Comenius University in Bratislava

Faculty of Mathematics, Physics and Informatics

Duality in Convex Optimization Problems

Dissertation Thesis

2024 Mgr. Jakub HRDINA

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Duality in Convex Optimization Problems

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Study program:	Applied Mathematics
Study field:	Mathematics
Department:	Department of Applied Mathematics and Statistics
Supervisor:	doc. RNDr. Mária Trnovská, PhD.

2024 **Mgr. Jakub HRDINA**

Comenius University Bratislava
Faculty of Mathematics, Physics and Informatics

45114766

THESIS ASSIGNMENT

Name and Surname: Mgr. Jakub Hrdina
Study programme: Applied Mathematics (Single degree study, Ph.D. III. deg., full time form)
Field of Study: Mathematics
Type of Thesis: Dissertation thesis
Language of Thesis: English
Secondary language: Slovak

Title: Duality in Convex Optimization Problems

Annotation: Convex optimization problems can be reformulated as conic linear problems, where the (possibly) nonlinear constraints are represented by a suitable convex cone. At the same time, the general conic-linear structure of the problems allows the formulation of many non-standard optimization problems as convex and their investigation with the same tools. Such problems are, for example, semi-definite or co-positive programming problems, or polynomial optimization problems. Duality in convex conic programming is a powerful tool that can be used in the design of the algorithms, modelling, or to examine the primal-dual relationships between optimal solutions.

Tutor: doc. RNDr. Mária Trnovská, PhD.
Department: FMFI.KAMŠ - Department of Applied Mathematics and Statistics
Head of department: doc. Mgr. Igor Melicherčík, PhD.

Assigned: 08.01.2020

Approved: 08.01.2020 prof. RNDr. Daniel Ševčovič, DrSc.
Guarantor of Study Programme

....
Student Tutor

45114766

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Mgr. Jakub Hrdina
Študijný program: aplikovaná matematika (Jednoodborové štúdium, doktorandské III. st., denná forma)
Študijný odbor: matematika
Typ záverečnej práce: dizertačná
Jazyk záverečnej práce: anglický
Sekundárny jazyk: slovenský

Názov: Duality in Convex Optimization Problems
Dualita v konvexných optimalizačných úlohách

Anotácia: Konvexné optimalizačné úlohy možno reformulovať ako kónické lineárne úlohy, pričom prípadné nelineárne ohraničenia sú reprezentované vhodným konvexným kužeľom. Všeobecná kónicko-lineárna štruktúra úloh zároveň umožňuje formulovať mnohé neštandardné úlohy ako konvexné a skúmať rovnakými nástrojmi. Takými úlohami sú napríklad úlohy semidefinitnej, kopozitívnej, či polynomiálnej optimalizácie. Dualita v konvexnom kónickom programovaní je silný nástroj, ktorý má využitie pri navrhovaní algoritmov, ale aj pri samotnom modelovaní, či skúmaní primárno-duálnych vzťahov medzi optimálnymi riešeniami.

Školiteľ: doc. RNDr. Mária Trnovská, PhD.
Katedra: FMFI.KAMŠ - Katedra aplikovanej matematiky a štatistiky
Vedúci katedry: doc. Mgr. Igor Melicherčík, PhD.

Dátum zadania: 08.01.2020

Dátum schválenia: 08.01.2020

prof. RNDr. Daniel Ševčovič, DrSc.
garant študijného programu

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študent školiteľ

I would like to express my most sincere gratitude to doc. RNDr. Mária Trnovská, PhD. for her precious time, consultations, valuable comments and assistance in the completion of this thesis. I would also like to thank my family and friends for supporting me, especially my parents and my fiancée.

Abstract

In this dissertation, we examine duality in convex conic optimization problems and its application in polynomial optimization. We derive new sufficient conditions for strong duality in convex conic programming and provide necessary and sufficient conditions for boundedness (or unboundedness) of nonempty sets of optimal solutions. We analyze the strong duality property in conic reformulations of standard convex programming problems and compare two versions of Slater conditions: the conic version for conic reformulations of standard convex programming problems and the generalized Slater condition for standard convex programming problems. Within the field of polynomial optimization, we concentrate on examining the properties of the cone of multivariate polynomials nonnegative on a given nonempty set and their respective dual cones. We analyze the strong duality property and its aspects in polynomial optimization problems.

Keywords: duality, convex conic optimization, polynomial optimization

Abstrakt

V tejto dizertačnej práci sa zaoberáme skúmaním duality v konvexných kónických optimalizačných úlohách a jej aplikácii v polynomiálnej optimalizácii. Odvádzame nové postačujúce podmienky na platnosť silnej duality v konvexných kónických optimalizačných úlohách spolu s nutnými a postačujúcimi podmienkami na ohraničenosť (neohraničenosť) neprázdnych množín optimálnych riešení. Analyzujeme silnú dualitu pre kónické reformulácie štandardných úloh konvexného programovania a porovnávame dve verzie Slaterovej podmienky: kónickú verziu Slaterovej podmienky pre kónické reformulácie štandardných úloh konvexného programovania a zovšeobecnenú verziu Slaterovej podmienky pre štandardné úlohy konvexného programovania. V oblasti polynomiálnej optimalizácie sa sústredíme na skúmanie vlastností kužeľov polynómov viacerých premenných nezáporných na danej neprázdnej množine a ich duálnych kužeľov. Analyzujeme vlastnosť silnej duality a jej aspektov pre úlohy polynomickej optimalizácie.

Kľúčové slová: dualita, konvexná kónická optimalizácia, polynomiálna optimalizácia

Preface

Duality is a philosophical term which can be defined as the quality or state of having two different or opposite parts or elements. Broadly speaking, duality means having two different views of the same object, which may but need not be equivalent. As a result, duality offers us the choice to work with the more convenient option.

There are many examples of duality in mathematics. For instance, there exist numbers and dual numbers; there exist vector spaces and dual vector spaces; a signal can be described in the time domain and, dually, in the frequency domain; a closed convex set can viewed as a union of points and, dually, as intersections of half-spaces containing the set; a convex function may be viewed through points and, dually, through nonvertical hyperplanes; there exist primal optimization problems and dual optimization problems.

Duality, particularly in mathematical optimization, is a fascinating concept and served as a source of motivation for completing this thesis.

The core of the thesis is organized into two chapters. The first chapter deals with Lagrangian duality in convex conic programming. It is divided into four sections as follows: in Section 1.1 the standard formulation of a primal-dual pair of convex conic programming problems is incorporated; in Section 1.2 the theorems of alternatives are included; in Section 1.3 the strong duality property and its aspects are elaborated; in Section 1.4 the strong duality property in convex programming is analyzed.

The second chapter focuses on the application of conic duality in polynomial optimization. It is divided into three sections as follows: in Section 2.1 the polynomial optimization problems, their equivalent conic reformulations and properties of the cone of polynomials nonnegative on the given set are introduced; in Section 2.2 the duality results in polynomial optimization problems are included; in Section 2.3 the application of the dual cone theorem is demonstrated.

The appendix, containing standard definitions, notations, and necessary and/or useful results, is divided into four sections: Section A contains the properties of cones and dual cones; Section B is devoted to the relative interior of a convex cone; Section C contains the results on closedness of the linear image of a convex cone; Section D contains the standard notations, definitions and basic results concerning the vector space of multivariate polynomials.

Contents

Table of symbols

$\mathcal{B}(x, r)$	an open ball centered at x and of radius r
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$ ($\mathbb{R}_{++}$)	the set of nonnegative (positive) real numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \ldots, x_n)^\top \mid x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$
$\mathbb{R}[x]$ ($\mathbb{R}[x]_d$)	the vector space of multivariate polynomials in variable $x \in \mathbb{R}^n$ (of degree at most d)
$(\cdot)^\top$	matrix transposition
$\lVert\cdot\rVert$, $\lVert\cdot\rVert_2$	a norm, the Euclidean norm
$\langle \cdot, \cdot \rangle$	an inner product
$\mathcal{O}$	an open set (with respect to a given topology)
$int(\cdot)$	the interior of a set
$relint(\cdot)$	the relative interior of a set
$\partial(\cdot)$	the boundary of a set
$cl(\cdot)$	the closure of a set
$lin(K)$	the smallest subspace containing K
$sub(K)$	the largest subspace contained in K
R_C	the recession cone of a nonempty set C
ℓ	the linear functional $\ell : \mathbb{R}[x] \to \mathbb{R}$
$m_d(x)$	$(1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, \ldots, x_1^d, \ldots, x_n^d)^\top$
$M_{m,n}(\mathbb{R})$	the set of $m \times n$ matrices with real entries
$\mathcal{N}(A)$	the nullspace of a matrix A

$\mathcal{S}(A)$	the column space of a matrix A
$A(S)$	the image of a set S in a linear transformation given by matrix A
$cone(S)$	the conic hull of a set S
$\mathbb{N}$ ($\mathbb{N}_0$)	the set of positive (nonnegative) integers
$\mathbb{N}^n$	$\{(\alpha_1,\dots,\alpha_n) \mid \alpha_i \in \mathbb{N}_0,\ i=1,\dots,n\}$
$\mathbb{N}^n_d$	$\{(\alpha_1,\dots,\alpha_n) \mid \alpha_i \in \mathbb{N}_0,\ \sum_{i=1}^n \alpha_i \le d,\ i=1,\dots,n\}$
α ($\lvert\alpha\rvert$)	the multi-index $\alpha \in \mathbb{N}^n$ or $\alpha \in \mathbb{N}^n_d$ (the sum $\sum_{i=1}^n \alpha_i$)
S^c	the complement of a set S with respect to a given universal set
$\mathcal{P}$ ($\mathcal{D}$)	the set of primal (dual) feasible points
$\mathcal{P}_0$ ($\mathcal{D}_0$)	the set of primal (dual) strictly feasible points
$\mathcal{P}^*$ ($\mathcal{D}^*$)	the set of primal (dual) optimal points
p^* (d^*)	the optimal value of the primal (dual) problem
$p(x)$	a multivariate polynomial p in variable $x = (x_1,\dots,x_n)^\top$
$1(x)$	a multivariate polynomial identically equal to 1
x^α	$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $\alpha \in \mathbb{N}^n_d$
$s(n,d)$	$\binom{n+d}{d} = \dim(\mathbb{R}[x]_d)$
K^*, $K^\perp$	the dual cone of K, the orthogonal complement of K
p_α, $(p_\alpha)_{\alpha\in\mathbb{N}^n_d}$	the coefficients of polynomial $p \in \mathbb{R}[x]_d$ with respect to the canonical basis of $\mathbb{R}[x]_d$, where $p(x) = \sum_{\alpha\in\mathbb{N}^n_d} p_\alpha x^\alpha$
$\mathcal{I}$	the isomorphism $\mathcal{I} : \mathbb{R}[x]_d \to \mathbb{R}^{s(n,d)}$ defined as $\mathcal{I}(p) = (p_\alpha)_{\alpha\in\mathbb{N}^n_d}$
$C_{n,d}(K)$	the cone of n-variate polynomials of degree at most d nonnegative on the set K
$\mathrm{rank}(A)$	the rank of matrix A
$A \times B$	the product[1] of sets A and B
$A + B$	the Minkowski sum[2] of sets A and B
$\simeq$	is isomorphic to

[1]for $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$, the product $A \times B$ is defined as $A \times B := \{(a^\top \mid b^\top)^\top \mid a \in A,\ b \in B\}$; the notation $(a^\top \mid b^\top)^\top$ may be contracted to (a,b)

[2]for $A, B \subseteq \mathbb{R}^n$, the Minkowski sum $A + B$ is defined as $A + B := \{a + b \mid a \in A,\ b \in B\}$

Introduction

Convex optimization

Convex optimization, or convex programming, is a subfield of mathematical optimization concerning the problem of minimizing a convex function over a convex set, or equivalently, maximizing a concave function over a convex set. Convex programming problems acquire three important and useful properties:

1. Every local minimum is a global minimum.
2. The optimal solution set is a convex set.
3. If the objective is strictly convex, then the problem has at most one optimal solution.

It is due to these properties that convex optimization has become an important part of the field of mathematical programming.

According to Dimitri Bertsekas [9], the prehistory of convex optimization is dated to the first half of the 20th century. During this period, mathematicians such as Caratheodry, Minkowski, Farkas and Steinitz concentrated on studying the properties of convex sets and convex functions, taking no interest in optimization.

The history of convex optimization[3] is considered to have started in the late 1940s with one of its simplest subclass. In 1947, George B. Dantzig proposed the simplex method for linear programming, see *e.g.* [21]. In 1949, Werner Fenchel published two works [27] and [28] dealing with the so-called Fenchel duality theory, min-max in game theory, subdifferentiability, optimality conditions, and sensitivity, however, not including algorithms. Moreover, by that time a group of mathematicians working in the field of mathematical optimization, including Fenchel, Von Neumann, Tucker, Kuhn and Nash,

[3]Note that terms “convex optimization” and “convex programming” were not used until much later.

had formed at Princeton University. The results from this period can be found in the book of Rockafellar in an extended and developed form, see [66].

Note that by the 1950s convex programming had not separated from nonlinear programming yet. Nevertheless, the following results from this period play an important role in convex programming nowadays. In 1950, Morton L. Slater in [69] discovered a sufficient condition for strong duality in convex programming. Moreover, owing to the Kuhn-Tucker theorem proved in 1951 by Harold W. Kuhn and Albert W. Tucker, the Karush-Kuhn-Tucker (KKT) conditions are, in case of differentiability, sufficient conditions for optimality, see *e.g.* [41]. These two results are connected within the theory of convex programming in the following way: if the Slater condition is satisfied, the KKT conditions are necessary and sufficient conditions for optimality in convex programming.

Convexity in nonlinear programming gained popularity with the invention of the interior point methods (IPM). In 1955, Ragnar Frisch proposed a logarithmic barrier method that was later analyzed by Fiacco and McCormick in the 1960s, see *e.g.* [30] and [29]. According to [76], while barrier methods were widely used in the 1960s, they suffered a severe decline in popularity in the 1970s due to various reasons, *e.g.* problems with inherent ill-conditioning. One of the first interior point methods for linear and quadratic programming, called the affine scaling method, was invented by Ilya I. Dikin in 1967, see [23]. However, this result received little attention until 1984.

In 1984, Karmarkar reinvented, developed and extended the result of Dikin, proposing an algorithm[4] for linear programming which runs in polynomial time and is approximately fifty times faster than the simplex method, see [39]. Karmarkar's algorithm quickly received publicity since it had outperformed the simplex method. In 1985, it was shown that Karmarkar's algorithm is formally equivalent to the classical logarithmic barrier method applied to a linear programming problem, see [76]. In 1988, Karmarkar's algorithm was extended by Yurii Nesterov and Arkadi Nemirovski to convex programming problems, based on a self-concordant barrier function used to encode the convex set, see *e.g.* [54], [55] and [49]. Note that nowadays there exist three basic types of interior point methods: potential reduction methods, path-following methods and primal-dual methods.

The development of interior point methods changed how linear and nonlinear programming problems were perceived. Traditionally, linear programming problems had

[4]It is sometimes referred to as "Karmarkar's projective method".

been regarded as easy problems to solve contrary to nonlinear programming problems. However, with the development of interior point methods, this view proved to be rather inappropriate. The appropriate view, according to [9], is to consider convex programming problems easy to solve and nonconvex programming problems difficult to solve. It is due to the fact that several classes of convex programming problems admit polynomial-time algorithms, whereas problems of mathematical programming are in general NP-hard, see *e.g.* [10].

An important outcome of the development of the interior point methods is the optimization over the cone of positive semidefinite matrices known as semidefinite programming (SDP). Semidefinite programming became a new class of convex programming problems in the early 1990s and soon became an attractive area of research. Primarily motivated by the problems of combinatorial optimization and control theory, semidefinite programming has numerous applications, including probability, statistics, machine learning, engineering, computational geometry, optimal experiment design, and many more, see [1]. Fortunately, despite being much more general than linear programming problems, semidefinite programming problems are not much harder to solve by means of the interior point methods, see *e.g.* [75], [55] and [1].

Another class of convex programming that became popular in the 1990s is second order cone programming (SOCP). Second order cone programming problems are convex optimization problems of minimizing a linear function over the intersection of an affine subspace and the second order cone[5]. Second order cone programming includes problems such as linear programming problems, quadratically constrained convex quadratic programming problems, problems involving fractional quadratic functions and the problem of finding the smallest ball containing a given set of ellipsoids, see *e.g.* [2]. Second order cone programming has various applications in combinatorial optimization, engineering, robust optimization, and many more. Second order cone programming problems can be formulated as semidefinite programming problems and therefore, second order cone programming is placed between linear programming, quadratic programming and semidefinite programming, see *e.g.* [46] and [2].

Linear programming, second order cone programming and semidefinite programming have a common structure. All three are problems of minimizing a linear function over

[5]The second order cone is sometimes referred to as the Lorentz cone.

the intersection of an affine subspace and a special convex cone. In the case of linear programming, the corresponding cone is the nonnegative orthant, in the case of second order cone programming, the corresponding cone is the second order cone, in the case of semidefinite programming, the corresponding cone is the cone of positive semidefinite matrices. Therefore, the natural conic generalization of these three classes lies in substituting the special convex cone with a general convex cone, which leads to convex conic programming.

Convex conic programming

Convex conic programming, or, more precisely, convex conic linear programming, is a special class of convex programming problems in which, in its standard form, a linear function is minimized over the intersection of an affine subspace and a convex cone. In addition to convexity, there might be other assumptions about the convex cone, such as closedness, having a nonempty interior and not containing a straight line, which make the class easier to analyze. The polyhedrality assumption leads to linear programming.

Geometric and topological properties of convex cones are essential aspects of convex conic programming. Fundamental results regarding the geometry and topology of convex cones can be found in standard convex analysis textbooks, *e.g.* [10], [66], [6], [28], or [15]. The characterization of the relative interior of a convex cone was studied in [47]. The closedness of the linear image of a closed convex cone was analyzed in [61].

In [6] it was shown that every epigraph constraint in the form of $f(x) \leq t$ can be embedded into a convex cone using the perspective mapping. Therefore, convex conic programming can be viewed as a superclass of standard convex programming and standard convex programming problems can obtain a different, conic, structure.

Furthermore, apart from linear programming, second order cone programming and semidefinite programming, convex conic programming encompasses new classes of convex programming. These classes typically pertain to the optimization over matrix cones, such as copositive programming and completely positive programming. Copositive programming problems are convex conic programming problems in which the corresponding cone is the cone of copositive matrices[6], see *e.g.* [40], [16], or [25]. In [16] it was shown

[6] A real symmetric matrix $A \in M_{n,n}(\mathbb{R})$ is called copositive if for all vectors $x \in \mathbb{R}^n_+$ it holds $x^\top Ax \geq 0$.

that the dual cone to the cone of copositive matrices is the cone of completely positive matrices[7], therefore, copositive programming and completely positive programming are linked through Lagrangian duality. Both copositive programming and completely positive programming have applications in control theory, graph theory, rigid body mechanics, mixed-binary optimization, and others, see *e.g.* [16].

Other new classes of convex conic programming include programming over the exponential cone and programming over the power cone, see [4]. The exponential cone is constructed by embedding the epigraph of the exponential function into a convex cone. Therefore, the exponential cone can be used to model a variety of constraints involving exponentials and logarithms, such as entropy, relative entropy, softplus function and Lambert W-function, see [4]. Problems of programming over the exponential cone are comprised of problems of geometric programming, risk parity portfolio problems, problems of maximization of entropy, problems of logistic regression, and others, see [4]. The power cone is the generalization of the second order cone, and hence, power cones are used to model constraints with powers other than two, such as p-norm cones and geometric mean, see [4]. Problems of programming over the power cone include the problem of portfolio optimization with market impact, the maximum volume cuboid problem, the p-norm geometric median problem, and the problem of maximum likelihood estimator of a convex density function, see [4].

Another active area of research within the field of convex conic optimization is focused on the development of effective methods and algorithms for solving convex conic optimization problems, see *e.g.* [17], [3], [18], and [51]. Company MOSEK ApS specializes in conic modeling together with the development of algorithms, based on interior point methods, especially for the programming over the exponential cone (see [20]) and the programming over the power cone. These algorithms prevalently rely on the duality theory which is crucial in primal-dual algorithms and is also central to understanding sensitivity analysis and infeasibility issues. It provides a simple and systematic way of obtaining nontrivial lower bounds on the optimal value, see [4].

[7]A real symmetrix matrix $A \in M_{n,n}(\mathbb{R})$ is completely positive if there exist $m \in \mathbb{N}$ and vectors $x_1, \ldots, x_m \in \mathbb{R}^n_+$ such that $A = \sum_{i=1}^m x_i x_i^\top$.

Duality in convex conic programming

Duality in convex conic programming was studied in a general setting, starting with [24], and followed by other authors, see also [1] for more references. In [14] and [13] the authors use a minimal cone to transform the original problem to an equivalent one, for which the Slater condition holds and, hence, the strong duality property is satisfied. However, as mentioned in [65], the facial reduction procedure to obtain a minimal cone is computationally unsatisfactory.

Duality theory in convex conic programming, with a focus on specific subclasses, has been revisited by many authors since the invention of interior point methods in 1984. One such paper is [32], where duality results for linear programming are obtained from the perspective of the interior point methodology. In particular, it is shown that the primal (dual) feasibility, together with the dual (primal) strict feasibility, is equivalent to the nonemptiness and boundedness of the primal (dual) optimal solution set, respectively (Theorem 3.2, [32]). Duality theory for semidefinite programming is studied in connection with interior point methods in [1], see also [75] and [70] for a survey. Simple proofs for the extension of the result of Theorem 3.2 in [32] to semidefinite programming are given in [71].

The papers mentioned in the paragraph above study the Lagrangian dual (considered in the convex optimization textbooks, such as [56], [6] or [15]) that requires the Slater condition for the strong duality to hold. Failure of strong duality motivated other authors, who attempted to construct a primal-dual pair satisfying strong duality without any constraint qualification. The extended Lagrange-Slater dual was proposed in [65] for semidefinite programming. In the paper [62], the facial reduction procedure of [13] was applied to obtain strong duality for convex conic problems over symmetric cones. Paper [73] dealt with general convex conic programs and it was shown that the minimal representation of the problem guarantees the Slater condition and, therefore, also strong duality. The same approach was applied in [40] for the class of copositive programming problems.

Contrary to the approaches mentioned in the paragraph above, we study the standard Lagrangian primal-dual pair of convex conic programming problems, see [1], [56], [6], [15], where strong duality may fail. We derive new sufficient conditions for strong duality in convex conic programming along with necessary and sufficient conditions for boundedness (or unboundedness) of nonempty sets of optimal solutions. In addition, we prove that if the generalized Slater condition for standard convex conic programs is satisfied, then the conic version of Slater condition for the respective conic reformulation is satisfied.

Polynomial optimization

Polynomial optimization is a subfield of mathematical optimization concerning the problem of minimizing a multivariate polynomial over a given nonempty set.

By introducing a new variable, which serves as a lower bound of the polynomial being minimized on the given set, the problem of polynomial optimization can be equivalently reformulated as a problem of finding the maximum lower bound of the polynomial on the given set. In maximizing the lower bound, it is required that the difference between the polynomial and the lower bound be a nonnegative polynomial on the given set.

This reformulation raises several questions, including whether one can optimize over the set of polynomials nonnegative on the given set. It also prompts inquiries into the structure of that set, whether one can test if a polynomial is nonnegative on the given set, and whether such testing can be done efficiently.

In [38] it was shown that testing whether a polynomial of degree at least 4 is nonnegative on a basic semialgebraic set is NP-hard, even if the given set is $\mathbb{R}^n$. Moreover, it was shown that unconstrained optimization of a quartic polynomial, optimization of a cubic polynomial over the sphere and optimization of a quadratic polynomial over the simplex are all NP-hard problems, see *e.g.* [53] and [22]. As a consequence, the mentioned reformulation provides motivation for examining the structure of a set of multivariate polynomials nonnegative on a given nonempty set.

The nonnegativity of polynomials on $\mathbb{R}^n$ has been widely examined for more than a hundred years. It is obvious that a polynomial is nonnegative on $\mathbb{R}^n$ if it can be represented as a finite sum of squares of other polynomials with a lower degree. However, it is not obvious whether the converse holds. In fact, David Hilbert in [34] proved that there are only three cases when the converse holds: for all odd-degree univariate polynomials, for all quadratic polynomials, and for all two-variable quartic polynomials. Thus, in general the set of sums-of-squares (SOS) polynomials is a proper subset of the set of polynomials nonnegative on $\mathbb{R}^n$.

Both of these sets are indeed proper cones, as discussed in *e.g.* [11] and [52]. The most important difference between these two cones is their structure. The cone of SOS polynomials is closely linked with semidefinite programming. More precisely, testing whether a given polynomial is SOS can be transformed into solving a feasibility problem of semidefinite programming, see [42]. On the other hand, no simple and tractable characterization of the cone of nonnegative polynomials is known, see [44]. Therefore, the SOS cone serves as a computational substitute for the cone of nonnegative polynomials. More details on the SOS cone, its geometry and applications can be found in [58] and [59].

The nonnegativity of polynomials and polynomial optimization have been studied within the context of convex and conic optimization and real algebra. In [68] it was proposed that a convex optimization technique be used to minimize an unconstrained multivariate polynomial. In [52] the author discussed the duality of cones of nonnegative polynomials and moment cones. More specifically, a moment cone was shown to be characterized by semidefinite constraints or, in other words, by linear matrix inequalities on the condition that the corresponding dual cone, the cone of nonnegative polynomials, was SOS-representable. The nonnegativity of polynomials with the use of real algebraic results was discussed in [60]. Finally, with the real algebra result of Putinar [63], J. B. Lasserre in [43] constructed a sequence of semidefinite program relaxations with optima converging to the optimum of a polynomial optimization problem, known as SOS or Lasserre hierarchy.

We focus on analyzing the properties of the set of polynomials nonnegative on a given nonempty set using convex analysis and linear algebra results. Additionally, we offer a representation of the dual cone corresponding to the cone of polynomials nonnegative on the given set. We formulate the dual cone theorem. Furthermore, we combine the results from Chapter 1 to derive findings pertaining to the zero duality gap in a primal-dual pair of polynomial optimization problems, as well as necessary and sufficient conditions for the nonemptiness and boundedness (or unboundedness) of sets of optimal solutions. Lastly, we demonstrate the application of the dual cone theorem when searching for the explicit characterizations of the set of univariate polynomials nonnegative on $[-1, 1]$.

Lagrangian duality in convex conic programming

In this chapter we will be focusing on the Lagrangian duality in convex conic programming. In Section 1.1 we include the standard formulation of a primal-dual pair of convex conic programs together with standard notions regarding convex conic programming problems. Additionally, we cover fundamental results concerning the weak duality property, along with a subsection on the recession cones associated with the primal and dual convex conic programs. In Section 1.2 we present four theorems of alternatives for linear systems over convex cones. Section 1.3 discusses strong duality results, including the zero duality gap and necessary and sufficient conditions for the nonemptiness and boundedness, as well as unboundedness, of sets of optimal solutions. It should be noted that Section 1.1, Section 1.2 and Section 1.3 were published in [72]. In Section 1.4 we concentrate on studying the strong duality property in convex programming, more precisely, on comparing the conic version of Slater condition with the generalized version of Slater condition for standard problems of convex programming.

In this chapter we will be using the basic terminology and properties concerning cones and their relative interior which can be found in Appendix A and Appendix B.

1.1 Primal and dual convex conic programs

In this section we introduce the primal-dual pair of convex conic programs in their standard form as it is formulated in *e.g.* [15, Section 4.6.1], [50, Section 4] or [6, Section 2.2].

In the following sections of this chapter we require that the cone K satisfy the following assumption.

Assumption 1. The cone K is a nontrivial convex cone. (see Definition A.2)

1.1.1 Primal convex conic program

Given vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, an $m \times n$ matrix A and a convex cone $K \subseteq \mathbb{R}^n$, the convex conic programming problem in standard form is formulated as

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \in K. \end{aligned} \tag{1.1}$$

The set of primal feasible points and the set of primal strictly feasible points are denoted by $\mathcal{P} = \{x \in K \mid Ax = b\}$ and $\mathcal{P}^0 = \{x \in relint(K) \mid Ax = b\}$, respectively. Furthermore, we define the optimal value of the problem (1.1) as $p^* = \inf\{c^\top x \mid x \in \mathcal{P}\}$ if $\mathcal{P} \neq \emptyset$ and $p^* = +\infty$ otherwise. The primal optimal solution set is then $\mathcal{P}^* = \{x \in \mathcal{P} \mid c^\top x = p^*\}$.

1.1.2 Dual convex conic program

The dual of problem of (1.1) is derived using the standard technique in the following way. The Lagrange function $L : K \times \mathbb{R}^m \to \mathbb{R}$ of the problem (1.1) is defined in the form

$$L(x, y) = c^\top x + y^\top (b - Ax) = (c - A^\top y)^\top x + b^\top y. \tag{1.2}$$

Due to the properties of the dual cone (see Appendix A, Definition A.4) it can be calculated that

$$\inf_{x \in K} L(x, y) = \begin{cases} b^\top y, & c - A^\top y \in K^*, \\ -\infty, & otherwise. \end{cases} \tag{1.3}$$

Introducing a slack variable $s := c - A^\top y$, one obtains the dual of problem of (1.1) in the form

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s \in K^*. \end{aligned} \tag{1.4}$$

The set of all dual feasible points of (1.4) is $\mathcal{D} = \{(y, s) \in \mathbb{R}^m \times K^* \mid A^\top y + s = c\}$ and the set of all dual strictly feasible points is $\mathcal{D}^0 = \{(y, s) \in \mathbb{R}^m \times relint(K^*) \mid A^\top y + s = c\}$. If

$\text{rank}(A) = m$, then there is one-to-one correspondence between the dual variables y and s, that is, if $(y_1, s), (y_2, s) \in \mathcal{D}$, then $y_1 = y_2$.

Remark 1.1. The assumption $rank(A) = m$ is only technical, but it is needed when analyzing the boundedness of $\mathcal{D}$. Note that if $rank(A) = m$, then for every s such that $c - s \in \mathcal{S}(A^\top)$ there exists a unique $y = (AA^\top)^{-1}A(c - s)$ such that $A^\top y + s = c$.

The optimal value of the problem (1.4) is defined as $d^* = \sup\{b^\top y \mid (y, s) \in \mathcal{D}\}$ if $\mathcal{D} \neq \emptyset$ and $d^* = -\infty$ otherwise. Finally, the dual optimal solution set is denoted by $\mathcal{D}^*$, *i.e.* $\mathcal{D}^* = \{(y, s) \in \mathcal{D} \mid b^\top y = d^*\}$.

1.1.3 Weak duality

Owing to the construction of the dual program (1.4), more precisely (1.3), the well-known *weak duality* property holds between the primal program (1.1) and the dual program (1.4). For the sake of completeness, we include the weak duality theorem together with its consequences. (see [15, Section 5.2.2])

Theorem 1.1 (Weak duality)**.** *Assume the primal-dual pair of convex conic programs* (1.1) *and* (1.4)*. For an arbitrary primal feasible point $x \in \mathcal{P}$ and for an arbitrary dual feasible point $(y, s) \in \mathcal{D}$ it holds that*

$$c^\top x - b^\top y = s^\top x \geq 0.$$

Corollary 1.1. *Assume the primal-dual pair of convex conic programs* (1.1) *and* (1.4)*.*

a) It holds that $p^* \geq d^*$.

b) If for some vectors $\bar{x} \in \mathcal{P}$ and $(\bar{y}, \bar{s}) \in \mathcal{D}$ holds $c^\top \bar{x} = b^\top \bar{y}$, then $\bar{x}$ is optimal for (1.1) *and $(\bar{y}, \bar{s})$ is optimal for* (1.4)*.*

c) If (1.1) *is unbounded from below, then* (1.4) *is infeasible. If* (1.4) *is unbounded from above, then* (1.1) *is infeasible.*

1.1.4 Recession cones related to the primal and dual convex conic programs

In the following sections we will be working with recession cones (see Appendix A, Definition A.3) of $\mathcal{P}$, $\tilde{\mathcal{D}} = \{s \mid (y, s) \in \mathcal{D}\}$, $\mathcal{P}^*$ and $\tilde{\mathcal{D}}^* = \{s^* \mid (y^*, s^*) \in \mathcal{D}^*\}$. We include their characterizations in the two following propositions.

Proposition 1.1. *Assume the primal-dual pair of convex conic programs* (1.1), *and* (1.4) *and assume that* $\mathcal{P}$ *and* $\tilde{\mathcal{D}}$ *are nonempty. Moreover, assume that* K *is closed, then*

$$R_{\mathcal{P}} = \{d \mid Ad = 0,\ d \in K\} = \mathcal{N}(A) \cap K, \tag{1.5}$$

$$R_{\tilde{\mathcal{D}}} = \{v \mid v \in \mathcal{S}(A^\top),\ v \in K^*\} = \mathcal{S}(A^\top) \cap K^*, \tag{1.6}$$

Proof. We only prove that $R_{\mathcal{P}} = \mathcal{N}(A) \cap K$, the second statement can be proved analogously. Take a $d \in \mathcal{N}(A) \cap K$ and an arbitrary $x \in \mathcal{P}$, then $x + \gamma d \in K$ for all $\gamma \geq 0$ and, furthermore, $A(x + \gamma d) = Ax + \gamma Ad = Ax = b$ for all $\gamma \geq 0$. Thus, $\{x + \gamma d \mid \gamma \geq 0\} \subseteq \mathcal{P}$, and, since x was chosen arbitrarily, we have that $d \in R_{\mathcal{P}}$. Now, take a $d \in R_{\mathcal{P}}$, then $\forall x \in \mathcal{P}$ and $\forall \gamma \geq 0$ we have that $x + \gamma d \in \mathcal{P}$, which means that $\forall \gamma \geq 0$ it holds that $A(x + \gamma d) = Ax + \gamma Ad = b + \gamma Ad$. Thus, $Ad = 0$. Moreover, $\forall \gamma > 0$ it holds that $x + \gamma d \in K$. Owing to the definition of a cone, we have that $\frac{1}{\gamma}x + d \in K$ for all $\gamma > 0$. Letting $\gamma \to +\infty$, we obtain that $d \in cl(K) = K$. □

Remark 1.2. If the assumption on closedness of K is omitted, one may obtain that $R_{\mathcal{P}} \neq \mathcal{N}(A) \cap K$, for instance

$$K = \{(x_1, x_2)^\top \mid x_1 > 0,\ x_2 \geq 0\} \cup \{(0,0)^\top\}, \quad A = (1,0), \quad b = 3 \tag{1.7}$$

Apparently, $\mathcal{N}(A) \cap K = \{(0,0)^\top\}$. However, $\mathcal{P} = \{(3,t)^\top \mid t \geq 0\}$, thus, $R_{\mathcal{P}} = \{(0,d_2)^\top \mid d_2 \geq 0\} = R_{\mathcal{P}} = \mathcal{N}(A) \cap cl(K)$.

However, it is incorrect to claim that $R_{\mathcal{P}} = \mathcal{N}(A) \cap cl(K)$, take K as in (1.7), $A = (0,0)$ and $b = 0$, thus $\mathcal{N}(A) = \mathbb{R}^2$. We have that $R_{\mathcal{P}} = K = \mathbb{R}^2 \cap K \neq \mathbb{R}^2 \cap cl(K)$.

Proposition 1.2. *Define the extended matrices* $\mathbf{A}_c = (A^\top\ c)^\top \in M_{m+1,n}(\mathbb{R})$ *and* $\mathbf{A}_b = (A\ \ -b) \in M_{n,n+1}(\mathbb{R})$. *Assume the primal-dual pair of convex conic programs* (1.1) *and* (1.4) *and assume that* $\mathcal{P}^*$ *and* $\tilde{\mathcal{D}}^* = \{s^* \mid (y^*, s^*) \in \mathcal{D}^*\}$ *are nonempty. Moreover, assume that* K *is closed, then*

$$R_{\mathcal{P}^*} = \{d \mid Ad = 0,\ c^\top d = 0,\ d \in K\} = \mathcal{N}(\mathbf{A}_c) \cap K, \tag{1.8}$$

$$R_{\tilde{\mathcal{D}}^*} = \{v \mid (v^\top, 0)^\top \in \mathcal{S}(\mathbf{A}_b^\top) \cap (K^* \times \{0\})\}. \tag{1.9}$$

Proof. The proof of these statements is analogous to the proof of Proposition 1.1. □

1.2 Theorems of alternatives

In this section, we present four theorems of alternatives for linear systems over cones. They are divided into two groups, depending on whether, regarding the (strict) feasibility, they are related to the primal or the dual conic program. Two of them are known as the Farkas lemma, and the alternatives presented in the theorems are weak in general. For strong alternatives, an additional assumption is required. Note that in the Farkas lemma, one alternative is exactly the feasibility of the primal (dual) convex conic program. We also formulate and prove a new different (primal-dual) pair of theorems of alternatives, where one alternative is the strict feasibility of the primal (dual) convex conic program. The alternatives in these theorems are strong (no additional assumption is required).

1.2.1 Primal theorems of alternatives

The first theorem is a generalization of the famous Farkas lemma for linear systems [26]. Various forms of the theorem have been studied within the last decades, also with the connection to linear matrix inequalities and semidefinite programming, see [74], [1]. For general conic programs, it was formulated by many authors in various forms; see *e.g.* [7], [5], or [19] in more general terms.

Theorem 1.2. *(Generalized Farkas lemma)*
Assume that $K \subseteq \mathbb{R}^n$ is a cone satisfying Assumption 1, A is a given $m \times n$, $(m \leq n)$ matrix, and $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given vectors. At most one of the following statements is true:

I $\exists x \in K : Ax = b$;

II $\exists z : A^\top z \in K^*$ *and* $z^\top b < 0$.

Moreover, if the convex cone $A(cl(K))$ (or alternatively the Minkowski sum $cl(K) + \mathcal{N}(A)$)[8] *is closed, then exactly one of the statements is true.*

Remark 1.3. Sufficient conditions for closedness of $A(cl(K))$ can be found in Theorem C.2 or Table 1 in Appendix C.

In the following, we establish and prove a new theorem of alternatives, which deals with the relative interior of the cone. It provides a strong alternative (and, therefore, also

[8] see Theorem C.1 in Appendix C

an equivalent condition) to the strict feasibility of the primal program (1.1).

Theorem 1.3. *Assume that $K \subseteq \mathbb{R}^n$ is a cone satisfying Assumption 1, A is a given $m \times n$, $(m \leq n)$ matrix, and $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given vectors. Exactly one of the following statements is true:*

I $\exists x \in relint(K) : Ax = b;$

II $\left[\exists z : A^\top z \in K^* \setminus sub(K^*) \text{ and } z^\top b \leq 0\right]$

or

$\left[\exists z : A^\top z \in sub(K^*) \text{ and } z^\top b \neq 0.\right]$

Proof. First, we will show that I and II cannot hold at once. Assume the opposite, then $\bar{z}^\top A\bar{x} \leq 0$ for some $\bar{x}$ and $\bar{z}$ that fulfill I and II, respectively. However, from the characterization (22) and (16), we obtain $\bar{z}^\top A\bar{x} > 0$, which is a contradiction. Thus, I implies $\neg II$.

Now, we will show that $\neg I$ implies II. Suppose that I does not hold, or equivalently $b \notin A(relint(K))$. With respect to vector b, there are two cases to consider:

1. $b \in lin(A(K)) \setminus A(relint(K))$,
2. $b \notin lin(A(K))$.

Case 1. implies $lin(A(K)) \setminus A(relint(K)) \neq \emptyset$, and hence $A(relint(K)) = relint(A(K))$ is nontrivial. This implies that $cl(A(K))$ is nontrivial and so is the dual cone $[A(K)]^* = \{z \mid A^\top z \in K^*\}$, thus $[A(K)]^* \setminus sub([A(K)]^*) = \{z \mid A^\top z \in K^* \setminus sub(K^*)\} \neq \emptyset$. Now, if $b \in cl(A(K))$, then from (22) we get that there exists a vector z such that $A^\top z \in K^* \setminus sub(K^*)$ and $z^\top b \leq 0$, which implies that the first part of II holds. If $b \notin cl(A(K))$, there exists a vector z such that $A^\top z \in K^*$ and $z^\top b < 0$. Since $b \in lin(cl(A(K))) = lin(A(K))$, it follows that $v^\top b = 0$ for all v such that $A^\top v \in sub(K^*)$ and, thus, $A^\top z \notin sub(K^*)$, which again implies that the first part of II holds.

Consider case 2. Since $b \notin lin(A(K))$, it follows that there exists a vector $z \in lin(A(K))^\perp = sub([A(K)]^*)$ (see (16)) such that $z^\top b \neq 0$. Thus, the second part of II holds. $\square$

Remark 1.4. Consider the primal-dual pair of programs (1.1) and (1.4). According to the proof of Theorem 1.3, if $\mathcal{P}^0 = \emptyset$, then there exists a vector $u = A^\top z \in R_{\tilde{\mathcal{D}}}$. Moreover, it can be said that

- $z^\top b \le 0$, if $b \in cl(A(K))$,
- $z^\top b < 0$, if $b \notin cl(A(K))$.

This means that, supposing that $\mathcal{D} \neq \emptyset$ and $b \notin cl(A(K))$ (which implies that the primal problem (1.1) is infeasible), we get that for any $(y, s) \in \mathcal{D}$ we have $\{(y, s) + \gamma(-z, A^\top z) \mid \gamma \ge 0\} \subseteq \mathcal{D}$ with $b^\top(y - \gamma z) \to +\infty$ as $\gamma \to +\infty$. Therefore, the dual problem (1.4) is unbounded.

Remark 1.5. It can be easily seen that if A is a full-rank $m \times n$ matrix ($m \le n$), *i.e.* the existence of the solution of $Ax = b$ is guaranteed, and the condition $\mathcal{S}(A^\top) \subseteq lin(K)$ holds, *i.e.* $\mathcal{S}(A^\top) \cap sub(K^*) = \{0\}$, then the alternatives in Theorem 1.3 simplify to

I $\exists x \in relint(K) : Ax = b$;

II $\exists z : A^\top z \in K^* \setminus sub(K^*)$ and $z^\top b \le 0$.

Moreover, for solid cones, the alternatives in Theorem 1.3 can be reduced to

I $\exists x \in int(K) : Ax = b$;

II $\exists z \neq 0 : A^\top z \in K^*$ and $z^\top b \le 0$.

This last special case was formulated in [7] and [8], and also for the semidefinite cone in [71].

Remark 1.6. From (16), it follows that if $\exists x \in K : \; Ax = b$, that is, the problem (1.1) is feasible, then the alternatives in Theorem 1.3 also can be simplified as stated in Remark 1.5.

1.2.2 Dual theorems of alternatives

In this subsection, we formulate the dual counterparts of Theorem 1.2 and Theorem 1.3.

The next theorem is the "dual variant" of the generalized Farkas lemma (Theorem 1.2). It is formulated in [31] for linear systems and is generalized to the case of symmetric matrices and linear matrix inequalities in [74]. A similar statement is included in [6], however, the strong alternative condition is formulated in terms of solvability of a perturbed system.

Theorem 1.4. *Assume that $K \subseteq \mathbb{R}^n$ is a cone satisfying Assumption 1, A is a given $m \times n$, ($m \le n$) matrix, and $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given vectors. At most one of the following statements is true:*

I $\exists y: \ c - A^\top y \in K^*$;

II $\exists z \in cl(K) : Az = 0$ *and* $c^\top z < 0$.

Moreover, if the cone $\mathcal{S}(A^\top) + K^*$ *is closed, then exactly one of the statements is true.*

Remark 1.7. Sufficient conditions for closedness of $\mathcal{S}(A^\top)+K^*$ can be found in Theorem C.2 or Table 1 in Appendix C.

Finally, we establish and prove a new theorem of alternatives, which deals with the relative interior of the cone K^*. It provides a strong alternative (and therefore also an equivalent condition) to the strict feasibility of the dual program (1.4).

Theorem 1.5. *Assume that* $K \subseteq \mathbb{R}^n$ *is a cone satisfying Assumption 1,* A *is a given* $m \times n$*,* $(m \leq n)$ *matrix, and* $b \in \mathbb{R}^m$*,* $c \in \mathbb{R}^n$ *are given vectors. Exactly one of the following statements is true:*

I $\exists y: \ c - A^\top y \in relint(K^*)$;

II $\left[\exists z \in cl(K) \setminus sub(cl(K)) : Az = 0 \text{ and } c^\top z \leq 0\right]$

or

$\left[\exists z \in sub(cl(K)) : Az = 0 \text{ and } c^\top z \neq 0\right]$.

Proof. First we will show that I implies $\neg II$. Assume by contradiction that I and II hold at once. Then $z^\top(c - A^\top \bar{y}) = \bar{z}^\top c \leq 0$ for some $\bar{y}$ and $\bar{z}$ that fulfill I and II, respectively. However, from characterization (23) and (17), we obtain $\bar{z}^\top c > 0$, which is a contradiction.

Now we will show that $\neg I$ implies II. Suppose that $\neg I$ holds or equivalently $\mathcal{S}(A^\top) + relint(K^*) = relint(\mathcal{S}(A^\top) + K^*) = \emptyset$. Regarding the vector c, there are two cases to consider.

1. $c \in lin(\mathcal{S}(A^\top) + K^*) \setminus relint(\mathcal{S}(A^\top) + K^*)$,
2. $c \notin lin(\mathcal{S}(A^\top) + K^*)$.

Case 1. Since $lin(\mathcal{S}(A^\top)+K^*) \setminus relint(\mathcal{S}(A^\top)+K^*) \neq \emptyset$, it follows that $relint(\mathcal{S}(A^\top)+K^*)$ and hence $cl(\mathcal{S}(A^\top) + K^*)$ and $(\mathcal{S}(A^\top) + K^*)^* = \mathcal{N}(A) \cap cl(K)$ are nontrivial. If $c \in cl(\mathcal{S}(A^\top) + K^*)$, then from (23) we get that there exists a vector $z \in cl(K) \setminus sub(cl(K))$ such that $Az = 0$ and $c^\top z \leq 0$, which implies that the first part of II holds. If $c \notin cl(\mathcal{S}(A^\top) + K^*)$, there exists a vector $z \in cl(K)$ such that $Az = 0$ such that $c^\top z < 0$.

Since $c \in lin(cl(\mathcal{S}(A^\top) + K^*)) = lin(\mathcal{S}(A^\top) + K^*)$, it follows that $c^\top z = 0$ for all $v \in \mathcal{N}(A) \cap sub(cl(K))$ and, thus, $z \notin sub(cl(K))$, which again implies that the first part of II is valid.

Case 2. Since $c \notin lin(\mathcal{S}(A^\top) + K^*)$, from (17) it follows that there exists a vector $z \in \mathcal{N}(A) \cap sub(cl(K))$ such that $c^\top z \neq 0$, which implies that the second part of II holds. □

Remark 1.8. Consider the primal-dual pair of programs (1.1) and (1.4). According to the proof of Theorem 1.5, if $\mathcal{D}^0 = \emptyset$, then there exists a vector $z \in \mathcal{N}(A) \cap cl(K)$ such that

- $c^\top z \leq 0$, if $c \in cl(\mathcal{S}(A^\top) + K^*)$,
- $c^\top z < 0$, if $c \notin cl(\mathcal{S}(A^\top) + K^*)$.

This means that, supposing that $\mathcal{P} \neq \emptyset$ and $c \notin cl(\mathcal{S}(A^\top) + K^*) = R^*_{\mathcal{P}}$ (which implies that the dual problem (1.4) is infeasible), we get that for any $x \in \mathcal{P}$ we have $\{x + \gamma z \mid \gamma \geq 0\} \subseteq \mathcal{P}$ with $c^\top(x + \gamma z) \to -\infty$ as $\gamma \to +\infty$. Therefore, the primal problem (1.1) is unbounded.

Remark 1.9. Theorems 1.4 and 1.5 can be obtained from Theorems 1.2 and 1.3, respectively, by rewriting the alternative I using the system of linear equations $c - A^\top y = s$ and the cone $\mathbb{R}^m \times K^*$. For the reader's convenience, we have included a straightforward proof of Theorem 1.5.

Remark 1.10. Analogously to the case of Theorem 1.3 and Remark 1.5, it can be seen that, requiring the condition $\mathcal{N}(A) \subseteq lin(K^*)$ to hold (implying $\mathcal{N}(A) \cap sub(cl(K)) = \{0\}$), the alternatives in Theorem 1.5 can be simplified to

I $\exists y: \; c - A^\top y \in relint(K^*)$;

II $\exists z \in cl(K) \setminus sub(cl(K)): Az = 0$ and $c^\top z \leq 0$.

Furthermore, if K^* is solid (or $cl(K)$ is pointed), the alternatives in Theorem 1.5 reduce to

I $\exists y: \; c - A^\top y \in int(K^*)$;

II $\exists z \in cl(K): Az = 0$ and $c^\top z \leq 0$.

This last special case has been considered for the semidefinite cone in [71].

Remark 1.11. From (17), it follows that if $\exists y: \; c - A^\top y \in K^*$, that is, the problem (1.4) is feasible, then the alternatives in Theorem 1.5 also can be simplified as stated in Remark 1.10.

1.3 Strong duality

1.3.1 Zero duality gap

The famous Slater result that the strict feasibility of the convex problem implies the strong duality property $d^* = p^*$ and, provided the optimal value is finite, also the existence of a dual optimal solution, is widely known. Its conic version was shown *e.g.* in [50] and [6] for proper cones. In [67], the strong duality property was studied for closed and solid, but not necessarily finite dimensional cones. Some duality results for general convex cones can be found in [47].

If one of the primal-dual pair of programs (1.1) and (1.4) is unbounded, the other is infeasible (see Corollary 1.1) and in this trivial case $p^* = d^*$. The basic idea behind the proof of the nontrivial strong duality property is linked with the generalized Farkas lemma and its dual counterpart (Theorem 1.2 and Theorem 1.4). In the generalized version of the theorems of alternatives, the assumption of closedness of the linear image of a convex cone (or closedness of the Minkowski sum of a convex cone and a linear subspace in the dual version, respectively) is needed. However, the closedness assumption is guaranteed by the existence of the interior point in the dual (primal) feasible set (see Appendix C). The known strong duality results for the convex conic problems are formulated in the next two theorems, see also [47] (Theorem 7) or, for conic programs with proper cones, in [6] (Theorem 2.4.1).

Theorem 1.6 (Strong duality). *Consider the primal-dual pair of programs* (1.1) *and* (1.4)*, where the cone K satisfies Assumption 1. Then*

a) if $\mathcal{D} \neq \emptyset$, $d^* < +\infty$ and $\mathbf{A}_c(cl(K))$ is closed, then $p^* = d^*$ and $\mathcal{P}^* \neq \emptyset$;

b) if $\mathcal{P} \neq \emptyset$, $p^* > -\infty$ and $\mathcal{S}(\mathbf{A}_b) + (K^* \times \{0\})$ is closed, then $p^* = d^*$ and $\mathcal{D}^* \neq \emptyset$,

where $\mathbf{A}_c = (A^\top \; c)^\top$ and $\mathbf{A}_b = (A \;\; -b)$.

Recall that the proof of Theorem 1.6 is based on Theorem 1.2, Theorem 1.4 and the weak duality property, and follows the standard scheme typically used in linear programming, or the one given *e.g.* in [6] for convex conic programs. The sufficient conditions that guarantee the closedness of $\mathbf{A}_c(cl(K))$, and $\mathcal{S}(\mathbf{A}_b) + (K^* \times \{0\})$ are $\mathcal{D}^0 \neq \emptyset$ and $\mathcal{P}^0 \neq \emptyset$, respectively, and the rest follows from Theorem C.2. This leads us to the following statement.

Theorem 1.7. *Consider the primal-dual pair of programs* (1.1) *and* (1.4)*, where the cone* K *satisfies Assumption 1.*

a) If $\mathcal{D}^0 \neq \emptyset$ *and* $\mathcal{P} \neq \emptyset$*, then* $p^* = d^*$ *and* $\mathcal{P}^* \neq \emptyset$*.*

b) If $\mathcal{P}^0 \neq \emptyset$ *and* $\mathcal{D} \neq \emptyset$*, then* $p^* = d^*$ *and* $\mathcal{D}^* \neq \emptyset$*.*

1.3.2 Necessary and sufficient conditions for nonemptiness and boundedness of sets of optimal solutions

The assumptions $\mathcal{D}^0 \neq \emptyset, \mathcal{P}^0 \neq \emptyset$ in statements a) and b) of Theorem 1.7, correspond to alternative I in Theorem 1.5 and Theorem 1.3, respectively. This gives us an opportunity to combine the results and establish necessary and sufficient conditions for boundedness of the optimal solution sets $\mathcal{P}^*$ and $\mathcal{D}^*$. We obtain a new result, stated in the next theorem.

Theorem 1.8. *Consider the primal-dual pair of programs* (1.1) *and* (1.4)*, where the cone* K *satisfies Assumption 1.*

a) Assume that K *is closed. The set* $\mathcal{P}^*$ *is nonempty and bounded if and only if* $\mathcal{P} \neq \emptyset$*,* $\mathcal{D}^0 \neq \emptyset$ *and* $sub(R_{\mathcal{P}}) = \{0\}$*.*

b) Suppose that $rank(A) = m$*. The set* $\mathcal{D}^*$ *is nonempty and bounded if and only if* $\mathcal{D} \neq \emptyset$*,* $\mathcal{P}^0 \neq \emptyset$ *and* $sub(R_{\tilde{\mathcal{D}}}) = \{0\}$*.*

Proof. a) First, assume that the set $\mathcal{P}^*$ is nonempty and bounded. Then clearly $\mathcal{P} \neq \emptyset$ and we only need to show that $\mathcal{D}^0 \neq \emptyset$ and $sub(R_{\mathcal{P}}) = \{0\}$. Take $x^* \in \mathcal{P}^*$ and assume by contradiction that the set $\mathcal{D}^0$ is empty. By applying Theorem 1.5 we obtain that:

- either there exists $z \in K \setminus sub(K)$ such that $Az = 0$ and $c^\top z \leq 0$ or

- there exists $z \in sub(K)$ such that $Az = 0$ and $c^\top z < 0$.

Consider the first case - then clearly for any $\gamma \geq 0$ we have $x^* + \gamma z \in \mathcal{P}^*$. We have constructed a ray in the optimal solution set $\mathcal{P}^*$, which contradicts its boundedness. Now consider the second case - then for any $\gamma \geq 0$ we have $x^* + \gamma z \in \mathcal{P}$, however

$c^\top(x^* + \gamma z) < p^*$, which contradicts the optimality of x^*.
Now assume that $\mathcal{D}^0 \neq \emptyset$ and $sub(R_\mathcal{P}) \neq \{0\}$. From (18) it follows that $sub(R_\mathcal{P}) = \mathcal{N}(A) \cap sub(K)$. We have that there exists $0 \neq z \in sub(K)$ such that $Az = 0$. This time, the strong alternatives in Theorem 1.5 imply $c^\top z = 0$. Again, we can construct a ray $\{x^* + \gamma z \mid \gamma \geq 0\} \subseteq \mathcal{P}^*$, which contradicts the boundedness of $\mathcal{P}^*$.

Conversely, suppose $\mathcal{P} \neq \emptyset$, $\mathcal{D}^0 \neq \emptyset$ and $sub(R_\mathcal{P}) = \{0\}$. From Theorem 1.7 a) we obtain that $\mathcal{P}^* \neq \emptyset$. Assume by contradiction that $\mathcal{P}^*$ is unbounded, *i.e.* $\hat{x} + \gamma w \in \mathcal{P}^* \subseteq \mathcal{P}$ $\forall \gamma \geq 0$. Hence, the equalities $c^\top w = 0$ and $Aw = 0$ hold and for an arbitrary $\hat{y} \in K^*$ we have that

$$\hat{x}^\top \hat{y} + \gamma w^\top \hat{y} \geq 0, \ \forall \gamma \geq 0. \tag{1.10}$$

Since the expression on the left in (1.10) is bounded below and $\gamma \geq 0$, it must hold $w^\top \hat{y} \geq 0$. Since $\hat{y} \in K^*$ was arbitrary, we get that $w \in K^{**} = K$. Recall that $w \in \mathcal{N}(A)$ and $c^\top w = 0$. If $w \notin sub(K)$, then by Theorem 1.5 we get a contradiction with the assumption $\mathcal{D}^0 \neq \emptyset$. On the other hand, $0 \neq w \in sub(K)$ contradicts the assumption $\mathcal{N}(A) \cap sub(K) = sub(R_\mathcal{P}) = \{0\}$.

b) This statement can be proved analogously, with the use of Theorem 1.3. The assumption $rank(A) = m$ is technical yet necessary to ensure the one-to-one correspondence between the dual variables y and s. It is only needed to argue that there would have to be a non-zero vector in $sub(R_{\hat{\mathcal{D}}}) = \mathcal{S}(A^\top) \cap sub(K^*)$ if we contradictorily assume that $\mathcal{D}^*$ is unbounded. □

The assumption in Theorem 1.8 a) that K is closed is necessary and cannot be left out as it is shown in the following example.

Example 1.1. Consider the primal convex conic program in the form (1.1)

$$\begin{aligned} \min\ & -x_1 + x_2 \\ \text{s.t. } & x_1 - x_2 = 0, \\ & x \in K = \{(x_1, x_2)^\top \mid x_1 - x_2 > 0,\ x_1 - 2x_2 \leq 0\} \cup \{0\}, \end{aligned} \tag{1.11}$$

and the corresponding dual program in the form (1.4)

$$\begin{aligned} &\max 0 \\ &\text{s.t. } s = (-1-y, 1+y)^\top \in K^*, \\ &\qquad K^* = \{(s_1, s_2)^\top \mid s_1 + s_2 \geq 0,\ 2s_1 + s_2 \geq 0\}, \end{aligned}$$

and

$$relint(K^*) = int(K^*) = \{(s_1, s_2)^\top \mid s_1 + s_2 > 0,\ 2s_1 + s_2 > 0\}.$$

We have that

$$\mathcal{P}^* = \mathcal{P} = \{(0,0)^\top\},$$

thus $\mathcal{P}^*$ is nonempty and unbounded. Obviously, $\mathcal{P} \neq \emptyset$ and $sub(R_{\mathcal{P}}) = \{0\}$, since K is pointed. However, $\mathcal{D}^0 = \emptyset$ and thus Theorem 1.8 a) fails to hold. Note that if we replace K with $cl(K)$ in the primal program (1.11), the optimal solution set will clearly be nonempty and unbounded, and thus Theorem 1.8 a) will hold.

If the cone K is pointed, then $sub(R_{\mathcal{P}}) = \{0\}$. Similarly, if the cone K^* is pointed (*i.e.* the cone K is solid), then $sub(R_{\tilde{\mathcal{D}}}) = \{0\}$. These special cases are covered in the following corollary. Clearly, if K is proper, then both equivalences a), b) in Corollary 1.2 hold.

Corollary 1.2.

a) *Suppose K is closed and pointed. The set $\mathcal{P}^*$ is nonempty and bounded if and only if $\mathcal{P} \neq \emptyset$ and $\mathcal{D}^0 \neq \emptyset$.*

b) *Suppose K is solid. The set $\mathcal{D}^*$ is nonempty and bounded if and only if $rank(A) = m$, $\mathcal{D} \neq \emptyset$ and $\mathcal{P}^0 \neq \emptyset$.*

Corollary 1.3. *Consider the primal-dual pair of programs* (1.1) *and* (1.4)*, where the cone K satisfies Assumption 1.*

a) *Suppose that K is closed. If the set $\mathcal{P}^*$ is nonempty and bounded, then $\mathbf{A}_c(cl(K))$ is closed.*

b) *Suppose that $rank(A) = m$. If the set $\mathcal{D}^*$ is nonempty and bounded, then $\mathcal{S}(\mathbf{A}_b) + (K^* \times \{0\})$ is closed.*

1.3.3 Discussion of the results in [47]

Remark 1.12. Consider the set $\tilde{\mathcal{D}} = \{s \mid (y, s) \in \mathcal{D}\}$ and the linear subspaces $sub(R_{\tilde{\mathcal{D}}})$ and $sub(R_{\tilde{\mathcal{D}}})^{\perp}$. Then for $\tilde{\mathcal{D}}$ it holds that $\tilde{\mathcal{D}} = (\tilde{\mathcal{D}} \cap sub(R_{\tilde{\mathcal{D}}})^{\perp}) + sub(R_{\tilde{\mathcal{D}}})$ (see Lemma A.1). The authors of [47] use this fact to define the so-called normalized dual feasible set $\tilde{\mathcal{D}}_N = \tilde{\mathcal{D}} \cap sub(R_{\tilde{\mathcal{D}}})^{\perp}$ and the normalized dual optimal solution set as $\tilde{\mathcal{D}}^*_N = \tilde{\mathcal{D}}^* \cap sub(R_{\tilde{\mathcal{D}}})^{\perp}$, where $\tilde{\mathcal{D}}^* = \{s^* \mid (y^*, s^*) \in \mathcal{D}^*\}$. They also study the boundedness of $\tilde{\mathcal{D}}^*_N$ and prove that $\mathcal{D} \neq \emptyset, \mathcal{P}^0 \neq \emptyset$ *if and only if the set* $\tilde{\mathcal{D}}^*_N$ *is nonempty and bounded.* (See Theorem 5 in [47].) Moreover, it is easy to show that under assumption $\mathcal{P}^0 \neq \emptyset$ it holds $sub(R_{\tilde{\mathcal{D}}}) = \{0\}$ iff $\tilde{\mathcal{D}}^* = \tilde{\mathcal{D}}^*_N$. Therefore, the result of Theorem 1.8 b), reformulated in terms of the normalized dual optimal solution set, states

- If $\mathcal{D} \neq \emptyset, \mathcal{P}^0 \neq \emptyset, sub(R_{\tilde{\mathcal{D}}}) = \{0\}$, then $\tilde{\mathcal{D}}^* = \tilde{\mathcal{D}}^*_N$ and it is nonempty and bounded.
- If $\tilde{\mathcal{D}}^*$ is nonempty and bounded, then $\mathcal{D} \neq \emptyset, \mathcal{P}^0 \neq \emptyset, sub(R_{\tilde{\mathcal{D}}}) = \{0\}$, *i.e.* $\tilde{\mathcal{D}}^* = \tilde{\mathcal{D}}^*_N$.

The authors of [47] do not explicitly formulate an analogous result dealing with the normalized primal optimal solution set. The main reason is that they consider the primal conic program with a general (not necessarily closed) convex cone. However, for a closed convex cone K we may consider the linear subspaces $sub(R_{\mathcal{P}})$ and $sub(R_{\mathcal{P}})^{\perp}$, and the normalized primal optimal solution set as $\mathcal{P}_N = \mathcal{P} \cap sub(R_{\mathcal{P}})^{\perp}$. Then the result of Theorem 1.8 a), reformulated in terms of the normalized primal optimal solution set, states

- If $\mathcal{P} \neq \emptyset, \mathcal{D}^0 \neq \emptyset, sub(R_{\mathcal{P}})^{\perp} = \{0\}$, then $\mathcal{P}^* = \mathcal{P}^*_N$ and it is nonempty and bounded.
- If $\mathcal{P}^*$ is nonempty and bounded, then $\mathcal{P} \neq \emptyset, \mathcal{D}^0 \neq \emptyset, sub(R_{\mathcal{P}})^{\perp} = \{0\}$, *i.e.* $\mathcal{P}^* = \mathcal{P}^*_N$.

As stated in Theorem 1.7 and Theorem 5 in [47] (see Remark 1.12) The assumption $\mathcal{P}^0 \neq \emptyset, \mathcal{D} \neq \emptyset$ guarantees that the sets $\tilde{\mathcal{D}}^*$ and $\tilde{\mathcal{D}}^*_N$ are nonempty. However, the boundedness of $\tilde{\mathcal{D}}^*$ is not guaranteed. This is demonstrated in the following simple example.

Example 1.2. Consider the primal convex conic program in the form (1.1)

$$\begin{array}{ll} \min & -5x_1 \\ \text{s.t.} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ & x \in K := \{(s, t, t)^\top \mid s \in \mathbb{R},\ t \geq 0\} \end{array}$$

and the corresponding dual program in the form (1.4)

$$\begin{aligned} \max \quad & 2y_1 + 2y_2 \\ \text{s.t.} \quad & s = (-5 - y_1 - y_2, -y_1, -y_2)^\top \in K^* \\ & K^* = \{(0, z_2, z_3)^\top \mid z_2 + z_3 \geq 0\}. \end{aligned}$$

Obviously $\mathcal{P}^0 \neq \emptyset$, $\mathcal{D} \neq \emptyset$, A is a full rank matrix, but $sub(R_{\tilde{\mathcal{D}}}) = \{(0, z_2, -z_2)^\top \mid z_2 \in \mathbb{R}\} \neq \{0\}$. From Theorem 5 in [47] we have that $\tilde{\mathcal{D}}_N^*$ is nonempty and bounded. It can be calculated that $\tilde{\mathcal{D}}_N^* = \{(0, 2.5, 2.5)^\top\}$. However, from Theorem 1.8 b) we have that $\mathcal{D}^*$ is unbounded or empty. In fact, it is unbounded since

$$\mathcal{D}^* = \{((-5 - r, r)^\top, (0, 5 + r, -r)^\top) \mid r \in \mathbb{R}\}.$$

and so is $\tilde{\mathcal{D}}^* = \{(0, 5 + r, -r)^\top \mid r \in \mathbb{R}\}$. Thus Theorem 5 in [47] does not guarantee the boundedness of $\tilde{\mathcal{D}}^*$.

1.3.4 Necessary and sufficient conditions for nonemptiness and unboundedness of sets of optimal solutions

Theorem 1.9. *Consider the primal-dual pair of programs* (1.1) *and* (1.4)*, where the cone K satisfies Assumption 1.*

a) Assume that $\mathcal{N}(\mathbf{A}_c) \cap relint(K) \neq \emptyset$. $\mathcal{D} \neq \emptyset$, $\mathcal{P} \neq \emptyset$ if and only if $p^* = d^*$, and the set $\mathcal{P}^*$ is nonempty and unbounded.

b) Assume that $\mathcal{S}(\mathbf{A}_b^\top) \cap relint(K^* \times \{0\}) \neq \emptyset$. $\mathcal{D} \neq \emptyset$, $\mathcal{P} \neq \emptyset$ if and only if $p^* = d^*$, and the set $\mathcal{D}^*$ is nonempty and unbounded.

Proof. a) Note that the assumption $\mathcal{N}(\mathbf{A}_c) \cap relint(K) \neq \emptyset$ is equivalent to the existence of a vector $v \in \mathcal{N}(A) \cap relint(K)$ such that $c^\top v = 0$.

Frist, assume that $\mathcal{D} \neq \emptyset$ and $\mathcal{P} \neq \emptyset$. The assumption $\mathcal{N}(\mathbf{A}_c) \cap relint(K) \neq \emptyset$ is equivalent to (ii-b) in Table 1 applied to the linear map $\mathbf{A}_c = (A^\top \; c)^\top$. Then, according to Theorem C.2 b), the cone $\mathbf{A}_c(cl(K))$ is a linear subspace (hence closed). Then from Theorem 1.6 a) we get that $\mathcal{P}^* \neq \emptyset$ and $p^* = d^*$. Thus, if $x^* \in \mathcal{P}^*$ and $v \in \mathcal{N}(\mathbf{A}_c) \cap relint(K)$, then clearly $x^* + \alpha v \in \mathcal{P}^*$, $\forall \alpha \geq 0$. Therefore $\mathcal{P}^*$ must be unbounded.

Now, suppose that $\mathcal{P}^*$ is nonempty and unbounded, and it holds that $p^* = d^*$. Clearly, $\mathcal{P} \neq \emptyset$ and it remains to show that $\mathcal{D} \neq \emptyset$. Since $\mathcal{N}(\mathbf{A}_c) \cap relint(K) \neq \emptyset$, we have that $\mathcal{N}(A) \cap relint(K) \neq \emptyset$ and thus (ii-b) in Table 1 holds. According to Theorem C.2

b) we obtain that $\mathcal{S}(A^\top) + K^*$ is closed. This means that the alternatives in Theorem 1.4 are strong: one and only one of them holds. Now, assume that $\mathcal{D} = \emptyset$, which is equivalent to $\neg I$. It follows that II holds and thus there exists a vector $z \in cl(K)$ such that $Az = 0$ and $c^\top z < 0$. Take $x^* \in \mathcal{P}^*$ and $v \in \mathcal{N}(\mathbf{A}_c) \cap relint(K)$. From (24) it follows that $v + z \in relint(K)$. We will show that the vector $x^* + v + z \in \mathcal{P}$. Again, from (24) we have that $x^* + v + z \in relint(K) \subseteq K$, moreover, it holds that $A(x^* + v + z) = Ax^* + A(v + z) = Ax^* = b$, thus $x^* + v + z \in \mathcal{P}$. However, we have that $c^\top(x^* + v + z) = c^\top x^* + c^\top z < c^\top x^* = p^*$, which is a contradiction with the optimality of x^*.

b) Note that the assumption $\mathcal{S}(\mathbf{A}_b^\top) \cap relint(K^* \times \{0\}) \neq \emptyset$ is equivalent to the existence of a vector z such that $A^\top z \in relint(K^*)$ and $b^\top z = 0$.

This statement can be proved analogously, with the use of (ii-a) in Table 1, Theorem C.2 a), Theorem 1.6 b) and Theorem 1.2. Note that the assumption $\mathcal{S}(\mathbf{A}_b^\top) \cap relint(K^* \times \{0\}) \neq \emptyset$ implies that condition (ii-a) in Table 1 holds, moreover, it is equivalent to condition (ii-a) in Table 1 applied to the linear map $\mathbf{A}_b = (A\ -b)$ and the cone $K \times \mathbb{R}$. $\square$

The following example demonstrates that the global assumption $\mathcal{N}(\mathbf{A}_c) \cap relint(K) \neq \emptyset$ in Theorem 1.9 a) is sufficient but not necessary for the equivalence to hold: the ray defined by $v \in \mathcal{N}(A) \cap relint(K)$ in part a) may fail to exist. Similarly, the global assumption $\mathcal{S}(\mathbf{A}_b^\top) \cap relint(K^* \times \{0\}) \neq \emptyset$ in Theorem 1.9 b) is sufficient but not necessary for the equivalence to hold: the vector z such that $A^\top z \in relint(K^*)$ in part b) may fail to exist.

Example 1.3. Consider the primal convex conic program in the form (1.1)

$$\begin{aligned} \min \quad & x_1 + x_3 \\ \text{s.t.} \quad & x_1 + x_3 = 0 \\ & x \in K := \{(x_1, x_2, x_3)^\top \mid \sqrt{x_1^2 + x_2^3} \leq x_3\} \end{aligned}$$

and the corresponding dual program in the form (1.4)

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & s = (1 - y, 0, 1 - y)^\top \in K^* \\ & K^* = \{(s_1, s_2, s_3)^\top \mid \sqrt{s_1^2 + s_2^2} \leq s_3\} = K. \end{aligned}$$

We have that

$$\mathcal{P}^* = \mathcal{P} = \{t(-1, 0, 1)^\top \mid t \geq 0\} \neq \emptyset,$$

thus $\mathcal{P}^*$ is nonempty and unbounded. Moreover, it holds $p^* = d^* = 0$. We also have that

$$\mathcal{D}^* = \mathcal{D} = \{(1-y, 0, 1-y)^\top \mid y \leq 1\} \neq \emptyset,$$

However, since $relint(K) = int(K) = \{(x_1, x_2, x_3)^\top \mid \sqrt{x_1^2 + x_2^3} < x_3\}$, we have that $\mathcal{N}(A) \cap relint(K) = \emptyset$, which implies that there does not exist $v \in \mathcal{N}(A) \cap relint(K)$ such that $c^\top v = 0$.

Similarly, there is no such $z \in \mathbb{R}$ for which it holds $z(1,0,1)^\top \in relint(K^*)$.

1.3.5 Summary of sufficient conditions for strong duality

If we put together results from Theorem 1.7, Theorem 1.8, Remark 1.12 and Theorem 1.9, we can list eight sufficient conditions for strong duality property $p^* = d^*$, see Table 1.1.

Table 1.1: List of sufficient conditions for zero optimal duality gap, *i.e.* $p^* = d^*$.

(P)	$\mathcal{P}^0 \neq \emptyset$ $\mathcal{P}^* \neq \emptyset$ and bounded provided that K is closed $\mathcal{P}^*_N \neq \emptyset$ and bounded provided that K is closed $\mathcal{D} \neq \emptyset$, $\mathcal{P} \neq \emptyset$, $\mathcal{N}(\mathbf{A}_c) \cap relint(K) \neq \emptyset$.
(D)	$\mathcal{D}^0 \neq \emptyset$ $\mathcal{D}^* \neq \emptyset$ and bounded $\tilde{\mathcal{D}}^*_N \neq \emptyset$ and bounded $\mathcal{D} \neq \emptyset$, $\mathcal{P} \neq \emptyset$, $\mathcal{S}(\mathbf{A}_b^\top) \cap relint(K^* \times \{0\}) \neq \emptyset$.

1.4 Strong duality in convex programming

In this section we concentrate on standard problems of convex programming. It is a known fact that, due to an equivalent conic reformulation, these problems form a special class of convex conic programming problems, see *e.g.* [15], [6], or [57]. Therefore, the generalized Slater condition (see [15, Section 5.2.3]) for convex programming problems and its conic version (Theorem 1.6) for convex conic problems are expected to be related in some way. In this section we examine the sufficient conditions for strong duality in the respective conic reformulation of a standard problem of convex programming and compare the results with the generalized Slater condition for convex programming.

1.4.1 Primal and dual convex program

The standard form of a primal convex program (see [15, Section 5.1.1]) is formulated as follows

$$\begin{aligned} \min\ & f_0(x) \\ \text{s.t. } & f_i(x) \leq 0,\ i = 1, 2, \dots, k, \\ & g_j(x) \leq 0,\ j = 1, 2, \dots, l, \\ & Ax = b, \end{aligned} \tag{1.12}$$

where $f_i : \emptyset \neq X_i \subseteq \mathbb{R}^n \to \mathbb{R}$, $i = 0, 1, \dots k$ are convex functions[9] defined on open convex sets X_i, $g_j(x) := \alpha_j^\top x + \beta_j$, $j = 1, 2, \dots, l$ are affine functions[10], A is an $m \times n$ matrix[11], and $b \in \mathbb{R}^m$. The common domain of functions f_i, $i = 0, 1, \dots, k$ is denoted $X := \bigcap_{i=0}^k X_i$, which is an open convex set. For problem (1.12) the following notions are usually defined

- the set of primal feasible points $\mathcal{P}_K := \{x \in X \mid Ax = b,\ f_i(x) \leq 0,\ g_j(x) \leq 0,\ i = 1, 2, \dots, k,\ j = 1, 2, \dots, l\}$;
- the primal optimal value $p_K^* := \inf_x \{f_0(x) \mid x \in \mathcal{P}_K\}$, if $\mathcal{P}_K \neq \emptyset$; and $p_K^* := +\infty$, if $\mathcal{P}_K = \emptyset$;
- the set of primal optimal points $\mathcal{P}_K^* := \{x \in \mathcal{P}_K \mid f_0(x) = p_K^*\}$.

The Lagrange function $L : X \times \mathbb{R}^m \times \mathbb{R}_+^k \times \mathbb{R}_+^l \to \mathbb{R}$ of problem (1.12) takes the following form

$$L(x, y, \lambda, \mu) = f_0(x) + y^\top (b - Ax) + \sum_{i=1}^k \lambda_i f_i(x) + \sum_{j=1}^l \mu_j g_j(x).$$

[9] We assume that $f_1, \dots, f_k$ are not affine functions.
[10] We assume that $\alpha_j \neq 0$, $j = 1, 2, \dots, l$.
[11] We assume that $A \neq 0_{m \times n}$.

The Lagrange dual problem of problem (1.12) takes the following form

$$\begin{aligned}
&\max\ G(y,\lambda,\mu) := \inf_{x\in X} L(x,y,\lambda,\mu) \\
&\text{s.t. } \lambda \geq 0,\ \ \mu \geq 0,\ \ y \in \mathbb{R}^m.
\end{aligned} \tag{1.13}$$

Analogously, we define

- the set of dual feasible points $\mathcal{D}_K := \{(y,\lambda,\mu) \mid \lambda,\mu \geq 0\}$;
- the dual optimal value $d_K^* := \sup_{y,\lambda,\mu}\{G(y,\lambda,\mu) \mid \lambda,\mu \geq 0\}$, if $\mathcal{D} \neq \emptyset$; and $d_K^* := -\infty$, if $\mathcal{D} = \emptyset$;
- the set of dual optimal points $\mathcal{D}_K^* := \{(y,\lambda,\mu) \mid G(y,\lambda,\mu) = d_K^*\}$.

1.4.2 Conic reformulation of a primal convex problem

In this section we present an equivalent reformulation of problem (1.12) as a convex conic problem. The idea of the reformulation lies in embedding a convex set into a convex cone, as it is described in [15, Section 3.2.6], [6, Section 3.3], or [36, Section 6.2].

$$\begin{aligned}
&\min\ t \\
&\text{s.t. } Ax = b, \\
&\qquad s = 1, \\
&\qquad (x^\top, t, s)^\top \in K,
\end{aligned} \tag{1.14}$$

where

$$\begin{aligned}
K := \Big\{&(x^\top,t,s)^\top \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid s > 0,\ \frac{x}{s} \in X,\ f_0\left(\frac{x}{s}\right) \leq \frac{t}{s},\ f_i\left(\frac{x}{s}\right) \leq 0, \\
&g_j\left(\frac{x}{s}\right) \leq 0,\ i = 1,2,\ldots,k,\ j = 1,2,\ldots,l\Big\} \cup \{(0^\top,0,0)^\top\}.
\end{aligned} \tag{1.15}$$

Thus, problem (1.12) can be equivalently reformulated as a convex conic problem (1.14), which is of the form (1.1). Therefore, the dual problem of (1.14) is

$$\begin{aligned}
&\max\ b^\top y + z \\
&\text{s.t. } (y^\top A, 0, z)^\top + S = (0^\top, 1, 0)^\top, \\
&\qquad S \in K^*,\ y \in \mathbb{R}^m,\ z \in \mathbb{R}.
\end{aligned} \tag{1.16}$$

1.4.3 The generalized Slater condition and the conic version of Slater condition

The generalized Slater condition (see [15, Section 5.2.3]) is a known sufficient condition for strong duality to hold in standard problems of convex programming. More precisely, if there exists a point $\tilde{x} \in X$ such that

$$f_i(\tilde{x}) < 0,\ i = 1, 2, \ldots, k, \tag{1.17}$$

$$g_j(\tilde{x}) \leq 0,\ j = 1, 2, \ldots, l, \tag{1.18}$$

$$A\tilde{x} = b, \tag{1.19}$$

then the strong duality in problems (1.12) and (1.13) holds, *i.e.* $p^*_K = d^*_K$. Moreover, if $d^*_K > -\infty$, then $\mathcal{D}^*_K \neq \emptyset$. In other words, the strong duality property is guaranteed if there exists a feasible point $\tilde{x} \in X$, for which non-affine constraints are satisfied with strict inequalities but affine constraints may not be satisfied with strict inequalities.

The conic version of Slater condition for problem (1.14) states that the strong duality property holds, *i.e.* $p^* = d^*$, if there exists a point $(\bar{x}^\top, \bar{t}, \bar{s})^\top$ such that

$$A\bar{x} = b, \tag{1.20}$$

$$\bar{s} = 1, \tag{1.21}$$

$$(\bar{x}^\top, \bar{t}, \bar{s})^\top \in relint(K). \tag{1.22}$$

We show that if the generalized Slater condition for problem (1.12) is satisfied, then the conic version of Slater condition for problem (1.14) is satisfied, and hence, the strong duality property holds for problems (1.14) and (1.16).

Theorem 1.10. *Consider the convex problem* (1.12)*, its equivalent conic reformulation* (1.14)*, and their dual problems* (1.13) *and* (1.16)*. If there exists* $\bar{x} \in X$ *such that* $\bar{x}$ *satisfies* (1.17)–(1.19)*, then*

a) there exists a point satisfying (1.20)–(1.22)*,*

b) it holds that $p^*_K = p^* = d^* = d^*_K$*.*

Before we prove Theorem 1.10, we need to include three auxiliary propositions.

We define

$$
\begin{aligned}
K_{\mathcal{F}} &:= \left\{(x^\top, t, s)^\top \mid s > 0,\ \frac{x}{s} \in X,\ f_0\left(\frac{x}{s}\right) \le \frac{t}{s},\ f_i\left(\frac{x}{s}\right) \le 0,\ i = 1, 2, \ldots, k\right\},\\
K_{\mathcal{G}} &:= \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \le 0,\ j = 1, 2, \ldots, l\},\\
K_{\mathcal{A}} &:= \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \le 0,\ j = 1, 2, \ldots, l,\ Ax - bs = 0\}.
\end{aligned}
$$

Problem (1.14) may then be equivalently formulated in the following form

$$
\begin{aligned}
&\min t\\
&\text{s.t. } s = 1,\\
&\quad (x^\top, t, s)^\top \in K_{\mathcal{F}} \cap K_{\mathcal{A}}.
\end{aligned}
\tag{1.23}
$$

The following essential proposition was formulated and proved in [36, Proposition 6.2].

Proposition 1.3. *Consider the convex problem* (1.12)*, its equivalent conic reformulation* (1.14)*, and their dual problems* (1.13) *and* (1.16)*. If there exists $\bar{x} \in X$ such that $\bar{x}$ satisfies* (1.17)*, then $int(K_{\mathcal{F}}) \neq \emptyset$ and $(\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top \in int(K_{\mathcal{F}})$.*

Now, we include a proposition dealing with the interior of cone $K_{\mathcal{G}}$. The proof can be found in Appendix B.

Proposition 1.4. *It holds that*

$$int(K_{\mathcal{G}}) = \{(x^\top, t, s) \mid \alpha_j^\top x + \beta_j s < 0,\ j = 1, 2, \ldots, l\}.$$

Constraint $Ax - bs = 0$ in the definition of $K_{\mathcal{A}}$ may be treated as $2m$ inequalities in the from $a_i^\top x - b_i s \le 0$ and $-a_i^\top x + b_i s \le 0$, where a_i represents the i-th row of matrix A, $i = 1, 2, \ldots, m$. We define $\alpha_{l+i} := a_i$, $\alpha_{l+m+i} = -a_i$, $\beta_{l+i} = -b_i$, $\beta_{l+m+i} = b_i$ for $i = 1, 2, \ldots, m$.

With this constraint modification we have two cases to consider. Either $K_{\mathcal{A}}$ is a vector subspace, *i.e.* $K_{\mathcal{A}} = lin(K_{\mathcal{A}})$, or not, *i.e.* $K_{\mathcal{A}} \neq lin(K_{\mathcal{A}})$.

Considering the first case, for any point $(\bar{x}^\top, \bar{t}, \bar{s})^\top \in K_{\mathcal{A}}$ it holds that $-(\bar{x}^\top, \bar{t}, \bar{s})^\top \in K_{\mathcal{A}}$, and hence, $\alpha_j^\top \bar{x} + \beta_j \bar{s} = 0$ for all $j = 1, 2, \ldots, l + 2m$. On the other hand, if for $(\bar{x}^\top, \bar{t}, \bar{s})^\top$ it holds that $\alpha_j^\top \bar{x} + \beta_j \bar{s} = 0$ for all $j = 1, 2, \ldots, l + 2m$ and $A\bar{x} - b\bar{s} = 0$, then $(\bar{x}^\top, \bar{t}, \bar{s})^\top \in K_{\mathcal{A}}$ and thus

$$K_{\mathcal{A}} = \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s = 0,\ j = 1, 2, \ldots, l + 2m\} = relint(K_{\mathcal{A}}). \tag{1.24}$$

Considering the second case, we will construct a new cone $K_{\mathcal{G}}^{(l+2m)}$ with a nonempty interior by eliminating some constraints from $K_{\mathcal{A}}$.

We will construct a subsequence $\{j_1, j_2, \dots, j_r\} \subseteq \{1, 2, \dots, l+2m\}$ of constraints defining $K_{\mathcal{G}}^{(l+2m)}$ and the complementary subsequence

$$(j_{r+1}, j_{r+2}, \dots, j_{l+2m}) = (j_k)_{k \in \{1,2,\dots,l+2m\} \setminus \{j_1,\dots,j_r\}},$$

which will be excluded from $K_{\mathcal{A}}$.

We set $j_1 := 1$ and $K_{\mathcal{G}}^{(1)} := \{(x^\top, t, s)^\top \mid \alpha_1^\top x + \beta_1 s \leq 0\}$. If

$$int(\{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \leq 0,\ j = 1, 2\}) \neq \emptyset,$$

we set $K_{\mathcal{G}}^{(2)} := \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \leq 0,\ j = 1, 2\}$ and $j_2 = 2$; otherwise, we set $K_{\mathcal{G}}^{(2)} := K_{\mathcal{G}}^{(1)}$ and $j_2 := \min_{i=2,\dots,l+2m} \left\{i \mid int(\{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \leq 0,\ j = 1, i\}) \neq \emptyset\right\}$.

We repeat the process with $K_{\mathcal{G}}^{(2)}$ to obtain $K_{\mathcal{G}}^{(3)}$ until we reach $K_{\mathcal{G}}^{(l+2m)}$. Note that this process ends after $l + 2m$ repetitions, and

$$K_{\mathcal{G}}^{(l+2m)} = \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \leq 0,\ j = j_1, j_2, \dots, j_r\},$$

with $int(K_{\mathcal{G}}^{(l+2m)}) \neq \emptyset$ and

$$j_k = \min_{i=k,k+1,\dots,l+2m} \{i \mid int(\{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \leq 0,\ j = j_1, j_2, \dots, j_{k-1}, i\}) \neq \emptyset\},$$

$k \in \{1, 2, \dots, l+2m\}$, $j_1 = 1$, and

$$(j_{r+1}, j_{r+2}, \dots, j_{l+2m}) = (j_k)_{k \in \{1,2,\dots,l+2m\} \setminus \{j_1,\dots,j_r\}}.$$

Moreover, it holds that $int(K_{\mathcal{G}}^{(l+2m)} \cap \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \leq 0\}) = \emptyset$ for all $j \in \{j_{r+1}, j_{r+2}, \dots, j_{l+2m}\}$. If for some $i \in \{j_{r+1}, \dots, j_{l+2m}\}$ it held that

$$int(K_{\mathcal{G}}^{(l+2m)} \cap \{(x^\top, t, s)^\top \mid \alpha_i^\top x + \beta_i s \leq 0\}) \neq \emptyset,$$

then also

$$int(\{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s \leq 0,\ j \in \{j_k \mid i > j_k,\ k \in \{1, 2, \dots, r\}\} \cup \{i\}\}) \neq \emptyset,$$

and thus $i \in \{j_1, \dots, j_r\}$, which would be a contradiction.

With this construction of $K_{\mathcal{G}}^{(l+2m)}$ we formulate and prove the following proposition.

Proposition 1.5. *a) If $K_{\mathcal{A}} \neq lin(K_{\mathcal{A}})$, it holds that*

$$relint(K_{\mathcal{A}}) = \big\{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s < 0,\ j \in J_1,\\ \alpha_j^\top x + \beta_j s = 0,\ j \in \{1, 2, \dots, l + 2m\} \setminus J_1\big\}, \quad (1.25)$$

for an appropriate set $J_1 \subseteq \{j_1, j_2, \dots, j_r\}$.

b) If $K_{\mathcal{A}} = lin(K_{\mathcal{A}})$, it holds that $relint(K_{\mathcal{A}}) = K_{\mathcal{A}}$, i.e. relation (1.24) *holds.*

Proof. The second part of the statement is trivial, we only prove the first part.

First assume that

$$int(K_{\mathcal{G}}^{(l+2m)}) \cap \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s = 0,\ j \in \{j_{r+1}, j_{r+2}, \dots, j_{l+2m}\}\} \neq \emptyset.$$

Then with the use of Proposition B.4 a) part ii) we have

$$relint(K_{\mathcal{A}}) = relint(K_{\mathcal{G}}^{(l+2m)} \cap \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s = 0,\ j \in \{j_{r+1}, j_{r+2}, \dots, j_{l+2m}\}\} =$$
$$= relint(K_{\mathcal{G}}^{(l+2m)}) \cap relint(\{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s = 0,\ j \in \{j_{r+1}, j_{r+2}, \dots, j_{l+2m}\}\}) =$$
$$= int(K_{\mathcal{G}}^{(l+2m)}) \cap \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s = 0,\ j \in \{j_{r+1}, j_{r+2}, \dots, j_{l+2m}\}\}.$$

Obviously, in this case $J_1 = \{j_1, \dots, j_r\}$.

Now assume that

$$int(K_{\mathcal{G}}^{(l+2m)}) \cap \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s = 0,\ j \in \{j_{r+1}, j_{r+2}, \dots, j_{l+2m}\}\} = \emptyset.$$

Let us denote

$$V := \{(x^\top, t, s)^\top \mid \alpha_j^\top x + \beta_j s = 0,\ j \in \{j_{r+1}, j_{r+2}, \dots, j_{l+2m}\}\}.$$

We need to eliminate constraints from $K_{\mathcal{G}}^{(l+2m)}$ to obtain a cone $K_{\mathcal{G}}^{(l+2m+r)}$ such that $int(K_{\mathcal{G}}^{(l+2m+r)})) \cap V \neq \emptyset$. Take j_1. If

$$int(\{(x^\top, t, s)^\top \mid \alpha_{j_1}^\top x + \beta_{j_1} s \leq 0\}) \cap V \neq \emptyset,$$

then we set $K_{\mathcal{G}}^{(l+2m+1)} := \{(x^\top, t, s)^\top \mid \alpha_{j_1}^\top x + \beta_{j_1} s \leq 0\})$; otherwise, $K_{\mathcal{G}}^{(l+2m+1)} := \mathbb{R}^{n+2}$. Take j_2, if $int(K_{\mathcal{G}}^{(l+2m+1)}) \cap V \neq \emptyset$, we set $K_{\mathcal{G}}^{(l+2m+2)} := K_{\mathcal{G}}^{(l+2m+1)} \cap \{(x^\top, t, s) \mid \alpha_{j_2}^\top x + \beta_{j_2} s \leq 0\}$; otherwise $K_{\mathcal{G}}^{(l+2m+2)} := K_{\mathcal{G}}^{(l+2m+1)}$. We repeat this process until we reach $K_{\mathcal{G}}^{(l+2m+r)}$ such that $int(K_{\mathcal{G}}^{(l+2m+r)}) \cap V \neq \emptyset$.

The set J_1 consits of indices j_i $i \in \{1, 2, \dots, r\}$ such that constraint $\alpha_{j_i}^\top x + \beta_{j_i} s \leq 0$ is included in $K_{\mathcal{G}}^{(l+2m+r)}$. Note that the case $J_1 = \emptyset$ corresponds with $K_{\mathcal{G}}^{(l+2m+r)} = \mathbb{R}^{n+2}$

and $K_{\mathcal{A}} = lin(K_{\mathcal{A}})$. On the other hand, the case $J_1 = \{j_1, \ldots, j_r\}$ corresponds with the case $K_{\mathcal{G}}^{(l+2m+r)} = K_{\mathcal{G}}^{(l+2m)}$.

Now we show that $K_{\mathcal{A}} = K_{\mathcal{G}}^{(l+2m+r)} \cap V$. It is easy to see that $K_{\mathcal{A}} \supseteq K_{\mathcal{G}}^{(l+2m+r)} \cap V$ now assume that $(\bar{x}^\top, \bar{t}, \bar{s})^\top \in K_{\mathcal{A}}$ but $(\bar{x}^\top, \bar{t}, \bar{s})^\top \notin K_{\mathcal{G}}^{(l+2m+r)} \cap V$. It means that there exists an index $i \in \{j_1, \ldots, j_r\} \setminus J_1$ such that $\alpha_i^\top \bar{x} + \beta_i \bar{s} < 0$, which implies that $int(\{(x^\top, t, s)^\top \mid \alpha_i^\top x + \beta_i s \leq 0\}) \cap V \neq \emptyset$, thus $int(K_{\mathcal{G}}^{(l+2m+r)}) \cap int(\{(x^\top, t, s)^\top \mid \alpha_i^\top x + \beta_i s \leq 0\}) \cap V \neq \emptyset$ and $i \in J_1$ which is a contradiction.

The use of Proposition B.4 a) part ii) to intersection $int(K_{\mathcal{G}}^{(l+2m+r)}) \cap V$ finishes the proof. $\square$

Proposition 1.5 is useful as it provides a characterization of the relative interior of all $l+2m$ affine constraints in the form of inequalities, which is now easy to manipulate with. We are now ready to prove Theorem 1.10.

Proof of Theorem 1.10. a) Suppose that $\bar{x} \in X$ satisfies (1.17), (1.18) and (1.19). According to Proposition 1.3, the point

$$(\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top \in int(K_{\mathcal{F}}).$$

Moreover, $(\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top \in K_{\mathcal{A}}$. Therefore, there are two cases to consider.

1.) $(\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top \in relint(K_{\mathcal{A}})$.

In this case we have that $int(K_{\mathcal{F}}) \cap relint(K_{\mathcal{A}}) \neq \emptyset$, therefore,

$$\begin{aligned}(\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top \in\ & int(K_{\mathcal{F}}) \cap relint(K_{\mathcal{A}}) = \\ & = relint(K_{\mathcal{F}}) \cap relint(K_{\mathcal{A}}) = relint(K_{\mathcal{F}} \cap K_{\mathcal{A}}).\end{aligned}$$

Relations (1.20) and (1.21) are automatically satisfied, and, since $K_{\mathcal{F}} \cap K_{\mathcal{A}} \subseteq K$, it holds $relint(K_{\mathcal{F}} \cap K_{\mathcal{A}}) \subseteq relint(K)$, and thus (1.22) is satisfied.

2.) $(\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top \notin relint(K_{\mathcal{A}})$.

Since $relint(K_{\mathcal{A}}) \neq \emptyset$, there exists a point $(\tilde{x}^\top, \tilde{t}, \tilde{s})^\top \in relint(K_{\mathcal{A}})$. By [10, Proposition 1.4.1 (a)] we have that

$$\{(\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top + \omega((\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top) \mid \omega \in (0, 1]\} \subseteq relint(K_{\mathcal{A}}).$$

Since $int(K_{\mathcal{F}}) \neq \emptyset$, there exists $\varepsilon > 0$ such that

$$(\bar{x}, f_0(\bar{x}) + 1, 1)^\top + \varepsilon(q_1^\top, q_2, q_3)^\top \in K_{\mathcal{F}},$$

for all $q = (q_1^\top, q_2, q_3)^\top$ with $q \in \mathcal{B}((0^\top, 0, 0)^\top, 1)$. Again, applying Proposition 1.4.1 (a) in [10], we get that $(\bar{x}, f_0(\bar{x}) + 1, 1)^\top + (1-\tau)\varepsilon(q_1^\top, q_2, q_3)^\top \in int(K_{\mathcal{F}})$ for all $\tau \in (0, 1]$.

We set

$$q_\gamma := \frac{(\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}, f_0(\bar{x}) + 1, 1)^\top}{\gamma \|(\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}, f_0(\bar{x}) + 1, 1)^\top\|_2},$$

obviously for $\gamma > 1$ we have $q_\gamma \in \mathcal{B}((0^\top, 0, 0)^\top, 1)$. Thus for all $\gamma > 1$ it holds that

$$(\bar{x}, f_0(\bar{x}) + 1, 1)^\top + \gamma\omega \|(\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}, f_0(\bar{x}) + 1, 1)^\top\|_2 q_\gamma \in relint(K_{\mathcal{A}}), \quad \forall \omega \in (0, 1].$$

We choose $\gamma > 0$, $\omega \in (0, 1]$ and $\tau \in (0, 1]$ so that it holds

$$\gamma \|(\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}, f_0(\bar{x}) + 1, 1)^\top\|_2 \omega = \varepsilon(1 - \tau).$$

Pick an arbitrary but fixed $\bar{\tau} \in (0, 1)$. We set

$$\bar{\omega} = \frac{\varepsilon(1 - \bar{\tau})}{\bar{\gamma} \|(\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}, f_0(\bar{x}) + 1, 1)^\top\|_2},$$

where

$$\bar{\gamma} > \max\left\{ \frac{\varepsilon(1 - \bar{\tau})}{\|(\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}, f_0(\bar{x}) + 1, 1)^\top\|_2}, -\frac{(\tilde{s} - 1)\varepsilon(1 - \bar{\tau})}{\|(\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}, f_0(\bar{x}) + 1, 1)^\top\|_2}, 1 \right\},$$

thus $\bar{\omega} \in (0, 1)$. It follows that

$$(\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top + \bar{\omega}((\tilde{x}^\top, \tilde{t}, \tilde{s})^\top - (\bar{x}^\top, f_0(\bar{x}) + 1, 1)^\top) \in int(K_{\mathcal{F}}) \cap relint(K_{\mathcal{A}}).$$

Moreover, $1 + \bar{\omega}(\tilde{s} - 1) > 0$, we have that

$$\left(\frac{\bar{x}^\top + \bar{\omega}(\tilde{x} - \bar{x})^\top}{1 + \bar{\omega}(\tilde{s} - 1)}, \frac{f_0(\bar{x}) + 1 + \bar{\omega}(\tilde{t} - f_0(\bar{x}) - 1)}{1 + \bar{\omega}(\tilde{s} - 1)}, 1 \right)^\top \in int(K_{\mathcal{F}}) \cap relint(K_{\mathcal{A}}).$$

Now, it is easy to see that relation (1.21) is satisfied. An analogous argument to case 1.) may be used to show that relation (1.22) is satisfied. We use Proposition 1.5 to show that relation (1.20) is satisfied.

If $K_{\mathcal{A}} = lin(K_{\mathcal{A}})$, then from (1.24) it follows that (1.20) is satisfied, *i.e.*

$$A \frac{\bar{x} + \bar{\omega}(\tilde{x} - \bar{x})}{1 + \bar{\omega}(\tilde{s} - 1)} = b,$$

since every affine constraint is satisfied with equality. If $K_{\mathcal{A}} \neq lin(K_{\mathcal{A}})$, then there exists a nonempty set $J_1 \subseteq \{j_1, \ldots, j_r\}$ such that $\alpha_j^\top x + \beta_j s = 0$, $\forall (x^\top, t, s)^\top \in relint(K_{\mathcal{A}})$ and $\forall j \in \{1, 2, \ldots, l + 2m\} \setminus J_1$. Since indices $l + m + 1, l + m + 2, \ldots, l + 2m \notin J_1$, it holds

$$\alpha_{l+m+i}^\top \frac{\bar{x} + \bar{\omega}(\tilde{x} - \bar{x})}{1 + \bar{\omega}(\tilde{s} - 1)} + \beta_{l+m+i} = 0, \quad i = 1, 2, \ldots, m,$$

from which it follows that $A\frac{\bar{x}+\bar{\omega}(\tilde{x}-\bar{x})}{1+\bar{\omega}(\tilde{s}-1)} = b$, and hence, (1.20) is satisfied.

b) From a) and Theorem 1.7 b) it follows $p^* = d^*$ for problems (1.14) and (1.16). The equality $p_K^* = p^*$ follows from the fact that (1.14) is an equivalent reformulation of (1.12) preserving optimal values. The relation $p_K^* = d_K^*$ follows from the generalized Slater condition for (1.12) and (1.13). We obtain that $p_K^* = p^* = d^* = d_K^*$. □

Remark 1.13. In general it holds that $p_K^* = p^* \geq d^* \geq d_K^*$. Further analysis of this relation may be found in [36, Section 6.4].

1.4.4 Example

In the following example, we show that a primal-dual pair of convex programs with a nonzero duality gap can be reformulated as a primal-dual pair of convex conic programs with a zero duality gap.

Example 1.4. Consider the convex program

$$\begin{aligned} &\min \frac{1}{x_1+1} \\ &\text{s.t. } \frac{x_1^2}{x_2} \leq 0, \end{aligned} \tag{1.26}$$

with $X = \{(x_1, x_2)^\top \mid x_1 + 1 > 0,\ x_2 > 0\}$. Apparently, the set of feasible points is equal to the set of optimal points

$$\{(0, x_2)^\top \mid x_2 > 0\},$$

and, therefore, $p^* = 1$. The Lagrange function for the problem (1.26) $L : (-1, +\infty) \times \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ takes the following form

$$L(x_1, x_2, \lambda) = \frac{1}{x_1+1} + \lambda\frac{x_1^2}{x_2}.$$

Note that if $\lambda = 0$, then $\inf_{x_1>-1,x_2>0} L(x_1, x_2, 0) = 0$. If $\lambda < 0$, then $L(n, n, \lambda) = \frac{1}{n+1} + \lambda n$, where $n \in \mathbb{N}$, and $\lim_{n\to+\infty} L(n, n, \lambda) = -\infty$. If $\lambda > 0$, note that $L(x_1, x_2, \lambda) \geq 0$ for all $x_1 > -1$, $x_2 > 0$. See that $L(n, n^3, \lambda) = \frac{1}{n+1} + \lambda\frac{1}{n}$, where $n \in \mathbb{N}$, and $\lim_{n\to+\infty} L(n, n^3, \lambda) = 0$. Therefore,

$$g(\lambda) = \inf_{x_1>-1,x_2>0} L(x_1, x_2, \lambda) = \begin{cases} 0, & \lambda \geq 0, \\ -\infty, & \textit{otherwise}. \end{cases} \tag{1.27}$$

Therefore, the corresponding Lagrange dual problem takes the form $\max\{0 \mid \lambda \geq 0\}$ and, therefore, $d^* = 0$. Now, we equivalently reformulate Problem (1.26) as a conic problem as follows.

$$\begin{aligned} &\min t \\ &\text{s.t. } s = 1 \\ &\quad (x_1, x_2, t, s)^\top \in K_{\mathcal{F}}, \end{aligned} \tag{1.28}$$

where

$$\begin{aligned} K_{\mathcal{F}} &:= \left\{ (x_1, x_2, t, s)^\top \mid \frac{x_2}{s} > 0,\ \frac{x_1}{s} > -1,\ s > 0,\ \frac{1}{\frac{x_1}{s} + 1} \leq \frac{t}{s},\ \frac{x_1^2}{x_2 s} \leq 0 \right\} \cup \{0\} \\ &= \left\{ (0, x_2, t, s)^\top \mid x_2 > 0,\ s > 0,\ s \leq t \right\} \cup \{0\}. \end{aligned}$$

with

$$relint(K_{\mathcal{F}}) = \left\{ (0, x_2, t, s)^\top \mid x_2 > 0,\ s > 0,\ s < t \right\}.$$

The corresponding conic dual problem takes the following form $\max\{z \mid (0, 0, 1, -z)^\top \in K_{\mathcal{F}}^*\}$, where

$$K_{\mathcal{F}}^* = \left\{ (u, v, w, z)^\top \mid u \in \mathbb{R},\ v \geq 0,\ w \geq 0,\ w + z \geq 0 \right\}.$$

Since $(0, 0, 1, -1)^\top \in \mathcal{D} \neq \emptyset$ and $(0, 1, 2, 1)^\top \in \mathcal{P}^0 \neq \emptyset$, according to Theorem 1.7 b) it holds $p^* = d^* = 1$ and, moreover, $(0, 0, 1, -1)^\top \in \mathcal{D}^*$.

Note that the Slater condition for the convex conic programs in (1.28), consisting in the existence of a feasible point from the relative interior of $K_{\mathcal{F}}$, is satisfied. However, the Slater condition for convex programs (see *e.g.* Section 5.2.3 in [15]) in (1.26) is not satisfied since the feasible set of (1.26) $\{(0, x_2)^\top \mid x_2 > 0\}$ has an empty interior.

Remark 1.14. A similar example as Example 1.4 can be found in [36, Example 6.4] or [15, Problem 5.21].

Application of conic duality in polynomial optimization

In this chapter we will be dealing with polynomial optimization. In Section 2.1 we introduce the standard polynomial optimization problem and its equivalent reformulations. Additionally, we demonstrate that polynomial optimization problems can be formulated equivalently as conic optimization problems over a cone of polynomials nonnegative on a specified nonempty set. We analyze properties of these cones, including their respective dual cones, formulate and present the dual cone theorem. Subsequently, in Section 2.2 we apply results on conic duality from Chapter 1 to derive results on duality theory in polynomial optimization problems. Finally, in Section 2.3, we demonstrate the application of the dual cone theorem. It should be noted that Section 2.1 and Section 2.3 were published in [37].

More details on (convex) cones can be found in Appendix A. Further information regarding the vector space of multivariate polynomials can be found in Appendix D.

2.1 Polynomial optimization problems

A polynomial optimization problem is typically regarded as a problem of optimizing a multivariate polynomial on a nonempty set $K \subseteq \mathbb{R}^n$. More precisely, it is a mathematical programming problem in the following form

$$\begin{aligned} \min\ & p(x) \\ & x \in K, \end{aligned} \tag{2.1}$$

where $p \in \mathbb{R}[x]$ is a multivariate polynomial in variable $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is a nonempty set. For more details see *e.g.* [42], [43], or [44].

By defining a new variable γ, which will serve as a lower bound of p on K, the problem (2.1) can be equivalently formulated as

$$\begin{aligned} &\max \gamma \\ &p(x) - \gamma \geq 0, \quad \forall x \in K. \end{aligned} \tag{2.2}$$

The constraint in problem (2.2) requires that polynomial $p(x) - \gamma$ be nonnegative on the given set K.

Formulation (2.2) relates two concepts: minimizing a polynomial p on K with requiring nonnegativity of a polynomial $p(x) - \gamma$ on K. Moreover, it gives rise to a few questions, such as whether one can optimize over the set of polynomials nonnegative on K, what the structure of that set is, whether one can test if a polynomial is nonnegative on K and whether such testing can be done efficiently.

Unfortunately, problem (2.1) and problem (2.2) in general are not convex programming problems without convexity assumptions placed on K and p, which disables one to benefit from the advantages of convex optimization. In fact, in [38] it was shown that testing whether a polynomial of degree at least 4 is nonnegative on a basic semialgebraic set K is NP-hard, even if $K = \mathbb{R}^n$. Moreover, it was shown that unconstrained optimization of a quartic polynomial, optimization of a cubic polynomial over the sphere and optimization of a quadratic polynomial over the simplex are all NP-hard problems (see [53], [22]).

As a consequence, formulation (2.2) provides motivation for examining the structure of a set of nonnegative multivariate polynomials on a nonempty set K. Furthermore, formulation (2.2) provides a way of formulating problem (2.1) (and problem (2.2)) as a convex conic program.

2.1.1 Conic formulation of polynomial optimization problems

In this section we include a convex conic formulation of problem (2.1) and problem (2.2) as it can be found in [44].

Note that polynomial p in problem (2.2) is a polynomial of degree at most d and so is polynomial $p(x) - \gamma$ for any $\gamma \in \mathbb{R}$. Denoting $C_{n,d}(K)$ the set of all n-variate polynomials with a degree at most d which are nonnegative on set K,

$$C_{n,d}(K) := \{p \in \mathbb{R}[x]_d \mid p(x) \geq 0, \quad \forall x \in K\},$$

constraint $p(x) - \gamma \geq 0$, $\forall x \in K$ may be formulated as follows

$$p(x) - \gamma \in C_{n,d}(K).$$

The following proposition states that the set $C_{n,d}(K)$ is a convex set with a conic structure, thus a convex cone.

Proposition 2.1. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, then $C_{n,d}(K)$ is a convex cone.*

Proof. For arbitrary $p \in C_{n,d}(K)$ and $c \geq 0$ it holds that $cp(x) \geq 0$ for all $x \in K$ and, therefore, $cp \in C_{n,d}(K)$. Moreover, for arbitrary $p, q \in C_{n,d}(K)$, it holds that $p(x) + q(x) \geq 0$ for all $x \in K$ and, therefore, $p + q \in C_{n,d}(K)$. We have shown that $C_{n,d}(K)$ is a convex cone. □

2.1.2 Properties of a cone of polynomials nonnegative on a nonempty set

In this section we concentrate on the properties of convex cone $C_{n,d}(K)$. We will show that this cone is closed and solid. Moreover, if $\operatorname{int}(K) \neq \emptyset$, it is a pointed cone, and hence, a proper cone.

An interesting property of cone $C_{n,d}(K)$ is the *nesting property*: one can observe that polynomials with lower degree than d which are nonnegative on K are also included in $C_{n,d}(K)$.

Proposition 2.2. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set. Then*

$$C_{n,d}(K) \supseteq C_{n,d-1}(K) \supseteq C_{n,d-2}(K) \supseteq \cdots \supseteq C_{n,0}(K).$$

Remark 2.1. In some cases it may happen that $C_{n,d}(K) = C_{n,d-1}(K)$, for example, $C_{1,3}(\mathbb{R}) = C_{1,2}(\mathbb{R})$.

In the following proposition we will show that $C_{n,d}(K)$ is a closed solid cone. Moreover, under the additional condition placed on the set K, it is also a pointed cone. Part a) and c) in Proposition 2.3 can be found in [59, Section 4.2] and [11, Section 1.1] for $K = \mathbb{R}^n$.

Proposition 2.3. *Let $d \in \mathbb{N}$. The convex cone $C_{n,d}(K)$ is*

a) closed in $\mathbb{R}[x]_d$,

b) pointed, if $int(K) \neq \emptyset$,

c) solid.

Proof. a) Consider an arbitrary sequence $\{p_j\}_{j=1}^{\infty} \subseteq C_{n,d}(K)$ such that $p_j \to p$ for $j \to \infty$. We have that $p_j(x) \geq 0$ for all $x \in K$ and all $j \in \mathbb{N}$ and thus $\lim_{j\to\infty} p_j(x) = p(x) \geq 0$ for all $x \in K$ which implies that $p \in C_{n,d}(K)$.

b) Assume by contradiction that $\mathrm{int}(K) \neq \emptyset$ but there exists a nonzero polynomial $p \in C_{n,d}(K)$ such that $-p \in C_{n,d}(K)$. It immediately follows that $p(x) = 0$ for all $x \in K$. Choose a point $\bar{x} \in \mathrm{int}(K)$. Then there exists $r > 0$ such that $\mathcal{B}(\bar{x}, r) \subset K$, which means that $p(x) = 0$, $\forall x \in \mathcal{B}(\bar{x}, r)$. Set

$$g(x) := p(\bar{x} - x), \quad \forall x \in \mathbb{R}^n.$$

It is obvious that $g(x)$ is a multivariate polynomial with a degree at most d and thus it can be expressed in the form

$$g(x) = \sum_{\alpha \in \mathbb{N}_d^{\alpha}} g_\alpha x^\alpha.$$

Moreover, g is infinitely many times differentiable on $\mathbb{R}^n$ and $g(x) = 0$, $\forall x \in \mathcal{B}(0, r)$. Note that

$$\frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}(0, 0, \ldots, 0) = \alpha_1! \cdot \alpha_2! \cdot \cdots \cdot \alpha_n! \cdot g_{(\alpha_1, \alpha_2, \ldots, \alpha_n)}, \quad \alpha \in \mathbb{N}_d^n.$$

Since $\frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}(0, 0, \ldots, 0) = 0$, we obtain $g_\alpha = 0$, $\forall \alpha \in \mathbb{N}_d^n$, which implies that $g \equiv 0$. Note that $p(x) = g(\bar{x} - x)$ and thus $p \equiv 0$, which is a contradiction.

c) We will show that $C_{n,2k}(K)$ is a solid cone for any $k \in \mathbb{N}$. Note that if d is divisible by 2, then set $d = 2k$ to show that $C_{n,d}(K)$ is a solid cone. If d is not divisible by 2, recall that by Proposition 2.2 we have $C_{n,d-1}(K) \subseteq C_{n,d}(K)$ with $d-1$ being divisible by 2. Since $\emptyset \neq \mathrm{int}(C_{n,d-1}(K)) \subseteq \mathrm{int}(C_{n,d}(K))$, we will eventually have $\mathrm{int}(C_{n,d}(K)) \neq \emptyset$.

We will show that a polynomial $q(x) = m_k(x)^\top m_k(x)$ is an interior point of $C_{n,2k}(K)$ for any nonempty set $K \subseteq \mathbb{R}^n$. Obviously, for any polynomial $p \in \mathbb{R}[x]_{2k}$ we have that

$$p(x) = q(x) + p(x) - q(x), \quad \forall x \in \mathbb{R}^n.$$

By Proposition D.1 we have $|p(x)-q(x)| = |(p-q)(x)| \leq \|p-q\| \|m_{2k}(x)\|_2$ for all $x \in \mathbb{R}^n$. We have that

$$\forall x \in \mathbb{R}^n : p(x) \geq q(x) - \|p-q\| \|m_{2k}(x)\|_2$$

We will choose $r > \|p-q\|$ such that $\forall x \in K$ we will have

$$p(x) \geq q(x) - r\|m_{2k}(x)\|_2 \geq 0,$$

for instance

$$r = \frac{1}{2} \inf_{x \in K} \left\{ \frac{m_k(x)^\top m_k(x)}{\sqrt{m_{2k}(x)^\top m_{2k}(x)}} \right\}.$$

Since $\frac{m_k(x)^\top m_k(x)}{\sqrt{m_{2k}(x)^\top m_{2k}(x)}} \geq 1$ for all $x \in \mathbb{R}^n$ (see Proposition D.2), $r > 0$ is indeed well-defined.

By this construction of $r > 0$ we have shown that $\mathcal{B}(q,r) = \{p \in \mathbb{R}[x]_{2k} \mid \|p-q\| < r\} \subset C_{n,2k}(K)$. □

Example 2.1. The assumption of a nonempty interior of K in Proposition 2.3 b) cannot be disposed of. Consider $K = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_2-1)^2 \leq 0\} = \{(x_1, 1) \mid x_1 \in \mathbb{R}\}$ and $C_{2,2}(K) = \{p \in \mathbb{R}[x_1, x_2]_2 \mid p(x_1, 1) \geq 0, \ \forall x_1 \in \mathbb{R}\}$. Clearly, $\mathrm{int}(K) = \emptyset$ the polynomial p, defined as $p(x_1, x_2) = (x_2-1)x_1$, is included in $C_{2,2}(K)$ but also $-p \in C_{2,2}(K)$. It shows that $C_{2,2}(K)$ is not pointed.

2.1.3 Dual cone and the dual cone theorem

In this section we introduce the dual cone of $C_{n,d}(K)$ and the dual cone theorem. Note that the (algebraic) dual cone of $C_{n,d}(K)$ by definition consists of linear functionals $\ell : \mathbb{R}[x]_d \to \mathbb{R}$ such that $\ell(p) \geq 0$ for all $p \in C_{n,d}(K)$ (see *e.g.* [11] or [44]), and thus $\ell \in \mathbb{R}[x]_d^*$, which is the dual vector space of $\mathbb{R}[x]_d$. However, since $\mathbb{R}[x]_d$ is finite dimensional, it holds $\mathbb{R}[x]_d \simeq \mathbb{R}[x]_d^*$. Therefore, the dual cone of $C_{n,d}(K)$ can be represented as follows

$$C_{n,d}(K)^* = \{q \in \mathbb{R}[x]_d \mid \langle p, q \rangle \geq 0, \quad p \in C_{n,d}(K)\}. \tag{2.3}$$

Note that the representation of $C_{n,d}(K)^*$ in [44] and [11] differs from the representation that we have introduced.

The properties of $C_{n,d}(K)^*$ directly follow from the general theory of dual cones (see Chapter 2.6.1 in [15] and the bipolar theorem (Theorem A.2)) and Proposition 2.3. They are included in the following proposition. Note that a similar statement to part c) is mentioned in [44, Lemma 4.6].

Proposition 2.4. *For the dual cone $C_{n,d}(K)^*$ the following statements hold.*

a) $C_{n,d}(K)^*$ is a closed convex cone in $\mathbb{R}[x]_d$,

b) $C_{n,d}(K)^*$ is pointed,

c) if int$(K) \neq \emptyset$, then $C_{n,d}(K)^*$ is solid,

d) $C_{n,d}(K)^{**} = C_{n,d}(K)$.

The dual cone $C_{n,d}(K)^*$, represented by (2.3), can be characterized as a closure of a conic hull of polynomials of the form $\sum_{\alpha \in \mathbb{N}_d^n} t^\alpha x^\alpha$, where $t \in K$. We state and prove this characterization of $C_{n,d}(K)^*$ in the following theorem. A similar characterization theorem of $C_{n,d}(\mathbb{R}^n)^*$ in terms of linear functionals was proved in [11, Lemma 2.1], or [59, Lemma 4.11]. Another characterization $C_{n,d}(K)^*$ in terms of vectors of $\mathbb{R}^{s(n,d)}$ having a finite representing measure with support contained in K was proved in [44, Lemma 4.7] with additional assumption on K being compact.

Theorem 2.1. *Let $C_{n,d}(K)$ be a cone of nonnegative polynomials on K of degree at most d ($d \in \mathbb{N}$). Then*

$$C_{n,d}(K)^* = cl\left(cone\left[\sum_{\alpha \in \mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K \right] \right).$$

Proof. Inclusion $\supseteq$: take a $q \in$ cone $\left[\sum_{\alpha \in \mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]$. It means that there exist a number $m \in \mathbb{N}$, coefficients $c_1, c_2, \ldots, c_m \geq 0$ and vectors $t_1, t_2, \ldots, t_m \in K$ such that

$$q(x) = \sum_{i=1}^m c_i \sum_{\alpha \in \mathbb{N}_d^n} t_i^\alpha x^\alpha = \sum_{\alpha \in \mathbb{N}_d^n} \sum_{i=1}^m c_i t_i^\alpha x^\alpha.$$

Now take an arbitrary $p \in C_{n,d}(K)$ to show that

$$\begin{aligned} \langle p, q \rangle = \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha q_\alpha &= \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha \sum_{i=1}^m c_i t_i^\alpha = \\ &= \sum_{i=1}^m c_i \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha t_i^\alpha = \sum_{i=1}^m c_i p(t_i) \geq 0. \end{aligned}$$

Note that $p(t_i) \geq 0$ for all $i = 1, 2, \ldots, m$ because $t_i \in K$ and p is nonnegative on K. Since p was chosen arbitrarily, we obtain that $q \in C_{n,d}(K)^*$.

Now suppose that $q \notin$ cone $\left[\sum_{\alpha \in \mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]$, but

$$q \in \text{cl}\left(\text{cone}\left[\sum_{\alpha \in \mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K \right] \right).$$

There exists a sequence of polynomials $\{q_j\}_{j=1}^{\infty} \subseteq \text{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]$ such that $\lim_{j\to\infty} q_j = q$. It is obvious, using the argument above, that for any $p \in C_{n,d}(K)$ it holds that $\langle p, q_j\rangle \geq 0$ for all $j \in \mathbb{N}$. With the inner product being continuous, by limit transition we have $\langle p, q\rangle \geq 0$ and hence $q \in C_{n,d}(K)$.

Inclusion $\subseteq$: we will use the separating hyperplane theorem (see *e.g.* [15, Chapter 2.5]): suppose that $q \in C_{n,d}(K)^*$ but

$$q \notin \text{cl}\left(\text{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]\right).$$

Since $\text{cl}\left(\text{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]\right)$ is a closed convex cone, there exists a separating polynomial $v \in \mathbb{R}[x]_d$ such that

$$\begin{aligned} &\langle v, q\rangle < 0 \quad \text{and} \\ &\langle v, r\rangle \geq 0 \quad \forall r \in \text{cl}\left(\text{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]\right). \end{aligned}$$

For arbitrary $\bar{t} \in K$ set $r(x) = \sum_{\alpha\in\mathbb{N}_d^n} \bar{t}^\alpha x^\alpha$. Note that $\langle v, r\rangle \geq 0$ and thus we have

$$\langle v, r\rangle = \sum_{\alpha\in\mathbb{N}_d^\alpha} v_\alpha \bar{t}^\alpha = v(\bar{t}) \geq 0.$$

Since $\bar{t}$ was chosen arbitrarily, we have $v(t) \geq 0$, $\forall t \in K$ and thus $v \in C_{n,d}(K)$. But this is in contradiction with $\langle v, q\rangle < 0$. $\square$

2.2 Duality results in polynomial optimization problems

In this section we synthesize the results from Chapter 1 with findings from Chapter 2. We will provide a representation of problem (2.2) in the form of a dual convex conic problem (1.4). We derive the corresponding primal convex conic problem and prove that it is equivalent to the original problem (2.1).

In this section we will consider the ordered canonical basis of $\mathbb{R}[x]_d$, which is concatenated in vector $m_d(x)$ (see Appendix D). Since $\mathbb{R}[x]_d \simeq \mathbb{R}^{s(n,d)}$, there exists an isomorphism

$$\mathcal{I} : \mathbb{R}[x]_d \to \mathbb{R}^{s(n,d)}$$

defined as follows

$$\mathcal{I} : p(x) = m_d(x)^\top (p_\alpha)_{\alpha\in\mathbb{N}^n_d} \mapsto \mathcal{I}(p) := (p_\alpha)_{\alpha\in\mathbb{N}^n_d}.$$

Note that since inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}[x]_d$ and inner product in $\mathbb{R}^{s(n,d)}$ are compatible, it holds that

$$\begin{aligned}\mathcal{I}(C_{n,d}(K)^*) &= \{(q_\alpha)_{\alpha\in\mathbb{N}^n_d} \mid (p_\alpha)^\top_{\alpha\in\mathbb{N}^n_d}(q_\alpha)_{\alpha\in\mathbb{N}^n_d} \geq 0,\ \forall (p_\alpha)_{\alpha\in\mathbb{N}^n_d} \in \mathcal{I}(C_{n,d}(K))\}\\ &= \{(p_\alpha)_{\alpha\in\mathbb{N}^n_d} \mid (p_\alpha)_{\alpha\in\mathbb{N}^n_d} \in \mathcal{I}(C_{n,d}(K))\}^* = (\mathcal{I}(C_{n,d}(K)))^*.\end{aligned}$$

2.2.1 Representation of polynomial optimization problems in the form of convex conic problems

In problem (2.2), which is of form

$$\begin{aligned}&\max \gamma\\ &\quad p(x) - \gamma \in C_{n,d}(K),\end{aligned}$$

we define $s(x) := p(x) - \gamma$, which leads to a problem in the following form

$$\begin{aligned}&\max \gamma\\ &\quad s(x) + \gamma = p(x),\\ &\quad s(x) \in C_{n,d}(K).\end{aligned} \tag{2.4}$$

Via isomorphism $\mathcal{I}$, we transform the given optimization problem (2.4) in $\mathbb{R}[x]_d$ to an optimization problem in $\mathbb{R}^{s(n,d)}$ as follows

$$\begin{aligned}&\max \gamma\\ &\quad (s_\alpha)_{\alpha\in\mathbb{N}^n_d} + \gamma(1,0,\ldots,0)^\top = (p_\alpha)_{\alpha\in\mathbb{N}^n_d},\\ &\quad (s_\alpha)_{\alpha\in\mathbb{N}^n_d} \in \mathcal{I}(C_{n,d}(K)),\end{aligned} \tag{2.5}$$

which is a problem of form (1.4) for $A = (1,0,0,\ldots,0)$, $b = 1$ and $c = (p_\alpha)_{\alpha\in\mathbb{N}^n_d}$.

Note that from Proposition 2.3 a) and the bipolar theorem (Theorem (A.2)) we have that $C_{n,d}(K) = C_{n,d}(K)^{**}$, which implies $\mathcal{I}(C_{n,d}(K)) = \mathcal{I}(C_{n,d}(K))^{**}$. The corresponding problem to problem (2.5) in the form of a primal convex conic problem takes the following form

$$\begin{aligned} \min\ & (p_\alpha)_{\alpha\in\mathbb{N}^n_d}^\top (q_\alpha)_{\alpha\in\mathbb{N}^n_d} \\ & (1,0,0,\ldots,0)(q_\alpha)_{\alpha\in\mathbb{N}^n_d} = 1, \\ & (q_\alpha)_{\alpha\in\mathbb{N}^n_d} \in \mathcal{I}(C_{n,d}(K))^*, \end{aligned} \tag{2.6}$$

or in the polynomial form

$$\begin{aligned} \min\ & \langle p, q\rangle \\ & \langle 1(x), q\rangle = 1, \\ & q \in C_{n,d}(K)^*. \end{aligned} \tag{2.7}$$

where $1(x) := 1$, $\forall x \in \mathbb{R}^n$. The constraint $\langle 1(x), q\rangle = 1$ is equivalent to $q(0) = 1$.

Note that problems (2.1) and (2.2) are equivalent. Problem (2.2) has a conic representation as a dual program (2.4). The question to be asked is whether or not the corresponding primal program (2.7) is equivalent to the original problem (2.1). We show that, supposing that K is a compact set, these two programs are equivalent.

Proposition 2.5. *Suppose that $K \subseteq \mathbb{R}^n$ is a compact set, then problems* (2.1) *and* (2.7) *are equivalent, i.e. they attain the same minimum value.*

Proof. Since K is a compact set, there exists a point $x^* \in K$ in which problem (2.1) attains its minimum. Take $q(x) = \sum_{\alpha\in\mathbb{N}^n_d}(x^*)^\alpha x^\alpha$. Since $x^* \in K$, we have that $q \in C_{n,d}(K)^*$. Moreover, $q(0) = 1$ and $\langle p, q\rangle = \sum_{\alpha\in\mathbb{N}^n_d} p_\alpha (x^*)^\alpha = p(x^*)$, which shows that

$$\inf\{\langle p, q\rangle \mid \langle 1(x), q\rangle = 1,\ q \in C_{n,d}(K)^*\} \le p(x^*) = \min\{p(x) \mid x \in K\}.$$

Now take any $q \in C_{n,d}(K)^*$ with $q(0) = 1$. First assume that

$$q \in \operatorname{cone}\left[\sum_{\alpha\in\mathbb{N}^n_d} t^\alpha x^\alpha \mid t \in K\right],$$

which means that there exist a number $m \in \mathbb{N}$, coefficients $c_1, c_2, \ldots, c_m \ge 0$ and vectors $t_1, t_2, \ldots, t_m \in K$ such that

$$q(x) = \sum_{i=1}^m c_i \sum_{\alpha\in\mathbb{N}^n_d} t_i^\alpha x^\alpha = \sum_{\alpha\in\mathbb{N}^n_d}\sum_{i=1}^m c_i t_i^\alpha x^\alpha.$$

The constraint $q(0) = 1$ translates to $\sum_{i=1}^m c_i = 1$. Moreover,

$$\langle p, q\rangle = \sum_{i=1}^m c_i p(t_i) \ge \sum_{i=1}^m c_i p(x^*) = p(x^*).$$

The case

$$q \notin \text{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]$$

is treated by a sequence-limit argument, similarly as in the proof of Theorem 2.1.

Therefore, we have

$$\min\{\langle p, q\rangle \mid \langle 1(x), q\rangle = 1,\ q \in C_{n,d}(K)^*\} = \min\{p(x) \mid x \in K\}.$$

□

Remark 2.2. A similar result can be found in [43] and [33] in terms of the cone of finite signed (Borel) measures nonnegative on K. However, owing to the representation (2.3) of $C_{n,d}(K)^*$ and the dual cone theorem, we are able to formulate problems (2.4), (2.7), Proposition 2.5 and the following results in a polynomial setting. Furthermore, our approach reveals the actual duality (not only equivalence) of problems (2.1) and (2.2), supposing that K is a compact set.

In the following sections we will assume that $K \subseteq \mathbb{R}^n$ in problem (2.1) is a compact set in order that problems (2.7) and (2.1) are equivalent as it is stated in Proposition 2.5.

Recall that $C_{n,d}(K)$ is a solid closed convex cone, moreover, if $\text{int}(K) \neq \emptyset$, then it is also a pointed cone. On the other hand, cone $C_{n,d}(K)^*$ is a pointed closed convex cone, moreover, if $\text{int}(K) \neq \emptyset$, then it is also a solid cone.

Note that for problems (2.7) and (2.4) it is defined

the set of primal feasible points

$\mathcal{P} = \{q \in C_{n,d}(K)^* \mid q(0) = 1\}$,

the set of primal strictly feasible points

$\mathcal{P}^0 = \{q \in relint(C_{n,d}(K)^*) \mid q(0) = 1\}$,

the set of dual feasible points

$\mathcal{D} = \{(\gamma, s) \mid s(x) + \gamma = p(x),\ s(x) \in C_{n,d}(K)\}$,

the set of dual strictly feasible points

$\mathcal{D}^0 = \{(\gamma, s) \mid s(x) + \gamma = p(x),\ s(x) \in relint(C_{n,d}(K))\}$,

the primal optimal value $p^* = \inf\{\langle p, q\rangle \mid q \in \mathcal{P}\}$, if $\mathcal{P} \neq \emptyset$ and $p^* = +\infty$, if $\mathcal{P} = \emptyset$. Similarly, the dual optimal value $d^* = \sup\{\gamma \mid (\gamma, s) \in \mathcal{D}\}$, if $\mathcal{D} \neq \emptyset$ and $d^* = -\infty$, if $\mathcal{D} = \emptyset$.

The following proposition states that problem (2.7) is always feasible and problem (2.4) is always strictly feasible.

Proposition 2.6. *Consider the primal-dual pair of programs* (2.7) *and* (2.4)*, where* $C_{n,d}(K)$ *satisfies Assumption 1. It holds that*

a) $\mathcal{P} \neq \emptyset$.

b) $\mathcal{D}^0 \neq \emptyset$.

Proof. a) The statement follows from the fact that $1(x) \in C_{n,d}(K)^*$ with $1(0) = 1$.

b) Note that $C_{n,d}(K)$ is a solid cone and $C_{n,d}(K)^*$ is a pointed cone, see Proposition 2.3 c) and Proposition 2.4 b). Using characterization (22) in Proposition B.2 we have that

$$\operatorname{int}(C_{n,d}(K)) = \{p \in C_{n,d}(K) \mid \langle p, q\rangle > 0,\ \forall q \in C_{n,d}(K)^* \ \setminus \{0\}\}.$$

Now choose any $\varepsilon > 0$ and set $\gamma := \min_{x\in K} p(x) - \varepsilon$, which is well-defined since K is a compact set. Consider any

$$0 \neq q \in \operatorname{cone}\left[\sum_{\alpha\in\mathbb{N}^n_d} t^\alpha x^\alpha \mid t \in K\right],$$

which means that there exist a number $m \in \mathbb{N}$, coefficients $c_1, c_2, \dots, c_m \geq 0$ and vectors $t_1, t_2, \dots, t_m \in K$ such that

$$q(x) = \sum_{i=1}^m c_i \sum_{\alpha\in\mathbb{N}^n_d} t_i^\alpha x^\alpha = \sum_{\alpha\in\mathbb{N}^n_d}\sum_{i=1}^m c_i t_i^\alpha x^\alpha.$$

Observe that

$$\langle p, q\rangle = \sum_{i=1}^m c_i p(t_i) - \gamma\sum_{i=1}^m c_i \geq (\min_{x\in K} p(x) - \gamma)\sum_{i=1}^m c_i = \varepsilon\sum_{i=1}^m c_i > 0,$$

since $q \neq 0$. The case

$$C_{n,d}(K)^* \ni q \notin \operatorname{cone}\left[\sum_{\alpha\in\mathbb{N}^n_d} t^\alpha x^\alpha \mid t \in K\right], \quad q \neq 0$$

is treated by a sequence-limit argument. □

2.2.2 Zero duality gap

In this section we apply results from Section 1.3.1 to primal-dual pair of polynomial optimization problems (2.1) and (2.2) with respect to the formulations (2.4) and (2.7) via isomorphic formulations (2.5) and (2.6).

Since problems (2.4) is a Lagrangian dual of (2.7), the weak duality property $p^* \geq d^*$ holds for these problems. By direct application of Theorem 1.7 we obtain the following result regarding the strong duality property.

Theorem 2.2. *Consider the primal-dual pair of programs* (2.7) *and* (2.4)*, where* $C_{n,d}(K)$ *satisfies Assumption 1. It holds that* $p^* = d^*$*,* $\mathcal{P}^* \neq \emptyset$ *and* $\mathcal{D}^* \neq \emptyset$*.*

2.2.3 Necessary and sufficient conditions for nonemptiness and (un-)boundedness of sets of optimal solutions

In this section we directly apply results from Corollary 1.2, Proposition 2.3 and Proposition 2.4 to provide necessary and sufficient conditions for nonemptiness and boundedness of sets of optimal solutions.

Theorem 2.3.

a) Suppose int$(K) \neq \emptyset$*. The set* $\mathcal{P}^*$ *is nonempty and bounded.*

b) The set $\mathcal{D}^*$ *is nonempty and bounded if and only if* $\mathcal{P}^0 \neq \emptyset$*.*

Furthermore, we directly apply Theorem 1.9 a), Proposition 2.3 and Proposition 2.4 to provide a necessary and sufficient condition for nonemptiness and unboundedness of sets of optimal solutions.

Theorem 2.4. *Consider the primal-dual pair of programs* (2.7) *and* (2.4)*, where cone* $C_{n,d}(K)$ *satisfies Assumption 1. Assume that there exists a polynomial* $v \in \mathbb{R}[x]_d$ *such that* $v(0) = 0$*,* $\langle p, v\rangle = 0$ *and* $v \in relint(C_{n,d}(K)^*)$*. The set* $\mathcal{P}^*$ *is nonempty and unbounded.*

2.3 Characterization of $C_{1,2}([-1,1])^*$ and $C_{1,2}([-1,1])$

In this section we will demonstrate the use of the dual cone theorem in finding explicit characterizations of the cones $C_{1,2}([-1,1])^*$ and $C_{1,2}([-1,1])$. Note that the general characterization of $C_{n,d}(K)$ is not known.

According to Theorem 2.1 we have

$$C_{1,2}([-1,1])^* = \mathrm{cl}\left(\mathrm{cone}\left[1+tx+t^2x^2 \mid t\in[-1,1]\right]\right).$$

It means that for every polynomial in cone $[1+tx+t^2x^2 \mid t\in[-1,1]]$ there exist a number $k\in\mathbb{N}$, $t_1,t_2,\ldots,t_k\in[-1,1]$ and $c_1,c_2,\ldots,c_k\geq 0$ such that

$$q(x) = \underbrace{\left(\sum_{i=1}^{k} c_i\right)}_{=:q_0} + \underbrace{\left(\sum_{i=1}^{k} c_i t_i\right)}_{=:q_1} x + \underbrace{\left(\sum_{i=1}^{k} c_i t_i^2\right)}_{=:q_2} x^2.$$

It can be easily verified that

$$\begin{aligned}&\mathrm{cone}\left[1+tx+t^2x^2 \mid t\in[-1,1]\right] \subseteq \\ &\subseteq \{q\in\mathbb{R}[x]_2 \mid q_2\geq 0,\ q_0\geq q_2,\ q_0q_2\geq q_1^2\}.\end{aligned}$$

Now, if $q\in C_{1,2}([-1,1])^*$, but $q\notin$ cone $[1+tx+t^2x^2 \mid t\in[-1,1]]$, there exists a sequence of polynomials $\{q^{(j)}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} q^{(j)} = q$. Note that for all $j\in\mathbb{N}$ it holds that $q_2^{(j)}\geq 0$, $q_0^{(j)}\geq q_2^{(j)}$ and $q_0^{(j)}q_2^{(j)}\geq (q_1^{(j)})^2$. Calculating the limits, we obtain that $q\in\{q\in\mathbb{R}[x]_2 \mid q_2\geq 0,\ q_0\geq q_2,\ q_0q_2\geq q_1^2\}$, which shows that $C_{1,2}([-1,1])^*\subseteq\{q\in\mathbb{R}[x]_2 \mid q_2\geq 0,\ q_0\geq q_2,\ q_0q_2\geq q_1^2\}$.

To show the converse inclusion, consider an arbitrary polynomial $q(x)=q_0+q_1x+q_2x^2\in\{q\in\mathbb{R}[x]_2 \mid q_2\geq 0,\ q_0\geq q_2,\ q_0q_2\geq q_1^2\}$. We need to show that $\langle p,q\rangle = p_0q_0+p_1q_1+p_2q_2\geq 0$ for all polynomials $p(x)=p_0+p_1x+p_2x^2\in C_{1,2}([-1,1])$.

Note that we may assume that $q_0\neq 0$; if $q_0=0$, then we also have $q_2=0$ and $q_1=0$ and thus $q\equiv 0$ and $\langle p,q\rangle=0$, $\forall p\in C_{1,2}([-1,1])$.

Also note that since $q_2\geq 0$, we have $q_0\geq 0$ and $q_0^2\geq q_1^2$, which implies that $q_0\geq |q_1|$. Hence, if $q_0>0$, we have $-1\leq \frac{q_1}{q_0}\leq 1$.

Now, take an arbitrary polynomial $p\in C_{1,2}([-1,1])$. There are three cases to consider.

1. $p_2\geq 0$. In this case we have

$$\begin{aligned}\langle p,q\rangle &= q_0\left(p_0+p_1\frac{q_1}{q_0}+p_2\frac{q_2}{q_0}\right)\geq \\ &\geq q_0\left(p_0+p_1\frac{q_1}{q_0}+p_2\frac{q_1^2}{q_0^2}\right) = \\ &= q_0 p\left(\frac{q_1}{q_0}\right)\geq 0.\end{aligned}$$

2. $p_2 < 0$ and $p_1 < 0$. In this case we define $P(x) = (-p_1 - p_2) + p_1 x + p_2 x^2$. It holds that $p(x) - P(x) = p_0 + p_1 + p_2 = p(1) \geq 0$, $\forall x \in \mathbb{R}$, and thus $\langle p - P, q\rangle = q_0 p(1) \geq 0$, from which we have $\langle p, q\rangle \geq \langle P, q\rangle + q_0 p(1)$. Now,
$$\begin{aligned}\langle P, q\rangle &= q_0(-p_1 - p_2) + q_1 p_1 + q_2 p_2 = \\ &= p_1(q_1 - q_0) + p_2(q_2 - q_0) \geq 0,\end{aligned}$$
since $p_1 < 0$, $q_1 - q_0 \leq 0$ and $p_2 < 0$, $q_2 - q_0 \leq 0$. Hence, $\langle p, q\rangle = \langle P, q\rangle + q_0 p(1) \geq 0$.
3. $p_2 < 0$ and $p_1 \geq 0$. In this case we define $P(x) = (p_1 - p_2) + p_1 x + p_2 x^2$. Again, $p(x) - P(x) = p_0 - p_1 + p_2 = p(-1) \geq 0$, $\forall x \in \mathbb{R}$, and again $\langle p - P, q\rangle = q_0 p(-1) \geq 0$, from which we have $\langle p, q\rangle = \langle P, q\rangle + q_0 p(-1)$. Now,
$$\begin{aligned}\langle P, q\rangle &= q_0(p_1 - p_2) + q_1 p_1 + q_2 p_2 = \\ &= p_1(q_0 + q_1) + p_2(q_2 - q_0) \geq 0,\end{aligned}$$
since $p_1 \geq 0$, $q_0 + q_1 \geq 0$ and $p_2 < 0$, $q_2 - q_0 \leq 0$. Hence, $\langle p, q\rangle = \langle P, q\rangle + q_0 p(-1) \geq 0$.

Since p was chosen arbitrarily, we have shown that $\langle p, q\rangle \geq 0$ for all $p \in C_{1,2}([-1,1])$ and thus $q \in C_{1,2}([-1,1])^*$.

We have found the explicit characterization of $C_{1,2}([-1,1])^*$. In fact,
$$C_{1,2}([-1,1])^* = \left\{q \in \mathbb{R}[x]_2 \,|q_2 \geq 0,\ q_0 \geq q_2,\ q_0 q_2 \geq q_1^2\right\}. \tag{2.8}$$

From the geometrical point of view, it can be said that $C_{1,2}([-1,1])^*$ is the intersection of a cone isomorphic to the cone of 2×2 symmetric positive semidefinite matrices and a polyhedral cone. More specifically,
$$\begin{aligned}&C_{1,2}([-1,1])^* = \\ &= \{q_0 + q_1 x + q_2 x^2 \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 q_2 \geq q_1^2\} \cap \\ &\cap \{q_0 + q_1 x + q_2 x^2 \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_1 \in \mathbb{R}\}.\end{aligned} \tag{2.9}$$

Since $C_{1,2}([-1,1])$ is a closed convex cone (see Proposition 2.1 and Proposition 2.3), it holds $C_{1,2}([-1,1]) = C_{1,2}([-1,1])^{**}$. Thus the explicit characterization of $C_{1,2}([-1,1])$ can be found by taking the dual of both sides in (2.9). More specifically,
$$\begin{aligned}C_{1,2}([-1,1]) = \mathrm{cl}\big(&\{q \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 q_2 \geq q_1^2\}^* + \\ &+ \{q \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_1 \in \mathbb{R}\}^*\big),\end{aligned} \tag{2.10}$$

or, after calculating the dual cones

$$C_{1,2}([-1,1]) = \mathrm{cl}\left(\{p \in \mathbb{R}[x]_2 \mid p_0 \geq 0,\ p_0p_2 \geq p_1^2/4\}+\right.$$
$$\left.+\{p \in \mathbb{R}[x]_2 \mid p_0 \geq 0,\ p_0 + p_2 \geq 0,\ p_1 = 0\}\right). \quad (2.11)$$

In fact, the closure operator in (2.10) and (2.11) is not needed since the sum of these two cones is closed. Note that the sum of two closed convex cones is closed if the intersection of their relative interiors is nonempty, for more details see Appendix C, or *e.g.* [61]. It can be easily verified that $1 + x^2$ belongs to the (relative) interiors of both cones.

We finally obtain the characterization of $C_{1,2}([-1,1])$ in the following form

$$C_{1,2}([-1,1]) = \big\{(p_0 + r_0) + p_1x + (p_2 + r_2)x^2 \in \mathbb{R}[x]_2 \mid$$
$$p_0, r_0 \geq 0,\ p_0p_2 \geq p_1^2/4,\ r_0 + r_2 \geq 0\big\}. \quad (2.12)$$

Note that in (2.12) one can write $r_0 + r_2x^2 = r_0(1 - x^2) + (r_0 + r_2)x^2$, with $r_0 \geq 0$ and $r_0 + r_2 \geq 0$. Using the convexity of $\{p \in \mathbb{R}[x]_2 \mid p_0 \geq 0,\ p_0p_2 \geq p_1^2/4\}$, one can rewrite (2.12) as follows

$$C_{1,2}([-1,1]) = \quad (2.13)$$
$$= \big\{p_0 + p_1x + p_2x^2 + r(1-x^2) \in \mathbb{R}[x]_2 \mid p_0 \geq 0,\ p_0p_2 \geq p_1^2/4,\ r \geq 0\big\}.$$

Note that from the characterization (2.13) it is possible to find the characterization of $C_{1,2}([a,b])$, where $a < b$ $(a, b \in \mathbb{R})$, by using an affine change of variables

$$x \mapsto \frac{2}{b-a}x - \frac{a+b}{b-a}.$$

It can be easily verified that if $p \in C_{1,2}([-1,1])$, then

$$q(x) := p\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) \geq 0, \quad \forall x \in [a,b], \quad (2.14)$$

and thus $q \in C_{1,2}([a,b])$. On the other hand, for every $q \in C_{1,2}([a,b])$ we may observe that

$$p(x) := q\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \geq 0, \quad \forall x \in [-1,1],$$

and thus $p \in C_{1,2}([-1,1])$. Thus $q \in C_{1,2}([a,b])$ if and only if q can be written in the form (2.14) for some $p \in C_{1,2}([-1,1])$.

Now from (2.13) it follows that $p \in C_{1,2}([-1,1])$ if and only if p can be written as

$$p(x) = s(x) + r(1 - x^2), \quad \forall x \in \mathbb{R},$$

where $s(x) := p_0 + p_1 x + p_2 x^2$ and $p_0 \geq 0,\ p_0 p_2 \geq p_1^2/4,\ r \geq 0$. Note that $s \in C_{1,2}(\mathbb{R})$ and thus

$$s\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) = s_0 + s_1 x + s_2 x^2 \geq 0, \quad \forall x \in \mathbb{R},$$

thus it also holds that $s_0 \geq 0$ and $s_0 s_2 \geq s_1^2/4$. We obtain that $q \in C_{1,2}([a,b])$ if and only if q can be written in the form

$$\begin{aligned} q(x) = p\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) &= s\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) + \frac{4r}{(b-a)^2}(b-x)(x-a) = \\ &= s_0 + s_1 x + s_2 x^2 + R(b-x)(a-x), \quad \forall x \in \mathbb{R}, \end{aligned}$$

where $s_0 \geq 0,\ s_0 s_2 \geq s_1^2/4,\ R \geq 0$. We conclude that

$$\begin{aligned} &C_{1,2}([a,b]) = \\ &= \left\{ s_0 + s_1 x + s_2 x^2 + R(b-x)(x-a) \in \mathbb{R}[x]_2 \mid s_0 \geq 0,\ s_0 s_2 \geq s_1^2/4,\ R \geq 0 \right\}. \end{aligned} \tag{2.15}$$

The characterizations (2.13) and (2.15) correspond to the result of Fekete, see *e.g.* Theorem 2.4 in [44], or [64].

Conclusion

The aim of this thesis was to examine duality in convex optimization problems. Our focus was on exploring Lagrangian duality in convex conic programming, as well as its application in polynomial optimization.

Chapter 1 was devoted to convex conic programming. In Section 1.1, we introduced the standard form of a primal convex conic program and its corresponding Lagrangian dual, as it is formulated in convex optimization textbooks, *e.g.* in [15]. To ensure comprehensive coverage of theoretical aspects, we included subsections on the weak duality property (see [15]) and recession cones related to both primal and dual convex conic programs.

Section 1.2 was devoted to primal and dual theorems of alternatives for linear systems over nontrivial convex cones, including the primal and dual variant of the generalized Farkas lemmas (Theorem 1.2 and Theorem 1.4), see *e.g.* [7] or [31]. Additionally, we introduced a new result: theorems of alternatives (Theorem 1.3 and Theorem 1.5) providing equivalent conditions to strict feasibility (primal or dual), which proved crucial in analyzing the boundedness of optimal solution sets. The new results obtained in this section extend the results formulated in [7], [8] and [71].

In Section 1.3 we analyzed the strong duality property and its aspects, in particular the zero duality gap and boundedness (or unboundedness) of the sets of optimal solutions, for convex conic programs. We included the established results (see *e.g* [47] or [6]) indicating that the satisfaction of the conic version of Slater condition for either the primal or dual convex conic program ensures a zero duality gap (Theorem 1.6 and Theorem 1.7). Subsection 1.3.2 presented new necessary and sufficient conditions for determining the nonemptiness and boundedness of optimal solution sets in both primal and dual convex conic programs (Theorem 1.8). Moreover, we obtained new sufficient conditions for strong duality. In Subsection 1.3.3 we compared our results with the results in [47] and the differences are illustrated in Example 1.2. In Subsection 1.3.4 we obtained new necessary and sufficient conditions for the nonemptiness and unboundedness of the sets of optimal solutions of primal and dual convex conic programs (Theorem 1.9). Similarly, we obtained new sufficient conditions for strong duality. In Table 1.1 in Subsection 1.3.5 we list eight sufficient conditions, six of which are new, for the zero duality gap, *i.e.* strong duality in convex conic programming problems. The new results might be useful in analyzing other subclasses of convex conic programs, in particular the primal-dual relations between optimal solutions, as well as in designing new algorithms.

In Section 1.4 we concentrated on the strong duality property in standard convex programming problems. In Subsection 1.4.2 we included the usual way of reformulating a standard convex programming problem as a convex conic programming problem by embedding the primal feasible set into a convex cone. This reformulation provided an opportunity to compare two versions of the Slater condition: the generalized version of Slater condition for a standard convex programming problem and the conic version of Slater condition for the corresponding conic reformulation. We proved (Theorem 1.10) that if the generalized version of Slater condition is satisfied for a standard convex programming problem, then the conic version of Slater condition is satisfied for the corresponding conic reformulation, which extends the results in [36]. We included Example 1.4 in which we demonstrated that Theorem 1.10 cannot be reversed, and hence, even if the generalized version of Slater condition for a standard convex programming problem is not satisfied and strong duality does not hold, the conic version of Slater condition for the corresponding conic reformulation is satisfied and strong duality in the corresponding primal-dual conic reformulations does hold.

Chapter 2 was devoted to the application of conic duality in polynomial optimization problems. In Section 2.1 we introduced the standard form of a polynomial optimization problem and its equivalent lower-bound reformulation, as it is formulated in various polynomial optimization textbooks, *e.g.* in [44]. In Subsection 2.1.1 we introduced the set of multivariate polynomials on a given nonempty set K and proved that this set is indeed a cone. In Subsection 2.1.2 we examined other properties of the cone of polynomials nonnegative on K which are included in Proposition 2.3, extending the results formulated in [11]. In Subsection 2.1.3 we introduced one possible representation of the dual cone to the cone of polynomials nonnegative on K, analyzed its properties, formulated and proved another equivalent representation included in the dual cone theorem (Theorem

2.1). In Section 2.2 we provided other representations of a standard polynomial optimization problem (different from those in [44]) and showed, supposing that K is compact, that the standard polynomial optimization problem and its lower-bound reformulation are in fact in a primal-dual relationship. We derived new results analogous to those in Section 1.3, particularly regarding the zero duality gap. (Theorem 2.2), necessary and sufficient conditions for nonemptiness and boundedness of sets of optimal solutions (Theorem 2.3) and a necessary and sufficient condition for nonemptiness and unboundedness of sets of optimal solutions (Theorem 2.4). In Section 2.3 we demonstrated the application of the dual cone theorem to find the explicit characterizations of $C_{1,2}([-1,1])^*$ and $C_{1,2}([-1,1])$. While these results had previously been discovered through algebraic manipulations with polynomials (see [44] and [64]), our approach utilized the dual cone theorem and conic duality to derive them.

In conclusion, the new results obtained in this thesis provide fertile ground for further research. The findings from Chapter 1 offer insights for analyzing other subclasses of convex conic programs and potentially designing new algorithms. Additionally, the duality results from Chapter 2 could be extended to a broader class of polynomial optimization problems. Lastly, the application of the dual cone theorem may prove useful in finding characterizations of other cones of multivariate polynomials nonnegative on a given set.

Appendix

A Properties of cones and dual cones

In this section we include the definitions of basic notions regarding the geometry of cones, together with various properties of cones. More details can be found in *e.g.* [6], [10] or [15]. It should be noted that a part of this section was published in [72].

Definition A.1 ([15], Section 2.1.5, Section 2.4)**.** Let K be a subset of $\mathbb{R}^n$.

a) A subset K is called a *cone* if $\forall x \in K$ and $\forall \alpha \geq 0$ it holds that $\alpha x \in K$.

b) A cone K is called a *convex cone* if K is a convex set.

c) A cone K is called *pointed* if it does not contain a straight line, *i.e.* $(x \in K) \wedge (-x \in K) \Rightarrow x = 0$.

d) A cone K is called *solid* if its interior is nonempty.

e) A cone K is called a *proper cone* if it is a convex, closed, pointed, and solid cone.

Remark A.1. Part a) in Definition A.1 translates that a cone is closed under nonnegative scalar multiplication. Part b) in Definition A.1 can be equivalently reformulated as follows: a convex cone is a cone closed under vector addition, *i.e.* $\forall x, y \in K$ it holds that $x+y \in K$.

We denote $lin(K) := K + (-K)$ the smallest linear subspace containing the cone K[12], and $sub(K) := K \cap (-K)$, the largest linear subspace contained in K. The following result immediately follows from these definitions.

Proposition A.1 ([36], Proposition 1.4)**.** *Let $K \subseteq \mathbb{R}^n$ be a cone.*

a) If K is pointed, then $sub(K) = \{0\}$.

b) If K is solid, then $lin(K) = \mathbb{R}^n$.

[12]or, equivalently, a set of all finite linear combinations of vectors contained in K, see [36]

Moreover, a convex cone is pointed if and only if $sub(K) = \{0\}$*; and it is solid if and only if* $lin(K) = \mathbb{R}^n$.

Definition A.2. A cone K is called trivial if it is a linear subspace, *i.e.* $K = sub(K) = lin(K)$, otherwise it is called nontrivial.

The following notion, the notion of a recession cone of a nonempty set, plays an important role in determining whether a given nonempty set is bounded or not.

Definition A.3 ([10], Section 1.5)**.** Let $C \subseteq \mathbb{R}^n$ be a nonempty set. The *recession cone* R_C of the set C is defined as

$$R_C = \{d \in \mathbb{R}^n \mid x + \gamma d \in C, \ \forall x \in C, \ \forall \gamma \geq 0\}.$$

Vectors d included in R_C are called *directions of recession* of the set C.

Remark A.2. The recession cone consists of directions of recessions. Points along any direction of recession $d \in R_C$ (in one direction) starting from any point $x \in C$ remain in the set C, *i.e.* for every vector $x \in C$ it holds that the ray $\{x + \gamma d \mid \gamma \geq 0\}$ lies in C.

We now include the proposition dealing with the properties of the recession cone. A similar result can be found in *e.g.* [10], Proposition 1.5.1.

Proposition A.2. *Let* $C \subseteq \mathbb{R}^n$ *be a nonempty set and* R_C *be the recession cone of* C.

a) If C *is a (closed) convex set, then* R_C *is a (closed) convex cone.*

b) If C *is bounded, then* $R_C = \{0\}$.

c) If C *is a closed set and* $R_C = \{0\}$*, then* C *is bounded.*

We now include the definition of a dual cone and its fundamental properties (see *e.g.* in [15, 6, 66]). Dual cone is an essential notion in conic programming.

Definition A.4 ([15], Section 2.6)**.** Let $K \subseteq \mathbb{R}^n$ be a cone. The *dual cone* of a cone K is the set

$$K^* = \{y \in \mathbb{R}^n \mid x^\top y \geq 0, \ \forall x \in K\}.$$

Remark A.3. Some authors work with the *polar cone* concept, typically denoted as K°. The relation between the dual and the polar cone is simply $K^* = -K^\circ$.

Dual cones have various important properties, some of which are listed in the following proposition.

Proposition A.3 ([15], Section 2.6). *a) Let $K \subseteq \mathbb{R}^n$ be a cone. K^* is a closed convex cone.*

b) Let $K \subseteq \mathbb{R}^n$ be a cone and a vector subspace in $\mathbb{R}^n$ ($K = lin(K) = sub(K)$), then $K^* = K^\perp$.

c) Let $K_1, K_2 \subseteq \mathbb{R}^n$ be cones and $K_1 \subseteq K_2$, then $K_2^* \subseteq K_1^*$.

d) Let $K \subseteq \mathbb{R}^n$ be a cone, then $K^* = (cl(K))^*$.

e) Let $K \subseteq \mathbb{R}^n$ be a solid cone, then K^* is a pointed cone.

f) Let $K_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, s$ be cones, then $(K_1 \times K_2 \times \cdots \times K_s)^* = K_1^* \times K_2^* \times \cdots \times K_s^*$.

g) Let $K_i \subseteq \mathbb{R}^n$, $i = 1, 2, \ldots, s$ be cones, then $(K_1 + K_2 + \cdots + K_s)^* = K_1^* \cap K_2^* \cap \cdots \cap K_s^*$.

An important tool in conic duality theory is the *bipolar theorem* and its consequences (see *e.g.* [66, Theorem 14.1]; [35, Proposition 4.2.6]). The bipolar theorem is usually proved using the conic version of a separating hyperplane theorem (for general concept see [15, Section 2.5], [10, Section 2.4] or [66, Section 11]).

Theorem A.1. *Let $K \subseteq \mathbb{R}^n$ be a convex cone and $\bar{x} \notin K$. Then there exists a separating hyperplane passing through the origin which separates $\bar{x}$ and K, i.e. $\exists v \neq 0$ such that*

$$v^\top \bar{x} < 0, \quad v^\top z \geq 0, \ \forall z \in K.$$

The bipolar theorem and its consequences are listed in a corollary below.

Theorem A.2. *(Bipolar theorem)*
If K is a convex cone, then $K^{**} = cl(K)$.

Corollary A.1. *Assume that $K, K_1, K_2 \subseteq \mathbb{R}^n$ are convex cones.*

a) If K is closed, then $K = K^{**}$.

b) If $cl(K)$ is pointed, then K^* is solid.

c) $cl(K)$ is a proper cone if and only if K^* is a proper cone.

d) If $cl(K_1) \subset cl(K_2)$, then $K_2^* \subset K_1^*$.

e) If $V \subseteq \mathbb{R}^n$ is a linear subspace such that $K \subset V$, then $V^\perp \subset K^*$.

f) $cl(K_1 + K_2) = (K_1^* \cap K_2^*)^*$.

Using the characterization of $lin(K)$ and $sub(K)$ and the bipolar theorem, it can be easily shown that the linear subspaces are linked in the following way (see [47, Corollary 1]).

Proposition A.4.

$$a)\ sub(K^*) = \{y \in K^*,\ \mid x^\top y = 0,\ \forall x \in K\} = lin(K)^\perp; \tag{16}$$

$$b)\ sub(cl(K)) = \{z \in cl(K),\ \mid z^\top y = 0,\ \forall y \in K^*\} = lin(K^*)^\perp. \tag{17}$$

Proof. a) Since $lin(K) = K + (-K)$, we have that $lin(K)^\perp = lin(K)^* = K^* \cap (-K)^* = sub(K^*)$. Part b) follows from part a) applied to K^* and the bipolar theorem. □

Note that when (17) is applied to K^* and combined with (16) and the bipolar theorem, it follows that $lin(K) = lin(cl(K))$.

In the following proposition, we list a few simple properties of $lin(\cdot)$ and $sub(\cdot)$ of a convex cone intersected with a linear subspace $V \subseteq \mathbb{R}^n$.

Proposition A.5.

$$a)\ sub(V \cap K) = V \cap sub(K), \tag{18}$$

$$b)\ V \cap [K \setminus sub(K)] = (V \cap K) \setminus sub(V \cap K), \tag{19}$$

$$c)\ lin(V + K) = V + lin(K), \tag{20}$$

$$d)\ lin(V \cap K) \subseteq V \cap lin(K). \tag{21}$$

Proof. a) $sub(V \cap K) = (V \cap K) \cap (-(V \cap K)) = (V \cap K) \cap ((-V) \cap (-K)) = (V \cap K) \cap (V \cap (-K))$, since $V = -V$. Finally, due to the properties of intersection we have that $(V \cap K) \cap (V \cap (-K)) = V \cap V \cap K \cap (-K) = V \cap sub(K)$.
b) $(V \cap K) \setminus sub(V \cap K) = (V \cap K) \cap (sub(V \cap K))^c = (V \cap K) \cap (V \cap sub(K))^c = (V \cap K) \cap (V^c \cup sub(K)^c) = (V \cap K \cap V^c) \cup (V \cap K \cap sub(K)^c) = \emptyset \cup (V \cap [K \setminus sub(K)]) = V \cap [K \setminus sub(K)]$.
c) In a) take $sub((V + K)^*) = sub(V^\perp \cap K^*) = V^\perp \cap sub(K^*)$, thus by taking orthogonal complement we get that $lin(V + K) = V + lin(K)$.
d) Take $x \in lin(V \cap K) = (V \cap K) + (-(V \cap K))$. Note that $-(V \cap K) = \{-z \mid z \in V,\ z \in K\} = \{z \mid -z \in V,\ -z \in K\} = \{z \mid z \in V\} \cap \{z \mid -z \in K\} = V \cap (-K)$. It follows that $x = x_1 + x_2$, where $x_1 \in V \cap K$ and $x_2 \in V \cap (-K)$. Now, we have $x \in V$ and $x \in K + (-K) = lin(K)$. □

Finally, we include a lemma that allows for the decomposition of an intersection of a convex cone with an affine subspace. Note that a similar result can be found in [10, Proposition 1.5.4].

Lemma A.1. *Let K be a cone satisfying Assumption 1, let V be a linear subspace and let c be an arbitrary but fixed vector. Then*

$$K^* \cap (c + V^\perp) = [K^* \cap (c + V^\perp)] \cap lin(V + K) + V^\perp \cap sub(K^*).$$

Proof. First suppose that $K^* \cap (c + V^\perp) = \emptyset$, then $[K^* \cap (c + V^\perp)] \cap lin(V + K) + V^\perp \cap sub(K^*) = \emptyset + V^\perp \cap sub(K^*) = \emptyset$.

Now suppose that $K^* \cap (c + V^\perp) \neq \emptyset$. Then there exists a vector $s \in K^* \cap (c + V^\perp)$. The vector s can be decomposed into two components, i. e. there exist vectors $s_1 \in lin(V + K)$ and $s_2 \in V^\perp \cap sub(K^*)$ such that $s = s_1 + s_2$. Obviously, $s - s_2 = s_1 \in K^* \cap (c + V^\perp)$ and thus $s_1 \in [K^* \cap (c + V^\perp)] \cap lin(V + K)$, which proves that $s \in [K^* \cap (c + V^\perp)] \cap lin(V + K) + V^\perp \cap sub(K^*)$.

Moreover, we have shown that $K^* \cap (c + V^\perp) \neq \emptyset$ iff $[K^* \cap (c + V^\perp)] \cap lin(V + K) + V^\perp \cap sub(K^*) \neq \emptyset$ and, therefore, in the following text we may assume that $[K^* \cap (c + V^\perp)] \cap lin(V + K) + V^\perp \cap sub(K^*) \neq \emptyset$.

Conversely, if $s \in [K^* \cap (c + V^\perp)] \cap lin(V + K) + V^\perp \cap sub(K^*)$ there exist vectors $s_1 \in [K^* \cap (c + V^\perp)] \cap lin(V + K)$ and $s_2 \in V^\perp \cap sub(K^*)$ such that $s = s_1 + s_2$. Obviously, $s \in K^*$. Moreover, since $s_1 \in (c + V^\perp)$ and $s_2 \in V^\perp$ we have that $s \in (c + V^\perp)$. $\square$

B Relative interior of a convex cone

In this section we include the standard definitions of a relative interior of a convex cone, provide various characterizations of this notion and list a few of its well-known properties. The relative interior of a convex cone and its characterizations play an important part in studying the strong duality property in convex conic programming and its various aspects, such as theorems of alternatives, Slater-like conditions for strong duality or (un)boundedness of the set of optimal solutions. For more information on relative interior see [10], [48] or [66]. It should be noted that a part of this section was published in [72].

For a general convex cone K, the relative interior $relint(K)$ is defined as the interior of K with respect to the subspace topology on $lin(K)$. A broader definition can be found in [48, Definition 2.72].

Definition B.1. Let $K \subseteq \mathbb{R}^n$ be a convex cone. The relative interior of K with respect to $lin(K)$ is defined as the set

$$relint(K) = \{x \in K \mid \exists r > 0 : \mathcal{B}(x, r) \cap lin(K) \subset K\}$$

The convexity property allows for an equivalent definition of the relative interior of K.

Definition B.2. Let $K \subseteq \mathbb{R}^n$ be a convex cone. The relative interior of K with respect to $lin(K)$ is defined as the set

$$relint(K) = \{x \in K \mid \forall v \in lin(K)\ \exists \lambda > 0 : x + \lambda v \in K\}.$$

This concept of the relative interior of a convex cone coincides with the notion of the (relative) algebraic interior (or core) of a convex cone in [48, Section 2.2.1] restricted to $X = lin(K)$ rather than $X = \mathbb{R}^n$. It can be shown that for convex sets in finite dimensional spaces the relative algebraic interior is equal to the relative interior, and, thus Definition B.1 is equivalent to Definition B.2. For proof see [48, Theorem 2.18] or [36, Theorem A.0.8, Theorem A.0.9]

The following proposition follows directly from Definition B.2.

Proposition B.1 ([36], Proposition 2.2)**.** *Let $K \subseteq \mathbb{R}^n$ be a convex cone and K^* be its dual cone. Then it holds* *a)* $relint(K^*) \subseteq K^* \setminus sub(K^*)$,

b) $relint(K) \subseteq K \setminus sub(cl(K))$.

An important characterization of the relative interior of K was introduced in [47, Theorem 2].

Proposition B.2. *For a convex cone $K \subseteq \mathbb{R}^n$ it holds that*

$$relint(K) = \{x \in K \mid x^\top y > 0,\ \forall y \in K^* \setminus sub(K^*)\}. \tag{22}$$

From characterization (22) and the bipolar theorem, we obtain a characterization of the relative interior of the dual cone K^*:

$$relint(K^*) = \{y \in K^* \mid x^\top y > 0,\ \forall x \in cl(K) \setminus sub(cl(K))\}. \tag{23}$$

The following result follows from the definition of the dual cone, characterization (22), and from the fact that $0 \in K$.

Proposition B.3. *For a convex cone $K \subseteq \mathbb{R}^n$ it holds that*

$$relint(K) = K + relint(K) = cl(K) + relint(cl(K)). \tag{24}$$

Proof. Since $0 \in K$, we have $relint(K) \subseteq K + relint(K)$. Now, take an $x \in K + relint(K)$, thus, $x = x_1 + x_2$, where $x_1 \in K$ and $x_2 \in relint(K)$. We observe that $y^\top x = y^\top x_1 + y^\top x_2 > 0$ for all $y \in K^* \setminus sub(K^*)$, showing $relint(K) \supseteq K + relint(K)$. Moreover, it holds that $relint(K) = relint(cl(K))$ (see Proposition 1.4.3 in [10]), which completes the proof. □

Now, we recall a few known properties (see [45], [10], and [66]).

Proposition B.4 ([10], Section 1.4)**.** *a) Assume that $K_1, K_2 \subseteq \mathbb{R}^n$ are convex cones. It holds that*

i) $relint(K_1 + K_2) = relint(K_1) + relint(K_2)$.

ii) if $relint(K_1) \cap relint(K_2) \neq \emptyset$, *then* $relint(K_1 \cap K_2) = relint(K_1) \cap relint(K_2)$.

b) Assume that $K_1 \subseteq \mathbb{R}^m$ and $K_2 \subseteq \mathbb{R}^n$, then $relint(K_1 \times K_2) = relint(K_1) \times relint(K_2)$.

c) Assume that $K \subseteq \mathbb{R}^n$ is a convex cone and $A \in M_{m,n}(\mathbb{R})$, then $A(relint(K)) = relint(A(K))$.[13]

Finally, we include the proof of Proposition 1.4 for completeness of theoretical results.

Proof of Proposition 1.4. Let $(\bar{x}^\top, \bar{t}, \bar{s})^\top \in int(K_\mathcal{G})$, then there exists $\varepsilon > 0$ such that

$$\mathcal{B}((\bar{x}^\top, \bar{t}, \bar{s})^\top, \varepsilon) \subset K_\mathcal{G}.$$

Thus, for all $q \in \mathcal{B}((0^\top, 0, 0)^\top, 1)$ we have that

$$(\bar{x}^\top, \bar{t}, \bar{s})^\top + \varepsilon(q_1^\top, q_2, q_3)^\top \in \mathcal{B}((\bar{x}^\top, \bar{t}, \bar{s})^\top, \varepsilon)$$

and it holds that

$$\alpha_j^\top \bar{x} + \beta_j \bar{s} + \varepsilon(\alpha_j^\top q_1 + \beta_j q_3) \leq 0, \quad j = 1, 2, \ldots, l.$$

Choosing $q = \frac{1}{\|(\alpha_j^\top, 0, \beta_j)^\top\|}(\alpha_j^\top, 0, \beta_j)^\top$ for each $j = 1, 2, \ldots, l$ sequentially, we obtain that

$$\alpha_j^\top \bar{x} + \beta_j \bar{s} \leq -\varepsilon\|(\alpha_j^\top, 0, \beta_j)^\top\| < 0, \quad j = 1, 2, \ldots, l.$$

Note that $\|(\alpha_j^\top, 0, \beta_j)^\top\| \neq 0$ since $\alpha_j \neq 0$ for any $j = 1, 2, \ldots, l$.

Now, suppose that $(\bar{x}^\top, \bar{t}, \bar{s})^\top$ satisfies $\alpha_j^\top \bar{x} + \beta_j \bar{s} < 0$ for all $j = 1, 2, \ldots, l$. We need to show that there exists $\varepsilon > 0$ such that $\mathcal{B}((\bar{x}^\top, \bar{t}, \bar{s})^\top, \varepsilon) \subset K_\mathcal{G}$. We set

$$\varepsilon := \min_{j=1,2,\ldots,l} \left\{ -\frac{\alpha_j^\top \bar{x} + \beta_j \bar{s}}{2\|(\alpha_j^\top, 0, \beta_j)^\top\|} \right\} > 0.$$

Take any arbitrary but fixed vector $q = (q_1^\top, q_2, q_3)^\top \in \mathcal{B}((0^\top, 0, 0)^\top, 1)$. We examine the vector $(\bar{x}^\top, \bar{t}, \bar{s})^\top + \gamma q \in \mathcal{B}((\bar{x}^\top, \bar{t}, \bar{s})^\top, \varepsilon)$, where $0 < \gamma < \varepsilon$ is an arbitrary number, we have that

$$\alpha_j^\top \bar{x} + \beta_j \bar{s} + \gamma(\alpha_j^\top q_1 + \beta_j q_3) \leq \alpha_j^\top \bar{x} + \beta_j \bar{s} + \gamma\|(\alpha_j^\top, 0, \beta_j)^\top\| \leq \frac{\alpha_j^\top \bar{x} + \beta_j \bar{s}}{2} < 0,$$

for all $j = 1, 2, \ldots, l$. Since γ and q were chosen arbitrarily, the proof is complete. $\square$

[13]For a set $S \subseteq \mathbb{R}^n$ and an $m \times n$ matrix A, we denote $A(S)$ the image of S in the linear transformation defined by matrix A, *i.e.* $A(S) = \{As \mid s \in S\}$.

C Closedness of the linear image of a convex cone

Linear programs, *i.e.* conic linear programs for which the cone K is polyhedral, are characterized by "ideal" duality theory. This is closely related to the famous Farkas theorem of alternatives [26] and the fact that convex polyhedral cones are finitely generated, and, hence, their linear images form closed cones. This guarantees that the alternatives appearing in Farkas lemma are strong, *i.e.* one and only one of the alternatives holds. However, in the generalized versions of the Farkas lemma, the alternatives are weak (*i.e.* at most one of the two holds), and the closedness of the linear image of the related convex cone becomes an additional assumption.

In this section we summarize the sufficient conditions for the closedness of the linear image of a convex cone. We start with the following lemma, which was formulated in [61] as Theorem 2.2, with the assumption of closedness of K, which, however, is not needed for the statement to hold. Note that in our formulation the assumption of closedness of K is omitted, therefore, we provide an alternative proof.

It should be noted that a part of this section was published in [72].

Lemma C.1. *Let $L \subseteq \mathbb{R}^n$ be a linear subspace and let $K \subseteq \mathbb{R}^n$ be a cone satisfying Assumption 1. Then the following statements are equivalent:*

(i) $L + K = L + lin(K)$;

(ii) $L \cap relint(K) \neq \emptyset$;

(iii) $L^\perp \cap [K^* \setminus sub(K^*)] = \emptyset$.

Proof. First we will show $(i) \Rightarrow (ii)$. From the assumption (i) and the definition of $lin(K)$ we have $L + K = L + lin(K) = (L + K) + (-K)$. Since $0 \in L + K$ it follows that $(-K) \subseteq L + K$. Take $\bar{k} \in -relint(K) \subseteq L + K$. Then $\bar{k} = l + k$ for some $l \in L$ and $k \in K$. However then $-l = (-\bar{k}) + k$ and $(-l) \in L$. From (24) it follows $(-l) \in relint(K)$. Therefore $(-l) \in L \cap relint(K)$.

Next, we will show $(ii) \Rightarrow (iii)$. Assume by contradiction that there exists $z \in L^\perp \cap [K^* \setminus sub(K^*)]$ and let $x \in L \cap relint(K)$. From the characterization (22) we get $z^\top x > 0$, however $x \in L, z \in L^\perp$ implies $z^\top x = 0$.

Finally, we will prove $(iii) \Rightarrow (i)$. It can be easily seen that (iii) is equivalent to $L^\perp \cap K^* = L^\perp \cap sub(K^*)$. Then, by applying the property (c6) (Section 2.1) we obtain that $cl(L+lin(K)) = cl(L+K)$. Then (i) holds since $L+lin(K)$ is a linear subspace. $\square$

Remark C.1. Note that if K is solid, then the statements $(i), (ii), (iii)$ can be simplified to $L + K = \mathbb{R}^n$, $L \cap int(K) \neq \emptyset$, $L^\perp \cap K^* = \{0\}$.

The paper [61] briefly discusses the appearance of the equivalent conditions in Lemma C.1 in literature, expressed in terms of $\mathcal{N}(A)$, or $\mathcal{S}(A^\top)$, *i.e.* L corresponding to the null space or the range of the $m \times n$ matrix A. For the reader's convenience, we formulate the alternative expressions of the equivalent conditions $(i) - (iii)$ of Lemma C.1 in Table 1.

Table 1: Equivalent conditions of Lemma C.1 formulated for specific linear subspaces and cones appearing in the primal and dual conic linear programs (1.1) and (1.4). Conditions (i-c)-(iii-c) correspond to the special case of $cl(K)$ being pointed, conditions (i-d)-(iii-d) correspond to the special case of K being solid.

(i-a)	$\mathcal{S}(A^\top) + K^* = \mathcal{S}(A^\top) + lin(K^*)$	(i-b)	$\mathcal{N}(A) + K = \mathcal{N}(A) + lin(K)$
(ii-a)	$\mathcal{S}(A^\top) \cap relint(K^*) \neq \emptyset$	(ii-b)	$\mathcal{N}(A) \cap relint(K) \neq \emptyset$
(iii-a)	$\mathcal{N}(A) \cap [cl(K) \setminus sub(cl(K))] = \emptyset$	(iii-b)	$\mathcal{S}(A^\top) \cap [K^* \setminus sub(K^*)] = \emptyset$
(i-c)	$\mathcal{S}(A^\top) + K^* = \mathbb{R}^n$	(i-d)	$\mathcal{N}(A) + K = \mathbb{R}^n$
(ii-c)	$\mathcal{S}(A^\top) \cap int(K^*) \neq \emptyset$	(ii-d)	$\mathcal{N}(A) \cap int(K) \neq \emptyset$
(iii-c)	$\mathcal{N}(A) \cap cl(K) = \{0\}$	(iii-d)	$\mathcal{S}(A^\top) \cap K^* = \{0\}$

Remark C.2. It can be easily seen that (i-a)–(iii-a) and (i-b)–(iii-b) (similarly (i-c)–(iii-d) and (i-d)–(iii-d) are weak alternatives: at most one of them holds. Clearly, if (i-a)–(iii-a) (or (i-c)–(iii-c)) holds, then (i-b)–(iii-b)) (or (i-d)–(iii-d)) does not hold. However, they are not strong alternatives, as demonstrated in the following example: let $A = (1\ 0\ 1)$ and

$$K = K^* = cl(K) := \{(x_1, x_2, x_3)^\top \mid \sqrt{x_1^2 + x_2^3} \leq x_3\}.$$

$\mathcal{N}(A)$ is generated by $(-1, 0, 1)^\top, (0, 1, 0)^\top$ and hence neither (iii-c), nor (iii-d) holds.

Remark C.3. Conditions (i-a), (iii-a), (i-b) and (iii-b) can be formulated in terms of recession cones of $\mathcal{P}$ and $\tilde{\mathcal{D}}$ (see (1.5) and (1.6)), provided that these sets are nonempty, as follows: condition (i-a) is equivalent to $R^*_{\mathcal{P}} = cl(S(A^\top) + K^*)$ being a linear subspace and condition (iii-a), under a condition of closedness of K, is equivalent to $R_{\mathcal{P}}$ being a linear subspace. Condition (i-b), with requirement that K be closed, is equivalent to $R^*_{\tilde{\mathcal{D}}} = cl(\mathcal{N}(A) + K)$ being a linear subspace and condition (iii-b) is equivalent to $R_{\tilde{\mathcal{D}}}$ being a linear subspace.

Table 1 lists conditions under which a linear image of a convex cone is closed: it was shown in [66, Theorem 9.1], that (iii-a) implies $cl(A(K)) = A(cl(K))$. On the other hand, since $A(\mathcal{N}(A)) = \{0\}$, it can be easily seen that (i-b) implies

$$A(K) = A(\mathcal{N}(A) + K) = A(\mathcal{N}(A) + lin(K)) = A(lin(K)),$$

and hence in this case $A(K)$ is also closed since it is a linear subspace.

We now include a known result often referred to as *Theorem of Abrams.*

Theorem C.1 ([12], Proposition 3.1). *Let $S \subseteq \mathbb{R}^n$ be a nonempty set. Assume a linear map given by matrix A. Then*

$$A(S) \text{ is closed } \Leftrightarrow \mathcal{N}(A) + S \text{ is closed,}$$

We summarize the results in the following theorem.

Theorem C.2. *Assume that K satisfies Assumption 1.*

a) If any of the conditions (i-a), (ii-a), (iii-a) holds, then $A(cl(K))$ is closed and $\mathcal{S}(A^\top) + K^*$ is a linear subspace.

b) If any of the conditions (i-b), (ii-b), (iii-b) holds, then $A(cl(K)) = A(K)$ is a linear subspace and $\mathcal{S}(A^\top) + K^*$ is closed.

Remark C.4. Consider the second order cone K and A from Remark C.2. It can be easily seen that in this case

$$A(K) = \{u + w \mid (u, v, w)^\top \in K\} = \mathbb{R}_+,$$

and hence it is closed. This shows that the conditions in Table 1 are not necessary.

For more results and references, we refer the reader to [61], where the sufficient conditions for the closedness of a linear image of a convex cone and the Minkowski sum were studied in a more general setting, and the conditions were shown to be also necessary for a special class of cones.

D Vector space of multivariate polynomials

In this section we include standard definitions, notations and basic results concerning multivariate polynomials. It should be noted that a part of this section was published in [37].

We denote $\mathbb{R}[x]_d$ the real vector space of n-variate polynomials ($x \in \mathbb{R}^n$) with degree at most d and $\mathbb{R}[x]$ the real vector space of n-variate polynomials. The standard (or canonical) basis of $\mathbb{R}[x]_d$ consists of all monomials of degree at most d, namely

$$1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_{n-1}x_n, \ldots, x_n^2, \ldots, x_1^d, \ldots, x_n^d.$$

For instance, for $n = 2$ and $d = 3$ the canonical basis consists of monomials

$$1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3.$$

In fact, there are $s(n,d) := \sum_{i=0}^{d} \binom{n+i-1}{i} = \binom{n+d}{d}$ monomials of degree at most d, where $d \in \mathbb{N}_0$. Thus, $\dim(\mathbb{R}[x]_d) = s(n,d)$ and clearly $\mathbb{R}[x]_d \simeq \mathbb{R}^{s(n,d)}$. Apparently, any polynomial $p \in \mathbb{R}[x]_d$ can be represented as a linear combination of canonical basis vectors.

Introducing the standard multi-index notation, for

$$\mathbb{N}^n \ni \alpha := \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \in \mathbb{N}_0,\ i = 1, 2, \ldots, n\},$$

we set $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n \mid |\alpha| \le d\}$. We set $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$. Now, every polynomial $p \in \mathbb{R}[x]_d$ can be expressed in the form

$$p(x) = \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha x^\alpha, \quad x \in \mathbb{R}^n$$

where $p_\alpha \in \mathbb{R}$ are coefficients.

The inner product $\langle \cdot, \cdot \rangle : \mathbb{R}[x]_d \times \mathbb{R}[x]_d \to \mathbb{R}$ is defined as follows

$$\langle p, q \rangle = \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha q_\alpha, \quad p, q \in \mathbb{R}[x]_d. \tag{25}$$

The norm induced by inner product (25) takes the following form

$$\|p\| = \left(\sum_{\alpha \in \mathbb{N}_d^n} p_\alpha^2 \right)^{\frac{1}{2}} \tag{26}$$

The norm (26) induces a topology on $\mathbb{R}[x]_d$. A set $\mathcal{O} \subseteq \mathbb{R}[x]_d$ is open if

$$\forall p \in \mathcal{O}\ \exists r > 0 : \mathcal{B}(p, r) := \{q \in \mathbb{R}[x]_d \mid \|p - q\| < r\} \subset \mathcal{O}.$$

Note that all norms on $\mathbb{R}[x]_d$ are equivalent and, therefore, they define the same open sets of $\mathbb{R}[x]_d$.

Note that a sequence $\{p_j\}_{j=1}^{\infty} \subseteq \mathbb{R}[x]_d$ converges to $p \in \mathbb{R}[x]_d$, denoted

$$\lim_{j\to\infty} p_j = p,$$

if

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall j > n_0 : \|p_j - p\| < \varepsilon.$$

Vector space $\mathbb{R}[x]_d$ equipped with the norm $\|\cdot\|$ is a normed space and, therefore, it is first-countable. It means that for any subset $S \subseteq \mathbb{R}[x]_d$ it holds that $x \in \mathrm{cl}(S)$ if and only if there exists a sequence $\{x_j\}_{j=1}^{\infty} \subseteq S$ such that $\lim_{j\to\infty} x_j = x$.

We denote

$$m_d(x) = \left(1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_{n-1}x_n, \ldots, x_n^2, \ldots, x_1^d, \ldots, x_n^d\right)^\top$$

for every $x \in \mathbb{R}^n$. Note that $m_d : \mathbb{R}^n \to \mathbb{R}^{s(n,d)}$.

Finally, we formulate two auxiliary propositions – Proposition D.1 and Preposition D.2 – which will be useful when dealing with multivariate polynomials. Their proofs can be found in the Appendix.

Proposition D.1. *Let $p \in \mathbb{R}[x]_d$. Then*

$$\forall x \in \mathbb{R}^n : |p(x)| \leq \|p\| \|m_d(x)\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

Proof. The claim follows from Cauchy-Schwarz inequality applied to a vector of coefficients $(p_\alpha)_{\alpha\in\mathbb{N}_d^n}$ and the vector of monomial basis $m_d(x)$, since

$$p(x) = m_d(x)^\top (p_\alpha)_{\alpha\in\mathbb{N}_d^n}, \quad \text{for all} \ \ x \in \mathbb{R}^n.$$

□

Proposition D.2. *For every $x \in \mathbb{R}^n$ and $d \in \mathbb{N}$ the following inequality holds*

$$\|m_d(x)\|_2^2 \geq \|m_{2d}(x)\|_2.$$

Proof. Since both right-hand side and left-hand side of the inequality are non-negative numbers, we can equivalently prove $(m_d(x)^\top m_d(x))^2 \geq m_{2d}(x)^\top m_{2d}(x)$ for all $x \in \mathbb{R}^n$ and $d \in \mathbb{N}$. Note that

$$m_{2d}(x)^\top m_{2d}(x) = \sum_{\gamma \in \mathbb{N}^n_{2d}} x_1^{2\gamma_1} x_2^{2\gamma_2} \dots x_n^{2\gamma_n}, \quad \forall x \in \mathbb{R}^n$$

and

$$(m_d(x)^\top m_d(x))^2 = \sum_{\alpha,\beta \in \mathbb{N}^n_d} x_1^{2\alpha_1+2\beta_1} x_2^{2\alpha_2+2\beta_2} \dots x_n^{2\alpha_n+2\beta_n},$$

$\forall x \in \mathbb{R}^n$.

Fix an arbitrary $x \in \mathbb{R}^n$ and $d \in \mathbb{N}$. We will show that every term included in $m_{2d}(x)^\top m_{2d}(x)$ is also included in $(m_d(x)^\top m_d(x))^2$. Since both $m_{2d}(x)^\top m_{2d}(x)$ and $(m_d(x)^\top m_d(x))^2$ are sums of non-negative numbers for any given $x \in \mathbb{R}^n$, we will prove that $(m_d(x)^\top m_d(x))^2 \geq m_{2d}(x)^\top m_{2d}(x)$ for all $x \in \mathbb{R}^n$ and $d \in \mathbb{N}$.

More specifically, we want to show that for any $\gamma \in \mathbb{N}^n_{2d}$ there exist $\alpha, \beta \in \mathbb{N}^n_d$ such that $\alpha + \beta = \gamma$. For an arbitrary but fixed $\gamma \in \mathbb{N}^n_{2d}$ we will construct $\alpha \in \mathbb{N}^n_d$ by setting

$$\alpha_i = \begin{cases} \frac{\gamma_i}{2}, & \gamma_i \equiv 0 \pmod 2, \\ \frac{\gamma_i - 1}{2}, & \gamma_i \equiv 1 \pmod 2 \ \wedge \ \sum_{j=1}^{i-1} \alpha_j > \sum_{j=1}^{i-1} (\gamma_j - \alpha_j), \\ \frac{\gamma_i + 1}{2}, & \gamma_i \equiv 1 \pmod 2 \ \wedge \ \sum_{j=1}^{i-1} \alpha_j \leq \sum_{j=1}^{i-1} (\gamma_j - \alpha_j), \end{cases}$$

$i = 1, 2, \dots, n$. Then $\beta_i = \gamma_i - \alpha_i$, $i = 1, 2, \dots, n$. We need to show that $\alpha, \beta \in \mathbb{N}^n_d$. It is obvious that $\alpha + \beta = \gamma$ and that $\alpha_i, \beta_i \in \mathbb{N}_0$, $i = 1, 2, \dots, n$ and, therefore, it suffices to show that $\sum_{i=1}^n \alpha_i \leq d$ and $\sum_{i=1}^n \beta_i \leq d$.

Denote o_1 the number of cases when $\gamma_i \equiv 1 \pmod 2 \ \wedge \ \sum_{j=1}^{i-1} \alpha_j \leq \sum_{j=1}^{i-1} (\gamma_j - \alpha_j)$, $i = 1, 2, \dots, n$ and o_2 the number of cases when $\gamma_i \equiv 1 \pmod 2 \wedge \sum_{j=1}^{i-1} \alpha_j > \sum_{j=1}^{i-1} (\gamma_j - \alpha_j)$, $i = 1, 2, \dots, n$. Then it follows that

$$\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \frac{\gamma_i}{2} + \frac{1}{2}(o_1 - o_2),$$
$$\sum_{i=1}^n \beta_i = \sum_{i=1}^n \frac{\gamma_i}{2} + \frac{1}{2}(o_2 - o_1).$$

Firstly, we will show that $o_1 - o_2 \in \{0, 1\}$. Suppose that there are k odd numbers among $\gamma_1, \gamma_2, \dots, \gamma_n$, where $k \in \{0, 1, 2, \dots, n\}$. If $k = 0$, then obviously $o_1 = o_2 = 0$ and thus $o_1 - o_2 \in \{0, 1\}$. If $k \neq 0$, denote these odd numbers $\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_k}$. If $k \equiv 1 \pmod 2$, then $k = 2l - 1$ for some $l \in \mathbb{N}$ and by construction of α we have $\alpha_{i_1} = \frac{\gamma_{i_1}+1}{2}$, $\alpha_{i_2} = \frac{\gamma_{i_2}-1}{2}$,

$\ldots$, $\alpha_{i_{2l-2}} = \frac{\gamma_{i_{2l-2}}-1}{2}$, $\alpha_{i_{2l-1}} = \frac{\gamma_{i_{2l-1}}+1}{2}$. Therefore, $o_1 = l$ and $o_2 = l-1$ and, therefore, $o_1 - o_2 = 1 \in \{0,1\}$. If $k \equiv 0 \pmod 2$, then $k = 2l$ for some $l \in \mathbb{N}$. Again, by construction of α we have $\alpha_{i_1} = \frac{\gamma_{i_1}+1}{2}$, $\alpha_{i_2} = \frac{\gamma_{i_2}-1}{2}$, $\ldots$, $\alpha_{i_{2l-1}} = \frac{\gamma_{i_{2l-1}}+1}{2}$, $\alpha_{i_{2l}} = \frac{\gamma_{i_{2l}}-1}{2}$. Therefore, $o_1 = o_2 = l$ and, therefore, $o_1 - o_2 = 0 \in \{0,1\}$.

Since $o_1 - o_2 \in \{0,1\}$, we have shown that $\sum_{i=1}^n \alpha_i \geq \sum_{i=1}^n \beta_i$. Now, we will show that $\sum_{i=1}^n \alpha_i \leq d$. It is evident that $\sum_{i=1}^n \alpha_i \leq \sum_{i=1}^n \frac{\gamma_i}{2} + \frac{1}{2}$. Moreover, since $\gamma \in \mathbb{N}^n_{2d}$, we have that $\sum_{i=1}^n \gamma_i \leq 2d$. There are two cases to consider:

1. $\sum_{i=1}^n \gamma_i \leq 2d-1 < 2d$. It automatically follows that $\sum_{i=1}^n \frac{\gamma_i}{2} + \frac{1}{2} \leq d$ and, therefore, $\sum_{i=1}^n \alpha_i \leq d$.
2. $\sum_{i=1}^n \gamma_i = 2d$. However, that is possible if and only if $k \equiv 0 \pmod 2$, which means that $o_1 - o_2 = 0$ and, therefore, $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \frac{\gamma_i}{2} = d$.

We have finally shown that $d \geq \sum_{i=1}^n \alpha_i \geq \sum_{i=1}^n \beta_i$ which completes the proof. $\square$

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