FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS
COMENIUS UNIVERSITY BRATISLAVA

Dynamic Stochastic Accumulation Model with Application to Portfolio Risk Management

Dissertation Thesis

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Declaration

I here declare that I have produced this Dissertation Thesis without the prohibited assistance of third parties and without making use of aids other than those specified; notions taken over directly or indirectly from other sources have been identified as such.

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This dissertation thesis was conducted from 2010 to 2014 under the supervision of prof. RNDr. Ševčovič, CSc at the Faculty of Mathematics, Physics and Informatics, Comenius University, Slovak Republic.

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Abstract

In our research we focus on the modeling of portfolio returns with fat-tailed distributions that have the property to exhibit extreme large skewness and kurtosis. We focus on the normal-inverse Gaussian distribution and analyze the impact of higher moments on the optimal choice of the portfolio composition. We consider a dynamic stochastic model of Bellman type and discuss the problem of optimal choice of portfolio composition with different level of risk dependent on proportion of risky - to - non-risky assets, especially in application to pension management.

We propose a numerical scheme for calculation and perform a sensitivity analysis of the descriptive statistics of asset returns on the accumulated sum at the final time as well as in each time step during the saving. We compare the results considering the normal distribution and NIG distribution with different skewness and kurtosis and discuss the distribution of accumulated sum at the final time.

Keywords: dynamic optimization, Bellman equation, fat tail distribution, normal inverse Gaussian distribution.

Abstrakt

V našej práci sa zameriavame na modelovanie výnosov portfólia pomocou rozdelení s ťažkými chvostami, ktoré sa vyznačujú nadobudaním vysokých hodnôt šikmosti a špičatosti. Zameriavame sa na normálne inverzné Gaussove rozdelenie a analyzujeme dopad vyšších momentov na optimálnu voľbu zloženia portfólia. Uvažujeme dynamický stochastický model Bellmanovho typu a rozoberáme problém optimálneho výberu skladby portfólia s rôznou mierou rizika popísanou pomocou podielu rizikových a bezrizikových aktív v portfóliu. Model bol diskutovaný najma v kontexte aplikácie dôchodkového sporenia.

Ponúkame numerickú schému pre výpočet formulovaného problému a analýzu citlivosti popisných štatistík výnosov aktív na kumulovanú sumu v konečnom čase, rovnako ako v každom časovom kroku v priebehu sporenia. Porovnáme výsledky pri uvažovaní normálneho rozdelenia a NIG rozdelenia výnosov pre rôzne úrovne šikmosti a špičatosti. Predovšetkým sa zameriavame na rozdelenie konečného bohatstva.

Kľúčové slová: dynamická optimalizácia, Bellmanova rovnica, rozdelenie s ťažkými chvostami, normálne inverzné Gaussove rozdelenie.

Contents

Chapter 1

Symbols

I_t	information about the history on the market
C_0	initial capital given in units
C	regular contribution value in units
θ	vector of asset allocation in portfolio composition
Θ	set of all optional portfolio composition θ
r_t^θ	return of the portfolio with θ composition at time t
X_t	wealth at time t in units
T	investment time horizon
ρ_t	inflation or wage growth at time t
φ	transaction costs
s_t	accumulated sum at time t
w_t	wage at time t
τ	regular contribution rate
d_t	ratio of accumulated sum and wage at time t, given as $\frac{s_t}{w_t}$
U	utility function
V_t	value function
$\mathbb{E}(.)$	expectation of a random variable
$\mathcal{S}td(.)$	standard deviation of a random variable
$\mathcal{V}(.)$	variance of a random variable
$\mathcal{S}(.)$	skewness of a random variable
$\mathcal{K}(.)$	kurtosis of a random variable
$e\mathcal{K}(.)$	excess kurtosis of a random variable
ρ	correlation of two random variables
Σ	correlation matrix

Chapter 2

Introduction

2.1 Introduction

One of the most important problems faced by investors involve the allocation of their wealth among different investment opportunities in a market consisting of risky assets. Determination of optimal portfolios is a rather complex problem depending on the objective of the investor. The problem of optimizing the portfolio returns has been discussed by many authors and the voluminous literature is devoted to solve such problem. The most known solution for portfolio optimization has been introduced by Markowitz (1952) and belongs to static portfolio optimization methods. The task to estimate the evolution on the financial market has shown to be very difficult. Especially in the recent years the financial markets are characterized by high volatility which causes extreme values of asset prices and their returns. The distributions of many financial quantities were shown to have heavy tails and exhibit skewness and other non Gaussian properties. These observations led to the new approach of modeling the asset returns by means of fat tailed probability distribution.

In the dissertation thesis we study the impact of the choice of fat-tailed distribution of asset returns when optimizing portfolio. We consider stochastic accumulation model in application saving management discussed by Kilianová, Melicherčík and Ševčovič in series of articles, see [29], [32], [43], [44]. We replace the modeling of returns by the normal distribution with non symmetrical distribution typical for their fatter tails.

2.2 Literature overview

By the portfolio optimization especially the expected return of the investment and its risk is taken into consideration. The optimizing criterion might involve also the other consequent financial risk attributions.The aim of the portfolio management is to optimize the performance by keeping the risk level relatively low depending on individual preferences. The portfolio management went through long time evolution from static and deterministic modeling to dynamic and stochastic modeling.

The mean-variance model is a optimization model for the single period portfolio selection problem providing analytical solution for maximizing expected utility from the wealth or minimizing the risk arising from portfolio. The analytic derivation of the meanvariance efficient portfolio frontier is given by Merton [46]. This single period is not sufficient for long-time investors and the necessity of adoption to economic market conditions and investors changing preferences over time led to introduction of multi-period methods with dynamic re-balancing.

2.2.1 Dynamic programming

Dynamic programming is the most powerful principle in optimization and is based on breaking complex problems to smaller subproblems. It is a method conceived to solve dynamic optimization models over time and can be applied in discrete and continuous time models, deterministic and stochastic models, and finite and infinite horizon models. These optimization approaches are also described by Halická et al. in [25]. The dynamic approach, however, is computationally quite expensive, since it is a brute-force method that goes through all possible solutions to pick the best one. Exploiting early exercise opportunities optimally requires going backward in time and the quantity at any previous time can be calculated by backward induction using the Bellman equation. Bringing uncertainty into problem transforms the problems into stochastic dynamic optimization problems. The portfolio management is thus a problem that can be formulated as multi-stage optimization problem and can be solved in different ways as discussed by Brandimarte in [6]. In the book he describes the several numerical approaches used to solve dynamic and stochastic problems using MATLAB.

Canakoğlu and Ozekici [10] used the dynamic programming approach to obtain an explicit characterization of the optimal policy for the optimal portfolio selection problem where the investor maximizes the expected utility of the terminal wealth. The utility function belongs to the HARA family. A tractable and realistic approach is provided by using a Markov chain representing the returns of the assets. The use of a modulating stochastic process as a source of variation in the model parameters and of dependence among the model components has proved to be quite useful in operations research and management science applications. They apply multi-period portfolio optimization by considering investors with logarithmic and power utility supposing that the asset returns all depend on a stochastic market depicted by a Markov chain.

2.2.2 Multi-stage stochastic programming

The multistage stochastic programming is a popular technique to deal with uncertainty in optimization models and therefore popular to solve the financial planning problems. However, the need to adequately capture the underlying distributions leads to large problems that are usually beyond the scope of general purpose solvers. The personal financial planning problem is characterized by three key elements. The problem is constrained optimal decision problem with objective to maximize multi-attribute objective function under possibly large number of constraints. The required inclusion of a risk preference parameter naturally leads to an expected utility problem. The necessity of achievement of intermediate targets as well as changing working conditions over time and the time distribution of liabilities and income changes induce a dynamic decision problem. The third key element is the stochastic behavior of the processes such as evolution of financial markets. The sequence of actions taken in the phase of uncertainty need to be taken into account within the given time frame. The effectiveness of any adopted strategy and the achievement of the individuals objectives depend on a sequence of random events. The long-term nature of the decision problem, furthermore, imposes a specific requirement on the model of uncertainty, and specifically on the properties of the generated economic and financial scenarios.

Celikyurt and Ozekici in [11] consider several multi-period portfolio optimization models where the market consists of a risk free asset and several risky assets. The returns in any period are random with a mean vector and a covariance matrix that depend on the prevailing economic conditions in the market during that period. The stochastic behavior is described by a Markov chain with perfectly observable states. They offer a solution for an auxiliary problem found by dynamic programming.

Numerical methods used to solve the problem of finding the optimal portfolio were discussed in a large number of papers. Campbell and Viceira [8] and Barberis [3] use numerical approximations to find optimal portfolios in a discrete setting. In continuous time, Brennan, Schwartz and Lagnado [7] solve numerically the PDE for specific parametrization of the utility function.

Dupačová and Sladký in [18] provide a comparison of multistage stochastic programs with recourse and stochastic dynamic programs with discrete time parameter and with a fixed finite horizon. The main distinction lies in the decision concept, in different structures of their formulation and, consequently, in different solution methods. On the contrary to the multistage stochastic programs, most of the motivation for the research on dynamic programming models come from a class of operations research and engineering applications where it is the decision rule that is primarily of interest and the horizon is very long. Dupačová modeled the asset-liability management problem applied to the specific model of a Czech pension fund using stochastic programming.

2.2.3 Analytical solutions

Merton [45] introduced the methodology to attack the problem of a rational investor with time additive preferences that chooses how to allocate his wealth between consumption and the existing securities. The computation requires PDE solution that have a closed form only in restrictive number of cases. Costa and Araujo in [16] derived analytically an optimal control policy for an auxiliary problem, as well as the expected value and variance of the investors wealth. The policy is obtained from the solution of a set of interconnected Riccati difference equations.They provide necessary and sufficient conditions for the solution of the generalized mean-variance problem, and a set of recursive equations to derive the solution of the problem.

2.2.4 Probabilistic approach

The work of Yin et al. [52] develops an approximation procedure for portfolio selection with bounded constraints. Based on the Markov chain approximation techniques, numerical procedures are constructed for the utility optimization task. The problem is defined as maximizing the terminal wealth under policy not allowing short positions. The wealth evolution over time is given by PDE. The numerical solution comes from approximation of the Markov chain by probabilistic methods, defining probability transition matrices between the states, and does not need any analytic properties of the solutions of the Hamilton-Jacobi-Bellman equations.

Detemple, Garcia and Rindisbacher [17] propose a simulation-based approach for optimal portfolio allocation in realistic environments with complex dynamics for the state variables and large numbers of factors and assets. Cvitanic, Goukasian and Zapatero introduced in [15] an optimization method purely based on Monte Carlo simulation. They rely on the fact that the optimal portfolio is part of the standard deviation of the wealth process and such standard deviation can be directly estimated. The method can be applied to any time additive utility function and any parametrization of the asset processes. The advantage of the Monte Carlo simulation is that it is very easy to implement and converges very fast. Computing standard deviation through Monte Carlo simulation has other applications in finance such as computation of hedge if an option.

2.2.5 Stochastic linear programming

Stochastic linear programming models are able to tackle portfolio management problems, however the models are more difficult than simulation. SLP searches for the best solution, given bounds on the variables, the constraints of the problem, and the objective function. The stochastic programming-based approaches for real space problems are often computationally not tractable. However, due to algorithmic progress and technological developments, nowadays relatively large models can be solved by SLP in reasonable time.

A common approach to reduce the scenario space to a manageable size for multi-stage stochastic programming problems is by using the Sample Average Approximation (SAA) method based on Monte Carlo simulation. An analysis of stochastic dual dynamic programming method applied to the constructed SAA problem is offered by Shapiro. A viable and practical alternative to stochastic programming can be provided by policy optimization where, given a set of possible scenarios, policy rules are directly associated with individual scenarios and the resulting simulated policy distributions can be directly evaluated. The range of possible decisions is thus limited through the introduction of policy rules.

The asset returns often exhibit of the portfolios and taking into account realistic properties of the assets returns exhibiting skewness and kurtosis.

2.2.6 Hamilton-Jacobi-Bellman approach

Munos and Moore in [47] discuss the necessity of introducing nonuniform discretization of probabilistic space when solving optimal control problems. They consider variable resolution discretizations to approximate the value function and the optimal control and compare experimentally several splitting criteria. They observed that this approach works well for 2D problems, however for more complex problems, they argue, uniform grids perform better.

Al-Tamini, Lewis and Abu-Khalaf in [1] proved the convergence of value-iteration-based heuristic dynamic programming (HDP) algorithm to the optimal control and the optimal value function that solves the HJB equation appearing in infinite horizon discrete-time nonlinear optimal control, assuming exact solution of the value update and the action update at each iteration. They used standard neural networks allowing the implementation of HDP without knowing the internal dynamics of the system.

The Halmilton-Jacobi-Bellman approach has been used for solving an optimization problem formulated as dynamic accumulation model with application to pension saving management, specially designed for the II. pillar of Slovak pension system, in series of articles by Kilianová, Melicherčík, Ševčovič. All the models consider normal distribution for the portfolio returns. The portfolios are usually represented by two asset types - a risky and a non-risky asset. The numerical procedure for solving respective HJB equation has been introduced and implemented [32]. Melicherčík, Ševčovič in cooperation with Jakubík provided a sensitivity analysis for the dynamic stochastic accumulation model for optimal pension saving management in [29]. Kilianová in her dissertation thesis in [31] offered a derivation of a partial differential equation for solving the optimal planning problem in pension model and Macová with Ševčovič showed how the portfolio management problem can be formulated in terms of the solution to the Hamilton-Jacobi-Bellman equation. Melicherčík and Ševčovič in [43] consider stock prices representing the risky asset to be driven by a Brownian motion and a non-risky asset to follow one factor Cox-Ingersoll-Ross model.

The optimization problem given by HJB differential equation with terminal condition can be transformed by Riccati transformation to a Cauchy problem and has been analyzed by Ishimura and Ševčovič [28]. Kilianová and Ševčovič in [34] offered an implicit iterative finite volume numerical approximation scheme for solving transformed Cauchy problem for the quasi-linear parabolic equation and optimize a portfolio consisting of $n = 30$ assets.

This set of articles create an inspiration for extension of the models for distributions exhibiting the fat tails.

2.3 Goals of the thesis

The main goal of the thesis is to study and implement the fat tailed distribution of the returns into financial planning problems. We adopt the dynamic accumulation model originally designed for the optimal saving management and consider fat tailed Normal-Inverse-Gaussian distribution for the asset returns and the portfolio return. We aim to analyze the pitfalls arising from considering the fat tailed distribution characterized by exhibiting large kurtosis and introduce a stable numerical method for solving formulated optimization problem.

We study the distribution of the final wealth and its properties for normal and for NIG distribution. Our following aim is to provide a sensitivity analysis of the model to the model parameters, especially to the parameters of the portfolio return distribution. We focus on the dependency of final accumulated sum on portfolio descriptive statistics and study the influence of a small change in one input parameters to final wealth descriptive statistics and optimal choice.

2.4 Structure of the thesis

The thesis is structured as follows. In chapter 3 we discuss the investment model, model with regular contributions and finally the model design for risk management in pension funds. We formulate the maximization utility problem by means of Belmann equation. The utility types and the risk aversion coefficients of the investor/saver are discussed in chapter 4.

The aim of the thesis is to study the impact of considering the fat tailed distributions for portfolio returns. In chapter 5 we introduce the family of generalized hyperbolic distributions and focus on normal inverse Gaussian distribution for modeling the asset returns. The motivation for the NIG distribution comes from Onalan [48] who analyzed the historical time series of some indices and the suitability of the NIG distribution for modeling their returns. We provide the similar study on the S&P 500 Index and show that the NIG can fit the empirical distribution in much better way than the normal distribution. In this chapter we analyze the NIG parameters, their influence on the density shape and discuss the NIG properties. The NIG distribution can depict the skewness and the kkurtosis of the asset returns in a very good way, however the convolution of more NIG distributed random variables is not defined, which is one of its main drawbacks leading to analysis of NIG mixtures in chapter 6. Here we propose the algorithms for replicating the NIG correlated time series and define the approximation of a NIG mixture.

The chapter 7 is devoted to the numerical solution of the formulated problem. We focus on the needs of the fat tailed distribution and discuss the appropriate method for integral approximation as well as method for interpolation and extrapolation the function values inside and outside the suitably chosen grid.

The chapter 8 to chapter 10 offer the results for a pension planning problem, comparison of optimal choice and expectation considering normal and NIG distributed portfolio returns with and without the regulations introduced by the Government. We provide a sensitivity analysis on the risk aversion coefficient and in chapter 10 we focus on sensitivity of skewness and kurtosis on optimal choice, expectation and risk.

Chapter 3

Model formulation

In this section we introduce a portfolio management problem requiring the investment solution leading to high performance with small risk. The model describes the problem of an investor starting from initial capital with infinite time horizon. A special case is related to pension fund management for clients who contribute regularly to selected funds in order to maximize their pension after certain time horizon, at the retirement time.

The model is based on the recovery of initial capital with the possibility of rebalancing at each time step. We sequentially introduce the investment model, model with regular contribution and model designed for pension planning. Finally, we formulate the stochastic dynamic problem.

We consider a general investment model discussed by Ferguson and Gilstein in [22], where an investor decides in every time step about the investment allocation among the asset structure. We assume that the investor has information about the history of the market denoted by I_t for each investment time $t \in \mathbb{R}^+$. The investor possesses capital of C_0 units, which he can invest in a finite set of investment opportunities, set of N different assets. Let X_t for $t \in \mathbb{R}^+$ denote the capital of the investor at the beginning of the time period. The decision of the investor is represented by N-dimensional vector $\theta_t = (\theta_t^1, \dots, \theta_t^N)$, where θ_t^i represents the part of capital invested in the opportunity i at time t. The investment vector θ_t is subject to constraints

$$\sum_{i=1}^{N} \theta_t^i \leq 1, \quad \text{for } t \in \mathbb{R}^+, \tag{3.1}$$

representing that the investment can not exceed current capital. The additional constraint $\theta_t^i \geq 0$, for $i = 1, 2, \dots, N$ and $t \in \mathbb{R}^+$ restricts the positions to long positions only.

Let r_t^f denote the risk free interest rate for the time period $[t, t+\Delta t)$ and r_t^i the return of the investment opportunity i for time period $[t, t+\Delta t)$. The interest rate r_t^f for period

$[t, t + \Delta t)$ is known at time t, however, the return r_t^i of the investment opportunity i behaves in stochastic way caused by the market volatility. The information I_t contains the history of interest rates r^f and the history of opportunities returns r^i for $i = 1, \ldots, N$ until time t. The return over time interval of length Δt is defined as $r^i_{t+\Delta t} = \dfrac{1}{\Delta t} \ln \dfrac{X_{t+\Delta t}}{X_t}$.

The investor's capital is evolving in time according to equation

$$X_{t+\Delta t} = X_t e^{r_t^\theta \Delta t}, \tag{3.2}$$

$$X_0 = C_0, \tag{3.3}$$

where r_t^θ represents the return of the portfolio composed from risky opportunities $i = 1, ..., N$ with weights θ_i and risk-free part $1 - |\theta_t|$ at time t expressed as

$$r_t^\theta = \left(1 - \sum_{i=1}^{N} \theta_t^i\right) r_t^f + \sum_{i=1}^{N} \theta_t^i r_t^i. \tag{3.4}$$

The new investment decision θ_t is taken at time t in regards to the current information I_t. The investment vector can be expressed as $\theta_t(C_0, I_0, I_1, \ldots, I_t)$, i.e. dependent on the initial capital and the sequence of historical information. Vector θ depends implicitly on the investor's sequence of capital to that time, i.e. $X_0, \ldots, X_t$.

Remark 3.1. *Two basic assumptions are made. It is assumed that the investor views the world as a Bayesian and thus that the joint distribution of $I_1, I_2, \ldots, I_t$ given I_0 is known to him. Furthermore, it is assumed that the amount he invests in various opportunities in period t has no influence on the course of the future market events I_τ for $\tau > t$. More precisely, it is assumed that the distribution of the future events is independent of his current and past choices of the investments.*

Remark 3.2. *The return of the portfolio Y (as by Merton) is considered as a linear combination of asset returns Y_i represented in portfolio with weights $w = w_1, ...w_N$, i.e.*

$$\sum_{i=1}^{N} w_i \frac{\mathrm{d}Y_i}{Y_i} =: \frac{\mathrm{d}Y}{Y}.$$

3.1 Model with regular contributions

The above introduced model considers one investment at the initial time and reinvestment according to portfolio performance. The portfolio management problem may contain the additional investments during the planned time period typical for saving accounts. We formulate the model allowing the contributions to current capital on the regular basis.

The problem is defined as a finite horizon problem with horizon T and with portfolio containing only long positions in assets from finite set of investment opportunities. We consider an investor possessing an initial capital C_0 and contributing regularly the value of value $C\Delta t$ in times corresponding to rebalance time points (the yearly contribution corresponds to the value C). The wealth evolution X_t can be described as following

$$
\begin{aligned}
X_{t+\Delta t} &= X_t e^{(r_t^{\theta_t}-\rho_t)\Delta t} + C\Delta t, \\
&\equiv F_t(X_t, \theta_t, r_t^{\theta_t}), \quad t \in [0,T), \\
X_0 &= C_0,
\end{aligned}
\tag{3.5}
$$

where r_t^{θ} represents the return of portfolio with weights θ_t in the time period $[t, t+\Delta t)$. The portfolio can consist from risky (stocks), low-risky (bonds) and non-risky (inflation linked bonds or cash) assets. The portfolio weights fulfill the condition $\sum_i \theta_t^i = 1$ for each t, i.e. the risk free assets are considered to be part of the portfolio with volatility close to zero. The $\rho_t > 0$ corresponds e.g. to inflation rate.

Considering the transaction costs φ expressed as a part of traded volume, the wealth evolution changes to

$$
\begin{aligned}
X_{t+\Delta t} &= X_t(e^{(r_t^{\theta_t}-\rho_t)\Delta t} - \varphi \sum_i |\Delta\theta_t^i|) + C\Delta t, \quad t \in [0,T), \\
X_0 &= C_0,
\end{aligned}
\tag{3.6}
$$

where $\Delta\theta_t = \theta_t - \theta_{t-\Delta t}$ represents the change in portfolio composition from time period $[t-\Delta t, t)$ to $[t, t+\Delta t)$. Expression $\varphi \sum\limits_i |\Delta\theta_t|$ represents the part of the current capital to be paid due to composition change of the portfolio.

3.2 Pension problem

Pension management problem is a special case of the model with regular contribution adapted to the needs of pension planning. The pension problem is a long-term horizon problem bound to the client's salary, economic prediction for the salary growth and restrictions introduced by Government. The pension problem suited for Slovak pension system has been discussed by Ševčovič, Melicherčík and Kilianová in set of articles [29], [32], [43], [44].

We suppose that the future pensioner deposits once a year a τ-part of his yearly salary w_t in a pension fund with composition expressed by θ with respect to the Governmental restriction. Let us denote the accumulated sum at time t by s_t, $t \in [0,T)$, where T is the expected retirement time. Then the equations read as follows:

$$\begin{aligned} s_{t+\Delta t} &= s_t e^{r_t^{\theta_t}\Delta t} + \tau w_{t+\Delta t}\Delta t, \quad t \in [0,T), \\ s_0 &= \tau w_0 \Delta t, \end{aligned} \tag{3.7}$$

where $r_t^{\theta_t}$ is the return dependent on portfolio composition θ_t in the time period $[t, t+\Delta t)$. The salary of the saver follows a deterministic process given by equation

$$w_{t+\Delta t} = w_t e^{\rho_t \Delta t},$$

where ρ_t denotes the wage growth at time t. We assume that the term structure of wage growth is known and can be estimated by means of an econometric model.

After the retirement time T the pensioner usually strives to maintain his living standard. Therefore the accumulated saved amount, from the pensioner's point of view, is not really what he is interested in. The ratio of the cumulative sum s_T and the yearly salary w_T, $d_T = \dfrac{s_T}{w_T}$, is more important. Using the quantity $d_t = \dfrac{s_t}{w_t}$ the budget-constraint equations (3.7) can be reformulated as

$$\begin{aligned} d_{t+\Delta t} &= d_t e^{(r_t^{\theta_t} - \rho_t)\Delta t} + \tau \Delta t, \\ &\equiv F_t(d_t, \theta, r), \quad t \in [0,T), \\ d_0 &= \tau \Delta t, \end{aligned} \tag{3.8}$$

where r_t^{θ} is the return of the fund with portfolio composition θ_t in time period $[t, t+1)$, and ρ_t denotes the wage growth in $[t, t+1)$. T is the expected retirement time, [29].

3.3 Problem formulation

Investor's satisfaction and risk attitude are often described by the utility function. The set of investor's opportunities is created by assets behaving in stochastic manner. We suppose that the investor's utility U with the risk aversion coefficient is known as well as the asset distribution.

The choice of the opportunity i from the set of N assets depends on the investor's profile and his attitude to risk. Intuitively, the investor's risk aversion decreases with his wealth. Therefore, we formulate the problem in the way: *at given level of investor's risk aversion we maximize the expected utility from his wealth at time T.*

We define set Θ of all possible portfolio compositions as

$$\Theta = \{\theta_t | \sum_{i=1}^{N} \theta_t^i = 1, \theta_t^i \geq 0, \ \forall i = 1, ..., N\}, \quad \forall t \in [0,T). \tag{3.9}$$

The investor decides at every time step t for portfolio composition θ_t according to information I_t containing the history of all asset returns. We formulate the problem as stochastic dynamic problem

$$\max_{\theta \in \Theta} \mathbb{E}(U(X_T)) \tag{3.10}$$

subject to budget constraint given by (3.5).

We apply a fact from the theory of conditional expectations that a sequence of non-decreasing information $\{I_t, t \in [0,T)\}$ may be considered as a sequence of non-decreasing σ-algebras. This allows the implementation of the Theorem 3.1 [50] of the tower law for conditional expectations on (3.10).

Theorem 3.1 (Tower law for conditional expectations)**.**
Let X be a random variable on a probabilistic space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}(|X|) < \infty$. Let $\mathcal{G}, \mathcal{H}$ be σ-algebras with $\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$. Then

$$\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}).$$

Using the law of iterated expectations

$$\mathbb{E}(U(X_T)) = \mathbb{E}(\mathbb{E}(U(X_T)|I_t)) = \mathbb{E}(\mathbb{E}(U(X_T)|I_t)) \tag{3.11}$$

we conclude $\mathbb{E}(\mathbb{E}(U(X_T)|X_t))$ to be maximal. Let us denote the investor's intermediate utility function at time t as

$$V_t(X) = \max_{\theta \in \Theta} \mathbb{E}(U(X_T)|X_t = X). \tag{3.12}$$

Applying the Theorem 3.1 to the problem (3.12), we obtain the Bellman equation

$$\begin{aligned} V_t(X) &= \max_{\theta} \mathbb{E}(V_{t+\Delta t}(F_t(X, \theta, r))) \\ V_T(X) &= U(X). \end{aligned} \tag{3.13}$$

The solution of (3.13) gives to the investor the information about the optimal portfolio composition θ in every time t in dependency on the random variable representing current wealth X_t. Supposing that the compound probability distribution of each portfolio composition is known and is represented by density f_t^θ, the equation (3.13) can be rewritten into form

$$\begin{aligned} V_t(X) &= \max_{\theta} \mathbb{E}(V_{t+\Delta t}(F_t(X, \theta, r))) \\ &= \max_{\theta} \int_{\mathbb{R}} V_{t+\Delta t}(F_t(X, \theta, r)) f_t^\theta(r) \mathrm{d}r. \end{aligned} \tag{3.14}$$

According to Proposition 3.1 for $F_t(X, \theta, r)$ increasing in X the optimal solution exists and is unique. The solution denotes the decisions process for the optimal opportunity choice in each decision time t. The proof for the normal distribution and two-dimensional space is provided in [43]. Analogically, the proof can be provided for NIG distribution and more dimensional space.

Proposition 3.1. *Let $U(X)$ be an increasing, strictly concave, $\mathcal{C}^2$ smooth function for $X > 0$. Then for any $t = 0, \Delta t, 2\Delta t \ldots, T - \Delta t$,*

1. *the function $V_t(X)$ is increasing and strictly concave in X-variable;*
2. *there exists the unique argument $\hat{\theta}_t(X)$ of the maximum in (3.14).*

Chapter 4

Utility function and risk aversion

The utility function is considered an appropriate measure of the usefulness of money, see the works of Pratt [49], Gerber [24] or Markowitz [41]. Typically, if x is the wealth or a gain of a decision-maker, $U(x)$ expresses the utility or “moral value” of x. An important part of the model is to define the right utility function U for the investor.

An utility function $U : \mathbb{R} \to \mathbb{R}$ exhibits two basic properties:

(i) $U(x)$ is an increasing function of x on $(0, \infty)$;

(ii) $U(x)$ is a concave function of x.

Usual assumption is that the function $U(x)$ is twice differentiable; then (i) and (ii) state that $U'(x) > 0$ and $U''(x) < 0$. The first property amounts to the evident requirement that more is better. The U' is referred to as a marginal utility. Several reasons are given for the second property. One way to justify it is to require that the marginal utility $U(x)$ is a decreasing function of wealth x. This property is referred to as a risk aversion.

4.1 Risk Aversion Functions

To a given utility function $U(x)$ we associate a measure of *Arrow-Pratt absolute risk aversion coefficient* given as a function

$$r_A(x) = -\frac{U''(x)}{U'(x)}. \tag{4.1}$$

We note that properties (i) and (ii) imply that $r_A(x) > 0$. If the absolute risk aversion coefficient does not depend on the wealth of the investor, a utility function U exhibits the constant absolute risk aversion (CARA). An increasing absolute risk aversion function $r_A(x)$ (IARA) denotes that the investor tends to invest in less risky portfolios when his wealth is increasing. The natural assumption is that most of the investors have decreasing absolute risk aversion (DARA) [49].

The *Arrow-Pratt relative risk aversion coefficient* to a utility function U is defined as

$$r_R(x) = -x\frac{U''(x)}{U'(x)}. \tag{4.2}$$

Utility functions with constant relative risk aversion are called CRRA utility functions. Similarly to absolute risk aversion function, DRRA denotes decreasing relative risk aversion and IRRA increasing relative risk aversion. The advantage of this measure is that it is still a valid measure of risk aversion, even if the utility function changes from risk-averse to risk-loving with varying α. A constant RRA implies a decreasing ARA, but the reverse is not always true.

CARA	ARA = const.	CRRA	RRA = const
IARA	ARA $\uparrow$	IRRA	RRA $\uparrow$
DARA	ARA $\downarrow$	DRRA	RRA $\downarrow$

4.2 Used utility functions

We suppose two utility functions in our research, both of CRRA type. The first CRRA power utility function with coefficient $\alpha > 0$ representing the investor's risk aversion is given by

$$U(x) = \frac{x^{1-\alpha}}{1-\alpha}. \tag{4.3}$$

The second is adopted from Ševčovič and Melicherčík and Kilianová [31], [44] designed for clients of pension management institutions. We suppose that the saver's satisfaction measure is represented by a CRRA utility U as a function of his wealth given as

$$U(x) = \frac{1}{1-\alpha}\left((\kappa x)^{1-\alpha} - 1\right), \tag{4.4}$$

where $\alpha > 0$ is relative risk aversion of the future pensioner and κ is a constant representing number of payments during specified time period, e.g. $\kappa = \frac{1}{12}$. Increasing α implies that the pensioner is looking for less risky funds.

4.3 Model property with CRRA utility functions

The model formulation (3.5) for $C_0 = C\Delta t$ allows to derive an interesting property that the level of wealth is proportional to the contribution rate, i.e.

$$\mathbb{E}(X_t^{\lambda C}) = \lambda \mathbb{E}(X_t^C), \tag{4.5}$$

for any t and $\alpha > 0$. Process X_t^C is evolving according to (3.5) with regular yearly contribution of value C.

Suppose the utility function from section 4.2 and evolution of the wealth according to (3.5). For $V_t(x)$ we have

$$\begin{aligned} V_t(X) &= \max_\theta \mathbb{E}(U(X_T)|X_t = X), \\ &= \max_\theta \mathbb{E}(V_{t+\Delta t}(F_t(X, \theta, r)), \\ V_T(X) &= U(X) \end{aligned}$$

and for the wealth evolution according to (3.5) we obtain that

$$F_t^{\lambda C}(\lambda X, \theta, r) = \lambda X_t e^{(r_t^\theta - \rho_t)\Delta t} + \lambda C \Delta t = \lambda F_t^C(X, \theta, r).$$

Proposition 4.1. *For any affine function F_t^C linear in C and utility function given by (4.3) or (4.4) it holds that*

$$\mathbb{E}(F_t^{\lambda C}) = \lambda \mathbb{E}(F_t^C).$$

The proof has been originally stated in [29] by Jakubík, Melicherčík and Ševčovič for normal distribution . Although the extension to arbitrary distribution is straightforward we give it for reader's convinience.

Proof 4.1. Investor utility

We first consider the utility (4.3)

$$U(x) = \frac{x^{1-\alpha}}{1-\alpha}.$$

By means of a backward mathematical induction for $t = T, T - \Delta t, \ldots, 0$ we prove that the value function $V_t(X)$ satisfies $V_t^{\lambda C}(\lambda X) = \lambda^{1-\alpha} V_t^C(X)$ for $\forall t, X$ and $\lambda > 0$.

For $t = T$ we have

$$U(\lambda X) = \frac{(\lambda X)^{1-\alpha}}{1-\alpha} = \lambda^{1-\alpha} U(X).$$

Now we suppose that $V^{\lambda C}_{t+\Delta t}(\lambda X) = \lambda^{1-\alpha} V^C_{t+\Delta t}(X)$. Then we have

$$\begin{aligned} V_t^{\lambda C}(\lambda X) &= \max_\theta \mathbb{E}[V^{\lambda C}_{t+\Delta t}(F_t^{\lambda C}(\lambda X, \theta, r))] \\ &= \max_\theta \mathbb{E}[V^{\lambda C}_{t+\Delta t}(\lambda F_t^{C}(X, \theta, r))] \\ &= \max_\theta \mathbb{E}[\lambda^{1-\alpha} V^{C}_{t+\Delta t}(\lambda F_t^{C}(X, \theta, r))] \\ &= \lambda^{1-\alpha} \max_\theta \mathbb{E}[V^{C}_{t+\Delta t}(\lambda F_t^{C}(X, \theta, r))] \\ &= \lambda^{1-\alpha} V_t^C(X), \end{aligned}$$

for the time t. The optimal argument $\hat{\theta}_t^{\lambda C}(\lambda X)$ as solution of maximization problem is independent of $\lambda > 0$, i.e. $\hat{\theta}_t^{\lambda C}(\lambda X) = \hat{\theta}_t^{C}(X)$ since

$$\begin{aligned} \max_\theta \mathbb{E}[V^{\lambda C}_{t+\Delta t}(F_t^{\lambda C}(\lambda X, \theta, r)] &= \max_\theta \mathbb{E}[V^{\lambda C}_{t+\Delta t}(\lambda F_t^{C}(X, \theta, r)] \\ &= \max_\theta \mathbb{E}[\lambda^{1-\alpha} V^{C}_{t+\Delta t}(F_t^{C}(X, \theta, r)] \\ &= \lambda^{1-\alpha} \max_\theta \mathbb{E}[V^{C}_{t+\Delta t}(F_t^{C}(X, \theta, r)]. \end{aligned}$$

The stochastic variable X_t^C defined recursively,

$$\begin{aligned} X^C_{t+\Delta t} &= F_t(X_t^C, \theta_t^C(X_t^C), r_t^{\theta_t^C}), \\ X_0^C &= C\Delta t, \end{aligned}$$

satisfies $X_t^{\lambda C} = \lambda X_t^C, \quad \forall t \in [0, T]$. The level of wealth as a stochastic variable is thus proportional to the contribution. The expected value $\mathbb{E}(X_t^C)$ of the accrued wealth X_t^C is a linear function of λ, i.e. (4.5) holds.

Pensioner utility

The pensioner's utility function is given by (4.4) characterized by a constant shift. Following the same procedure it can be shown that $V_t^{\lambda C}(\lambda X) = \lambda^{1-\alpha} V_t^C(X) - \frac{1}{1-\alpha}(1 - \lambda^{1-\alpha}), \quad \forall t, X$ and $\lambda > 0$.

For $V_T(x) = U(x)$ we have

$$\begin{aligned} U(\lambda x) &= \frac{1}{1-\alpha}\left((\kappa \lambda x)^{1-\alpha} - 1\right) \\ &= \frac{1}{1-\alpha}\left((\kappa x)^{1-\alpha} \lambda^{1-\alpha} - 1\right) \\ &= \frac{1}{1-\alpha}\left((\kappa x)^{1-\alpha} - \frac{1}{\lambda^{1-\alpha}} + 1 - 1\right)\lambda^{1-\alpha} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-\alpha}\left((\kappa x)^{1-\alpha} - 1\right)\lambda^{1-\alpha} + \frac{1}{1-\alpha}\left(1 - \frac{1}{\lambda^{1-\alpha}}\right)\lambda^{1-\alpha} \\
&= \lambda^{1-\alpha}U(x) - \frac{1}{1-\alpha}(1-\lambda^{1-\alpha}).
\end{aligned}
$$

We now assume that $V^{\lambda C}_{t+\Delta t}(\lambda X) = \lambda^{1-\alpha} V^{C}_{t+\Delta t}(X) - \frac{1}{1-\alpha}(1-\lambda^{1-\alpha})$, so for time t we obtain

$$
\begin{aligned}
V^{\lambda C}_t(\lambda X) &= \max_\theta \mathbb{E}[V^{\lambda C}_{t+\Delta t}(F^{\lambda C}_t(\lambda X, \theta, r))] \\
&= \max_\theta \mathbb{E}[V^{\lambda C}_{t+\Delta t}(\lambda F^{C}_t(X, \theta, r))] \\
&= \max_\theta \mathbb{E}\left[\lambda^{1-\alpha} V^{C}_{t+\Delta t}(\lambda F^{C}_t(X, \theta, r)) - \frac{1}{1-\alpha}(1-\lambda^{1-\alpha})\right] \\
&= \lambda^{1-\alpha} V^{C}_t(X) - \frac{1}{1-\alpha}(1-\lambda^{1-\alpha}),
\end{aligned}
$$

where optimal portfolio composition $\hat{\theta}$ does not depend on λ. Therefore, for stochastic variable x^C_t we have $X^{\lambda C}_t = \lambda X^C_t$ and so (4.5). □

Chapter 5

Modeling of returns with fat-tailed distribution

Portfolio optimization is based on trading of risk and return. The construction of portfolios with minimum risk for a given return depends on two inputs: the choice of the risk measure, and the probability distribution used to model returns. Although the normal distribution for modeling returns was widely used for many years, we can find a voluminous literature concerning modeling returns with probability distribution which may better take in account more extreme changes. It can be shown that for many assets the returns do not behave "normally". It has been observed that the fat tails are one of the features of the asset returns distribution. Andersen, Bollerslev, Diebold, Ebens in [2] show that the daily DJIA (Dow Jones Industrial Average) returns, have fatter tails than the normal and, for the majority of the stocks, are also skewed.

Markowitz suggested to measure the risk of the portfolio returns by means of their variances which involve the joint distribution of returns of all assets, see Chapman [13] or Lederman [37]. Despite its simplicity and tractability, the Markowitz model has two pitfalls:

- First, the probability distribution of each asset return is characterized only by its first two moments.
- Second, it is generally insensitive to extreme events which may lead to losses caused by their underestimation.

Value at Risk is considered to describe the extreme events much better. It turns out, by a result of Embrechts, McNeil, and Straumann [19], that by using elliptical distributions for modeling asset return, managing risk with VaR is entirely equivalent to managing risk with the variance of the portfolio. The optimized portfolio composition given a certain return will be the same as the traditional Markowitz portfolio composition. Only the choice of distribution can affect the optimized portfolio. Elliptical distributions are generalizations of the multivariate normal distributions and share many of their tractable properties [36]. This generalization of the normal family provides an attractive tool for actuarial and financial risk management. However, the managers cannot neglect the deviation from multivariate normal distribution. To model financial returns series also other heavy tailed elliptical distributions can be used, such as t-Student and symmetric generalized hyperbolic distribution, or non-elliptical distribution, such as the skewed t distribution.

A fat-tailed distribution is a probability distribution that has the property to exhibit extremely large skewness and kurtosis. The comparison is often made relative to the ubiquitous normal distribution, which is considered to be a thin tail distribution, or to the exponential distribution. Fat tail distributions have been empirically encountered in a fair number of areas, not only finance and economics, but also physics, and earth sciences. Fat tail distributions have power law decay in the tail of the distribution, but do not necessarily follow a power law everywhere.

Definition 1. *Random variable X with probability density function, $f_X(x)$, in form*

$$f_X(x) \sim x^{-(1+\alpha)} \text{ as } x \to \infty, \qquad \alpha > 0$$

with $0 < \alpha < 2$, is said to have a fat tail if

$$\Pr[X > x] \sim x^{-\alpha} \text{ as } x \to \infty, \qquad \alpha > 0.$$

5.1 Generalized hyperbolic distribution

In this part we present the class of generalized hyperbolic distributions which has been introduced in Barndorff-Nielsen. The GH distributions create a wide class of interesting distributions including the normal inverse Gaussian distribution, the hyperbolic distribution, the normal distribution, the skew t and the variance gamma distribution, see Scott in [51]. Many of them are known as fat tailed distributions and are widely used in many branches.

Definition 2 (Modified Bessel function of the third kind)**.** *The integral presentation of the modified Bessel function of the third kind with index λ is given as*

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty y^{\lambda-1} e^{-\frac{x}{2}(y+y^{-1})} \mathrm{d}y, \quad x > 0. \tag{5.1}$$

The useful property of Bessel function is that $K_\lambda(x) = K_{-\lambda}(x)$.

Definition 3 (Generalized hyperbolic distribution)**.** *The random variable X has the generalized hyperbolic distribution, $X \sim GH(\lambda, \alpha, \beta, \mu, \delta)$, if its probability density function* *(introduced by Barndorff-Nielsen in 1977) is given by*

$$f_{GH}(x|\lambda,\alpha,\beta,\mu,\delta) = \frac{(\frac{\gamma}{\delta})^{\lambda}}{\sqrt{2\pi}K_{\lambda}(\delta\gamma)} e^{\beta(x-\mu)} \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\frac{\sqrt{\delta^2+(x-\mu)^2}}{\alpha}\right)^{\frac{1}{2}-\lambda}} \tag{5.2}$$

where $\gamma = \sqrt{\alpha^2-\beta^2}$, K_λ *is the modified Bessel function of the third kind, and* $x \in \mathbb{R}$. *The domain of variation of the parameters is* $\mu, \alpha \in \mathbb{R}$, *and*

$$\delta \geq 0, |\beta| < \alpha \quad \text{if } \lambda > 0 \tag{5.3}$$
$$\delta > 0, |\beta| < \alpha \quad \text{if } \lambda = 0 \tag{5.4}$$
$$\delta > 0, |\beta| \leq \alpha \quad \text{if } \lambda < 0 \tag{5.5}$$

Tails of the generalized hyperbolic distribution are considered as semi-heavy and follow

$$f_{GH}(x|\lambda,\alpha,\beta,\mu,\delta,\gamma) \sim |x|^{\lambda-1}e^{(\mp\alpha+\beta)x} \quad \text{as } x \to \pm\infty. \tag{5.6}$$

Among special and limiting cases of the GH distribution we include hyperbolic distributions in particular for $\lambda = 1$ and the normal inverse Gaussian distributions when $\lambda = -1/2$. The most useful representation of the GH distribution is a mean-variance mixture of the generalized inverse Gaussian distribution, as discussed by Folks and Chikara in [23] or by Hu [27].

We will focus on especially on normal inverse Gaussian (NIG) distribution inspired by Eriksson, Ghysels and Wang [20] and Onalan [48].

5.2 The normal inverse Gaussian distribution

The NIG distribution is a special case of generalized hyperbolic distribution. It has four parameters $\alpha, \beta, \mu, \delta$ specifying the shape of the density function. Barndorff-Nielsen [4] defined the NIG distribution as a normal variance-mean mixtures when the mixture distribution is a inverse Gaussian distribution.

Definition 4. *The random variable* X *is normal inverse Gaussian distributed* $NIG(\alpha,\beta,\mu,\delta)$ *if its probability density function is given by*

$$f(x) = \frac{\alpha}{\pi}\exp\{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\}\frac{K_1(\alpha\delta\sqrt{1+(\frac{x-\mu}{\delta})^2})}{\sqrt{1+(\frac{x-\mu}{\delta})^2}} \tag{5.7}$$

where K_1 *denotes the modified Bessel function of the third kind, and the conditions for* *the parameters are* $\alpha > 0, \delta > 0, \mu \in \mathbb{R}, 0 \leq |\beta| \leq \alpha$.

Although the probability density function is fairly complicated, its moment generating function takes a simple form [39]

$$M_X(t) = \exp[t\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + t)^2})].$$

Depending on the role of the parameters, we can divide them into two groups. The first group of the parameters affects the shape. Here belong α and β. The other two parameters μ and δ are scaling parameters of the distribution. The parameter α refers to flatness of the density function. The greater α, the greater concentration of the probability mass around mean and the greater the peak of density function. The parameter β determines a kind of skewness, for $\beta = 0$ we obtain symmetric distribution around mean.. The scaling parameter μ is responsible for the shift of the density function and the last parameter δ corresponds to the scale of the distribution. Small values narrow the distribution down and larger ones make it wider as it can be seen on Figure 5.1. Some values of NIG distribution parameters with corresponding moments are given in Table 5.1.

Theorem 5.1. *The first four moments, mean* $\mathbb{E}$*, variance* $\mathcal{V}$*, skewness* $\mathcal{S}$ *and excess kurtosis* $e\mathcal{K}$*, of the NIG distribution can be expressed using the four parameters as follows*

$$\mathbb{E}(X) = \mu + \delta \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \tag{5.8}$$

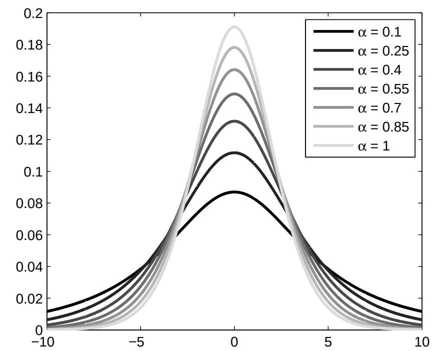
$$\mathcal{V}(X) = \delta \frac{\alpha^2}{(\alpha^2 - \beta^2)^{\frac{3}{2}}} \tag{5.9}$$

$$\mathcal{S}(X) = 3 \frac{\beta}{\alpha \sqrt{\delta \sqrt{\alpha^2 - \beta^2}}} \tag{5.10}$$

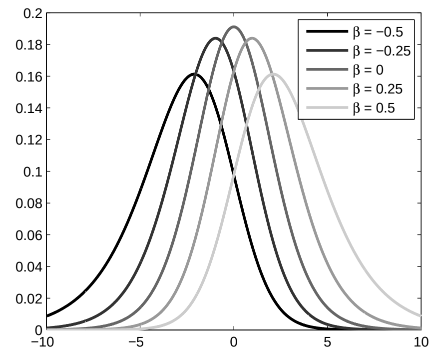
$$e\mathcal{K}(X) = 3 \frac{(1 + 4\frac{\beta^2}{\alpha^2})}{\delta(\alpha^2 - \beta^2)^{\frac{1}{2}}}. \tag{5.11}$$

Correspondingly, the parameters of the NIG distribution using the moments can be expressed as stated in Theorem 5.2.

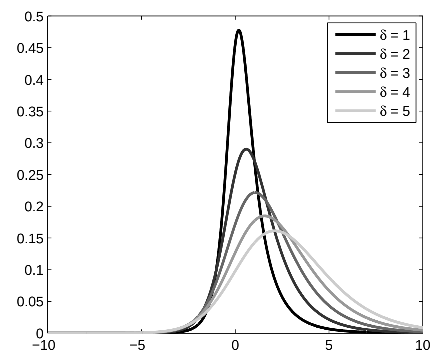
Theorem 5.2. *Suppose that random variable* X *is* $NIG(\alpha, \beta, \mu, \delta)$ *distributed and its mean, variance, skewness and excess kurtosis are denoted as* $\mathbb{E}, \mathcal{V}, \mathcal{S}$ *and* $e\mathcal{K}$*, respectively.*



(a) varying α and $\beta = 0, \mu = 0, \delta = 5$



(b) varying β and $\alpha = 1, \mu = 0, \delta = 5$



(c) varying δ and $\alpha = 1, \beta = 0, 5, \mu = 0$

Figure 5.1 Shape of density function of NIG distribution for different values of parameters α, β and δ.

Then the parameters are related to the moments by

$$\alpha = \frac{3}{\sqrt{\mathcal{V}}} \frac{(3e\mathcal{K} - 4\mathcal{S}^2)^{\frac{1}{2}}}{(3e\mathcal{K} - 5\mathcal{S}^2)} \tag{5.12}$$

$$\beta = \alpha \frac{\mathcal{S}}{(3e\mathcal{K} - 4\mathcal{S}^2)^{\frac{1}{2}}} \tag{5.13}$$

$$\delta = \alpha \mathcal{V} \left(1 - \frac{\beta^2}{\alpha^2}\right)^{\frac{3}{2}} \tag{5.14}$$

$$\mu = \mathbb{E} - \delta \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \tag{5.15}$$

under condition that $3e\mathcal{K} - 5\mathcal{S}^2 > 0$.

The proof of the Theorem 5.2 can be find in section A.2.

Table 5.1 The values of four moments of NIG distribution with different parameters.

α	β	μ	δ	$\mathbb{E}$	$\mathcal{V}$	$\mathcal{S}$	$e\mathcal{K}$
0.1	0	0	1	0	10	0	30
1	-0.8	0	1	-1.33	4.63	-3.10	17.8
1	0	0	5	0	5	0	0.6
1	0	0	1	0	1	0	3
7.746	0	0	7.746	0	1	0	0.05

5.3 Properties of normal inverse Gaussian distribution

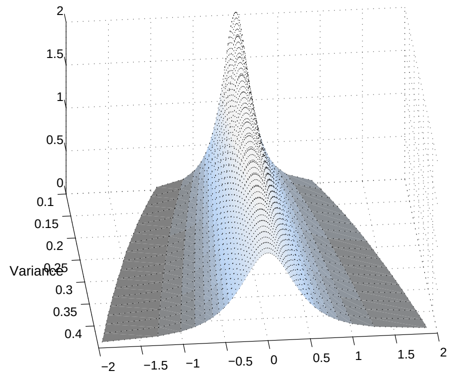
Proposition 5.1. *The NIG class of densities has the following properties:*

1. *Scaling property: If* $X \sim NIG(\alpha, \beta, \mu, \delta)$, *then* $Y = cX \sim NIG(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta)$.
2. *Convolution property: If* $X_1 \sim NIG(\alpha, \beta, \mu_1, \delta_1)$ *and* $X_2 \sim NIG(\alpha, \beta, \mu_2, \delta_2)$ *are independent, then the sum* $Y = X_1 + X_2 \sim NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2)$.
3. *Standardization: If* $X \sim NIG(\alpha, \beta, \mu, \delta)$, *then variable* $Y = \dfrac{X - \mu}{\delta}$ *has the Standard Normal Inverse Gaussian Distribution* $NIG(\alpha\delta, \beta\delta, 0, 1)$.

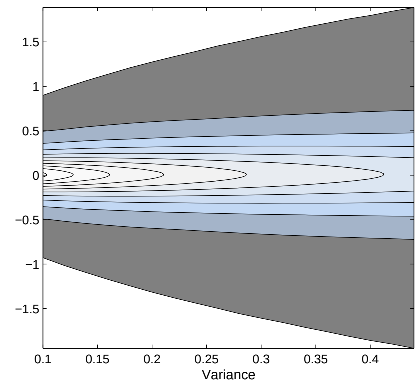
5.4 Sensitivity of the NIG density shape on descriptive statistics

For better illustration of the density shape in dependence of moments, we introduce the small sensitivity analysis. We study influence of each of four moments $\mathbb{E}, \mathcal{V}, \mathcal{S}, \mathcal{K}$ on the distribution characteristics such as mean, median and different quantiles while the other three stay fixed. We propose Figure 5.2 depicting the change of the distribution shape. The shape of the distribution can be seen on the cross-sections for each level of variance, skewness and kurtosis.

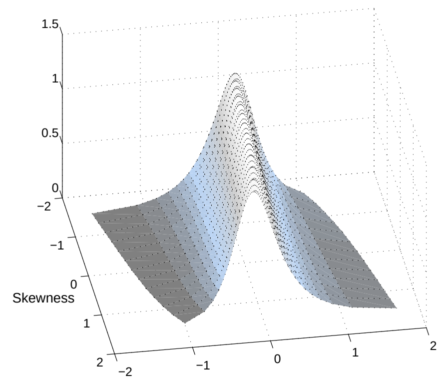
The change of mean causes only the shift of the peak settlement while the shape remains constant. On the other hand, the next three moments influence the shape in a crucial way. Higher value of variance (Figure 5.2 (a), (b)) causes flattening of the distribution. The median does not change while the other quantiles are increasing in absolute value. The higher skewness (Figure 5.2 (c), (d)) shifts the median value down and makes the right tail longer and left tail shorter. The kurtosis (Figure 5.2 (e),(f)) influences the height of the distribution peak, the higher value of kurtosis implies the thinner and higher peak.



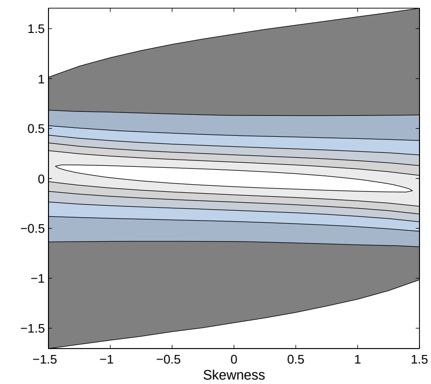
(a) varying $\mathcal{V}$ and $\mathbb{E} = 0, \mathcal{S} = -0.12, e\mathcal{K} = 7$



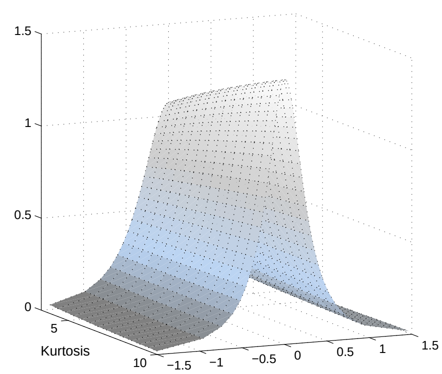
(b) contour graph



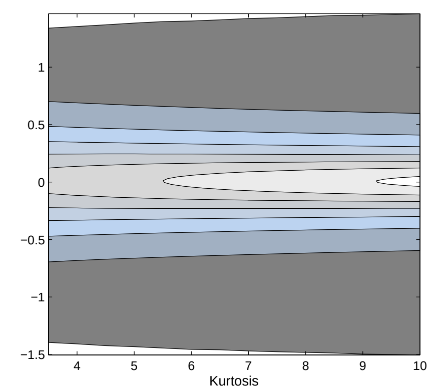
(c) varying $\mathcal{S}$ and $\mathbb{E} = 0, \mathcal{V} = 0.25, e\mathcal{K} = 7$



(d) contour graph



(e) varying $\mathcal{K}$ and $\mathbb{E} = 0, \mathcal{V} = 0.25, \mathcal{S} = -0.12$



(f) contour graph

Figure 5.2 Shape of density function of NIG distribution for different moments $\mathbb{E}, \mathcal{V}, \mathcal{S}, \mathcal{K}$ with the corresponding contour graphs on the right hand side.

5.5 Motivation for using normal inverse Gaussian distribution

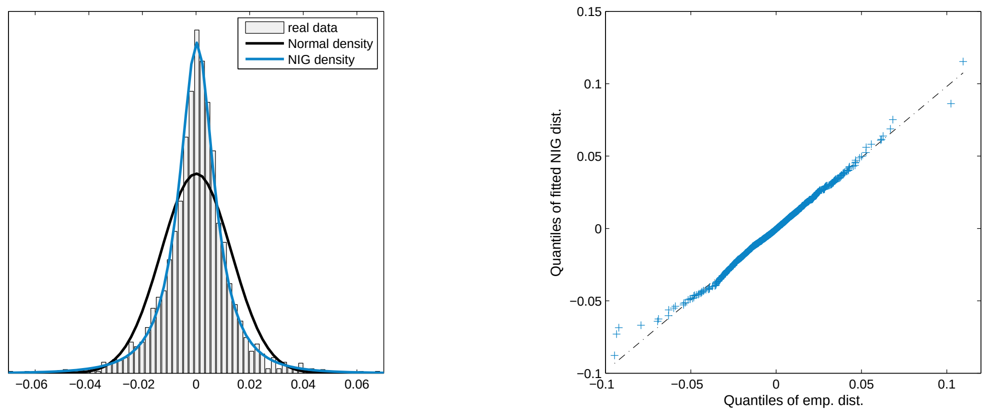
Onalan in his article [48] provided an empirical analysis of financial data, especially he focused on S&P Index and VIX Index. He investigated the use of normal inverse Gaussian distribution in financial risk management. He showed that the log-returns for both indices strongly deviate from the normal distribution, since the skewness exhibits non-zero value and kurtosis gains higher value than 3 which is typical for normal distribution. In the paper, author estimated the values of distribution parameters and used them to estimate Expected Shortfall and Value at Risk to the VIX and S&P500 Index. The density of NIG distribution with estimated parameters values was close to the empirical density of log returns, and in addition the tail behavior of probability distributions revealed that the NIG density represents the fat tails of empirical data better than normal density. He concluded that in the risk measurement the normal inverse Gaussian model performs better than normal and historical Value at Risk (VaR) and Expected Shortfall calculation methods.

5.5.1 Historical data of S&P500 Index

We have adopted the NIG model and analyze the fitness on the real data. In Figure 5.3 (a) we offer an illustration how the real data are actually fitted by the normal distribution and the fat tailed distribution NIG. We have considered the daily data for S&P500 Index in time period 02/01/2000 - 02/01/2014. The values of the first four moments were obtained from the historical data series (as in section A.1). The annual mean value of asset returns is 1.65% , the annual standard deviation is 20.87%, the skewness is negative with value -0.17 and the kurtosis 10.7. The parameters of the NIG distribution were calculated based on the values of the moments (using Theorem 5.2). As we can see, the density of the NIG probability distribution seems to better fit the tails and the peak than the normal distribution. The Figure 5.3 (b) represents Q-Q plot to compare the empirical distribution with fitted NIG distribution by plotting their quantiles against each other.

We can conclude that the log-returns of S&P500 Index follow the distribution with higher kurtosis than 3, and more important, the distribution with non zero skewness. We have computed the values of the first four moments on the yearly basis and showed their values in Table 5.2 corresponding to time periods illustrated on Figure 5.4. We present the average yearly values for the different time periods characterized by various trends.

Analysis of the distribution of S&P500 Index returns and the results in Table 5.2 motivate us to provide the sensitivity analysis of different levels of skewness and kurtosis on the final value of the accumulated sum.



(a) normal vs. NIG distribution

(b) Q-Q plot

Figure 5.3 Histogram of log returns for daily data for S&P Index and density of probability distribution for time period Jan 2000 - Jan 2014 and corresponding Q-Q plot of log returns for S&P index vs fitted NIG distribution. (The descriptive statistics obtained from daily historical data are: yearly mean value is 1.65%, yearly standard deviation 20.87%, skewness -0.17 and excess kurtosis 7.7.)

Table 5.2 Descriptive statistics obtained from historical data series of S&P500 Index for the time period 02/01/2000 - 02/01/2014.

	2000 - 2003	2003 - 2008	2008 - 2009	2009 - 2014	2000 - 2014
Mean	-15.77%	9.31%	-43.85%	13.54%	1.65%
Std. dev.	23.39%	13.16%	41.13%	19.44%	20.87%
Skewness	0.1951	-0.1747	-0.0457	-0.2740	-0.1748
Kurtosis	4.1154	4.6763	6.6140	7.1379	10.7006

5.6 Drawbacks of NIG distribution

NIG distribution has the great property to describe the density shape of the variable in very good way and depict its skewness and kurtosis. The replication is straightforward from NIG distribution with parameters obtained by Theorem 5.2.

One of the main drawbacks follows directly from NIG convolution property (Theorem 5.1) defined only for independent variables exhibiting same shape parameters which is a very restrictive condition. Considering more dimensional space of uncorrelated variables leads to non-problematic replication exhibiting the pre-desribed values of mean, variance, skewness and kurtosis. The problem appears when considering more dimensional space of correlated NIG distributed variables. *The higher the correlation and the higher the skewness and kurtosis of the variables, the less precise replication.*

Therefore we introduce the algorithms for generating mixtures created by correlated

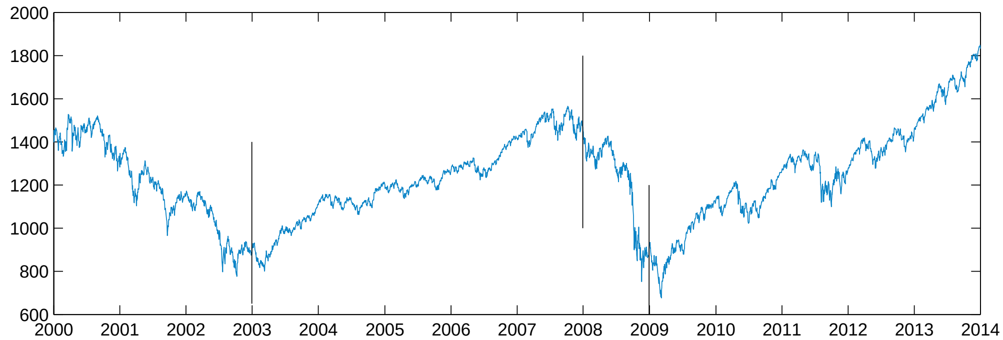


Figure 5.4 Evolution of S&P Index during time period 02/01/2000 - 02/01/2014 graphically divided into four time periods (details in Table 5.2) characterized by different trends.

NIG distributed variables in chapter 6.

Chapter 6

Mixture of NIG distributed time series

The normal inverse Gaussian distribution possesses the property to approximate the empirical distribution of real returns to a high degree. Considering a portfolio composed from assets with NIG distributed historical returns it is necessary to create a mixture.

The normal inverse Gaussian distribution does not belong to class of infinitely divisible distributions, i.e. the mixture of two or more NIG distributed variables does not have to create NIG distributed variable unless the mixing variables share the shape parameters α and β. This feature of NIG distribution causes difficulties when assuming a portfolio composed from assets with NIG distributed returns. In general, it is not usual that more assets follow distributions with same shape parameters. Therefore, we introduce a procedure for approximation of parameters of NIG mixtures.

$$NIG + NIG \neq NIG$$
$$NIG + NIG \approx NIG$$

Another issue arising by creating NIG mixtures follows from the properties of the historical time series often exhibiting high correlation and their modeling (replication) requires to take this correlation into consideration. In the next sections we offer two algorithms for replicating the time series with pre-described four moments and correlation.

6.1 Replication of time series: Choleski approach

Replication of the time series from pre-defined moments and correlation create an important part by generating the NIG mixtures. We first present an algorithm based on Choleski decomposition in Table 6.1 commonly used by generating correlated normal random variables. Despite of the difference between the NIG and normal distribution the algorithm gives for some set of parameters sufficiently precise results.

6.2 Properties of the replicated time series

If we consider independent NIG distributed time series with given first four moments, the introduced algorithm can secure that the replicated time series will carry the same features - same four moments. However, if the time series are correlated, the replicated times series do not have to exhibit the pre-described values of the four moments and correlation. We have provided an analysis regarding the correctness of replication of asset returns when considering mixture of two correlated NIG distributed variables.

The differences between pre-described moments and moments of replicated time series arise by correlating independent NIG time series in step 3. The correlation of replicated time series is obtained by multiplying the independent time series by lower triangular matrix calculated by Choleski decomposition, i.e. the realization of the second time series is transformed to copy the influence caused by correlation. However, this transformation degenerates the realization and so its features. We observe the following aspects.

- Higher level of correlation causes higher degeneration of replicated features, i.e. of skewness and kurtosis of the second time series. Additionally, it is not possible to transfer high correlation into replicated time series. The reproduced time series are therefore less correlated than in original.
- Higher values of kurtosis of time series lower the ability to replicate the distributions and correlation correctly. If both time series share the kurtosis value, the replication disposes with much better fit ability of correlation, skewness and kurtosis, than if the kurtosis of time series differs. In that case, usually the resulting correlation does not correspond to the pre-described one.
- Skewness of the assets does not seem to influence the fitness of the moments.

The more detailed analysis of the observed dependencies of pre-described and replicated time series properties by Choleski decomposition based algorithm is presented in Table 6.2.

The presented algorithm for replicating two NIG processes is suitable only for assets which returns' distributions are not strongly correlated and do not dispose very high kurtosis. Replication of the time series is thus not exact and requires an optimization approach. For relatively low correlation the objective should be to find correlation satisfying the pre-described correlation and for higher correlation to find NIG parameters for the second asset and correlation generating the time series satisfying the pre-described moments and correlation.

Table 6.1 CHOLESKI DECOMPOSITION BASED ALGORITHM

`INPUT`

`Return of asset` i (A_i): $\mathbb{E}_i = \mathbb{E}(A_i), \mathcal{V}_i = \mathcal{V}(A_i), \mathcal{S}_i = \mathcal{S}(A_i), \mathcal{K}_i = \mathcal{K}(A_i)$;

`correlation matrix` $\Sigma = corr(i,j)$ `for` $i, j = 1, ..., N$.

`ALGORITHM`

1. `calculate NIG parameters` $\alpha_i, \beta_i, \mu_i, \delta_i$ `for each asset` i `in portfolio based on its four moments` $\mathbb{E}_i, \mathcal{V}_i, \mathcal{S}_i, e\mathcal{K}_i$ `according to Theorem 5.2;`

2. `generate the random vectors` Z_i `from the standard normal inverse Gaussian distribution` $NIG(\alpha_i\delta_i, \beta_i\delta_i, 0, 1)$ `for each asset` $i = 1, \ldots, N$;

3. `correlate the random vectors` Z_i `by means of Choleski decomposition as`

$$\widetilde{Z} = LZ$$

   `where columns of matrix` Z `are created by vectors` Z_i `and` L `denotes the lower triangular matrix from Choleski decomposition` ($\Sigma = LL^\top$) `of the correlation matrix`

$$\Sigma = corr(i,j) \quad \texttt{for} \quad i, j = 1, \ldots, N;$$

   `Remark: It is important to note that the random vectors` Z_i `constructed in regards to` Z_1 `loose their original properties, i.e. the skewness and the kurtosis are not equal to initial moments. The correlation is degenerated as well.`

4. `construct the correlated random vectors (sufficiently large) so that the first two moments are exact as`

$$X_i = \hat{\delta}_i \widetilde{Z}_i + \hat{\mu}_i$$

   `where coefficients` $\hat{\delta}_i$ `and` $\hat{\mu}_i$ `are defined as (according to section A.3)`

$$\hat{\delta}_i = \sqrt{\frac{\mathcal{V}_i}{\mathcal{V}(\widetilde{Z}_i)}}$$
$$\hat{\mu}_i = \mathbb{E}_i - \hat{\delta}_i \mathbb{E}(\widetilde{Z}_i)$$

   `The vectors` X_i `for` $i = 1, \ldots, N$ `represent the replicated asset returns.`

Table 6.2 Table of observed dependencies of pre-described and replicated time series properties by Choleski decomposition based algorithm.

Skewness	Kurtosis	Corr $\|\rho\|$	Consequence
$\mathcal{S}(A_1)=0$ $\mathcal{S}(A_2)=0$	$\mathcal{K}(A_1)=\mathcal{K}(A_2)$	low	✓
		higher	the higher correlation causes worse fit of $\mathcal{K}(A2)$, higher $\mathcal{K}$ worse $\mathcal{K}(A_2)$
$\mathcal{S}(A_1)\neq 0$ $\mathcal{S}(A_2)\neq 0$	$\mathcal{K}(A_1)=\mathcal{K}(A_2)$	low	the higher the skewness of asset 2 $\|\mathcal{S}(A_2)\|$, the worse correlation
		higher	the higher correlation $\|\rho\|$, the worse $\mathcal{S}(A_2)$ (and $\|\mathcal{K}(A_2)\|$)
$\mathcal{S}(A_1)=0$ $\mathcal{S}(A_2)=0$	$\mathcal{K}(A_1)\neq\mathcal{K}(A_2)$	low	correlation does not correspond to pre-described correlation ($\approx \rho/2$)
		higher	worse correlation and worse $\mathcal{K}(A_2)$
$\mathcal{S}(A_1)\neq 0$ $\mathcal{S}(A_2)\neq 0$	$\mathcal{K}(A_1)\neq\mathcal{K}(A_2)$	low	correlation does not correspond to pre-described correlation
		higher	worse correlation and worse $\mathcal{K}(A_2)$

6.3 Replication of time series: Optimization approach

We assume a NIG mixture consisting from two assets, whose properties are given by their first four moments and correlation. The aim is to generate the time series satisfying these inputs. However, the replicated time series do not maintain the input properties and thus the time series have to be generated by means of other parameters. The objective is to find the set of parameters, that would replicate time series exhibiting pre-described properties.

Since the transformation of the independent random variables to correlated variables with given correlation by means of Choleski decomposition keeps the first variable unchanged, we fix the realization of the first asset generated by following the steps in Table 6.1 while the moments of the second asset and the correlation will be a subject of optimization. The algorithm searches a set of 4 parameter - variance, skewness and kurtosis for the second asset and correlation of asset 1 and asset 2 using the genetic algorithm, that would replicate time series using the Choleski approach. This algorithm is presented in Table 6.3.

The genetic algorithm is used because of the presence of the random generator for Z_2 in the objective function. The residual value can thus exhibit different values for the same parameters. Any optimizing algorithm based on gradient method could be therefore misleading. The boundaries for the optimizing parameters are set as $\pm 25\%$ around the required moments for Asset 2 and $[\rho, \max(2\rho, 1)]$ for $\rho > 0$ and $[\min(-2\rho, -1), \rho]$ for $\rho < 0$.

Remark 6.1. *The use of the optimization based algorithm seem to be effective for time*

Table 6.3 OPTIMIZATION BASED ALGORITHM

INPUT

Return of asset 1 (A_1):

$\mathbb{E}_1 = \mathbb{E}(A_1), \mathcal{V}_1 = \mathcal{V}(A_1), \mathcal{S}_1 = \mathcal{S}(A_1), \mathcal{K}_1 = \mathcal{K}(A_1)$;

Return of asset 2 (A_2):

$\mathbb{E}_2 = \mathbb{E}(A_2), \mathcal{V}_2 = \mathcal{V}(A_2), \mathcal{S}_2 = \mathcal{S}(A_2), \mathcal{K}_2 = \mathcal{K}(A_2)$;

correlation of assets: $\rho = corr(A_1, A_2)$.

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optimizing parameters: $\tilde{\rho}$, $\tilde{\mathcal{V}}_2, \tilde{\mathcal{S}}_2, \tilde{\mathcal{K}}_2$;

Remark: the mean value does not influence the shape of the distribution and its optimization is not relevant ($\tilde{\mathbb{E}}_2 = 0$). The other three moments influence the shape (as presented in section 5.4) and therefore belong to optimizing parameters.

the objective function:

$$\min_{\tilde{\mathcal{V}}_2, \tilde{\mathcal{S}}_2, \tilde{\mathcal{K}}_2, \tilde{\rho}} R := (\mathcal{S}_2 - \tilde{\mathcal{S}}_2)^2 + (\mathcal{K}_2 - \tilde{\mathcal{K}}_2)^2 + (\rho - \tilde{\rho})^2. \tag{6.1}$$

INITIALIZATION

initial moments for replicated Asset 2 ($\tilde{A}_2$):

$\tilde{\mathbb{E}}_2 = \mathbb{E}(A_2), \tilde{\mathcal{V}}_2 = \mathcal{V}(A_2), \tilde{\mathcal{S}}_2 = \mathcal{S}(A_2), \tilde{\mathcal{K}}_2 = e\mathcal{K}(A_2)$;

initial replicated correlation $\tilde{\rho} = \rho$;

ALGORITHM

1. compute NIG parameters for $A_1 \Rightarrow \alpha_1, \beta_1, \mu_1, \delta_1$ for A_1 generate Z_1 as $Z_1 \sim NIG(\alpha_1\delta_1, \beta_1\delta_1, 0, 1)$ and keep the realization fixed;
2. compute NIG parameters for $\tilde{A}_2$ $\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\mu}_2, \tilde{\delta}_2$ for $\tilde{A}_2$ generate Z_2 as $Z_2 \sim NIG(\alpha_2\delta_2, \beta_2\delta_2, 0, 1)$;
3. correlate vectors Z_1, Z_2 as
$$\overline{Z} = LZ \quad \text{with} \quad L = \begin{pmatrix} 1 & 0 \\ \tilde{\rho} & \sqrt{1-\tilde{\rho}^2} \end{pmatrix};$$
4. calculate the skewness and kurtosis of $\overline{Z}_2$ and correlation $\overline{\rho}$ of $\overline{Z}_1 (\equiv Z_1), \overline{Z}_2$;

```
Remark: any scaling of vectors Z̄1, Z̄2 does not influence skewness,
kurtosis and correlation, see section A.3.
```

5. `compute the residuals` R `of input and replicated properties and find the` $\min R$ `(6.1) by repeating steps 2-5;`

6. `construct the correlated random vectors as`

$$X_i = \hat{\delta}_i \widetilde{Z}_i + \hat{\mu}_i$$

`where coefficients` $\hat{\delta}_i$ `and` $\hat{\mu}_i$ `are defined as (according to section A.3);`

$$\hat{\delta}_i = \sqrt{\frac{\mathcal{V}_i}{\mathcal{V}(\widetilde{Z}_i)}}, \quad \hat{\mu}_i = \mathbb{E}_i - \hat{\delta}_i \mathbb{E}(\widetilde{Z}_i)$$

`The vectors` X_i `for` $i = 1, \ldots, N$ `represent the replicated asset returns.`

series with relatively high correlation ($|\rho| > 0.4$) and different kurtosis. The effectiveness of use of the genetic algorithm vanishes with decreasing correlation.

For $|\rho|$ low the optimization based algorithm works effectively (sufficient replication properties and short time) if the optimizing parameter is only correlation ρ and other model parameters are fixed.

6.4 Generating NIG mixtures with predefined moments

We consider a portfolio consisting from N assets with NIG distributed returns, $i = 1, \ldots, N$, with weights θ, where $\theta = (\theta_1, \ldots, \theta_N)^T$. We presume that the portfolio returns are also NIG distributed, i.e. $r_P^\theta \sim NIG(\alpha^\theta, \beta^\theta, \mu^\theta, \delta^\theta)$. To estimate the parameters $\alpha^\theta, \beta^\theta, \mu^\theta, \delta^\theta$ of the portfolio returns, we first define an approximation of a NIG mixture.

Definition 5 (NIG approximation of mixture). *Assume that $r_i \sim NIG(\alpha_i, \beta_i, \mu_i, \delta_i)$ for $i = 1, 2, \ldots, N$. Then for weighted mixture $r = \sum_{i=1}^N \theta_i r_i$ the aim is to find parameters $\alpha^\theta, \beta^\theta, \mu^\theta, \delta^\theta$ of NIG distribution dependent on vector θ such that it holds*

1. *for $\theta = \mathbf{e}$ the parameters agree exactly ($\mathbf{e}$ is a unit vector);*
2. *first four parameters of convex combination $r = \sum_{i=1}^N \theta_i r_i$ agree with four moments of r for each $\theta_i \in [0, 1], \sum_i \theta_i = 1$.*

Table 6.4 NIG MIXTURE ALGORITHM

```
INPUT

    Return of asset i (A_i):  E_i = E(A_i), V_i = V(A_i), S_i = S(A_i), K_i = K(A_i);

    correlation matrix Σ = corr(i,j)  for i,j = 1,...,N.

ALGORITHM

  1. replicate the asset returns X_i for i = 1,...,N by Choleski
     decomposition based algorithm in Table 6.1 or by Optimization
     based algorithm in Table 6.3;

  2. compute the portfolio return as linear combination of asset
     returns weighted according to composition θ, i.e.  r^θ = Σ_{i=1}^N θ_i X_i;

  3. compute the values of the first four moments E(r^θ), V(r^θ), S(r^θ) and
     K(r^θ) for the vector r^θ using Theorem A.2;

  4. calculate NIG parameters using Theorem A.

     Remark:  It is assumed that the random vector r^θ representing the
     portfolio returns is also NIG distributed.
```

Assumption 6.1. *We assume that for each asset i, $i = 1, \ldots, N$ the values of first four moments of asset returns ($\mathbb{E}_i, \mathcal{V}_i, \mathcal{S}_i, e\mathcal{K}_i$) and the corresponding correlation matrix Σ are known or can be calculated from real time series using Theorem A.2.*

The algorithm is presented in Table 6.4.

6.5 Example

For the demonstration of the introduced algorithms we consider two time series of asset returns represented by their first four moments and correlation given in Table 6.5 in column 'pre-described'. Both time series are skewed and by one asset by observe high excess kurtosis. The time series are correlated. This causes deviation of replicated characteristics from original when applying the Choleski decomposition based algorithm from Table 6.1. Especially the correlation is strongly deviated from required value. Applying the Optimization based algorithm from Table 6.3, the replicated correlation as well as the replicated skewness and kurtosis of returns of the second asset are closer to pre-described values. The result of replication of both algorithms are presented in Table 6.5.

Table 6.5 Comparison of replicated characteristic using the Choleski algorithm from Table 6.1 and Optimization algorithm from Table 6.3.

		Pre-described	Replicated Choleski algo	Replicated Optimization algo
Asset 1	$\mathbb{E}$	5%	5%	5%
	$\mathcal{S}td$	8.5%	8.5%	8.5%
	$\mathcal{S}$	-0.10	-0.0968	-0.0971
	$\mathcal{K}$	3.5	3.4856	3.5014
Asset 2	$\mathbb{E}$	10%	10%	10%
	$\mathcal{S}td$	25%	25%	25%
	$\mathcal{S}$	-0.45	-0.4392	-0.4017
	$\mathcal{K}$	6	5.8804	6.1251
Correlation	ρ	-0.4	-0.1791	-0.41

The Figure 6.2 illustrates the dependence of the NIG parameters $\alpha, \beta, \mu, \delta$ of portfolio return r^θ composed from two assets - stock and bond - defined by their first four moments. The composition is expressed through parameter θ representing the proportion of a stock in portfolio. By $\theta = 0$ we understand the portfolio consisting only from asset 1 and vice verca $\theta = 1$ indicates portfolio composed only from asset 2. The time series are replicated by extended algorithm and exhibit the features from Table 6.5. Figure 6.3 illustrate the dependence of portfolio return moments on asset proportion θ. The change in shape of the distribution depicts Figure 6.1. The shape of portfolio return distribution for $\theta = 0$ is the shape of the asset 1 return distribution and equally shape for $\theta = 1$ is the asset 2 return distribution. Mean of the portfolio return is a linear combination of asset mean returns. Variance (or standard deviation) is a concave function indicating existence of a certain combination of asset 1 and asset 2 minimizing the volatility of the portfolio. The extreme of skewness and kurtosis functions are acquired for same value of θ however different from θ minimizing variance.

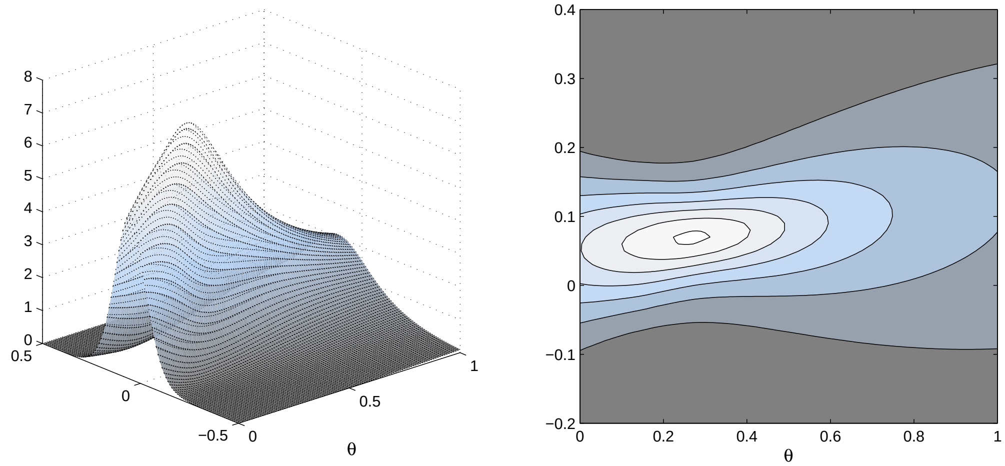


Figure 6.1 Evolution of θ-mixture distribution. For $\theta = 0$ the mixture consists only from asset 1 and $\theta = 1$ indicates 100% representation of asset 2. (The replicated time series exhibit the features from Table 6.5 in last column.)

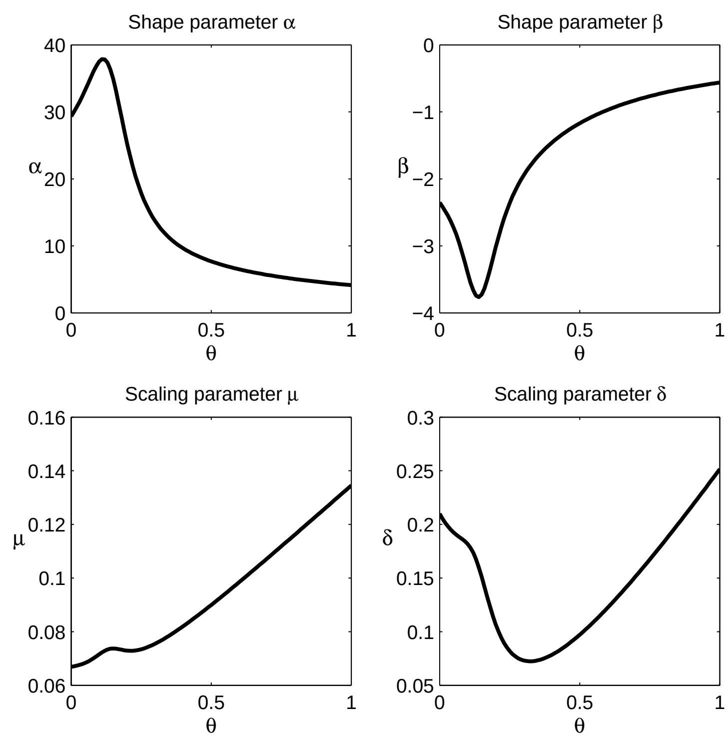


Figure 6.2 Development of the NIG parameters $\alpha, \beta, \mu, \delta$ of portfolio return r^{θ} as a function of asset proportion θ. (The replicated time series exhibit the features from Table 6.5 in last column.)

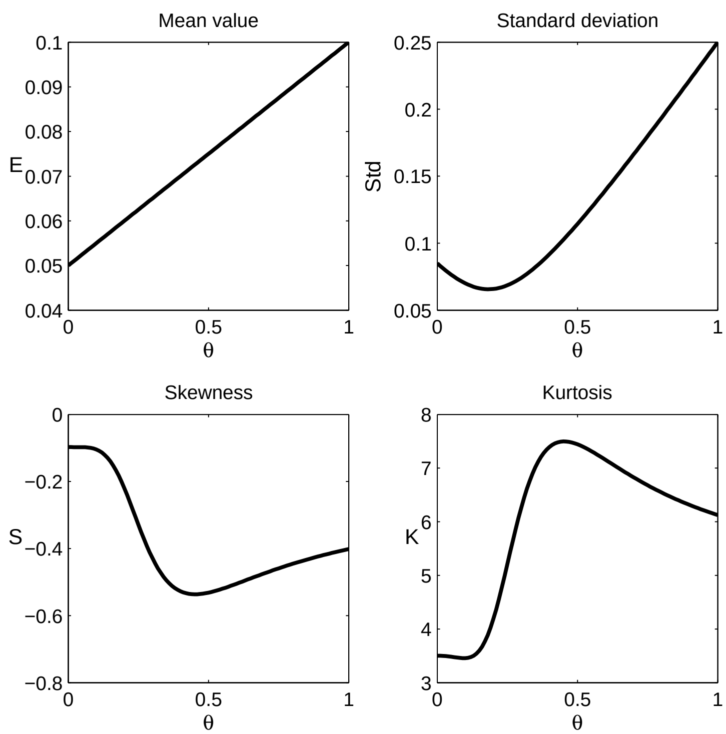


Figure 6.3 Development of the moment parameters $\mathbb{E}, \mathcal{V}, \mathcal{S}$ and $e\mathcal{K}$ of portfolio return r^θ as a function of asset proportion θ. (The replicated time series exhibit the features from Table 6.5 in last column.)

Chapter 7

Numerical approximation of stochastic dynamic optimization problem

In this section we describe the numerical approximation procedure for solving the formulated maximization problem (3.10) with dynamic constraints given by (3.5). The numerical scheme for solving this problem has been described in details by Kilianová, Melicherčík and Ševčovič e.g. in [32]. The proposed scheme has been constructed for needs of implementation of normal distribution. The NIG distribution exhibits a specific shape and requires also a special treatment for its numerical approximation. We describe the numerical process with a focus on needs of use of NIG distribution or generally distribution with fat tails and high kurtosis.

The problem is formulated as a decision problem. In each time t the investor decides for the stock-to-bond proportion θ according to the problem given by Bellman equation. The problem to be solved in each time step t is to find the maximum over θ of the integral

$$\int_{\mathbb{R}} V_{t+\Delta t}(x_{t+\Delta t}) f^{\theta}(r) \mathrm{d}r. \tag{7.1}$$

We construct an equidistant division $\{\theta_j, j = 1, ..., n_\theta\}$ of the interval $[0, 1]$ where n_θ is sufficiently large. The restriction for non-negative θ follows from not allowed short positioning in portfolio. By Proposition 3.1 there exists a unique argument of maximum $\hat{\theta}_t(x)$ of (7.1). Hence we can find a unique θ_j such that the value of the integral (7.1) is maximal. Further, we define an interval $[x_{\min}, x_{\max}]$, where the final wealth would most likely be. We construct an equidistant division of this interval $\{x_i, i = 1, ..., n_x\}$, where $x_1 = x_{\min}$ and $x_{n_x} = x_{\max}$. The NIG distribution exhibiting fatter tails than the normal distribution requires to consider wider space interval since the extreme return values are more likely to occur. The interval division however should stay fine enough to depict the difference of the value function affected by the shape of the density function in two neighboring points.

The algorithm for finding the optimal solution θ is as follows. The function $V_T(x) = U(x)$ is given by (4.3). We compute the functional V_t recurrently from $t = T - \Delta t$ down to $t = 0$. In each time step we compute an approximate value of function $V_t^{\theta_j}(x_i)$ in discrete points x_i and discrete points θ_j and set the maximal value over all θ_j as $V_t(x_i)$ and the optimal $\hat{\theta}_t(x_i) = \theta_j$. We repeat this computation for all points $\{x_i | i = 1, ..., n_x\}$.

The density function f^θ is considered to be the density function of the portfolio returns. When considering the NIG density function f^θ the NIG parameters $\alpha^\theta, \beta^\theta, \mu^\theta$ and δ^θ are as discussed in chapter 6. For normal distribution the formula for parameters μ^θ and σ^θ is known and no approximation algorithm is needed.

The peak of the NIG density function can be due to the high kurtosis thin and high. The classical trapezoidal rule with uniform spacing grid for approximating the definite integral is not sufficient, since the grid should be denser in neighborhood of the center of the peak. For the Bellman type integral with NIG distribution densities f^θ we therefore use the Legendre-Gauss quadrature[1] with $n > 0$ nodes. The quadrature rule is based on approximation of the integral by the weighted sum of function values at specified points within the domain of integration. We consider interval I_r to be sufficiently large to cover all non zero values, $I_r = (\overline{r^\theta} - 15\sigma^\theta, \overline{r^\theta} + 15\sigma^\theta)$, using $n = 50$ nodes. The integral (7.1) is approximated by

$$\sum_{k=1}^{n_r} V_{t+\Delta t}(x_{t+\Delta t}) w_k^\theta f^\theta(r_k^\theta), \tag{7.2}$$

where $r_k^\theta \in I_r, k = 1, \dots, n_r$ represent the nodes and w_k^θ corresponding weights according to Gauss-Lengedre quadrature for density function f^θ of θ-mixture. The efficient interval I_r for normal distribution or distributions with thinner tails can be much smaller.

As far as the values of function V_t are computed only in discrete points x_i, we need to define a proper approximation of the values $\xi \in [x_{\min}, x_{\max}]$ as well as for values outside of the defined interval. Especially, the approximation of $V_t(\xi)$ in points outside the space grid $\xi > x_{\max}$ require a special attention. We define the approximation of $V_t(\xi), \forall \xi \in [0, \infty]$ as following

1. for $\xi < x_{\min}$ we set $V_t(\xi) = V_t(x_{\min})$ (this restriction can be viewed as the bottom value that has to be ensured);

2. for $\xi > x_{\max}$, $V_t(\xi)$ is set as (7.9);

3. for $\xi \in [x_{\min}, x_{\max}]$ an interpolation of nearest neighboring grid points is calculated as (7.10) or as (7.11).

[1] see Appendix B for more information

7.1 Boundary conditions

The correct setting of the boundary condition creates a crucial part of the numerical solution. The error caused by incorrect valuation of the value function outside the grid accumulates by backward calculation and thus strongly influences the solution. In the portfolio optimization problem the contour lines of optimal choice θ could be therefore incorrectly curved.

The boundary condition for $\xi > x_{\max}$ set as $V_t(\xi) = V_t(x_{\max})$ would require a large value of $x_{\max}$ which would be very time consuming since the division of the interval has to be dense. Another option is to use an extrapolation for approximation of $V_t(\xi)$. However, a decline in the rate of growth of the value function V_t (can be seen already for $V_T(.) = U(.)$) causes that the values obtained by extrapolation are strongly deviated from the true values. The error accumulates therefore from the first step, for $t = T$. These facts leaded to derivation of the heuristic of a boundary condition specific for our type of problem.

For each space grid point x_i and for each time grid point t we calculate the approximation of the functional given as

$$V_t(x) = \max_{\theta} \int_{\mathbb{R}} V_{t+\Delta t}(xe^{(r^\theta-\rho)\Delta t} + C\Delta t) f_t^\theta(r)\mathrm{d}r. \tag{7.3}$$

For the final time horizon and utility function in form (4.3) we have the terminal condition

$$V_T(x) = U(x) = \frac{x^{1-\alpha}}{1-\alpha}.$$

We now assume that the value function $V_t(x)$ is for each t a multiple of utility value $U(x)$ and can be expressed as

$$V_t(x) = \beta_t \frac{x^{1-\alpha}}{1-\alpha}, \tag{7.4}$$

with terminal $\beta_T = 1$ and $\beta_t > 0$.

Substituting (7.4) into (7.3) we obtain

$$\beta_t V_t(x) = \max_{\theta} \int_{\mathbb{R}} \beta_{t+\Delta t} V_{t+\Delta t}(xe^{(r^\theta-\rho)\Delta t} + C\Delta t) f_t^\theta(r)\mathrm{d}r,$$

$$\gamma \equiv \frac{\beta_t}{\beta_{t+\Delta t}} = \min_{\theta} \int_{\mathbb{R}} \frac{(xe^{(r^\theta-\rho)\Delta t} + C\Delta t)^{1-\alpha}}{x^{1-\alpha}} f_t^\theta(r)\mathrm{d}r. \tag{7.5}$$

For evolution of β_t we thus obtain a process

$$\begin{aligned} &\beta_t = \gamma\beta_{t+\Delta t}, \quad \text{for } t = 0, ..., T-\Delta t, \\ &\beta_T = 1, \end{aligned}$$

that can be simply expressed as

$$\begin{aligned} &\beta_t = \gamma^{T-t}\beta_T, \\ &\beta_T = 1, \end{aligned}$$

implying that

$$V_t(x) = \beta_t \frac{x^{1-\alpha}}{1-\alpha} = \gamma^{T-t}\frac{x^{1-\alpha}}{1-\alpha} = \gamma^{T-t}U(x). \tag{7.6}$$

Lemma 7.1. *For utility function in form (4.3) and wealth evolution given by (3.5), for $x \to \infty$ the integral in (7.5) can be approximated as*

$$\lim_{x\to\infty}\int_{\mathbb{R}}\left(\frac{xe^{(r^\theta-\rho)\Delta t}+C\Delta t}{x}\right)^{1-\alpha} f(r)\mathrm{d}r = \int_{\mathbb{R}} e^{(r^\theta-\rho)\Delta t(1-\alpha)} f(r)\mathrm{d}r. \tag{7.7}$$

For model without the contributions, i.e. $C = 0$, the solution is exact.

Proof 7.1. To prove the statement in Lemma 7.1 we use Lebesgue's Dominated Convergence Theorem. Assuming the utility function (4.3) with risk aversion coefficient $\alpha > 1$ $(1-\alpha < 0)$, for each x and $k \geq 1$ it holds

$$\left(e^{(r^\theta-\rho)\Delta t} + \frac{C\Delta t}{x}\right)^{1-\alpha} f(r) \leq k(e^{(r^\theta-\rho)\Delta t})^{1-\alpha} f(r).$$

By simple operations we obtain that the inequality holds for all $k \geq 1$

$$\begin{aligned} e^{(r-\rho)\Delta t} + \frac{C\Delta t}{x} &\geq k^{\frac{1}{1-\alpha}} e^{(r^\theta-\rho)\Delta t}, \\ \underbrace{e^{(r^\theta-\rho)\Delta t}}_{>0}\underbrace{\left(1-k^{\frac{1}{1-\alpha}}\right)}_{\geq 0 \text{ for } k\geq 1} &\geq -\frac{C\Delta t}{x}. \end{aligned}$$

$\square$

We approximate the value of $V_t(\xi)$ for $\xi > x_{\max}$ as

$$\begin{aligned} V_t(\xi) &= \beta_t U(\xi) \\ \beta_t &= \gamma^{T-t}\beta_T \\ \gamma &= \min_\theta \int_{\mathbb{R}} e^{(r^\theta-\rho)\Delta t(1-\alpha)} f^\theta(r)\mathrm{d}r. \end{aligned} \tag{7.8}$$

For time dependent inflation/growth ρ_t, the function γ_t is also time dependent, i.e. the approximation of $V_t(\xi)$ is defined as

$$\begin{aligned} V_t(\xi) &= \beta_t U(\xi) \\ \beta_t &= \gamma_T\gamma_{T-\Delta t}\cdots\gamma_{t+\Delta t}\beta_T \\ \gamma_t &= \min_\theta \int_{\mathbb{R}} e^{(r^\theta-\rho_t)\Delta t(1-\alpha)} f^\theta(r)\mathrm{d}r. \end{aligned} \tag{7.9}$$

7.2 Interpolation of the value function

The value function of the point not lying on the grid has to be approximated by means of the adjacent grid points. The value of the integral (7.3) for any $\xi \in [x_{\min}, x_{\max}]$ can be expressed by means of the interpolation of values $V_t(x_i)$ and $V_t(x_{i+1})$ corresponding to the closest grid points fulfilling $\xi \in [x_i, x_{i+1}]$. One option is to use the linear interpolation for the approximation of the value $V_t(\xi)$ as

$$V_t(\xi) \approx V_t(x_i) + \frac{V_t(x_{i+1}) - V_t(x_i)}{x_{i+1} - x_i}(\xi - x_i), \tag{7.10}$$

where ξ here is generated by (3.5) and lies in some interval $[x_i, x_{i+1}]$.

We offer another option based on copying the shape of the utility function and its mapping to calculated values. We use the shape of the utility function and compute $V_t(\xi)$ as

$$\begin{aligned} V_t(\xi) &= c_\xi U(\xi), \\ c_\xi &\approx c_{x_i} + \frac{c_{x_{i+1}} - c_{x_i}}{x_{i+1} - x_i}(\xi - x_i), \end{aligned} \tag{7.11}$$

where c_ξ is calculated as a linear interpolation of ratios of V_t and U scaled according to β_t, i.e. $c_{x_i} = \dfrac{V_t(x_i)}{\beta_t U(x_i)}$ and $c_{x_{i+1}} = \dfrac{V_t(x_{i+1})}{\beta_t U(x_{i+1})}$. This approach enables the use of lower space division n_x than by the linear interpolation while achieving the same result.

Remark 7.1. *Since the values inside and outside the space grid point should create a continuous function, the values $V_t(\xi)$ for $\xi > x_{\max}$ have to be parallel shifted such that* *the value $V_t(x_{\max})$ equals to $V_t(x_{\max})$ obtained by (7.9).*

7.3 Numerical algorithm scheme

INPUT

Asset 1 (A_1): $\mathbb{E}_1 = \mathbb{E}(A_1), \mathcal{V}_1 = \mathcal{V}(A_1), \mathcal{S}_1 = \mathcal{S}(A_1), e\mathcal{K}_1 = e\mathcal{K}(A_1)$

Asset 2 (A_2): $\mathbb{E}_2 = \mathbb{E}(A_2), \mathcal{V}_2 = \mathcal{V}(A_2), \mathcal{S}_2 = \mathcal{S}(A_2), e\mathcal{K}_2 = e\mathcal{K}(A_2)$

correlation between Asset 1 and Asset 2 $\rho = corr(A_1, A_2)$

time horizon T, initial capital C_0 and value of regular contribution C

time- , θ- and space- discretization

ALGORITHM

1. generate replicating time series for returns of asset 1 and asset 2 according to algorithm presented in Table 6.4 (based on Choleski algorithm in Table 6.1 or on Optimization algorithm in Table 6.3) and compute NIG parameters for each $\theta_j, j = 1, ..., n_\theta$
2. compute the optimal nodes (r_k^j) and weights (w_k^j) for density function of θ_j-mixture for each $j = 1, ..., n_\theta$ applying the Gauss-Lengedre quadrature with 50 nodes $k = 1, ..., 50$
3. calculate the terminal value $V_T(X) = U(x)$ and set $\beta_T = 1$ and $t = T - \Delta t$
4. for each $x_i, i = 1, ..., n_x$ and for each $\theta_j, j = 1, ..., n_\theta$ calculate $V_t^{\theta_j}(x_i)$ applying following set of steps
   (a) calculate $x_{t+\Delta t} = x_t e^{(r^{\theta_j} - \rho_t)\Delta t} + C\Delta t \equiv \xi^{\theta_j}$ for each $j = 1, ..., n_\theta$
   (b) find interval $[x_i, x_{i+1}]$ such that $\xi^{\theta_j} \in [x_i, x_{i+1}]$
   (c) set approximation value of $V_{t+\Delta t}(\xi^{\theta_j})$ as
      i. $V_{t+\Delta t}(x_{\min})$ for $\xi^{\theta_j} \leq x_{\min}$;
      ii. (7.9) for $\xi^{\theta_j} > x_{\max}$
      iii. (7.11) for $\xi^{\theta_j} \in [x_{\min}, x_{\max}]$

(d) compute approximation for $V_t^{\theta_j}(x_i)$ as

$$V_t^{\theta_j}(x_i) = \sum_{k=1}^{n_r} V_{t+\Delta t}(\xi^{\theta_j}) w_k^j f^{\theta_j}(r_k^j)$$

(e) compute $\gamma_t^{\theta_j}$ as

$$\gamma_t^{\theta_j} = \sum_{k=1}^{n_r} \frac{1}{1-\alpha} U(e^{(r^{\theta_j} - \rho_t)}) w_k^j f^{\theta_j}(r_k^j)$$

set $V_t(x_i)$ and optimal $\hat{\theta}$ at time t as

$$V_t(x_i) = \max_{\theta_j, j=1,\ldots,n_\theta} V_t^{\theta_j}(x_i)$$

$\hat{\theta}(t, x_i) = \theta_j$ maximizing the above defined relation

set $\gamma_t = \min_{\theta_j, j=1,\ldots,n_\theta} \gamma_t^{\theta_j}$ and β_t as $\beta_t = \gamma_t \beta_{t+\Delta t}$

set $t = t - \Delta t$

5 if $t \geq 0$ go to step 4

otherwise go to step 6

6 simulate (Monte Carlo) wealth evolution X_t according to (3.5) and optimal decision $\hat{\theta}(t, X_t)$

Chapter 8

Results

In this chapter we examine the proposed numerical scheme on the saving management problem designed for the II. pillar of the Slovak pension system. We follow the discussed dynamic model in section 3.2 given by (3.7) and maximize the expected utility of the final accumulated wealth of the pensioner. The numerical schemes for solving this problem considering normally distributed portfolio returns have been already discussed in several articles ([29], [32], [43], [44]).

To prove the correctness of our algorithm we adopt the model parameters and first analyze the achieved results for normally distributed asset returns. The aim of this chapter is not be actual, but to show that the achieved results are comparable to results from the literature. We further calculate the required values for skewness and kurtosis of the asset returns and repeat the procedure for the normal inverse Gaussian distribution.

We compare the trajectories of the expected wealth evolution during the investment time and specially the optimal decision for portfolio composition for normal and for NIG distributed portfolio returns. We analyze the distribution of the final wealth and its properties and discuss the impact of the considered skewness and kurtosis of the asset returns.

The considered portfolio is composed from two asset types. Non-risky assets are represented by US 10Y Governmental bond and the risky assets (stocks) by S&P 500 Index. The model is suited for pension planning characterized by the Governmental restriction on ratio of risky to non-risky assets during the saving period. We apply these restrictions for both types of return distribution, i.e. normal and NIG, and conclude our observations.

8.1 Historical time series

We consider the same time period Jan 1996 - Jun 2002 as in Kilianová et al. [32] for time series representing the portfolio assets. The bond yield as a non-risky asset is characterized by small volatility which is 0.82% with expected return 5.16% per year. On the other hand, the risky-assets offer higher yield but under higher risk. The S&P 500 Index in considered time period yields to 10.28% per year with volatility almost 17%. The correlation of their returns is -0.1151.

To apply the NIG distribution we calculate the other two moments, skewness and kurtosis, for both assets. As we expected the bond returns are close to normal distribution, with small skewness and small excess kurtosis, while the stock returns exhibit typical negative skewness and high kurtosis.

The characteristics of the asset returns together with their replication obtained using the Optimization based algorithm with one optimizing parameter ρ (Table 6.3) are presented in Table 8.1. The graphical illustration of asset evolution is drawn on Figure 8.1.

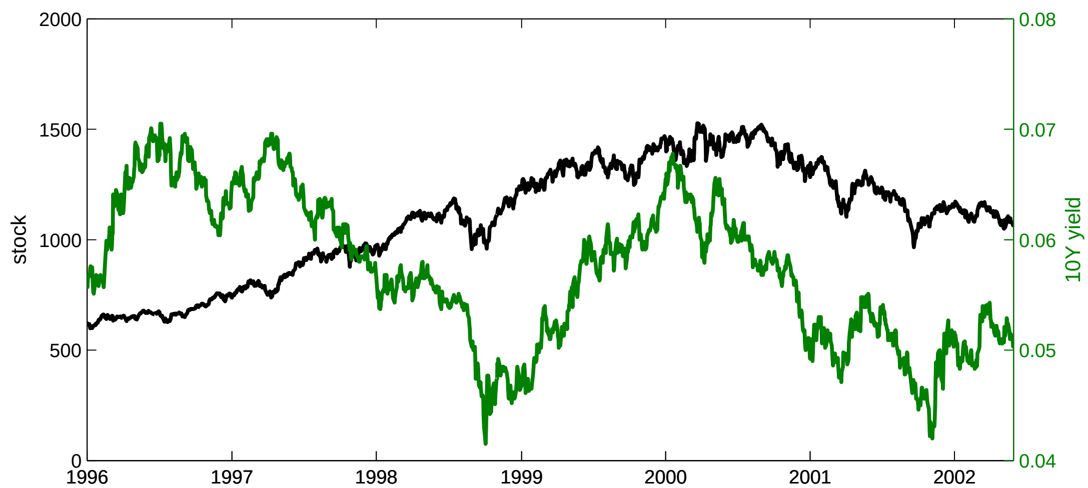


Figure 8.1 Evolution of S&P Index and 10Y US Governmental bond yield during the time period 02/01/1996 - 01/06/2002. (Source: yahoo finance)

The portfolio returns are constructed based on the asset returns and portfolio composition θ. The Figure 8.2 illustrates the dependence of the NIG parameters $\alpha, \beta, \mu, \delta$ of the portfolio return r^{θ}. The composition is expressed through parameter θ representing the proportion of stocks in portfolio. By $\theta = 0$ we understand the portfolio consisting only from bonds and vice verca $\theta = 1$ indicates portfolio composed only from stocks. Figure 8.3 illustrates the dependence of portfolio return moments on stock-to-bond proportion θ. The change in shape of the distribution depicts Figure 8.4. The shape of portfolio return distribution for $\theta = 0$ is the shape of the bond return distribution characterized by a high peak since the variance is very low. Shape for $\theta = 1$ is the stock return distribution which is on the contrary much flatter.

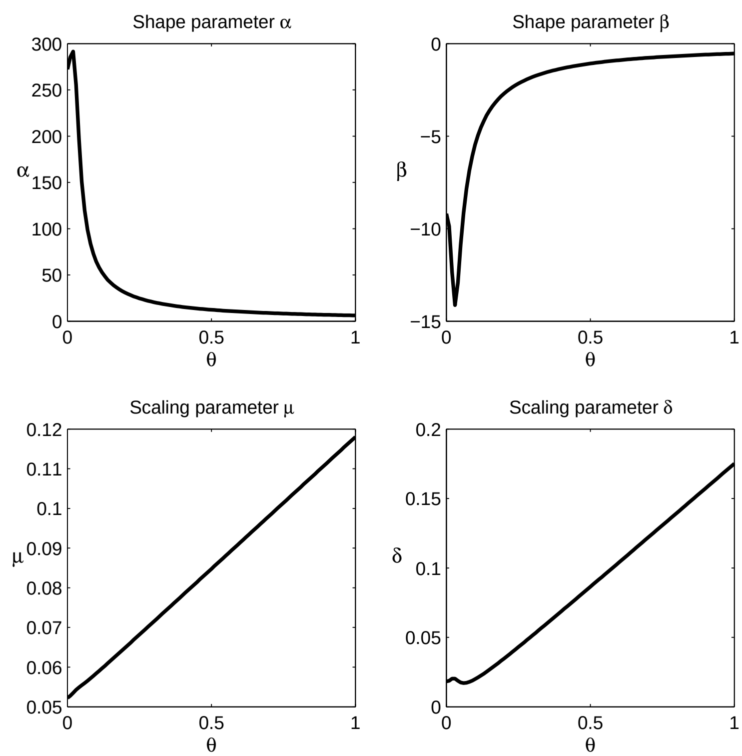


Figure 8.2 Development of the NIG parameters $\alpha, \beta, \mu, \delta$ of portfolio return r^{θ} as a function of stock-to-bond proportion θ. (The replicated time series exhibit the features from Table 8.1 in last column.)

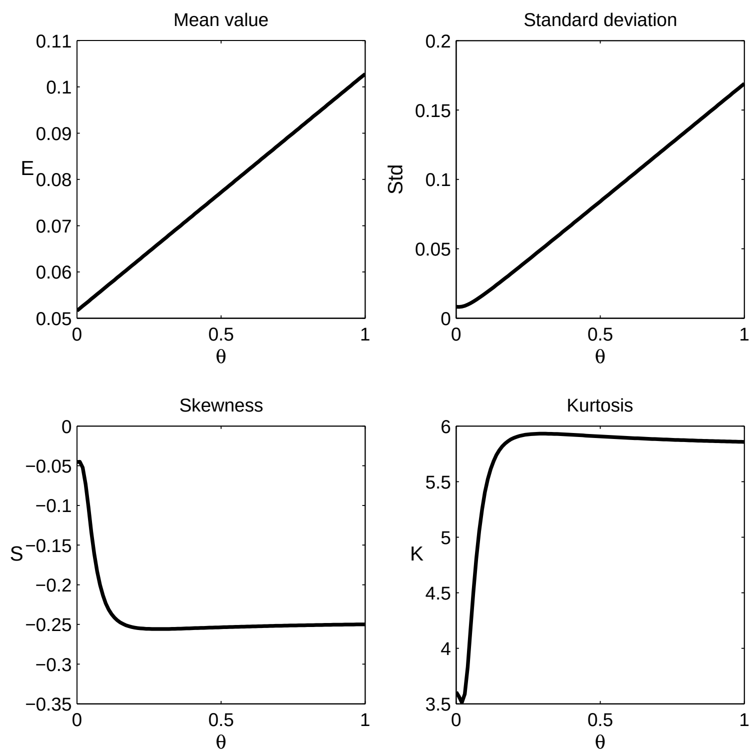


Figure 8.3 Development of the moment parameters $\mathbb{E}, \mathcal{V}, \mathcal{S}$ and $e\mathcal{K}$ of portfolio return r^{θ} as a function of stock-to-bond proportion θ. (The replicated time series exhibit the features from Table 8.1 in last column.)

Table 8.1 Descriptive statistics obtained from historical data series of US 10Y Governmental bonds and S&P500 Index for the time period 02/01/1996 - 01/06/2002 and their replicated characteristic using the Optimization based algorithm with one optimizing parameter ρ (Table 6.3).

		Pre-described	Replicated by Table 6.3
Bond	$\mathbb{E}$	5.16%	5.16%
	$\mathcal{S}td$	0.82%	0.82%
	$\mathcal{S}$	-0.05	-0.0503
	$\mathcal{K}$	3.6	3.5993
Stock	$\mathbb{E}$	10.28%	10.28%
	$\mathcal{S}td$	16.90%	16.90%
	$\mathcal{S}$	-0.24	-0.2448
	$\mathcal{K}$	5.92	5.9229
Correlation	ρ	-0.1151	-0.1153

8.2 Numerical parameters

The numerical parameters are adopted from [32] and are summarized in Table 8.2. We consider a Slovak future pensioner whose retirement time T is in 40 years. According to Slovak pension system (valid in 2006), he contributes into his saving account in II. pillar every month 9% of his salary (the effective contribution is 8.91%). Without loss of generality, we assume that he contributes only once a year 8.91% of his yearly salary. Further we assume that the pension management institutions invest only into two assets - stock and bond with characteristics stated in Table 8.1. The composition of the portfolio is restricted by the Slovak Government by set Θ of all possible portfolio compositions as

$$\theta_t = \begin{cases} [0, 0.8] & \text{if } T - t > 15 \text{ (last 15 years of saving)}, \\ [0, 0.5] & \text{if } T - t > 7 \text{ (last 7 years of saving)}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

The wage growth ρ_t in Slovakia was taken from a paper by Kvetan et al. [35]. The term structure in shown in Figure 8.5.

The mean value $\mathbb{E}(d_t)$ is obtained as an average from 10 000 simulated paths generated for a portfolio with computed optimal stock-to-bond proportion $\hat{\theta}_t$ for each $t = 0, \ldots, T - \Delta t$.

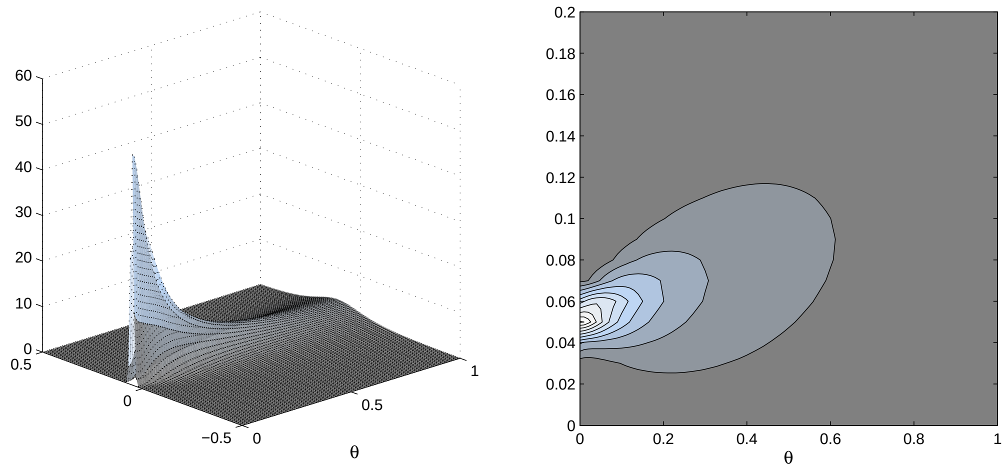


Figure 8.4 Evolution of θ-mixture distribution. For $\theta = 0$ the mixture consists only from bond and $\theta = 1$ indicates 100% representation of stock. (The replicated time series exhibit the features from Table 8.1 in last column.)

Period	Wage growth ρ_t (%)
2006-2008	7.0
2009-2014	7.5
2015-2021	6.5
2022-2024	6.0
2025-2050	5.0

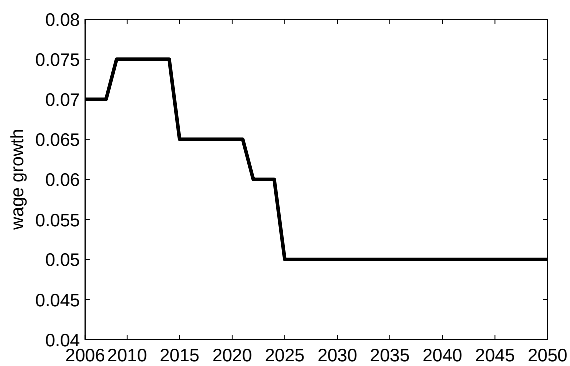


Figure 8.5 Prediction of wage growth in SR (Source: Predictions of SAS).

Table 8.2 Input parameters for the numerical computation.

Parameter	Value
Time horizon T	40
Contribution rate τ	0.0891
Rebalancing time step (Δt)	1
Wage growth ρ_t	Figure 8.5
Utility function type	(4.3)
Risk aversion coefficient α	5
Discretization of θ (n_θ)	100
Discretization of d (n_x)	1 200
Range of wealth d	[0.01,50]

8.3 Normal distribution

We consider the normally distributed asset returns

$$\begin{aligned}
\text{Bonds } r_B \sim N(\mu_B, \sigma_B^2) &\quad \mu_B = 0.0516; \sigma_B = 0.0082;\\
\text{Stocks } r_S \sim N(\mu_S, \sigma_S^2) &\quad \mu_S = 0.1028; \sigma_S = 0.1690;\\
corr(\text{Bonds, Stocks}) &\quad \rho_{BS} = -0.1151;
\end{aligned}$$

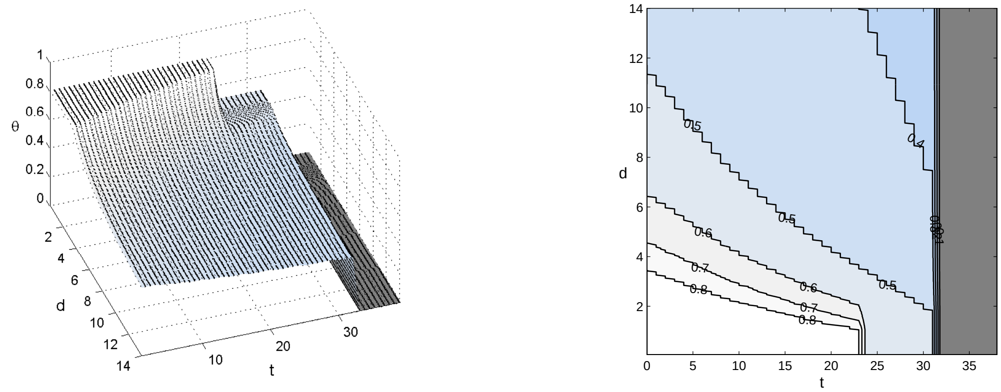
and normally distributed portfolio returns fulfilling the relations

$$\begin{aligned}
\text{Portfolio} \quad & r^\theta \sim N(\mu^\theta, (\sigma^\theta)^2)\\
& \mu^\theta = \theta\mu_S + (1-\theta)\mu_B;\\
& \sigma^\theta = \sqrt{\theta^2\mu_S^2 + 2\rho_{BS}\theta(1-\theta)\mu_S\mu_B + (1-\theta)^2\mu_B^2}.
\end{aligned}$$

We have applied the numerical algorithm to obtain the results for the optimal choice of stock to bond proportion $\hat{\theta}$ in portfolio in each time step $t = 0, \dots, T-1$. The optimal choice for portfolio composition of the future pensioner has the characteristics that in the early years, he prefers a high proportion of risky assets, which decreases with shortening of time to retirement. The pensioner tends to decide for more conservative portfolios in the last years of savings. The preferences change also in dependence on the current accumulated wealth, i.e the higher accumulated sum the lower proportion of the risky assets in portfolio. Due to the Governmental restrictions, the optimal choice is regulated and the decision might be strongly affected.

The Figure 8.6 and Figure 8.7 capture this phenomenon in pair of graphs, 3D graph and a respective contour graph, for optimal choice of proportion θ in dependence on time t and accumulated sum d. As we can observe, by the current model setting, the first Governmental restriction influences the choice only for some range of accumulated sum d, however the second restriction affects the whole range. From contour graphs on the right side it can be easily seen that the pensioner would rather decide for a mix of risky and non-risky assets.

Figure 8.12 (a) illustrates the paths of the mean value $\mathbb{E}(d_t)$ at time t for both cases, applying the restrictions and not. The blue line represents the expected path for restricted case leading to value 5.17, while the black line leading to value 6.13 illustrates the path without applying any restriction. The dashed lines represent the 10% and 90% qunatiles, light blue lines belong to restricted case and light gray to non-restricted case. The paths of optimal choice θ_t evolution are depicted on the right hand graph (b). Again, the blue line represents the restricted case and black line the non-restricted case. One can see that in both cases the saver starts with the most risky investment. Note that the highest possible value of θ, when imposing the governmental regulation, is 0.8. The biggest difference is observed in last 7 years, where by imposing the restrictions, the

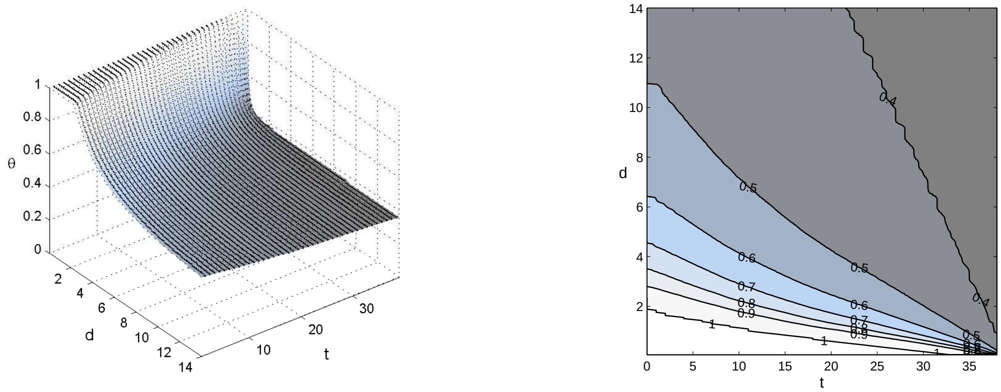


(a) optimal θ in 3D representation (b) optimal θ in contour representation

Figure 8.6 The optimal stock-to-bond proportion θ as a function of time and wealth with normally distributed bond and stock returns applying the Governmental restrictions.

saver is not allowed to invest any part of his savings into risky assets. Here, we observe a shift in $\mathbb{E}(d_t)$.

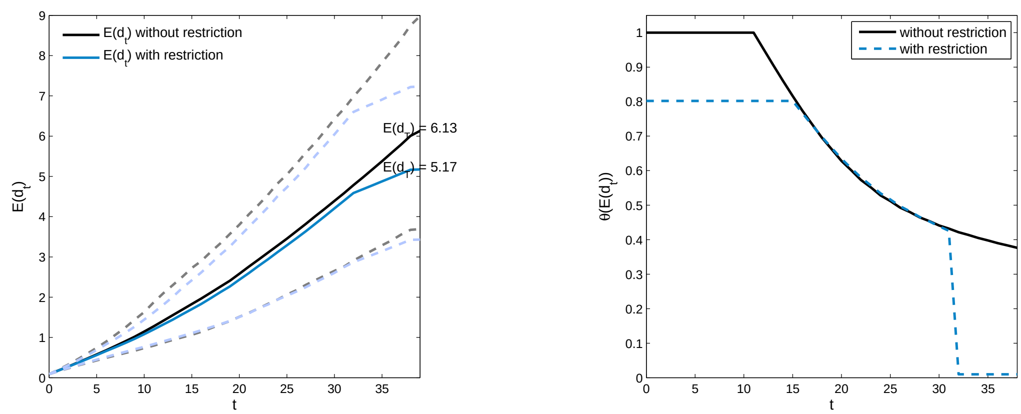
The expected final wealth is higher for non-regulated investments and lead to higher volatility. The distribution of the final accumulated wealth is characterized by high positive skewness since the yearly contribution shifts the savings always in positive sense. The empirical distribution is depicted on Figure 8.9 (a) while the right table (b) states corresponding statistics. We keep the color marking - blue for case with restriction and black for non-restricted case. Since both distributions are by model definition skewed to the right, the volatility is not an appropriate risk measure. From risk point of view it is more interesting to analyze the left tail of the distribution. We can see from the distribution shape and from characteristics listed in the table, that already the 10% quantile is higher for non-regulated case. Additionally, the mean value and the right tail support the advantage of non-regulated investment decisions in comparison to introduced regulations.



(a) optimal θ in 3D representation

(b) optimal θ in contour representation

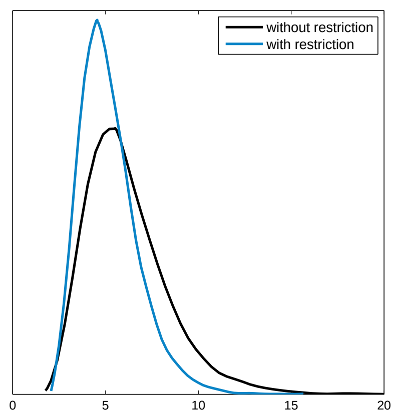
Figure 8.7 The optimal stock-to-bond proportion θ as a function of time and wealth with normally distributed bond and stock returns without restrictions.



(a) evolution of $\mathbb{E}(d_t)$

(b) evolution of optimal θ_t

Figure 8.8 Evolution of the expected accumulated wealth with 10% and 90% quantiles and the optimal θ proportion evolution ($\theta_t(\mathbb{E}(d_t))$) considering normally distributed bond and stock returns with (blue line) and without (black line) applying the Governmental restrictions.



(a) empirical distribution of d_T

	with	without
$\mathbb{E}(d_T)$	5.17	6.13
$\mathcal{S}td(d_T)$	1.55	2.22
$\mathcal{S}(d_T)$	1.03	1.23
$\mathcal{K}(d_T)$	4.85	5.79
$Q_{10\%}(d_T)$	3.43	3.69
$Q_{50\%}(d_T)$	4.92	5.75
$Q_{90\%}(d_T)$	7.23	8.98

(b) statistics of d_T

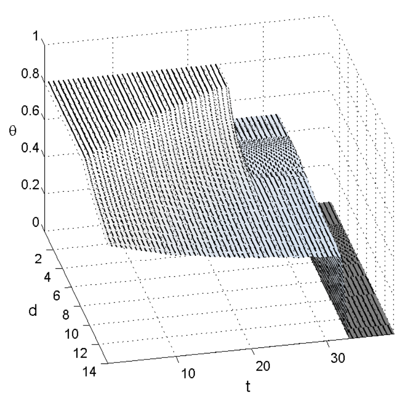
Figure 8.9 Empirical distribution of the wealth at the final time horizon d_T and corresponding statistics when considering normally distributed bond and stock returns with (blue line) and without (black line) applying the Governmental restriction.

8.4 NIG distribution

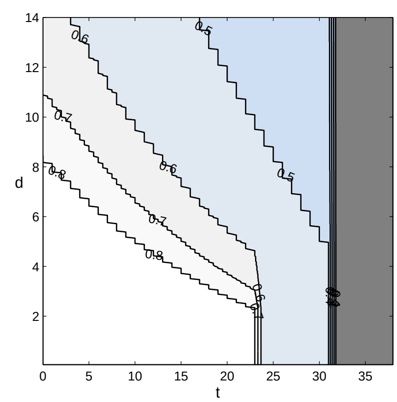
We now consider normal inverse Gaussian distribution for asset returns and for portfolio returns discussed in section 8.1. We apply the same numerical procedure as for the normal distribution and analyze the behavior of the observed characteristic.

The NIG distribution is characterized by fatter tails than the normal distribution with higher the concentration of the observations around the median. The negative skewness present in stock representation indicates longer left tail and higher probability of occurrence of a return much lower than the expectation.

The optimal θ choice with and without applying the regulation can be seen on Figure 8.10 and Figure 8.11, respectively. By both cases we can observe qualitatively same behavior as by normal distribution. The pensioner prefers to invest in the first years higher part of his savings in risky portfolio and with time closer to retirement and higher accumulated sum, he tends to take more conservative decisions and increases the ratio of non-risky assets. However, in comparison to normal distribution keeps higher ratio of risky assets in first years and later reduces its ratio with faster pace.



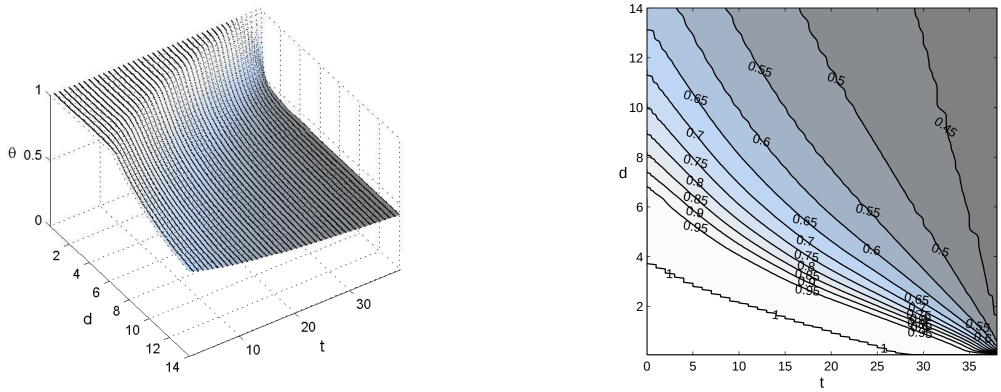
(a) optimal θ in 3D representation



(b) optimal θ in contour representation

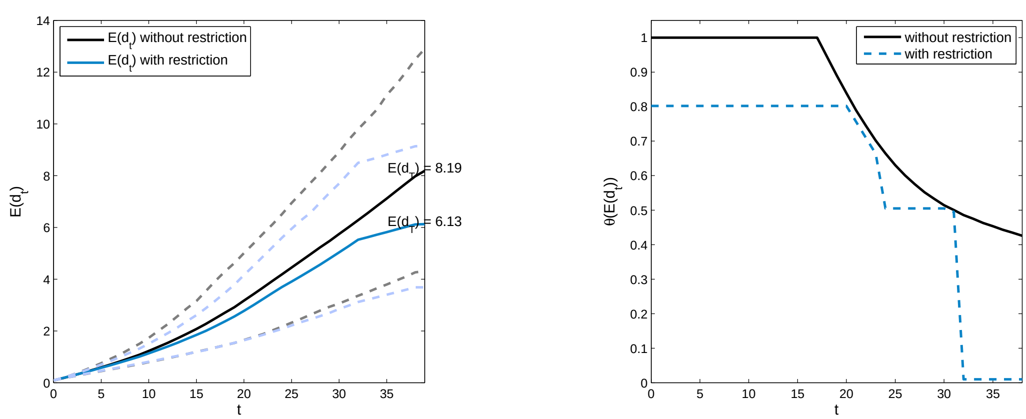
Figure 8.10 The optimal stock-to-bond proportion θ as a function of time and wealth with NIG distributed bond and stock returns applying the Governmental restriction.

Figure 8.12 depicts the expected wealth evolution when applying the regulations, drawn in blue line, and when not, drawn in black line. From the right hand graph we can see the strong effect of the regulations on the decision. Again the regulation lead to lower expected wealth but the value-at-risk on 10% level is higher for non-regulated case. It suggests that the regulations do not lower the risk in this model setting and by considering the NIG distribution for asset returns. The detailed statistics together with the empirical distributions for both cases are stated in Figure 8.13.



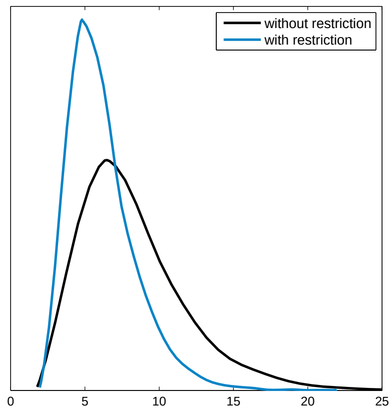
(a) optimal θ in 3D representation (b) optimal θ in contour representation

Figure 8.11 The optimal stock-to-bond proportion θ as a function of time and wealth with NIG distributed bond and stock returns without any regulations.



(a) evolution of $\mathbb{E}(d_t)$ (b) evolution of optimal θ_t

Figure 8.12 Evolution of the expected accumulated wealth with 10% and 90% quantiles and the optimal θ proportion evolution $(\theta_t(\mathbb{E}(d_t)))$ considering NIG distributed bond and stock returns with (blue line) and without (black line) applying the Governmental restrictions.



(a) empirical distribution of d_T

	with	without
$\mathbb{E}(d_T)$	6.13	8.19
$\mathcal{S}td(d_T)$	2.27	3.62
$\mathcal{S}(d_T)$	1.30	1.37
$\mathcal{K}(d_T)$	5.91	6.5
$Q_{10\%}(d_T)$	3.68	4.31
$Q_{50\%}(d_T)$	5.70	7.51
$Q_{90\%}(d_T)$	9.15	12.9

(b) statistics of d_T

Figure 8.13 Empirical distribution of the wealth at the final time horizon d_T and corresponding statistics when considering NIG distributed bond and stock returns with (blue line) and without (black line) applying the Governmental restriction.

From Figure 8.12 (b) we can see another interesting observation coming from comparison of the optimal choice trajectory for normal and for NIG distribution. When considering the NIG distribution, the investor tends to keep the maximal stock-to-bond proportion much longer than by assuming normal distribution, however the choice in the last year of saving is very similar.

Chapter 9

Sensitivity to risk aversion coefficient

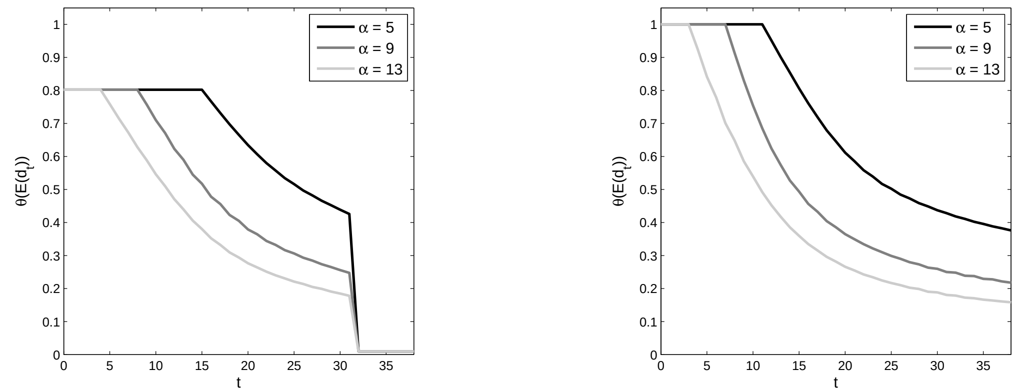
The risk aversion coefficient reflects the investor's attitude to the risk. With increasing risk aversion, we can expect that the investor will tend to take more conservative decisions for same time and level of accumulated sum. In this chapter we provide a sensitivity analysis of the model to level of risk aversion of the future pensioner. We analyze the results for normal and for NIG distribution with and without applying the regulations.

In this chapter we want to show, how the risk aversion coefficient influences the optimal choice of the saver and especially, how it influences the distribution of the final wealth. In the previous chapter, we have already discussed the impact of the regulations on the optimal choice, expectation and on the final wealth distribution properties, which showed the advantage of non-regulated cases for normal and for NIG distribution. Here we continue with comparison of regulated and non-regulated cases and focus on the distribution properties to better understand the point of the regulations.

9.1 Normal distribution

As we expected higher risk coefficient leads to more conservative decisions. Figure 9.1 illustrates the trajectory of the optimal choice of the saver for tree different coefficient levels $\alpha = 5, 9, 13$. By both cases, with (a) and without (b) the regulations we observe that the saver tends to reduce the risky part of the portfolio sooner for higher α. The proportion of stocks reduces with time for all α.

Smaller α indicates riskier portfolio and thus implies a higher expected value of the final wealth $\mathbb{E}(d_t)$, see Figure 9.2. The mean value $\mathbb{E}(d_T)$ is always higher when there are no governmental limits. The empirical distribution of the final wealth for different coefficients α is illustrated on Figure 9.3. Generally, for both cases (a) and (b) the decreasing α flattens the distribution shape. The shape for high α is characterized by

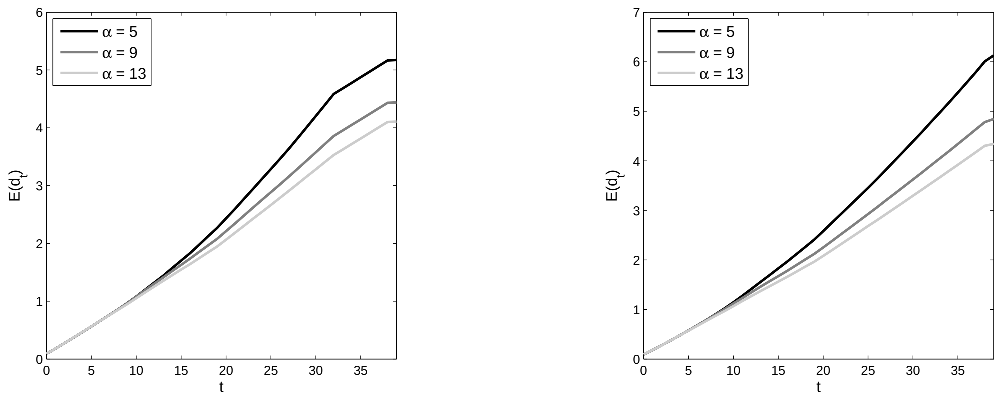


(a) with regulation (b) without regulation

Figure 9.1 Evolution of optimal choice θ for different levels of risk aversion coefficient during the saving period considering normal distribution with and without applying regulations.

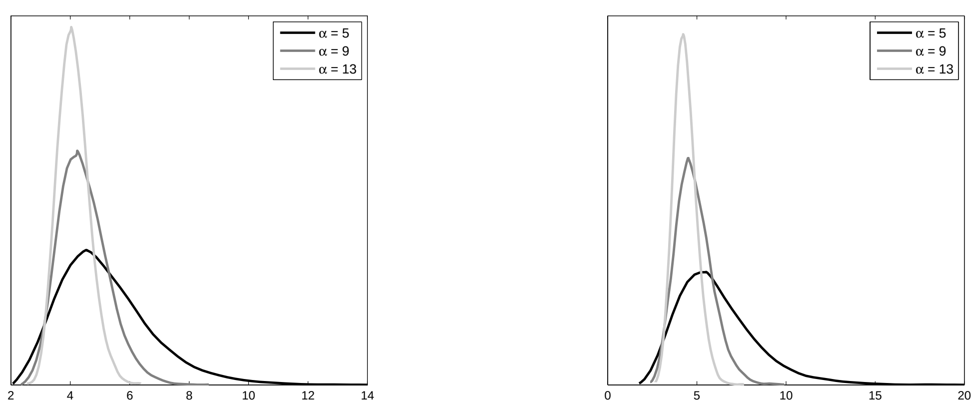
higher and sharper peak implying higher concentration of the observations around the median. The standard deviation is therefore lower as well as the skewness. On the hand, the 10% quantile is comparable for all α coefficients, however the median and 90% quantile are much higher for lower α.

According to the distribution properties listed in Table 9.1, the regulations and also the high risk aversion do not help to reduce the risk from the view of value-at-risk (VaR) in this model setting. If the risk would be measured as an absolute difference between the expectation and VaR, then the regulation can be understood as a defense from saver's disappointment of not succeeding the expectation and especially achieving low value compared to expectation.



(a) with regulation (b) without regulation

Figure 9.2 Evolution of the expected accumulated wealth for different levels of risk aversion coefficient during the saving period considering normal distribution with and without applying regulations.



(a) with regulation (b) without regulation

Figure 9.3 Shape of empirical distribution of the final wealth for different levels of risk aversion coefficient considering normal distribution with and without applying regulations.

Table 9.1 Properties of the final wealth distribution for different levels of risk aversion coefficient considering normal distribution.

	$\alpha = 5$	$\alpha = 9$	$\alpha = 13$
$\mathbb{E}(d_T)$	5.17	4.44	4.11
$\mathcal{S}td(d_T)$	1.55	0.81	0.52
$\mathcal{S}(d_T)$	1.03	0.57	0.42
$\mathcal{K}(d_T)$	4.86	3.45	3.28
$Q_{10\%}(d_T)$	3.43	3.47	3.47
$Q_{50\%}(d_T)$	4.92	4.36	4.07
$Q_{90\%}(d_T)$	7.23	5.50	4.79

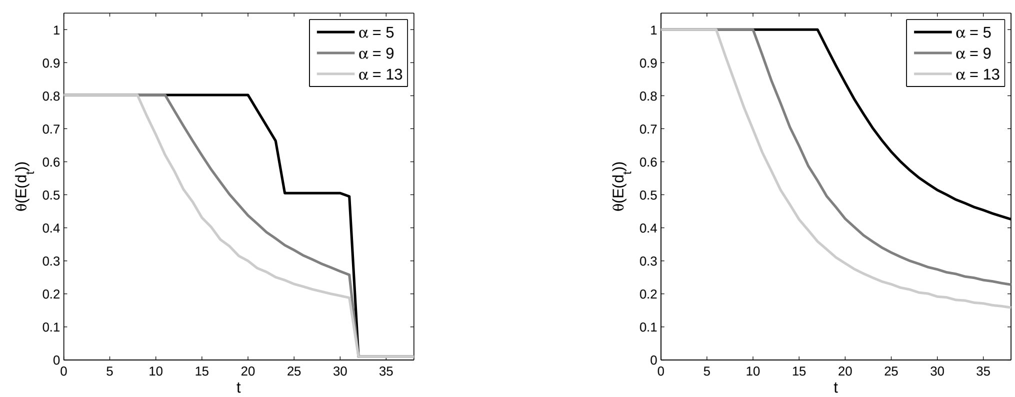
(a) with regulation

	$\alpha = 5$	$\alpha = 9$	$\alpha = 13$
$\mathbb{E}(d_T)$	6.13	4.85	4.34
$\mathcal{S}td(d_T)$	2.22	1.02	0.63
$\mathcal{S}(d_T)$	1.23	0.62	0.48
$\mathcal{K}(d_T)$	5.79	3.65	3.44
$Q_{10\%}(d_T)$	3.69	3.63	3.57
$Q_{50\%}(d_T)$	5.75	4.74	4.30
$Q_{90\%}(d_T)$	8.98	6.19	5.18

(b) without regulation

9.2 NIG distribution

The results for the NIG distribution showed qualitatively same results as by considering the normal distribution. The evolution of the optimal choice θ depicts the saver's preference to switch faster to less risky portfolios when considering higher risk aversion coefficient α, Figure 9.4. This behavior strongly affects the evolution of the expected wealth that exhibits therefore smaller values, Figure 9.5. Figure 9.6 with Table 9.2 similarly as by normal distribution imply, that the reduction in standard deviation with increasing α (or with introducing regulations) is a consequence of a higher concentration of observations around the peak and smaller skewness. However again the 10% quantile is comparable for all α but median and 90% quantile are increasing for decreasing α. On the other hand, the difference of expectation and Var(10%) decreases with higher risk aversion or by introducing regulations, Figure 9.7.

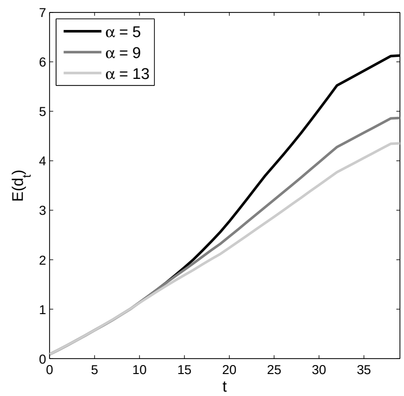


(a) with regulation (b) without regulation

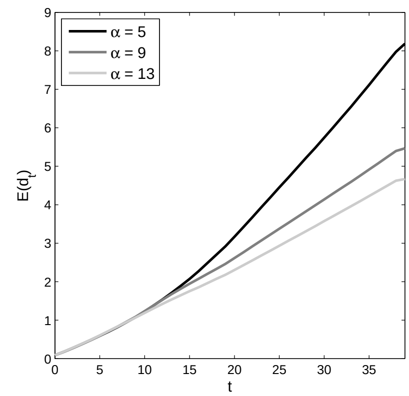
Figure 9.4 Evolution of optimal choice θ for different levels of risk aversion coefficient during the saving period considering NIG distribution with and without applying regulations.

9.3 Comparison: Normal vs. NIG distribution

NIG distribution can better map the asset returns distribution and thus better describe the probabilities of occurrence of individual observations. The asset returns concentrate more around the peak than the normal distribution is able to describe. The difference in shape of the normal and NIG distribution causes the difference of optimal portfolio choice in time and space. Considering NIG distribution the saver prefers to invest in more risky portfolios especially in first years as it can be seen from trajectories of the



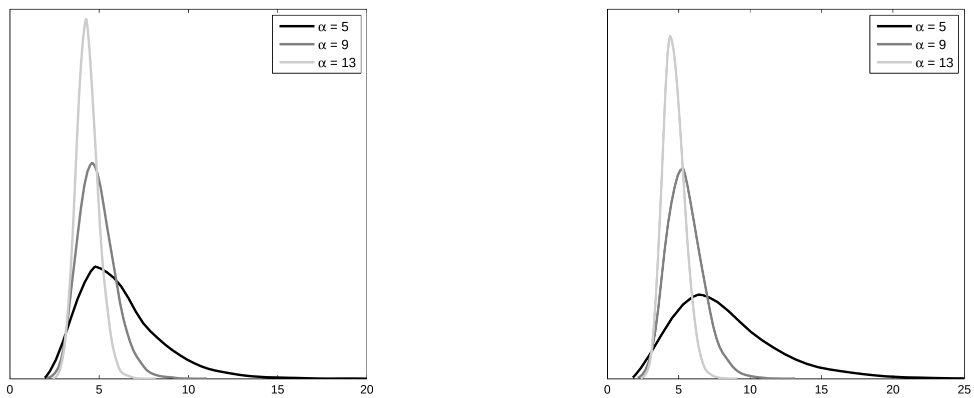
(a) with regulation



(b) without regulation

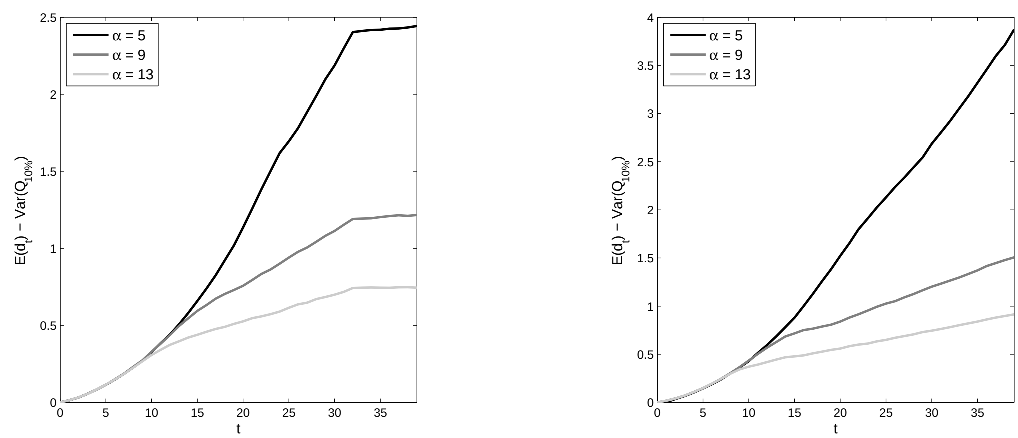
Figure 9.5 Evolution of the expected accumulated wealth for different levels of risk aversion coefficient during the saving period considering NIG distribution with and without applying regulations.

$\theta_t(\mathbb{E}(d_t))$ for all α and also from the contour graphs (Figure 8.7 (b) and Figure 8.11 (b)). If we compare Figure 9.1 and Figure 9.4, we observe that the optimal choice in final time is almost the same for normal and for NIG distribution (for respective α), but the pure stock portfolio is held longer when considering NIG distribution. This is the reason for faster growth of $\mathbb{E}(d_t)$ in first years. The reduction of risky assets follows in subsequent years faster. The comparison of results for normal and NIG distribution (for $\alpha = 9$) can be seen on Figure 9.8.



(a) with regulation (b) without regulation

Figure 9.6 Shape of empirical distribution of the final wealth for different levels of risk aversion coefficient considering NIG distribution with and without applying regulations.



(a) with regulation (b) without regulation

Figure 9.7 Difference between the expectation and Var(10%) for different levels of risk aversion coefficient considering NIG distribution with and without applying regulations.

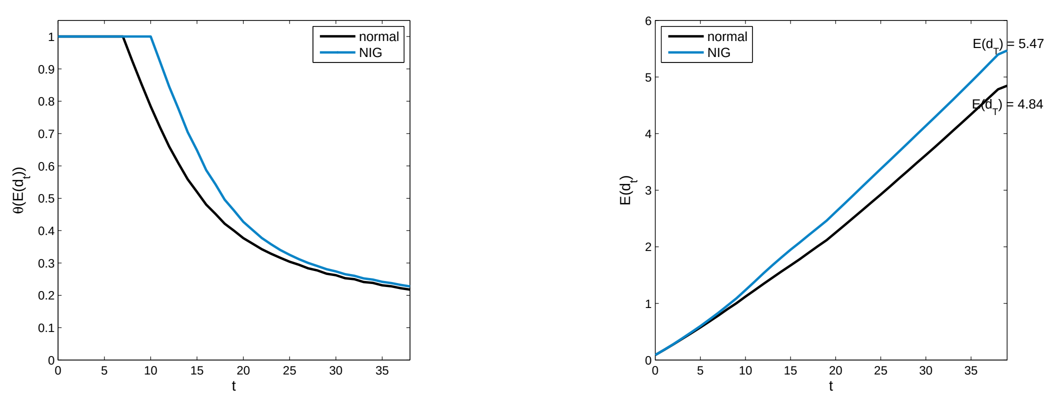
Table 9.2 Properties of the final wealth distribution for different levels of risk aversion coefficient considering NIG distribution.

	$\alpha = 5$	$\alpha = 9$	$\alpha = 13$
$\mathbb{E}(d_T)$	6.13	4.87	4.35
$\mathcal{S}td(d_T)$	2.28	1.01	0.62
$\mathcal{S}(d_T)$	1.30	0.65	0.44
$\mathcal{K}(d_T)$	5.91	3.89	3.57
$Q_{10\%}(d_T)$	3.68	3.65	3.61
$Q_{50\%}(d_T)$	5.70	4.77	4.31
$Q_{90\%}(d_T)$	9.15	6.18	5.16

(a) with regulation

	$\alpha = 5$	$\alpha = 9$	$\alpha = 13$
$\mathbb{E}(d_T)$	8.19	5.47	4.67
$\mathcal{S}td(d_T)$	3.62	1.27	0.75
$\mathcal{S}(d_T)$	1.37	0.67	0.48
$\mathcal{K}(d_T)$	6.50	3.92	3.48
$Q_{10\%}(d_T)$	4.31	3.96	3.75
$Q_{50\%}(d_T)$	7.51	5.34	4.60
$Q_{90\%}(d_T)$	12.90	7.11	5.67

(b) without regulation



(a) $\theta_t(\mathbb{E}(d_t))$ evolution

(b) $\mathbb{E}(d_t)$ evolution

Figure 9.8 Comparison of optimal θ evolution and expected wealth $\mathbb{E}(d_t)$ evolution for normal (black line) and NIG (blue line) distribution for $\alpha = 9$ without regulations.

Chapter 10

Sensitivity to skewness and kurtosis

Considering the NIG distribution of the asset returns and portfolio returns influences the optimal choice of the saver in comparison to considering the normal distribution. We have showed that the saver tends to decide for more risky portfolios in the first years and then fast change the portfolio into more conservative.

In this chapter we analyze the impact of the second two moments of the assets on optimal choice and the final wealth properties. We fix the properties of the asset that is close to normal, such that we can better observe a change caused by just one model parameter. We consider the bonds characteristic - mean, volatility, skewness and kurtosis and stocks characteristics - mean and volatility to be fixed while the stock's skewness and stock's kurtosis will change. We thus analyze the impact of the stock's skewness - negative to positive - for different levels of stock's kurtosis on the optimal choice of the portfolio and properties of the final wealth.

We provide this analysis to show that considering four moments really do influence the saver's preferences during the saving time period.

10.1 Numerical parameters

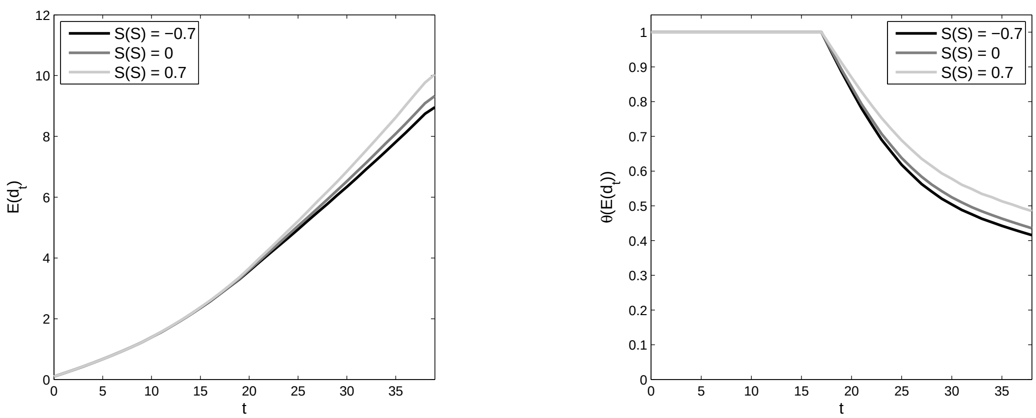
The used numerical parameters are the same as in chapter 8, i.e. we apply parameters from Table 8.2. We do not consider any regulation. Statistics of bonds and stocks are as in Table 10.1.

10.2 Sensitivity on stock's skewness and kurtosis

In this analysis we focus on influence of one model parameter on the optimal choice and wealth evolution and characteristics of the final wealth. We first analyze the sensitivity of the stock's skewness while the other parameters are kept fixed.

Table 10.1 Properties of bonds and stocks used in sensitivity analysis.

	Bond	Stock
$\mathbb{E}$	5.16%	10.28%
$\mathcal{S}td$	0.82%	16.90%
$\mathcal{S}$	-0.05	[-0.7, 0.7]
$\mathcal{K}$	3.6	[4,9]
ρ	-0.1151	



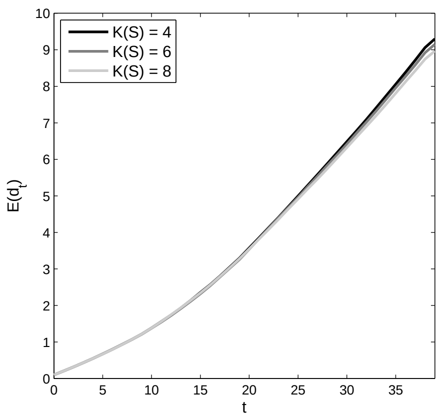
(a) evolution of $\mathbb{E}(d_t)$ (b) evolution of $\theta(\mathbb{E}(d_t))$

Figure 10.1 Evolution of the expected wealth and optimal choice θ for different levels of stock's skewness, while the other parameters are fixed ($\mathcal{K}(S) = 6$).

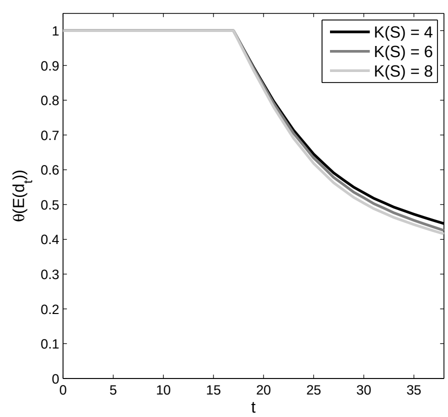
Figure 10.1 illustrates the wealth and optimal choice evolution for different values of stock's skewness, $\mathcal{S}(S) = -0.7, 0, 0.7$ for $\mathcal{K}(S) = 6$. The lower value of skewness leads to more conservative decisions which directly relates to lower expected value of the wealth. We can observe that the time point when the saver starts to add the bonds to portfolio is the same for all skewness values.

The same behavior but for dependence of stock's kurtosis is depicted on Figure 10.2. The skewness is set to $\mathcal{S}(S) = -0.2$ and kurtosis sequentially to $\mathcal{K}(S) = 4, 6, 8$. The higher is the kurtosis the more conservative portfolios are preferred.

The characteristics on the final wealth are listed in Table 10.2. Higher skewness leads to higher expected wealth while the kurtosis lowers it. The 10% quantile increases with skewness and decreases with kurtosis. The same hold for the 90% quantile as it is also shown on Figure 10.3.



(a) evolution of $\mathbb{E}(d_t)$

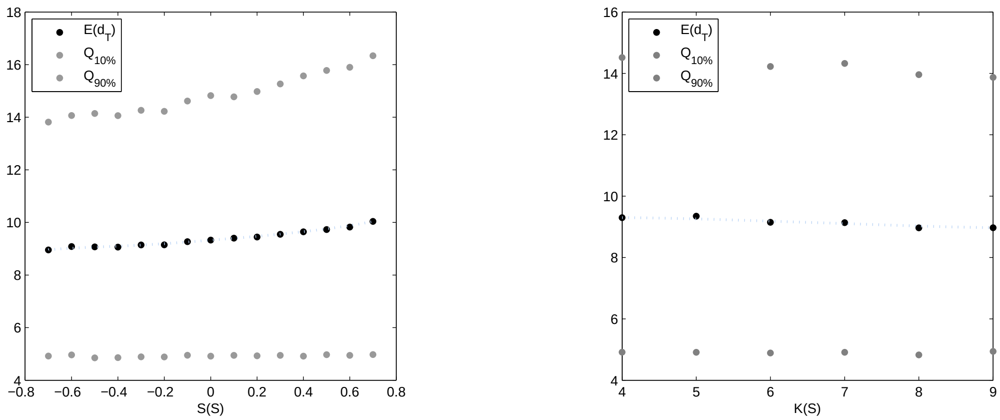


(b) evolution of $\theta(\mathbb{E}(d_t))$

Figure 10.2 Evolution of the expected wealth and optimal choice θ for different levels of stock's kurtosis, while the other parameters are fixed ($\mathcal{S}(S) = -0.2$).

Table 10.2 Properties of the final wealth distribution for different levels of stock's skewness and stock's kurtosis.

	$\mathcal{S}(S) = -0.7$	$\mathcal{S}(S) = 0$	$\mathcal{S}(S) = 0.7$	$\mathcal{K}(S) = 4$	$\mathcal{K}(S) = 6$	$\mathcal{K}(S) = 8$
$\mathbb{E}(d_T)$	8.96	9.33	10.04	9.23	9.15	8.97
$\mathcal{S}td(d_T)$	3.74	4.27	5.17	4.10	4.01	3.89
$\mathcal{S}(d_T)$	1.23	1.53	1.76	1.35	1.45	1.41
$\mathcal{K}(d_T)$	5.84	7.16	7.74	6.29	6.83	6.87
$Q_{10\%}(d_T)$	4.91	4.92	4.98	4.91	4.89	4.83
$Q_{50\%}(d_T)$	8.30	8.45	8.88	8.54	8.40	8.23
$Q_{90\%}(d_T)$	13.81	14.82	16.34	14.52	14.22	13.96



(a) dependence on skewness

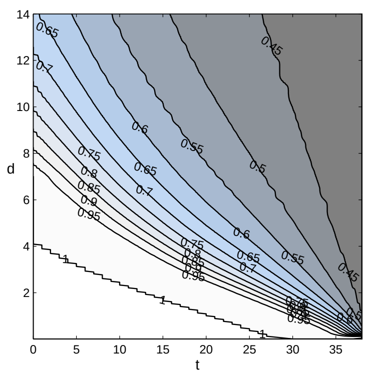
(b) dependence on kurtosis

Figure 10.3 Evolution of the expected wealth at the final time horizon (with 10% and 90% quantile) in dependence of the stock's skewness and kurtosis.

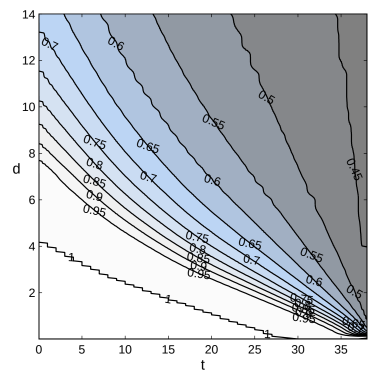
10.3 Optimal choice evolution

In this section we offer a series of contour graphs depicting the dependence of the optimal choice θ on time t and level of savings d.

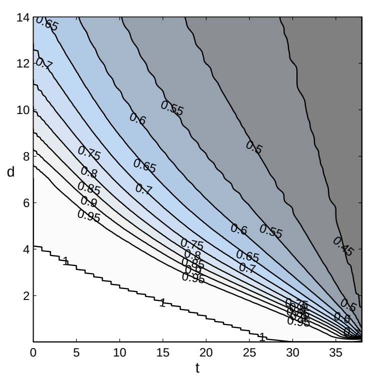
We can observe that for skewness the contours move clockwise, Figure 10.4 on the left side, while for the kurtosis the contours move counterclockwise, Figure 10.4 on the right. It indicates that higher skewness supports higher proportion of stocks in portfolio and on the contrary higher kurtosis supports higher proportion of bonds.



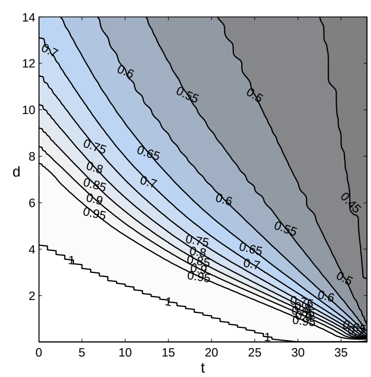
(a) skewness of stock $\mathcal{S} = -0.7$



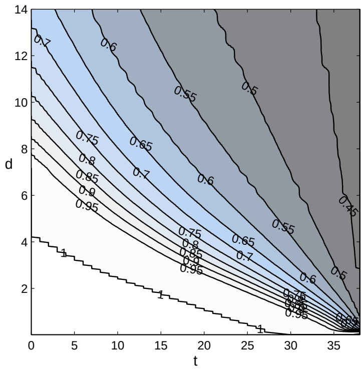
(b) kurtosis of stock $\mathcal{K} = 4$



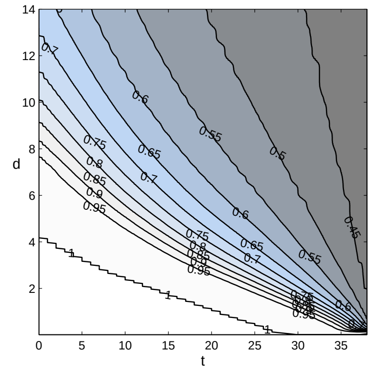
(c) skewness of stock $\mathcal{S} = -0.4$



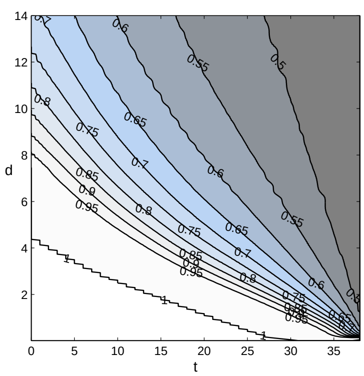
(d) kurtosis of stock $\mathcal{K} = 5$



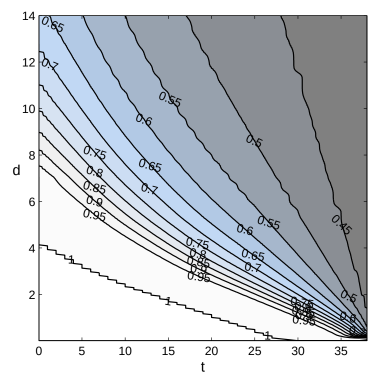
(e) skewness of stock $\mathcal{S} = 0$



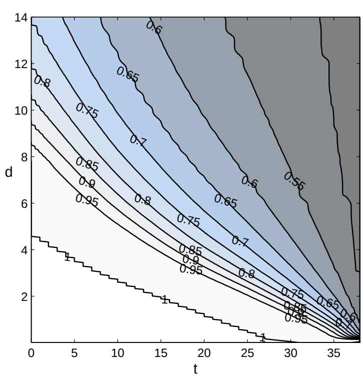
(f) kurtosis of stock $\mathcal{K} = 6$



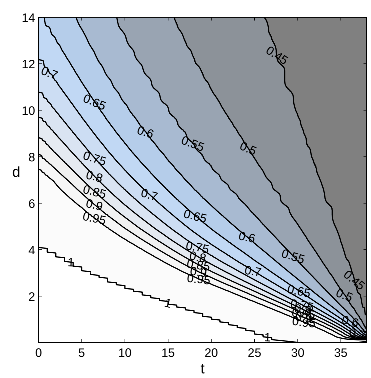
(g) skewness of stock $\mathcal{S} = 0.4$



(h) kurtosis of stock $\mathcal{K} = 7$



(i) skewness of stock $\mathcal{S} = 0.7$



(j) kurtosis of stock $\mathcal{K} = 8$

Figure 10.4 Evolution of the optimal θ in dependence of the stock's skewness and kurtosis.

10.4 Sensitivity on skewness and kurtosis in 3D

This section illustrates the sensitivity of the characteristics of the final wealth on stock's skewness and kurtosis while keeping the other parameters fixed. The following graphs depict the dependencies that have been observed and discussed in this chapter but allow to view for different combinations of skewness and kurtosis of the stock.

All the following graphs illustrate the estimated curve of order 3 through the calculated points. Figure 10.5 offers the view on the $\mathbb{E}(d_T)$. Generally, the expectation grows with skewness and declines with kurtosis. The relations are almost linear. The very same behavior shows also the volatility $\mathcal{V}(d_T)$, Figure 10.6. Skewness of the final wealth on Figure 10.7 and kurtosis of the final wealth on Figure 10.8 grow with the stock's skewness and also grow with the stock's kurtosis.

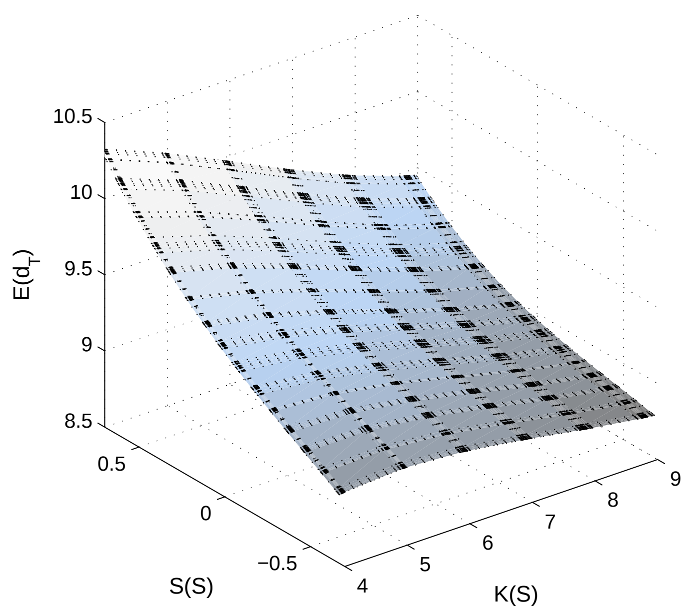


Figure 10.5 Dependence of expected wealth at the final time horizon $\mathbb{E}(d_T)$ in dependence of the stock's skewness and kurtosis.

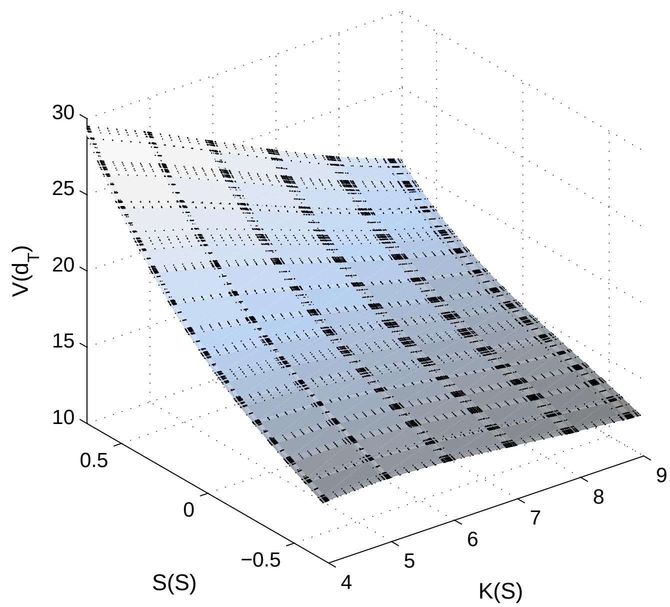


Figure 10.6 Dependence of wealth variane at the final time horizon $\mathcal{V}(d_T)$ in dependence of the stock's skewness and kurtosis.

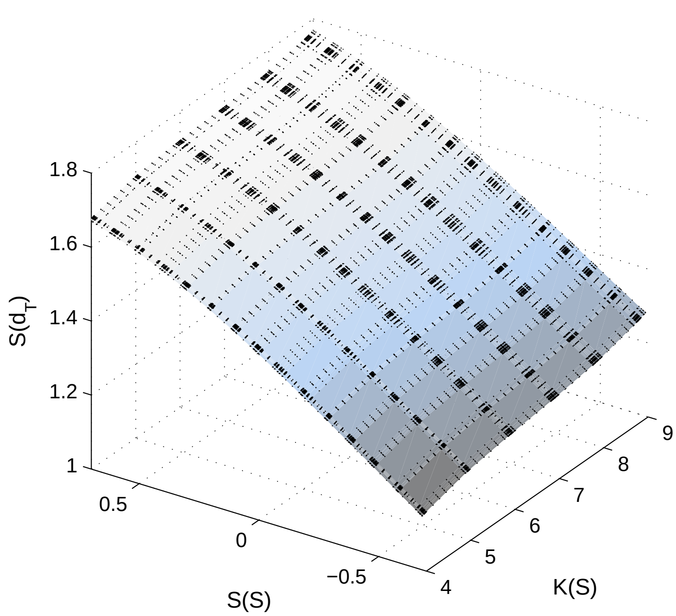


Figure 10.7 Dependence of wealth skewness at the final time horizon $\mathcal{S}(d_T)$ in dependence of the stock's skewness and kurtosis.

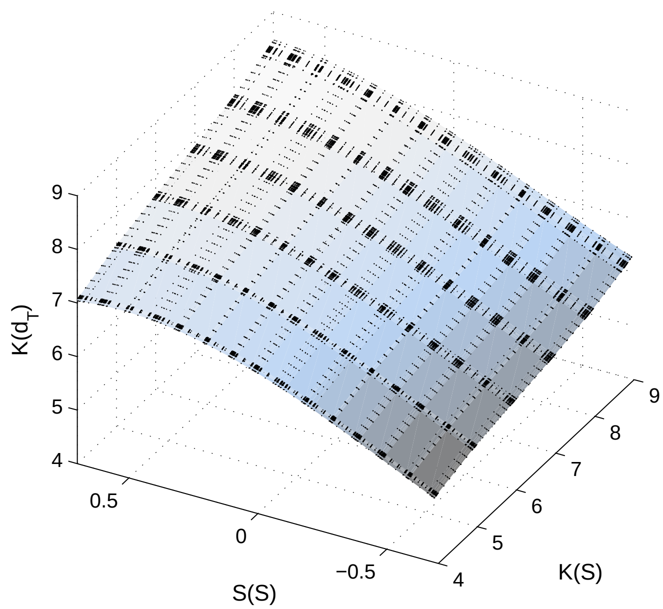


Figure 10.8 Dependence of wealth kurtosis at the final time horizon $\mathcal{K}(d_T)$ in dependence of the stock's skewness and kurtosis.

Chapter 11

Conclusion

In the thesis we have presented an investment model with regular contributions for determining the optimal investment opportunity. We have analyzed the model for needs of the risk management in pension system formulated as a dynamic stochastic accumulation model for determining the optimal value of the stock to bond proportion in the pension saving decision. The model has been formulated by Kilianová, Melicherčík, Ševčovič in series of articles [29], [31], [32], etc. The adopted model has been analyzed considering the normally distributed returns. However, voluminous literature alerts that by modeling returns more extreme changes should be taken into account. We aimed to introduce the fat tailed distributions for asset returns' modeling and analyze the impact on the saver's preferences.

We have focused on the normal inverse Gaussian distribution among the generalized hyperbolic distributions and studied the S&P500 Index in more detail and similarly to findings of Onalan in [48], we have concluded that the log returns do not follow the normal distribution. They are characterized by higher kurtosis than is typical for the normal distribution and are skewed to the left. The NIG distribution has four parameters which can be used to express first four moments. We have showed how the parameters as well as the moments affect the density shape.

The biggest drawback of NIG distribution follows directly from its convolution property defined only for independent variables exhibiting same shape parameters which is a very restrictive condition. We therefore study the behavior of generating the dependent NIG random variables using the approach usually applied for generating the dependent normally distributed variables, i.e by means of Choleski decomposition. We have analyzed the properties of replicated time series defined by their moments and based on our observations we have introduced the algorithm for generating such mixtures. The crucial parameters showed to be the correlation and the kurtosis.

The NIG distribution exhibits a specific shape and requires also a special treatment by numerical approximation of the formulated Belmann problem. The numerical procedure has to be specified with a focus on needs of NIG distribution or generally distribution with fat tails and high kurtosis. Due to the high kurtosis the peak is thin and high and the numerical approximation of the integral requires different approach than the classical trapezoidal rule with uniform spacing grid. We offered the Legendre-Gauss quadrature based on approximation of the integral by the weighted sum of function values at specified points within the domain of integration. The next NIG property - a fat tail requires bigger domain for the space grid which might be computationally expensive. The need of the proper boundary condition is therefore very strong. Since the linear extrapolation or the constant for points outside the grid showed to be not sufficient, we have derived a heuristic of a boundary condition specific for our type of problem. We have also offered a special interpolation inside the grid based on the shape of the utility function allowing the use of a less dense grid.

The results of our algorithm for the formulated problem suitable for distributions with fat tails were provided with the numerical parameters adopted from [32] in order to demonstrate the algorithm's correctness. We calculated the next two moments of the assets representing the stocks and bonds in portfolio and showed the impact of adding the skewness and the kurtosis on the optimal choice of the saver during the saving time period.

Considering the NIG distribution for the asset returns, the saver tends to keep the maximal stock-to-bond proportion much longer than by assuming normal distribution, however the choice in the last decision year is comparable. This causes that the expected wealth grows faster in first years and in years, when the portfolio is more conservative, the growth does not have to be so high to higher the absolute value of the savings. The expected wealth by considering the NIG distribution is therefore higher than by normal distribution.

The regulation introduced by the Government strongly influences the choice of the saver who would by this parameter setting decide for more risky portfolio. The expected final wealth is lower for regulated investments but lead to lower volatility. In our study we have focus mainly on the distribution of the final accumulated wealth which is characterized by positive skewness since the yearly contribution shifts the savings always in a positive sense. The volatility is therefore not an appropriate risk measure. From risk point of view it is more interesting to analyze the left tail of the distribution, the value-at-risk measure. From the distribution shape for regulated and non-regulated case and calculated quantiles we can observe that the regulation does not lower the risk. However the difference between the expected value and VaR is higher for non-regulated case. Considering this risk measure the regulation can be understood as a defense from saver's disappointment of not succeeding the expectation and especially of achieving low value compared to expectation. The more regulations influence the choice of the saver the higher is the difference in peak settlement and quantiles of the final wealth distributions.

The same qualitative behavior can be observed by both examined distributions and also for different risk aversion coefficients. As expected, with increasing risk aversion the investor tends to take more conservative decisions for same time and level of accumulated sum. More conservative portfolio leads to lower expected final wealth and lower volatility.

The aim of the thesis was to study the impact of the skewness and the kurtosis of the portfolio assets on the optimal choice of the investor and on his expectation. In our sensitivity analysis we consider again the same numerical and model parameters. We fix the properties of one asset and change the skewness and kurtosis of the second asset in order to better observe a change caused by just one model parameter. We consider the bonds characteristic - mean, volatility, skewness and kurtosis and stocks characteristics - mean and volatility to be fixed while the stock's skewness and stock's kurtosis change. We have showed that the lower value of skewness leads to more conservative decisions leading to lower expected value of the wealth. The time point when the saver starts to add the bonds to portfolio keeps the same for all skewness values. On the other hand lower kurtosis asks for more risky portfolio. The results of the sensitivity analysis are depicted in detailed tables and set of figures illustrating the evolution of the optimal choice, expected wealth, final wealth distribution and its properties. The optimal decision in dependence on level of accumulated sum and on time are illustrated on the contour graphs for different stock's skewness and kurtosis and show the above described behavior. The results are supported by 3D graphs expressing the dependence on different combinations of stock's skewness and kurtosis.

In our thesis we defined an algorithm for formulated problem suitable for distribution also exhibiting fatter tails. We have showed that considering higher moments of the portfolio assets influence the optimal decision of an investor and thus his expectations.

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Appendix A

NIG distribution and its parameters

The goal of this chapter is to analyze the NIG distribution. The NIG distribution belongs to a family of generalized hyperbolic distributions. It is characterized by four parameters $\alpha, \beta, \mu, \delta$ specifying the shape of the density function. Barndorff-Nielsen [4] defined the NIG distribution as a normal variance-mean mixtures when the mixture distribution is a inverse Gaussian distribution.

Definition 6. *The random variable X is normal inverse Gaussian distributed $NIG(\alpha, \beta, \mu, \delta)$ if its probability density function is given by*

$$f(x) = \frac{\alpha}{\pi} \exp\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\} \frac{K_1(\alpha\delta\sqrt{1 + (\frac{x-\mu}{\delta})^2})}{\sqrt{1 + (\frac{x-\mu}{\delta})^2}} \tag{A.1}$$

where K_1 denotes the modified Bessel function of the third kind, and the conditions for the parameters are $\alpha > 0, \delta > 0, \mu \in \mathbb{R}, 0 \leq |\beta| \leq \alpha$.

The first four moments of the NIG distributed time series can be simply calculated from the four NIG parameters as in

Theorem A.1. *The first four moments, mean $\mathbb{E}$, variance $\mathcal{V}$, skewness $\mathcal{S}$ and excess kurtosis $e\mathcal{K}$, of the NIG distribution can be expressed using the four parameters as follows*

$$\mathbb{E}(X) = \mu + \delta \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \tag{A.2}$$

$$\mathcal{V}(X) = \delta \frac{\alpha^2}{(\alpha^2 - \beta^2)^{\frac{3}{2}}} \tag{A.3}$$

$$\mathcal{S}(X) = 3\frac{\beta}{\alpha\sqrt{\delta\sqrt{\alpha^2-\beta^2}}} \tag{A.4}$$

$$e\mathcal{K}(X) = 3\frac{(1+4\frac{\beta^2}{\alpha^2})}{\delta(\alpha^2-\beta^2)^{\frac{1}{2}}}. \tag{A.5}$$

A.1 Moment computation

The first four moments can be calculated from the historical time series data. We state the relations here for reader's convenience.

Theorem A.2. *We compute the four moments from the observed data x_i for $i = 1, ..., N$ as*

$$\mathbb{E} = \frac{1}{N}\sum_{i=1}^{N} x_i = \bar{x} \tag{A.6}$$

$$\mathcal{V} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 = \bar{\sigma}^2 \tag{A.7}$$

$$\mathcal{S} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{x_i-\bar{x}}{\sigma}\right)^3 \tag{A.8}$$

$$e\mathcal{K} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{x_i-\bar{x}}{\sigma}\right)^4 - 3 \tag{A.9}$$

A.2 Derivation of NIG parameters

The first four moments are defined by means of the first four moments of the modeled time series. Similarly, the four NIG parameters can be calculated under some conditions from the first four moments. We offer here the derivation of NIG parameters from 5.2.

Proof A.1 (Derivation of NIG paramters)**.** The relation for skewness in form (A.4) can be transformed to

$$\mathcal{S}^2 = 9\frac{\beta^2}{\alpha^2\delta(\alpha^2-\beta^2)^{\frac{1}{2}}}.$$

Now we define $\xi = \frac{\beta}{\alpha}$, i.e. $(\alpha^2 - \beta^2)^{1/2} = \alpha(1-\xi^2)^{\frac{1}{2}}$ and substituting into previous relation we obtain

$$\mathcal{S}^2 = 9\frac{\beta^2}{\alpha\delta(1-\xi^2)^{\frac{1}{2}}}. \tag{A.10}$$

The relation (A.5) by means of ξ can be modified as

$$e\mathcal{K} = 3\frac{1+4\xi^2}{\alpha\delta(1-\xi^2)^{\frac{1}{2}}} \quad \Rightarrow \quad \frac{3}{\alpha\delta(1-\xi^2)^{\frac{1}{2}}} = \frac{e\mathcal{K}}{1+4\xi^2} \tag{A.11}$$

Combining (A.10) and (A.11) we can write

$$\mathcal{S}^2 = 3\xi^2\frac{3}{\alpha\delta(1-\xi^2)^{\frac{1}{2}}} = 3\xi^2\frac{e\mathcal{K}}{1+4\xi^2}$$

It follows easily that

$$\xi = \frac{\mathcal{S}}{\sqrt{3e\mathcal{K}-4\mathcal{S}^2}} \tag{A.12}$$

Relation (A.3) can be expressed by ξ and (A.10) as

$$\begin{aligned}\mathcal{V} &= \frac{\delta}{\alpha(1-\xi^2)^{\frac{3}{2}}} = \frac{\delta}{\alpha(1-\xi^2)}\frac{1}{(1-\xi^2)^{\frac{1}{2}}}\\ &= \frac{\delta^2 S^2}{9\xi^2(1-\xi^2)}\end{aligned}$$

Than the δ is

$$\begin{aligned}\delta &= \frac{\sqrt{\mathcal{V}}}{\mathcal{S}}3\xi(1-\xi^2)^{\frac{1}{2}} = 3\sqrt{\mathcal{V}}\frac{1}{\sqrt{3e\mathcal{K}-4\mathcal{S}^2}}\frac{\sqrt{3e\mathcal{K}-5\mathcal{S}^2}}{\sqrt{3e\mathcal{K}-4\mathcal{S}^2}}\\ \delta &= 3\sqrt{\mathcal{V}}\frac{\sqrt{3e\mathcal{K}-5\mathcal{S}^2}}{3e\mathcal{K}-4\mathcal{S}^2}\end{aligned}$$

Substituting δ to (A.3) we obtain relation for α

$$\alpha = \frac{\delta}{\mathcal{V}}\frac{1}{(1-\xi^2)^{\frac{3}{2}}} \tag{A.13}$$

$$= \frac{3}{\sqrt{\mathcal{V}}}\frac{(3e\mathcal{K}-5\mathcal{S}^2)^{\frac{1}{2}}}{(3e\mathcal{K}-4\mathcal{S}^2)}\frac{(3e\mathcal{K}-4\mathcal{S}^2)^{\frac{3}{2}}}{(3e\mathcal{K}-5\mathcal{S}^2)^{\frac{3}{2}}} \tag{A.14}$$

$$= \frac{3}{\sqrt{\mathcal{V}}}\frac{(3e\mathcal{K}-4\mathcal{S}^2)^{\frac{1}{2}}}{(3e\mathcal{K}-5\mathcal{S}^2)} \tag{A.15}$$

Finally we have α given by (A.15), δ from (A.13) given as $\delta = \alpha\mathcal{V}\left(1-\frac{\beta^2}{\alpha^2}\right)^{\frac{3}{2}}$, β as product of $\alpha\xi$, where ξ is given by (A.12) and μ easily from (A.2) as $\mathcal{M}(X) - \delta\frac{\beta}{\sqrt{\alpha^2-\beta^2}}$. $\square$

A.3 Fitness of mean and variance

We aim to construct a random variable Y by means of a random variable Z, $Z \sim NIG(\alpha\delta, \beta\delta, 0, 1)$ with pre-defined $\mathbb{E}_Y$ and $\mathcal{V}_Y$. For Y defined as $Y = Z\sigma + r$ and for arbitrary r, σ it holds

$$\begin{aligned}
\mathbb{E}(Y) =& \mathbb{E}(Z\sigma + r) = \mathbb{E}(Z)\sigma + r,\\
\mathcal{V}(Y) =& \mathcal{V}(Z\sigma + r) = \mathcal{V}(Z)\sigma^2,\\
\mathcal{S}(Y) =& \mathcal{S}(Z\sigma + r) = \mathcal{S}(Z),\\
e\mathcal{K}(Y) =& e\mathcal{K}(Z\sigma + r) = e\mathcal{K}(Z).
\end{aligned}$$

The constructed variable Y keeps the skewness and the kurtosis of Z and pre-defined mean and variance of Y are obtained with the appropriate scaling and shifting of variable Z as following

$$\begin{aligned}
\mathcal{V}_Y =& \mathcal{V}(Y) = \mathcal{V}(Z)\hat{\sigma}^2,\\
\mathbb{E}_Y =& \mathbb{E}(Y) = \mathbb{E}(Z)\sigma + \hat{r}.
\end{aligned}$$

The parameters $\hat{r}, \hat{\sigma}$ take the form

$$\begin{aligned}
\hat{\sigma} =& \sqrt{\frac{\mathcal{V}_Y}{\mathcal{V}(Y)}},\\
\hat{r} =& \mathbb{E}_Y - \hat{\sigma}\mathbb{E}(Y).
\end{aligned}$$

Appendix B

Legendre-Gauss Quadrature

In this appendix we recall the details of Lengedre-Gauss quadrature for the reader's convenience. Legendre-Gauss quadrature is a numerical integration method also called "the" Gaussian quadrature or Legendre quadrature described by Hildebrand in [26]. In a general Gaussian quadrature rule, an definite integral of $f(x)$ is first approximated over the interval $[1, 1]$ by a polynomial approximate function $g(x)$ and a known weighting function $W(x)$, i.e.

$$\int_{-1}^{1} f(x)\mathrm{d}x = \int_{-1}^{1} W(x)g(x)\mathrm{d}x.$$

In a case of Legendre-Gauss quadrature, the weighting function $W(x) = 1$ over the interval $[-1, 1]$, i.e.

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} w_i f(x_i).$$

The abscissas for quadrature order n are given by the roots of the Legendre polynomials $P_n(x)$, which occur symmetrically about 0. The weights are

$$w_i = -\frac{A_{n+1}\gamma_n}{A_n P_n'(x_i) P_{n+1}(x_i)} = \frac{A_n}{A_{n-1}} \frac{\gamma_{n-1}}{P_{n-1}(x_i) P_n'(x_i)},$$

where A_n is the coefficient of x^n in $P_n(x)$. For Legendre polynomials,

$$A_n = \frac{(2n)!}{2^n (n!)^2},$$

so

$$\frac{A_{n+1}}{A_n} = \frac{[2(n+1)]!}{2^{n+1}[(n+1)!]^2} \frac{2^n (n!)^2}{(2n)!} = \frac{2n+1}{n+1}.$$

Additionally,

$$\gamma_n = \int_{-1}^{1} [P_n(x)]^2 \mathrm{d}x = \frac{2}{2n+1}$$

implying that

$$w_i = -\frac{2}{(n+1)P_{n+1}(x_i)P_n'(x_i)} = \frac{2}{nP_{n-1}(x_i)P_n'(x_i)}.$$

Using the recurrence relation

$$(1-x^2)P_n'(x) = -nxP_n(x) + nP_{n-1}(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x)$$

gives

$$w_i = \frac{2}{(1-x_i^2)[P_n'(x_i)]^2} = \frac{2(1-x_i^2)}{(n+1)^2[P_{n+1}(x_i)]^2}.$$

The weights w_i satisfy $\sum_{i=1}^{n} w_i = 2$, which follows from the identity

$$\sum_{\nu=1}^{n} \frac{1-x_\nu^2}{(n+1)^2[P_{n+1}(x_\nu)]^2} = 1.$$

The error term is

$$E = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi).$$