Risk adjusted dynamic hedging strategies

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Abstract The aim of the paper is to develop a dynamic portfolio hedging strategy leading to an optimal wealth policy in a finite investment horizon while obeying a risk constraint. The utility maximization problem is restricted by an upper bound applied on the Conditional Value-at-Risk (CVaR) measure. We investigate the strategy dynamics and properties in terms of the desired wealth distribution and risky assets exposure.

Key words: dynamic strategy, Conditional Value-at-Risk, complete market

1 Market Settings

We consider a financial market with N *risky* assets with random returns and one risk-free asset with deterministic yield. The dynamics of market prices follow the system of N stochastic and one ordinary differential equations $dS(t) = S(t)\mu(t)dt + S(t)\sigma(t)dw(t)$ and $dB(t) = B(t)r(t)dt$, where $\mu(t)$ is the vector of drifts, $\sigma(t)$ is the volatility matrix and $r(t)$ is the deterministic bond yield. The process $w(t)$ is an N-dimensional standardised Brownian motion.

Let $W(t)$ be a value of the portfolio at time t. The investor chooses the investment horizon T and the investment strategy $\theta(t)$, which represents the fraction of wealth invested in each risky asset at time $t \in (0,T]$. Then the portfolio value follows the stochastic differential equation $dW(t) = W(t)\theta(t)^\top \left(\mu(t)dt + \sigma dw(t)\right) + W(t)\left(1 - \theta(t)^\top \underline{1}\right) r(t)dt$, where $\underline{1} \equiv (1,1,\cdots,1)^\top$. We assume that market is *complete*. Such an assumption implies (by Ito's lemma) the existence of a unique state-price density process $\xi(t)$ given by $d\xi(t) = -\xi(t)r(t)dt - \xi(t)\kappa(t)^\top dw(t)$, where $\xi(0) = 1$ and $\kappa(t) = \sigma(t)^{-1}\left(\mu(t) - r(t)\underline{1}\right)$ is the *Sharpe ratio* process.

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2 Problem Statement

The coherent [1] and convex [3] risk measure *Conditional Value-at-Risk* (CVaR) is defined as a conditional expectation of losses greater than the Value-at-Risk (VaR) threshold. VaR is a widely used risk measure, technically it is equal to $(1-\alpha)$-quantile of the portfolio loss distribution (e.g. 99%)

$$\begin{aligned} CVaR_\alpha(W_0-W_T) &= \mathbb{E}\big[W_0-W_T \mid W_0-W_T \geq VaR_\alpha(W_0-W_T)\big] \leq \delta W_0 \\ VaR_\alpha(W_0-W_T) &= \{c\in\mathbb{R} : \mathbb{P}(W_0-W_T \leq c) = 1-\alpha\}\,. \end{aligned}$$

As the above statement is relatively complex, we substitute it by a more convenient representation

$$G_\alpha(W_0-W_T,c) = c + \frac{1}{\alpha}\int_{-\infty}^{\infty}(W_0-W_T-c)^+\,dP(W_0-W_T). \tag{1}$$

The way of CVaR substitution is well described in [4]. We incorporate the risk constraint in terms of the terminal portfolio CVaR in the utility maximization problem, hence define the *CVaR Investor Optimization Problem*:

$$\max_{W_T,c}\mathbb{E}\left[u\left(W_T\right)\right] \quad \text{s.t.} \quad \mathbb{E}\left[\xi_T W_T\right] \leq W_0 \quad \text{and} \quad G_\alpha(W_0-W_T,c) \leq \delta W_0, \tag{2}$$

where $G_\alpha(W_0-W_T,c)$ is given by (1), $u(\cdot)$ is the utility function, W_0 is the initial wealth, α and δ are given exogenously, ξ_T is defined in previous section and $c\in\mathbb{R}$ is a variable to be optimized.

3 Optimal Investment Strategy

We solve the problem as a two-stage optimization procedure. The solution of the first stage (Theorem 1) defines an optimal portfolio choice in the $W_T\times\xi_T$ space for each given c. As a result of the second stage optimization, we obtain an optimum through all possible settings of c by solving $\max_{c\in\mathbb{R}}\mathbb{E}\left[u\left(\hat{W}_T(c)\right)\right]$. In our practical calculations we suppose that the investor's preferences are well described by an iso-elastic utility function given by $u(x)=\frac{x^p}{p}, p<0$ and the exogenous model parameters r and κ are constant in time.

Theorem 1 (T-Time Optimal Portfolio Choice). *Define*

$$W_T(c,y_1,y_2) = I(y_1\xi_T)1_{\{\xi_T<\underline{\xi}\}} + (W_0-c)1_{\{\underline{\xi}\leq\xi_T<\overline{\xi}\}} + I\left(y_1\xi_T - y_2/\alpha\right)1_{\{\overline{\xi}\leq\xi_T\}},$$

where $c\in\mathbb{R}$, $y_1>0$, $y_2\geq 0$, $I(\cdot)$ is the inverse function of $u'(\cdot)$, $1_{\{\cdot\}}$ is the indicator function, $\underline{\xi}=u'(W_0-c)/y_1$ and $\overline{\xi}=\left(u'(W_0-c)+\frac{y_2}{\alpha}\right)/y_1$. Denote $\hat{y}_1$ and $\hat{y}_2$ to be a solution of equation system

$$\mathbb{E}\left[\xi_T W_T(c,\hat{y}_1,\hat{y}_2)\right] = W(0)$$
$$c + \frac{1}{\alpha}\mathbb{E}\left[(W_0 - W_T(c,\hat{y}_1,\hat{y}_2) - c)^+\right] = \delta W_0 \quad \text{or} \quad \hat{y}_2 = 0.$$

Then $\forall c$ *the problem (2) attains maximum at the point* $\hat{W}_T(c) \equiv W_T(c,\hat{y}_1,\hat{y}_2)$.

Theorem 2 (t-Time Optimal Portfolio Choice). *The wealth process of the solution* $\hat{W}_T$ *given by Theorem (1) is*

$$W(t) = \frac{y_1^{\frac{1}{p-1}}}{\xi_t} e^{\frac{p}{p-1}\left(\ln\xi_t + \left(\frac{\|\kappa\|^2}{2p-2} - r\right)(T-t)\right)}\Phi(d_3)$$
$$+ \frac{W_0 - c}{\xi_t} e^{\ln\xi_t - r(T-t)}\left(\Phi(d_2) - \Phi(d_1)\right) + \frac{1}{\xi_t}\int_{\bar{\xi}}^{\infty} \xi_T \left(y_1\xi_T - \frac{y_2}{\alpha}\right)^{\frac{1}{p-1}} d\mathbb{P}(\xi_T),$$

where $\Phi(\cdot)$ *is the cumulative distribution function of* $N(0,1)$,
$d_1 = \frac{\ln\underline{\xi} - \ln\xi_t + \left(r - \frac{1}{2}\|\kappa\|^2\right)(T-t)}{\|\kappa\|\sqrt{T-t}}$, $d_2 = d_1 + \frac{\ln\overline{\xi} - \ln\underline{\xi}}{\|\kappa\|\sqrt{T-t}}$ *and* $d_3 = d_1 - \frac{p}{p-1}\frac{\|\kappa\|}{2}\sqrt{T-t}$.

By definition of the portfolio wealth process W_t and the optimal process of the solution $\hat{W}_T$ given by Theorem 2, the optimal dynamic strategy is given by $\theta(t) = -\frac{\left(\sigma^\top\right)^{-1}\kappa^\top}{W(t)}\frac{\partial W(t)}{\partial\xi(t)}\xi(t) = -\frac{1-p}{W(t)}\theta^B(t)\frac{\partial W(t)}{\partial\xi(t)}\xi(t)$, where $\theta^B(t)$ stands for the *benchmark investor* strategy defined in [2] as $\theta^B(t) = \frac{1}{1-p}\left(\sigma^\top\right)^{-1}\kappa^\top$. Finally, we can define the process $q(t)$ as the exposure to risky assets *relative* to the benchmark portfolio, that we use for further analyses: $\theta^B(t)\,q(t) = \theta(t)$.

4 Numerical Results

To represent a real world market we set the exogenous model parameters as follows: $\alpha = 0.05, \delta = 0.15, p = -1.5$, $\kappa = 0.4$, $\xi(0) = 1$, $W(0) = 1$, $r = 0.03$ and $T = 1$. In common model applications we observe three market-state intervals in which the portfolio manager behaves differently. In *good* market states (low ξ_T) the CVaR portfolio payoff is similar to the benchmark payoff. In *intermediate* states the CVaR portfolio is fully hedged to the level $W_0 - \hat{c}$ and in *the worst* states (high ξ_T) the CVaR portfolio is only partially secured. As a result of our hedging strategy we observe an *adjusted* distribution of the terminal portfolio value, indicating lower probability mass concentrated in the left tail, i.e. the probability of attaining the most severe losses is *lower* than that of the benchmark investor (Fig. 1).

In good states the exposition to risky assets is very similar to the benchmark investor. As the market goes down, CVaR investor is selling out risky positions in order to retain the portfolio value above the acceptable level of loss. In the worst cases we observe a *leverage* effect: the investor opens relatively large positions in the risky assets with the intention to rise the portfolio value back to the acceptable level. The portfolio values before investment horizon can be evaluated as a response to the

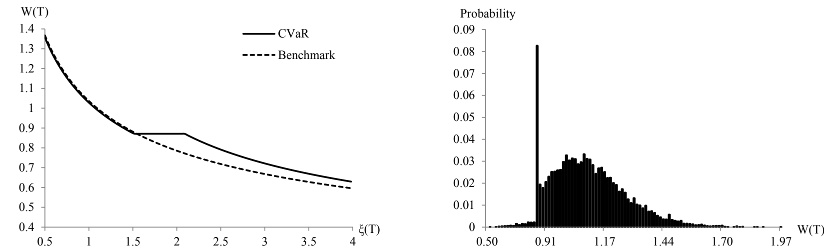

Fig. 1 Left: CVaR terminal portfolio payoff and benchmark terminal portfolio payoff, both as functions of the state variable $\xi(T)$. Right: Distribution of the CVaR terminal value payoff for the initial wealth $W_0 = 1$.

dynamic investment strategy process $\theta(t)$. As time t approaches T, the convergence to the terminal payoff is a necessary condition for the models consistency (Fig. 2).

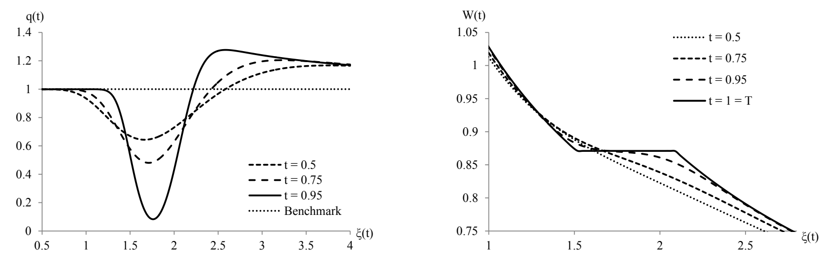

Fig. 2 Left: The dynamics of the relative risky assets exposition as a function of $\xi(t)$. Right: CVaR portfolio payoffs convergence to the terminal time shape; as functions of the state variable $\xi(t)$.

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