THE RAMSEY MODEL OF ECONOMIC GROWTH AS AN OPTIMAL CONTROL PROBLEM

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We study the Ramsey model of economic growth with the irreversible investment constraint. The Ramsey model is one of the basic concepts used when solving the problem of the optimal allocation of the production. The objective is to maximize the discounted value of utility of consumption across the whole planning horizon. To solve the model, the Pontriagin maximum principle is used. The model is formulated as an optimal control problem with infinite horizon. The solution of the model is well known providing that the constraint is not binding. However, only a little we know about the behaviour of the solution if the constraint is binding, especially for large values of the initial level of capital. In this paper, we extend the result that are known considering this case and show that there are several types of the solution plausible.

Keywords: Ramsey Model, optimal control, Pontriagin maximum principle

2000 *Mathematics Subject Classification*:

1 INTRODUCTION

The Ramsey model of optimal economic growth was firstly introduced in [8] in 1928. Today it is considered to be one of the basic models in the modern dynamic macroeconomics. In this paper, we deal with a specific version of this model including irreversible investment constraint. Although the solution of the model is widely available in the prevalent literature regarding the theory of the economic growth (see e.g. [2], [3]), the solution is given only in case that the constraint is not actually binding. Some results that are already known consider the limit behaviour of the optimal solution for infinitely large level of initial resources (see [1]) or a very local behaviour at some points (see [4]). Our purpose is to bring further contribution to the description of the optimal solution if the constraint of irreversible investment is binding.

In section 2 we formulate the Ramsey model as an optimal control problem and introduce the assumptions of the model. The necessary conditions implied by the Pontriagin maximum principle are given in section 3. Section 4 introduced the well-known solution of the model for "standard" level of initial condition. In section 5 we deal with the situation when the constraint in the model is binding. Section 6 concludes.

2 FORMULATION OF THE MODEL

To introduce the model, consider a closed economy, which produces only one product. The amount of production depends on the level of capital per capita denoted by k. This dependency is given by the production function $f(k)$. We assume that

(A1) $f \in C^2(\langle 0,\infty))$,

(A2) f is strictly increasing, strictly concave and nonconstrained on $\langle 0,\infty)$,

(A3) $f(0)=0$, $\lim_{k\to 0^+} f'(k)=\infty$, $\lim_{k\to\infty} f'(k)=0$.

This production can be divided between current consumption c and investment in the capital. The investment increases the disposable capital in the future and hence increases the future consumption. As the resources are scarce, in every moment a decision has to be made: We have to determine how much of the production should be consumed now and how much should be invested to make future consumption higher. To solve this dilemma, we have to find the optimal distribution of the consumption in the time in terms of maximizing the utility of this consumption U discounted by $r>0$ across the whole time horizon $\langle 0,\infty)$. The assumptions on the utility functions are as follows:

(A4) $U \in C^2((0,\infty))$,

(A5) U is strictly increasing and strictly concave on $(0,\infty)$,

(A6) $\lim_{c\to 0^+} U'(c)=\infty$, $\lim_{c\to\infty} U'(c)=0$.

Moreover, at every moment we assume that the amortisation of the capital per capita is given by the factor $\lambda>0$, which is the sum of the growth rate of the population and the rate of amortisation of the total capital. Finally, the consumption cannot be negative and it is not possible to consume more than there was the production at the given time. This constraint is called constraint of irreversible investment.

To sum it up, the Ramsey model can be formulated as an optimal control problem in the following form:

$$\begin{aligned}
&\max_{\{c(t)\}} \int_0^\infty e^{-rt} U\big(c(t)\big)\,dt\\
&\dot{k} = f(k) - \lambda k - c, \quad k(0)=k_0>0 \text{ given}\\
&0 \le c(t) \le f((k(t)). \qquad \text{(RM)}
\end{aligned}$$

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3 THE NECESSARY CONDITIONS

To solve the problem (RM), we define the Hamiltonian function H and Lagrangian function L as follows

$$H(k, c, \psi_0, \psi) = \psi_0 U(c) + \psi[f(k) - \lambda k \ - c]$$

$$L(k, c, \psi_0, \psi, \mu_1, \mu_2) = H(k, c, \psi_0, \psi) + \mu_1 c + \mu_2[f(k) - c].$$

The Pontriagin maximum principle (PPM) yields (cf. [4, Th. 7.4]) that if $(c^*(t), k^*(t))$ is the optimal solution to (RM), then there exists a constant $\psi_0 = 0$ or $\psi_0 = 1$, a continuous function $\psi(t)$ and piecewise continuous functions $\mu_1(t)$, $\mu_2(t)$ such that following conditions are satisfied:

(i) For all $t \geq 0$ it holds $(\psi_0, \psi, \mu_1, \mu_2) \neq (0, 0, 0, 0)$.

(ii) With the possible exception of the discontinuity points of c^* we have

$$\frac{\partial L}{\partial c} = \psi_0 U'(c^*) - \psi + \mu_1 - \mu_2 = 0, \qquad (1)$$

$$\begin{aligned} &\mu_1 \geq 0, \quad \mu_1 c^* = 0, \\ &\mu_2 \geq 0, \quad \mu_2[f(k^*) - c^*] = 0. \end{aligned} \qquad (2)$$

(iii) The function ψ has a continuous derivative at all discontinuity points of c^* and

$$\dot{\psi} = r\psi - \frac{\partial L}{\partial k} = [r + \lambda - f'(k^*)]\psi - \mu_2 f'(k^*). \qquad (3)$$

Moreover, we use the condition introduced and proved in [7]

$$e^{-rt} H(k^*(t), c^*(t), \psi_0, \psi(t)) = r\psi_0 \int_t^\infty e^{-rs} U(c^*(s))\, ds, \qquad (4)$$

again for all continuity points of c^*. Although this condition is derived in [7] only for problems without constraints on state or control variables, as it is stated in this paper the proof can be easily extended for discounted autonomous problems with mixed constraints.

Although the necessary conditions state that $\psi_0 = 0$ or $\psi_0 = 1$, the former case can be actually excluded. For contradiction, assume that $\psi_0 = 0$. If $\psi(\tau) = 0$ at any $\tau \geq 0$, then from the condition (i) and the Equation (1) we have $\mu_1(\tau) = \mu_2(\tau) \neq 0$. The Equation (3) then implies $c^*(\tau) = 0$ and $c^*(\tau) = f(k^*(\tau))$, hence $k^*(\tau) = 0$ using (A2) and (A3). Recalling the formulation of the model, from the equation $\dot{k}^* = f(k^*) - \lambda k^* - c^* \geq -\lambda k$ using $k^*(0) > 0$ we can derive $k^*(t) > 0$ for all $t \geq 0$, a contradiction.

So far we know that ψ is continuous function which is different from zero everywhere. As a result, employing the necessary condition we get that two plausible cases can be distinguished: $c^*(t) \equiv 0$ if $\psi(t) > 0$, and $c^*(t) \equiv f(k^*(t))$ if $\psi(t) < 0$.

First, we shall exclude the latter case using (4). The condition $c^* = f(k^*) > 0$ implies $\mu_1 = 0$ and from (1) we have $\psi = -\mu_2$. Hence, the Equation (2) takes the form $\dot{\psi} = (r + \lambda)\psi$, where the solution is given by $\psi(t) = Ae^{(r+\lambda)t}$, $A < 0$. On the other hand, for the state variable k we have $\dot{k}^* = -\lambda k^*$, suggesting $k^*(t) = k_0 e^{-\lambda t}$. Finally we shall verify whether the condition (4) is satisfied: For $\psi_0 = 0$ we get that

$$e^{-rt} H(k, c, \psi_0, \psi) = e^{-rt}\psi[f(k^*) - \lambda k^* - c^*] = -\lambda A k_0$$

should be zero, a contradiction with $\lambda > 0$, $A < 0$, $k_0 > 0$.

It is now easy to exclude $c^* \equiv 0$ and so to complete the proof. Actually, the value of the objective function is strictly greater for $c = f(k)$, which is admissible but non-optimal, as we have just shown.

4 THE SOLUTION TO THE MODEL

The solution of the Ramsey model (RM) is well-known in case that the constraint $c \leq f(k)$ is not binding at any $t \geq 0$. We shall describe it in short in this section. In addition, using (A6) it can be easily concluded that the second constraint $c \geq 0$ cannot be binding at any $t \geq 0$ (cf. [4, Ex. 8.5]), thus $\mu_1 = 0$. The interpretation of this fact tell us that it cannot be optimal to consume nothing and save the whole production, even for arbitrary small level of the initial capital.

First, consider that the constraint $c \leq f(k)$ is not binding at any $t \geq 0$. Recall that in this case each optimal solution (k^*, c^*) has to satisfy a system of non-linear differential equations in the following form:

$$\begin{aligned} \dot{k}^* &= f(k^*) - \lambda k^* - c^*, \\ \dot{c}^* &= \frac{U'(c^*)}{U''(c^*)}[r + \lambda - f'(k^*)]. \end{aligned}$$

It can be shown that this system has a saddle point $(\hat{k}, \hat{c})$, for which it holds $0 < \hat{c} < f(\hat{k})$ (for the precise derivation of these results we can refer e.g. to [4], [5], [6]).

In our model, the Hamiltonian function H is concave in both state and control variable. To show this, we use $\psi > 0$ implied by (1) and (2). Moreover, the constraints are quasi-concave. Therefore we also have sufficient conditions for the optimal solution that read (cf. [9, Th. 6.11]): A pair of admissible solution to (RM) $((k^*(t), c^*(t))$ is optimal if there exist functions ψ, μ_1 and μ_2 such that conditions (i) – (iii) from the previous section are satisfied with $\psi_0 = 1$ and, in addition,

$$\liminf_{t \to \infty} e^{-rt}\psi(t)(\tilde{k}(t) - k^*(t)) \geq 0$$

for all admissible $\tilde{k}(t)$. Furthermore, as H is strictly concave, the solution satisfying this conditions is unique. The precise derivation specifically for the Ramsey model is given also in [6, Th. 2.5]. However, if $(k^*(t), c^*(t))$ is the solution of the necessary conditions that converges (along the stable saddle path) to $(\hat{k}, \hat{c})$, these sufficient conditions hold. This claim is based on the fact that each admissible solution $\tilde{k}(t)$ is restricted from bellow and $k^*(t)$ is restricted from above.

5 THE SOLUTION TO THE MODEL FOR LARGE k_0

From the previous chapter we know that the optimal solution to the Ramsey model is the stable saddle path for any initial level of capital k_0, providing that this saddle path lies everywhere under the constraint $c = f(k)$, i.e. the constraint is not binding. However, there arises a question concerning the optimal solution in the case that the intersection of the stable path and the constraint $c = f(k)$ exists. Denote by k_p the lowest value such that $k_p \geq \hat{k}$ and $(k_p, f(k_p))$ is the intersection of the constraint $c = f(k)$ and the stable saddle path; define $k_p = \infty$ if such an intersection does not exist. There are two types of problems: First, we can study whether it can be optimal to consume the whole production as k_0 approaches to infinity (note that this can be the case only if $k_p < \infty$). Another interesting question is if the constraint is binding for $k > k_p$.

The first problem can be presented as a problem of limit behaviour of the optimal solution when the initial level of capital approaches to infinity. As $k_0 \to \infty$, at the beginning of the time horizon it can be optimal to consume the whole production, not to consume the whole production, or neither of these cases (i.e. the limit behaviour cannot be determined). This problem was dealt by Arrow and Kurz (see [1]). Considering the specific form of the function U given by

$$U_\theta(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{for } \theta > 0,\ \theta \neq 1, \\ \ln c & \text{for } \theta = 1, \end{cases}$$

they suggested and proved a sufficient condition providing that $c^*(0) = f(k^*(0))$ as k_0 approaches to infinity. This condition reads

$$\limsup_{k\to\infty} \left(\lambda\theta \frac{f'(k)k}{f(k)}\right) < r + \lambda.$$

Expressed in economic terms, it is optimal to consume the whole production for the sufficient level of capital if the rate of impatience r is sufficient large, the elasticity of production is small and the elasticity of marginal utility given by $-\theta$ is sufficiently high.

Now we provide the answer to the second question. We want to distinguish two types of solutions: (a) The constraint is binding for all $k > k_p$, and (b) the constraint is not binding for all $k > k_p$, although it is binding on some right neighborhood of k_p. Actually, we already know that the constraint is always binding on some right neighborhood of k_p (see [4, Ex. 8.5]). We shall extend this result and prove that if the stable saddle path lies above the constraint on (k_p, k_p'), the constraint is not binding on some right neighborhood of k_p'. In addition, the optimal solution surprisingly does not lie on the saddle path for $k > k_p'$ anymore, but it lies under this path.

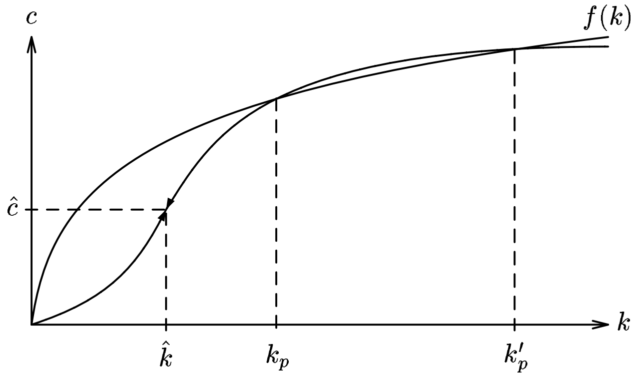

Fig. 1. The stable saddle path lies above the constraint on (k_p, k_p').

Lemma 1. *Let $(k^*(t), c^*(t))$ be the optimal solution to (RM) with the initial condition $k(0) = k_0 > k_p$. Then $(c^*(t), k^*(t))$ lies for all $t \in \langle t_0, t_p)$ under the stable saddle path, where $k(t_p) = k_p$.*

Proof. Let us choose arbitrary $\tau_0 \in \langle t_0, t_p)$. Consider the stable path approaching to $(\hat{k}, \hat{c})$ as a parameterized curve $(\bar{k}(t), \bar{c}(t))$, where $\bar{k}(\tau_0) = k^*(\tau_0)$. Furthermore, let α be arbitrary number satisfying

$$\alpha > \max_{\tau_0 \leq t \leq \tau_p} \frac{\bar{c}(t)}{f(\bar{k}(t))},$$

where τ_p is such that $\bar{k}(\tau_p) = k_p$. It holds $\bar{c}(t) < \alpha f(\bar{k}(t))$ for all $t \in \langle \tau_0, t_p)$. Hence $(\bar{k}(t), \bar{c}(t))$ is an optimal solution to the problem

$$\begin{aligned} &\max_{\{c(t)\}} \int_{\tau_0}^{\infty} e^{-r(t-\tau_0)} U\big(c(t)\big)\, dt \\ &\dot{k} = f(k) - \lambda k - c,\ k(\tau_0) = k^*(\tau_0), \\ &0 \leq c(t) \leq \alpha f((k(t)). \end{aligned} \qquad (\mathrm{RM}_\alpha)$$

The necessary conditions implied by the Pontriagin maximum principle state that there exist a constant $\bar{\psi}_0$ and functions $\bar{\psi}(t)$, $\bar{\mu}_1(t)$ and $\bar{\mu}_2(t)$ satisfying conditions very similar to (i) – (iii) from section 4 for the problem (RM_α). Moreover, the proof that $\psi_0 = 1$ can be straightforwardly rewritten to derive that $\bar{\psi}_0 = 1$.

Notice that the constraint in (RM_α) is not binding at any $t \geq \tau_0$, thus $\bar{\mu}_1(t) = \bar{\mu}_2(t) = 0$. On the other hand, $(k^*(t), c^*(t))$ is the unique optimal solution to (RM), where the constraint is binding on a non-trivial interval. Therefore we have the following inequality for the values of the objective functions:

$$\int_{\tau_0}^{\infty} e^{-r(t-\tau_0)} U\big(\bar{c}(t)\big)\, dt > \int_{\tau_0}^{\infty} e^{-r(t-\tau_0)} U\big(c^*(t)\big)\, dt. \quad (5)$$

Recall that the necessary conditions for both problems (RM) and (RM$_\alpha$) are satisfied with $\psi_0 = 1$ and $\bar{\psi}_0 = 1$, respectively. Using this fact together with (4) and the definition $k^*(\tau_0) = \bar{k}(\tau_0) =: \bar{k}_0$ we get

$$\int_{\tau_0}^{\infty} e^{-r(t-\tau_0)} U\big(c^*(t)\big)\, dt = \frac{1}{r} U(c^*(\tau_0)) +$$

$$+\frac{1}{r}\left[U'(c^*(\tau_0)) - \mu_2(\tau_0)\right]\left[f(\bar{k}_0) - \lambda\bar{k}_0 - c^*(\tau_0)\right]$$

and analogously

$$\int_{\tau_0}^{\infty} e^{-r(t-\tau_0)} U\big(\bar{c}(t)\big)\, dt =$$

$$= \frac{1}{r}\left\{ (U(\bar{c}(\tau_0)) + U'(\bar{c}(\tau_0))\left[f(\bar{k}_0) - \lambda\bar{k}_0 - \bar{c}(\tau_0)\right] \right\}.$$

Let us denote

$$G(c_0) := U(c_0) + U'(c_0)\left(f(\bar{k}_0) - \lambda\bar{k}_0 - c_0\right). \qquad (6)$$

Using the derived formulae, the Inequality (5) takes the following form:

$$G(\bar{c}(\tau_0)) > G(c^*(\tau_0)) - \mu_2(\tau_0)\left(f(\bar{k}_0) - \lambda\bar{k}_0 - c^*(\tau_0)\right).$$

Further we can use (2) twice to get

$$G(\bar{c}(\tau_0)) > G(c^*(\tau_0)) + \mu_2(\tau_0)\lambda\bar{k}_0 \geq G(c^*(\tau_0)). \qquad (7)$$

This directly implies $\bar{c}(\tau_0) \neq c^*(\tau_0)$.

Suppose for contradiction that $\bar{c}(\tau_0) < c^*(\tau_0)$. Differentiating of (6) leads to

$$G'(\bar{c}(\tau_0)) = U''(\bar{c}(\tau_0))\left(f(\bar{k}_0) - \lambda\bar{k}_0 - \bar{c}(\tau_0)\right) =$$
$$= U''(\bar{c}(\tau_0))\dot{\bar{k}}(\tau_0).$$

For $\bar{k} > \hat{k}$ lies the stable path in the region with $\dot{k} < 0$, $\dot{c} < 0$. Moreover, the Assumption (A5) yields $U''(\bar{c}(\tau_0)) < 0$, thus

$$G'(\bar{c}(\tau_0)) > 0.$$

For arbitrary $c_0 > \bar{c}(\tau_0)$ we get

$$G'(c_0) = U''(c_0)\left(f(\bar{k}_0) - \lambda\bar{k}_0 - c_0\right) =$$
$$= U''(c_0)\left(f(\bar{k}_0) - \lambda\bar{k}_0 - \bar{c}(\tau_0)\right) - U''(c_0)(c_0 - \bar{c}(\tau_0) =$$
$$= U''(c_0)\dot{\bar{k}}(\tau_0) - U''(c_0)(c_0 - \bar{c}(\tau_0)) > 0.$$

The function G is therefore increasing on $\langle \bar{c}(\tau_0), c^*(\tau_0)\rangle$, hence $G(\bar{c}(\tau_0)) < G(c^*(\tau_0))$, a contradiction with (7). We can conclude that $\bar{c}(\tau_0) > c^*(\tau_0)$. ■

6 CONCLUSION AND DISCUSSION

The contribution of this paper was the introduction of some extension to the standard solution of the Ramsey model. We proved that the necessary conditions implied by the Pontriagin maximum principle has to be satisfied with $\psi_0 = 0$ and that the well-known solution of the Ramsey model, which converges along the stable path to the saddle point $(\hat{k}, \hat{c})$ is really unique. We used this properties to extend the description of the solution that has been known so far in case that the stable path intersect the constraint $c = f(k)$ at k_p but lies under the constraint for $k > k_p'$, where $k_p' > k_p$. To be more specific, we have shown that in this case the constraint is not binding at the beginning of the planning horizon, than $c^* = f(k^*)$ on a non-trivial interval and finally it converges along the stable saddle path to $(\hat{k}, \hat{c})$. Translated into economic terms: If the irreversible investment constraint is binding on a non-trivial time interval, we cannot consume such an amount as it would be optimal if we had not considered this constraint. However, it is optimal to consume less that it would be optimal without this constraint also before this interval, even though the constraint is not binding.

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