-\tau)} v^- - \sup_{|x|=R, t\in(\tau,\tau_4)} e^{\beta_0(\tau_4-\tau)} v^-(x,t) - \kappa_1 \|f\|_{X_{(\tau,\tau_4)}(Q)}\,,$$

where κ and κ_1 are as in Lemma 2.3.9. In particular they are independent of R. Passing $R \to \infty$ and using (2.43), we obtain the desired result. □

We mostly use Corollary 2.3.10 with

$$\tau = \tau_1 - \vartheta \quad \text{and} \quad \tau_i = \tau + i\vartheta, \qquad (i = 1,2,3,4)\,. \tag{2.44}$$

With this choice we obtain the following result.

Corollary 2.3.11. *For given $d > 0$, $\lambda \in \mathbb{R}\cup\{-\infty\}$, $\vartheta \in (0,1)$ and $\tau > 1$. Denote $Q := \mathbb{R}^N_\lambda \times (\tau, \tau + 4\vartheta)$ and fix a bounded domain $D \subset\subset \mathbb{R}^N_\lambda$ with dist$(\bar{D}, H_\lambda) \geq d$. If $v \in C_{loc}(\bar{Q}) \cap W^{2,1}_{N+1,loc}(Q)$ satisfies (2.35) with $L \in E(\alpha_0, \beta_0, Q)$, $\|c\|_{L^\infty(Q)} \leq \beta_0$, $f \in X_{(\tau,\tau+4\vartheta)}(Q)$, and*

$$\lim_{M\to\infty} \sup_{(x,t)\in Q, |x|\geq M} |v(x,t)| = 0\,,$$

then there exist constants κ and κ_1 depending on N, α_0, β_0, d, diam (D), ϑ such that

$$\inf_{\bar{D}\times(\tau+3\vartheta,\tau+4\vartheta)} v \geq \kappa \|v^+\|_{L^\infty(D\times(\tau+\vartheta,\tau+2\vartheta))} - \sup_{\partial_P Q} e^{4\beta_0\vartheta} v^- - \kappa_1 \|f\|_{X_{(\tau,\tau+4\vartheta)}(Q)}\,.$$

2.4 Proof of Theorem 2.2.2

In this section the notation and assumptions are as in Section 2.2. In particular $(A_{ij})_{1\leq i,j\leq N}$ and F satisfy (N1) – (N4) and G satisfies (G1). At some places, where explicitly stated, we also assume (G2). Let u be a positive, global, classical solution of (2.1) satisfying (2.15) and (2.16).

In addition we denote

$$X^{\lambda}_{(s,t)} := X_{(s,t)}(\mathbb{R}^N_\lambda \times (s,t)) \qquad (\lambda \in \mathbb{R}, s, t \in (0,\infty), s < t)\,,$$

where $X_{(s,t)}(Q)$, for general $Q \subset \mathbb{R}^{N+1}$, was defined at the beginning of Subsection 2.3.2.

To start the proof, we assume

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} > 0\,, \tag{2.45}$$

otherwise $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ the the theorem follows.

Lemma 2.4.1. *Given any ball $B \subset \mathbb{R}^N$, there exists $k(B) > 0$ and $\tilde{T} > 0$ depending on N, α_0, β_0, and B such that*

$$u(x,t) \geq k(B) \qquad ((x,t) \in \bar{B} \times (\tilde{T},\infty))\,. \tag{2.46}$$

Proof. We claim that

$$\liminf_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} > 0\,. \tag{2.47}$$

Suppose not, that, is suppose

$$\liminf_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = 0\,. \tag{2.48}$$

We find a contradiction by showing that there exists $\tau > 0$ with

$$u(x,t) < 3\varepsilon := \frac{1}{2}\min\left\{\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}, \varepsilon^*_\gamma\right\} \qquad ((x,t) \in \mathbb{R}^N \times (\tau,\infty))\,, \tag{2.49}$$

where ε^*_γ was defined in Remark 2.2.1.

According to Example 2.3.2, u satisfies (2.25) with $L \in E(\alpha_0, \beta_0, \mathbb{R}^N \times (0,\infty))$

such that (2.27), (2.29), (2.31), and (2.32) hold. Let $C = C(\alpha_0, \beta_0, N)$ be a constant from Corollary 2.3.6 b). Then by (2.29) (or (G1)) and (2.48) there is τ with

$$\max\left\{\frac{C}{1-e^{-\gamma}}\sup_{s\geq\tau}\|G\|_{X_{(s,s+1)}}, \|u(\cdot,\tau)\|_{L^\infty(\mathbb{R}^N)}\right\} \leq \varepsilon\,.$$

We prove that (2.49) holds for such τ. Suppose not, that is, suppose that

$$T := \inf\{t > \tau : \sup_{x\in\mathbb{R}^N} u(x,t) = 3\varepsilon\} < \infty\,.$$

Since $3\varepsilon < \varepsilon^*_\gamma$, by (2.31) one has $c(x,t) \leq -\gamma$ for any $(x,t) \in \mathbb{R}^N \times [\tau, T]$. An application of Corollary 2.3.6 b) with $Q = \mathbb{R}^N \times (\tau, T)$ yields

$$\sup_{\mathbb{R}^N} u(\cdot, T) \leq e^{-\gamma(T-\tau)}\|u(\cdot,\tau)\|_{L^\infty(\mathbb{R}^N)} + \frac{C}{1-e^{-\gamma}}\sup_{s\geq\tau}\|G\|_{X_{(s,s+1)}} \leq 2\varepsilon < 3\varepsilon\,,$$

a contradiction. Thus, (2.47) holds true, or equivalently there are constants $s, T > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} > s \qquad (t \in (T,\infty))\,.$$

By (2.16), we can replace $\mathbb{R}^N$ in the previous inequality by $B_R \cup B$ for a sufficiently large R independent of T. Then, an application of Corollary 2.3.11 with $(d, \lambda, D, \tau, \vartheta) = (1, -\infty, B_R \cup B, t, 1)$ yields

$$u(x,t) \geq \kappa s - \kappa_1\|G\|_{X_{(t-4,t)}} \qquad ((x,t) \in (\bar{B}_R \cup B) \times (T+4, \infty))\,,$$

where κ, κ_1 depend on R, N, α_0, β_0. Since the second term in the previous inequality converges to 0 as $t \to \infty$, we obtain for sufficiently large $\tilde{T}$

$$u(x,t) \geq k(B_R \cup B) := \frac{\kappa s}{2} \qquad ((x,t) \in (\bar{B}_R \cup B) \times (\tilde{T}, \infty))\,. \qquad \square$$

Recall, that for any $x = (x_1, x') \in \mathbb{R}^N_\lambda$ we already defined $x^\lambda = (2\lambda - x_1, x')$.

Now, for any function $g : \mathbb{R}^N \to \mathbb{R}$, let

$$V_\lambda g(x) := g(x^\lambda) - g(x) \qquad (x \in \mathbb{R}^N_\lambda, \lambda \in \mathbb{R}),$$

and for the solution u of (2.1) let

$$w^\lambda(x,t) := V_\lambda u(x,t) := u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \mathbb{R}^N_\lambda \times (0,\infty), \lambda \in \mathbb{R}).$$

As shown in Example 2.3.4, the function w^λ satisfies (2.25) such that (2.27), (2.29), (2.33), and (2.34) hold. Hence, the results of Subsection 2.3.2 are applicable to w^λ. We use this observation below, often without notice.

In the process of the moving hyperplanes we examine the following statement

$$\liminf_{t\to\infty} \inf_{x\in D} w^\lambda(x,t) \geq 0 \qquad \text{for all bounded } D \subset \mathbb{R}^N_\lambda, \tag{2.50}$$

which by the compactness of $\{u(\cdot,t) : t \geq 0\}$ in $C^1_0(\mathbb{R}^N)$ is equivalent to

$$V_\lambda z(x) \geq 0 \qquad (x \in \mathbb{R}^N_\lambda, z \in \omega(u)). \tag{2.50*}$$

The next lemma states an criterion for (2.50) to hold.

Lemma 2.4.2. *Consider g and $\delta = \delta(N, \alpha_0, \beta_0, \gamma) > 0$ such that Lemma 2.3.7 is satisfied with $(\Theta, \varepsilon) = (\frac{\beta_0}{\alpha_0} + 1, \frac{\gamma}{2\alpha_0})$. For fixed $\lambda > 0$ consider a domain $D_0 \subset\subset \mathbb{R}^N_\lambda$ such that*

$$B_{\rho^*_\gamma + 2|\lambda|} \cap \{x \in \mathbb{R}^N_\lambda : x_1 > \lambda + \delta\} \subset D_0, \tag{2.51}$$

where ρ^*_γ was defined in (2.18). Then (2.50) holds, provided there exist $\eta > 0$ and $t_0 > 0$ with

$$(w^\lambda)^+(x,t) \geq \eta \qquad ((x,t) \in D_0 \times (t_0, \infty)). \tag{2.52}$$

Remark 2.4.3. Notice that (2.52) is equivalent to assumptions in [49, Lemma

3.2]:

$$\liminf_{t\to\infty} \|w^\lambda(\cdot,t)\|_{L^\infty(D_0)} > 0\,, \tag{2.53}$$

$$w^\lambda(x,t) > 0 \qquad ((x,t) \in D_0 \times (t_0,\infty))\,. \tag{2.54}$$

Proof of Lemma 2.4.2. Fix a bounded domain $D^* \subset\subset \mathbb{R}^N_\lambda$ with $D_0 \subset D^*$ and denote $d := \text{dist}\,(D^*, H_\lambda)$.

If we transform (2.34) as described in Remark 2.3.8, then

$$\tilde{w}^\lambda(x,t) := \frac{w^\lambda(x,t)}{g(x_1-\lambda)} \qquad ((x,t) \in \mathbb{R}^N_\lambda \times (0,\infty))$$

satisfies (2.41) with $\hat{L}^\lambda \in E(\alpha_0, 5\beta_0, \mathbb{R}^N_\lambda \times (0,\infty))$, $\|\hat{c}^\lambda\|_{L^\infty(\mathbb{R}^N_\lambda\times(0,\infty))} \le 5\beta_0$, and $\hat{f}^\lambda$ satisfies (2.29). Moreover, by (2.33) one has $c(x,t) < -\gamma$ for each $(x,t) \in (\mathbb{R}^N_\lambda \setminus B_{\rho^*_\gamma+2|\lambda|}) \times (0,\infty)$, and consequently (2.42) yields

$$\hat{c}(x,t) \le -\frac{\gamma}{2} \qquad ((x,t) \in (\mathbb{R}^N_\lambda \setminus D_0) \times (0,\infty))\,.$$

By (2.52), any connected component Q of the set $\{(x,t) : \tilde{w}^\lambda(x,t) < 0, t \ge t_0\}$ is contained in $(\mathbb{R}^N_\lambda \setminus D_0) \times (t_0,\infty)$, and in particular $\hat{c}(x,t) \le -\gamma/2$ for any $(x,t) \in Q$. Then Corollary 2.3.6 b) implies

$$\|(\tilde{w}^\lambda)^-\|_{L^\infty(Q_t)} \le e^{-\frac{\gamma}{2}(t-t^*)}\|(\tilde{w}^\lambda)^-\|_{L^\infty(Q_{t^*})} + \frac{C}{1-e^{-\frac{\gamma}{2}}} \sup_{s\ge t^*} \|f^\lambda\|_{X^\lambda_{(s,s+1)}} \qquad (t_0 < t^* < t)\,, \tag{2.55}$$

where C depends on N, α_0, and β_0. This, (2.52), and an application of Corollary

2.3.11 with $\vartheta = \frac{1}{4}$ imply

$$
\begin{aligned}
\tilde{w}^\lambda(x,t+1) \geq \kappa \|(\tilde{w}^\lambda)^+\|_{L^\infty(D^*\times(t+\frac{1}{4},t+\frac{1}{2}))} - e^{\beta_0} \sup_{\partial_P(\mathbb{R}^N_\lambda\times(t,t+1))} (\tilde{w}^\lambda)^- \\
- \kappa_1 \|f^\lambda\|_{X^\lambda_{(t,t+1)}} \\
\geq \kappa\eta - e^{\beta_0} e^{-\frac{\gamma}{2}(t-t^*)} \|(\tilde{w}^\lambda)^-(\cdot,t^*)\|_{L^\infty(\mathbb{R}^N_\lambda)} \\
- \left(\kappa_1 + \frac{e^{\beta_0}C}{1-e^{\frac{\gamma}{2}}}\right) \sup_{s\geq t^*} \|f^\lambda\|_{X^\lambda_{(t,t+1)}} \qquad ((x,t)\in D^*\times(t^*,\infty)),
\end{aligned}
\tag{2.56}
$$

where $\kappa, \kappa_1 > 0$ depends on N, α_0, β_0, d and diam D^*. Then, by (2.15) and (2.29), one can choose large enough t^* such that $\tilde{w}^\lambda(x,t+1) \geq \kappa\frac{\eta}{2}$ for any $(x,t) \in D^* \times (2t^*,\infty)$. Since $D^* \subset \mathbb{R}^N_\lambda$ was arbitrary, (2.50) follows. □

The following lemma shows that the method of moving hyperplanes can get started, that is, (2.50) is true for large λ. The proof is similar to [49, Lemma 3.3], and we omit it here.

Lemma 2.4.4. *There exists λ_1 such that* (2.50) *holds for all $\lambda > \lambda_1$.*

Now, we move the hyperplane H_λ to the left (decrease λ) as far as (2.50) is satisfied and we investigate properties of the limiting position:

$$
\lambda_\infty := \inf\{\mu : (2.50) \text{ holds for all } \lambda \geq \mu\}. \tag{2.57}
$$

Lemma 2.4.5. *Let λ_1 be as in Lemma 2.4.4. Then:*

(i) $-\infty < \lambda_\infty \leq \lambda_1$.

(ii) $V_{\lambda_\infty} z \geq 0$ *for all* $z \in \omega(u)$.

(iii) There exists $\hat{z} \in \omega(u)$ *such that* $V_{\lambda_\infty}\hat{z} \equiv 0$.

(iv) For each $z \in \omega(u)$ *one has* $\partial_{x_1} z < 0$ *in* $\mathbb{R}^N_{\lambda_\infty}$.

Proof. The proofs of (i) and (ii) are analogous to [49, Lemma 3.4 (i), (ii)].

To prove (iii), we proceed by contradiction, that is, we assume $V_{\lambda_\infty} z \not\equiv 0$ for each $z \in \omega(u)$. By (ii), one has $V_{\lambda_\infty} z \geq 0$ for each $z \in \omega(u)$. By the compactness of $\omega(u)$ we can assume the existence of a bounded open set $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ and $b > 0$ such that

$$\|(V_{\lambda_\infty} z)^+\|_{L^\infty(D_0)} > 2b \qquad (z \in \omega(u))\,. \tag{2.58}$$

This remains valid if we enlarge $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$. We make D_0 so large that it satisfies the assumptions of Lemma 2.4.2 for any $\lambda < \lambda_\infty$ sufficiently close to λ_∞. By (2.20) and (2.58), there is $t^* > 0$ such that

$$\|(w^{\lambda_\infty})^+(\cdot, t)\|_{L^\infty(D_0)} > b \qquad (t \geq t^*)\,.$$

Consequently, Corollary 2.3.11 with $\vartheta = \frac{1}{4}$ yields

$$w^{\lambda_\infty}(x,t) \geq \kappa b - e^{\beta_0} \sup_{\mathbb{R}^N_{\lambda_\infty}} (w^{\lambda_\infty})^-(\cdot, t-1) - \kappa_1 \|f^{\lambda_\infty}\|_{X^\lambda_{(t-1,t)}}$$
$$(x \in D_0, t \geq t^*)\,,$$

where κ and κ_1 depend on N, α_0, β_0, $\operatorname{dist}(D_0, H_\lambda)$ and $\operatorname{diam}(D_0)$. Since $V_{\lambda_\infty} z \geq 0$ for each $z \in \omega(u)$ and (2.29) holds true, the last two terms decay to 0 as $t \to \infty$. Hence, for any sufficiently large t

$$w^{\lambda_\infty}(x,t) \geq \frac{1}{2}\kappa b \qquad (x \in D_0)\,.$$

Since ∇u is bounded, the previous inequality holds with λ_∞ replaced by λ for any $\lambda < \lambda_\infty$ sufficiently close to λ_∞. Then, Lemma 2.4.2 implies (2.50) for any λ sufficiently close to λ_∞, a contradiction.

The statement (iv) is proved by analogous arguments as in [49, Proposition 3.5]. We only modify the application of the Harnack inequality in the same way as we did in the proof of (iii). $\square$

This lemma finishes the proof of the first part of Theorem 2.2.2.

Before we proceed we state a lemma analogous to Lemma 2.4.5. Define $v : \mathbb{R}^N \times (0,\infty) \to \mathbb{R}$ as $v(x,t) := u(-x_1, x', t)$ for all $(x_1, x', t) = (x,t) \in \mathbb{R}^N \times (0,\infty)$, and observe that v satisfies (2.1), (2.15), and (2.16) with G changed to $\tilde{G}(x,t) := G(-x_1, x', t)$. Then $\tilde{G}$ satisfies (G1), and Lemma 2.4.5 applied to v yields to following result.

Lemma 2.4.6. *There exists λ_∞^- such that*

(i) $-\infty < \lambda_\infty^- \leq \lambda_\infty$,

(ii) $V_{\lambda_\infty^-} z \leq 0$ *for all* $z \in \omega(u)$,

(iii) There exists $\tilde{z} \in \omega(u)$ *such that* $V_{\lambda_\infty^-} \tilde{z} \equiv 0$,

(iv) For each $z \in \omega(u)$ *one has* $\partial_{x_1} z > 0$ *in* $(\mathbb{R}^N_{\lambda_\infty^-})^- := \{x = (x_1, x') \in \mathbb{R}^N : x_1 < \lambda_\infty^-\}$.

To prove the second part of Theorem 2.2.2, it suffices to show $\lambda_\infty = \lambda_\infty^-$. Indeed, then Lemma 2.4.5 (ii), (iv) and Lemma 2.4.6 (ii), (iv) imply that all functions $z \in \omega(u)$ are symmetric with respect to H_{λ_∞} and decreasing in x_1 for $x_1 > \lambda_\infty$.

Lemma 2.4.7. *If (G2) holds, then* $\lambda_\infty = \lambda_\infty^-$.

The basic idea of the proof, already introduced in [49], is to move a hyperplane H_λ beyond the natural limit H_{λ_∞}, that is, to consider $\lambda < \lambda_\infty$, and investigate the behavior of sign-changing functions w^λ. One of the crucial steps is to estimate $(w^\lambda)^+$ from below. This is done by the comparison of w^λ with a subsolution, similar to one constructed in [49, Lemma 3.8]. Its properties are listed in the following lemma.

Lemma 2.4.8. *Given any domain* $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ *and any* $\theta > 0$*, there exist* $\lambda_2 < \lambda_\infty$*,* $t_0 > 0$*, a domain* D*, and a function* $\phi : \bar{D} \times [t_0, \infty) \to \mathbb{R}$ *with the following properties:*

(i) $D_0 \subset\subset D \subset\subset \mathbb{R}^N_{\lambda_\infty}$*,*

(ii) $\phi \in C^{2,1}(\bar{D} \times [t_0, \infty))$*,*

(iii) $e^{\theta t}\phi(x,t) \geq C_2 > 0$ *for any* $(x,t) \in D_0 \times (t_0, \infty)$ *and some* C_2 *independent of* t_0 *and* t*,*

(iv) $\phi < 0$ *in* $\partial D \times (t_0, \infty)$*,*

(v) one has

$$\frac{\|\phi^+(\cdot,t)\|_{L^\infty(D)}}{\|\phi^+(\cdot,s)\|_{L^\infty(D)}} \geq Ce^{-\theta(t-s)} \qquad (t \geq s \geq t_0), \tag{2.59}$$

for some constant $C > 0$ *independent of* t *and* s*,*

(vi) for each $\lambda \in [\lambda_2, \lambda_\infty]$*,* ϕ *satisfies*

$$\phi_t < a_{ij}(x,t)\phi_{x_i x_j} + b_i^\lambda(x,t)\phi_{x_i} + c^\lambda(x,t)\phi + C' e^{-\theta t}|f^\ell(x,t)|, \qquad (x,t) \in D \times (t_0, \infty)\,,$$

where $\ell > \lambda_\infty$*, is a fixed number close to* λ_∞*, and* C' *depends on the* L^∞ *bound of* u *and* ℓ*.*

Sketch of the proof of Lemma 2.4.8. Since the proof closely follows the proof of [49, Lemma 3.8], we only outline differences. We define

$$\phi(x,t) = e^{-\theta t}v^\alpha(x,t) + s(-e^{-\theta t}(x_1 - \ell)^\beta) = w_1 + sw_2\,, \tag{2.60}$$

where $v := w^\ell$, $\ell > \lambda_\infty$ is sufficiently close to λ_∞, and $\alpha > 1 > \beta$ with α, β sufficiently close to 1. We remark that [49, Lemma 3.8] uses μ instead of ℓ. Then by calculations similar to those in [49, Lemma 3.8] one obtains for any $\lambda < \lambda_\infty$

sufficiently close to λ_∞:

$$e^{\theta t}(\partial_t w_1 - a_{ij}(x,t)(w_1)_{x_i x_j} + b_i^\lambda(x,t)(w_1)_{x_i} + c^\lambda(x,t)w_1) \le -\frac{\theta}{8}v^\alpha + v^{\alpha-1}f^\ell$$
$$\le -\frac{\theta}{8}v^\alpha + Cf^\ell\,.$$

The rest of the proof remains unchanged. Notice that (iii) immediately follows from [49, (3.31)]. □

Proof of Lemma 2.4.7. We proceed by contradiction, that is, we assume $\lambda_\infty > \lambda_\infty^-$. Since (G2) holds, f^λ satisfies (2.30) (and in particular (2.29)) for each $\lambda \in \mathbb{R}$. Then, by Lemma 2.4.5 and Lemma 2.4.6, there exist $\hat{z}$ and $\tilde{z} \in \omega(u)$ monotone in $\mathbb{R}^N_{\lambda_\infty}$ and $\mathbb{R}^N_{\lambda_\infty^-}$ respectively, with $V_{\lambda_\infty}\hat{z} \equiv V_{\lambda_\infty^-}\tilde{z} \equiv 0$. Hence,

$$\begin{aligned} &V_\lambda \hat{z}(x) < 0 \qquad (x \in \mathbb{R}^N_\lambda, \lambda \in (\lambda_\infty^-, \lambda_\infty))\,,\\ &V_\lambda \tilde{z}(x) > 0 \qquad (x \in \mathbb{R}^N_\lambda, \lambda \in (\lambda_\infty^-, \lambda_\infty))\,. \end{aligned} \tag{2.61}$$

Fix sequences $(\hat{t}_n)_{n\in\mathbb{N}}$ and $(\tilde{t}_n)_{n\in\mathbb{N}}$ such that $\tilde{t}_n < \hat{t}_n < \tilde{t}_{n+1}$ for all $n \in \mathbb{N}$ and

$$u(\hat{t}_n,\cdot) \to \hat{z}, \qquad u(\tilde{t}_n,\cdot) \to \tilde{z} \quad \text{with the convergence in } C^1(\mathbb{R}^N)\,.$$

Let $\delta = \delta(N,\alpha_0,\beta_0,\gamma) > 0$ be such that Lemma 2.3.7 is satisfied with $(\Theta,\varepsilon) = (\frac{\beta_0}{\alpha_0}+1, \frac{\gamma}{2\alpha_0})$. Fix a domain $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ with $B^{\lambda_\infty+\frac{\delta}{2}}_{\rho_\gamma^*+2|\lambda_\infty|} \subset D_0$. Consequently,

$$B^{\lambda+\delta}_{\rho_\gamma^*+2|\lambda|} \subset D_0 \qquad \left(\lambda \in \left[\lambda_\infty - \frac{\delta}{2}, \lambda_\infty\right]\right)\,. \tag{2.62}$$

Let $\lambda_2 < \lambda_\infty$ and D be such that Lemma 2.4.8 holds with D_0 and

$$\theta := \frac{1}{4}\min\left\{\frac{\gamma}{2}, \alpha_0, \mu\right\}\,, \tag{2.63}$$

where μ and γ are defined in (G2) and (N4) respectively. Fix any λ with

$$\max\{\lambda_2, \lambda_\infty - \frac{\delta}{2}\} < \lambda < \lambda_\infty \,.$$

Then by (2.61), there is $q > 0$ such that

$$\begin{aligned} &V_\lambda \hat{z}(x) < -q \qquad && (x \in \bar{D})\,, \\ &V_\lambda \tilde{z}(x) > q \qquad && (x \in \bar{D})\,, \end{aligned} \tag{2.64}$$

and therefore for large $n \in \mathbb{N}$ we have

$$\begin{aligned} &w^\lambda(x, \hat{t}_n) < -q \qquad && (x \in \bar{D})\,, \\ &w^\lambda(x, \tilde{t}_n) > q \qquad && (x \in \bar{D})\,. \end{aligned} \tag{2.65}$$

Then, there exists $T_n \in (\tilde{t}_n, \hat{t}_n)$ with

$$\begin{aligned} w^\lambda(x,t) > 0 \qquad & ((x,t) \in \bar{D} \times (\tilde{t}_n, T_n))\,, \\ w^\lambda(x_0, T_n) = 0 \qquad & \text{for some } x_0 \in \bar{D}\,. \end{aligned} \tag{2.66}$$

We claim that the following three statements are true.

(C1) $\lim_{n\to\infty} T_n - \tilde{t}_n = \infty$.

(C2)

$$\lim_{n\to\infty} \sup_{t\in[\tilde{t}_n, T_n]} e^{2\theta(t-\tilde{t}_n)} \|(w^\lambda)^-(\cdot,t)\|_{L^\infty(\mathbb{R}^N_\lambda)} = 0\,.$$

(C3) For any sufficiently large n and any $\tilde{t}_n \le t_1 < t_2 \le T_n$ one has

$$\begin{aligned} \sup_{x\in\bar{D}} w^\lambda(x,t) \ge C_0 e^{-\theta(t-t_1)} &\inf_{x\in\bar{D}} w^\lambda(x,t_1) \\ &- C_1 e^{\beta_0(t-t_1)} e^{-\mu t_1} \qquad (t \in [t_1, t_2])\,, \end{aligned}$$

where C_0 is independent of t_1, t_2 and n, C_1 depends on $t_2 - t_1$, but it is independent of t_1 and n.

Let us first prove (C1). Fix $M > 0$ and for each $n \in \mathbb{N}$ we define

$$w_n^\lambda(x,t) := w^\lambda(x,t) - w^\lambda(x,\tilde{t}_n) \qquad ((x,t) \in \mathbb{R}^N_\lambda \times (\tilde{t}_n,\infty))\,.$$

Then, w_n^λ is a classical bounded solution of

$$\begin{aligned} (w_n^\lambda)_t &= L^\lambda(x,t)w_n^\lambda + c^\lambda(x,t)w_n^\lambda + h_n^\lambda(x,t), \quad && (x,t) \in \mathbb{R}^N_\lambda \times (\tilde{t}_n,\infty)\,, \\ w_n^\lambda(x,t) &= 0, && (x,t) \in \partial_P(\mathbb{R}^N_\lambda \times (\tilde{t}_n,\infty))\,, \end{aligned}$$

where $L^\lambda \in E(\alpha_0, \beta_0, \mathbb{R}^N \times (\tilde{t}_n,\infty))$, $\|c^\lambda\|_{L^\infty(\mathbb{R}^N \times (\tilde{t}_n,\infty))} \leq \beta_0$ and

$$h_n^\lambda(x,t) := f^\lambda(x,t) + L^\lambda(x,t)w^\lambda(x,\tilde{t}_n) + c^\lambda(x,t)w^\lambda(x,\tilde{t}_n) \qquad ((x,t) \in \mathbb{R}^N_\lambda \times (\tilde{t}_n,\infty))\,.$$

Consequently, by Corollary 2.3.6 a) and the boundedness of coefficients of L^λ, one has

$$\begin{aligned} \sup_{\mathbb{R}^N_\lambda \times (\tilde{t}_n, \tilde{t}_n+\vartheta)} (w_n^\lambda)^- &\leq C\|h_n^\lambda\|_{X^\lambda_{(\tilde{t}_n,\tilde{t}_n+\vartheta)}} \\ &\leq C(\|f^\lambda\|_{X^\lambda_{(\tilde{t}_n,\tilde{t}_n+1)}} + \vartheta^{\frac{1}{N+1}}\beta_0\|w^\lambda(\cdot,\tilde{t}_n)\|_{C^2(\mathbb{R}^N_\lambda)} \qquad (\vartheta \in [0,1])\,, \end{aligned} \tag{2.67}$$

where C depends on N, α_0 and β_0. Now, choosing ϑ sufficiently small (independent of n) and n sufficiently large, we can by (2.15) and (2.30) achieve $(w_n^\lambda)^- \leq \frac{q}{2}$ in $\mathbb{R}^N_\lambda \times [\tilde{t}_n, \tilde{t}_n + 4\vartheta]$. Then, by the definition of w_n^λ and (2.65) one has

$$w^\lambda(x,t) \geq \frac{q}{2} \qquad ((x,t) \in \bar{D} \times [\tilde{t}_n, \tilde{t}_n + 4\vartheta])\,. \tag{2.68}$$

Next, an application of Corollary 2.3.10 with constants $(D, \tau, \theta, \tau_1, \tau_2, \tau_3, \tau_4) =$

$(D, t_n, \vartheta, t_n + 2\vartheta, t_n + 3\vartheta, t_n + 4\vartheta, t_n + M)$ yields

$$\begin{aligned} w^\lambda(x,t) \geq \kappa \|(w^\lambda)^+\|_{L^\infty(D\times(t_n+2\vartheta, t_n+3\vartheta))} - e^{2\beta_0} \sup_{\mathbb{R}^N_\lambda} (w^\lambda)^-(\cdot, t_n) \\ - \kappa_1 \|f^\lambda\|_{X^\lambda_{(t_n, t_n+M)}} \qquad ((x,t) \in \bar{D} \times (\tilde{t}_n + 4\vartheta, \tilde{t}_n + M))\,, \end{aligned}$$

where κ and κ_1 depend on N, α_0, β_0, diam D, ϑ and M. By (2.61) and (2.30), the last two terms in the previous inequality converge to 0 as $n \to \infty$, whereas the first one is bounded from below by $\kappa q/2$. Therefore $w^\lambda(x,t) \geq \kappa \frac{q}{8}$ for all $(x,t) \in \bar{D} \times [\tilde{t}_n + 4\vartheta, \tilde{t}_n + M]$ and sufficiently large n. This and (2.68) yields the desired result, since M was arbitrary.

To prove (C2) it is enough to show that for any $\varepsilon' > 0$, there is n_0 such that

$$\sup_{(x,t)\in\mathbb{R}^N_\lambda \times [\tilde{t}_n, T_n]} v_n(x,t) \leq \varepsilon' \qquad (n \geq n_0)\,, \tag{2.69}$$

where

$$v_n(x,t) := e^{2\theta(t-\tilde{t}_n)} \frac{(w^\lambda)^-(x,t)}{g(x_1 - \lambda)} \qquad ((x,t) \in \mathbb{R}^N_\lambda \times [\tilde{t}_n, T_n])\,,$$

and g is as in Lemma 2.3.7 with $(\Theta, \varepsilon) = (\frac{2\beta_0}{\gamma} + 1, \frac{\gamma}{2})$. Since $w^\lambda > 0$ in $D \times [\tilde{t}_n, T_n)$,

$$U^n := \{(x,t) : v_n(x,t) > 0, t \in [\tilde{t}_n, T_n)\} \subset (\mathbb{R}^N_\lambda \setminus D) \times [\tilde{t}_n, T_n)\,. \tag{2.70}$$

Observe that $(w^\lambda)^- = -w^\lambda$ on $\bar{U}^n$ for each $n \in \mathbb{N}$. Thus Remark 2.3.8 yields

$$\begin{aligned} (v_n)_t - \hat{L}^\lambda(x,t) v_n &= \tilde{c}^\lambda(x,t) v_n + \tilde{f}^\lambda(x,t), && (x,t) \in U^n\,, \\ v_n(x,t) &= 0, && (x,t) \in \partial_P U^n \setminus (U^n)_{\tilde{t}_n}\,, \\ v_n(x, \tilde{t}_n) &= \frac{(w^\lambda)^-(x, \tilde{t}_n)}{g(x_1 - \lambda)}, && x \in (U^n)_{\tilde{t}_n}\,, \\ \lim_{|x|\to\infty} \sup_{t\in(0,\infty)} |v(x,t)| &= 0 \end{aligned} \tag{2.71}$$

where $\hat{L}^\lambda \in E(\alpha_0, 5\beta_0, U^n)$,

$$\tilde{c}^\lambda := \hat{c}^\lambda + 2\theta \quad \text{and} \quad \tilde{f}^\lambda(x,t) := -e^{2\theta(t-\tilde{t}_n)} \frac{f^\lambda(x,t)}{g(x_1-\lambda)} .$$

By Remark 2.3.8 we have $\|\hat{c}^\lambda\|_{L^\infty(U^n)} \le 5\beta_0$, and therefore

$$\|\tilde{c}^\lambda\|_{L^\infty(U^n)} \le 7\beta_0 .$$

Moreover, by (2.42) and (2.63)

$$\tilde{c}^\lambda(x,t) \le \hat{c}^\lambda(x,t) + 2\theta \le -\frac{\gamma}{2} + 2\theta \le -\theta \qquad (x_1 \in [\lambda, \lambda+\delta], t > 0) , \tag{2.72}$$

and by (2.33) one has

$$\tilde{c}^\lambda(x,t) \le -\gamma + 2\theta \le -\theta \qquad (|x| \ge \rho^*_\lambda + 2\lambda, t > 0) .$$

Since $B_{\rho^*_\gamma+2|\lambda|} \subset D$ and (2.70) holds true, $c^\lambda < -\gamma$ for any $(x,t) \in U^n$, $n > 0$. Also, $\mu > 2\theta$ and (2.30) implies, that there exists t'_ε such that

$$\|\tilde{f}^\lambda\|_{X^\lambda_{(t_{\varepsilon'},\infty)}} < \varepsilon' \frac{1-e^{-\vartheta}}{2C} ,$$

where $C = C(N, \alpha_0, 7\beta_0)$ is the constant from Corollary 2.3.6 b). Then Corollary 2.3.6 b) yields

$$\begin{aligned} \sup_{U^n_t} v_n &\le e^{-\theta(t-\tilde{t}_n)} \sup_{U^n_{\tilde{t}_n}} v_n + C\frac{1}{1-e^{-\theta}} \|\tilde{f}^\lambda\|_{X^\lambda_{(t_\varepsilon,\infty)}} \\ &\le \sup_{U^n_{\tilde{t}_n}} v_n + \frac{\varepsilon'}{2} \qquad (t \in [\tilde{t}_n, T_n], \tilde{t}_n > t_\varepsilon) . \end{aligned}$$

Since $\|v_n(\cdot, \tilde{t}_n)\|_{L^\infty(U^n_{\tilde{t}_n})} \to 0$ as $n \to \infty$, we obtain that (2.69) holds for sufficiently large n_0.

Let us prove (C3). Recall that D was fixed such that Lemma 2.4.8 holds with

D_0 and θ. Let ϕ be the corresponding subsolution. Denote

$$\eta := \frac{\inf_{x\in\bar D} w^\lambda(x,t_1)}{\|\phi^+(\cdot,t_1)\|_{L^\infty(D)}} > 0 \qquad \text{and} \qquad v := w^\lambda - \eta\phi\,.$$

Lemma 2.4.8 (iii) and (2.15) imply that $e^{-\theta t}\eta$ is bounded by a constant independent of t_1.

Then

$$\begin{aligned}
&v_t \geq L^\lambda(x,t)v + c^\lambda(x,t)v + (f^\lambda - C'e^{-\theta t}\eta|f^\ell|), && (x,t)\in D\times(t_1,t_2)\,,\\
&0 < v(x,t), && (x,t)\in \partial D\times(t_1,t_2)\,,\\
&0 \leq v(x,t_1), && x\in\bar D\,.
\end{aligned}$$

Consequently, (2.30), Corollary 2.3.6 a), and positivity of w^λ in $D\times[t_1,t_2]$ yield

$$\begin{aligned}
C_1 e^{\beta_0(t-t_1)}e^{-\mu t_1} &\geq C e^{\beta_0(t-t_1)}\|f^\lambda - C'e^{-\theta t}\eta|f^\ell|\|_{X^\lambda_{(t_1,t)}} \geq \sup_{x\in D}(v(x,t))^-\\
&\geq -\sup_{x\in D} w^\lambda(x,t) + \eta\sup_{x\in D}\phi(x,t) \qquad (t\in[t_1,t_2])\,,
\end{aligned}$$

where C_1 depends on t_2-t_1, but is independent of t_1. Since by Lemma 2.4.8 (v) and the definition of η one has

$$\eta\sup_{x\in D}\phi(x,t) \geq \eta C e^{-\theta(t-t_1)}\|\phi(\cdot,t_1)\|_{L^\infty D} \geq C e^{-\theta(t-t_1)}\inf_{x\in\bar D} w^\lambda(x,t_1)\,,$$

(C3) follows.

We will complete the proof of the lemma by showing that (C1)–(C3) lead to a contradiction.

By (C1) we have $T_n - \tilde t_n \to \infty$ as $n\to\infty$. Let C_0 be as in (C3), let κ, κ_1 be as in Corollary 2.3.11 for already fixed D and $\vartheta = \frac14$. Denote

$$\hat C := \frac{\kappa C_0 e^{\frac{\theta}{2}}}{2}\,.$$

Fix $K > 2$ such that

$$e^{-\theta K} \leq \hat{C} \tag{2.73}$$

and let $C_1 := C_1(t_2 - t_1)$ be as in (C3) with $t_2 - t_1 = K$. By (C2) there is $n_0 > 0$ such that

$$e^{2\theta} e^{\beta_0} e^{-2\theta(t-\tilde{t}_n)} \|(w^\lambda)^-(\cdot,t)\|_{L^\infty(\mathbb{R}^N_\lambda)} \leq \frac{q}{2} \qquad (t \in [\tilde{t}_n, T_n], n \geq n_0)\,. \tag{2.74}$$

Enlarge n_0 if necessary, such that $T_n - \tilde{t}_n > K$ and

$$(\kappa_1 + C_1(K)e^{\beta_0 K})C_\mu e^{\theta K} e^{-\mu \tilde{t}_n} \leq \frac{1}{2} q\hat{C} \qquad (n \geq n_0)\,. \tag{2.75}$$

Now, fix $n \geq n_0$. We prove by the mathematical induction that for any $i \in \mathbb{N} \cup \{0\}$ with $i \leq \frac{T_n - \tilde{t}_n}{K}$, one has

$$w^\lambda(x, \tau_i) \geq q e^{-\theta i K} \hat{C}^i \qquad (x \in \bar{D})\,, \tag{2.76}$$

where $\tau_i := iK + \tilde{t}_n$.

For $i = 0$ the statement follows from (2.65). Next assume that (2.76) is true for some $i \in \mathbb{N} \cup \{0\}$ such that $(i+1)K \leq T_n - \tilde{t}_n$. We show that (2.76) holds with i replaced by $i+1$. Indeed, Corollary 2.3.11 with $(\tau, \vartheta) = (\tau_{i+1} - 1, \frac{1}{4})$, (C3), (2.74), and (2.30) yield

$$\begin{aligned} w^\lambda(x, \tau_{i+1}) &\geq \kappa \left\|(w^\lambda)^+\right\|_{L^\infty(D\times(\tau_{i+1}-\frac{3}{4}, \tau_{i+1}-\frac{1}{2}))} - e^{\beta_0} \|(w^\lambda)^-\|_{L^\infty(\mathbb{R}^N_\lambda \times(\tau_{i+1}-1, \tau_{i+1}))} \\ &- \kappa_1 \|f^\lambda\|_{X^\lambda_{(\tau_{i+1}-1, \tau_{i+1})}} \\ &\geq \kappa e^{-\theta K} C_0 e^{\frac{\theta}{2}} \inf_{x\in\bar{D}} w^\lambda(x, \tau_i) - \frac{q}{2} e^{-2\theta(\tau_{i+1}-\tilde{t}_n)} - (\kappa_1 + C_1 e^{\beta_0 K}) C_\mu e^{-\mu\tau_i} \quad (x \in \bar{D}) \end{aligned}$$

Consequently, (2.76), (2.73), and (2.75) imply

$$\begin{aligned}
w^\lambda(x,\tau_{i+1}) &\geq 2\hat{C}e^{-\theta K} q e^{-\theta i K}\hat{C}^i - \frac{q}{2}e^{-\theta(i+1)K}(e^{-\theta K})^{i+1} \\
&\quad - (\kappa_1 + C_1 e^{\beta_0 K})C_\mu e^{-\theta i K}(e^{-\theta K})^i e^{-\mu \tilde{t}_n} \\
&\geq q e^{-\theta(i+1)K}\hat{C}^{i+1}\left(2 - \frac{1}{2} - \frac{(\kappa_1 + C_1 e^{\beta_0 K})C_\mu e^{\theta K}}{q\hat{C}} e^{-\mu\tilde{t}_n}\right) \\
&\geq q e^{-\theta(i+1)K}\hat{C}^{i+1} \qquad (x \in \bar{D})\,.
\end{aligned}$$

Thus, if $i_0 \in \mathbb{N}$ is such that $i_0 K \leq T_n - \tilde{t}_n < (i_0+1)K$, then (2.76) holds with $i = i_0$. If we replace τ_{i+1} by T_n and τ_i by τ_{i_0} in the previous calculation, we obtain by the same reasoning

$$w^\lambda(x, T_n) \geq q e^{-\theta(i+1)K}\hat{C}^{i+1} > 0 \qquad (x \in \bar{D})\,,$$

a contradiction to the definition of T_n. This finishes the proof of the lemma. □

Summary

In this work, we studied qualitative properties of the second order parabolic partial differential equations. We showed that blow up rated of nonnegative solutions to semilinear indefinite equations is controlled by the explicitly calculated function. We also stated several propositions that yield optimal blow up rate for the solution. As a consequence, we obtained a refinement of existing results on the complete blow up and on the a priori estimates for the nonnegative solutions of the indefinite elliptic problems. The proofs required new nonlinear Liouville type theorems for semilinear equations on the whole space and on half spaces. We also showed optimality of the stated Liouville theorems.

In the second part, we employed the method of moving hyperplanes and the maximum principle in the study of asymptotic properties of solutions to asymptotically symmetric equations on the whole space. We proved that if the problem converges exponentially to a quasilinear symmetric problem, then the omega limit set of the solution is a subset of radially symmetric functions with the common origin. We also provided an example that showed that this conclusion does not hold true in general, if the convergence is not exponential.

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