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SUSTAINABILITY IN MODELS OF OPTIMAL ECONOMIC GROWTH

Dissertation Thesis
in Applied Mathematics

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Preface

Motto:

"Treat the Earth well.
It is not inherited from your parents,
it is borrowed from your children."

(old Kenyan proverb)

In recent years, sustainability has become an often-used concept of the economic development, influencing the policy debates and actions even on its highest level. However, its definition is far from being agreed unequivocally. Indeed, the sustainability includes great variety of different perspectives. Strong increase in population facing restricted land and space, impact of industrial growth on environment and climate and growing energy consumption with finite stock of non-renewable resources are only several examples. In addition, the sustainability often comprises a social aspect of the economic growth. Stated generally, this aspect requires the so-called intergenerational equity which means that needs of the present generation should be met without compromising the ability of future generations to meet their own needs. However, designing a policy which takes into account rights and desicions of people not yet born might be quite a challenging task. In this thesis, we will focus on models of an efficient extraction of exhaustible resources taking into consideration the requirement of intergenerational equity.

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Pavol Jurča

Abstract

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In this thesis, we deal with models of sustainable economic growth in an economy with renewable as well as exhaustible capital resources, formulated as optimal control problems. For this type of models, one of the most important rules is Hartwick's rule. It states that all revenues from the extraction of exhaustible capital goods should be reinvested to reproducible capital. Based on the rigorous formulation of results from optimal control theory, we summarize the most important results on Hartwick's rule in a unified framework. We consider models of economic growth with discounted utility criterion as well as maximin criterion and shed light on the relationship between them. We provide a new or generalized formulation of some results and simplify some proofs. Then, we propose a novel model with two types of mutually substitutable exhaustible goods with different productivities. Using necessary conditions of optimality for problems with binding pure state constraints, we provide a qualitative analysis of solutions to this model. In particular, we find that it is not optimal to further exploit the resource with a constant productivity after the extraction of the resource with a growing productivity started. Moreover, we extend some results on application of Noether's theorem in optimal control problems and use them for formulating conservation laws which represent quantities that remain sustained along trajectories of optimal solutions. Finally, we also make a contribution to the optimal control theory itself. In particular, we shed light on comparison between two different sets of necessary conditions of optimality for problems with pure state constraints.

Keywords: Sustainability, Harwick's rule, optimal control, conservation law, exhaustible resources

Abstrakt

Jurča, P. (2010): Udržateľnosť v modeloch optimálneho ekonomického rastu [Dizertačná práca]. Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky. Školiteľka: doc. RNDr. Margaréta Halická, CSc. Bratislava.

V práci sa zaoberáme modelmi udržateľného ekonomického rastu v ekonomike, v ktorej existujú obnoviteľné ako aj vyčerpateľné zdroje, formulovaných v tvare úloh optimálneho riadenia. Pre tento typ modelov je jedným z najdôležitejších pravidiel Hartwickovo pravidlo. Podľa tohto pravidla by všetky výnosy z ťažby vyčerpateľných zdrojov mali byť reinvestované do obnoviteľného kapitálu. Na základe presnej formulácie výsledkov teórie optimálneho riadenia zhrnieme v jednotnej forme najvýznamnejšie výsledky týkajúce sa Hartwickovho pravidla. Okrem toho sa budeme zaoberať modelmi ekonomického rastu s účelovou funkciou založenou na diskontovanej užitočnosti, ako aj s účelovou funkciou typu max-min a objasníme vzťah medzi nimi. Pri niektorých výsledkoch uvedieme ich novú alebo všeobecnejšiu formuláciu, prípadne zjednodušíme ich dôkaz. Navrhneme nový model s dvoma navzájom substituovateľnými vyčerpateľnými zdrojmi, ktoré majú rôznu produktivitu. Pomocou nutných podmienok optimality pre úlohy s aktívnymi čistými ohraničeniami na stavové premenné uvedieme kvalitatívnu analýzu riešení tohto modelu. Konkrétne ukážeme, že nie je optimálne ďalej využívať zdroj s konštantnou produktivitou, ak už začala ťažba zdroja s rastúcou produktivitou. Ďalej rozšírime niektoré výsledky týkajúce sa aplikácie Noetherovej vety v optimálnom riadení a využijeme ich na formuláciu zákonov zachovania, ktoré reprezentujú veličiny, ktoré zostávajú konštantné pozdĺž trajektórií optimálnych riešení. Napokon uvedieme niekoľko príspevkov aj do samotnej teórie optimálneho riadenia. Konkrétne sa budeme zaoberať porovnaním dvoch rôznych množín nutných podmienok optimality pre úlohy zahrnujúce čisté ohraničenia na stavové premenné.

Kľúčové slová: Udržateľnosť, Hartwickovo pravidlo, optimálne riadenie, zákon zachovania, vyčerpateľné zdroje

Contents

Chapter 1

Introduction

1.1 Motivation

The world economy is heavily dependent on extraction of exhaustible resources such as oil, coal, natural gas, uranium or minerals. Hence, a natural question arises whether the current level of consumption is sustainable despite its dependence on these non-renewable resources. This question is studied in the natural resources economy. However, the sustainability of economic growth based on depletion of non-reproducible resources should be viewed well beyond the horizon of the current generation. This is captured by the social aspect of the sustainability concept which requires that the current state of the economy and its consumption possibilities should be preserved to all future generations.

In this thesis, we study models of sustainable economic growth. These models will be formulated as optimal control problems. Several approaches how to incorporate the sustainability requirement have been adopted in the literature.

The first approach comes up from the utilitarian tradition. The conventional models of economic growth were traditionally based on maximizing of the discounted utility of consumption. The discount factor imposes a higher weight on the current utility at the expense of utility enjoyed by future generations. In addition, it expresses the preferences of the current generation. In this approach, the conventional objective criterion of maximizing discounted utility is amended by an additional requirement that some (undiscounted) level of utility is maintained forever. In this context, one of the most important results was formulated by Hartwick (1977): The utility stays at a constant level provided that all the revenues from exhaustible capital depletion are being invested into the reproducible capital. This rule became known as Hartwick's rule. The second approach which is called the maximin approach incorporates the sustainability requirement directly to the objective function by maximizing the utility level of the least advantaged generation. Both of these approaches focus on the sustainability of the utility level. On the other hand, some authors regard this definition of sustainability as rather narrow and search for any invariant quantities that stay preserved, using a theory based on Noether's theorem.

1.2 Goals of this thesis

The main goals of this thesis and their motivation are summarized in the following paragraphs:

1. The literature on this topic is rather wide and the research has been still quite lively in recent years. However, the variety of different frameworks, models and assumptions is also rather diverse. Optimal control models can be employed to study the sustainability topic, but still many researches use other approaches. Hence, the first goal is to summarize the most important results on Hartwick's rule and to present them in a unified framework.

2. Secondly, the level of mathematical rigor varies across the literature. Our presentation of known results builds on rigorous formulation of results from optimal control theory, which are described in detail in the last chapter. We will show that some results can be generalized and some proofs can be presented in a simpler way. We also provide new formulation of some results.

3. In addition to known results, we propose a new model with two exhaustible resources. These resources are mutually perfectly substitutable, although the rate of this substitutability changes in time. Employing the optimal control theory, we are able to derive interesting qualitative properties of the solution to this model. Moreover, models used in the context of sustainable economic growth naturally impose non-negativity constraints on the economic variables. In the analysis of these models, the non-negativity constraints are then prevalently neglected and only interior solutions are studied. In our newly proposed model, these constraints become binding along an optimal solution, including constraints on state variables. Hence, we have to use optimal control theory for problems with binding pure state constraints, which is more difficult than the theory for problems without these constraints.

4. As mentioned earlier, one of the approaches to sustainability is to study invariant quantities which stay constant along optimal trajectories. This approach is based on Noether's theorem which states that if a model is invariant with respect to a transformation of variables, then an invariant quantity can exist. The method of searching for such invariant quantities was proposed by Torres (2004). Since this method cannot be directly applied to our models of optimal economic growth, we formulate and prove an extension of results obtained by Torres which are applicable to our models.

5. Besides studying of models of sustainable growth, this thesis also provides some insights into optimal control theory for problems with pure state constraints. We present the necessary conditions of optimality for these problems based on two different sources – Feichtinger and Hartl (1986) and Seierstad and Sydsæter (1987). We provide proofs of some properties of solutions to these models which were not found in the literature. We study in detail the relationship between these two sets of necessary conditions. The proposed transformation allows us to use theoretical results from both sources.

1.3 Structure of this thesis

The structure of this thesis is as follows:

In Chapter 2, we formulate the models of sustainable economic growth as optimal control problems for both approaches – the discounted utility approach and the maximin approach. We summarize all assumptions on parameters and functions included in these models.

In Chapter 3, we present the necessary conditions of optimality which will be used later in the analysis. We also provide some known existence results and the explicit form of optimal solution for some specific cases.

Chapter 4 summarizes the results on Hartwick's rule in a unified and rigorous form. This chapter also provides precise formulation and mathematical justification of the relationship between the two approaches mentioned above. We show how Hartwick's rule can be extended for models with population growth.

In Chapter 5, we propose a new model with two exhaustible resources, where the non-negativity constraints on state variables become binding. We describe some qualitative properties of the solution of this model and formulate Hartwick's rule in this context.

Chapter 6 studies the applications of conservation laws (or invariant quantities) to models of sustainable growth. We summarize and extend the relevant theory and apply it to our models.

The results of optimal control theory which are used in the thesis are summarized in great details in the last chapter. However, we not only formulate known results, but we also shed new light on the relationship between two different sources, which is a novel result of this thesis. The two sets of necessary conditions and their relationship is illustrated on several examples. Chapter 7 deals with optimal control problems with pure state as well as mixed control and state constraints.

The models used in the thesis involve a number of variables, parameters and multipliers. A comprehensive list of all symbols together with their meaning can be found at the end of the thesis. A list of common mathematical notations and an index of specific terms which are used within the thesis are also included.

Chapter 2

Models of sustainable economic growth

2.1 Basic models

In this chapter, we formulate basic models of intertemporal optimization of consumption in an economy with exhaustible and renewable goods.[1] Exhaustible goods can be depleted but not produced, renewable ones can also be produced. In particular, we will study models of optimal economic growth taking into account a sustainability criterion. This criterion requires that some level of the utility should be preserved forever. The models will be formulated as optimal control problems. We will use the optimal control theory introduced in Chapter 7 to formulate necessary conditions of optimality for these models.

Later in this thesis, we will use these models as a common framework in which we will introduce, study and extend known results. The goal is to use these common models for all results, irrespective of their original framework. Furthermore, it will be a basis for developing new results in this field.

2.1.1 Capital goods and production

Assume that there are n renewable capital goods corresponding to different sectors of the economy and m exhaustible capital goods. Let us denote the level of renewable capital

[1] In this thesis, exhaustible capital goods are equivalently called non-renewable or non-reproducible capital goods. Analogously, reproducible or man-made capital goods are considered to be equivalents to renewable capital goods.

goods at time t by $k(t) \in \mathbb{R}^n_+$ and the level of exhaustible capital goods by $s(t) \in \mathbb{R}^m_+$.[2] Suppose that the initial level of endowments (k_0, s_0) is positive for all capital goods (we use the notation $(k_0, s_0) > 0$). The amount of production per a time unit is given by the production function f which depends on the actual amount of the renewable capital goods $k(t)$ and on the actual rates of extraction $r(t) \in \mathbb{R}^m_+$ of the exhaustible capital goods per a time unit.

Part of the production can be invested in the capital ($\dot{k}(t)$) in order to increase its future volume. Other part of the production can be consumed ($c \in \mathbb{R}^n_+$). The last part of the production can be used to restore the amortized capital ($\delta(k)$), where $\delta : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is an exogenously given function representing the amortization of the renewable capital. Hence,

$$f(k(t), r(t)) = \dot{k}(t) + c(t) + \delta(k(t)), \tag{2.1}$$

where

$$r(t) = -\dot{s}(t) \tag{2.2}$$

and

$$k(t) \geq 0, \quad s(t) \geq 0, \tag{2.3}$$

$$r(t) \geq 0, \quad c(t) \geq 0 \tag{2.4}$$

on $\langle 0, \infty)$.

We assume that the production function satisfies the following assumption:

(A1) The production function $f : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^n_+$ is a vector function $f = (f_1, \ldots, f_n)^T$ such that each f_i, $i = 1, \ldots, n$ is an increasing function in any variable on $\mathbb{R}^n_+$ and a strictly concave and twice differentiable function w.r.t. each variable on $\mathbb{R}^n_{++} \times \mathbb{R}^m_{++}$ and such that $f_i(k, r) = 0$ if $k_i = 0$ or if any component of the vector r is zero.

Later in Chapter 5 we will propose a new model where this assumption will be partly weakened.

In some cases we will add the assumption that $f(k, r)$ is a homogeneous function of degree 1, i.e. $f(\xi k, \xi r) = \xi f(k, r)$ for all $\xi \in \mathbb{R}_+$ and for all $(k, r) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$.

The well-known example of a production function which is often used in case with one renewable and one exhaustible capital good (i.e. $n = 1$ and $m = 1$) is the Cobb-Douglas production function

$$f(k, r) = k^\alpha r^\beta, \tag{2.5}$$

[2] We use the following notation: $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n;\ x_i \geq 0,\ i = 1, \ldots, n\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n;\ x_i > 0,\ i = 1, \ldots, n\}$.

where α and β are given positive constants such that $\alpha+\beta \leq 1$. They can be interpreted as output elasticities (i.e. the percentage change of output (or production) divided by the percentage change of an input) of renewable and exhaustible goods, respectively. Note that the production function (2.5) is homogeneous of degree 1 if and only if $\alpha+\beta=1$.

Regarding the amortization function δ, we suppose that the following property is satisfied:

(A2) The function $\delta : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a vector function $\delta(k) = (\delta_1(k_1), \dots, \delta_n(k_n))^T$ such that $\delta_i(k_i)$, $i = 1, \dots, n$ is an increasing and concave function in k_i on $\mathbb{R}_+$ and it is differentiable w.r.t. k_i on $\mathbb{R}_{++}$. In addition, $\delta_i(k_i) = 0$ if $k_i = 0$.

2.1.2 Preferences

Let us assume that the preferences of agents depend on the level of consumption of reproducible capital goods and are described by a utility function. In line with the common economic theory, we assume that the following property is valid:

(A3) The function $U : \mathbb{R}^n_{++} \to \mathbb{R}$ is increasing in each variable, strictly concave and twice continuously differentiable on $\mathbb{R}^n_{++}$. In addition, whenever $\bar{c} \in \mathbb{R}^n_+$ is such that a component of c vanishes, then $U(c) \geq U(\bar{c})$ for all $c \in \mathbb{R}^n_{++}$.[3]

An example of a typical utility function often used in the literature which satisfies (A3) is the following utility function with constant relative risk aversion (i.e. constant intertemporal elasticity of substitution) and constant (unit) elasticity of substitution between consumer goods:

$$U_{\theta,\rho}(c) = \begin{cases} \frac{\bar{C}_\rho^{1-\theta}}{1-\theta}, & \text{if } \theta > 0,\ \theta \neq 1, \\ \ln \bar{C}_\rho, & \text{if } \theta = 1, \end{cases} \tag{2.6}$$

where

$$\bar{C}_\rho = \prod_{i=1}^{n} c_i^{\rho_i}, \tag{2.7}$$

is an aggregated consumer good using an aggregator with the constant elasticity of substitution between consumer goods. Parameter $\theta > 0$ is a given relative risk aversion

[3] If U is not defined at $\bar{c}$, we define $U(\bar{c}) := -\infty$.

and n-dimensional vector parameter ρ is a vector of weights such that $\rho_i \in (0,1)$ and $\rho_1 + \cdots + \rho_n = 1$. Note that the formulations of $U_{\theta,\rho}$ in (2.6) for $\theta = 1$ is obtained by taking the respective limit in the first row of (2.6).

2.1.3 Discounted utility approach

The most common approach to set the objective criterion expressing the intertemporal optimization is the discounted utility approach. In this approach, the total present value of the utility over the whole time horizon $(0,\infty)$ is maximized. We assume that the discount factor $\pi(t)$ satisfies the following property:

(A4) The discount factor $\pi(t)$ is positive, continuous and $\int_0^\infty \pi(t)\,\mathrm{d}t < \infty$.

We now formulate the model of the economy with the renewable and exhaustible capital goods with dynamics described by (2.1) and (2.2) where the agents maximize their discounted utility function over the infinite time horizon as follows:

$$\begin{aligned}
&\max_{\{c(t),r(t)\}} \int_0^\infty \pi(t)U\big(c(t)\big)\,\mathrm{d}t,\\
&\dot{k}(t) = f(k(t),r(t)) - \delta(k(t)) - c(t),\\
&\dot{s}(t) = -r(t),\\
&k(0) = k_0 > 0 \text{ given},\\
&s(0) = s_0 > 0 \text{ given},\\
&k(t) \geq 0, \quad s(t) \geq 0,\\
&r(t) \geq 0, \quad c(t) \geq 0.
\end{aligned} \tag{2.8}$$

From the point of view of the optimal control theory, Problem (2.8) is a standard infinite horizon optimal control problem with a discount factor and with both pure constraints on state as well as control variables.

There are two special cases of this problem, both widely studied in the literature:

- Ramsey model (see Ramsey (1928)), where $m = 0$ and $n = 1$ (i.e. only one renewable but no non-renewable resource is considered) and
- Dasgupta-Heal-Solow (DHS) model (see Dasgupta and Heal (1974) and Solow (1974)), where $m = 1$ and $n = 1$ (i.e. one renewable and one non-renewable capital is considered).

Note however, that model (2.8) does not include any requirement of sustainability. To have a sustainable economic growth, one may for example require that utility remains at some constant forever. However, including such a constraint into model (2.8) leads to a non-standard optimal control problem. Hence we will rather consider model (2.8) without this constraint and then we will analyze conditions which ensure that this constraint is satisfied. As will be shown later in Section 4.3, another possibility how to incorporate a requirement of sustainability into model (2.8) is to prescribe a specific form of the intertemporal preferences represented by the discount factor.

As noted by Endress and Roumasset (2000), the utility can be flattered over the optimal consumption path also by modifying the elasticity of marginal utility. In particular, if the elasticity approaches infinity, a solution with constant utility arises naturally, without imposing any other requirements. However, we do not deal with this approach in this thesis.

2.1.4 Maximin approach

In its broadest sense, sustainable development means a development which gives an equal opportunity to current and future generations.[4] If the utility level is the right characteristics of the well-being that should be preserved forever, then it is natural to include the maximal sustainable level of utility directly into the objective function. In this context, the objective criterion is actually the maximization of the infimum of utility along the whole time horizon[5] instead of the maximization of the integral of discounted value of utility as in (2.8). Although the latter was prevalent in the relevant literature until recently, it is rather artificial in the original context, since the objective criterion might not correspond to the sustainability criterion. As will be shown later in Section 4.3, this is confirmed also by the fact that to ensure the sustainability of optimal paths in the discounted utility framework, the discount factor has to be chosen properly.

Hence, it is reasonable to study a model analogous to (2.8) but with the objective function replaced by

$$\max_{\{c(t),r(t)\}} \inf_{t\geq 0} U(c(t)). \tag{2.9}$$

This is not a standard form of the objective function for optimal control problems.

[4]See Chichilnisky (1996) [p. 232].
[5]Cf. Solow (1974) [p. 35].

Using a new variable w, we can transform the problem into form

$$
\begin{aligned}
&\max_{\{c(t),r(t),w\}} w, \quad \text{where } t \in \langle 0, \infty),\\
&\dot{k}(t) = f(k(t), r(t)) - \delta(k(t)) - c(t),\\
&\dot{s}(t) = -r(t),\\
&k(0) = k_0 > 0 \text{ given},\\
&s(0) = s_0 > 0 \text{ given},\\
&k(t) \geq 0, \quad s(t) \geq 0,\\
&r(t) \geq 0, \quad c(t) \geq 0,\\
&U(c(t)) \geq w.
\end{aligned}
\tag{2.10}
$$

Problem (2.10) is formulated as an optimal control problem with parameter w. To formulate this problem without including a parameter, let us consider w as a new state variable.[6] Hence, (2.10) can be rewritten as follows

$$
\begin{aligned}
&\max_{\{c(t),r(t)\}} w(0), \quad \text{where } t \in \langle 0, \infty),\\
&\dot{k}(t) = f(k(t), r(t)) - \delta(k(t)) - c(t),\\
&\dot{s}(t) = -r(t),\\
&\dot{w}(t) = 0,\\
&k(0) = k_0 > 0 \text{ given},\\
&s(0) = s_0 > 0 \text{ given},\\
&w(0) \text{ free},\\
&k(t) \geq 0, \quad s(t) \geq 0,\\
&r(t) \geq 0, \quad c(t) \geq 0,\\
&U(c(t)) \geq w(t).
\end{aligned}
\tag{2.11}
$$

This is an autonomous optimal control problem with infinite time horizon, with pure constraints on state as well as control variables and with one mixed constraint on both state and control variables. The objective function is in the form of maximizing the initial state of the variable w. Obviously, $w(0)$ is free and hence Problem (2.11) is in the form of Problem (7.115) which is discussed later in Chapter 7.

Note that if c is one-dimensional, the constraint $U(c(t)) \geq w(t)$ can be equivalently replaced by constraint $c(t) \geq w(t)$, since $U(c)$ is strictly increasing in c (see Assumption (A3)).

[6] Cf. Berkovitz (1974) or Leonard and Long (1992).

The maximin approach was already introduced by Solow (1974), but the discounted utility approach was then dominating for almost three decades. The maximin approach was re-established by Cairns and Tian (2003), Cairns (2003), Cairns and Long (2006) and is further analyzed by Asheim et al. (2007) and Martinet (2007).

2.2 Extensions to the basic models

In the basic models formulated in the previous section, the population and technology was supposed to be constant over the time horizon. In this section, we describe some extensions of the basic models, which can be found in the current literature. We consider extensions in the following three directions:

- It is assumed that the population $n(t)$ increases over time with an exogenously given (and possibly non-constant) positive growth rate $\vartheta(t)$. In addition, the population enters as a production factor into the production function.
- An exogenously given technological progress is considered resulting in a non-autonomous production function $f(t,k,r,n)$.
- The utility function $U(c,s)$ is a function of both the level of consumption $c(t)$ and the level of exhaustible capital goods $s(t)$.

In this case, we modify Assumption (A3) as follows:

(A3') The function $U:\mathbb{R}^n_{++}\times\mathbb{R}^m_{++}\to\mathbb{R}$ is increasing in each variable, strictly concave and twice continuously differentiable on $\mathbb{R}^n_{++}\times\mathbb{R}^m_{++}$. In addition, whenever the vector $(c,s)\in\mathbb{R}^n_{++}\times\mathbb{R}^m_{++}$ is such that a component of c or s vanishes, then $U(c,s)\geq U(\bar{c},\bar{s})$ for all $(c,s)\in\mathbb{R}^n_{++}\times\mathbb{R}^m_{++}$.

As the framework for studying these extensions, we can use the following model:

$$\max_{\{c(t),r(t)\}}\int_0^\infty \pi(t)U(c(t),s(t))\,\mathrm{d}t \tag{2.12}$$

or

$$\begin{aligned}&\max_{\{c(t),r(t)\}} w(0), \quad \text{where } t\in\langle 0,\infty),\\ &U(c(t),s(t))\geq w(t),\\ &\dot{w}(t)=0,\\ &w(0)\ \text{free}\end{aligned} \tag{2.13}$$

and subject to

$$\begin{aligned}
&\dot{k}(t) = f(t, k(t), r(t), n(t)) - \delta(k(t)) - c(t), \\
&\dot{s}(t) = -r(t), \\
&\dot{n}(t) = \vartheta(t)n(t), \\
&k(0) = k_0 > 0 \text{ given}, \\
&s(0) = s_0 > 0 \text{ given}, \\
&n(0) = n_0 > 0 \text{ given}, \\
&k(t) \geq 0, \quad s(t) \geq 0, \\
&r(t) \geq 0, \quad c(t) \geq 0,
\end{aligned} \tag{2.14}$$

where the following assumption is met:

(A5) The rate $\vartheta(t)$ of population growth is positive.

As we have already mentioned, we will mainly analyze conditions for constant utility along the solutions.

The model with population growth is studied e.g. by Mitra (1983), Farzin (2006), Asheim et al. (2007) and Mitra (2008). Asheim et al. (2007) postulate that the population growth is quasi-arithmetic, i. e.

$$\vartheta(t) = \frac{\kappa_1 \kappa_2}{1 + \kappa_1 t}, \tag{2.15}$$

hence $n(t) = n_0(1 + \kappa_1 t)^{\kappa_2}$, where κ_1 and κ_2 are given positive constants. Mitra (2008) imposes no specific restriction on the growth rate.

Models extended by a technological progress were initiated by Solow (1974) and studied by several authors since then, e.g. by Hartwick and Long (1999), Cairns and Long (2006), Farzin (2006) and d'Autume and Schubert (2008), among others. In the most easiest way, the exogenous technological progress is implemented as exponentially growing production given the same value of renewable capital goods and extraction rates, i.e. $f(t, k, r, n) = e^{\gamma t} \tilde{f}(k, r, n)$. Later in Chapter 5, we shall deal with a model with two non-renewable goods where the technological progress applied to these non-renewable goods is different. In particular, we consider a production function

$$f(t, k, r_1, r_2) = k^{\alpha} (r_1 + d(t) r_2)^{1-\alpha}, \tag{2.16}$$

where $d(t)$ is a given technological progress applicable to the second exhaustible good.

Finally, a model with the utility function which includes the level of non-renewable goods is analyzed in a recent paper by d'Autume and Schubert (2008), but it is also referred to by Farzin (2006) and Heijnen (2008).

2.3 Alternative approaches

2.3.1 Competitive paths

In several papers, the model of the economy with renewable and exhaustible capital goods as described by (2.8) is not formulated as an optimal control problem. Instead of that, a general concept of so-called competitive paths is defined as follows:

Definition 2.1. *An admissible path* $(c^*(t), r^*(t), k^*(t), s^*(t))$ *is called a competitive path on* $(0, \infty)$ *at discount factor* $\pi(t) > 0$ *and prices* $(\psi_c, \psi_k(t), \psi_s(t)) \geq 0$ *if the following conditions are satisfied for all* $t \geq 0$*:*

(i) Instantaneous discounted utility is maximized, i.e. c^* *maximizes* $\pi U(c) - \psi_k^T c$ *and*

(ii) instantaneous profit is maximized, i.e. (k^*, s^*, c^*, r^*) *maximizes*

$$\psi_c^T c + \psi_k^T (f(k, r) - \delta(k) - c) - \psi_s^T r + \dot{\psi}_k^T k + \dot{\psi}_s^T s$$

on the set of all admissible solutions.

This framework was first proposed by Dixit et al. (1980) [p. 552] and is hence called Dixit-Hammond-Hoel model (DHH model). It is also used by Asheim et al. (2003) [Definition 1, p. 132], Buchholz et al. (2005) [p. 552] and Heijnen (2008) [p. 3].

Later we show that there is a close relationship between the admissible solutions to Problem (2.8) which satisfy the necessary conditions of optimality and the competitive paths (see Remark 4.2).

2.3.2 Viability theory approach

Martinet (2004) and Martinet and Doyen (2007) attempt to solve the problem to find the maximal level of utility under the condition that the total depletion of the exhaustible resource does not exceed its initial level. Their papers are not based on the optimal control framework but on the so-called viable control analysis. This approach is based on the fact that for arbitrary utility level a *viability kernel* can be found. The viability kernel represents the minimal initial level of renewable and non-renewable capital stocks that enables to sustain the required level of utility provided that the depletion of exhaustible capital is optimal.

Chapter 3

Characteristics of optimal solutions

3.1 Applications of necessary conditions of optimality

In this section, we apply the necessary conditions of optimality to models, which have been presented in the previous chapter. As these models include pure state constraints, the complete set of necessary conditions taking these constraints into account is quite complicated. However, we now show that if we are interested only in solutions with constant utility, then it suffices to consider only solutions satisfying $k(t) > 0$, $s(t) > 0$ and $c(t) > 0$ for all $t \geq 0$. Indeed, we show that such assumption do not restrict the set of optimal values of the objective criterion for the set of solutions such that $U(c(t))$ is constant in t.

First, assume that $c_i(t) = 0$ on some non-trivial interval for some $i = 1, \ldots, n$. Then it follows from Assumption (A3) that $U(c(t))$ attains on this interval its lowest possible value (possibly minus infinity). Hence, the discounted utility objective criterion (2.12) together with the condition $U(c) =$ const. as well as the maximin criterion (2.9) both attain their lowest possible value. The constraint $c(t) > 0$ therefore does not restrict the set of maximal attainable values of the objective criterion neither in the model (2.8) (for solutions such that $U(c(t))$ is constant) nor in the model (2.11).

Second, note that if $k_i(\tau) = 0$ for some $\tau \geq 0$ and $i = 1, \ldots, n$, Assumptions (A1) and (A2) imply that $f_i(k, r) = 0$ and $\delta_i(k) = 0$. Equality (2.1) then states $\dot{k}_i(t) = -c_i(t)$. Since we require $k_i \geq 0$ and $c_i \geq 0$, we have that $k_i(t) = 0$ and $c_i(t) = 0$ for all $t \geq \tau$. Analogously to the previous case, the constraint $k(t) > 0$ does not restrict the set of maximal attainable values of the objective function.

Furthermore, assume that $s_i(\tau) = 0$ at some τ, hence $r_i(t) = 0$ for all $t \in (\tau, \infty)$. Assumption (A1) then implies that $f_j(k, r) = 0$ for all $j = 1, \ldots, n$ on (τ, ∞), hence $\dot{k} = -\delta(k) - c \geq 0$, implying that $c(t) = 0$ for $t \geq \tau$, which is a result analogous to the previous two cases.

In addition, in Chapters 2 – 4 we will only deal with solutions satisfying $r_j(r) > 0$ for all $j = 1, \ldots, m$ and for all $t \geq 0$. This assumption is necessary because we have not assumed that the derivative of the production function f exists w.r.t. r_j at $t = 0$. However, necessary conditions of optimality formulated later in Chapter 7 require this assumption.

To summarize, in the discounted utility framework, Chapters 2 – 4 only deal with interior solutions to (2.8). In case of the maximin framework, we study only solutions to (2.11) satisfying $k(t) > 0$, $s(t) > 0$, $c(t) > 0$ and $r(t) > 0$ for all $t \geq 0$, although constraint $U(c(t)) \geq w(t)$ in (2.13) might be binding. For simplicity, let us call these solutions also as interior solutions.[1]

Later in Chapter 5, we release some of the assumptions by modifying Assumption (A1). For example, the production function will not be assumed to be zero if only some (not all) r_j, $j = 1, \ldots, m$ are zero. In this case, we will replace the above-mentioned constraint that $r(t) > 0$ by $r(t) \geq 0$. Similarly, the utility function might not attain its minimal value if only some of the components of the consumption vector c are zero.

3.1.1 Discounted utility approach

Firstly, we present the necessary conditions of optimality for an interior solution to Problem (2.8) with the discounted utility objective according to the theory discussed later in Chapter 7.

In accordance with (7.8), the Hamiltonian is defined by

$$H(k, s, c, r, \psi^0, \psi_k, \psi_s) = \psi^0 \pi U(c) + \psi_k^T (f(k, r) - \delta(k) - c) + \psi_s^T (-r). \tag{3.1}$$

Now we formulate the necessary conditions of optimality according to Theorem 7.6: If (k^*, s^*, c^*, r^*) is an interior optimal solution to Problem (2.8), then there exist a constant ψ^0 and continuous vector functions $\psi_k(t) : \mathbb{R} \to \mathbb{R}^n$ and $\psi_s(t) : \mathbb{R} \to \mathbb{R}^m$ such that

[1] More specifically, we require that one-sided limits of all control variables at points of discontinuity are positive as well.

following conditions are met:

$$\text{(i)}\quad \psi^0 = 0 \text{ or } \psi^0 = 1, \tag{3.2}$$

$$\text{(ii)}\quad (\psi^0, \psi_k(t), \psi_s(t)) \neq (0,0,0) \text{ for all } t \geq 0, \tag{3.3}$$

$$\begin{aligned}\text{(iii)}\quad & \psi^0 \pi U(c^*) + \psi_k^T(f(k^*, r^*) - \delta(k^*) - c^*) - \psi_s^T r^* \geq \\ & \geq \psi^0 \pi U(c) + \psi_k^T(f(k^*, r) - \delta(k^*) - c) - \psi_s^T r \\ & \text{for all } (c, r) \in \mathbb{R}_{++}^{n+m} \text{ almost everywhere on } (0, \infty),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\text{(iv)}\quad & \frac{\partial H}{\partial c}(k^*, s^*, c^*, r^*, \psi^0, \psi_k, \psi_s) = \psi^0 \pi \frac{\mathrm{d}U}{\mathrm{d}c}(c^*) - \psi_k^T = 0 \text{ for all } t \geq 0 \\ & \text{with possible exception of discontinuity points of } c^* \text{ or } r^*,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\text{(v)}\quad & \frac{\partial H}{\partial r}(k^*, s^*, c^*, r^*, \psi^0, \psi_k, \psi_s) = \psi_k^T \frac{\partial f}{\partial r}(k^*, r^*) - \psi_s^T = 0 \text{ for all } t \geq 0 \\ & \text{with possible exception of discontinuity points of } c^* \text{ or } r^*,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\text{(vi)}\quad & \dot{\psi}_k^T = -\frac{\partial H}{\partial k}(k^*, s^*, c^*, r^*, \psi^0, \psi_k, \psi_s) = \psi_k^T \frac{\mathrm{d}\delta}{\mathrm{d}k}(k^*) - \psi_k^T \frac{\partial f}{\partial k}(k^*, r^*) \\ & \text{almost everywhere on } (0, \infty),\end{aligned} \tag{3.7}$$

$$\begin{aligned}\text{(vii)}\quad & \dot{\psi}_s^T = -\frac{\partial H}{\partial s}(k^*, s^*, c^*, r^*, \psi^0, \psi_k, \psi_s) = 0 \\ & \text{almost everywhere on } (0, \infty).\end{aligned} \tag{3.8}$$

The set of conditions (3.2) – (3.8) will be further referred to as necessary conditions of optimality for an interior solution of (2.8). Recall that a detailed list of all variables and multipliers can be found at the end of this thesis.

In addition, Theorem 7.9 implies that for almost all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\psi^0 \pi(t) U(c^*(t)) + \psi_k(t)^T \dot{k}^*(t) - \psi_s(t)^T r^*(t)\big) = \frac{\partial}{\partial t}\big(\psi^0 \pi(t) U(c^*)\big).$$

This can be rewritten to

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\psi_k(t)^T \dot{k}^*(t) - \psi_s(t)^T r^*(t)\big) = -\psi^0 \pi(t) \frac{\mathrm{d}U}{\mathrm{d}t}(c^*(t)). \tag{3.9}$$

Moreover, Theorem 7.10 implies that if

$$\int_1^\infty \pi(t+\delta) U^*(c^*(t))\,\mathrm{d}t < \infty \qquad \text{for all } \delta \in (-1, 1) \tag{3.10}$$

and if there exists a piecewise continuous function $\xi(t)$ such that

$$|\dot{\pi}(t)(t+\delta) U(c^*(t))| \leq \xi(t) \qquad \text{for all } \delta \in (-1,1) \text{ and } t \geq 1 \tag{3.11}$$

and

$$\int_1^\infty \xi(t)\,\mathrm{d}t < \infty, \tag{3.12}$$

then

$$\lim_{t\to\infty} \psi^0 \pi(t) U(c^*(t)) + \psi_k(t)^T \dot{k}^*(t) - \psi_s(t)^T r^*(t) = 0. \tag{3.13}$$

It might be difficult to verify whether conditions (3.10) – (3.12) are satisfied. To provide a specific example when they are met consider a solution and a discount factor such that $U(c^*(t)) \equiv$ const. and $\dfrac{\dot{\pi}(t)}{\pi(t)} \leq p$, where p is a constant. Indeed, it is straightforward to verify that conditions (3.10) – (3.12) are satisfied (with $\xi(t) := p\pi(t)U(c^*(t))$) due to Assumption (A4).

Remark 3.1. Note that if $\psi^0 = 0$, then conditions (3.5) and (3.6) imply that $\psi_k(t) = 0$ and $\psi_s(t) = 0$ for all $t \geq 0$ with possible exception of discontinuity points of $c^*(t)$ or $r^*(t)$, which is a contradiction with (3.3). Hence, condition (3.2) implies $\psi^0 = 1$. In this case, if follows from conditions (3.5) and (3.6) and Assumptions (A1), (A3) and (A4) that both costate variables ψ_k and ψ_s are positive almost everywhere. We can conclude that both functions ψ_k and ψ_s are positive everywhere since they are continuous. To sum up, one has

$$\psi^0 = 1,\ \psi_k(t) > 0 \text{ and } \psi_s(t) > 0 \text{ for all } t \geq 0. \tag{3.14}$$

Remark 3.2. It can be easily shown that condition (3.4) follows from conditions (3.5) and (3.6). Indeed, note that the function on the right-hand side of Inequality (3.4) is a concave function of c and r at any fixed value of $\psi_k > 0$ and $\pi > 0$ (see (3.14) and Assumption (A4)). The desired result follows from the fact that in this case, (3.5) and (3.6) represent sufficient conditions to maximum of this concave function.

3.1.2 Maximin approach

Analogously to the discounted utility approach, for Problem (2.11) we formulate necessary conditions of optimality only for interior solutions, i.e. for solutions satisfying $k > 0$, $s > 0$, $c > 0$ and $r > 0$. Recall that for interior solutions to (2.11), the constraint $U(c) \geq w$ still might be binding.

In accordance with (7.8) and (7.9), the Hamiltonian is defined by

$$\bar{H}(k, s, w, c, r, \psi^0, \psi_k, \psi_s, \psi_w) = \psi_k^T (f(k, r) - \delta(k) - c) - \psi_s^T r. \tag{3.15}$$

and the Lagrangian by

$$\bar{L}(k, s, w, c, r, \psi^0, \psi_k, \psi_s, \psi_w, \mu_w)$$
$$= \psi_k^T(f(k,r) - \delta(k) - c) - \psi_s^T r + \mu_w(U(c) - w). \tag{3.16}$$

Again we can formulate the necessary conditions of optimality based on Theorem 7.6: If $(k^*, s^*, w^*, c^*, r^*)$ is an interior optimal solution to Problem (2.11), then there exist a constant ψ^0, continuous vector functions $\psi_k(t) : \mathbb{R} \to \mathbb{R}^n$, $\psi_s(t) : \mathbb{R} \to \mathbb{R}^m$ and $\psi_w(t) : \mathbb{R} \to \mathbb{R}$ and a function $\mu_w(t) : \mathbb{R} \to \mathbb{R}$ which has one-sided limits everywhere, such that following conditions are met:

(i) $\psi^0 = 0$ or $\psi^0 = 1$, (3.17)

(ii) $(\psi^0, \psi_k(t), \psi_s(t), \psi_w(t)) \neq (0,0,0,0)$ for all $t \geq 0$, (3.18)

(iii) $\bar{H}\left(k^*, s^*, w^*, c^*, r^*, \psi^0, \psi_k, \psi_s, \psi_w\right) \geq \bar{H}\left(k^*, s^*, w^*, c, r, \psi^0, \psi_k, \psi_s, \psi_w\right)$ for all $(c, r) \in \mathbb{R}_{++}^{m+n}$ and $U(c) \geq w^*$ almost everywhere on $(0, \infty)$, (3.19)

(iv) $\dfrac{\partial \bar{L}}{\partial c} = \mu_w \dfrac{\mathrm{d}U}{\mathrm{d}c}(c^*) - \psi_k^T = 0$ for all $t \geq 0$ with possible exception of discontinuity points of c^* or r^*, (3.20)

(v) $\dfrac{\partial \bar{L}}{\partial r} = \psi_k^T \dfrac{\partial f}{\partial r}(k^*, r^*) - \psi_s^T = 0$ for all $t \geq 0$ with possible exception of discontinuity points of c^* or r^*, (3.21)

(vi) $\dot{\psi}_k^T = -\dfrac{\partial \bar{L}}{\partial k} = \psi_k^T \dfrac{\mathrm{d}\delta}{\mathrm{d}k}(k^*) - \psi_k^T \dfrac{\partial f}{\partial k}(k^*, r^*)$ almost everywhere on $(0, \infty)$, (3.22)

(vii) $\dot{\psi}_s^T = -\dfrac{\partial \bar{L}}{\partial s} = 0$ almost everywhere on $(0, \infty)$, (3.23)

(viii) $\dot{\psi}_w = -\dfrac{\partial \bar{L}}{\partial w} = \mu_w, \quad \psi_w(0) = -\psi^0$, (3.24)

(ix) $\mu_w(U(c^*) - w^*) = 0, \quad \mu_w \geq 0$. (3.25)

The set of conditions (3.17) – (3.25) will be further referred to as necessary conditions of optimality for an interior solution to (2.11).

In addition, Theorem 7.9 implies that for almost all $t \geq 0$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_k(t)^T \dot{k}^*(t) - \psi_s(t)^T r^*(t)\right) = 0. \tag{3.26}$$

Now, let us introduce the following definition of a regular solution in accordance with Cairns and Tian (2003) [p. 8], which will be useful in the analysis later:[2]

Definition 3.1. *A solution (k,s,w,c,r) to Problem (2.11) is called regular if it satisfies the necessary conditions of optimality together with $\mu_w(t) > 0$ for all $t \geq 0$.*

Remark 3.3. Note that μ_w is a Lagrange multiplier which is associated with the constraint $U(c) \geq w$. It can be interpreted as a price for relaxing this constraint. If $\mu_w > 0$ for all $t \geq 0$, then (3.25) implies that this constraint is binding for all $t \geq 0$. We then have that $U(c)$ is constant since w is constant due to the state equation $\dot{w} = 0$. However, the converse of this implication might not be true, i.e. if $U(c)$ is constant, then there still might be the case that $\mu_w = 0$ on some non-trivial interval. In our results, we need to exclude this case, hence we directly assume that $\mu_w > 0$ for all $t \geq 0$.

3.2 Existence of optimal solutions for some specific cases

In this section we summarize and extend some known results on the existence and explicit form of an optimal solution. We consider the general model (2.14) with the maximin objective function (2.13). We restrict ourselves to the case with one renewable and one exhaustible capital good. We assume that there is no population growth ($\vartheta = 1$ and $n_0 = 1$) and the utility function does not depend explicitly on s. In addition, we assume that the production function is in the Cobb-Douglas form with an exponentially growing productivity, i.e.

$$\tilde{f}(t,k,r,n) = e^{\gamma t} k^{\alpha} r^{\beta}, \tag{3.27}$$

where $\gamma > 0$ is a given constant and output elasticities w.r.t. both capital stocks are positive ($\alpha > 0$ and $\beta > 0$). The production function is assumed to have decreasing or constant returns to scale ($\alpha + \beta \leq 1$). To sum up these assumptions, we formulate the

[2]Cf. also Cairns and Long (2006) [p. 279], who define a regular solution as a solution which satisfies the necessary conditions of optimality for an interior solution, the utility is constant and the solution is Pareto efficient (i.e. for any interval $A = (t_1, t_2) \subsetneqq (0, \infty)$, the utility can only be increased outside A by relaxing the constraint $U(c) \geq w$ on A). They note that for any such solution we have $\mu_w(t) > 0$ for all $t \geq 0$.

model once more:

$$\begin{aligned}
&\max_{\{c(t),r(t)\}} w(0), \quad \text{where } t \in \langle 0, \infty), \\
&\dot{k}(t) = e^{\gamma t} k(t)^{\alpha} r(t)^{\beta} - \delta(k(t)) - c(t), \\
&\dot{s}(t) = -r(t), \\
&\dot{w}(t) = 0, \\
&k(0) = k_0 > 0 \text{ given}, \\
&s(0) = s_0 > 0 \text{ given}, \\
&w(0) \text{ free}, \\
&k(t) \geq 0, \quad s(t) \geq 0, \\
&r(t) \geq 0, \quad c(t) \geq 0, \\
&U(c(t)) \geq w(t).
\end{aligned} \tag{3.28}$$

3.2.1 Existence results

When formulating the existence results, we make an additional assumption that $\gamma = 0$. We consider three cases which are different in the amortization function:

(a) If there is no amortization of the reproducible capital ($\delta(k) = 0$), it is known that a positive level of consumption can be sustained forever if and only if the relative elasticity of production w.r.t. reproducible capital is greater than the relative elasticity of production w.r.t. the exhaustible capital ($\alpha > \beta$). [3]

(b) In case of a linear amortization function ($\delta(k) = \bar{\delta} k$, where $\bar{\delta} > 0$ is a given constant), no constant utility path exists. [4]

(c) In case of an amortization function in the form $\delta(k) = \delta_1 k^{\delta_2}$, where $\delta_1 \in (0, 1)$ and $\delta_2 \in \langle 0, 1)$, a sufficient condition for an existence of a constant utility path is:[5]

$$\alpha - \delta_2 > \beta \quad \text{and} \quad k_0 > \max\{1, \bar{k}_0\}, \tag{3.29}$$

where

$$\bar{k}_0 := \left[\left(\frac{\delta_1}{1-\beta} \right)^{1-\beta} \frac{1}{(s_0(\alpha - \delta_2 - \beta))^{\beta}} \right]^{\frac{1}{\alpha - \delta_2 - \beta}}. \tag{3.30}$$

[3] See e.g. Solow (1974) [p. 37], Buchholz et al. (2005) [p. 553] and Martinet and Doyen (2007) [Proposition 3, p. 24].

[4] See Martinet and Doyen (2007) [Proposition 4, p. 27].

[5] See Buchholz et al. (2005) [p. 553].

More extensive set of results based on the viability analysis comprising also other types of the production function is provided by Martinet and Doyen (2007).

3.2.2 Explicit form of optimal solutions

In some specific cases, explicit solutions to the models given above are known. For the restricted model (3.28) described above with no amortization of capital ($\delta(k) = 0$), an explicit solution is known at least in two specific cases:

(a) In case of no productivity growth ($\gamma = 0$) and if the relative elasticity of production w.r.t. reproducible capital is greater than the relative elasticity w.r.t. the exhaustible capital ($\alpha > \beta$), the optimal solution to this model has the following form:[6]

$$\begin{aligned} c^*(t) &\equiv (1-\beta)\big(s_0(\alpha-\beta)\big)^{\frac{\beta}{1-\beta}} k_0^{\frac{\alpha-\beta}{1-\beta}}, && (3.31)\\ k^*(t) &= \frac{c^*(t)\beta}{1-\beta} t + k_0, && (3.32)\\ r^*(t) &= \left(\frac{c^*(t)}{1-\beta}\right)^{\frac{1}{\beta}} k^*(t)^{-\frac{\alpha}{\beta}}, && (3.33)\\ w^*(t) &= U(c^*(t)), && (3.34)\\ s^*(t) &= s_0 \left(\frac{k^*(t)}{k_0}\right)^{1-\frac{\alpha}{\beta}}. && (3.35) \end{aligned}$$

(b) In case of exponentially growing productivity ($\gamma > 0$) and no amortization ($\delta(k) = 0$), we further assume that the initial levels of reproducible and exhaustible capital stocks satisfy the following relationship:

$$s_0 = \left(\frac{1}{\alpha}\right)^{\frac{1}{\beta}} \left(\frac{\gamma}{\beta}\right)^{\frac{1-\beta}{\beta}} k_0^{\frac{1-\alpha}{\beta}}. \qquad (3.36)$$

In this case, the optimal solution is known to have the following form:[7]

$$\begin{aligned} c^*(t) &= \left(\frac{\gamma}{\beta}\right)^{\beta} k_0^{\alpha} s_0^{\beta}, && (3.37)\\ r^*(t) &= s_0 \frac{\gamma}{\beta} e^{-\frac{\gamma}{\beta} t}, && (3.38) \end{aligned}$$

[6] Cf. Solow (1974) [p. 38–39] and Martinet and Doyen (2007) [Proposition 3, p. 24].
[7] Cf. Cairns and Long (2006) [Proposition 7, p. 298].

$$s^*(t) = s_0 e^{-\frac{\gamma}{\beta}t}, \quad (3.39)$$
$$k^*(t) = k_0, \quad (3.40)$$
$$w^*(t) = U(c^*(t)). \quad (3.41)$$

Remark 3.4. The results given above are only related to solutions whose initial conditions for k_0 and s_0 satisfy (3.36). However, the optimal level of c for other initial conditions has not been derived yet. For example, a question arises whether an increase of only one of the initial endowments k_0 or s_0 increases the sustainable level of consumption provided that (k_0, s_0) initially satisfies Equality (3.36).

Our preliminary results on this topic show that it is possible to derive an expression for the level of maximal sustainable consumption, but this expression involves an integral which does not have a closed solution for general values of α and has to be computed numerically. As an example, Figure 3.1 illustrates the numerically calculated values of maximal sustainable consumption for different values of k_0 and s_0. However, we do not deal with this subject in more details in this thesis since furher research is needed in this topic.

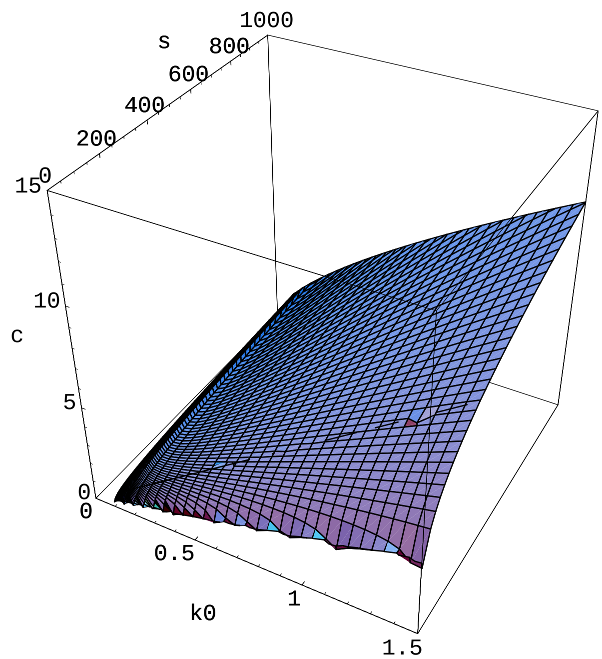

Figure 3.1: Illustration of the value of sustainable consumption c for different values s_0 and k_0 if the value of parameter α is $\frac{3}{4}$. The straight line illustrates the values of c along (k_0, s_0) satisfying (3.36).

Chapter 4

Hartwick's rule

In the previous chapters, we have presented a detailed formulation of a model of economy with renewable and non-renewable resources. We have also formulated the sets of necessary conditions of optimality for interior solutions. Two basic frameworks were considered: the discounted utility approach and the maximin approach.

Whereas the relationship between the maximin objective criterion with the requirement of sustainability is clear, this is not the case for the discounted utility approach. Hence, we need to impose a condition on the solutions of this model. This is addressed by so-called Hartwick rule, which is the main topic of this chapter. First, we briefly describe the historical context.

4.1 Overview of the most relevant literature

The utilitarian conception of the theory of optimal economic growth was introduced by Ramsey (1928) and thereafter developed by many other economists. However, it came under fundamental critique in early 70s, especially represented by John Rawls (see Rawls (1971)). His theory of justice abandoned the classical principle of utility criterion as unethical. The main reason was that in this conception, a loss of one generation's utility can be offset by an increase of utility of another generation. Rawls even gave rise to doubts about applicability of any optimal principle approach when considering intergenerational allocation of resources.

As a response to these remarks, Solow (1974) formulated a problem of maximising the level of consumption which can be maintained forever, even if one of the essential inputs of the production function is a non-renewable capital. In a framework of a model with dynamics analogous to (2.8), however, with a criterion of minimal use of resources and with Cobb-Douglas production function, he came to the solution satisfying the Pontryagin necessary conditions of optimality. His solution is identical to the one given in Section 3.2.2 (a). However, he only assumed the optimal consumption to be constant.

These results were further extended by Hartwick (1977) who introduced a rule (later called Hartwick's rule) which prescribes to invest all the revenues from exhaustible capital depletion into the reproducible capital (i.e. zero net investment). In other words, Hartwick's rule requires the total value of net investments to both reproducible and exhaustible capital goods priced at shadow prices to be zero. Hartwick himself formulates this rule as a sufficient condition for constant consumption paths, which is now known as Hartwick's result. Hartwick (1978a) extends the rule for the case of several capital stocks.

Substantial contribution to the further research on Hartwick's result is brought by Dixit et al. (1980). They argue that Hartwick's rule of a zero net investment as a sufficient condition for constant consumption can be generalized to a constant net investment. In addition, they formulate and prove the converse of generalized Hartwick's result. Although they do not prove that converse of Hartwick's result as it is given later in Theorem 4.7, they formulate a weaker version of this claim. They prove that if an admissible interior solution meets the sufficient conditions of optimality and the value of net investment is zero, then the utility is constant.

These seminal papers have been followed by a large amount of other papers which further clarify and extend the results on Hartwick's rule and its converse. The reference to many of them will be made later in this thesis where the results will be described in more details. However, this has brought a divergence of models, assumptions and techniques that were used to describe these results. In addition, some of the papers were more practically oriented and less focused on the mathematical rigour.

Additionally, the maximin approach has arisen to a greater extent recently. Although the approach of Solow himself can be regarded closer to the maximin approach than the utilitarian framework, the model analogous to (2.11) and its relationship to the discounted utility approach is first discussed by Cairns and Tian (2003) and further analyzed mainly by Cairns and Long (2006).

Hence, the aim of this chapter is threefold:

(a) The most important results are summarized and described in the unified framework based on precise using of the optimal control theory.

(b) In several cases, the known results are reformulated or extended and some new results are given.

(c) Several results are formulated in both discounted utility and maximin framework and these two approaches are compared.

4.2 Hartwick's result

4.2.1 Discounted utility approach

Now we formulate Hartwick's result for Problem (2.8) with the discounted utility objective function.

Theorem 4.1 (Hartwick's result, discounted utility approach)**.** [1] *Let (k, s, c, r) be an admissible interior solution to Problem (2.8) which fulfills the necessary conditions of optimality (3.2) – (3.8) and condition (3.9) together with (ψ^0, ψ_k, ψ_s). In addition, suppose that $\psi_k^T \dot{k} = \psi_s^T r$ (Hartwick's rule) for all $t \geq 0$. Then $U(c(t)) \equiv$ const. for all $t \geq 0$.*

Remark 4.1. Note that the assumption that the necessary conditions of optimality (3.2) – (3.8) and condition (3.9) are satisfied for an admissible solution (k, s, c, r) are clearly satisfied in case that (k, s, c, r) is an optimal solution to Problem (2.8) (see Theorem 7.6 and Theorem 7.9).

Proof of Theorem 4.1. One has $\psi^0 = 1$ in accordance with (3.14). Moreover, if the Hartwick's rule $\psi_k^T \dot{k} = \psi_s^T r$ is satisfied, we obtain from (3.9) that

$$\pi(t) \frac{\mathrm{d}U}{\mathrm{d}t}(c(t)) = 0 \tag{4.1}$$

almost everywhere. Since $\pi(t) > 0$ in accordance with Assumption (A4), Equality (4.1) implies that $U(c(t))$ is piecewise constant. According to Lemma 7.5, the Hamiltonian is continuous everywhere, hence $\pi(t)U(c(t))$ is continuous. As $\pi(t)$ is continuous, this implies that $U(c(t))$ is continuous everywhere and hence constant. □

The common proof of Hartwick's result is as follows:[2] We differentiate the equation for $\dot{k}$ in (2.8) w.r.t. t to obtain

$$\ddot{k} = \frac{\partial f}{\partial k}(k, r)\dot{k} + \frac{\partial f}{\partial r}(k, r)\dot{r} - \frac{\mathrm{d}\delta}{\mathrm{d}k}(k)\dot{k} - \dot{c}. \tag{4.2}$$

[1] Cf. e.g. Hartwick (1977) [p. 973], Solow (1986) [p. 144], Aronsson et al. (1995), Cairns and Yang (2000) [Theorem 1, p. 8], Mitra (2002) [p. 367], Asheim et al. (2003) [Proposition 1, p. 134] and Buchholz et al. (2005) [Proposition 3, p. 556], among others.

[2] The proof given here is a shorter version of the proof presented by Mitra (2002).

We multiply it by ψ_k^T and finally we use (3.5) – (3.8) to obtain:

$$\begin{aligned}\psi^0\pi\frac{\mathrm{d}U}{\mathrm{d}t} &= \psi^0\pi\frac{\mathrm{d}U}{\mathrm{d}c}\dot{c} \overset{(3.5)}{=} \psi_k^T\dot{c} \overset{(4.2)}{=} \psi_k^T\frac{\partial f}{\partial k}\dot{k} + \psi_k^T\frac{\partial f}{\partial r}\dot{r} - \psi_k^T\frac{\mathrm{d}\delta}{\mathrm{d}k}\dot{k} - \psi_k^T\ddot{k} \\ &\overset{(3.6)}{=} \left(\psi_k^T\frac{\partial f}{\partial k} - \psi_k^T\frac{\mathrm{d}\delta}{\mathrm{d}k}\right)\dot{k} + \psi_s^T\dot{r} - \psi_k^T\ddot{k} \\ &\overset{(3.7,3.8)}{=} -\dot{\psi}_k^T\dot{k} + \psi_s^T\dot{r} - \psi_k^T\ddot{k} + \dot{\psi}_s^T r = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_k^T\dot{k} - \psi_s^T r\right). \end{aligned} \tag{4.3}$$

Hartwick's rule implies that the term on the right-hand side of (4.3) is zero. Since $\psi^0 = 1$ and $\pi(t)$ is positive for all $t \geq 0$, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}t}(c(t)) = 0,$$

hence $U(c(t)) \equiv$ const. for all $t \geq 0$, analogously to the arguments given above. Note that this proof requires the differentiability of c and r w.r.t. t.

Remark 4.2. (Economic interpretation of Hartwick's rule) Hartwick's rule means that the net investment has to be zero, i.e. that any decrease in a non-renewable or renewable capital has to be compensated by an increase in (other) renewable capital. The necessary conditions ensure that ψ_k and ψ_s can be interpreted as present values of shadow prices of k and s, respectively. Hence, the value of net investment in Hartwick's rule should be calculated using the shadow values (competitive prices) of capital goods (net of possible extraction costs) instead of using actual market prices which might be different. This is the main reason why Hartwick's rule is considered to be more like a descriptive property of the sustainable path, not as a prescriptive policy for ensuring sustainability (Withagen (1996), Martinet (2007)).

Remark 4.3. (Relationship with competitive paths) It is easy to show that if an admissible solution (k, s, c, r) to Problem (2.8) with prices (ψ_k, ψ_s) is an interior competitive path (see Definition 2.1) such that ψ_k and ψ_s are continuous everywhere, then it satisfies the necessary conditions of optimality for an interior solution to (2.8) together with $\psi^0 = 1$. Indeed, condition (i) in Definition 2.1 implies (3.5), whereas condition (ii) implies (3.6), (3.7) and (3.8). As noted in Remark 3.2, condition (3.4) follows from (3.5) and (3.6). Conditions (3.2) and (3.3) are satisfied trivially.

If we review the proof of Hartwick's result, it becomes obvious that it suffices to assume that the present value of net investment stays constant, i.e. $\psi_k^T\dot{k} - \psi_s^T r = I$, where I is a constant (possibly non-zero). This result is called generalized Hartwick's result and is formulated as follows:

Theorem 4.2 (Generalized Hartwick's result). [3] *Let (k,s,c,r) be an admissible interior solution to Problem (2.8) which fulfills the necessary conditions of optimality (3.2) – (3.8) and condition (3.9) together with (ψ^0,ψ_k,ψ_s). In addition, suppose that there exists a constant I such that $\psi_k^T\dot{k} - \psi_s^T r = I$ (generalized Hartwick's rule) for all $t \geq 0$. Then $U(c(t)) \equiv$ const. for all $t \geq 0$.*

Proof. If $\psi_k^T\dot{k} - \psi_s^T r = I$, Equality (3.9) again implies (4.1). The proof of equality $U(c^*(t)) \equiv$ const. can be concluded analogously as in proof of Theorem 4.1. □

The following lemma is formulated by Dixit et al. (1980) [Theorem 3, p. 554] in case of no amortization ($\delta(k) = 0$):[4]

Lemma 4.1. *Let (k,s,c,r) be an admissible interior solution to Problem (2.8) with no amortization ($\delta(k) = 0$) which fulfills the necessary conditions of optimality (3.2) – (3.8) together with (ψ^0,ψ_k,ψ_s). In addition, suppose that there exists a constant I such that condition $\psi_k^T\dot{k} - \psi_s^T r = I$ (generalized Hartwick's rule) is met for all $t \geq 0$. Then $I \geq 0$.*

Proof. We will prove that $I < 0$ is inadmissible. Indeed, if $\delta(k) = 0$ and $I < 0$, then for all $t \geq 0$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_k^T k + \psi_s^T s) = \psi_k^T\dot{k} - \psi_s^T r + \dot{\psi}_k^T k + \dot{\psi}_s^T s \tag{4.4}$$

$$\overset{(3.7),(3.8)}{=} I - \psi_k^T \frac{\partial f}{\partial k}(k,r)k \tag{4.5}$$

$$\overset{(3.5)}{=} I - \psi^0 \pi \frac{\mathrm{d}U}{\mathrm{d}c}(c)\frac{\partial f}{\partial k}(k,r)k \leq I < 0 \tag{4.6}$$

which implies that $\psi_k^T k + \psi_s^T s$ will become negative at a finite time. However, this is a contradiction, because we assume k and s to be non-negative and ψ_k and ψ_s are non-negative as well (see (3.14)).

4.2.2 Maximin approach

It was shown in the previous section that Hartwick's rule is a sufficient condition for constant utility paths satisfying necessary conditions of optimality in case that the model

[3] The first who proved generalized Hartwick's result were Dixit et al. (1980) [Theorem 1, p. 553], hence it is sometimes called Dixit-Hammond-Hoel's result. However, it was formulated in several other papers, cf. e.g. Heijnen (2008) [p. 4].

[4] Buchholz et al. (2005) introduce a more general conditions for the non-negativity of I in case of $n = 1$ (i.e. one reproducible capital stock): For all $r > 0$ there exists a $\bar{k}_r > 0$ such that the function $f(k,r) - \delta(k)$ is increasing for all $k > \bar{k}_r$ (cf. Buchholz et al. (2005) [Proposition 2, p. 555]).

with discounted utility objective is considered. We now formulate a sufficient condition for constant consumption paths in the framework of the model with maximin objective.

Theorem 4.3 (Hartwick-like result, maximin approach). *Let (k, s, c, r) be an admissible interior solution to Problem (2.11) which fulfills the necessary conditions of optimality together with (ψ^0, ψ_k, ψ_s). In addition, suppose that this solution is regular (see Definition 3.1). Then $U(c) \equiv$ const. for all $t \geq 0$.*

Proof. The proof follows directly from (3.25). □

As seen in this theorem, Hartwick's rule is replaced by the assumption of regularity.

4.3 A comparison of both approaches

The aim of this section is to study the relationship between the necessary conditions of optimality for discounted utility approach and the maximin approach which were described above. As it will be clear later, Hartwick rule and the concept of regular solution are crucial in this context.

Some aspects of this relationship are discussed by Cairns and Tian (2003), Cairns (2003) and Cairns and Long (2006). Their main conclusion is that the discount factor in the discounted utility approach can be interpreted as a shadow value of the constraint $U(c) \geq w$ in the maximin approach.

To present a precise formulation and mathematical justification of this relationship, we use the fact that the necessary conditions of optimality for an interior solution to Problem (2.11) are very similar to those for Problem (2.8). Indeed, the following theorem is true:

Theorem 4.4. [5] *Let $(\bar{k}, \bar{s}, \bar{c}, \bar{r})$ be an admissible interior solution to Problem (2.8) that satisfies the necessary conditions (3.2) – (3.8) and condition (3.9) together with (ψ^0, ψ_k, ψ_s), where $\psi_k^T \dot{\bar{k}} = \psi_s^T \bar{r}$. Then there exist $\bar{w}$, ψ_w and μ_w, where $\mu_w(t) = \pi(t)$, such that $(\bar{k}, \bar{s}, \bar{w}, \bar{c}, \bar{r})$ satisfies the necessary conditions of optimality for Problem (2.11) together with $(\psi^0, \psi_k, \psi_s, \psi_w, \mu_w)$.*

Proof. Under the assumptions of this theorem, Theorem 4.1 implies that $U(\bar{c}(t))$ is con-

[5] This theorem is an original result of this paper. Cairns and Tian (2003) [Proposition 3] introduced the converse of this theorem but only for the case $n = 1$ and $m = 0$.

stant. Denote this constant by $\bar{w}$. Further define

$$\psi_w(t) := -\psi^0 + \int_0^t \pi(\tau)\,\mathrm{d}\tau. \tag{4.7}$$

Note that ψ_w is well-defined since the integral in (4.7) converges for $t \to \infty$ due to Assumption (A4). Given these definitions, it is straightforward to verify that $(\bar{k}, \bar{s}, \bar{w}, \bar{c}, \bar{r})$ is an interior solution to Problem (2.11) which satisfies the necessary conditions of optimality for this problem given by (3.17) – (3.25) together with $(\psi^0, \psi_k, \psi_s, \psi_w, \mu_w)$. The only necessary condition that needs to be shown in detail is (3.19). We prove that (3.4) implies (3.19). Recall that (3.4) reads

$$\psi^0 \pi U(\bar{c}) + \psi_k^T(f(\bar{k}, \bar{r}) - \delta(\bar{k}) - \bar{c}) - \psi_s^T \bar{r} \geq \psi^0 \pi U(c) + \psi_k^T(f(\bar{k}, r) - \delta(\bar{k}) - c) - \psi_s^T r$$

for all $(c, r) \in \mathbb{R}^{n+m}_{++}$. Since $U(c) \geq \bar{w}$, one has

$$\psi^0 \pi U(c) + \psi_k^T(f(\bar{k}, r) - \delta(\bar{k}) - c) - \psi_s^T r \geq \psi^0 \pi \bar{w} + \psi_k^T(f(\bar{k}, r) - \delta(\bar{k}) - c) - \psi_s^T r.$$

Substituting $\bar{w}$ by $U(\bar{c})$ and combining the last two inequalities we obtain

$$\psi_k^T(f(\bar{k}, \bar{r}) - \delta(\bar{k}) - \bar{c}) - \psi_s^T \bar{r} \geq \psi_k^T(f(\bar{k}, r) - \delta(\bar{k}) - c) - \psi_s^T r$$

for all $(c, r) \in \mathbb{R}^{n+m}_{++}$ such that $U(c) \geq \bar{w}$. This proves (3.19). □

The converse of Theorem 4.4 can be formulated as follows:

Theorem 4.5. *Let $(\bar{k}, \bar{s}, \bar{w}, \bar{c}, \bar{r})$ be an admissible interior solution to Problem (2.11) satisfying the necessary conditions of optimality together with $(\psi^0, \psi_k, \psi_s, \psi_w, \mu_w)$. In addition, assume that this solution is regular. Then $(\bar{k}, \bar{s}, \bar{c}, \bar{r})$ together with (ψ^0, ψ_k, ψ_s), where $\psi^0 = 1$, satisfy the necessary conditions of optimality for an interior solution to Problem (2.8) with the discount factor $\pi(t) := \mu_w(t)$.*

Proof. If an admissible interior solution $(\bar{k}, \bar{s}, \bar{w}, \bar{c}, \bar{r})$ to Problem (2.11) satisfies the necessary conditions of optimality together with $(\psi^0, \psi_k, \psi_s, \psi_w, \mu_w)$, then conditions (3.2) – (3.3) and (3.6) – (3.8) are satisfied trivially. Condition (3.5) follows directly from (3.23) since $\pi(t) = \mu_w(t)$ and $\psi^0 = 1$. Condition (3.4) follows from (3.5) and (3.6) (see Remark 3.2). □

Remark 4.4. (Interpretation of the relationship between both approaches) The importance of Theorem 4.4 and Theorem 4.5 is based on the fact that it links the discounted utility approach and the maximin approach. Actually, it highlights the artificiality behind the discounted utility approach: This discount factor can be considered as implicitly included in the maximin approach. Actually, it is the shadow value of the constraints

$U(c) \geq w$. Moreover, as Cairns (2003) noted, the relationship between these two approaches can be interpreted also from another point of view: The appropriate chosen discount factor $\pi(t)$ can be considered as a coefficient in the infinitely-dimensional hyperplane given by equation

$$\int_0^\infty \pi(t)U(c^*(t))\,\mathrm{d}t = w^*. \tag{4.8}$$

4.4 Converse of Hartwick's result

As mentioned earlier, Hartwick's rule was first formulated as a sufficient conditions for paths along which the utility stays at a constant level (sustainable paths). Later, a question arised whether Hartwick's rule can be stated also as a necessary condition for a constant utility, i.e. whether a converse of Hartwick's result is valid.

Let us begin with the converse of generalized Hartwick's result, which is straightforward. It states that for a sustainable interior path satisfying necessary conditions, the value of net investment represented by

$$I(t) := \psi_k(t)^T \dot{k}(t) - \psi_s(t)^T r(t) \tag{4.9}$$

is constant for all $t \geq 0$.

Theorem 4.6 (Converse of generalized Hartwick's result). [6] *Let (k, s, c, r) be an admissible interior solution to (2.8) which satisfies the necessary conditions of optimality (3.2) – (3.8) and condition (3.9) together with (ψ^0, ψ_k, ψ_s). Then from $U(c(t)) \equiv$ const. for all $t \geq 0$ it follows that there exists a constant I such that $\psi_k(t)^T \dot{k}(t) - \psi_s(t)^T r(t) \equiv I$ for all $t \geq 0$.*

Proof. If $U(c(t)) \equiv$ const. for all $t \geq 0$, then we obtain from (3.9)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\psi_k(t)^T \dot{k}(t) - \psi_s(t)^T r(t)\Big) = 0 \tag{4.10}$$

almost everywhere. This equality implies that $\psi_k^T \dot{k} - \psi_s^T r$ is piecewise constant. Since the Hamiltonian (3.1) is continuous everywhere (in accordance Lemma 7.5) and the term $\psi^0 \pi(t) U(c(t))$ is also continuous everywhere, we obtain that $\psi_k^T \dot{k} - \psi_s^T r$ is continuous and hence constant. □

[6] Cf. e.g. Mitra (2002) [Proposition 1, p. 369], Asheim et al. (2003) [Correct claim 1, p. 144], Mitra (2002) [Theorem 2] and Buchholz et al. (2005) [Proposition 3, p. 556].

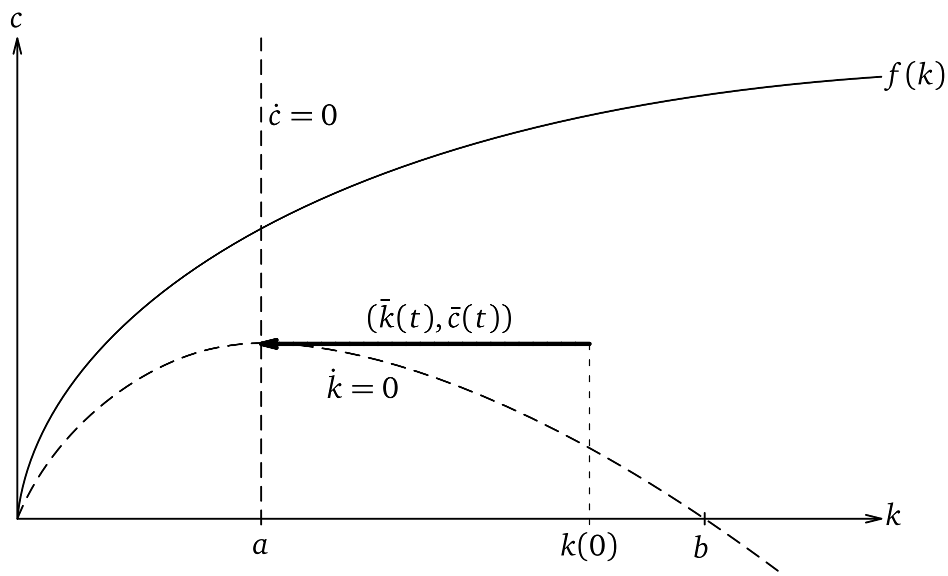

Figure 4.1: Counterexample that I defined by (4.9) can be nonzero (see Example 4.1).

Unlike the converse of generalized Hartwick's result, the converse of Hartwick's result is not quite straightforward. Indeed, under the assumption of Theorem 4.6, it can be shown that the constant I defined by (4.9) might not be zero. To show this, consider the following example:

Example 4.1. [7] We consider the model (2.8) with $n=1$, $m=0$ and $\delta(k)=\bar{\delta}k$, where $\bar{\delta}>0$ is a given constant. In addition, assume that the production function satisfies

$$\lim_{k\to 0^+} f'(k)=\infty \qquad \text{and} \qquad \lim_{k\to\infty} f'(k)=0. \tag{4.11}$$

Let the points a and b be as follows: $f'(a)=\bar{\delta}$ and $f(b)=\bar{\delta}b$ (see Figure 4.1). It is easy to show that such points exist, they are unique and $a<b$. Indeed, the existence and uniqueness of b is implied by Equalities (4.11) and Assumption (A1), as $f'(k)$ is supposed to be positive and decreasing on $(0,\infty)$. Consider now the function

$$\tilde{f}(k) := f(k)-\bar{\delta}k. \tag{4.12}$$

One has $\tilde{f}(0)=0$ and $\tilde{f}(b)=0$. According to (A1), $\tilde{f}$ is differentiable on $(0,b)$. Rolle's theorem then implies that there exists $a\in(0,b)$ such that $\tilde{f}'(a)=0$, i.e. $f'(a)=\bar{\delta}$. In addition, the function $\tilde{f}(k)$ has a maximum at point a. Since $\tilde{f}(k)$ is strictly concave, the point a is unique.

[7] Cf. Mitra (2002) [p. 369-370].

Further we assume that $k(0) \in (a, b)$. We denote

$$\bar{c}(t) := f(a) - \bar{\delta} a \equiv \text{const.} \tag{4.13}$$

and $\bar{k}(t)$ satisfying $\dot{\bar{k}}(t) = f(\bar{k}(t)) - \bar{\delta}\bar{k}(t) - \bar{c}(t)$, $\bar{k}(0) = k(0)$. Finally let the discount factor[8] and the costate variable be in the form

$$\pi(t) := e^{\int_0^t (\bar{\delta} - f'(\bar{k}(\tau))\, d\tau} \tag{4.14}$$

and

$$\psi_k(t) := \pi(t) \frac{\mathrm{d}U}{\mathrm{d}c}(\bar{c}) > 0. \tag{4.15}$$

Then it can be easily verified that:

(i) $(\bar{k}(t), \bar{c}(t))$ is an admissible solution to the given problem,

(ii) $(\bar{k}(t), \bar{c}(t))$ together with $\psi^0 = 1$ and $\psi_k(t)$ defined by (4.15) satisfies the necessary conditions of optimality (3.2) – (3.8) and condition (3.9) and

(iii) $\psi_k(t)\dot{\bar{k}}(t)$ equals to a negative constant.

As $(\bar{k}(t), \bar{c}(t))$ satisfies the state equation $\dot{k} = f(k) - \bar{\delta} k - c$ together with the initial condition, to prove (i) it suffices to verify that $\bar{k} \geq 0$ and $\bar{c} \geq 0$. To show this, consider again the function $\tilde{f}(k)$ defined by (4.12). According to Assumption (A1), this function is strictly concave. Equalities $\tilde{f}(0) = \tilde{f}(b) = 0$ then imply that $\tilde{f}(k) > 0$ on $(0, b)$. As $\bar{c} = \tilde{f}(a)$ and $a \in (0, b)$, we have that $\bar{c} > 0$. In addition, a is a stacionary point of the state equation. Hence, $\bar{k}(t) > a > 0$ since $\bar{k}(0) > a$.

Regarding (ii), note that conditions (3.2), (3.3) and (3.6) – (3.8) are satisfied trivially or are empty, condition (3.5) follows directly from (4.15). Condition (3.4) follows from (3.5) and (3.6) due to the fact that Hamiltonian (3.1) is concave in both c and r (see Remark 3.2). In addition, condition (3.9) states

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_k \dot{\bar{k}}) = -\psi^0 \pi \frac{\mathrm{d}}{\mathrm{d}t} U(\bar{c}) = 0 \tag{4.16}$$

[8]Note that the discount factor defined by (4.14) does not satisfy Assumption (A4).

since $\dot{\bar{c}} = 0$. This statement can be proved as follows:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\psi_k \dot{\bar{k}}) &= \dot{\psi}_k \dot{\bar{k}} + \psi_k \ddot{\bar{k}} \\ &\overset{(4.15)}{=} \dot{\pi} \frac{\mathrm{d}U}{\mathrm{d}c}(\bar{c})\dot{\bar{k}} + \pi \frac{\mathrm{d}^2 U}{\mathrm{d}c^2}(\bar{c})\dot{\bar{c}}\dot{\bar{k}} + \pi \frac{\mathrm{d}U}{\mathrm{d}c}(\bar{c})(f'(\bar{k})\dot{\bar{k}} - \bar{\delta}\dot{\bar{k}} - \dot{\bar{c}}) \\ &\overset{(4.14)}{=} \pi(\bar{\delta} - f'(\bar{k}))\frac{\mathrm{d}U}{\mathrm{d}c}(\bar{c})\dot{\bar{k}} + \pi \frac{\mathrm{d}U}{\mathrm{d}c}(\bar{c})(f'(\bar{k})\dot{\bar{k}} - \bar{\delta}\dot{\bar{k}}) \\ &= 0. \end{aligned}$$

Rgearding the proof of (iii), we already know that $\frac{\mathrm{d}}{\mathrm{d}_t}(\psi_k \dot{\bar{k}}) = 0$. In addition, inequality $\psi_k \dot{\bar{k}} < 0$ directly follows from the fact the ψ_k defined by (4.15) is positive and

$$\dot{\bar{k}} = f(\bar{k}) - \bar{\delta}\bar{k} - \bar{c} = \tilde{f}(\bar{k}) - \tilde{f}(a) < 0 \tag{4.17}$$

since a is the point of a global maximum of $\tilde{f}$ and $\bar{k} > a$. [9] ■

As we have seen in the previous example, it does not suffice that the quadruple (k, s, c, r) meets conditions (3.2) – (3.9) for the converse of Hartwick's result to be valid; additional assumptions have to be made. One possibility is to assume that it satisfies also condition (3.13), in addition to other conditions of optimality. Alternatively, in case of no amortization of capital, this assumption can be replaced by the assumption that at least one of the exhaustible capital goods is an essential input to the production of all renewable capital goods. More precisely, the relative elasticity of production w.r.t. the non-renewable capital is greater than some positive constant for all renewable capital goods.

Theorem 4.7 (Converse of Hartwick's result)**.** [10] *Let (k^*, s^*, c^*, r^*) be an admissible interior solution to (2.8) which satisfies the necessary conditions of optimality (3.2) – (3.8) and condition (3.9) with $(\psi^0, \psi_k(t), \psi_s(t))$. In addition, let us assume that at least one of the following conditions is met:*

[9] Another example that I can be nonzero was given by Asheim et al. (2003) [p. 135]. However, their example is based on the assumption of existence of solution satisfying the necessary conditions of optimality together with $\dot{c} = 0$ at least on an interval $(0, T)$. The proof of the validity of this assumption is missing.

[10] To our knowledge, this formulation of the theorem cannot be found in the literature. The theorem with the condition (i) is only given e.g. by Mitra (2002) [Theorem 2, p. 375], Withagen et al. (2003) [Proposition 2, p. 223], Asheim et al. (2003) [Proposition 4, p. 136], Withagen and Asheim (1998) [Proposition 2, p. 163], Cairns and Yang (2006) [Proposition 1 (i), p. 280], cf. also Martinet (2004) [Proposition 3, p. 9]. On the other hand, the theorem fomulated with condition (ii) only was stated by Buchholz et al. (2005) [Theorem 1, p. 556], albeit only for the case $n = m = 1$ with an additional assumption involving the production and amortization function which is actually not necessary.

(i) (k^*, s^*, c^*, r^*) *satisfies condition (3.13) as well*[11]*, or*

(ii) $\delta(k) = 0$ *for all* $k \geq 0$ *and there exists a* $j = \{1, \ldots, m\}$ *and a positive number* ϱ *such that*

$$\frac{r_j^*(t)\frac{\partial f_i}{\partial r_j}(k^*(t), r^*(t))}{f_i(k^*(t), r^*(t))} \geq \varrho \tag{4.18}$$

for all $t \geq 0$ *and for all* $i = 1, \ldots, n$.

Then from $U(c^*(t)) \equiv$ *const. for all* $t \geq 0$ *it follows* $\psi_k^T \dot{k}^* = \psi_s^T r^*$.

Proof. Since (k^*, s^*, c^*, r^*) is an admissible interior solution to (2.8) which satisfies the necessary conditions of optimality (3.2) – (3.8), condition (3.9) and $U(c^*(t)) \equiv$ const. for all $t \geq 0$, Theorem 4.6 implies that there exists a constant I such that one has $\psi_k(t)^T \dot{k}^*(t) - \psi_s(t)^T r^*(t) = I$ for all $t \geq 0$. If condition (i) is met, then condition (3.13) yields

$$\lim_{t\to\infty} \pi(t) U(c^*(t)) + I = 0, \tag{4.19}$$

since $\psi^0 = 1$ (see (3.14)). In addition, the necessary condition of the convergence of the integral in the objective function states that

$$\lim_{t\to\infty} \pi(t) U(c^*(t)) = 0. \tag{4.20}$$

By combining Equalities (4.19) and (4.20) one has $I = 0$.

On the other hand, if condition (ii) is met, it follows from Lemma 4.1 that the constant I is non-negative since we assume no amortization. Suppose that $I > 0$. Equality (3.6) and condition (ii) imply that there exists a $j \in \{1, \ldots, m\}$ such that

$$\frac{\psi_{s_j} r_j^*}{\varrho} \overset{(3.6)}{=} \psi_k^T \frac{\partial f}{\partial r_j}(k^*, r^*) \frac{r_j^*}{\varrho} = \sum_{i=1}^n \psi_{k_i} \frac{\partial f_i}{\partial r_j}(k^*, r^*) \frac{r_j^*}{\varrho} \overset{(4.18)}{\geq} \sum_{i=1}^n \psi_{k_i} f_i(k^*, r^*)$$

$$= \psi_k^T f(k^*, r^*) = \psi_k^T(\dot{k}^* + \delta(k^*) + c^*) \geq \psi_k^T \dot{k}^* = I + \psi_s^T r^* \geq I > 0 \tag{4.21}$$

(we have used that $c^* > 0$, $r^* > 0$ and $\psi_{s_j} > 0$). From (3.14) we have that ψ_{s_j} is a positive constant. Hence,

$$r_j^*(t) \geq \frac{I\varrho}{\psi_{s_j}} > 0 \tag{4.22}$$

[11] Recall that Theorem 7.10 implies that condition (3.13) is satisfied if (k^*, s^*, c^*, r^*) is an interior optimal solution to Problem (2.8) satisfying conditions (3.10) – (3.12).

for all $t \geq 0$, i.e. r_j^* is separated from zero. The state equation $\dot{s}(t) = -r(t)$ then implies that $s^*(t)$ is unbounded from below, which is a contradiction with the non-negativity condition on $s^*(t)$. Therefore we have that $I = 0$ and Hartwick's rule is satisfied.[12] □

Remark 4.5. Note that in Example 4.1, neither condition (i) nor condition (ii) given in Theorem 4.7 is satisfied. Indeed, regarding condition (i), we have that $\psi_k \dot{\bar{k}}$ is a negative constant (denote it by A, $A < 0$). Note that this does not contradicts Lemma 4.1 since the model in Example 4.1 involves amortization of the capital. In addition, Equality (4.14) implies that π is increasing, since

$$\lim_{t\to\infty} \bar{k}(t) = \bar{k}(a) \tag{4.23}$$

and $\bar{k}(t) > \bar{k}(a)$, hence $f'(\bar{k}(t)) > f'(\bar{k}(a))$ for all $t \geq 0$. Therefore

$$\lim_{t\to\infty} \pi(t)U(\bar{c}(t)) + \psi_k(t)(f(\bar{k}(t)) - \bar{\delta}\bar{k} - \bar{c}(t)) \tag{4.24}$$

cannot be zero and the Equality (3.13) is not satisfied. This is not in contradiction with Theorem 7.10, since conditions (3.11) is not satisfied, because $\pi(t)$ is increasing and $U(\bar{c})$ is constant in t. Moreover, condition (ii) in Theorem 4.7 is also not satified, since f does not depend on r.

4.5 Hartwick's rule in case of population growth

In a recent paper, Mitra (2008) formulated the converse of Hartwick's result generalized for the case of exponential population growth in a model with the following dynamics:

$$\begin{aligned}
\dot{k} &= f(k,r,n) - c, \quad k(0) = k_0 > 0, \\
\dot{s} &= -r, \quad s(0) = s_0 > 0, \\
\dot{n} &= \vartheta n, \quad \vartheta > 0,
\end{aligned}$$

where $f(k,r,n) = k^\alpha r^\beta n^{1-\alpha-\beta}$, α and β are given constants from $(0,1)$, $\alpha + \beta < 1$.

Theorem 4.8 (Mitra (2008)). [13] *If (k,c,s,r,n) is a path satisfying:*

(i) k, c, s, r, n are all positive for all $t \geq 0$,

[12] The proof is a straightforward generalization to a multidimensional case of the proof given by Buchholz et al. (2005) [p. 556-557].

[13] See Mitra (2008) [Theorem 1, p. 65].

(ii) $\dfrac{c}{n}$ *is constant (denoted by* $\bar{c}$*),*

(iii) $\dfrac{\frac{d}{dt}\frac{\partial f}{\partial r}(k,r,n)}{\frac{\partial f}{\partial r}(k,r,n)} = \dfrac{\partial f}{\partial k}(k,r,n)$

then

$$\frac{d}{dt}(p\dot{k}+\dot{s}) = (wn - p\bar{c}n)\vartheta \tag{4.25}$$

for all $t \geq 0$*, where*

$$p = \frac{1}{\frac{\partial f}{\partial r}(k,r,n)} \qquad \textit{and} \qquad w = p\frac{\partial f}{\partial n}(k,r,n).$$

The proof of this theorem introduced by Mitra (2008) is quite long and technical. In this section, we deal with the converse of Hartwick's result in the context of a simplified model given by (2.13) and (2.14). In the optimal control framework, Hartwick's rule can be easily derived and we show its relationship to Equality (4.25). Contrary to Mitra (2008) who restricted himself only to a case of Cobb-Douglas production function, we consider any production function satisfying the general Assumption (A1).

Consider the following optimal control problem:

$$\begin{aligned}
&\max_{\{r(t)\}} \bar{c}(0), \quad t \in \langle 0, \infty),\\
&\dot{k} = f(k,r,n) - \bar{c}n, \quad k(0) = k_0 > 0\\
&\dot{s} = -r, \quad s(0) = s_0 > 0,\\
&\dot{n} = \vartheta n, \quad n(0) = n_0 > 0,\\
&\dot{\bar{c}} = 0, \quad \bar{c}(0)\text{free},\\
&k(t) \geq 0, \quad s(t) \geq 0,\\
&r(t) \geq 0, \quad \bar{c}(t) \geq 0.
\end{aligned} \tag{4.26}$$

Suppose that the production function f satisfies Assumption (A1). Note that any admissible solution to Problem (4.26) satisfies that $\bar{c}$ is constant, which is a condition analogous to the condition (ii) in Theorem 4.8.

We formulate the necessary conditions of optimality for an interior optimal solution. Denote the costate variable associated with n by ψ_n and the costate variable associated with $\bar{c}$ by ψ_c. The Hamiltonian is

$$H(k,s,n,\bar{c},r,\psi_k,\psi_s,\psi_n,\psi_c) = \psi_k(f(k,r,n) - \bar{c}n) - \psi_s r + \psi_n \vartheta n. \tag{4.27}$$

Let $(k^*, s^*, n^*, \bar{c}^*, r^*)$ be an admissible interior solution to (4.26). Then there exist a constant $\psi^0 = 0$ or $\psi^0 = 1$ and continuous functions ψ_k, ψ_s, ψ_n and ψ_c such that:

$$(\psi^0, \psi_k, \psi_s, \psi_n, \psi_c) \neq 0 \text{ for all } t \geq 0, \tag{4.28}$$

$$\frac{\partial f}{\partial r}(k^*, r^*, n^*)\psi_k = \psi_s, \tag{4.29}$$

$$\dot{\psi}_k = -\psi_k \frac{\partial f}{\partial k}(k^*, r^*, n^*), \tag{4.30}$$

$$\dot{\psi}_s = 0, \tag{4.31}$$

$$\dot{\psi}_n = -\psi_k \frac{\partial f}{\partial n}(k^*, r^*, n^*) + \psi_k \bar{c} - \psi_n \vartheta, \tag{4.32}$$

$$\dot{\psi}_c = -\psi_k n, \tag{4.33}$$

$$\psi_c(0) = -\psi^0. \tag{4.34}$$

In addition, Theorem 7.9 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_k \dot{k}^* + \psi_s \dot{s}^* + \psi_n \dot{n}^* + \psi_c \dot{\bar{c}}^*) = 0 \tag{4.35}$$

for almost all $t \geq 0$.

Now we can formulate the following theorem, which is an original result of this thesis:

Theorem 4.9. *Let $(k^*, s^*, r^*, n^*, \bar{c}^*)$ be an admissible interior solution to (4.26), which satisfies the necessary conditions of optimality together with $(\psi^0, \psi_k, \psi_s, \psi_n, \psi_c)$ and with $\psi_s \neq 0$. Then Equality (4.25) is satisfied where p and w are defined by*

$$p := \frac{\psi_k}{\psi_s} \overset{(4.29)}{=} \frac{1}{\frac{\partial f}{\partial r}(k^*, r^*, n^*)} \qquad \textit{and} \qquad w := p\frac{\partial f}{\partial n}(k^*, r^*, n^*). \tag{4.36}$$

Proof. Since $\dot{\bar{c}} = 0$, Equality (4.35) can be rewritten to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_k \dot{k}^* + \psi_s \dot{s}^*) = -\dot{\psi}_n \dot{n}^* - \psi_n \ddot{n}^* = \vartheta(-\dot{\psi}_n n^* - \psi_n \vartheta n^*). \tag{4.37}$$

After substituting (4.32) into (4.37) one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_k \dot{k}^* + \psi_s \dot{s}^*) = \vartheta\left(\psi_k \frac{\partial f}{\partial n}(k^*, r^*, n^*)n^* - \psi_k \bar{c} n^* + \psi_n \vartheta n^* - \psi_n \vartheta n^*\right). \tag{4.38}$$

Note that (4.31) implies that ψ_s is a constant. Since we have assumed $\psi_s \neq 0$, we can divide (4.37) by ψ_s to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi_k}{\psi_s}\dot{k}^* + \dot{s}^*\right) = \vartheta\left(\frac{\psi_k}{\psi_s}\frac{\partial f}{\partial n}(k^*, r^*, n^*)n^* - \frac{\psi_k}{\psi_s}\bar{c} n^*\right). \tag{4.39}$$

We can use (4.36) to simplify (4.39) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p\dot{k}^* + \dot{s}^*\right) = \vartheta\left(wn^* - p\bar{c}n^*\right) \tag{4.40}$$

which is the same Equality as (4.25). □

Remark 4.6. Note that if $(k^*, s^*, r^*, n^*, \bar{c}^*)$ is an admissible interior solution to (4.26), which satisfies the necessary conditions of optimality with $(\psi^0, \psi_k, \psi_s, \psi_n, \psi_c)$, where $\psi_s \neq 0$, then condition (iii) in Theorem 4.8 can be easily derived as well. To show this, differentiate (4.29) w.r.t. t totally:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial r}(k^*, r^*, n^*)\,\psi_k + \frac{\partial f}{\partial r}(k^*, r^*, n^*)\,\dot{\psi}_k = \dot{\psi}_s. \tag{4.41}$$

Substituting (4.30) and (4.31) into (4.41) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial r}(k^*, r^*, n^*)\,\psi_k - \frac{\partial f}{\partial r}(k^*, r^*, n^*)\frac{\partial f}{\partial k}(k^*, r^*, n^*)\,\psi_k = 0. \tag{4.42}$$

From the assumption that $\psi_s \neq 0$ and from (4.29) it follows $\psi_k \neq 0$. Moreover, by Assumption (A1) one has $\frac{\partial f}{\partial r}(k^*, r^*, n^*) > 0$ for any interior solution, hence (4.42) yields condition (iii) in Theorem 4.8.

Chapter 5

Perfectly substitutable exhaustible resources

In Chapters 2 – 4, we have assumed that all exhaustible resources are essential for the production of reproducible capital goods (see Assumption (A1)), although they were considered as imperfect substitutes. One of the most important implications of this assumption for the mathematical analysis of these models is that we do not need to consider the case of binding non-negativity constraints imposed on stocks of reproducible resources along optimal paths.[1] This is the reason why the analysis in majority of relevant papers was focused mainly on internal paths i.e. paths for which it was assumed that these constraints are not binding. Hence, it was possible to avoid technical difficulties which may arise in the presence of pure state constraints.

In this chapter, we formulate a new model of an economy with two exhaustible resources which are mutually perfectly substitutable. We will assume that although the extraction of at least one of the exhaustible capital goods is necessary for the production, it is not required that both exhaustible resources have to be used at each time. Moreover, we assume that the productivity of one resource is constant in time, whereas the productivity of the other one is increasing. We believe that this assumption is quite realistic, because in many cases a non-renewable resource is beeing gradually replaced by another one for which the process technology and hence productivity has improved. As an example, coal was broadly substituted by uranium in electricity plants.

We formulate the model in the maximin framework given by (2.13) and (2.14). We focus on the qualitative analysis of optimal solutions to this model based on the results from the optimal control theory. However, in this case we cannot restrict ourselves only to interior solutions, since zero extraction rate and even zero level of one of the

[1] See the discussion in Section 3.1 and in Cairns and Long (2006) [p. 285].

exhaustible capital goods might still allow some level of consumption and possibly might be optimal.

To our knowledge, only three papers have considered the case that pure state constraints are binding in models of sustainable economic growth so far:

First, Dixit et al. (1980) introduce few comments on this issue without any further analysis in conclusion of their article. They expect that generalized Hartwick's rule is valid separately before and after the moment when one of the resources becomes depleted, but the value of net investment might exhibit a jump at this moment, while the value of consumption stays continuous.

Second, Cairns and Long (2006) present a brief discussion on the possibility of binding non-negativity stock constraints in the context of the DHS model (2.11) with a maximin objective function. However, they do not provide any deeper analysis on this issue. They merely propose some conjectures regarding discontinuity of shadow values of capital stock at points where these constraints become or cease to be binding. They assume that at these points, the level of net investment of at least one stock must exhibit a jump.

Third, the issue of binding constraints on state variables is dealt with in a recent paper by Martinet (2009). He considers constraints in form of some thresholds on exhaustible resources. His approach contains two steps: First he applies a viability analysis (see Section 2.3.2) to determine the set of achievable objective values given the initial endowments of the economy and second he uses a static optimization taking into account preferences. He does not use optimal control theory. Therefore, he does not study the paths of optimal solutions.

All results in this section are new.

5.1 The model and necessary conditions of optimality

We will consider the following version of the model given in (2.13) and (2.14):

$$
\begin{aligned}
&\max_{\{c(t),r_1(t),r_2(t)\}} w(0), \quad \text{where } t \in \langle 0,\infty), \\
&\dot{k}(t) = f(t,k(t),r_1(t),r_2(t)) - c(t), \quad k(0) = k_0 > 0 \text{ given}, \\
&\dot{s}_1(t) = -r_1(t), \quad s_1(0) = \bar{s}_1 > 0 \text{ given}, \\
&\dot{s}_2(t) = -r_2(t), \quad s_2(0) = \bar{s}_2 > 0 \text{ given}, \\
&\dot{w}(t) = 0, \quad w(0) \text{ free}, \qquad\qquad (5.1) \\
&k(t) \geq 0, \quad s_1(t) \geq 0, \quad s_2(t) \geq 0,
\end{aligned}
$$

$$r_1(t) \geq 0, \quad r_2(t) \geq 0, \quad c(t) \geq 0,$$
$$U(c(t)) \geq w(t).$$

The production function is assumed to be in the following Cobb-Douglas form with homogeneity of degree one:

$$f(t, k(t), r_1(t), r_2(t)) = k(t)^{\alpha}(r_1(t) + d(t) r_2(t))^{1-\alpha}, \tag{5.2}$$

where $d(t)$ is a given function which represents the (time dependent) marginal rate of substitution between the extraction rates of the exhaustible capital goods in the production process. Since it does not depend neither on r_1 nor on r_2, both ehaustible capital goods are considered to be mutually perfectly substitutable. Alternatively, the interpretation of $d(t)$ can be viewed as exogeneously given increasing factor of productivity of the second exhaustible capital good. It is supposed that $d(t)$ satisfies the following assumption:

(A6) The function $d(t) : \mathbb{R}_+ \to \mathbb{R}_{++}$ is positive, strictly increasing and continuous for all $t \geq 0$.

We assume that $\alpha \in (\frac{1}{2}, 1)$ which is equivalent to $\alpha > 1 - \alpha$ (see Remark 5.1 below).

Problem (5.1) is a non-autonomous optimal control problem with infinite time horizon, mixed and pure state constraints and with an initial scrap value function. Note that there is no initial condition imposed on $w(0)$.

Remark 5.1. There is a close relationship between Problem (5.1) and Problem (3.28) where $\gamma = 0$ and $\delta = 0$. In fact, it is easy to see that if (k, s, c, r) is an admissible solution to the latter problem, then $(c, r_1, r_2, k, s_1, s_2, w)$ where $r_1 = r$, $r_2 \equiv 0$, $s_1 = s$ and $s_2 \equiv \bar{s}_2$ is an admissible solution to Problem (5.1). As it was stated in Section 3.2.1 (a), the condition $\alpha > 1 - \alpha$ is a necessary and sufficient condition for the existence of optimal solution in the model (3.28) where $\gamma = 0$ and $\delta = 0$. Moreover, the explicit form of solution was given in Section 3.2.2 (a). It will be used later in the proof of the main result of this chapter.

In accordance with (7.8), (7.9) and (7.10), define for (5.1) the Hamiltonian by

$$H(t, k, s_1, s_2, w, c, r_1, r_2, \psi^0, \psi_k, \psi_1, \psi_2, \psi_w) = \psi_k(f(t, k, r_1, r_2) - c) - \psi_1 r_1 - \psi_2 r_2,$$

the Lagrangian by

$$L(t, k, s_1, s_2, w, c, r_1, r_2, \psi^0, \psi_k, \psi_1, \psi_2, \psi_w, \mu_c, \mu_1, \mu_2, \mu_w, \nu_k, \nu_1, \nu_2)$$

$$= (\psi_k+\nu_k)(f(t,k,r_1,r_2)-c)-(\psi_1+\nu_1)r_1-(\psi_2+\nu_2)r_2+\mu_c c+\mu_1 r_1+\mu_2 r_2+\mu_w(U(c)-w)$$

and the simplified Lagrangian by

$$\breve{L}(t,k,s_1,s_2,w,c,r_1,r_2,\psi^0,\psi_k,\psi_1,\psi_2,\psi_w,\mu_c,\mu_1,\mu_2,\mu_w)$$

$$= \psi_k(f(t,k,r_1,r_2)-c)-\psi_1 r_1-\psi_2 r_2+\mu_c c+\mu_1 r_1+\mu_2 r_2+\mu_w(U(c)-w).$$

According to Definition 7.3, a solution $(k^*,s_1^*,s_2^*,w^*,c^*,r_1^*,r_2^*)$ to Problem (5.1) satisfies the weak constraint qualification, if the matrix

$$\begin{pmatrix} r_1^* & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & r_2^* & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & c^* & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & U(c^*)-w^* & 0 & 0 & \frac{\mathrm{d}U}{\mathrm{d}c}(c^*) \end{pmatrix} \tag{5.3}$$

has a full rank when it is evaluated at $(r_1^*(t^-),r_2^*(t^-),c^*(t^-),w^*(t))$ for any $t\in(0,\infty)$ and at $(r_1^*(t^+),r_2^*(t^+),c^*(t^+),w^*(t))$ for any $t\in\langle 0,\infty)$. This condition is not satisfied only if $c^*(t)=0$ and $U(c^*(t))=w^*(t)$ simultaneously on a non-trivial interval. However, then the value of the objective function attains its lowest possible value. Hence, excluding such solutions does not restrict the set of maximal attainable values of the objective function.

Using Theorem 7.6, we obtain the following necessary conditions of optimality: If $(k^*,s_1^*,s_2^*,c^*,w^*,r_1^*,r_2^*)$ is an optimal solution to Problem (5.1), then there exist a constant $\psi^0=0$ or $\psi^0=1$, piecewise continuous functions ψ_k, ψ_1, ψ_2, ψ_w, μ_c, μ_1, μ_2 and μ_w and non-increasing piecewise continuous functions ν_k, ν_1 and ν_2 such that[2]

$$(\psi^0,\psi_k(t^+),\psi_1(t^+),\psi_2(t^+),\psi_w(t^+))\neq 0 \text{for all } t\geq 0, \tag{5.4}$$

$$(\psi^0,\psi_k(t^-)\psi_1(t^-),\psi_2(t^-),\psi_w(t^-))\neq 0 \text{ for all } t>0 \tag{5.5}$$

and the following conditions are satisfied almost everywhere:

$$\psi_k(f(t,k^*,r_1^*,r_2^*)-c^*)-\psi_1 r_1^*-\psi_2 r_2^*\geq \psi_k(f(t,k^*,r_1,r_2)-c)-\psi_1 r_1-\psi_2 r_2 \tag{5.6}$$

[2]Recall that the list of variables and multipliers can be found at the end of this thesis.

for all (c, r_1, r_2) such that $c \geq 0$, $r_1 \geq 0$, $r_2 \geq 0$,

$$\frac{\partial \breve{L}}{\partial c} = -\psi_k + \mu_c + \mu_w \frac{\mathrm{d}U}{\mathrm{d}c} = 0, \tag{5.7}$$

$$\frac{\partial \breve{L}}{\partial r_1} = \psi_k \frac{\partial f}{\partial r_1} - \psi_1 + \mu_1 = 0, \tag{5.8}$$

$$\frac{\partial \breve{L}}{\partial r_2} = \psi_k \frac{\partial f}{\partial r_2} - \psi_2 + \mu_2 = 0, \tag{5.9}$$

$$\dot{\psi}_k - \dot{\nu}_k = -\frac{\partial L}{\partial k} = -(\psi_k + \nu_k)\frac{\partial f}{\partial k}, \tag{5.10}$$

$$\dot{\psi}_1 - \dot{\nu}_1 = -\frac{\partial L}{\partial s_1} = 0, \tag{5.11}$$

$$\dot{\psi}_2 - \dot{\nu}_2 = -\frac{\partial L}{\partial s_2} = 0, \tag{5.12}$$

$$\dot{\psi}_w = -\frac{\partial L}{\partial w} = \mu_w, \quad \psi_w(0) = -1, \tag{5.13}$$

$$\nu_k k = 0, \quad \nu_k \geq 0, \tag{5.14}$$

$$\nu_1(0) = 0, \nu_1 \text{ is constant on any interval where } s_1^* > 0, \tag{5.15}$$

$$\nu_2(0) = 0, \nu_2 \text{ is constant on any interval where } s_2^* > 0, \tag{5.16}$$

$$\mu_c c^* = 0, \quad \mu_c \geq 0, \tag{5.17}$$

$$\mu_1 r_1^* = 0, \quad \mu_1 \geq 0, \tag{5.18}$$

$$\mu_2 r_2^* = 0, \quad \mu_2 \geq 0, \tag{5.19}$$

$$\mu_w(U(c^*) - w^*) = 0, \quad \mu_w \geq 0, \tag{5.20}$$

$$\nu_k \text{ is continuous if } k^* = 0 \text{ and } f(t, k^*, r_1^*, r_2^*) - \delta(k^*) - c^* \text{ is discontinuous}, \tag{5.21}$$

$$\nu_1 \text{ is continuous if } s_1^* = 0 \text{ and } r_1^* \text{ is discontinuous}, \tag{5.22}$$

$$\nu_2 \text{ is continuous if } s_2^* = 0 \text{ and } r_2^* \text{ is discontinuous}. \tag{5.23}$$

The functions $\psi_k - \nu_k$, $\psi_1 - \nu_1$ and $\psi_2 - \nu_2$ are continuous everywhere and it follows from Lemma 7.5 that

$$H[t] := H(t, k^*(t), s_1^*(t), s_2^*(t), c^*(t), r_1^*(t), r_2^*(t), w^*(t), \psi_k(t), \psi_1(t), \psi_2(t), \psi_w(t))$$
$$\text{is continuous for all } t \geq 0. \tag{5.24}$$

In addition, Theorem 7.9 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_k(f(t, k^*, r_1^*, r_2^*) - c^*) - \psi_1 r_1^* - \psi_2 r_2^*\right) = \psi_k \frac{\partial f}{\partial t}(t, k^*, r_1^*, r_2^*). \tag{5.25}$$

5.2 Main result

Using conditions (5.5) – (5.24), we can proceed with the qualitative analysis of the solutions to these conditions. However, we restrict ourselves only to regular solutions (see Definition 3.1).

As mentioned above, we assume that the economy is able to produce even if a stock of one of the exhaustible capital goods is zero, since the exhaustible resources are supposed to be perfect substitutes. Hence, instead of interior solutions used in the previous chapters, we introduce the notion of a weakly interior solution as follows:

Definition 5.1. *A solution $(k, s_1, s_2, w, c, r_1, r_2)$ is called a weakly interior solution to Problem (5.1) if it is an admissible solution to this problem, $k(t) > 0$ for all $t \geq 0$, $c(t^-) > 0$ and $r_1(t^-) + d(t)r_2(t^-) > 0$ for all $t > 0$, $c(t^+) > 0$ and $r_1(t^+) + d(t)r_2(t^+) > 0$ for all $t \geq 0$.*

The main result of the qualitative analysis is that the stock of the resource s_1 with constant productivity will be extracted first and only then the second resource s_2 with increasing productivity will be used. It is quite interesting that this remains true even if $d(t) < 1$ on some interval, i.e. the productivity of the second resource is at the beginning smaller than the productivity of the first resource. More precisely, the following theorem can be formulated:

Theorem 5.1. *Let $(k^*, s_1^*, s_2^*, w^*, c^*, r_1^*, r_2^*)$ be an optimal solution to Problem (5.1) with $\alpha > \frac{1}{2}$, which is regular (see Definition 3.1) and weakly interior. Then there exists $T \geq 0$ such that the paths of extraction rates of non-renewable resources have the following form:*

$$\begin{array}{llll} \text{For every } t \in \langle 0, T\rangle : & r_1^*(t) > 0 & \text{and} & r_2^*(t) = 0, \\ \text{for every } t \in (T, \infty) : & r_1^*(t) = 0 & \text{and} & r_2^*(t) > 0. \end{array} \tag{5.26}$$

To prove this theorem, we first formulate and prove the following lemma:

Lemma 5.1. *Let $(k^*, s_1^*, s_2^*, w^*, c^*, r_1^*, r_2^*)$ be an optimal solution to Problem (5.1) with $\alpha > \frac{1}{2}$, which is regular and weakly interior. Then one has:*

(i) Equality $r_2^*(t) = 0$ is not satisfied for all $t \geq 0$.

(ii) Inequalities $r_1^*(t) > 0$, $r_2^*(t) > 0$ are not satisfied simultaneously on any non-trivial interval.

(iii) There is no $\tau > 0$ such that $r_1^*(\tau^-) = 0$, $r_2^*(\tau^-) > 0$ and $r_1^*(\tau^+) > 0$, $r_2^*(\tau^+) = 0$.

Proof of (i). If $r_2^*(t) = 0$ for all $t \geq 0$, then we known from Section 3.2.2(a) (see Remark 5.1 and Equalities (3.31) – (3.33)) that the maximal value of consumption which can be sustained forever is

$$c^* \equiv \alpha(\bar{s}_1(2\alpha - 1))^{\frac{1-\alpha}{\alpha}} k_0^{\frac{2\alpha-1}{\alpha}} \tag{5.27}$$

and the rate of extraction of the first exhaustible capital good obeys the equation

$$r_1^*(t) = \left(\frac{c^*}{\alpha}\right)^{\frac{1}{1-\alpha}} \left(\frac{c^*(1-\alpha)}{\alpha} t + k_0\right)^{\frac{\alpha}{\alpha-1}}. \tag{5.28}$$

It is easy to show that the control given by (5.27), (5.28) and $r_2^*(t) = 0$ together with its response is a weakly interior solution to (5.1), i.e. inequalities $s_1^* \geq 0$, $s_1^* \geq 0$, $k^* > 0$ and $r_1^*(t) + d(t) r_2^*(t) > 0$ are satisfied for all $t \geq 0$.

Now, we show that the sustainable level of consumption given by (5.27) can be increased if r_2 is not prescribed to be zero and hence we obtain a contradiction with optimality. Let us define

$$A_i := \left[\bar{s}_i(2\alpha - 1)\right]^{\frac{1-\alpha}{\alpha}} k_0^{\frac{2\alpha-1}{\alpha}} \tag{5.29}$$

and

$$\tilde{r}_i(t) := A_i^{\frac{1}{1-\alpha}} \left[(1-\alpha)A_i t + k_0\right]^{-\frac{\alpha}{1-\alpha}}, \tag{5.30}$$

where $i = 1, 2$. Both A_1 and A_2 are positive numbers, since we have assumed that $\alpha > \frac{1}{2}$. Note that combining (5.27) and (5.29) for $i = 1$ yields

$$A_1 = \frac{c^*}{\alpha}. \tag{5.31}$$

Substituting this into (5.30) for $i = 1$ implies that $\tilde{r}_1$ is the same as r_1^* defined by (5.28).

We now prove that for $\tilde{r}_1$ and $\tilde{r}_2$ defined by (5.30), $\tilde{c}$ defined by

$$\tilde{c} := \alpha A_1 + A_1 \left((1+B)^{1-\alpha} - 1\right) \tag{5.32}$$

is a sustainable level of consumption greater than c^* defined by (5.27) which is the maximal sustainable level of consumption for $(r_1^*, 0)$, where

$$B := \inf_{t \geq 0} d(t) \left(\frac{A_2[(1-\alpha)A_2 t + k_0]^{-\alpha}}{A_1[(1-\alpha)A_1 t + k_0]^{-\alpha}}\right)^{\frac{1}{1-\alpha}}. \tag{5.33}$$

Note that combining (5.31) and (5.32) yields

$$\tilde{c} = c^* + A_1 \left((1+B)^{1-\alpha} - 1\right). \tag{5.34}$$

Hence, to prove that $\tilde{c}$ is a sustainable level of consumption which is greater than c^*, we need to prove the following statements:

(a) B is positive (Equality (5.34) then implies that $\tilde{c} > c^*$) and

(b) $(\tilde{r}_1, \tilde{r}_2, \tilde{c})$ together with its response is a weakly interior solution.

To prove (a), note that (5.33) together with (A6) imply

$$B \geq d_0 \left(\frac{A_2}{A_1}\right)^{\frac{1}{1-\alpha}} \left[\inf_{t\geq 0} \frac{(1-\alpha)A_1 t + k_0}{(1-\alpha)A_2 t + k_0}\right]^{\frac{\alpha}{1-\alpha}}. \tag{5.35}$$

The term in brackets is an infimum of a division of two linear functions on $\langle 0, \infty)$. Its value can be obtained for $t = 0$ or $t \to \infty$. Hence one has

$$\inf_{t\geq 0} \frac{(1-\alpha)A_1 t + k_0}{(1-\alpha)A_2 t + k_0} = \min\left\{1, \frac{A_1}{A_2}\right\}. \tag{5.36}$$

Substituting (5.36) into (5.35) yields

$$B \geq d_0 \left(\frac{A_2}{A_1}\right)^{\frac{1}{1-\alpha}} \left[\min\left\{1, \frac{A_1}{A_2}\right\}\right]^{\frac{\alpha}{1-\alpha}} > 0, \tag{5.37}$$

since d_0 and both A_1 and A_2 are positive numbers.

Now, it remains to prove (b). Note that condition $r_1(t) + d(t)r_2(t) > 0$ is satisfied trivially, since both $\tilde{r}_1(t) > 0$ and $\tilde{r}_2(t) > 0$ for all $t \geq 0$. In addition, it follows from (5.34) that $\tilde{c} > 0$ since $B > 0$. Furthermore, it can be easily verified that

$$\tilde{s}_i(t) := \frac{1}{2\alpha - 1} A_i^{\frac{\alpha}{1-\alpha}} \left[(1-\alpha)A_i t + k_0\right]^{\frac{1-2\alpha}{1-\alpha}} > 0, \quad \text{where } i = 1, 2 \tag{5.38}$$

is a solution to the differential equation $\dot{\tilde{s}}_i = -\tilde{r}_i$ and

$$\tilde{s}_i(0) \overset{(5.38)}{=} \frac{1}{2\alpha - 1} A_i^{\frac{\alpha}{1-\alpha}} k_0^{\frac{1-2\alpha}{1-\alpha}} \overset{(5.29)}{=} \bar{s}_i, \tag{5.39}$$

where $i = 1, 2$. Hence, it suffices to prove that the solution $\tilde{k}(t)$ to equation

$$\dot{\tilde{k}} = \tilde{k}^\alpha (\tilde{r}_1 + d\tilde{r}_2)^{1-\alpha} - \tilde{c} \tag{5.40}$$

is positive for all $t \geq 0$.

We have

$$(\tilde{r}_1 + d(t)\tilde{r}_2)^{1-\alpha} \overset{(5.30)}{=} \left(A_1^{\frac{1}{1-\alpha}} \left[(1-\alpha)A_1 t + k_0\right]^{-\frac{\alpha}{1-\alpha}} + d(t) A_2^{\frac{1}{1-\alpha}} \left[(1-\alpha)A_2 t + k_0\right]^{-\frac{\alpha}{1-\alpha}}\right)^{1-\alpha}$$

$$= A_1\left[(1-\alpha)A_1 t + k_0\right]^{-\alpha}\left[1 + d(t)\left(\frac{A_2[(1-\alpha)A_2 t + k_0]^{-\alpha}}{A_1[(1-\alpha)A_1 t + k_0]^{-\alpha}}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}$$

$$\overset{(5.33)}{\geq} A_1\left[(1-\alpha)A_1 t + k_0\right]^{-\alpha}(1+B)^{1-\alpha}, \tag{5.41}$$

where B is defined by (5.33).

Therefore one has

$$\dot{\tilde{k}} = \tilde{k}^{\alpha}(\tilde{r}_1 + d\tilde{r}_2)^{1-\alpha} - \tilde{c}$$

$$\overset{(5.32),(5.41)}{\geq} \tilde{k}^{\alpha}A_1\left[(1-\alpha)A_1 t + k_0\right]^{-\alpha}(1+B)^{1-\alpha} - A_1(1+B)^{1-\alpha} + (1-\alpha)A_1. \tag{5.42}$$

It can be easily verified that the solution (denoted by $\bar{k}(t)$) of (5.42) taken as equality with initial condition $\bar{k}(0) = k_0$ is

$$\bar{k}(t) = (1-\alpha)A_1 t + k_0. \tag{5.43}$$

Hence, the solution of (5.42) with initial condition $\tilde{k}(0) = k_0$ satisfies

$$\tilde{k}(t) \geq \bar{k}(t) \geq k_0 > 0. \tag{5.44}$$

Proof of (ii). The form of the production function (5.2) which is used in the model (5.1) implies

$$\frac{\partial f}{\partial r_2}(t, k^*, r_1^*, r_2^*) = \frac{\partial f}{\partial r_1}(t, k^*, r_1^*, r_2^*)\, d(t) \tag{5.45}$$

for all $t \geq 0$. Conditions (5.8) and (5.9) then imply

$$(\psi_1 - \mu_1)d = \psi_2 - \mu_2 \tag{5.46}$$

almost everywhere.

Suppose now that there exists a non-trivial open interval I where $r_1^*(t) > 0$ and $r_2^*(t) > 0$. From (5.18) and (5.19) we obtain $\mu_1 = 0$ and $\mu_2 = 0$ on I. In addition, positive values of r_1^* and r_2^* on I imply also positive values of s_1^* and s_2^* on I. Hence, from (5.15) and (5.16) we obtain that ν_1 and ν_2 are both piecewise constant on I. Therefore, from (5.11) and (5.12) also ψ_1 and ψ_2 are both piecewise constant on I. Finally, (5.46) than implies that d is also piecewise constant on I. This is a contradiction, because we have assumed that d is a strictly increasing function (see Assumption (A6)).

Proof of (iii). Suppose that there exists $\tau > 0$ such that $r_1^*(\tau^-) = 0$, $r_2^*(\tau^-) > 0$ and $r_1^*(\tau^+) > 0$, $r_2^*(\tau^+) = 0$. Then (5.18) implies $\mu_1(\tau^-) \geq 0$ and $\mu_1(\tau^+) = 0$, (5.19) implies

$\mu_2(\tau^-) = 0$ and $\mu_2(\tau^+) \geq 0$. Hence, we obtain from (5.46)

$$(\psi_1(\tau^-) - \mu_1(\tau^-))d(\tau) = \psi_2(\tau^-), \tag{5.47}$$

$$\psi_1(\tau^+)d(\tau) = \psi_2(\tau^+) - \mu_2(\tau^+). \tag{5.48}$$

By subtracting (5.48) from (5.47) we obtain

$$(\psi_1(\tau^-) - \psi_1(\tau^+) - \mu_1(\tau^-))d(\tau) = \psi_2(\tau^-) - \psi_2(\tau^+) + \mu_2(\tau^+). \tag{5.49}$$

Because we have assumed that $r_1^*(\tau^+) > 0$, we have $r_1^*(t) > 0$ on some right neighbourhood of τ (denoted by $\mathcal{O}(\tau^+)$). Thus $\mu_1(t) = 0$ on $\mathcal{O}(\tau^+)$ and from (5.46) we obtain

$$\psi_1(t)d(t) = \psi_2(t) - \mu_2(t) \tag{5.50}$$

on $\mathcal{O}(\tau^+)$. In addition, $r_1^*(t) > 0$ implies $s_1^*(t) > 0$ on $\mathcal{O}(\tau^+)$. From (5.15) ν_1 is constant on $\mathcal{O}(\tau^+)$ and from the continuity of $\psi_1 - \nu_1$ one has that ψ_1 is continuous on $\mathcal{O}(\tau^+)$ as well. Furthermore, Equality (5.11) then implies that ψ_1 is constant on $\mathcal{O}(\tau^+)$. We can prove that this constant is positive. Indeed, from condition (5.7) we obtain $\psi_k(t) > 0$ for all $t \geq 0$ because $\mu_w > 0$ (because we only consider regular solutions). Hence, condition (5.8) implies the positivity of ψ_1.

As ν_2 is non-increasing everywhere, from continuity of $\psi_2 - \nu_2$ and Equality (5.12) we have that also ψ_2 is non-increasing everywhere. From (5.50) we obtain

$$\mu_2(t) = \psi_2(t) - \psi_1(t)d(t) \tag{5.51}$$

on $\mathcal{O}(\tau^+)$, where $\psi_2(t) - \psi_1(t)d$ is a strictly decreasing function (because ψ_2 is non-increasing, ψ_1 is a positive constant and d is strictly increasing), which means $\mu_2(\tau^+) > 0$, because μ_2 is non-negative everywhere.

Because the function $\psi_1 - \nu_1$ is continuous everywhere and ν_1 is non-increasing, Equality (5.11) implies that the function ψ_1 is also non-increasing. This together with $\mu_2(\tau^+) > 0$ implies that the right-hand side of (5.49) is positive. Because d was assumed to be positive, we have

$$\psi_1(\tau^-) - \psi_1(\tau^+) - \mu_1(\tau^-) > 0 \tag{5.52}$$

or, equivalently

$$\psi_1(\tau^-) - \mu_1(\tau^-) > \psi_1(\tau^+) - \mu_1(\tau^+) \tag{5.53}$$

due to $\mu_1(\tau^+) = 0$. Then, using (5.8) we obtain

$$\frac{\partial f}{\partial r_1}(\tau, k^*(\tau), r_1^*(\tau^-), r_2^*(\tau^-)) > \frac{\partial f}{\partial r_1}(\tau, k^*(\tau), r_1^*(\tau^+), r_2^*(\tau^+)). \tag{5.54}$$

We have used that ψ_k is positive and it is continuous everywhere because $\psi_k - \nu_k$ is continuous and $\nu_k = 0$ for all t since we have assumed that $k^*(t) > 0$ for all $t \geq 0$ (recall that we only deal with weakly interior solutions). From (5.54) we obtain

$$(1-\alpha)\left(\frac{k^*(\tau)}{r_1^*(\tau^-) + d(\tau) r_2^*(\tau^-)}\right)^{\alpha} > (1-\alpha)\left(\frac{k^*(\tau)}{r_1^*(\tau^+) + d(\tau) r_2^*(\tau^+)}\right)^{\alpha} \tag{5.55}$$

which can be simplified to

$$d(\tau) r_2^*(\tau^-) < r_1^*(\tau^+) \tag{5.56}$$

due to $k^*(t) > 0$ for all t, $r_1^*(\tau^-) = 0$ and $r_2^*(\tau^+) = 0$.

Now we make use of condition (5.24) which states that the Hamiltonian (5.24) is continuous in t for all $t \geq 0$. This implies

$$\begin{aligned} &\psi_k(\tau) k^*(\tau)^{\alpha} (d(\tau) r_2^*(\tau^-))^{1-\alpha} - \psi_2(\tau^-) r_2^*(\tau^-) = \\ &= \psi_k(\tau) k^*(\tau)^{\alpha} r_1^*(\tau^+)^{1-\alpha} - \psi_1(\tau^+) r_1^*(\tau^+). \end{aligned} \tag{5.57}$$

Again, we have used that $r_1^*(\tau^-) = 0$, $r_2^*(\tau^+) = 0$, ψ_k and k^* are continuous everywhere and also c^* is continuous everywhere because we have assumed that c^* is regular and hence constant (see Remark 3.3). From (5.8) and (5.9) we have

$$\psi_k(\tau) \frac{\partial f}{\partial r_1}(\tau, k^*(\tau), r_1^*(\tau^+), r_2^*(\tau^+)) = \psi_1(\tau^+) \tag{5.58}$$

and

$$\psi_k(\tau) \frac{\partial f}{\partial r_2}(\tau, k^*(\tau), r_1^*(\tau^-), r_2^*(\tau^-)) = \psi_2(\tau^-) \tag{5.59}$$

because $\mu_1(\tau^+) = 0$ and $\mu_2(\tau^-) = 0$. Putting the last two equalities into (5.57) we obtain

$$\begin{aligned} \psi_k(\tau) k^*(\tau)^{\alpha} \Big((d(\tau) r_2^*(\tau^-))^{1-\alpha} - r_1^*(\tau^+)^{1-\alpha} \Big) &= \psi_2(\tau^-) r_2^*(\tau^-) - \psi_1(\tau^+) r_1^*(\tau^+) = \\ = \psi_k(\tau)(1-\alpha) k^*(\tau)^{\alpha} (d(\tau) r_2^*(\tau^-))^{1-\alpha} &- \psi_k(\tau)(1-\alpha) k^*(\tau)^{\alpha} r_1^*(\tau^+)^{1-\alpha}. \end{aligned}$$

This simplifies to

$$d(\tau) r_2^*(\tau^-) = r_1^*(\tau^+) \tag{5.60}$$

using that ψ_k and k^* are positive everywhere. However, (5.60) contradicts (5.56). Lemma 5.1 is proved. □

Proof of Theorem 5.1. We recall (see Definition 7.1 (i) and the text below it) that the optimal paths of any control variable is supposed to have one-sided limits everywhere. At discountinuiuty points, its value is equal to its left-hand limit and its value at 0 is equal to its right-hand limit. Hence, one has $r_i^*(0) := \lim_{t \to 0^+} r_i^*(t)$, $i = 1, 2$. We can distinguish four cases:

(a) If $r_1^*(0) > 0$ and $r_2^*(0) = 0$, then there exists an $\varepsilon > 0$ such that $r_1^*(t) > 0$ on $\langle 0, \varepsilon)$. Lemma 5.1 (ii) then implies that $r_2^*(t) = 0$ on $\langle 0, \varepsilon)$. On the other hand, it follows from Lemma 5.1 (i) that $r_2^*(t)$ cannot be zero everywhere on $(0, \infty)$. Since it is assumed to be piecewise continuous, it has be positiveon a non-trivial interval. Recall that the solution was assumed to be weakly interior, hence r_1^* and r_2^* cannot vanish simultaneously. In addition, Lemma 5.1 (ii) implies that r_1^* and r_2^* cannot be simultaneously positive on a non-trivial interval. As a result, there exists T such that $r_1^*(T^-) > 0$, $r_2^*(T^-) = 0$ and $r_1^*(T^+) = 0$, $r_2^*(T^+) > 0$.

   Now we prove that $r_1^*(t) = 0$ and $r_2^*(t) > 0$ everywhere on (T, ∞). Indeed, suppose that there exists a non-trivial interval I such that $r_2^*(t) = 0$ on I. Since the solution is weakly interior, we have that $r_1^*(t) > 0$ everwhere on I. However, then there exists τ such that $r_1^*(\tau^-) = 0$, $r_2^*(\tau^-) > 0$ and $r_1^*(\tau^+) > 0$, $r_2^*(\tau^+) = 0$, which is a contradiction with Lemma 5.1 (iii).

(b) If $r_1^*(0) = 0$ and $r_2^*(0) > 0$, then there exists an $\varepsilon > 0$ such that $r_2^*(t) > 0$ on $\langle 0, \varepsilon)$ and Lemma 5.1 (ii) implies $r_1^*(t) = 0$ on $\langle 0, \varepsilon)$. Analogously to the previous case, it can be proved that $r_1^*(t) = 0$ and $r_2^*(t) > 0$ everywhere on $\langle 0, \infty)$.

(c) If $r_1^*(0) > 0$ and $r_2^*(0) > 0$, then there exists an $\varepsilon > 0$ such that $r_1^*(t) > 0$ and $r_2^*(t) > 0$ on $\langle 0, \varepsilon)$, which is a contradiction with Lemma 5.1 (ii).

(d) The case $r_1^*(0) = 0$ and $r_2^*(0) = 0$ is a contradiction with the assumption that the solution is weakly interior, because this assumption implies

$$r_1^*(0) + d(0) r_2^*(0) = r_1^*(0^+) + d(0) r_2^*(0^+) = 0. \tag{5.61}$$

The proof of the theorem is completed. □

Remark 5.2. (Continuity of the value of net investment.) Recall that at the beginning of this chapter, we have cited Dixit et al. (1980) who stated a conjecture that the value of net investment exhibits a jump at the moment of depletion of one of the exhaustible resources. Note that in our model, this conjecture is not valid. The reason is that according to Lemma 7.5, the Hamiltonian stays constant along the solutions even at junction times. However, the Hamiltonian expresses the value of net investment in this model.

5.3 Hartwick's result for this model

Consider now Problem (5.1), but let the maximin objective function be replaced by the discounted utility objective function:

$$\begin{aligned}
&\max_{\{c,r_1,r_2\}} \int_0^\infty \pi(t)U(c(t))\,\mathrm{d}t,\\
&\dot{k}(t) = k^\alpha(r_1(t)+d(t)r_2(t))^{1-\alpha} - c(t), \quad k(0)=k_0>0 \text{ given},\\
&\dot{s}_1(t) = -r_1(t), \quad s_1(0)=\bar{s}_1>0 \text{ given},\\
&\dot{s}_2(t) = -r_2(t), \quad s_2(0)=\bar{s}_2>0 \text{ given},\\
&k(t)\ge 0, \quad s_1(t)\ge 0, \quad s_2(t)\ge 0,\\
&r_1(t)\ge 0, \quad r_2(t)\ge 0, \quad c(t)\ge 0,
\end{aligned} \tag{5.62}$$

where d satisfies (A6).

The necessary conditions of optimality for this problem follow from Theorems 7.6 and 7.9 and from Lemma 7.5.

In particular, Theorem 7.9 implies that if $(k^*, s_1^*, s_2^*, c^*, r_1^*, r_2^*)$ is an admissible solution to Problem (5.62) which fulfills the necessary conditions of optimality given in Theorem 7.6 together with $(\psi^0, \psi_k, \psi_1, \psi_2, \mu_c, \mu_1, \mu_2, \nu_k, \nu_1, \nu_2)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi^0\pi U(c^*) + \psi_k\dot{k}^* + \psi_1\dot{s}_1^* + \psi_2\dot{s}_2^*\right) = \psi^0\dot{\pi}U(c^*) + \psi_k\frac{\partial f}{\partial t}(t,k^*,r_1^*,r_2^*) \tag{5.63}$$

for all $t \ge 0$ with the possible exceptions of junction times or discontinuity points of c^*, r_1^* or r_2^*. In addition, Lemma 7.5 implies that the Hamiltonian is continuous everywhere. It is important in this context that all these propositions remain valid for infinite horizon models with pure state constraints.

The proof of Hartwick's result for Problem (2.8) (Theorem 4.1) was based on condition (3.9) and continuity of the Hamiltonian. Hence, it is crucial for derivation of Hartwick's result for Problem (5.62) that the condition (5.63) (which is an analogous condition to (3.9)) together with continuity of the Hamiltonian are valid. Therefore we can formulate and prove the following theorem:

Theorem 5.2 (Hartwick's result, perfectly substitutable exhaustible resources)**.** [3] *Let*

[3] Comparing to the results in Chapter 4, this theorem extend the formulation of Hartwick's result for (a) non-autonomous models and (b) models with binding pure state constraints. Hartwick's result for non-autonomous models is treated by Farzin (2006), who also describes the role of pure time effect on production (clarified below). However, to our knowledge there is no formulation of Hartwick's result taking into account binding pure state constraints.

$(k^*, s_1^*, s_2^*, c^*, r_1^*, r_2^*)$ be an admissible weakly interior solution to Problem (5.62) which fulfills the necessary conditions of optimality together with $(\psi^0, \psi_k, \psi_1, \psi_2, \mu_c, \mu_1, \mu_2, \nu_k, \nu_1, \nu_2)$, where $\psi^0 = 1$. In addition, suppose that

$$\psi_k(t)\dot{k}^*(t) + \psi_1(t)\dot{s}_1^*(t) + \psi_2(t)\dot{s}_2^*(t) = \int_0^t \psi_k(\tau)\frac{\partial f}{\partial t}(\tau, k^*(\tau), r_1^*(\tau), r_2^*(\tau))\,d\tau \quad (5.64)$$

(Hartwick's rule) for all $t \geq 0$. Then $U(c) \equiv$ const. for all $t \geq 0$.

Proof. If Equality (5.64) is satisfied, from (5.63) we obtain

$$\psi^0\pi(t)\frac{\mathrm{d}U}{\mathrm{d}t}(c^*(t)) = 0. \quad (5.65)$$

Assumption (A4) and $\psi^0 = 1$ then yields that $U(c^*(t))$ is piecewise constant. Furthermore, the Hamiltonian

$$H(t, k^*, s_1^*, s_2^*, c^*, r_1^*, r_2^*, \psi^0, \psi_k, \psi_1, \psi_2) = \psi^0\pi U(c^*) + \psi_k\dot{k}^* + \psi_1\dot{s}_1^* + \psi_2\dot{s}_2^* \quad (5.66)$$

is continuous everywhere in t and it follows from Equality (5.64) that

$$\psi_k(t)\dot{k}^*(t) + \psi_1(t)\dot{s}_1^*(t) + \psi_2(t)\dot{s}_2^*(t) \quad (5.67)$$

is continuous in t. As a result, $\psi^0\pi(t)U(c^*(t))$ is continuous, hence $U(c^*(t))$ is constant. □

Remark 5.3. (Economic interpretation.) Note that Equality (5.64) representing Hartwick's rule for the model (5.62) can be rewritten to

$$\psi_k(t)\dot{k}^*(t) + \psi_1(t)\dot{s}_1^*(t) + \psi_2(t)\dot{s}_2^*(t) =$$

$$= \int_0^t \psi_k(\tau)(1-\alpha)(k^*(\tau))^\alpha(r_1^*(\tau) + d(\tau)r_2^*(\tau))^{-\alpha}r_2^*(\tau)\dot{d}(\tau)\,\mathrm{d}\tau =$$

$$= \int_0^t \psi_k(\tau)\frac{\partial f}{\partial r_2}\frac{\dot{d}(\tau)}{d(\tau)}r_2^*(\tau)\,\mathrm{d}\tau. \quad (5.68)$$

This equality states that the total value of net investment to all capital stocks priced at shadow values at time t has to be equal to the total change of the production solely due to the time (i.e. time effect on production[4]) over the time interval $(0, t)$ priced at shadow value. This pure time effect on production equals to the actual unit marginal production multiplied by the stock of the exhaustible capital r_2 and further multiplied by the growth rate of productivity of r_2.

[4]Cf. Farzin (2006) [p. 525].

Chapter 6

Conservation laws

6.1 Conservation laws in the sustainability framework

So far, we have assumed that the concept of sustainability of economic growth requires maintaining some level of utility that can be sustained for all future generations. In other words, it is set a priori which kind of quantity has to be preserved. In this chapter, we present another kind of analysis that might be interesting when examining sustainability of economies in models of economic growth. In this type of analysis, we explore the structure of the particular model and search for some conservation laws, i.e. for quantities that can be constant along trajectories. We further examine how these quantities are interpreted in the sustainability framework.

The analysis will be based on Noether's theorem (see Noether (1918)). Roughly speaking, Noether's theorem states that if a dynamic system has a continuous symmetry property, then there exist corresponding quantities whose values are conserved in time. In optimal control framework, the interpretation of Noether's theorem might be as follows: If an optimal control model has some kind of symmetry, then there exists an invariant quantity, i.e. a quantity that is constant along solutions of necessary conditions of optimality. A more precise formulation of this result is given later in this chapter.

Using conservation laws (or invariant quantities) along paths satisfying necessary conditions of optimality for solutions to DHS model was first proposed by Sato and Kim (2002). However, they do not directly refer to Noether's theorem. Instead they consider an autonomous DHS model where they only find one simple conservation law which states that the Hamiltonian is constant along optimal paths.

Recently, the concept of using conservation laws based on Noether's theorem was broadly studied by Martinet and Rotillon (2007). However, their results are rather sceptical: They tried to derive conservation laws in a general version of DHS model with utility objective function and with a general discount factor and exogenous growth of technology. They require that the results obtained should be valid without imposing any constraints on the utility function and the production function. Under these assumptions they were able to derive some conservation laws, but only under rather restricting assumptions on the discount factor and exogenous growth.

One of the reasons why these assumptions are needed is that they use a theory based on invariant problems. However, Torres (2004a) showed that it is possible to find conservation laws even in case when the problem is not fully invariant, but only quasi-invariant.

Hence, the motivation of this chapter is to explore the usage of ideas presented by Torres in models (2.8) and (2.11). In particular, we use the general theoretical results on conservation laws in the context of an invariant optimal control problem presented by Torres (2002) and their further extensions (separately) for quasi-invariant optimal control problems (Torres (2004a)) and for problems involving mixed constraints on both state and control variables (Torres (2004b)).

We combine and extend the above-mentioned results formulated by Torres in order to be applicable to our problem. Then, we extend the results introduced by Martinet and Rotillon (2007) in several ways:

- We use a concept of "quasi-invariant optimal control problems" introduced by Torres (2004a). Using this concept, we are able to find other invariant quantities not found previously by Martinet and Rotillon (2007).

- We include population growth (see Asheim et al. (2007) and Mitra (2008)) and exhaustible resources with an amenity value (d'Autume and Schubert (2008)).

- We consider not only models with the discounted utility criterion, but also models with the maximin objective function. In addition, we show that both types of objective criterions lead to the same conservation law.

- Instead of trying to find results which remain valid for rather general utility and production function, we assume that the production function is homogeneous of degree one. This is a reasonable assumption which allows us to extend the set of possible conservation laws.

Similarly to Martinet and Rotillon (2007), we only consider paths where all control and state variables are positive, hence no non-negativity constraint is binding.

6.2 Theory

Following ideas presented by Torres (2002), Torres (2004a) and Torres (2004b), we first provide precise definitions of invariant and quasi-invariant optimal control problems.

Consider a standard optimal control problem in the following form:

$$\begin{aligned}
&\max_{\{u(t)\}} \int_{t_0}^{t_1} f^0(t,x(t),u(t))\,\mathrm{d}t, \qquad t_0, t_1 \text{ fixed,} \\
&\dot{x}(t) = f(t,x(t),u(t)), \\
&g(x(t),u(t)) \geq 0,
\end{aligned} \tag{6.1}$$

with any type of initial and terminal conditions, where f, f^0 and g are C^1-functions in each variable.

The definition of an invariant problem is as follows:

Definition 6.1. *Let there exist an $\varepsilon > 0$ and a C^2-smooth one-parameter family of transformations $(t,x,u) \to (\tilde{t},\tilde{x},\tilde{u})$ depending on a parameter $\xi \in (-\varepsilon,\varepsilon)$ where*

$$\begin{aligned}
\tilde{t}(t,x,u,\xi) &: \langle t_0,t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r \times (-\varepsilon,\varepsilon) \to \mathbb{R}, \\
\tilde{x}(t,x,u,\xi) &: \langle t_0,t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r \times (-\varepsilon,\varepsilon) \to \mathbb{R}^n, \\
\tilde{u}(t,x,u,\xi) &: \langle t_0,t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r \times (-\varepsilon,\varepsilon) \to \mathbb{R}^r
\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}
\tilde{t}(t,x,u,0) &= t, \\
\tilde{x}(t,x,u,0) &= x, \\
\tilde{u}(t,x,u,0) &= u
\end{aligned} \tag{6.3}$$

for all $(t,x,u) \in \langle t_0,t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r$. In addition, let the following conditions be satisfied:

(i) $f^0(t,x,u) = f^0(\tilde{t}(t,x,u,\xi),\tilde{x}(t,x,u,\xi),\tilde{u}(t,x,u,\xi))\dfrac{d\tilde{t}}{dt}(t,x,u,\xi) + o(\xi)$,

(ii) $\dfrac{d\tilde{x}}{dt}(t,x,u,\xi) = f(\tilde{t}(t,x,u,\xi),\tilde{x}(t,x,u,\xi),\tilde{u}(t,x,u,\xi))\dfrac{d\tilde{t}}{dt}(t,x,u,\xi) + o(\xi)$,

(iii) $g(x,u) = g(\tilde{x}(t,x,u,\xi), \tilde{u}(t,x,u,\xi))\dfrac{d\tilde{t}}{dt}(t,x,u,\xi) + o(\xi)$,

for all $(t,x,u,\xi) \in \langle t_0, t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r \times (-\varepsilon, \varepsilon)$, *where* $o(\xi)$ *is a differentiable function such that*

$$\lim_{\xi \to 0} \frac{o(\xi)}{\xi} = 0. \tag{6.4}$$

Problem (6.1) is then said to be invariant up to first-order terms in parameter ξ *under the transformation (6.2).*

The concept of invariance is rather strict, because it requires invariance in the objective function, in the state differential equation and in constraints as well. For our purposes, it will suffice to use a concept of quasi-invariance which releases the requirement of the invariance in the objective function or in constraints.

Definition 6.2. *Let there exist an* $\varepsilon > 0$ *and a* C^2*-smooth one-parameter family of transformations* $(t,x,u) \to (\tilde{t}, \tilde{x}, \tilde{u})$ *depending on a parameter* $\xi \in (-\varepsilon, \varepsilon)$ *which satisfies Definition 6.1 with conditions (i) and (iii) modified as follows (arguments for* $\tilde{t}$, $\tilde{x}$, $\tilde{u}$ *are dropped):*

(i') *there exist a* C^1*-function* $\Gamma(t,x,u,\xi) : \langle t_0, t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ *such that*

$$f^0(t,x,u) + \Gamma(t,x,u,\xi) = f^0(\tilde{t}, \tilde{x}, \tilde{u})\frac{d\tilde{t}}{dt} + o(\xi) \tag{6.5}$$

for all $(t,x,u,\xi) \in \langle t_0, t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r \times (-\varepsilon, \varepsilon)$ *and*

(iii') *there exist a* C^1*-function* $\Delta(t,x,u,\xi) : \langle t_0, t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ *such that*

$$g(x,u) + \Delta(t,x,u,\xi) = g(\tilde{x}, \tilde{u})\frac{d\tilde{t}}{dt} + o(\xi) \tag{6.6}$$

for all $(t,x,u,\xi) \in \langle t_0, t_1\rangle \times \mathbb{R}^n \times \mathbb{R}^r \times (-\varepsilon, \varepsilon)$.

Problem (6.1) is then said to be quasi-invariant up to a residual term Γ *in objective function, a residual term* Δ *in mixed constraints and first-order terms in parameter* ξ *under the transformation (6.2).*

Generally speaking, Noether's theorem states that if Problem (6.1) is invariant, then there exists a quantity which stays constant along the paths of solutions to the necessary conditions of optimality. Torres (2002) found this quantity in case of invariant problem without constraints. This result was further extended to quasi-invariant problems without constraints by Torres (2004a) [Theorem 5.1, p. 105] and to constrained invariant problems by Torres (2004b) [Theorem 3.1, p. 3]. In the next theorem, we combine these results to obtain a new Noether-type theorem for constrained quasi-invariant problems. Moreover, we will see in Remark 6.2 that this result can be extended to more general optimal control problems.

Theorem 6.1. *If Problem (6.1) is quasi-invariant up to a residual term Γ in objective function, a residual term Δ in constraints and first-order terms in parameter ξ with respect to the transformation $(t,x,u) \to (\tilde{t},\tilde{x},\tilde{u})$, then for any admissible solution $(x(t),u(t))$ to Problem (6.1) satisfying the Pontryagin necessary conditions together with (ψ^0,ψ,μ), the following equality is satisfied:*

$$\frac{d}{dt}\left(L(t,x,u,\psi^0,\psi,\mu)\left[\frac{\partial \tilde{t}}{\partial \xi}(t,x,u,\xi)\right]_{\xi=0} - \psi^T\left[\frac{\partial \tilde{x}}{\partial \xi}(t,x,u,\xi)\right]_{\xi=0}\right) - \\ - \psi^0\left[\frac{\partial \Gamma}{\partial \xi}(t,x,u,\xi)\right]_{\xi=0} - \mu^T\left[\frac{\partial \Delta}{\partial \xi}(t,x,u,\xi)\right]_{\xi=0} = 0 \tag{6.7}$$

for all $t \in \langle t_0, t_1\rangle$ possibly except the discontinuity points of $u(t)$, where L is the Lagrangian defined in accordance with (7.9) as follows:

$$L(t,x,u,\psi^0,\psi,\mu) = \psi^0 f^0(t,x,u) + \psi^T f(t,x,u) + \mu^T g(x,u). \tag{6.8}$$

Proof. Denote

$$\tilde{f}^0(t,x,u,\xi) := f^0(\tilde{t}(t,x,u,\xi),\tilde{x}(t,x,u,\xi),\tilde{u}(t,x,u,\xi)). \tag{6.9}$$

In this notation, condition (6.3) implies

$$\tilde{f}^0(t,x,u,0) = f^0(\tilde{t}(t,x,u,0),\tilde{x}(t,x,u,0),\tilde{u}(t,x,u,0)) = f^0(t,x,u) \tag{6.10}$$

and Equality (6.5) has the following form:

$$f^0(t,x,u) = \tilde{f}^0(t,x,u,\xi)\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi) - \Gamma(t,x,u,\xi) + o(\xi). \tag{6.11}$$

In addition, (6.4) implies

$$\left[\frac{\mathrm{d}o}{\mathrm{d}\xi}(\xi)\right]_{\xi=0} = 0 \tag{6.12}$$

and we also have

$$\left[\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi)\right]_{\xi=0} = \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,0) = \frac{\mathrm{d}t}{\mathrm{d}t} = 1 \tag{6.13}$$

since ξ does not depend on t.

Now, we differentiate Equality (6.11) totally w.r.t. ξ and take $\xi = 0$ (for simplicity, we drop the arguments (t,x,u) or (t,x,u,ξ) in the last two rows):

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}f^0}{\mathrm{d}\xi}(t,x,u)\right]_{\xi=0} \\
&= \left[\frac{\mathrm{d}\tilde{f}^0}{\mathrm{d}\xi}(t,x,u,\xi)\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi) + \tilde{f}^0(t,x,u,\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi) - \frac{\partial \Gamma}{\partial \xi}(t,x,u,\xi)\right]_{\xi=0} \\
&= \left[\frac{\mathrm{d}\tilde{f}^0}{\mathrm{d}\xi}(t,x,u,\xi)\right]_{\xi=0} + f^0(t,x,u)\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi)\right]_{\xi=0} - \left[\frac{\partial \Gamma}{\partial \xi}(t,x,u,\xi)\right]_{\xi=0} \\
&= \left[\frac{\partial \tilde{f}^0}{\partial t}\frac{\partial \tilde{t}}{\partial \xi} + \frac{\partial \tilde{f}^0}{\partial x}\frac{\partial \tilde{x}}{\partial \xi} + \frac{\partial \tilde{f}^0}{\partial u}\frac{\partial \tilde{u}}{\partial \xi}\right]_{\xi=0} + f^0\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{t}}{\partial \xi}\right]_{\xi=0} - \left[\frac{\partial \Gamma}{\partial \xi}\right]_{\xi=0} \\
&= \frac{\partial f^0}{\partial t}\left[\frac{\partial \tilde{t}}{\partial \xi}\right]_{\xi=0} + \frac{\partial f^0}{\partial x}\left[\frac{\partial \tilde{x}}{\partial \xi}\right]_{\xi=0} + \frac{\partial f^0}{\partial u}\left[\frac{\partial \tilde{u}}{\partial \xi}\right]_{\xi=0} + f^0\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{t}}{\partial \xi}\right]_{\xi=0} - \left[\frac{\partial \Gamma}{\partial \xi}\right]_{\xi=0}.
\end{aligned} \tag{6.14}$$

We have used (6.11) and (6.12) in the second equality and (6.10), (6.13) in the third one.

Analogously, we denote

$$\tilde{f}(t,x,u,\xi) := f(\tilde{t}(t,x,u,\xi), \tilde{x}(t,x,u,\xi), \tilde{u}(t,x,u,\xi)). \tag{6.15}$$

Condition (ii) in Definition 6.1 can then be rewritten to

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}(t,x,u,\xi) = \tilde{f}(t,x,u,\xi)\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi) + o(\xi). \tag{6.16}$$

If we differentiate this equality totally with respect to ξ and take $\xi = 0$, we obtain

$$\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{x}}{\partial \xi}\right]_{\xi=0} &= \left[\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi}(t,x,u,\xi)\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi) + \tilde{f}(t,x,u,\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi)\right]_{\xi=0} \\
&= \left[\frac{\partial \tilde{f}}{\partial t}\frac{\partial \tilde{t}}{\partial \xi} + \frac{\partial \tilde{f}}{\partial x}\frac{\partial \tilde{x}}{\partial \xi} + \frac{\partial \tilde{f}}{\partial u}\frac{\partial \tilde{u}}{\partial \xi}\right]_{\xi=0}\left[\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right]_{\xi=0} + \left[\tilde{f}\right]_{\xi=0}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{t}}{\partial \xi}\right]_{\xi=0} \\
&= \frac{\partial f}{\partial t}\left[\frac{\partial \tilde{t}}{\partial \xi}\right]_{\xi=0} + \frac{\partial f}{\partial x}\left[\frac{\partial \tilde{x}}{\partial \xi}\right]_{\xi=0} + \frac{\partial f}{\partial u}\left[\frac{\partial \tilde{u}}{\partial \xi}\right]_{\xi=0} + f\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{t}}{\partial \xi}\right]_{\xi=0}.
\end{aligned} \tag{6.17}$$

Again, we have used (6.13), (6.12) and

$$\left[\tilde{f}(t,x,u,\xi)\right]_{\xi=0} = f(t,x,u). \tag{6.18}$$

Finally, we use the notation

$$\tilde{g}(t,x,u,\xi) := g(\tilde{x}(t,x,u,\xi),\tilde{u}(t,x,u,\xi)) \tag{6.19}$$

to rewrite Equality (6.6) into the following form:

$$g(x,u) = \tilde{g}(t,x,u,\xi)\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi) - \Delta(t,x,u,\xi) + o(\xi). \tag{6.20}$$

By differentiating this equality totally with respect to ξ and taking $\xi = 0$ we obtain

$$\begin{aligned} 0 &= \left[\frac{\mathrm{d}g}{\mathrm{d}\xi}(x,u)\right]_{\xi=0} \\ &= \left[\frac{\mathrm{d}\tilde{g}}{\mathrm{d}\xi}(t,x,u,\xi)\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi) + \tilde{g}(t,x,u,\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}(t,x,u,\xi) - \frac{\partial\Delta}{\partial\xi}(t,x,u,\xi)\right]_{\xi=0} \\ &= \frac{\partial g}{\partial x}\left[\frac{\partial\tilde{x}}{\partial\xi}\right]_{\xi=0} + \frac{\partial g}{\partial u}\left[\frac{\partial\tilde{u}}{\partial\xi}\right]_{\xi=0} + g\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{t}}{\partial\xi}\right]_{\xi=0} - \left[\frac{\partial\Delta}{\partial\xi}\right]_{\xi=0}. \end{aligned} \tag{6.21}$$

When we add Equality (6.14) multiplied by ψ^0, Equality (6.17) multiplied by ψ^T and Equality (6.21) multiplied by μ^T, we obtain after some rearranging of terms that

$$\left(\psi^0\frac{\partial f^0}{\partial t} + \psi^T\frac{\partial f}{\partial t}\right)\left[\frac{\partial\tilde{t}}{\partial\xi}\right]_{\xi=0} + \left(\psi^0\frac{\partial f^0}{\partial x} + \psi^T\frac{\partial f}{\partial x} + \mu^T\frac{\partial g}{\partial x}\right)\left[\frac{\partial\tilde{x}}{\partial\xi}\right]_{\xi=0} + \tag{6.22a}$$

$$+\left(\psi^0\frac{\partial f^0}{\partial u} + \psi^T\frac{\partial f}{\partial u} + \mu^T\frac{\partial g}{\partial u}\right)\left[\frac{\partial\tilde{u}}{\partial\xi}\right]_{\xi=0} + \tag{6.22b}$$

$$+\left(\psi^0 f^0 + \psi^T f + \mu^T g\right)\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{t}}{\partial\xi}\right]_{\xi=0} - \tag{6.22c}$$

$$-\psi^T\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{x}}{\partial\xi}\right]_{\xi=0} - \psi^0\left[\frac{\partial\Gamma}{\partial\xi}\right]_{\xi=0} - \mu^T\left[\frac{\partial\Delta}{\partial\xi}\right]_{\xi=0} = 0. \tag{6.22d}$$

We have assumed that $(x(t),u(t))$ is an admissible solution to Problem (6.1) satisfying the necessary conditions of optimality. Hence, we obtain from Theorems 7.2 and 7.3[1] that at continuity points of u, the following simplifications can be done:

[1]Note that we use here the necessary conditions by Feichtinger and Hartl (1986) whereas we have used the necessary conditions by Seierstad and Sydsæter (1987) in other part of the thesis. However, the costate variables are the same as for both sets of necessary conditions in this case since we do not have binding pure state constraints.

- the first term in (6.22a) equals to $\frac{\mathrm{d}L}{\mathrm{d}t}\left[\frac{\partial\tilde{t}}{\partial\xi}\right]_{\xi=0}$ (implied by Theorem 7.3),
- the second term in (6.22a) equals to $-\dot{\psi}^T\left[\frac{\partial\tilde{x}}{\partial\xi}\right]_{\xi=0}$ (implied by condition (vi) in Theorem 7.2),
- the term in (6.22b) vanishes (due to condition (iv) in Theorem 7.2) and
- the term in (6.22c) equals to $L\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\xi}\right]_{\xi=0}$, where L is the Lagrangian defined by (6.8).

Therefore, we can write

$$\frac{\mathrm{d}L}{\mathrm{d}t}\left[\frac{\partial\tilde{t}}{\partial\xi}\right]_{\xi=0}-\dot{\psi}^T\left[\frac{\partial\tilde{x}}{\partial\xi}\right]_{\xi=0}+L\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{t}}{\partial\xi}\right]_{\xi=0}-$$

$$-\psi^T\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{x}}{\partial\xi}\right]_{\xi=0}-\psi^0\left[\frac{\partial\Gamma}{\partial\xi}\right]_{\xi=0}-\mu^T\left[\frac{\partial\Delta}{\partial\xi}\right]_{\xi=0}=$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}\left(L\left[\frac{\partial\tilde{t}}{\partial\xi}\right]_{\xi=0}-\psi^T\left[\frac{\partial\tilde{x}}{\partial\xi}\right]_{\xi=0}\right)-\psi^0\left[\frac{\partial\Gamma}{\partial\xi}\right]_{\xi=0}-\mu^T\left[\frac{\partial\Delta}{\partial\xi}\right]_{\xi=0}=0. \quad \square \tag{6.23}$$

Remark 6.1. Note that (6.7) is actually not a conservation law, because we have not found a quantity which stays constant along the trajectories. We have just employed the same technique which are used by Torres (2004a) and Torres (2004b) to find conservation laws in simpler problems. However, if $\Gamma\equiv 0$ and $\Delta\equiv 0$, then Theorem 6.1 implies the following conservation law:

$$L(t,x,u,\psi^0,\psi,\mu)\left[\frac{\partial\tilde{t}}{\partial\xi}(t,x,u,\xi)\right]_{\xi=0}-\psi(t)^T\left[\frac{\partial\tilde{x}}{\partial\xi}(t,x,u,\xi)\right]_{\xi=0}=\text{const.} \tag{6.24}$$

on $\langle t_0,t_1\rangle$.

Remark 6.2. It is easy to show that the proof of Theorem 6.1 remains valid without any change even if we extend Problem (6.1) to an infinite time horizon problem or to a problem with free boundary conditions or with a scrap value function, because we have used neither the transversality conditions for costate variables nor the boundary conditions for the state variables in the proof.

6.3 Conservation laws in model with a discounted utility objective

Now we apply the theory introduced above to a model of an economy with renewable and exhaustible resources, with a population growth and a linear amortization function in the following form (argument t is dropped):

$$\begin{aligned}
&\max_{\{c,r\}} \int_0^\infty \pi U\left(c,s\right) \mathrm{d}t, \\
&\dot{k} = d f(k,r,n) - \delta k - c, \quad k(0) = k_0, \\
&\dot{s} = -r, \quad s(0) = s_0, \\
&\dot{n} = \vartheta n, \quad n(0) = n_0, \\
&k \geq 0, \quad s \geq 0, \\
&r \geq 0, \quad c \geq 0,
\end{aligned} \tag{6.25}$$

where $k_0 > 0$, $s_0 > 0$, $n_0 > 0$ and $\delta > 0$ are given vectors or constants and $d(t)$ and $\vartheta(t)$ are given positive continuous functions. We assume that the production function $f(k,r,n)$ is homogeneous of degree one.

To find quasi-invariant transformations, notice that the dynamics of the model is scale invariant (i.e. the state equations for $\dot{k}$, $\dot{s}$ and $\dot{n}$ remain true if we multiply all state and control variables by a positive constant). This leads to the following transformation:

$$\begin{aligned}
\tilde{k} &= (1+\xi)k, \\
\tilde{s} &= (1+\xi)s, \\
\tilde{n} &= (1+\xi)n, \\
\tilde{c} &= (1+\xi)c, \\
\tilde{r} &= (1+\xi)r, \\
\tilde{t} &= t.
\end{aligned} \tag{6.26}$$

For $\xi = 0$ we have an identity. It is straightforward to verify that conditions (ii) and (iii) in Definition 6.1 are satisfied. For example,

$$\begin{aligned}
\dot{\tilde{k}} &= (1+\xi)\dot{k} \\
&= (1+\xi)(d(t)f(k,r,n) - \delta k - c) \\
&= d(t)f((1+\xi)k,(1+\xi)r,(1+\xi)n) - \delta(1+\xi)k - (1+\xi)c \\
&= \left(d(\tilde{t})f(\tilde{k},\tilde{r},\tilde{n}) - \delta\tilde{k} - \tilde{c}\right)\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}.
\end{aligned}$$

Regarding condition (i'), we can use the MacLaurin polynomial of U to obtain

$$\begin{aligned}\pi(\tilde{t})U(\tilde{c},\tilde{s}) &= \pi(t)U(c+\xi c, s+\xi s)\\ &= \pi(t)\Big(U(c,s)+\frac{\partial U}{\partial c}(c,s)\,\xi c+\frac{\partial U}{\partial s}(c,s)\,\xi s+o(\xi)\Big)\\ &= \pi(t)U(c,s)+\Gamma(t,c,s,\xi)+o(\xi),\end{aligned}$$

where

$$\Gamma(t,c,s,\xi)=\xi\,\pi(t)\,\Big(\frac{\partial U}{\partial c}(c,s)c+\frac{\partial U}{\partial s}(c,s)s\Big).$$

Now, if (k,s,n,c,r) is an interior admissible solution satisfying the necessary conditions of optimality together with $(\psi^0,\psi_k,\psi_s,\psi_n)$, then Theorem 6.1 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\psi_k^T k+\psi_s^T s+\psi_n^T n\big)+\psi^0\pi\Big(\frac{\partial U}{\partial c}(c,s)c+\frac{\partial U}{\partial s}(c,s)s\Big)\equiv 0. \tag{6.27}$$

Note that this equality together with the assumption that the utility function U is increasing in both variables and the discount factor π is positive everywhere (see (A4)) implies that $\psi_k(t)^T k(t)+\psi_s(t)^T s(t)+\psi_n(t)^T n(t)$, which is the total present value of all reproducible, exhaustible and human capital goods at shadow prices, is decreasing in time.

Integrating (6.27) yields[2]

$$\begin{aligned}&\psi_k(t)^T k(t)+\psi_s(t)^T s(t)+\psi_n(t)^T n(t)\\ =\ &\text{const.}+\psi^0\int_t^\infty \pi(\tau)\Big(\frac{\partial U}{\partial c}(c(\tau),s(\tau))c(\tau)+\frac{\partial U}{\partial s}(c(\tau),s(\tau))s(\tau)\Big)\,\mathrm{d}\tau.\end{aligned} \tag{6.28}$$

We show that for particular type of the utility function, it is possible to obtain rather insightful interpretation of (6.27) and (6.28). We choose the utility function given by (2.6) and (2.7), where

$$U(c,s)=\frac{\bar{C}(c,s)^{1-\theta}}{1-\theta}, \tag{6.29}$$

where $\bar{C}(c,s)$ is a homogeneous function of degree 1, i.e. $\bar{C}(\zeta c,\zeta s)=\zeta\bar{C}(c,s)$ for all $\zeta\geq 0$. Differentiating this equality w.r.t. ζ and taking $\zeta=1$ yields

$$\bar{C}(c,s)=\frac{\partial\bar{C}}{\partial c}(c,s)c+\frac{\partial\bar{C}}{\partial s}(c,s)s \tag{6.30}$$

[2]The term in the left-hand side of (6.28) is continuous since we do not consider binding pure state constraints.

for all $(c,s) \in \mathbb{R}^n_{++} \times \mathbb{R}^m_{++}$.[3] Furthermore, assume $\psi^0 = 1$. We have

$$\begin{aligned}
\psi_k^T k + \psi_s^T s + \psi_n^T n &\overset{(6.28)}{=} \text{const.} + \int_t^\infty \pi(\tau)\bar{C}(c,s)^{-\theta} \left(\frac{\partial \bar{C}}{\partial c}(c,s)c + \frac{\partial \bar{C}}{\partial s}(c,s)s \right) \mathrm{d}\tau \\
&\overset{(6.30)}{=} \text{const.} + \int_t^\infty \pi\bar{C}(c,s)^{-\theta}\,\bar{C}(c,s)\,\mathrm{d}\tau \\
&\overset{(6.29)}{=} \text{const.} + (1-\theta)\int_t^\infty \pi U(c,s)\,\mathrm{d}\tau. \qquad (6.31)
\end{aligned}$$

This equality states that the difference between the total present value at time t of all capital goods at shadow prices and the total discounted value of utility over the remaining time horizon multiplied by $1-\theta$ is constant. Moreover, if the utility is constant, Equality (6.31) can be rewritten to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_k^T k + \psi_s^T s + \psi_n^T n) = -(1-\theta)\pi U(c,s). \qquad (6.32)$$

This equality establishes a relationship between the change of total value of capital goods and the discounted instantaneous utility.

6.4 Conservation laws in model with a maximin objective

It is possible to show that (6.28) remains a conservation law even if we replace the utility objective function in Problem (6.25) by a maximin objective. Recall that we still assume that the production function $f(k,r,n)$ is homogeneous of degree 1. Let Problem (6.25) be modified to the following problem (again, argument t is dropped):

$$\begin{aligned}
&\max_{\{c,r\}} w(0), \quad \text{where } t \in \langle 0,\infty), \\
&\dot{k} = df(k,r,n) - \delta k - c, \quad k(0) = k_0 > 0 \text{ given}, \\
&\dot{s} = -r, \quad s(0) = s_0 > 0 \text{ given}, \\
&\dot{n} = \vartheta n, \quad n(0) = n_0 > 0 \text{ given}, \\
&\dot{w} = 0, \quad w(0) \text{ free}, \\
&k \geq 0, \quad s \geq 0, \\
&r \geq 0, \quad c \geq 0, \\
&U(c,s) \geq w.
\end{aligned} \qquad (6.33)$$

[3] This result is known as Euler's theorem for homogeneous functions.

This is an infinite horizon problem with free initial condition on w and a scrap value function. We will only consider solutions for which the non-negativity constaints are not binding, i.e. $k > 0$, $s > 0$, $r > 0$, $c > 0$ (interior solutions). In this case, it is easy to show that transformation (6.26) together with $\tilde{w} = w$ is a quasi-invariant transformation, which satisfies conditions (i), (ii) and (iii') in Definitions 6.1 and 6.2 with $f^0 = 0$ and $g(k,s,w,c,r) = U(c,s) - w$. Indeed, condition (iii') has the form

$$\begin{aligned} U(\tilde{c},\tilde{s}) - \tilde{w} &= U(c+\xi c, s+\xi s) - w \\ &= U(c,s) + \frac{\partial U}{\partial c}(c,s)\,\xi c + \frac{\partial U}{\partial s}(c,s)\,\xi s + o(\xi) - w \\ &= U(c,s) - w + \Delta(c,s,\xi) + o(\xi), \end{aligned} \tag{6.34}$$

where

$$\Delta(c,s,\xi) = \xi\left(\frac{\partial U}{\partial c}(c,s)\,c + \frac{\partial U}{\partial s}(c,s)\,s\right) \tag{6.35}$$

is a residual term in constraint. Hence, Theorem 6.1 (used with $\Gamma \equiv 0$) can be applied to this problem for any admissible interior solution (k,s,n,w,c,r) satisfying the necessary conditions, according to Remark 6.1. Hence we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_k^T k + \psi_s^T s + \psi_n^T n\right) + \mu_w\left(\frac{\partial U}{\partial c}(c,s)\,c + \frac{\partial U}{\partial s}(c,s)\,s\right) \equiv 0, \tag{6.36}$$

where μ_w is the Lagrange multiplier associated with the constraint $U(c) \geq w$. Note that $\psi_w w$ is not included in the first term in parentheses since the transformation $\tilde{w} = w$ yields $\frac{\partial \tilde{w}}{\partial \xi} = 0$ for all ξ. Integrating (6.36) yields

$$\psi_k(t)^T k(t) + \psi_s(t)^T s(t) + \psi_n(t)^T n(t) = \tag{6.37}$$

$$= \text{const.} + \int_t^\infty \mu_w(\tau)\left(\frac{\partial U}{\partial c}(c(\tau),s(\tau))\,c(\tau) + \frac{\partial U}{\partial s}(c(\tau),s(\tau))\,s(\tau)\right)\mathrm{d}\tau. \tag{6.38}$$

We know that if c is constant and regular (i.e. $\mu_w > 0$ for all $t \geq 0$), then a solution to Pontryagin necessary conditions for (6.33) with $\psi^0 = 1$ satisfies Pontryaging necessary conditions for (6.25) with $\pi(t) = \mu_w(t)$ (see Theorem 4.5). Hence, (6.38) becomes (6.28).

6.5 An example with several conservation laws

Now we will show that if we assume specific forms of the discount factor, the utility function and the production function in Problem (6.25) taken with one-dimensional control and state variables, we may obtain even several conservation laws. In particular, assume that the discount factor is exponentially decreasing, the utility function is a utility function with constant relative risk aversion and the production function has the Cobb-Douglas form with constant returns to scale, i.e.

$$\pi(t) = e^{-\gamma t}, \qquad U(c) = \frac{c^{1-\theta}}{1-\theta} \qquad \text{and} \qquad f(k,r,n) = k^{\alpha} r^{\beta} n^{1-\alpha-\beta}, \tag{6.39}$$

where $\alpha \in (0,1)$, $\beta \in (0,1)$ and $\gamma > 0$ are given constants and $\alpha+\beta \leq 1$. In addition, we again assume that (k,s,n,c,r) is an interior solution to this problem satisfying necessary conditions of optimality.

Consider the following set of transformations:[4]

$$\begin{aligned}
\tilde{k} &= k\, e^{\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi}, \\
\tilde{s} &= s\, e^{A_1 \xi} + A_3 \xi, \\
\tilde{n} &= n\, e^{A_2 \xi}, \\
\tilde{c} &= c\, e^{\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi}, \\
\tilde{r} &= r\, e^{A_1 \xi}, \\
\tilde{t} &= t + \frac{1-\theta}{\gamma}\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi,
\end{aligned}$$

where A_1, A_2 and A_3 are arbitrary constants. Note that we obtain an identity if $\xi = 0$. It is easy to show that condition (i) in Definition 6.1 is satisfied:

$$\begin{aligned}
e^{-\gamma \tilde{t}} U(\tilde{c}) &= \frac{1}{1-\theta} e^{-\gamma t - (1-\theta)\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi} \left[c\, e^{\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi}\right]^{1-\theta} \\
&= e^{-\gamma t} \frac{c^{1-\theta}}{1-\theta} \\
&= e^{-\gamma t} U(c).
\end{aligned}$$

It is also straightforward to prove that condition (ii) is met for $\tilde{k}$:

$$\begin{aligned}
\tilde{k}^{\alpha} \tilde{r}^{\beta} \tilde{n}^{1-\alpha-\beta} - \delta \tilde{k} - \tilde{c} &= e^{\alpha\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi} k^{\alpha}\, e^{\beta A_1 \xi}\, r^{\beta}\, e^{(1-\alpha-\beta)A_2 \xi}\, n^{1-\alpha-\beta} - \\
&\quad - \delta e^{\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi} k - e^{\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi} c \\
&= e^{\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi} \left[k^{\alpha} r^{\beta} n^{1-\alpha-\beta} - \delta k - c\right] \\
&= e^{\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\xi} \dot{k} \\
&= \dot{\tilde{k}}
\end{aligned}$$

[4] These transformations were found using a Maple program by Gouveia and Torres (2005).

and also for $\tilde{s}$ and $\tilde{n}$:

$$\begin{aligned}\dot{\tilde{s}} &= e^{A_1\xi}\,\dot{s} = -e^{A_1\xi}\,r = -\tilde{r},\\ \dot{\tilde{n}} &= e^{A_2\xi}\,\dot{n} = e^{A_2\xi}\,\vartheta n = \vartheta\tilde{n}.\end{aligned}$$

Theorem 6.1 (used with $\Gamma = 0$ and $\Delta = 0$) then implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\psi_k k + A_1\psi_s s + A_3\psi_s + A_2\psi_n n - \right.$$
$$\left. - H(t,k,s,n,c,r,\psi^0,\psi_k,\psi_s,\psi_n)\,\frac{1-\theta}{\gamma}\left(\frac{\beta}{1-\alpha}A_1 + \frac{1-\alpha-\beta}{1-\alpha}A_2\right)\right) \equiv 0,$$

where H is the Hamiltonian for the given problem, which is equal to the Lagrangian, since we only consider interior solutions.

Hence, for interior solutions satisfying the necessary conditions of optimality such that both control variables are continuous everywhere and for (A_1, A_2, A_3) equal to $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, respectively, we obtain three independent conservation laws:

$$\begin{aligned}\frac{\beta}{1-\alpha}\,\psi_k k + \psi_s s - H(t,k,s,n,c,r,\psi^0,\psi_k,\psi_s,\psi_n)\,\frac{1-\theta}{\gamma}\,\frac{\beta}{1-\alpha} &\equiv \text{const.},\\ \frac{1-\alpha-\beta}{1-\alpha}\,\psi_k k + \psi_n n - H(t,k,s,n,c,r,\psi^0,\psi_k,\psi_s,\psi_n)\,\frac{1-\theta}{\gamma}\,\frac{1-\alpha-\beta}{1-\alpha} &\equiv \text{const.},\\ \psi_s &\equiv \text{const.}\end{aligned}$$

Chapter 7

Optimal control theory for standard problems with mixed and pure state constraints

We will now introduce the optimal control theory which provides the main background for the qualitative analysis of models of sustainable economic growth. We will also deal with the presence of non-negativity constraints on variables included in the models. As it has been already mentioned earlier, non-negativity restrictions on state variables such as the level of capital cause some difficulties in the formulation of necessary conditions of optimality. Generally, constraints involving state variables only (called *pure state constraints*) are more difficult to handle comparing to constraints involving both state and control variables (called *mixed constraints*).

Therefore, in this chapter we will present a detailed study of necessary conditions of optimality for a standard optimal control problem with mixed and pure state constraints on finite time horizon. In addition, we will extend this problem in several directions to create a framework needed to the rigorous analysis of the economic models.

The main objective of this chapter is to formulate the standard problem together with some basic definitions and assumptions and to introduce the necessary conditions of optimality. However, the formulation of these conditions is not unified in the literature. If we want to use results from several sources, we have to find the relationship between different formulations. In this thesis, we shall mainly utilize two distinct approaches: The first one was introduced by Seierstad and Sydsæter (1987) and the second one by Feichtinger and Hartl (1986). We demonstrate the main differencies between these two approaches on simple examples. Furthermore, we propose transformation rules for the multipliers involved in both approaches. Based on these transformations, we provide a detailed analysis of the relationship between both formulations of the necessary conditions. The main results are summarized in Theorems 7.4 and 7.5. This analysis, together with proof of several auxiliary results, is one of the original results of this thesis.

7.1 Formulation of the standard problem and basic definitions

Let us begin with the formulation of the standard optimal control problem on finite time horizon. One of the important characteristics of this problem is the presence of mixed and pure state constraints.

Let $x(t) = (x_1(t), \dots, x_n(t))^T$ be an n-dimensional vector of state variables and $u(t) = (u_1(t), \dots, u_r(t))^T$ an r-dimensional vector of control variables. The time horizon of the problem is $\langle t_0, t_1 \rangle$, where both t_0 and t_1 are given. In the standard problem, the initial values of $x(t)$ at t_0 are given. The terminal values at t_1 may be free or may by constrained by some equalities or non-strict inequalities. The dynamics of the system of state variables is described by a vector function $f : \langle t_0, t_1 \rangle \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$. The evolution of the system can be influenced by suitable choices of the control variables according to the objective function (criterion functional). This objective function depends on a real function $f^0 : \langle t_0, t_1 \rangle \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$. Finally, the feasible region of control and state variables is restricted by inequalities on these variables using functions $g : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^q$. We assume that f, f^0 and g are C^1-functions in each variable and h is a C^2-function.

The standard problem with mixed and pure state constraints can then be stated as follows:

$$\begin{aligned} &\max_{\{u(t)\}} \int_{t_0}^{t_1} f^0(t, x(t), u(t))\,\mathrm{d}t, \qquad t_0, t_1 \text{ fixed}, \\ &\text{subject to } \dot{x}(t) = f(t, x(t), u(t)), \qquad x(t_0) = x^0 \ (x^0 \text{ fixed in } \mathbb{R}^n), \end{aligned} \tag{7.1a}$$

the terminal conditions

$$\begin{array}{lll} x_i(t_1) = x_i^1, & i = 1, \dots, l & x_i^1 \text{ all fixed}, \\ x_i(t_1) \geq x_i^1, & i = l+1, \dots, m & x_i^1 \text{ all fixed}, \\ x_i(t_1) \text{ free}, & i = m+1, \dots, n & \end{array} \tag{7.1b}$$

and the constraints

$$g(x(t), u(t)) \geq 0, \tag{7.1c}$$

$$h(x(t)) \geq 0. \tag{7.1d}$$

Definition 7.1. *Any solution $(x(t),u(t))$, $t \in \langle t_0, t_1\rangle$ to Problem (7.1a) – (7.1d) is called an admissible solution if it satisfies following conditions:*

(i) $u(t)$ is piecewise continuous on $\langle t_0, t_1\rangle$, i.e. it has a finite number of discontinuity points and at each such a point it has finite one-sided limits,

(ii) $x(t)$ is a solution to the differential equation and initial condition in (7.1a) for the given $u(t)$ such that it is continuous and piecewise differentiable on $\langle t_0, t_1\rangle$ and together with $u(t)$ satisfies (7.1b) – (7.1d).

Note that the value of the objective function does not depend on values of $u(t)$ at discontinuity points. Let us decide that the value of $u(t)$ at a point of discontinuity equals to the left-hand limit at this point.[1] Moreover, the value of u at t_0 is equal to the right-hand limit.

Problem (7.1a) – (7.1d) is a non-autonomous control problem with mixed as well as pure state constraints and with a fixed time horizon. The constraints are autonomous. We want to formulate the necessary conditions of optimality using the so-called indirect adjoining approach. This approach is based on the fact that for each fixed admissible solution $(x(t),u(t))$, the function $t \to h_i(x(t))$, $i = 1,\dots,q$ has a global minimum at any $t \in (t_0, t_1)$ at which the i-th pure state constraint is binding, i.e. $h_i(x(t)) = 0$. Hence,

$$\frac{\mathrm{d}h_i}{\mathrm{d}t}(x(t)) = 0 \quad \text{whenever} \quad h_i(x(t)) = 0, \tag{7.2}$$

provided that this total derivative exists, i.e. at continuity points of u. We have

$$\frac{\mathrm{d}h_i}{\mathrm{d}t}(x(t)) = \frac{\mathrm{d}h_i}{\mathrm{d}x}(x(t))\,f(t,x(t),u(t)). \tag{7.3}$$

Recall that both functions h_i and f are continuous at all variables and x is also assumed to be continuous. Therefore the derivative in (7.2) exists at least at all continuity points of u. Moreover, if we define

$$k_i(t,x(t),u(t)) := \frac{\mathrm{d}h_i}{\mathrm{d}x}(x(t))\,f(t,x(t),u(t)) \tag{7.4}$$

for all $t \in \langle t_0, t_1\rangle$, from (7.2) – (7.4) it follows

$$k_i(t,x(t),u(t)) = 0 \quad \text{whenever} \quad h_i(x(t) = 0 \tag{7.5}$$

[1] The definition of values of $u(t)$ at discontinuity points is not unified across the literature. For example, Seierstad and Sydsæter (1987) [p. 73] and Feichtinger and Hartl (1986) [Note 2.1(c), p. 19] set the value as the left-hand limit. On the other hand, Brunovský (1980) [p. 66] uses the right-hand side limit.

at continuity points of $u(t)$.

We assume that the constraint $h(x(t)) \geq 0$ is a constraint of the first order, i.e. each of the functions $k_i(t, x(t), u(t))$, $j = i, \ldots, q$ depends on $u(t)$. In order to assure that both the constraints (7.1c) and (7.5) are trully mixed for an admissible solution to Problem (7.1a) – (7.1d), we introduce the following constraint qualification:

Definition 7.2. *An admissible solution $(x^*(t), u^*(t))$ to Problem (7.1a) – (7.1d) satisfies the strong constraint qualification, if the $(p+q) \times (r+p+q)$ matrix*

$$\begin{pmatrix} \frac{\partial g_1}{\partial u} & g_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial u} & 0 & \cdots & g_p & 0 & \cdots & 0 \\ \frac{\partial k_1}{\partial u} & 0 & \cdots & 0 & k_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial k_q}{\partial u} & 0 & \cdots & 0 & 0 & \cdots & k_q \end{pmatrix} \tag{7.6}$$

has a full rank when it is evaluated at $(x^*(t), u^*(t^-))$ for any $t \in (t_0, t_1\rangle$ and at $(x^*(t), u^*(t^+))$ for any $t \in \langle t_0, t_1)$.

The strong constraint qualification means that the gradients w.r.t. u of all the active constraints g_i, $i = 1, \ldots, p$ and k_j, $j = 1, \ldots, q$ are linearly independent. This constraint qualification is called "strong" because for the necessary conditions that will be formulated later in Theorems 7.1 and 7.2, it is sufficient to assume that the following weaker form of the constraint qualification is met: [2]

Definition 7.3. *An admissible solution $(x^*(t), u^*(t))$ to Problem (7.1a) – (7.1d) satisfies the weak constraint qualification, if the $p \times (r+p)$ matrix*

$$\begin{pmatrix} \frac{\partial g_1}{\partial u} & g_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial u} & 0 & \cdots & g_p \end{pmatrix} \tag{7.7}$$

has a full rank when it is evaluated at $(x^*(t), u^*(t^-))$ for any $t \in (t_0, t_1\rangle$ and at $(x^*(t), u^*(t^+))$ for any $t \in \langle t_0, t_1)$.

In addition, we introduce the following definition:

[2] According to Hartl et al. (1995), it is common in the literature on indirect adjoint approach that the stronger form of constraint qualification is required (see Hartl et al. (1995) [footnote 8 at page 197]). However, this is not the case neither in the formulation of necessary conditions by Seierstad and Sydsæter (1987), nor by Feichtinger and Hartl (1986) which are given below.

Definition 7.4. [3] *A* $\tau \in \langle t_0, t_1 \rangle$ *is called*

(i) an entry time if there exists $j \in \{1, \dots, q\}$ *and* $\varepsilon > 0$ *such that* $h_j(x^*(t)) > 0$ *for all* $t \in (\tau - \varepsilon, \tau)$ *and* $h_j(x^*(t)) = 0$ *for all* $t \in \langle \tau, \tau + \varepsilon)$*,*

(ii) a contact time if there exists $j \in \{1, \dots, q\}$ *and* $\varepsilon > 0$ *such that* $h_j(x^*(t)) > 0$ *for all* $t \in (\tau - \varepsilon, \tau) \cup (\tau, \tau + \varepsilon)$ *and* $h_j(x^*(\tau)) = 0$*,*

(iii) an exit time if there exists $j \in \{1, \dots, q\}$ *and* $\varepsilon > 0$ *such that* $h_j(x^*(t)) = 0$ *for all* $t \in (\tau - \varepsilon, \tau\rangle$ *and* $h_j(x^*(t)) > 0$ *for all* $t \in (\tau, \tau + \varepsilon)$*.*

If $\tau \in \langle t_0, t_1 \rangle$ *is an entry, contact or exit time, it is called a junction time.*

7.2 Necessary conditions of optimality for standard problems

Although there is a large amount of literature regarding problems with pure state constraints, the inconvenience is that the variety of formulations of the necessary conditions of optimality is rather diverse.[4] Hence, caution should by taken when combining results from different papers. As we want to build a comprehensive basis of results which can be used in economic applications, we introduce the necessary conditions of optimality from two different sources: The necessary conditions given by Seierstad and Sydsæter (1987) and by Feichtinger and Hartl (1986). Later, the relationship between these two sets of necessary conditions will be shown.

7.2.1 Necessary conditions by Seierstad and Sydsæter

First, we formulate the set of necessary conditions for Problem (7.1a) – (7.1d) as they are given in Seierstad and Sydsæter (1987) [Theorem 6.5, p. 372]. Define the Hamiltonian

$$H(t, x, u, \psi^0, \psi) = \psi^0 f^0(t, x, u) + \psi^T f(t, x, u), \tag{7.8}$$

the Lagrangian

$$L(t, x, u, \psi^0, \psi, \mu, \nu) = \psi^0 f^0(t, x, u) + \psi^T f(t, x, u) + \mu^T g(x, u) + \nu^T \frac{\mathrm{d}h}{\mathrm{d}x}(x) f(t, x, u) \tag{7.9}$$

[3] See also Feichtinger and Hartl (1986) [p. 165].

[4] See Table 5.1 in Hartl et al. (1995).

and the "simplified" Lagrangian

$$\breve{L}(t,x,u,\psi^0,\psi,\mu) = \psi^0 f^0(t,x,u) + \psi^T f(t,x,u) + \mu^T g(x,u). \tag{7.10}$$

For simplicity, we will mostly omit the argument ψ^0 from the list of arguments of functions H, L and $\breve{L}$, since it is a constant equal to 0 or 1. Then the following theorem can be formulated :

Theorem 7.1 (Necessary conditions by Seierstad and Sydsæter)**.** *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.1a) – (7.1d), which fulfills the weak constraint qualification. Then there exist a constant ψ^0, vector functions $\psi(t) : \mathbb{R} \to \mathbb{R}^n$ and $\mu(t) : \mathbb{R} \to \mathbb{R}^p$ and a non-increasing[5] vector function $\nu(t) : \mathbb{R} \to \mathbb{R}^q$, all having one-sided limits everywhere such that the following conditions are satisfied:*

(i) $\psi^0 = 0$ *or* $\psi^0 = 1$*,*

(ii) $(\psi^0, \psi(t), \nu(t_1) - \nu(t_0)) \neq (0,0,0)$ *for all* t*,*

(iii) *for all* $t \in (t_0, t_1)$*,* $u^*(t^+)$ *and* $u^*(t^-)$ *maximize both* $H(t, x^*(t), u, \psi(t^+))$ *and* $H(t, x^*(t), u, \psi(t^-))$ *for all* u *such that* $g(x^*(t), u) \geq 0$*,*[6]

(iv) $\dfrac{\partial \breve{L}}{\partial u}\Big(t, x^*(t), u^*(t^+), \psi(t^+), \mu(t^+)\Big) = 0$ *for all* $t \in \langle t_0, t_1)$ *and*
$\dfrac{\partial \breve{L}}{\partial u}\Big(t, x^*(t), u^*(t^-), \psi(t^-), \mu(t^-)\Big) = 0$ *for all* $t \in (t_0, t_1\rangle$*,*[7]

(v) $\nu_j(t)$ *is constant (not necessarily zero) on any interval where* $h_j(x^*(t)) > 0$ *for* $j = 1, \ldots, q$*; in addition,* $\nu_j(t)$ *is continuous at all* $t \in (t_0, t_1)$ *at which* $h_j(x^*(t)) = 0$ *and* $\dfrac{dh_j}{dx}(x^*(t)) f(t, x^*(t), u^*(t))$ *is discontinuous,*

(vi) *defining*

$$\bar{\psi}(t)^T := \psi(t)^T - \nu(t)^T \frac{dh}{dx}(x^*(t)), \tag{7.11}$$

$\bar{\psi}(t)$ *is continuous everywhere and has a continuous derivative satisfying*

$$\dot{\bar{\psi}}(t)^T = -\frac{\partial L}{\partial x}\Big(t, x^*(t), u^*(t), \bar{\psi}(t), \mu(t), \nu(t)\Big) \tag{7.12}$$

[5]In the original result published by Seierstad and Sydsæter (1986), it is assumed that ν is non-decreasing, but it has an opposite sign in (7.8) – (7.10).

[6]Seierstad and Sydsæter (1987) present only a weaker version of this claim: for almost all $t \in (t_0, t_1)$, $H(t, x^*(t), u^*(t), \psi(t)) \geq H(t, x^*(t), u, \psi(t))$ for all u such that $g(x^*(t), u) > 0$. For the claim presented here, see Seierstad and Sydsæter (1987) [Note 6.4(a), p. 374].

[7]See Seierstad and Sydsæter (1987) [Note 6.4(a), p. 373].

at all points of continuity of $u^*(t)$ and $\nu(t)$.

(vii) $\mu(t) \geq 0$ *and* $\mu(t)^T g(x^*(t), u^*(t)) = 0$ *for all* t,

(viii) $\nu(t_1) = 0$,

(ix) $\psi(t)$ *satisfies*

$$\begin{array}{ll} \text{no condition for } \psi_i(t_1), & i = 1, \dots, l, \\ \psi_i(t_1) \geq 0,\ \psi_i(t_1)\left(x_i^*(t_1) - x_i^1\right) = 0, & i = l+1, \dots, m, \\ \psi_i(t_1) = 0, & i = m+1, \dots, n. \end{array}$$

Remark 7.1. Note that the condition (viii) is not directly listed among conditions given in Seierstad and Sydsæter (1987) [Theorem 6.5]. However, all other conditions in this theorem are also satisfied if ν is replaced by $\nu + c$, where c is an arbitrary constant vector. Hence we can assume $\nu(t_1) = 0$ without loss of generality.[8] Indeed, besides condition (viii) the multiplier ν actually enters only into conditions (ii), (v) and (vi). If we replace ν by $\nu_c := \nu + c$, conditions (ii) and (v) remain satisfied trivially. To show that also condition (vi) is satisfied, some calculations are needed. Define

$$\bar{\psi}_c^T := \psi^T - \nu_c^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*) = \psi^T - (\nu + c)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*) \overset{(7.11)}{=} \bar{\psi}^T - c^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*). \tag{7.13}$$

We show that $\bar{\psi}_c$ satisfies

$$\dot{\bar{\psi}}_c(t)^T = -\frac{\partial L}{\partial x}\left(t, x^*(t), u^*(t), \bar{\psi}_c(t)_c, \mu(t), \nu(t)_c\right) \tag{7.14}$$

at all points of continuity of $u^*(t)$ and $\nu(t)$. We have

$$\begin{aligned} -\frac{\partial L}{\partial x}&\left(t, x^*, u^*, \bar{\psi}_c, \mu, \nu_c\right) = \\ \overset{(7.9),(7.13)}{=} \quad & -\psi^0 \frac{\partial f^0}{\partial x}(t, x^*, u^*) - \left[\bar{\psi}^T - c^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*)\right] \frac{\partial f}{\partial x}(t, x^*, u^*) - \mu^T \frac{\partial g}{\partial x}(x^*, u^*) - \\ & -(\nu + c)^T \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(x^*) f(t, x^*, u^*) - (\nu + c)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*) \frac{\partial f}{\partial x}(t, x^*, u^*) \\ \overset{(7.9)}{=} \quad & -\frac{\partial L}{\partial x}\left(t, x^*, u^*, \bar{\psi}, \mu, \nu\right) - c^T \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(x^*) f(t, x^*, u^*) \\ \overset{(7.12)}{=} \quad & \dot{\bar{\psi}}^T - \frac{\mathrm{d}}{\mathrm{d}t}\left[c^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*)\right] \\ \overset{(7.13)}{=} \quad & \dot{\bar{\psi}}_c^T \end{aligned}$$

which proves (7.14).

[8] See Seierstad and Sydsæter (1987) [Note 6.4(b), p. 374].

Furthermore, note that the conditions in Theorem 7.1 do not determine the values of ψ and ν at junction times. The only exception is condition (v) which states that ν_j is continuous at junction times if $\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(t))f(t,x^*(t),u^*(t))$ is discontinuous, $j=1,\ldots,q$. Moreover, if the value of one of the functions ψ and ν at junction times is given, the value of the other one is determined through the condition (vi), since we know that $\bar{\psi}$ is continuous everywhere. Thus, let us determine that the values of ν_j and ψ_j, $j=1,\ldots,q$, at those junction times t where $\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(t))f(t,x^*(t),u^*(t))$ is continuous, are equal to the left-hand limits. This is in accordance with the convention that the value of u is equal to its left-hand limit at discontinuity points. Recall that Theorem 7.1 ensures that both the functions ψ and ν have finite one-sided limits everywhere. In addition, the values of ψ and ν at t_0 are equal to right-hand limits.

As stated in the following lemma, the Hamiltonian is continuous in t. Moreover, the second part of the condition (v) needs not be verified as it is a consequence of other conditions of Theorem 7.1.

Lemma 7.1. [9] *Given the assumptions of Theorem 7.1, the following is true:*

(a) For all $\tau\in(t_0,t_1)$ one has

$$H\big(\tau^-,x^*(\tau^-),u^*(\tau^-),\psi(\tau^-)\big)=H\big(\tau^+,x^*(\tau^+),u^*(\tau^+),\psi(\tau^+)\big). \tag{7.15}$$

(b) The second part of the condition (v) is implied by other conditions.

Proof of (a). Let us denote

$$G(\tau):=H\big(\tau^-,x^*(\tau^-),u^*(\tau^-),\psi(\tau^-)\big)-H\big(\tau^+,x^*(\tau^+),u^*(\tau^+),\psi(\tau^+)\big). \tag{7.16}$$

We have to prove that $G(\tau)=0$. Recall that $x^*(t)$ is assumed to be a continuous function (see Definition 7.1 (ii)), hence we can write $x^*(\tau):=x^*(\tau^-)=x^*(\tau^+)$. In addition, note that Hamiltonian is continuous in the first variable, which is implied by (7.8) and the continuity of functions $f^0(t,x,u)$ and $f(t,x,u)$ in the first variable. Therefore, one has

$$G(\tau)=H\big(\tau,x^*(\tau),u^*(\tau^-),\psi(\tau^-)\big)-H\big(\tau,x^*(\tau),u^*(\tau^+),\psi(\tau^-)\big)+ \tag{7.17a}$$

[9] Part (a) of the Lemma is stated without proof in Seierstad and Sydsæter (1987) [Note 3(c), p. 333] for problems without mixed constraints. Part (b) is stated in Seierstad and Sydsæter (1987) [Exercise 5.3.4, p. 344] for problems without mixed constraints; the proof is only sketched.

$$+H\big(\tau,x^*(\tau),u^*(\tau^+),\psi(\tau^-)\big)-H\big(\tau,x^*(\tau),u^*(\tau^+),\psi(\tau^+)\big). \tag{7.17b}$$

We have assumed that $(x^*(t),u^*(t))$ is an optimal solution. Thus we have that the condition $g(x^*(\tau),u^*(\tau^-))\geq 0$ is satisfied and the condition (iii) in Theorem 7.1 states

$$H\big(\tau,x^*(\tau),u^*(\tau^-),\psi(\tau^-)\big)\geq H\big(\tau,x^*(\tau),u^*(\tau^+),\psi(\tau^-)\big). \tag{7.18}$$

Hence, the term in (7.17a) is greater than or equal to zero. Regarding the term in (7.17b), we can use the definition of $\bar{\psi}$ in (vi) in Theorem 7.1 and the continuity of $\bar{\psi}$ to obtain

$$\begin{aligned}H\big(\tau,x^*(\tau),u^*(\tau^+),\psi(\tau^-)\big)&-H\big(\tau,x^*(\tau),u^*(\tau^+),\psi(\tau^+)\big)\\&=\big(\psi(\tau^-)-\psi(\tau^+)\big)^T f(\tau,x^*(\tau),u^*(\tau^+))\\&=\big(\nu(\tau^-)-\nu(\tau^+)\big)^T\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau))f(\tau,x^*(\tau),u^*(\tau^+)).\end{aligned} \tag{7.19}$$

Combining these results, one has

$$G(\tau)\geq\big(\nu(\tau^-)-\nu(\tau^+)\big)^T\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau))f(\tau,x^*(\tau),u^*(\tau^+)). \tag{7.20}$$

Similarly, we can write

$$G(\tau)=H\big(\tau,x^*(\tau),u^*(\tau^-),\psi(\tau^-)\big)-H\big(\tau,x^*(\tau),u^*(\tau^-),\psi(\tau^+)\big)+ \tag{7.21a}$$

$$+H\big(\tau,x^*(\tau),u^*(\tau^-),\psi(\tau^+)\big)-H\big(\tau,x^*(\tau),u^*(\tau^+),\psi(\tau^+)\big). \tag{7.21b}$$

Now the term in (7.21b) is less than or equal to zero (using condition (iii)) and the term in (7.21a) can be rewritten using (vi). We obtain

$$G(\tau)\leq\big(\nu(\tau^-)-\nu(\tau^+)\big)^T\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau))f(\tau,x^*(\tau),u^*(\tau^-)). \tag{7.22}$$

It is straightforward to prove that

$$\big(\nu_j(\tau^-)-\nu_j(\tau^+)\big)\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(\tau))f(\tau,x^*(\tau),u^*(\tau^-))=0 \tag{7.23}$$

and

$$\big(\nu_j(\tau^-)-\nu_j(\tau^+)\big)\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(\tau))f(\tau,x^*(\tau),u^*(\tau^+))=0 \tag{7.24}$$

for all $j=1,\ldots,q$ such that $h_j(x^*(\tau))>0$. Indeed, as h_j is assumed to be a continuous function, we have that $h_j(x^*(t))>0$ on some neighbourhood of τ (denoted by $\mathcal{O}_j(\tau)$). From condition (vi) we have that ν_j is then constant on $\mathcal{O}_j(\tau)$. Hence $\nu_j(\tau^-)=\nu_j(\tau^+)$ which implies (7.23).

Now we will prove that

$$\big(\nu_j(\tau^-)-\nu_j(\tau^+)\big)\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(\tau))\,f(\tau,x^*(\tau),u^*(\tau^-))\le 0 \tag{7.25}$$

and

$$\big(\nu_j(\tau^-)-\nu_j(\tau^+)\big)\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(\tau))\,f(\tau,x^*(\tau),u^*(\tau^+))\ge 0 \tag{7.26}$$

for all $j=1,\dots,q$ such that $h_j(x^*(\tau))=0$. Note that $h_j(x^*(\tau))=0$ together with $h_j(x)\ge 0$ for all x then implies

$$\frac{\mathrm{d}h_j}{\mathrm{d}t}(x^*(\tau^-))\le 0 \quad \text{and} \quad \frac{\mathrm{d}h_j}{\mathrm{d}t}(x^*(\tau^+))\ge 0. \tag{7.27}$$

Hence

$$\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(\tau))\,f(\tau,x^*(\tau),u^*(\tau^-))\le 0 \quad \text{and} \quad \frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(\tau))\,f(\tau,x^*(\tau),u^*(\tau^+))\ge 0. \tag{7.28}$$

This directly implies (7.25) and (7.26), because $\nu(\tau^-)-\nu(\tau^+)\ge 0$ as $\nu(t)$ is a non-increasing function.

Using (7.23) and (7.25) in (7.22) implies

$$G(\tau)\le \big(\nu(\tau^-)-\nu(\tau^+)\big)^T\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau))\,f(\tau,x^*(\tau),u^*(\tau^-))\le 0. \tag{7.29}$$

On the other hand, using (7.24) and (7.26) in (7.20) implies

$$G(\tau)\ge \big(\nu(\tau^-)-\nu(\tau^+)\big)^T\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau))\,f(\tau,x^*(\tau),u^*(\tau^+))\ge 0. \tag{7.30}$$

Hence we conclude that $G(\tau)=0$ and Equality (7.15) is proved. In addition, both (7.29) and (7.30) are satisfied as equalities.

Proof of (b). Let us assume that $\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(t))\,f(t,x^*(t),u^*(t))$ is discontinuous at points $\tau\in(t_0,t_1)$ and $h_j(x^*(\tau))=0$ for some $j=1,\dots,q$. Recall that $h_j(x^*(\tau))=0$ implies

$$\frac{\mathrm{d}h_j}{\mathrm{d}t}(x^*(\tau^-))\le 0 \tag{7.31}$$

and

$$\frac{\mathrm{d}h_j}{\mathrm{d}t}(x^*(\tau^+))\ge 0. \tag{7.32}$$

Since

$$\frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(t))\,f(\tau,x^*(t),u^*(t))=\frac{\mathrm{d}h_j}{\mathrm{d}t}(x^*(t)), \tag{7.33}$$

we have that $\frac{\mathrm{d}h_j}{\mathrm{d}t}(x^*(t))$ is discontinuous at τ. Hence, at least one of Inequalities (7.31) and (7.32) has to be a strict inequality.

Now we use that both (7.29) and (7.30) are satisfied as equalities which was proved at the end of the proof of part (a). Together with (7.23), (7.24), (7.25) and (7.26) this implies

$$\big(\nu_i(\tau^-) - \nu_i(\tau^+)\big)\frac{\mathrm{d}h_i}{\mathrm{d}x}(x^*(\tau))f(\tau, x^*(\tau), u^*(\tau^-)) = 0 \tag{7.34}$$

and

$$\big(\nu_i(\tau^-) - \nu_i(\tau^+)\big)\frac{\mathrm{d}h_i}{\mathrm{d}x}(x^*(\tau))f(\tau, x^*(\tau), u^*(\tau^+)) = 0 \tag{7.35}$$

for all $i = 1, \ldots, q$. If (7.31) is a strict inequality then (7.34) implies $\nu_j(\tau^-) = \nu_j(\tau^+)$. On the other hand, if (7.32) is a strict inequality then the continuity of ν_j at τ is implied by (7.35). □

Example 7.1. To illustrate the necessary conditions for a specific problem, let us consider the following example:

$$\max_{u(t)} \int_0^2 x(t)\,\mathrm{d}t, \tag{7.36}$$

$$\dot{x}(t) = 1 - u(t)^2, \tag{7.37}$$

$$x(0) = 0, \tag{7.38}$$

$$x(2) \text{ is free}, \tag{7.39}$$

$$1 - x(t) \geq 0. \tag{7.40}$$

Clearly, the optimal solution is such that $x^*(t)$ is first increased by a maximum rate until the constraint (7.40) becomes binding and then it remains at this level. Hence,

$$x^*(t) = \begin{cases} t & \text{for } t \in \langle 0, 1\rangle, \\ 1 & \text{for } t \in (1, 2\rangle \end{cases} \quad \text{and} \quad u^*(t) = \begin{cases} 0 & \text{for } t \in \langle 0, 1\rangle, \\ \pm 1 & \text{for } t \in (1, 2\rangle. \end{cases} \tag{7.41}$$

We show that $(x^*(t), u^*(t))$ defined by (7.41) satisfies necessary conditions of optimality given in Theorem 7.1 together with $\psi^0 = 1$,

$$\psi(t) = \begin{cases} 1 - t & \text{for } t \in \langle 0, 1\rangle, \\ 0 & \text{for } t \in (1, 2\rangle \end{cases} \quad \text{and} \quad \nu(t) = \begin{cases} 1 & \text{for } t \in \langle 0, 1\rangle, \\ 2 - t & \text{for } t \in (1, 2\rangle. \end{cases} \tag{7.42}$$

For this problem, we have

$$H(x, u, \psi^0, \psi) = \check{L}(x, u, \psi^0, \psi) = \psi^0 x + \psi(1 - u^2) \tag{7.43}$$

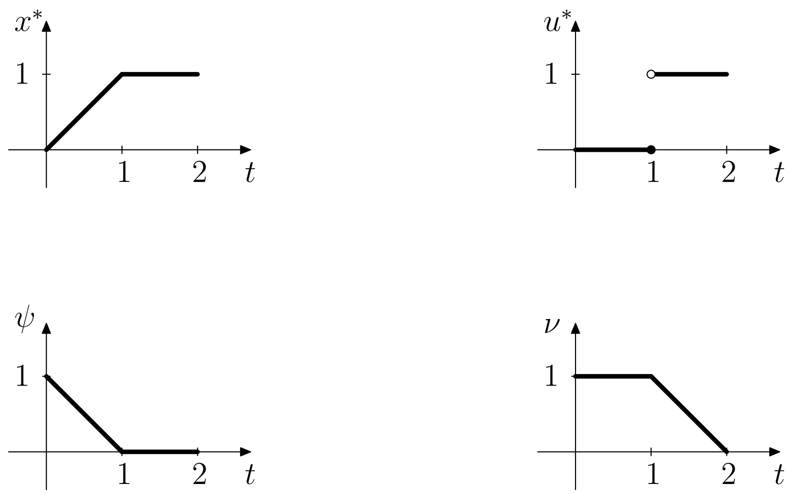

Figure 7.1: Optimal solution and functions ψ and ν for Problem (7.36) – (7.40).

and

$$L(x,u,\psi^0,\psi,\nu) = \psi^0 x + (\psi - \nu)(1-u^2). \tag{7.44}$$

Functions ψ and ν have one-sided limits everywhere on $\langle 0,2\rangle$ and ν is non-increasing. We formulate and verify conditions (i) – (ix):

(i) $\psi^0 = 1$ is satisfied,

(ii) $(\psi^0, \psi(t), \nu(t_1) - \nu(t_0)) = (1, \psi(t), 1) \neq (0,0,0)$ for all t,

(iii) the condition

$$\psi^0 x^* + \psi(1-u^{*2}) \geq \psi^0 x^* + \psi(1-u^2) \quad \text{for all} u \tag{7.45}$$

is satisfied on $\langle 0,1)$ where $\psi(t) > 0$ because $1 - u^*(t)^2 = 1 \geq 1 - u^2$ for all u and it is also trivially satisfied at $\langle 1,2\rangle$ where $\psi(t) = 0$,

(iv) $-2\psi(t)u^*(t) = 0$ for all $t \in \langle 0,2\rangle$,

(v) $\nu(t)$ is constant on $\langle 0,1)$ where $1 - x(t) > 0$ and $\nu(t)$ is continuous at $\tau = 1$ where

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)) f(\tau, x^*(\tau), u^*(\tau)) = u^*(\tau)^2 - 1 \tag{7.46}$$

is discontinuous and $1 - x(\tau) = 0$,

(vi) $\bar{\psi}(t) = \psi(t) + \nu(t) = 2 - t$ is continuous everywhere and has a continuous derivative $\dot{\bar{\psi}}(t) = -\psi^0 = -1$,

(vii) condition is empty,

(viii) $\nu(2) = 0$ is satisfied,

(ix) $\psi(2) = 0$ is satisfied. ■

7.2.2 Necessary conditions by Feichtinger and Hartl

Now we state the necessary conditions (using tilded variables) according to Feichtinger and Hartl (1986) [Theorem 6.3, p. 169] (neglecting the discount factor r in their formulation of necessary conditions). The main difference is that in these conditions, $\tilde{\nu}_j$ is zero on intervals where $h_j(x^*(t)) > 0$, $j = 1, \dots, q$, the Hamiltonian si maximized over different region in condition (iii) and the standard form of the Lagrangian (7.9) is used instead of the simplified Lagrangian (7.10) in (iv).

Theorem 7.2 (Necessary conditions by Feichtinger and Hartl). *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.1a) – (7.1d), which fulfills the weak constraint qualification. Then there exists a constant $\tilde{\psi}^0$, piecewise continuous and piecewise differentiable vector functions $\tilde{\psi}(t) : \mathbb{R} \to \mathbb{R}^n$, $\tilde{\mu}(t) : \mathbb{R} \to \mathbb{R}^p$ and $\tilde{\nu}(t) : \mathbb{R} \to \mathbb{R}^q$, such that for all $t \in (t_0, t_1)$ with the possible exception of the discontinuity points of $u^*(t)$ and the junction times, the following conditions are satisfied:*

(i) $\tilde{\psi}^0 = 0$ *or* $\tilde{\psi}^0 = 1$,

(ii) $(\tilde{\psi}^0, \tilde{\psi}(t)) \neq (0, 0)$ *for all* $t \in (t_0, t_1)$,

(iii) $H(t, x^*(t), u^*(t), \tilde{\psi}(t)) \geq H(t, x^*(t), u, \tilde{\psi}(t))$ *for all u such that* $g(x^*(t), u) \geq 0$ *and* $\dfrac{dh_i}{dx}(x^*(t)) f(t, x^*(t), u) = 0$ *whenever* $h_i(x^*(t)) = 0$, $i = 1, \dots, q$,

(iv) $\dfrac{\partial L}{\partial u}\Big(t, x^*(t), u^*(t), \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big) = 0$,

(v) $\tilde{\nu}(t) \geq 0$, $\tilde{\nu}(t)^T h(x^*(t)) = 0$ *and* $\dot{\tilde{\nu}}(t) \leq 0$,

(vi) $\dot{\tilde{\psi}}(t)^T = -\dfrac{\partial L}{\partial x}\Big(t, x^*(t), u^*(t), \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big)$,

(vii) $\tilde{\mu}(t) \geq 0$ *and* $\tilde{\mu}(t)^T g(x^*(t), u^*(t)) = 0$ *for all* t,

(viii) $\tilde{\nu}(t)$ *satisfies* $\tilde{\nu}(t_1) = 0$,

(ix) $\tilde{\psi}(t)$ *satisfies*

$$\begin{array}{ll} \text{no condition for } \tilde{\psi}_i(t_1), & i = 1, \dots, l, \\ \tilde{\psi}_i(t_1) \geq 0,\ \tilde{\psi}_i(t_1)\left(x_i^*(t_1) - x_i^1\right) = 0, & i = l+1, \dots, m, \\ \tilde{\psi}_i(t_1) = 0, & i = m+1, \dots, n, \end{array}$$

(x) *one has for all* $\tau \in (t_0, t_1)$ *that*

$$H\big(\tau^-, x^*(\tau), u^*(\tau^-), \tilde{\psi}(\tau^-)\big) = H\big(\tau^+, x^*(\tau), u^*(\tau^+), \tilde{\psi}(\tau^+)\big). \tag{7.47}$$

(xi) *At an entry or contact time* τ *of the constraint* $h_j(x^*(t)) \geq 0$ *for any* $j = 1, \dots, q$, *the costate trajectory* $\tilde{\psi}_j(t)$ *may have a discontinuity of the form*

$$\tilde{\psi}_j(\tau^-) = \tilde{\psi}_j(\tau^+) + \tilde{\eta}_j(\tau)\frac{dh_j}{dx}(x^*(\tau)), \tag{7.48}$$

where

$$\tilde{\eta}_j(\tau) \geq 0 \quad \text{and} \quad \tilde{\eta}_j(\tau)h_j(x^*(\tau)) = 0. \tag{7.49}$$

In addition to these conditions, another necessary condition is given by Feichtinger and Hartl (1986):

Theorem 7.3. [10] *Let* $(x^*(t), u^*(t))$ *be an optimal solution to Problem (7.1a) – (7.1d), which fulfills the weaker form of constraint qualification and let* $\tilde{\psi}$, $\tilde{\mu}$ *and* $\tilde{\nu}$ *be functions according to Theorem 7.2. Then for all* $t \in (t_0, t_1)$ *with the possible exception of the discontinuity points of* $u^*(t)$ *and the junction times the following condition is satisfied:*

$$\frac{dL}{dt}\Big(t, x^*(t), u^*(t), \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big) = \frac{\partial L}{\partial t}\Big(t, x^*(t), u^*(t), \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big). \tag{7.50}$$

None of the conditions in Theorem 7.2 determines the value of $\tilde{\psi}$ and $\tilde{\nu}$ at junction times and discontinuity points of u^*. In particular, note that conditions (i) – (ix) might not be satisfied at these points and conditions (x) and (xi) refer only to one-sided limits. Hence we are free to choose the values of $\tilde{\psi}$ and $\tilde{\nu}$ at these points. Let us determine that these values are equal to the left-hand limits. This is in accordance with the convention for values of u^*, ψ and ν given above.

[10]Feichtinger and Hartl (1986) [Theorem 6.3, p. 169].

Example 7.2. To illustrate conditions given in Theorems 7.2 and 7.3 and their relationship to conditions given in Theorem 7.1, consider again Problem (7.36) – (7.40) formulated in Example 7.1. We show that $(x^*(t), u^*(t))$ defined by (7.41) satisfies necessary conditions of optimality given in Theorem 7.2 together with $\tilde{\psi}^0 = 1$, $\tilde{\eta}(1) = 1$,

$$\tilde{\psi}(t) = \begin{cases} 1-t & \text{for } t \in \langle 0,1\rangle, \\ 2-t & \text{for } t \in (1,2\rangle \end{cases} \quad \text{and} \quad \tilde{\nu}(t) = \begin{cases} 0 & \text{for } t \in \langle 0,1\rangle, \\ 2-t & \text{for } t \in (1,2\rangle. \end{cases} \tag{7.51}$$

Figure 7.2: Functions $\tilde{\psi}$ and $\tilde{\nu}$ for Problem (7.36) – (7.40).

The Hamiltonian and Lagrangian for this problem are defined by

$$H(x,u,\tilde{\psi}^0,\tilde{\psi}) = \tilde{\psi}^0 x + \tilde{\psi}(1-u^2) \tag{7.52}$$

and

$$L(x,u,\tilde{\psi}^0,\tilde{\psi},\tilde{\nu}) = \tilde{\psi}^0 x + (\tilde{\psi}-\tilde{\nu})(1-u^2). \tag{7.53}$$

Functions $\tilde{\psi}$ and $\tilde{\nu}$ are piecewise continuous and piecewise differentiable on $\langle 0,2\rangle$. We now formulate and verify conditions (i) – (xi) given in Theorem 7.2:

(i) $\tilde{\psi}^0 = 1$ is satisfied,

(ii) $(\tilde{\psi}^0, \tilde{\psi}(t)) = (1, \tilde{\psi}(t)) \neq (0,0)$ for all t,

(iii) this condition states that $\tilde{\psi}^0 x^* + \tilde{\psi}(1-u^{*2}) \geq \tilde{\psi}^0 x^* + \tilde{\psi}(1-u^2)$ for all $u \in \mathbb{R}$ if $1-x^* > 0$ and for $u \in \{-1,1\}$ if $1-x^* = 0$; it is satisfied on $\langle 0,1)$ where $\tilde{\psi}(t) > 0$ because $1-u^*(t)^2 = 1 \geq 1-u^2$ for all $u \in \mathbb{R}$ and it is also trivially satisfied at $(1,2\rangle$ where $u^* = \pm 1$,

(iv) condition $-2(\tilde{\psi}(t)-\tilde{\nu}(t))u^*(t) = 0$ is satisfied for all $t \in \langle 0,2\rangle$, because $u^*(t) = 0$ on $\langle 0,1\rangle$ and $\tilde{\psi}(t)-\tilde{\nu}(t) = 0$ on $(1,2\rangle$.

(v) $\tilde{\nu}(t) = 0$ on $\langle 0,1)$ where $1-x(t) > 0$ and $h(x^*(t)) = 0$ on $\langle 1,2\rangle$; in addition, $\tilde{\nu}(t) \geq 0$ everywhere, $\dot{\tilde{\nu}}(t) = 0$ on $(0,1)$ and $\dot{\tilde{\nu}}(t) = -1 \leq 0$ on $(1,2)$,

(vi) $\dot{\tilde{\psi}}(t) = -\tilde{\psi}^0 = -1$ is satisfied everywhere on $(0,2)$ except the entry time $t=1$,

(vii) condition is empty,

(viii) $\tilde{\nu}(2) = 0$ is satisfied,

(ix) $\tilde{\psi}(2) = 0$ is satisfied,

(x) for $t = 1$ we have

$$H(x^*(1), u^*(1^-), \tilde{\psi}(1^-), \tilde{\nu}(1^-)) = H(x^*(1), u^*(1^+), \tilde{\psi}(1^+), \tilde{\nu}(1^+)) = 1;$$

Equality (7.47) is trivially satisfied for all other $t \in \langle 0, 2 \rangle$,

(xi) Equality (7.48) states that $\tilde{\psi}(1^-) = \tilde{\psi}(1^+) - \tilde{\eta}(1)$ which is satisfied since we have $\tilde{\psi}(1^-) = 0$, $\tilde{\psi}(1^+) = 1$ and $\tilde{\eta}(1) = 1$; moreover, the constraint $1 - x^*(t) \geq 0$ is active at $t = 1$. ■

In addition, Equality (7.50) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\psi}^0 x + (\tilde{\psi} - \tilde{\nu})(1 - u^2)) = 0. \tag{7.54}$$

This equality is indeed satisfied, since one has

$$\tilde{\psi}^0 x + (\tilde{\psi} - \tilde{\nu})(1 - u^2) = 1 \tag{7.55}$$

for all $t \in \langle 0, 2 \rangle$.

7.2.3 Relationship between the two types of necessary conditions

The mean feature of the necessary conditions of optimality for a problem with active pure state constraints is that the costate variable ψ might not be continuous everywhere on $\langle t_0, t_1 \rangle$, unlike in problems without any pure state constraints. Hence, the necessary conditions provide some "additional" information in order to restrict the set of potential candidates on optimal solutions. Note that this additional information is different in both types of necessary conditions given in Theorems 7.1 and 7.2.

If we use the set of necessary conditions by Seierstad and Sydsæter (1987), we know that the function $\bar{\psi}$ defined by (7.11) is continuous everywhere. In addition, Hamiltonian is continuous everywhere as it was proved in Lemma 7.1. On the other hand, we only know that $\nu_j(t)$ is constant on intervals where $h_j(x^*(t)) > 0$, but we do not know the values of these constants.

Regarding the necessary conditions by Feichtinger and Hartl (1986), they ensure that $\tilde{\psi}$ is continuous at exit times. However, they provide no information about the value of jumps of $\tilde{\psi}$ at entry and contact times except their signs. Moreover, they state that $\tilde{\nu}_j(t) = 0$ on intervals where $h_j(x^*(t)) > 0$. Another disadvantage of these conditions compared with conditions by Seierstad and Sydsæter is that condition (iii) is weaker in Theorem 7.2 than in Theorem 7.1. This does not matter if the Hamiltonian has only global maxima. If however there are also local maxima different from the global maxima, Theorem 7.2 may fail to exclude solutions which are not optimal and which can be excluded by Theorem 7.1, as will be shown later in Example 7.5.

Now let us introduce how the conditions stated in Theorem 7.1 can be translated into the conditions given in Theorem 7.2. However, we restrict our attention only to problems with the following property:

(P) The interval (t_0, t_1) can be split up into finite number of subintervals such that on any one of them the set of binding pure state constraints is constant.

This property allows us to define for all $t \in \langle t_0, t_1 \rangle$ and $j = 1, \dots, q$

$$\tilde{\nu}_j(t) := \begin{cases} 0 & \text{if } h_j(x^*(t)) > 0 \text{ or } t \text{ is an entry or contact time,} \\ \nu_j(t) - \nu_j(\tau^+) & \text{if } h_j(x^*(t)) = 0 \text{ and } t \text{ is not an entry or contact time,} \end{cases} \tag{7.56}$$

where τ is an exit time which is the nearest to t and which is greater or equal to t or $\tau = t_1$ if no such exit time exists, where $\nu_j(t_1^+) := 0$.

Define further

$$\begin{aligned} \tilde{\psi}^T(t) &:= \psi^T(t) - \tilde{\nu}^T(t)\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)), && (7.57) \\ \tilde{\psi}^0 &:= \psi^0, && (7.58) \\ \tilde{\eta}(t) &:= \big(\nu(t^-) - \tilde{\nu}(t^-)\big) - \big(\nu(t^+) - \tilde{\nu}(t^+)\big), && (7.59) \\ \tilde{\mu}(t) &:= \mu(t). && (7.60) \end{aligned}$$

Note that transformation (7.56) is a composition of two transformations:

- Transformation for $h_j(x^*(t)) > 0$ and at entry and contact times ensures that the respective variable $\tilde{\nu}_j$ will be zero, whereas ν_j is a constant (possibly non-zero).
- Transformation for $h_j(x^*(t)) = 0$ ensures that $\tilde{\psi}$ is continuous at exit times, as will be shown later in Theorem 7.4.

Both of these transformations are based on the fact that ν can be replaced by $\nu + c$, where c is an arbitrary piecewise constant function with possible discountinuity points at junction times (see Remark 7.1). However, ψ has to be replaced in accordance with (7.13).

Example 7.3. Let us briefly verify that transformations (7.56) – (7.60) can be applied in case of the relationship between the two sets of multipliers in Example 7.1 and Example 7.2. First, note that (7.56) applied to ν yields that $\tilde{\nu}(t) = 0$ for $t \in \langle 0, 1\rangle$ since we have $h(x^*(t)) > 0$ on $\langle 0, 1)$ and $t = 1$ is an entry time. On the other hand, $\tilde{\nu}(t) = \nu(t) - \nu(\tau^+)$ on $(1, 2\rangle$, where $\tau = 2$ and $\nu(\tau^+) = 0$ because there is no exit time. Hence, $\tilde{\nu}(t) = \nu(t) = 2 - t$ on $(1, 2\rangle$. We can conclude that the values of $\tilde{\nu}$ obtained by (7.56) are the same as those given by (7.51). Next, applying Equality (7.57) to ψ and ν defined by (7.42) yields

$$\tilde{\psi}(t) = \psi(t) - \tilde{\nu}(t)\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) = \psi(t) + \tilde{\nu}(t) = \begin{cases} 1 - t & \text{for } t \in \langle 0, 1\rangle, \\ 2 - t & \text{for } t \in (1, 2\rangle, \end{cases} \tag{7.61}$$

which is in accordance with (7.51). Finally, it follows from combining (7.42) and (7.51) with (7.59) that

$$\tilde{\eta}(1) = \big(\nu(1^-) - \tilde{\nu}(1^-)\big) - \big(\nu(1^+) - \tilde{\nu}(1^+)\big) = (1 - 0) - (1 - 1) = 1. \tag{7.62}$$

Hence, $\tilde{\eta}(1)$ calculated by (7.59) attains the same value as given in Example 7.2. ■

Example 7.4. To better illustrate transformations (7.56) and (7.57), consider the following example: Let us assume that a virtual problem involving a pure state constraint $h(x(t)) := x(t) \geq 0$ is given. For simplicity, we do not formulate the problem, we just sketch the function $h(x^*(t))$ along the supposed optimal solution (see the chart of $h(x^*)$ on Figure 7.3). We assume that there are two entry times ($t = 1$ and $t = 4$) and two exit times ($t = 2$ and $t = 5$). Furthermore, we assume that $\frac{\mathrm{d}h}{\mathrm{d}t}(x^*(t)) = \dot{x}^*(t)$ is continuous at $t = 1$ and $t = 2$, whereas it is discontinuous at $t = 4$ and $t = 5$. Hence, $\nu(t)$ is continuous at $t = 4$ and $t = 5$ (see condition (v) in Theorem 7.1), although it may be discontinuous at $t = 1$ and $t = 2$. We also know that $\nu(t)$ is a non-increasing function and $\nu(6) = 0$ (see the chart of ν on Figure 7.3). Regarding $\tilde{\nu}$, transformation (7.56) ensures that it is zero on interval where $h(x^*(t)) > 0$, in accordance with condition (v) in Theorem 7.2. This transformation also implies that the distance labelled as d is the same in both charts of ν as well as of $\tilde{\nu}$.

Now, let as assume that $\psi(t)$ has the form depicted in Figure 7.3 (see chart of ψ). Since the given problem is not specified enough, we are free to set $\psi(t)$ arbitrarily,

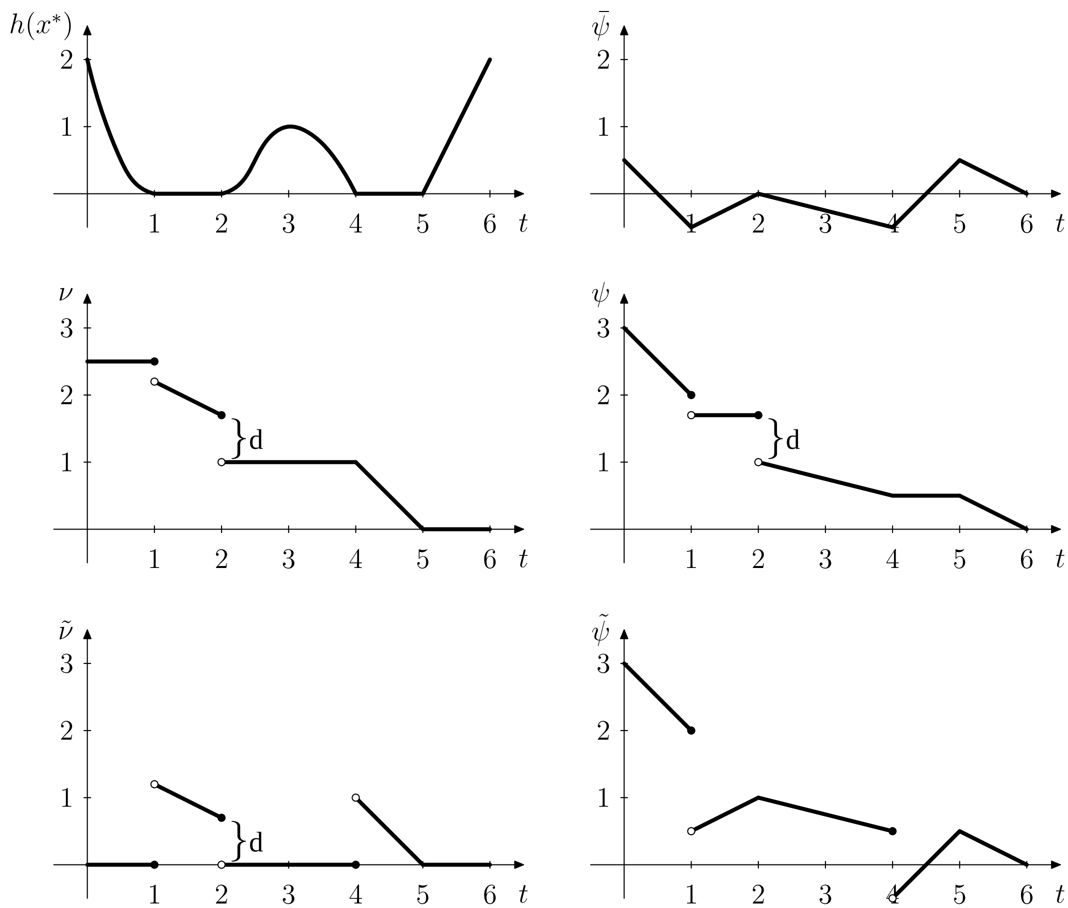

Figure 7.3: Functions $h(x(x^*)$, ν, $\tilde{\nu}$, $\bar{\psi}$, ψ and $\tilde{\psi}$ for Example 7.4.

provided that

$$\bar{\psi}(t) := \psi(t) - \nu(t)\frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) = \psi(t) - \nu(t) \tag{7.63}$$

is continuous everywhere (see condition (vi) in Theorem 7.1). Therefore, the distance labelled as d on chart of ψ is again the same as on charts on ν and $\tilde{\nu}$. The path of $\bar{\psi}$ is also depicted in Figure 7.3 (note that it is continuous). Finally, $\tilde{\psi}$ was calculated as $\psi - \tilde{\nu}$ in accordance with (7.57). Note that it is indeed discontinuous and non-increasing at entry times but it is continuous at exit times, as stated in condition (xi) of Theorem 7.2. ■

Using these definitions, we can formulate the following auxiliary result, which will be useful later:

Lemma 7.2. *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.1a) – (7.1d), which* *fulfills the necessary conditions of optimality given in Theorem 7.1 together with ψ, μ and ν. Assume that Property (P) is satisfied. Let $\bar{\psi}$ be a function defined by (7.11) and let $\tilde{\psi}$, $\tilde{\mu}$ and $\tilde{\nu}$ be functions defined by (7.56), (7.57) and (7.60). Then we have*

$$\frac{\partial L}{\partial x}\Big(t, x^*(t), u^*(t), \psi^0, \bar{\psi}(t), \mu(t), \nu(t)\Big) - \frac{\partial L}{\partial x}\Big(t, x^*(t), u^*(t), \tilde{\psi}^0, \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big)$$
$$= \frac{d}{dt}\left[\big(\nu(t) - \tilde{\nu}(t)\big)^T \frac{dh}{dx}(x^*(t))\right] \tag{7.64}$$

almost everywhere, where the function L is defined by (7.9).

Proof. Using the definition of $\bar{\psi}$ (7.11) and Equality (7.57) we obtain

$$\bar{\psi}(t)^T = \tilde{\psi}(t)^T - \big(\nu(t) - \tilde{\nu}(t)\big)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)). \tag{7.65}$$

Using definition of Lagrangian (7.9) and Equalities (7.65) – (7.58) and (7.60) we obtain

$$\frac{\partial L}{\partial x}\Big(t, x^*, u^*, \psi^0, \bar{\psi}, \mu, \nu\Big) - \frac{\partial L}{\partial x}\Big(t, x^*, u^*, \tilde{\psi}^0, \tilde{\psi}, \tilde{\mu}, \tilde{\nu}\Big)$$

$$\begin{aligned}
= \;& \psi^0 \frac{\partial f^0}{\partial x}(t, x^*, u^*) + \left[\tilde{\psi}^T - (\nu - \tilde{\nu})^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*)\right] \frac{\partial f}{\partial x}(t, x^*, u^*) + \mu^T \frac{\partial g}{\partial x}(x^*, u^*) + \\
& + \nu^T \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(x^*) f(t, x^*, u^*) + \nu^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*) \frac{\partial f}{\partial x}(t, x^*, u^*) - \\
& - \tilde{\psi}^0 \frac{\partial f^0}{\partial x}(t, x^*, u^*) - \tilde{\psi}^T \frac{\partial f}{\partial x}(t, x^*, u^*) - \tilde{\mu}^T \frac{\partial g}{\partial x}(x^*, u^*) - \\
& - \tilde{\nu}^T \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(x^*) f(t, x^*, u^*) - \tilde{\nu}^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*) \frac{\partial f}{\partial x}(t, x^*, u^*) \\
= \;& (\nu - \tilde{\nu})^T \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(x^*) f(t, x^*, u^*) \\
= \;& \frac{\mathrm{d}}{\mathrm{d}t}\left[(\nu - \tilde{\nu})^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*)\right] - \left[\frac{\mathrm{d}}{\mathrm{d}t}(\nu - \tilde{\nu})^T\right] \frac{\mathrm{d}h}{\mathrm{d}x}(x^*),
\end{aligned} \tag{7.66}$$

where the first and second equality are satisfied everywhere and the third equality is satisfied everywhere with the possible exception of junction times. However, Equality (7.56) implies that $\nu_j - \tilde{\nu}_j$ is piecewise constant, hence the last term in (7.66) is zero almost everywhere. ☐

Before introducing the relationship between Theorems 7.1 and 7.2, we prove another auxiliary result:

Lemma 7.3. *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.1a) – (7.1d), which fulfills the strong constraint qualification, let ψ, μ and ν be functions according to Theorem 7.1. Assume that Property (P) is satisfied and let $\tilde{\psi}$, $\tilde{\mu}$ and $\tilde{\nu}$ be defined by (7.56), (7.57) and (7.60). Then the functions $\psi(t)$, $\mu(t)$, $\nu(t)$ and $\tilde{\mu}(t)$ are continuous at the points $t = \tau \in (t_0, t_1)$, if $u^*(t)$ is continuous at τ. In addition, functions $\tilde{\psi}(t)$ and $\tilde{\nu}(t)$ are continuous at $t = \tau \in (t_0, t_1)$, if $u^*(t)$ is continuous at τ and τ is not an entry time.*

Proof.[11] Assume that $u^*(t)$ is continuous at $\tau \in (t_0, t_1)$, i.e. $u^*(\tau^-) = u^*(\tau^+) =: u^*(\tau)$. Theorem 7.1 (iv) implies

$$\frac{\partial \breve{L}}{\partial u}\Big(\tau, x^*(\tau), u^*(\tau), \psi(\tau^-), \mu(\tau^-)\Big) = \frac{\partial \breve{L}}{\partial u}\Big(\tau, x^*(\tau), u^*(\tau), \psi(\tau^+), \mu(\tau^+)\Big) = 0. \quad (7.67)$$

We used that $\breve{L}(t, x, u, \psi, \mu)$ is continuous in the first variable due to continuity of functions $f^0(t, x, u)$ and $f(t, x, u)$ in the first variable. In accordance with (7.10), the difference between the two terms in (7.67) can be written as

$$\begin{aligned} 0 &= \frac{\partial \breve{L}}{\partial u}\Big(\tau, x^*(\tau), u^*(\tau), \psi(\tau^-), \mu(\tau^-)\Big) - \frac{\partial \breve{L}}{\partial u}\Big(\tau, x^*(\tau), u^*(\tau), \psi(\tau^+), \mu(\tau^+)\Big) \\ &= \big(\psi(\tau^-) - \psi(\tau^+)\big)^T \frac{\partial f}{\partial u}(\tau, x^*(\tau), u^*(\tau)) + \big(\mu(\tau^-) - \mu(\tau^+)\big)^T \frac{\partial g}{\partial u}(x^*(\tau), u^*(\tau)). \end{aligned} \quad (7.68)$$

Now we use the definition of $\bar{\psi}$ in (vi) in Theorem 7.1 and the continuity of $\bar{\psi}$ to obtain

$$\big(\psi(\tau^-) - \psi(\tau^+)\big)^T = \big(\nu(\tau^-) - \nu(\tau^+)\big)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)). \quad (7.69)$$

By combining (7.69) with (7.68) one has

$$\begin{aligned} &\big(\nu(\tau^-) - \nu(\tau^+)\big)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)) \frac{\partial f}{\partial u}(\tau, x^*(\tau), u^*(\tau)) + \\ &\qquad + \big(\mu(\tau^-) - \mu(\tau^+)\big)^T \frac{\partial g}{\partial u}(x^*(\tau), u^*(\tau)) = 0. \end{aligned} \quad (7.70)$$

Recall that $h_i(x^*(t)) = 0$ implies $\frac{\mathrm{d}h_i}{\mathrm{d}x}(x^*(t)) f(t, x^*(t), u^*(t)) = 0$ at continuity points of u^*. Therefore, if $\frac{\mathrm{d}h_i}{\mathrm{d}x}(x^*(\tau)) f(\tau, x^*(\tau), u^*(\tau)) \neq 0$, $i = 1, \ldots, q$ and τ is a continuity point of u^*, we have $h_i(x^*(\tau)) > 0$. Then $h_i(x^*(t)) > 0$ on some neighbourhood $\mathscr{O}(\tau)$

[11] In the first part of the proof we follow the proof given in Feichtinger and Hartl (1986) [p. 168] for the direct adjoining approach.

of τ, since both h and x^* are continuous functions. Equality (7.56) then implies that $\tilde{\nu}_i(t) = 0$ on $\mathcal{O}(\tau)$, hence $\tilde{\nu}_i$ is continuous at τ. Moreover, for $g_j(x^*(\tau), u^*(\tau)) > 0$, $j = 1, \dots, p$, we have that $g_j(x^*(t), u^*(t)) > 0$ on some neighbourhood $\mathcal{O}(\tau)$ of τ. It then follows from condition (vii) in Theorem 7.1 that $\mu_j(t) = 0$ on $\mathcal{O}(\tau)$, implying that $\mu_j(t)$ is continuous at τ. As a result, defining

$$I(\tau) := \{i \,|\, \frac{\mathrm{d}h_i}{\mathrm{d}x}(x^*(\tau)) f(\tau, x^*(\tau), u^*(\tau)) \neq 0\} \tag{7.71}$$

and

$$J(\tau) := \{j \,|\, g_j(x^*(\tau), u^*(\tau)) > 0\} \tag{7.72}$$

we have just shown that $\nu_i(\tau)$ is continuous for all $i \in I(\tau)$, $\mu_j(\tau)$ is continuous for all $j \in J(\tau)$ and

$$\sum_{i \in I(\tau)} \left(\left(\nu(\tau^-) - \nu(\tau^+)\right)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)) \frac{\partial f}{\partial u}(\tau, x^*(\tau), u^*(\tau)) \right) + \\ + \sum_{j \in J(\tau)} \left(\left(\mu(\tau^-) - \mu(\tau^+)\right)^T \frac{\partial g}{\partial u}(x^*(\tau), u^*(\tau)) \right) = 0. \tag{7.73}$$

Hence, Equality (7.70) can be rewritten to

$$\sum_{i \notin I(\tau)} \left(\left(\nu(\tau^-) - \nu(\tau^+)\right)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)) \frac{\partial f}{\partial u}(\tau, x^*(\tau), u^*(\tau)) \right) + \\ + \sum_{j \notin J(\tau)} \left(\left(\mu(\tau^-) - \mu(\tau^+)\right)^T \frac{\partial g}{\partial u}(x^*(\tau), u^*(\tau)) \right) = 0. \tag{7.74}$$

The strong constraint qualification implies that vectors

$$\frac{\mathrm{d}h_i}{\mathrm{d}x}(x^*(\tau)) \frac{\partial f}{\partial u}(\tau, x^*(\tau), u^*(\tau)) \quad \text{for } i \text{ such that} \quad \frac{\mathrm{d}h_i}{\mathrm{d}x}(x^*(\tau)) f(\tau, x^*(\tau), u^*(\tau)) = 0 \tag{7.75}$$

and

$$\frac{\partial g_j}{\partial u}(x^*(\tau), u^*(\tau)) \quad \text{for } j \text{ such that} \quad g_j(x^*(\tau), u^*(\tau)) = 0 \tag{7.76}$$

are linearly independent. Hence, we have that $\nu_i(\tau^-) = \nu_i(\tau^+)$ for all $i \notin I(\tau)$ and $\mu_j(\tau^-) = \mu_j(\tau^+)$ for all $j \notin J(\tau)$.

To sum up, we can conclude that $\nu(\tau^-) = \nu(\tau^+)$ and $\mu(\tau^-) = \mu(\tau^+)$ at all continuity points of u^*. Condition (vi) in Theorem 7.1 then implies that also $\psi(\tau^-) = \psi(\tau^+)$.

From $\mu(\tau^-) = \mu(\tau^+)$ and (7.60) we immediately have that $\tilde{\mu}$ is continuous everywhere with the possible exception of the discontinuity points of u^*. Moreover, we show that $\tilde{\nu}_j$, $j = 1, \ldots, q$ is continuous at τ if τ is not a discontinuity point of u^* or an entry time of h_j. Indeed, if τ is an exit time of h_j, we have that $\tilde{\nu}_j(\tau) = \tilde{\nu}_j(\tau^+) = 0$ because $h_j(x^*(t)) > 0$ on a punctured right neighbourhood of τ. According to Equality (7.56), we have

$$\tilde{\nu}_j(\tau^-) - \tilde{\nu}_j(\tau^+) = \left(\nu_j(\tau^-) - \nu_j(\tau^+)\right) - 0 = 0. \tag{7.77}$$

If τ is a contact time of h_j, we have that $\tilde{\nu}_j(\tau^-) = \tilde{\nu}_j(\tau^+) = 0$ because $h_j(x^*(t)) > 0$ on a punctured neighbourhood of τ. Finally, it is straightforward that $\tilde{\nu}_j(\tau)$ is continuous at τ if $\nu_j(\tau)$ is continuous at τ and τ is not a junction time. As a result, $\tilde{\nu}$ is a continuous function everywhere with the possible exception of the discontinuity points of u^* or entry times.

To show that the lemma is true also for $\tilde{\psi}$, we use (7.65) and the continuity of $\bar{\psi}$ to obtain

$$\left(\tilde{\psi}(\tau^-) - \tilde{\psi}(\tau^+)\right)^T = \left(\nu(\tau^-) - \tilde{\nu}(\tau^-) - (\nu(\tau^+) - \tilde{\nu}(\tau^+))\right)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)). \tag{7.78}$$

Because ν is continuous everywhere with the possible exception of the discontinuity points of u^* and $\tilde{\nu}$ is continuous everywhere possibly except the discontinuity points of u^* and the entry times, we immediately have that their difference is also continuous everywhere possibly except the discontinuity points of u^* and the entry times. This completes the proof of this lemma. □

Let us now prove that if an admissible solution to Problem (7.1a) - (7.1d) satisfies necessary conditions formulated by Seierstad and Sydsæter (1987) and the strong constraint qualification, then it satisfies necessary conditions formulated by Feichtinger and Hartl (1986), possibly except the "non-triviality condition" (ii). Note that the formulation of both types of necessary conditions assumes only the weak constraint qualification to be fulfilled. The requirement of the strong constraint qualification was used in Lemma 7.3 which is subsequently used in the proof of Theorem 7.4.

Theorem 7.4. *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.1a) – (7.1d) satisfying the strong constraint qualification, which fulfills the conditions (i) – (ix) in Theorem 7.1 together with ψ^0, ψ, μ and ν. Suppose further that the Assumption (P) is satisfied. Then the conditions (i) and (iii) – (xi) of Theorem 7.2 are met, where $\tilde{\psi}^0$ and the functions $\tilde{\psi}$, $\tilde{\mu}$, $\tilde{\nu}$ and $\tilde{\eta}$ are defined by (7.56) – (7.60).*

Proof of (i). This is exactly the same as the condition (i) in Theorem 7.1.

Proof of (iii). Note first that from (7.8), (7.57) and (7.58) one has

$$H(t,x^*(t),u,\tilde{\psi}^0,\tilde{\psi}(t)) = H(t,x^*(t),u,,\psi^0,\psi(t)) - \tilde{\nu}(t)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t))f(t,x^*(t),u). \tag{7.79}$$

However, the second term on the right-hand side is zero with the possible exception of the junction times: The reason is, that if $h_j(x^*(t)) > 0$, then $\tilde{\nu}_j(t) = 0$ (from (7.57)). On the other hand, if $h_j(x^*(t)) = 0$ and t is not a junction time, then

$$\frac{\mathrm{d}h_j}{\mathrm{d}t}(x^*(t)) = \frac{\mathrm{d}h_j}{\mathrm{d}x}(x^*(t))\,f(t,x^*(t),u^*(t)) = 0. \tag{7.80}$$

Hence, we have that the inequality in (iii) in Theorem 7.1 is the same as the inequality in (iii) in Theorem 7.2 (possibly except junction times), but the Hamiltonian is maximized in Theorem 7.2 with respect to a subset of a set given in condition (iii) in Theorem 7.1. This implies that condition (iii) in Theorem 7.2 is satisfied everywhere with the possible exception of discontinuity points of u^* and entry times, since $\psi(t)$ is continuous everywhere with the possible exception of discontinuity points of u^* and entry times (see Lemma 7.3).

Proof of (iv). Using the definitions of Lagrangian (7.9) and simplified Lagrangian (7.10) combined with (7.57), (7.58) and (7.60), we obtain

$$\begin{aligned} & L(t,x^*(t),u^*(t),\tilde{\psi}^0,\tilde{\psi},\tilde{\mu},\tilde{\nu}) - \breve{L}(t,x^*(t),u^*(t),\psi^0,\psi,\mu) \\ &= \tilde{\psi}^0 f^0(t,x^*(t),u^*(t)) + \left(\psi^T - \tilde{\nu}^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t))\right) f(t,x^*(t),u^*(t)) + \mu^T g(x^*(t),u^*(t)) + \\ &\quad + \tilde{\nu}^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t))f(t,x^*(t),u^*(t)) - \psi^0 f^0(t,x^*(t),u^*(t)) - \\ &\quad - \psi^T f(t,x^*(t),u^*(t)) - \mu^T g(x^*(t),u^*(t)) = 0 \end{aligned} \tag{7.81}$$

for all $t \in \langle t_0, t_1 \rangle$. Both the functions L and $\breve{L}$ are C^1-functions w.r.t. u, therefore we can write for all t

$$\frac{\partial L}{\partial u}\Big(t,x^*(t),u^*(t),\tilde{\psi}^0,\tilde{\psi}(t),\tilde{\mu}(t),\tilde{\nu}(t)\Big) = \frac{\partial \breve{L}}{\partial u}\Big(t,x^*(t),u^*(t),\psi^0,\psi(t),\mu(t)\Big). \tag{7.82}$$

We have already proved in Lemma 7.3 that for each $t \in (t_0, t_1)$ such that it is not a discontinuity point of u^* neither an entry time, the functions ψ and μ are both continuous. Hence, combining (7.82) together with condition (iv) in Theorem 7.1 implies

$$\frac{\partial L}{\partial u}\Big(t,x^*(t),u^*(t),\tilde{\psi}^0,\tilde{\psi}(t),\tilde{\mu}(t),\tilde{\nu}(t)\Big) = 0 \tag{7.83}$$

for each $t \in (t_0, t_1)$ such that it is not a discontinuity point of u^* neither an entry time.

Proof of (v). According to (7.56), one has that $\tilde{\nu}_j(t) = 0$, $j = 1, \ldots, q$ for all t such that $h_j(x^*(t)) > 0$. Moreover, for all t such that t is not a junction time and $h_j(x^*(t)) = 0$ it follows from (7.56) that $\tilde{\nu}_j(t) = \nu_j(t) - \nu_j(\tau^+) \geq 0$ because $t \leq \tau$ and ν_j is a non-increasing function. Hence, the first equality in (v) is proved. The second equality is implied directly by (7.56). Finally the third equality in (v) follows from the fact that $\tilde{\nu}$ is differentiable almost everywhere since it is piecewise monotonic and $\dot{\tilde{\nu}} \leq 0$ almost everywhere since $\tilde{\nu}$ is non-increasing at all points of continuity (because ν is non-increasing everywhere).

Proof of (vi). One has for all $t \in \langle t_0, t_1 \rangle$ with the possible exception of the discontinuity points of $u^*(t)$ and the junction times that

$$\begin{aligned}
\dot{\tilde{\psi}}(t)^T &\overset{(7.65)}{=} \dot{\bar{\psi}}(t)^T + \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\nu(t) - \tilde{\nu}(t)\right)^T \frac{\mathrm{d}h(x^*(t))}{\mathrm{d}x}\right] \\
&\overset{\text{(vi)}}{=} -\frac{\partial L}{\partial x}\Big(x^*(t), u^*(t), \psi^0, \bar{\psi}(t), \mu(t), \nu(t)\Big) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\nu(t) - \tilde{\nu}(t)\right)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t))\right] = \\
&\overset{(7.64)}{=} -\frac{\partial L}{\partial x}\Big(x^*(t), u^*(t), \tilde{\psi}^0, \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big),
\end{aligned}$$

where (vi) refers to the condition (vi) in Theorem 7.1.

Proof of (vii). This condition is implied directly by (7.60) and condition (vii) in Theorem 7.1.

Proof of (viii). This is implied directly by (7.56).

Proof of (ix). The conditions in (ix) can be obtained from the conditions (ix) in Theorem 7.1 using (7.57) and the previous condition (viii) in Theorem 7.1.

Proof of (x). Recalling (7.57) and (7.58), one has for all $\tau \in (t_0, t_1)$ that

$$\begin{gathered}
H\big(\tau, x^*(\tau), u^*(\tau^-), \tilde{\psi}^0, \tilde{\psi}(\tau^-)\big) - H\big(\tau, x^*(\tau), u^*(\tau^+), \tilde{\psi}^0, \tilde{\psi}(\tau^+)\big) \\
= H\big(\tau, x^*(\tau), u^*(\tau^-), \psi^0, \psi(\tau^-)\big) - \tilde{\nu}(\tau^-)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)) f(\tau, x^*(\tau), u^*(\tau^-)) - \\
-H\big(\tau, x^*(\tau), u^*(\tau^+), \psi^0, \psi(\tau^+)\big) + \tilde{\nu}(\tau^+)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)) f(\tau, x^*(\tau), u^*(\tau^+)).
\end{gathered}$$

Now we prove

$$\tilde{\nu}(\tau^-)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)) f(\tau, x^*(\tau), u^*(\tau^-)) = 0 \tag{7.84a}$$

and

$$\tilde{\nu}(\tau^+)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau)) f(\tau, x^*(\tau), u^*(\tau^+)) = 0, \tag{7.84b}$$

which proves (xi), because

$$H\big(\tau, x^*(\tau), u^*(\tau^-), \psi^0, \psi(\tau^-)\big) - H\big(\tau, x^*(\tau), u^*(\tau^+), \psi^0, \psi(\tau^+)\big) = 0 \tag{7.85}$$

due to Equality (7.15). Under the assumption (P), for all $j = 1, \ldots, q$ there exists $\varepsilon > 0$ such that either $h_j(x^*(t)) = 0$ for all $t \in (\tau - \varepsilon, \tau)$ or $h_j(x^*(t)) > 0$ for all $t \in (\tau - \varepsilon, \tau)$. In the former case we have

$$\frac{\mathrm{d}h_j}{\mathrm{d}t}(x^*(\tau^-)) = \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\tau))\, f(\tau, x^*(\tau), u^*(\tau^-)) = 0 \tag{7.86}$$

On the other hand, $\tilde{\nu}_j(\tau^-) = 0$ in the latter case. To sum up, we can conclude that Equality (7.84a) is satisfied. Equality (7.84b) can be proved in an analogous way.

Proof of (xi). We show that $\tilde{\eta}$ defined by (7.59) fulfills all conditions given in (xi) in Theorem 7.2.

One has

$$\begin{aligned}
\tilde{\psi}(t^-) - \tilde{\psi}(t^+) &\overset{(7.65)}{=} \Big[(\bar{\psi}(t^-) - \bar{\psi}(t^+)) + \big(\nu(t^-) - \tilde{\nu}(t^-)\big) - \\
&\quad - \big(\nu(t^+) - \tilde{\nu}(t^+)\big)\Big] \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) \\
&\overset{(vi)}{=} \Big[\big(\nu(t^-) - \tilde{\nu}(t^-)\big) - \big(\nu(t^+) - \tilde{\nu}(t^+)\big)\Big] \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) \\
&\overset{(7.59)}{=} \tilde{\eta}_j(t)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)),
\end{aligned} \tag{7.87}$$

where (vi) refers to condition (vi) in Theorem 7.1. This proves (7.48).

According to (7.56), we have for all $j = 1, \ldots, q$ that

$$\nu_j(\tau^-) - \tilde{\nu}_j(\tau^-) = \begin{cases} \nu_j(\tau^-) & \text{if } h_j(x^*(\tau)) > 0 \text{ or } \tau \text{ is an entry or contact time,} \\ \nu_j(\tau_1^+) & \text{if } h_j(x^*(\tau)) = 0 \text{ and } \tau \text{ is not an entry or contact time,} \end{cases} \tag{7.88}$$

where τ_1 is an exit time for h_j which is the nearest to τ and which is greater or equal to τ or $\tau_1 = t_1$ if no such exit time exists. Indeed, if $h_j(x^*(\tau)) = 0$ and τ is not an entry or contact time, we have $h_j(x^*(t)) = 0$ on some left neighbourhood $\mathscr{O}(\tau^-)$ of τ. Equality (7.56) then implies $\nu_j(t) - \tilde{\nu}_j(t) = \nu_j(\tau_1^+)$ on $\mathscr{O}(\tau^-)$, where τ_1 is an exit time for h_j which is the nearest to t and which is greater or equal to τ or $\tau_1 = t_1$ if no such exit time exists. On the other hand, if τ is an entry or contact time or $h_j(x^*(\tau)) > 0$, we have $h_j(x^*(t)) > 0$ on some left punctured neighbourhood $\mathscr{O}(\tau^-)$ of τ. In this case, Equality (7.56) implies that $\nu_j(t) - \tilde{\nu}_j(t) = \nu_j(t)$ on $\mathscr{O}(\tau^-)$, hence $\nu_j(\tau^-) - \tilde{\nu}_j(\tau^-) = \nu_j(\tau^-)$.

Similarly,

$$\nu_j(t^+) - \tilde{\nu}_j(t^+) = \begin{cases} \nu_j(t^+) & \text{if } h_j(x^*(t)) > 0 \text{ or } t \text{ is a contact time,} \\ \nu_j(\tau_2^+) & \text{if } h_j(x^*(t)) = 0 \text{ and } t \text{ is not a contact time,} \end{cases} \tag{7.89}$$

where τ_2 is an exit time for h_j which is the nearest to t and which is greater or equal to t or $\tau_2 = t_1$ if no such exit time exists.

Suppose first that $\tau \in (t_0, t_1)$ is not a junction time for any h_j, $j = 1, \ldots, q$. We can consider two cases:

(a) If $h_j(x^*(\tau)) = 0$, the second condition in (7.49) is fulfilled. Equality (7.88) then implies that $\nu_j(\tau^-) - \tilde{\nu}_j(\tau^-) = \nu_j(\tau_1^+)$ (because τ is not an entry time), where τ_1 is an exit time for h_j which is the nearest to τ and which is greater or equal to τ or $\tau_1 = t_1$ if no such exit time exists. Equality (7.89) implies $\nu_j(\tau^+) - \tilde{\nu}_j(\tau^+) = \nu_j(\tau_1^+)$ because an exit time for h_j which is the nearest to τ and which is greater or equal to τ, is again τ_1. Hence, $\tilde{\eta}_j(\tau) = 0$ and the first condition in (7.49) is satisfied.

(b) If $h_j(x^*(\tau)) > 0$ then $h_j(x^*(t)) > 0$ also on some neighbourhood $\mathscr{O}(\tau)$. Equalities (7.88) and (7.89) then imply that $\tilde{\eta}_j(\tau) = \nu_j(\tau^-) - \nu_j(\tau^+)$. From condition (v) in Theorem 7.1 we know that ν_j is constant on $\mathscr{O}(\tau)$. Hence $\tilde{\eta}_j(\tau) = 0$ which implies that both conditions in (7.49) are satisfied.

Now, suppose that $\tau \in (t_0, t_1)$ is a junction time for some h_j. Hence, $h_j(\tau) = 0$ and the second equality in (7.49) is satisfied. It remains to prove that also the first equality in (7.49) is satisfied. Let us distinguish three cases:

(a) If τ is an entry time for h_j, then (7.88) implies that $\nu_j(\tau^-) - \tilde{\nu}_j(\tau^-) = \nu_j(\tau^-)$ and (7.89) implies that $\nu_j(\tau^+) - \tilde{\nu}_j(\tau^+) = \nu_j(\tau_2^+) \geq 0$, where τ_2 is an exit time for h_j which is the nearest to τ and which is greater or equal to τ or $\tau_2 = t_1$ if no such exit time exists. Hence $\tilde{\eta}_j(\tau) = \nu_j(\tau^-) - \nu_j(\tau_2^+) \geq 0$ because $\tau < \tau_2$ and ν_j is a non-increasing function.

(b) If τ is a contact time for h_j, then (7.88) implies that $\nu_j(\tau^-) - \tilde{\nu}_j(\tau^-) = \nu_j(\tau^-)$ and (7.89) implies that $\nu_j(\tau^+) - \tilde{\nu}_j(\tau^+) = \nu_j(\tau^+)$. Hence $\tilde{\eta}_j(\tau) = \nu_j(\tau^-) - \nu_j(\tau^+)$ which is non-negative again because ν_j is a non-increasing function.

(c) If τ is an exit time for h_j, then (7.88) implies that $\nu_j(\tau^-) - \tilde{\nu}_j(\tau^-) = \nu_j(\tau^+)$ because τ itself is an exit time and (7.89) implies that $\nu_j(\tau^+) - \tilde{\nu}_j(\tau^+) = \nu_j(\tau^+)$. Hence $\tilde{\eta}_j(\tau) = \nu_j(\tau^+) - \nu_j(\tau^+) = 0$ and we can conlude that $\tilde{\psi}$ is continuous at exit times. □

In Theorem 7.4, we have proved that if an admissible solution to Problem (7.1a) – (7.1d) satisfies the strong constraint qualification and the necessary conditions stated by Seierstad and Sydsæter (1987) are met, then it satisfies also necessary conditions formulated by Feichtinger and Hartl (1986) (possibly except condition (ii)). However, the "converse" of this result can also be proved in the following form: If an admissible solution to Problem (7.1a) -Ű (7.1d) satisfies necessary conditions formulated by Feichtinger and Hartl (1986) and the weak constraint qualification, then it satisfies necessary conditions formulated by Seierstad and Sydsæter (1987), possibly except the condition (iii). Hartl et al. (1995) and Feichtinger and Hartl (1986) suggest that this condition is not implied by conditions given in Theorem 7.2. They argue that if the Kuhn-Tucker conditions (iv) and (vii) in Theorem 7.1 are not sufficient for maximizing the Hamiltonian according to (iii), then Theorem 7.1 provides more information than does Theorem 7.2. However, no example to support this conclusion was given.

Before we prove the above-mentioned "converse" of Theorem 7.4, we introduce the transformation of multipliers which are reversed to those given in (7.56) – (7.60).

Lemma 7.4. *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.1a) – (7.1d), let ψ^0, ψ, μ and ν be multipliers according to Theorem 7.1 and let $\tilde{\psi}^0$, $\tilde{\psi}$, $\tilde{\mu}$, $\tilde{\eta}$ and $\tilde{\nu}$ be defined by (7.56) – (7.60). Then ψ^0, $\psi(t)$, $\mu(t)$ and $\nu(t)$ satisfy the following equalities:*

$$\psi(t)^T = \tilde{\psi}(t)^T + \tilde{\nu}(t)^T \frac{dh}{dx}(x^*(t)), \tag{7.90}$$

$$\psi^0 = \tilde{\psi}^0, \tag{7.91}$$

$$\mu(t) = \tilde{\mu}(t), \tag{7.92}$$

$$\nu(t) = \tilde{\nu}(t) + \sum_{\tau \geq t} \tilde{\eta}(\tau) \tag{7.93}$$

for all $t \in \langle t_0, t_1 \rangle$.

Proof. The proof is straightforward for Equalities (7.90), (7.91) and (7.92). Indeed, (7.90) is directly implied by (7.57), (7.91) follows from (7.58) and (7.92) follows from (7.60). To show that also (7.93) follows from (7.56) and (7.59), a more deeper analysis is needed: For a given $j = 1, \ldots, q$, let us define

$$\Lambda_j(\bar{t}) := \left\{ \tau \geq \bar{t} \,\middle|\, \tau \text{ is an entry or a contact time for } h_j \right\} \tag{7.94}$$

and

$$\xi_j(\bar{t}) := \tilde{\nu}_j(\bar{t}) + \sum_{\tau \in \Lambda_j(t)} \tilde{\eta}_j(\tau). \tag{7.95}$$

We have to prove that $\xi_j(\bar{t}) = \nu_j(\bar{t})$.

Note that the set $\Lambda_j(\bar{t})$ is finite (and possibly empty) due to Property (P). If it is empty, it means that there is no entry or contact time which is greater or equal to $\bar{t}$. In this case, Equality (7.56) together with condition (viii) in Theorem 7.1) implies that $\tilde{\nu}(\bar{t}) = \nu(\bar{t})$. In addition, (7.95) implies that $\xi_j(\bar{t}) = \tilde{\nu}_j(\bar{t})$, hence $\xi_j(\bar{t}) = \nu_j(\bar{t})$ and the proof is concluded in this case.

Let us now assume that $\Lambda_j(\bar{t})$ is not empty. Given j and t fixed, we have that

$$\Lambda_j(\bar{t}) =: \{\tau_i\}_{i=1}^{n}. \tag{7.96}$$

Using (7.59), we obtain

$$\xi_j(\bar{t}) = \tilde{\nu}_j(\bar{t}) + \sum_{i=1}^{n} \Big[\big(\nu_j(\tau_i^-) - \tilde{\nu}_j(\tau_i^-)\big) - \big(\nu_j(\tau_i^+) - \tilde{\nu}_j(\tau_i^+)\big) \Big]. \tag{7.97}$$

According to (7.88), we have

$$\nu_j(\tau_i^-) - \tilde{\nu}_j(\tau_i^-) = \nu_j(\tau_i^-). \tag{7.98}$$

Similarly, (7.89) implies

$$\nu_j(\tau_i^+) - \tilde{\nu}_j(\tau_i^+) = \begin{cases} \nu_j(\tau_i^+) & \text{if } \tau_i \text{ is a contact time,} \\ \nu_j(\hat{\tau}_i^+) & \text{if } \tau_i \text{ is an entry time,} \end{cases} \tag{7.99}$$

where $\hat{\tau}_i$ is an exit time for h_j which is the nearest to t and which is greater than τ_i or $\hat{\tau}_i = t_1$ if no such exit time exists.

For a given $i = 1, \ldots, n-1$, let us now distinguish two cases:

(a) If τ_i, $i = 1, \ldots, n-1$, is a contact time, the nearest junction time τ_{i+1} which is greater than τ_i is either an entry time or a contact time again. Hence, $h_j(x^*(t)) > 0$ on (τ_i, τ_{i+1}). It follows from condition (v) in Theorem 7.1 that ν_j is constant on (τ_i, τ_{i+1}), which implies $\nu_j(\tau_i^+) = \nu_j(\tau_{i+1}^-)$. This together with (7.98) and (7.99) allows us to simplify Equality (7.97) using

$$\begin{aligned} \big(\nu_j(\tau_i^-) - \tilde{\nu}_j(\tau_i^-)\big) - \big(\nu_j(\tau_i^+) - \tilde{\nu}_j(\tau_i^+)\big) &= \nu_j(\tau_i^-) - \nu_j(\tau_i^+) \\ &= \nu_j(\tau_i^-) - \nu_j(\tau_{i+1}^-), \end{aligned} \tag{7.100}$$

where $i = 1, \ldots, n-1$ is such that τ_i is a contact time.

(b) If τ_i, $i = 1, \ldots, n-1$, is an entry time, the nearest junction time which is greater than τ_i is an exit time which is denoted by $\hat{\tau}_i$ in (7.99). Furthermore, the nearest greater junction time to $\hat{\tau}_i$ is an entry time or a contact time (denoted by τ_{i+1} and one has $h_j(x^*(t)) > 0$ on $(\hat{\tau}_i, \tau_{i+1})$. Again, condition (v) in Theorem 7.1 implies that $\hat{\nu}_j(\hat{\tau}_i^+) = \nu_j(\tau_{i+1}^-)$ since ν_j is constant on $(\hat{\tau}_i, \tau_{i+1})$. In this case, we can use (7.98) and (7.99) to write

$$\begin{aligned}\big(\nu_j(\tau_i^-) - \tilde{\nu}_j(\tau_i^-)\big) - \big(\nu_j(\tau_i^+) - \tilde{\nu}_j(\tau_i^+)\big) &= \nu_j(\tau_i^-) - \nu_j(\hat{\tau}_i^+) \\ &= \nu_j(\tau_i^-) - \nu_j(\tau_{i+1}^-), \quad (7.101)\end{aligned}$$

where $i = 1, \ldots, n-1$ is such that τ_i is an entry time.

Furthermore, we can consider four cases for τ_n:

(a) If $\tau_n < t_1$ is a contact time, we have that $h_j(x^*(t)) > 0$ on $(\tau_n, t_1\rangle$. Conditions (v) and (viii) in Theorem 7.1 then implies that $\nu_j(\tau_n^+) = 0$.

(b) If $\tau_n < t_1$ is an entry time and $h_j(x^*(t_1^-)) > 0$ (i.e. these exists an exit time $\hat{\tau}_n$ between τ_n and t_1), it follows from conditions (v) and (viii) in Theorem 7.1 that $\nu(\hat{\tau}_n^+) = 0$.

(c) If $\tau_n < t_1$ is an entry time and $h_j(x^*(t)) = 0$ on $\langle\tau_n, t_1\rangle$, then $\hat{\tau}_n$, which is refered to in (7.99), is equal to t_1 and $\nu_j(\hat{\tau}_n^+) = 0$.

(d) If $\tau_n = t_1$, then Equality (7.99) together condition (viii) in Theorem 7.1 directly imply that $\nu_j(\tau_n^+) - \tilde{\nu}_j(\tau_n^+) = 0$.

In all four cases we have obtained

$$\nu_j(\tau_n^+) - \tilde{\nu}_j(\tau_n^+) = 0. \quad (7.102)$$

Hence, according to (7.98) and (7.102) we have

$$\big(\nu_j(\tau_n^-) - \tilde{\nu}_j(\tau_n^-)\big) - \big(\nu_j(\tau_n^+) - \tilde{\nu}_j(\tau_n^+)\big) = \nu_j(\tau_n^-), \quad (7.103)$$

Now we can combine (7.100), (7.101) and (7.103) with (7.97), which yields

$$\xi_j(\bar{t}) = \tilde{\nu}_j(\bar{t}) + \sum_{i=1}^{n-1} \left[\nu_j(\tau_i^-) - \nu_j(\tau_{i+1}^-)\right] + \nu_j(\tau_n^-) = \tilde{\nu}_j(\bar{t}) + \nu_j(\tau_1^-). \quad (7.104)$$

Once again we have to distinguish several cases:

(a) If $h_j(x^*(\bar{t})) > 0$, then there exists $\tau_0 \in \langle t_0, \bar{t})$ such that $h_j(x^*(t)) > 0$ on (τ_0, τ_1). Then $\nu_j(\bar{t}) = \nu_j(\tau_1^-)$, since condition (v) in Theorem 7.1 states that $\nu_j(t)$ is constant on (τ_0, τ_1). In addition, (7.56) implies that $\tilde{\nu}_j(\bar{t}) = 0$. As a results, it follows from (7.104) that $\xi_j(\bar{t}) = \nu_j(\bar{t})$.

(b) If $\bar{t}$ is an entry time or a contact time, then actually $\bar{t} = \tau_1$. Equality (7.104) yields

$$\xi_j(\bar{t}) = \xi_j(\tau_1) = \tilde{\nu}_j(\tau_1) + \nu_j(\tau_1^-) = \nu_j(\tau_1), \tag{7.105}$$

since $\tilde{\nu}_j(\tau_1) = \tilde{\nu}_j(\tau_1^-) = 0$ at any entry or contact time and $\nu_j(\tau_1^-) = \nu_j(\tau_1)$.

(c) If $h_j(x^*(\bar{t})) = 0$ and $\bar{t}$ is neither an entry time nor a contact time, then (7.56) states that $\nu_j(\bar{t}) = \tilde{\nu}_j(\bar{t}) + \nu_j(\hat{\tau}^+)$, where $\hat{\tau}$ is an exit time nearest to $\bar{t}$ which is greater or equal to $\bar{t}$ (or t_1 if no such exit time exists). Moreover, condition (v) in Theorem 7.1 implies that $\nu_j(\hat{\tau}^+) = \nu_j(\tau_1^-)$, where τ_1 is the entry or contact time which is the nearest greater to $\bar{t}$ and $h_j(x^*(t) > 0$ for $t \in (\hat{\tau}, \tau_1)$. By combining these results we obtain that $\nu_j(\bar{t}) = \tilde{\nu}_j(\bar{t}) + \nu_j(\tau_1^-) = \xi_j(\bar{t})$.

In all three cases (a) – (c) we have obtained that $\xi_j(\bar{t}) = \nu_j(\bar{t})$, hence the proof is concluded. □

Using Equalities (7.90) – (7.93), the following theorem can be formulated:

Theorem 7.5. *Let $(x^*(t), u^*(t))$ be an admissible solution to Problem (7.1a) – (7.1d) satisfying the weak constraint qualification, which satisfies the conditions (i) – (xi) in Theorem 7.2 together with $\tilde{\psi}^0$, $\tilde{\psi}$, $\tilde{\mu}$, $\tilde{\nu}$ and $\tilde{\eta}$. Suppose further that Assumption (P) is satisfied. Then the conditions (i), (ii) and (iv) – (ix) of Theorem 7.1 are met, where the multipliers ψ^0, ψ, μ and ν are defined by (7.90) – (7.93).*

Proof of (i). This is exactly the same as condition (i) in Theorem 7.2.

Proof of (ii). Assume for contradiction that $(\tilde{\psi}^0, \tilde{\psi}(t)) \neq (0,0)$ for all $t \in (t_0, t_1)$ but $\psi^0 = 0$,

$$\psi(\bar{t})^T = \tilde{\psi}(\bar{t})^T + \tilde{\nu}(\bar{t})^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(\bar{t})) = 0 \tag{7.106}$$

for some $\bar{t}$ (using (7.90)) and $\nu(t_1) - \nu(t_0) = 0$. The last equality implies that $\nu(t) = 0$ for all t because ν is a non-increasing function and $\nu(t_1) = 0$. Then also $\tilde{\nu}(t) = 0$ for all t since (7.93) then implies

$$0 = \nu(t) = \tilde{\nu}(t) + \sum_{\tau \geq t} \tilde{\eta}(\tau) \tag{7.107}$$

and both $\tilde{\nu}$ and $\tilde{\eta}$ are non-negative functions according to conditions (vii) and (xi) in Theorem 7.2. However, (7.106) then implies that $\tilde{\psi}(\bar{t}) = 0$. However, it follows from (7.91) that $\tilde{\psi}^0 = 0$ since we have assumed that $\psi^0 = 0$. Hence $(\tilde{\psi}^0, \tilde{\psi}(\bar{t})) = (0, 0)$, a contradiction.

Proof of (iv). Using the definitions of Lagrangian (7.9) and simplified Lagrangian (7.10) combined with (7.90) – (7.92), we obtain

$$L(t, x^*(t), u^*(t), \tilde{\psi}^0, \tilde{\psi}, \tilde{\mu}, \tilde{\nu}) - \breve{L}(t, x^*(t), u^*(t), \psi^0, \psi, \mu) =$$

$$= \tilde{\psi}^0 f^0(t, x^*(t), u^*(t)) + \tilde{\psi}^T f(t, x^*(t), u^*(t)) + \mu^T g(x^*(t), u^*(t)) +$$

$$+ \tilde{\nu}^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) f(t, x^*(t), u^*(t)) - \psi^0 f^0(t, x^*(t), u^*(t)) -$$

$$- \Big(\tilde{\psi}(t)^T + \tilde{\nu}(t)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t))\Big) f(t, x^*(t), u^*(t)) - \mu^T g(x^*(t), u^*(t)) = 0 \qquad (7.108)$$

for all $t \in \langle t_0, t_1 \rangle$. Both the functions L and $\breve{L}$ are C^1-functions w.r.t. u, therefore we can write for all t

$$\frac{\partial L}{\partial u}\Big(t, x^*(t), u^*(t), \tilde{\psi}^0, \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big) = \frac{\partial \breve{L}}{\partial u}\Big(t, x^*(t), u^*(t), \psi^0, \psi(t), \mu(t)\Big). \qquad (7.109)$$

Hence, it follows from condition (iv) in Theorem 7.2

$$\frac{\partial \breve{L}}{\partial u}\Big(t, x^*(t), u^*(t), \psi^0, \psi(t), \mu(t)\Big) = 0 \qquad (7.110)$$

for all $t \in (t_0, t_1)$ with the possible exception of the discontinuity points of $u^*(t)$ and the junction times. Moreover, functions $\psi(t)$, $\mu(t)$ and $\nu(t)$ have one-sided limits everywhere, thus we have that condition (iv) in Theorem 7.1 is satisfied.

Proof of (v). The first part of this condition is implied by (7.93) and condition (v) in Theorem 7.2 which states that $\tilde{\nu}_j(t) = 0$ almost everywhere whenever $h_j(x^*(t)) > 0$. Indeed, one has $\nu_j(t) = \sum_{\tau > t} \tilde{\eta}_j(\tau)$ on intervals where $h_j(x^*(t)) > 0$, which is a constant on these intervals because they do not contain a junction point. The second part is a consequence of other conditions in Theorem 7.1 (see Lemma 7.1(b)).

Proof of (vi). Let us denote

$$\bar{\psi} := \psi - \nu^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*). \qquad (7.111)$$

Then we have

$$\begin{aligned}
\dot{\bar{\psi}}^T &\overset{(7.111)}{=} \dot{\psi}^T - \frac{\mathrm{d}}{\mathrm{d}t}\left[\nu^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*)\right] \\
&\overset{(7.90)}{=} \dot{\tilde{\psi}}^T + \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\tilde{\nu} - \nu\right)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*)\right] \\
&\overset{(vi)}{=} -\frac{\partial L}{\partial x}\Big(t, x^*, u^*, \tilde{\psi}, \tilde{\mu}, \tilde{\nu}\Big) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\tilde{\nu} - \nu\right)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*)\right] \\
&\overset{(7.64)}{=} -\frac{\partial L}{\partial x}\Big(t, x^*, u^*, \psi, \mu, \nu\Big),
\end{aligned}$$

almost everywhere where (vi) refers to the condition (vi) in Theorem 7.2.

Further, we prove that $\bar{\psi}$ is a continuous function. One has

$$\begin{aligned}
\bar{\psi}(t^-)^T - \bar{\psi}(t^+)^T &\overset{(7.111)}{=} \psi(t^-)^T - \nu(t^-)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) - \psi(t^+)^T + \nu(t^+)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) \\
&\overset{(7.90)}{=} \tilde{\psi}(t^-)^T + \tilde{\nu}(t^-)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) - \nu(t^-)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) - \\
&\quad -\tilde{\psi}(t^+)^T - \tilde{\nu}(t^+)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) + \nu(t^+)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) \\
&\overset{(7.93)}{=} \tilde{\psi}(t^-)^T - \tilde{\psi}(t^+)^T + \left(\tilde{\nu}(t^-) - \left(\tilde{\nu}(t^-) + \sum_{\tau \geq t} \tilde{\eta}(\tau)\right) - \right. \\
&\quad \left. - \tilde{\nu}(t^+) + \left(\tilde{\nu}(t^+) + \sum_{\tau > t} \tilde{\eta}(\tau)\right) - \right)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) \\
&= \tilde{\psi}(t^-)^T - \tilde{\psi}(t^+)^T - \tilde{\eta}(t)^T \frac{\mathrm{d}h}{\mathrm{d}x}(x^*(t)) \\
&\overset{(7.48)}{=} 0.
\end{aligned}$$

Proof of (vii). This is exactly the same as condition (vii) in Theorem 7.2 .

Proof of (viii). This is implied directly by (7.93) and condition (viii) in Theorem 7.2.

Proof of (ix). The conditions in (ix) can be obtained from the previous condition (viii) in Theorem 7.1 and the conditions (ix) in Theorem 7.2 using (7.90). □

Theorem 7.5 states that if an admissible solution to Problem (7.1a) – (7.1d) satisfies the necessary conditions of optimality by Feichtinger and Hartl (1986), then it satisfies also necessary condtions by Seierstad and Sydsæter (1987), possibly except the condition (iii) in Theorem 7.1, given that the weak constraint qualification and Assumption (P) are met. Now we provide an example of a problem such that an admissible solution to this problem satisfies all conditions of optimality by Feichtinger and Hartl together with the weak constraint qualification and Assumption (P), but it does not satisfy the condition (iii) in Theorem 7.1. According to Theorem 7.1, such a solution cannot be an optimal solution to the given problem. Hence, necessary conditions by Seierstad and Sydsæter are stronger in this case in the sense that they are able to exclude an admissible solution from the set of potential candidates for optimal solution. On the other hand, necessary conditions by Feichtinger and Hartl do not exclude this solution from the set of candidates.

Example 7.5. Let us consider the following problem:

$$\begin{aligned}
&\max_{\{u(t)\}} \int_0^1 x(t)\,\mathrm{d}t,\\
&\dot{x}(t) = -u(t)^2(u(t)-1)(u(t)-3),\\
&x(0) = 0,\\
&x(1) \text{ free},\\
&x(t) \geq 0,\\
&u \in \mathbb{R}.
\end{aligned}$$

It is easy to derive that the optimal solution is $u^*(t) = \frac{3+\sqrt{3}}{2}$ and $x^*(t) = \frac{9+6\sqrt{3}}{4}t$. Indeed, it is clearly optimal to increase $x(t)$ by a maximum rate. Note that the function $-u^2(u-1)(u-3)$ has two local maxima: the first one is 0 at $u=0$ and the second one is $\frac{9+6\sqrt{3}}{4}$ at $u = \frac{3+\sqrt{3}}{2}$, which is the global one. However, we show that $\bar{u}(t) = 0$ and $\bar{x}(t) = 0$ also satisfies all conditions of Theorem 7.2. Let us formulate these conditions. We set $\psi^0 = 1$, hence conditions (i) and (ii) are satisfied trivially.

(iii) For almost all $t \in (0,1)$, $\bar{x} + \tilde{\psi}[-\bar{u}^2(\bar{u}-1)(\bar{u}-3)] \geq \bar{x} + \tilde{\psi}[-u^2(u-1)(u-3)]$ for all u such that $-u(t)^2(u(t)-1)(u(t)-3) = 0$ whenever $\bar{x}(t) = 0$, i.e. $u \in \{0,1,3\}$,

(iv) $(-4\bar{u}(t)^3 + 12\bar{u}(t)^2 - 6\bar{u}(t))(\tilde{\psi}(t) + \tilde{\nu}(t)) = 0$,

(v) $\tilde{\nu}(t) \geq 0$, $\tilde{\nu}(t)^T\bar{x}(t) = 0$ and $\dot{\tilde{\nu}}(t) \leq 0$,

(vi) $\dot{\tilde{\psi}}(t) = -1$,

(viii) $\tilde{\nu}(t)$ satisfies $\tilde{\nu}(1) = 0$,

(ix) $\tilde{\psi}(t)$ satisfies $\tilde{\psi}(1) = 0$,

(x) $\bar{x} - \tilde{\psi}[-\bar{u}^2(\bar{u}-1)(\bar{u}-3)]$ is continuous for all $t \in (0,1)$,

(xi) $\tilde{\psi}(t)$ is continuous.

It is easy to verify that all conditions are satisfied for all $t \in \langle 0,1\rangle$ with $\tilde{\psi}(t) = 1-t$ and $\tilde{\nu}(t) = 0$. Note that the weak constraint qualification is trivially satisfied since there are no mixed constraints. In addition, Assumption (P) is also satisfied.

We now show that conditions (i) – (ix) in Theorem 7.1 cannot be satisfied for $(\bar{x}, \bar{u})$. Assume that these conditions are satisfied. Condition (iii) then states that for almost all $t \in \langle 0,1\rangle$,

$$\bar{x}(t) + \psi(t)[-\bar{u}(t)^2(\bar{u}(t)-1)(\bar{u}(t)-3)] \geq \bar{x}(t) + \psi(t)[-u^2(u-1)(u-3)] \tag{7.112}$$

for all $u \in \mathbb{R}$. Hence one has $\psi(t) = 0$ almost everywhere on $\langle 0,1\rangle$, since the function $-u^2(u-1)(u-3)$ attains neither a global maximum nor a global minimum at $\bar{u} = 0$. Furhermore, condition (vi) states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\psi(t) - \nu(t)\big) = -\psi^0 \tag{7.113}$$

almost everywhere. This implies $\dot{\nu}(t) = \psi^0$ almost everywhere, since $\psi(t) = 0$ almost everywhere. However, Theorem 7.1 states that $\nu(t)$ is a non-increasing function on $\langle 0,1\rangle$, hence $\psi^0 \leq 0$ and we obtain from condition (i) that $\psi^0 = 0$. Equality (7.113) now states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\psi(t) - \nu(t)\big) = 0 \tag{7.114}$$

almost everywhere and $\psi(t) - \nu(t)$ is continuous everywhere on $\langle 0,1\rangle$ againg according to the condition (vi). In addition, the terminal condition for (7.114) is $\psi(1) - \nu(1) = 0$ due to conditions (viii) and (ix). Hence we can conclude that $\psi(t) - \nu(t) = 0$ everywhere on $\langle 0,1\rangle$. As we have already noted, $\psi(t) = 0$ almost everywhere and $\nu(t)$ is a non-increasing function. This implies that $\psi(t) = \nu(t) = 0$ for all $t \in \langle t_0, t_1\rangle$. Accordingly, we obtain that $(\psi^0, \psi(t), \nu(0) - \nu(1)) = (0,0,0)$ for all $t \in \langle t_0, t_1\rangle$, which is a contradiction with the condition (ii). ■

7.3 Extensions to the standard problem

In the previous section, we have introduced the standard optimal control problem with mixed and pure state constraints. One of the main results was that we have to deal with discontinuity in costate variables if the pure state constraints become or cease to be active. We have provided two useful forms of necessary conditions of optimality for such problems. However, economic models require some extensions of the standard problem. These extensions include mainly the infinite time horizon. Furthermore, some types of the models also require a scrap value function in the objective criterion and free initial conditions in some state variables. Hence we present the necessary conditions of optimality for these problems.

Let us consider the problem in the following form:

$$
\begin{aligned}
&\max_{\{u(t)\}} \int_{t_0}^{\infty} f^0(t, x(t), u(t))\,\mathrm{d}t + \varphi(x(t_0)),\\
&\dot{x}(t) = f(t, x(t), u(t)),\\
&g(x(t), u(t)) \geq 0,\\
&h(x(t)) \geq 0,\\
&\begin{array}{lll}
x_i(t_0) = x_i^0, \quad i = 1, \dots, l_0 & x_i^0 \text{ all fixed}, & h(x_i^0) > 0,\\
x_i(t_0) \geq x_i^0, \quad i = l_0 + 1, \dots, m_0 & x_i^0 \text{ all fixed}, & h(x_i^0) > 0,\\
x_i(t_0) \text{ free}, \quad i = m_0 + 1, \dots, n & &
\end{array}\\
&\begin{array}{ll}
\liminf_{t\to\infty} x_i(t_1) = x_i^1, \quad i = 1, \dots, l & x_i^1 \text{ all fixed},\\
\liminf_{t\to\infty} x_i(t_1) \geq x_i^1, \quad i = l + 1, \dots, m & x_i^1 \text{ all fixed},\\
\liminf_{t\to\infty} x_i(t_1) \text{ free}, \quad i = m + 1, \dots, n. &
\end{array}
\end{aligned}
\tag{7.115}
$$

In addition to the standard assumptions on f^0, f, g and h, we assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function. Furthermore, we restrict the set of admissible solutions (see Definition 7.1) only for those pairs $(x(t), u(t))$, for which the integral in (7.115) converges. We also set $t_1 = \infty$ in Definitions 7.2 and 7.3 which are modified accordingly.

As stated by Seierstad and Sydsæter (1987) [Theorem 6.9, p. 381 and Theorem 6.16, p. 397], the necessary conditions for Problem (7.115) are as follows:

Theorem 7.6 (Necessary conditions by Seierstad and Sydsæter, infinite horizon problem). *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.115), which fulfills the weak constraint qualification. Then it satisfies conditions in Theorem 7.1 on (t_0, ∞) provided that condition (ii) is replaced by*

$$
\begin{aligned}
&(\psi^0, \psi(t^+)) \neq (0, 0), \text{ for all } t \geq t_0,\\
&(\psi^0, \psi(t^-)) \neq (0, 0), \text{ for all } t > t_0,
\end{aligned}
\tag{7.116}
$$

condition (viii) is deleted and condition (ix) is replaced by *(ix') $\psi(t)$ satisfies*

$$
\begin{array}{ll}
\text{no condition for } \psi_i(t_0), & i = 1, \dots, l_0, \\
\psi_i(t_0) \le -\psi^0 \frac{\partial \varphi^*}{\partial x_i}, \ \left(\psi_i(t_0) + \psi^0 \frac{\partial \varphi^*}{\partial x_i}\right)\left(x_i^*(t_0) - x_i^0\right) = 0, & i = l_0 + 1, \dots, m_0, \\
\psi_i(t_0) = -\psi^0 \frac{\partial \varphi^*}{\partial x_i}, & i = m_0 + 1, \dots, n
\end{array}
$$

where

$$
\frac{\partial \varphi^*}{\partial x_i} = \frac{\partial \varphi}{\partial x_i}(x^*(t_0)), \quad i = l_0 + 1, \dots, n. \tag{7.117}
$$

Now we formulate necesary conditions of optimality for an infinite horizon problem in accordance with Feichtinger and Hartl (1986) [Theorem 7.4, p. 187]:

Theorem 7.7 (Necessary conditions by Feichtinger and Hartl, infinite horizon problem)**.** *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.115) with $l_0 = n$ (i.e. initial values of all state variables are given), which fulfills the weak constraint qualification. Then it satisfies conditions in Theorem 7.2 on (t_0, ∞) provided that conditions (viii) and (ix) are deleted.*

In addition to these conditions, it is useful to have another condition, which is not mentioned by Seierstad–Sydsæter (1987), but is given by Feichtinger and Hartl (1986) for Problem (7.115). It is an analogous conditions to the condition (7.50) given in Theorem 7.3.

Theorem 7.8. [12] *Let $(x^*(t), u^*(t))$ be an optimal solution to Problem (7.115), which fulfills the weak constraint qualification and let $\tilde{\psi}^0$, $\tilde{\psi}$, $\tilde{\mu}$, $\tilde{\nu}$ and $\tilde{\eta}$ be functions according to Theorem 7.2. Then for all $t > t_0$ with the possible exception of the discontinuity points of $u^*(t)$ and the junction times the following condition is satisfied:*

$$
\frac{dL}{dt}\Big(t, x^*(t), u^*(t), \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big) = \frac{\partial L}{\partial t}\Big(t, x^*(t), u^*(t), \tilde{\psi}(t), \tilde{\mu}(t), \tilde{\nu}(t)\Big). \tag{7.118}
$$

In this thesis, we primarily use the necessary conditions by Seierstad–Sydsæter which, contrary to conditions formulated by Feichtinger and Hartl, also include the case of free initial state. On the other hand, Equality (7.118) is only mentioned by Feichtinger and Hartl. However, we know from Theorem 7.5 that if an optimal solution $(x^*(t), u^*(t))$ to Problem (7.115) together with $\tilde{\psi}^0$, $\tilde{\psi}$, $\tilde{\mu}$, $\tilde{\nu}$ and $\tilde{\eta}$ satisfy conditions of Theorem 7.7, then $(x^*(t), u^*(t))$ together with ψ^0, ψ, μ and ν defined by transformations (7.90) - (7.93) satisfy conditions of Theorem 7.6, possibly except condition (iii) (the extention of

[12] See Feichtinger and Hartl (1986) [Theorem 7.4, p. 187].

the proof of Theorem 7.5 to infinite time horizon problems is straightforward). Hence, we can use the same transformations to write Equality (7.118) in terms of Seierstad–Sydsæter type:

Theorem 7.9. [13] *Let $(x^*(t), u^*(t))$ be an admissible solution to Problem (7.115) satisfying the weaker constraint qualification, which satisfies the necessary conditions of optimality given in Theorem 7.6 together with ψ^0, ψ, μ and ν. Then for all $t > t_0$ with the possible exception of the discontinuity points of $u^*(t)$ and the junction times the following condition is satisfied:*

$$\frac{dH}{dt}\Big(t, x^*(t), u^*(t), \psi(t)\Big) = \frac{\partial H}{\partial t}\Big(t, x^*(t), u^*(t), \psi(t)\Big). \tag{7.119}$$

Proof. Recall that from (7.81) one has

$$L(t, x^*, u^*, \tilde{\psi}, \tilde{\mu}, \tilde{\nu}) = \breve{L}(t, x^*, u^*, \psi, \mu) \tag{7.120}$$

for all $t \geq 0$. In addition, from (7.8) and (7.10) one has

$$\breve{L}(t, x^*, u^*, \psi, \mu) = H(t, x^*, u^*, \psi) + \mu^T g(x^*, u^*). \tag{7.121}$$

Hence

$$\frac{\partial}{\partial t}\breve{L}(t, x^*, u^*, \psi, \mu) = \frac{\partial}{\partial t}H(t, x^*, u^*, \psi) \tag{7.122}$$

In addition, we know that $\mu^T(t)g(x^*(t), u^*(t)) = 0$ for all $t \geq 0$, hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mu^T(t)g(x^*(t), u^*(t))\big) = 0 \tag{7.123}$$

which together with (7.121) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\breve{L}(t, x^*, u^*, \psi, \mu) = \frac{\mathrm{d}}{\mathrm{d}t}H(t, x^*, u^*, \psi) \tag{7.124}$$

As a result, (7.118) implies (7.119), using (7.122) and (7.124). □

Another useful condition is the continuity of the Hamiltonian along an optimal solution. For the finite time horizon, this condition was proved in Lemma 7.1(a). However, if we review the proof of this condition, we might check that it only uses conditions (iii) and (iv) which are valid also in the case on infinite horizon. Hence, we can state the following lemma:

[13] To our knowledge, this theorem is original. Seierstad and Sydsæter (1987) formulate an analogous result only for problem with pure state constraints (i.e. without mixed constraints) and finite time horizon only (cf. Seierstad and Sydsæter (1987) [Note 3(f), p. 334]).

Lemma 7.5. *Given the assumptions of Theorem 7.6, one has for all* $\tau \in (t_0, \infty)$

$$H\big(\tau^-, x^*(\tau^-), u^*(\tau^-), \psi(\tau^-)\big) = H\big(\tau^+, x^*(\tau^+), u^*(\tau^+), \psi(\tau^+)\big). \tag{7.125}$$

As stated in the Theorem 7.6, the transversality conditions on $\nu(t_1)$ and $\psi(t_1)$ given in conditions (viii) and (ix) in Theorem 7.1 are missing in the set of the necessary conditions for a problem with infinite horizon. However, there are some conditions under which the transversality conditions are still valid. One condition that might partly replace the missing transversality conditions is the condition that the Hamiltonian vanishes as time goes to infinity. However, to our knowledge this result was only proved for problems without any constraints and with autonomous state equation so far. The precise formulation of this result is given by Seierstad and Sydsæter (2009) [Theorem 1, p. 508]:

Theorem 7.10. *Let* $(x^*(t), u^*(t))$ *be an optimal solution to Problem (7.115) with* f *not depending explicitly on* t *and without any constraints. Assume that this solution satisfies the necessary conditions of optimality together with* ψ^0 *and* $\psi(t)$*. In addition, assume that there exists an* $\varepsilon > 0$ *such that*

(i) the integral $\displaystyle\int_{t_0+1}^{\infty} |f^0(t+\delta, x^*(t), u^*(t))|\,dt$ *exists for all* $\delta \in (-1,1)$ *and*

(ii) there exists a piecewise continuous function $\xi(t)$ *such that*

$$\left|\frac{\partial f^0}{\partial t}(t+\delta, x^*(t), u^*(t))\right| \leq \xi(t) \tag{7.126}$$

for all $\delta \in (-1,1)$ *and* $t \geq t_0 + 1$ *and* $\displaystyle\int_{t_0+1}^{\infty} \xi(t)\,dt < \infty$.

Then

$$\lim_{t\to\infty} H(t, x^*(t), u^*(t), \psi^0, \psi(t)) = 0. \tag{7.127}$$

Note that condition (7.127) together with condition (7.119) and the continuity of Hamiltonian implies

$$H\Big(t, x^*(t), u^*(t), \psi^0, \psi(t)\Big) = -\int_t^{\infty} \frac{\partial H}{\partial t}\Big(\tau, x^*(\tau), u^*(\tau), \psi^0, \psi(\tau)\Big)\,\mathrm{d}\tau. \tag{7.128}$$

It can be easily shown that this Equality is a generalization of the often-cited condition introduced by Michel (1982) for autonomous optimal control problems with a discount factor r without any mixed or pure state constraints:

$$H\Big(t, x^*(t), u^*(t), \psi(t)\Big) = r\psi^0 \int_t^\infty e^{-r\tau} f^0(x^*(\tau), u^*(\tau))\, \mathrm{d}\tau. \tag{7.129}$$

Chapter 8

Conclusion

In this thesis, we have studied models of optimal sustainable economic growth. Besides this, we provide some novel insights into the optimal control theory, particularly into the formulation of necessary conditions of optimality for problems involving pure state constraints and into some other properties of the optimal solutions. These results were used for the rigorous analysis of the economic models.

Regarding the optimal control theory, we have found a relationship between two sets of necessary conditions of optimality formulated by Seierstad and Sydsæter (1987) and those formulated by Feichtinger and Hartl (1986). This relationship is summarized in Theorems 7.4 and 7.5. Finding this relationship has been a key point for application of the optimal control theory to our models, mainly in Chapter 5. Indeed, the formulation of necessary conditions by Seierstad and Sydsæter (1987) includes a useful property formulated in condition (vi) in Theorem 7.1 which is not directly included in the set of necessary conditions by Feichtinger and Hartl (1986). On the other hand, the latter set of necessary conditions contains a condition formulated in Theorem 7.3 which has been the basis for derivation of Hartwick's result but was not formulated by Seierstad and Sydsæter (1987) for relevant problems. By interconnecting both sets of conditions, we have been able to use both these results in the analysis of the model with two exhaustible resources with binding state constraints as presented in Chapter 6.

The analysis of the model with two exhaustible resources can be regarded as the most important contribution of this thesis. Based on the precise formulation of necessary conditions, we have been able to provide an interesting description of properties of the solution to this model, which is summarized in Theorem 5.1. According to this theorem, in case of two mutually substitutable non-renewable goods, it is not optimal to futher exploit the one with a constant productivity after the extraction of the second resource with a growing productivity started. In addition, Hartwick's rule has been derived for this model in Theorem 5.2.

We have also provided a summary regarding Hartwick's rule and its converse for models with discounted utility as well as maximin objective criterion. In addition, we have studied the relationship between these two approaches. We have also brought a new insight into the converse of Hartwick's result. Its comprehensive formulation is given in Theorem 4.7. Some of these results were published earlier in an author's paper (see Jurča (2007)). We have also proposed a new derivation of Hatwick's result for model with population growth, which was studied by Mitra (2008).

Following Martinet and Rotillon (2007), we have applied the concept of Noether's theorem to study the conservation laws in models with both types of objective criterion. For this purpose, we have extended the results provided by Torres (2002), Torres (2004a) and Torres (2004b) in Theorem 6.1. Using this approach, it has been possible to establish a relationship between the total value of all capital goods at shadow prices and the total discounted value of utility over the remaining time horizon (see (6.31)).

Some topics given in this thesis remain open for further research. For example, the formulation of Theorem 5.1 allows a possibility that the first exhaustible resource with constant productivity is not exhausted and even that it is not beeing extracted at all. A question arises whether these possibilities can be excluded. Additionally, further research can be conducted for extending the result given in Section 3.2.1(b), particularly on better description of the values of optimal sustainable consumption in cases that (k_0, s_0) does not satisfy (3.36). Moreover, the existence of solution satisfying all necessary conditions by Seierstad and Sydsæter (1987) and all necessary conditions by Feichtinger and Hartl (1986) except condition (ii) remains an open question.

Symbols and notation

General symbols and abbreviations

□	end of proof
■	end of example
p.	page
w.r.t.	with respect to
DHS	Dasgupta-Heal-Solow model
DHH	Dasgupta-Hammond-Heal model

Mathematical notation

$\mathbb{R}$	set of real numbers
$\mathbb{Z}$	set of whole numbers
$\mathbb{N}$	set of natural numbers
$\mathbb{R}^n$	n-dimensional Euclidian space
$\langle a, b\rangle$	closed interval
$x \in \mathbb{R}^n$	n-dimensional column vector
x^T	x transposed to a row vector i.e. if $x \in \mathbb{R}^n$ then $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1, \dots, x_n)^T$,
x_i	i-th component of vector x
$\mathbb{R}^n_+$	set $\{x \in \mathbb{R}^n;\ x_i \geq 0,\ i = 1, \dots, n\}$
$\mathbb{R}^n_{++}$	set $\{x \in \mathbb{R}^n;\ x_i > 0,\ i = 1, \dots, n\}$
$\mathscr{C}^n$	set of n-times continuously differentiable functions
$\mathscr{O}(\tau^-)$	left neighbourhood of τ
$\mathscr{O}(\tau^+)$	right neighbourhood of τ

$f(\tau^-)$	$f(\tau^-) = \lim_{t\to\tau^-} f(t)$; if f is not defined on $\mathcal{O}(\tau^-)$, then $f(\tau^-) := f(\tau)$
$f(\tau^+)$	$f(\tau^+) = \lim_{t\to\tau^+} f(t)$; if f is not defined on $\mathcal{O}(\tau^+)$, then $f(\tau^+) := f(\tau)$
$[f(x)]_{x=a}$	function f evaluated at point $x = a$, i.e. $f(x)\big\|_{x=a}$
$\frac{\mathrm{d}f}{\mathrm{d}x}$	Jacobian of the function f (or gradient taken as a row vector, if f is a real function), i.e. $\frac{\mathrm{d}f}{\mathrm{d}x} = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$

Notations in economic models

Basic variables

t	time
c	consumption (n-dimensional vector)
r	rate of extraction of exhaustible capital goods (m-dimensional vector)
k	stock of renewable capital goods (n-dimensional vector)
s	stock of exhaustible capital goods (m-dimensional vector)
n	population

Functions

f	production function (satisfies Assumption (A1))
δ	amortization function (satisfies Assumption (A2))
U	utility function (satisfies Assumption (A3))
π	discount factor (satisfies Assumption (A4))
ϑ	population growth (satisfies Assumption (A5))
d	productivity of the second exhaustible capital in the model with two exhaustible capital goods (satisfies Assumption (A6))

Parameters

α	output elasticity of the renewable capital good
β	output elasticity of the exhaustible capital good
γ	parameter of the exponential growth of productivity
θ	relative risk aversion in the utility function
ρ	n-dimensional vector parameter of weights of consumer goods

Multipliers and others

ψ_k	n-dimensional vector of costate variables associated with k
ψ_s	m-dimensional vector of costate variables associated with s
ψ_i	costate variable associated with s_i (a simplified notation of ψ_{s_i}, $i = 1, 2$)
ψ_w	costate variable associated with w
ψ_n	costate variable associated with n
μ_c	n-dimensional vector of Lagrange multipliers associated with the constraint $c \geq 0$
μ_r	m-dimensional vector of Lagrange multipliers associated with the constraint $r \geq 0$
μ_i	Lagrange multiplier associated with the constraint $r_i \geq 0$ (a simplified notation of μ_{r_i}, $i = 1, 2$)
μ_w	Lagrange multiplier associated with the constraint $U(c) \geq w$
ν_k	n-dimensional vector of Lagrange multipliers associated with the constraint $k \geq 0$
ν_s	m-dimensional vector of Lagrange multipliers associated with the constraint $s \geq 0$
ν_i	Lagrange multiplier associated with the constraint $s_i \geq 0$ (a simplified notation of ν_{s_i}, $i = 1, 2$)
I	value of net investment, i.e. $I = \psi_k^T \dot{k} + \psi_s^T \dot{s} = \psi_k^T \dot{k} - \psi_s^T r$

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Bibliography

[1] Aronsson, T., Johansson, P.O. and Löfgren, K.G. (1995): A Result on Investment Decisions, Future Consumption and Sustainability under Optimal Growth, Working Paper No. 49, Stockholm School of Economics

[2] Asheim, G.B., Buchholz, W. and Withagen, C. (2003): The Hartwick Rule: Myths and Facts, Environmental and Resource Economics, Vol. 25, 129–150

[3] Asheim, G. B., Buchholz, W., Hartwick, J. M., Mitra, T. and Withagen C. (2007): Constant savings rates and quasi-arithmetic population growth under exhaustible resource constraints, Journal of Environmental Economics and Management, Vol. 53, Issue 2, 213–229

[4] d'Autume, A. and Schubert, K. (2008): Hartwick's rule and maximin paths when the exhaustible resource has an amenity value, Journal of Environmental Economics and Management, Vol. 56, 260–274

[5] Berkovitz, L.D. (1974): Optimal Control Theory, Springer-Verlag, New York

[6] Brunovský, P. (1980): Matematická teória optimálneho riadenia, Alfa, Bratislava (in Slovak).

[7] Buchholz, W., Dasgupta, S., Mitra T. (2005): Intertemporal Equity and Hartwick's Rule in an Exhaustible Resource Model, Scandinavian Journal of Economics, Vol. 107, No. 3, 547–561

[8] Cairns, R.D. and Yang, Z. (2000): The converse of the Hartwick's Rule and Uniqueness of the Sustainable Path, Natural Resource Modeling, Vol. 13, No. 4, 493–502

[9] Cairns, R.D. and Tian, H. (2003): Evolution of Population and Resource with Maximin Objective, proceeding at the Twelfth Annual Meeting of the Canadian Resource and Environmental Economics Study Group, Montréal

[10] Cairns, R.D. (2003): Maximin without Discounting, Natural Resource Modeling, Vol. 16, No. 2, 121–126

[11] Cairns, R. D. and Long, N. V. (2006): Maximin: a direct approach to sustainability, Environment and Development Economics, Vol. 11, 275–300

[12] Cairns, R. D. (2008): Value and income, Ecological Economics, Vol. 66, 417–424

[13] Chichilnisky, G. (1996): An axiomatic approach to sustainable development, Social Choice and Welfare, Vol. 13, No. 2, 231–257

[14] Dasgupta, p. and Heal. G. (1974): The Optimal Depletion of Exhaustible Resources, The Review of Economic Studies, Vol. 41, (symposium issue), 3–28

[15] Dixit, A., Hammond, P. and Hoel, M. (1980): On Hartwick's Rule for Regular Maximin Paths of Capital Accumulation and Resource Depletion, The Review of Economic Studies, Vol. 47, No. 3, 551–556

[16] Endress, L.H. and Roumasset, J. (2000): Sustainable Development without Constraints, Working Paper 2000-9, University of Hawaii

[17] Farzin Y. H. (2006): Conditions for Sustainable Optimal Economic Development, Review of Development Economics, Vol. 10, No. 3, 518–534

[18] Feichtinger, G. and Hartl, R. (1986): Optimale Kontrolle ökonomischer Prozesse, W. Gruyter, Berlin – New York (in German).

[19] Gouveia, P. D. F. and Torres, D. F. M. (2005): Automatic Computation of Conservation Laws in the Calculus of Variations and Optimal Control, Computational Methods in Applied Mathematics, Vol. 5, No. 4, 387–409

[20] Hamilton, K., Hartwick, J.M. (2005): Investing Exhaustible Resource Rents and the Path of Consumption, Canadian Journal of Economics, Vol. 38, No. 2, 615–621

[21] Hartl, F.H., Sethi, S.P. and Vickson, R.G. (1995): A Survey of the Maximum Principles for Optimal Control Problems with State Constraints, SIAM Review, Vol. 37, No. 2, 181–218

[22] Hartwick, J.M. (1977): Intergenerational Equity and the Investing of Rents from Exhaustible Resources, The American Economic Review, Vol. 67, No. 5., 972–974

[23] Hartwick, J.M. (1978a): Exploitation of Many Deposits of an Exhaustible Resource, Econometrica, Vol. 46, No. 1, 201–207

[24] Hartwick, J.M. (1978b): Substitution among Exhaustible Resources and Intergenerational Equity, The Review of Economic Studies, Vol. 45, No. 2, 347–354

[25] Hartwick, J. M. and Long, N. V. (1999): Constant Consumption and the Economic Depreciation of Natural Capital: The Non-Autonomous Case, International Economic Review, Vol. 40, No. 1, 53–62

[26] Heijnen, P. (2008): The Hartwick rule as a conservation law, CeNDEF Working Papers 08-11, Universiteit van Amsterdam, Center for Nonlinear Dynamics in Economics and Finance

[27] Jurča, P. (2007): On sustainability constraint in models with non-renewable resources, Proccedings of Mathematical Methods In Economics And Industry 2007

[28] Leonard, D. and Long, N. V. (1992): Optimal Control Theory and Static Optimization in Economics, Cambridge University Press, Cambridge

[29] Martinet, V. (2004): The Hartwick Rule and the Characterization of Constant Consumption Paths in the Presence of an Exhaustible Resource, Working Paper, THEMA

[30] Martinet, V. (2007): A Step beside the Maximin Path: Can We Sustain the Economy by Following Hartwick's Investment Rule? Ecological Economy, Vol. 64, 103–108

[31] Martinet, V. and Doyen, L. (2007): Sustainability of an economy with an exhaustible resource: A viable control approach, Resource and Energy Economics, Vol. 29, 17–39

[32] Martinet, V. and Rotillon, G. (2007): Invariance in growth theory and sustainable development, Journal of Economic Dynamics and Control, Vol. 31, 2827-2846

[33] Martinet, V. (2009): Defining sustainability objectives, Working paper 2009-07, Université de Paris Ouest Nanterre La Défense

[34] Michel, P. (1982): On the Transversality Condition in Infinite Horizon Optimal Problems, Econometrica, Vol. 50, No. 4, 975–984

[35] Mitra, T. (1983): Limits on population growth under exhaustible resource constraints, International economic review, Vol. 24, No. 1, 155–168

[36] Mitra, T. (2002): Intertemporal Equity and Efficient Allocation of Resources, Journal of Economic Theory, Vol. 107, 356–376

[37] Mitra, T. (2008): On competitive equitable paths under exhaustible resource constraints: The case of a growing population, International Journal of Economic Theory, Vol. 4, Issue 1, 53–76

[38] Noether, E. (1918): Invariante Variationsprobleme, Nachrichten der Koeniglichen Gesellschaft der Wissenschaften, Goettingen, 235–257 (in German)

[39] Ramsey, F. P. (1928): A mathematical Theory of Saving, Economic Journal, Vol. 38, 543–559

[40] Rawls, J. (1971): A Theory of Justice, Harvard University Press, Cambridge

[41] Sato, R. and Kim, Y. (2002): Hartwick's Rule and Economic Conservation Laws, Journal of Economic Dynamics and Control, Vol. 26, 437–449

[42] Seierstad, A. and Sydsæter, K. (1987): Optimal Control Theory with Economic Applications, North-Holland, Amsterdam

[43] Seierstad, A. and Sydsæter, K. (2009): Conditions implying the vanishing of the Hamiltonian at the infinite horizon in optimal control problems. Optimization Letters, No. 3, 507–512

[44] Solow, R. M. (1974): Intergenerational Equity and Exhaustible Resources, The Review of Economic Studies, Vol. 41 (symposium issue), 29–45

[45] Solow, R. M. (1986): On the Intergenerational Allocation of Natural Resources, Scandinavian Journal of Economics, Vol. 88, No. 1, 141–149

[46] Stiglitz, J. (1974): Growth with Exhaustible Natural Resources: Efficient and Optimal Growth Paths, The Review of Economic Studies, Vol. 41 (symposium issue), 123–137

[47] Torres, D. F. M. (2002): On the Noether Theorem for Optimal Control, European Journal of Control, Vol. 8, No. 1, 56–63.

[48] Torres, D. F. M. (2004a): Quasi-invariant optimal control problems, Portugaliæ Mathematica, Vol. 61, No. 1, 97–114

[49] Torres, D. F. M. (2004b): On the Noether Invariance Principle for Constrained Optimal Control Problems, WSEAS Transactions on Mathematics, Issue 3, Vol. 3, 620–624

[50] Withagen, C. (1996): Sustainability and investment rules, Economic letters, Vol. 53, 1–6

[51] Withagen, C. and Asheim, G. (1998): Characterizing Sustainability: The Converse of Hartwick's Rule, Journal of Economic Dynamics and Control, Vol. 23, 159–165

[52] Withagen, C., Asheim, G.B. and Buchholz, W. (2003): On the Sustainable Program in Solow's Model, Natural Resource Modeling, Vol. 16, No. 2, 219–231