Stochastic Dynamic Optimization Models for Pension Planning

Soňa Kilianová

Dissertation Thesis

Comenius University

Faculty of Mathematics, Physics
and Informatics

Department of Applied Mathematics and Statistics
Faculty of Mathematics, Physics, and Informatics
Comenius University
Mlynská dolina
842 48 Bratislava
Slovakia
http://www.iam.fmph.uniba.sk/institute/kilianova/

Doctoral Thesis in Applied Mathematics 2008

Preface

In this thesis mathematical models and their numerical implementation have been employed to study and simulate pension saving within the second fully funded pillar of multi-pillar pension systems. The models help future pensioners understand the factors affecting the amount saved in their pension accounts and they give them a helping tool for determining the optimal fund selection strategy. Although pension saving is a long-term investment and stochastic behavior of the financial markets is only hardly predictable for a long period, we hope that our models provide a good guidance for savers' decisions.

I am indebted to many great people from my community who made this work possible. Therefore, I would like to express my gratitude towards them. First of all, I would like to thank my supervisor Daniel Ševčovič for his excellent collaboration, support, worthy ideas and guidance in my research. Secondly, I would like to express my gratitude to co-authors Georg Pflug, Igor Melicherčík and again Daniel Ševčovič for their great team-work that empowered our fruitful collaboration. Furthermore, I am grateful to Margaréta Halická and Mária Trnovská for their valuable discussions on interior point methods. I am also thankful to Martin Proksa for providing me with some experimental data. I would also like to thank Alan Wallace for his language corrections. I am very grateful to all collegues from the Department of Applied Mathematics and Statistics, for the pleasant and friendly atmosphere that ensured my time spent at the Department during my PhD studies was a beautiful and joyful experience. Last but not least I would like to thank my family and friends for their continuing support and understanding. Thank you all.

Bratislava, March 2008 Soňa Kilianová

Contents

Chapter 1

Introduction

In the last decades, many European countries underwent several social and economic reforms. Pension reform was and in some countries still is one of the most important topics for political discussions. The main reason for the reform of pension systems in most countries of the European Union (EU) and also other parts of the world is the forecasted rapid aging of population. According to the World Bank and International Monetary Fund projections ([32], [13]), the region's old-age dependency ratio (percentage ratio of people over the age 64 to the working age population) is projected to double to 54 percent by 2050, meaning that the EU will move from having four persons of working age for every elderly citizen to only two. The pension reforms are designed to lower the burden on a shrinking number of workers, responsible of providing for an increasing number of pensioners, and also to reduce the strain on public budgets. The next reason for pension reforms in European countries lies in European economic integration, which will prompt higher levels of internal and external migration. The labor mobility has to be supported by a justified pension system. The reforms include strengthening the link between pension contributions and benefits, prolonging the contribution period by raising the retirement age, and diversification of sources of retirement pension benefits. The most favored approach has been to gradually replace the pay-as-you-go system with a fully funded system so that retirement income will be fully financed by investing the pension plan members' contributions in financial assets ([14]).

Two reform styles have emerged ([32]): a parametric style and a paradigmatic style. A *parametric reform* is an attempt to rationalize the pension system by seeking more revenues and reducing expenditures while expanding voluntary private pension provisions. A pay-as-you-go (PAYG) pillar is downsized by raising the retirement age, reducing pension indexation and curtailing sector privileges. A development of voluntary pension funds beyond the mandatory social security system is promoted through tax advantages, organizational assistance, tripartite agreements and other means of administrative and public information facilitation. This type of reform is taking place in Austria, Czech Republic, France, Germany, Greece, and Slovenia.

Some countries have decided to change the model in which pension systems operate – that is, to move away from the monopoly of a PAYG pillar within the mandatory social security system. A *paradigmatic reform* is a deep change in the fundamentals of pension provision. It is typically based on the introduction of a mandatory funded pension pillar, along with an essentially reformed PAYG pillar and the expansion of opportunities for voluntary retirement saving. Most transition economies have kept a reformed and downsized public and unfunded (first) pillar and added a second and funded pillar. All countries undertook efforts to introduce a regulated third pillar to handle voluntary individual savings. This type of reform has been introduced for example in Bulgaria, Croatia, Denmark, Hungary, Latvia, Lithuania, the Netherlands, Poland, Slovak Republic, Sweden, and the United Kingdom. Arguments commonly used to support paradigmatic as well as parametric reforms are discussed in [32].

In general, the systems after a paradigmatic reform are based on three main pillars, therefore we refer to these reforms as *three pillar reforms*. The first pillar is represented by the traditional PAYG mechanism which is a social insurance based on regular contributions from workers, immediately redistributed to pensioners. Typically, the unfunded first pillar was reformed and downsized to make a room for an earnings-related funded second pillar. The scope and structure of the newly introduced funded second pillar differs significantly amongst the countries. In general, the fully funded second pillar, or some part of it, is based on private savings of citizens in pension funds, managed by commercial pension fund administrators. The third pillar comprises voluntary individual saving programmes in supplementary pension accounts. The ratio of population participating in this pillar is low in most countries.

Synopsis

In this thesis, we focus on the second, fully funded pillar. In particular, we consider the position of a working person who is a future pensioner and participates in the second pillar of the pension system of the corresponding country. Our research was motivated by the pension system of Slovak Republic, the second pillar of which is based on savers paying regular contributions to their pension account and investing the savings to one of three, by government strictly defined pension funds. The funds differ in their risk profiles. The saver is given a possibility to choose the fund they wish to invest in and is allowed to change their choice periodically. Since the funds are invested in financial markets, their returns have stochastic character and the investments are more or less risky. We develop two types of models which give the saver a proposal, which fund it is optimal for them to choose at each rebalancing period during their active life, depending on certain parameters.

Although the exact form of the second pillar varies significantly among countries, the above described principle is present at least as an element of pension systems in several countries, for example Denmark or the United States of America. Therefore, the proposed models may be applied to these countries also. Moreover, even outside the context of pension saving, the problem may be viewed generally as a problem of investing money into several funds with different characteristics and rebalancing periodically.

In Chapter 2 we give a brief description of pension systems of selected countries to which our models may be applied. Chapter 3 is dedicated to clarifying the goals of the thesis. The next two chapters review the basic knowledge required for the models: theory of utility functions and the concept of risk measures. Chapter 6 presents the proposed mathematical models. The first two models are the dynamic accumulation model and the proportional investment allocation model, both leading to a problem of stochastic dynamic programming. The third and fourth models are based on risk minimization: model minimizing single-period risk and model minimizing multi-period risk. The models are numerically implemented for the case of Slovak Republic in Chapter 7 and the sensitivity of results to varying parameters is investigated. The thesis is closed in Chapter 8 with conclusions.

Chapter 2

Overview of pension systems in selected countries

Pension reforms and pension systems adopted in different countries vary a lot in details. It is not our concern to repeat all the complicated descriptions of various systems in particular countries. We refer to the vast works of literature that have already been written, see for example the publications of The World Bank ([2], [5], [15], [19], [22], [32], [33], [54], [67]), The International Monetary Fund ([13], [14]), Observatoire Social Européen ([46]) or publicly available information of Ministries of Social Affairs of individual countries.

In this chapter, we give a very brief overview of pension systems of selected countries. In particular, we focus on countries with multi-pillar pensions systems. Among them we choose Slovak Republic, Denmark and USA and we describe the fully funded second pillar of their pension systems in more detail.

2.1 The scope of the second pillar

A multi-pillar pension system was adopted in many countries all around the world: in most countries of Eastern and also Western Europe, Latin America, USA, or even Russia. Multi-pillar systems combine the unfunded, partially funded, or funded public "solidarity systems" (the first pillar) and funded individual account systems (the second pillar and voluntary third pillar).

The scope and structure of all pillars vary among countries. In the second and fourth column of Table 2.1 we give an overview of contribution rates to the second pillar in selected countries. Usually, the pension fund management institutions are obliged to offer at least two types of pension funds to their members. "Type 1" pension fund may be

Country	2^{nd} pillar	Country	2^{nd} pillar
Bulgaria *	2%	Lithuania ***	5.50%
Croatia	5%	Macedonia	7%
Denmark	12 – 15%	Poland	7.3%
Estonia	6%	Romania	8%
Hungary	6%	Russia	2 – 6%
Kazakhstan	10%	Slovakia	9%
Kosovo	10%	Sweden	2 – 5%
Latvia **	10%	Ukraine	7%

Table 2.1: Different contribution rates to the second pillar of pension systems of selected countries, as a percentage of gross earnings.
Note: the data contained in the table have an informative character, since more complicated rules for determining the contributions rates apply in many cases.
* to be increased from 2% in 2006 to 6%
** to be increased from 2% in 2006 to 10% in 2010 and thereafter.
*** voluntary second pillar.
Source: Observatoire Social Européen 2004 ([46]) and publicly available information of Ministries of Social Affairs of individual countries.

invested in fixed income instruments and stocks, whereas "Type 2" pension fund must be placed exclusively in fixed income securities.

2.2 Slovak Republic

The "three-pillar" reform in Slovak Republic was adopted in the year 2003 as a part of several crucial social and economic reforms. The unfunded first pillar is mandatory and based on the pay-as-you-go system. The second pillar is based on saving in private pension accounts, that is, paying contributions towards the investors' own future benefits. The voluntary, fully funded third pillar is designed for supplementary pension savings and has a small size.

People who were in active employment before January 2004 were given the possibility to decide whether they wish to stay in the first pillar only, or to split their contributions between the first and the second pillar. The new workforce entrants after January 2004 are obliged to participate in the latter form. The regular contribution rate 18% which was formerly paid to the pay-as-you-go pillar is then split to 9% of the gross salary in both the first and the second pillar.

Assets in the second pillar are managed by pension fund management institutions. Each of them is obliged to create and offer three types of funds with different risk profiles: a Growth fund with the highest ratio of stocks in the portfolio, a Balanced fund and a Conservative fund that is allowed to invest to secure financial instruments only. The investment limits defining the funds are specified in Table 2.2. The participants choose one of the funds and they are allowed to revise their decision during the period of saving and switch to another fund eventually. Hence, future pensioners are able to partially influence

Fund type	Stocks	Bonds and money market instruments
Growth Fund (1)	up to 80%	at least 20%
Balanced Fund (2)	up to 50%	at least 50%
Conservative Fund (3)	no stocks	100%

Table 2.2: Limits for investment for the pension funds in Slovak Republic.

the amount of their savings at their retirement time and also its risk by balancing between the three fund types. Furthermore, there are additional governmental regulatory restrictions imposed on the fund selection: investment in the Growth fund is not allowed during the 15 years prior to retirement and the last 7 years are reserved for the Conservative fund only. A detailed study of the pension system of Slovak Republic is given in e.g. [28] and [43].

2.3 Denmark

The pension system in Denmark is based on three pillars as well, although it is very different from the Slovak system. The first pillar is represented by public pension schemes. These cover two schemes that are administered by public sector institutions and aim to provide universal or near-universal benefits. The main scheme is unfunded and financed from general tax revenues, but the main supplementary scheme is financed from the employer's and employee's contributions and is fully funded. In addition to the flat pension, a supplement is paid to low-income pensioners.

In 1964, the authorities introduced a supplementary pension scheme, because the level of the social pension was rather modest. ATP (Arbejdsmarkedets Tillægspension, Labor Market Supplementary Pension Scheme) is an independent, self-supporting institution which is a part of Denmark's overall social security scheme. ATP covers all wage earners in Denmark. Members pay contributions during the years they are actively participating in the labor market. The ATP is funded by the employer's (2/3) and employee's (1/3) contributions that are subject to relatively low ceilings (maximum of DKK 2,684 per year in 2004), corresponding to less than 0.9 percent of the average wage. Contributions to ATP are not related to income, but are set as fixed amounts. These depend on a few broad categories that have been defined on the basis of the number of working hours. ATP benefits are payable at age 65.

The second pillar of the Danish pension system is based on occupational pensions. Most workers are covered by private occupational pension schemes that have been promoted by collective bargaining, because both the social pension scheme and ATP pay modest benefits. Participation is not mandatory by law, but is effectively imposed by collective labor agreements. The contribution rate varies with the specific plan. It has steadily grown over the past decade and the average contribution rate now exceeds 10 percent of wages. Occupational pension plans offer a variety of retirement products, ranging from

Fund type	Stocks	Bonds and money market instruments
AP Profil 35	up to 100%	at least 0%
AP Profil 25	up to 75%	at least 25%
AP Profil 15	up to 50%	at least 50%
AP Profil 7	up to 25%	at least 75%

Table 2.3: Funds with different risk profiles, AP Pension, Denmark. *Source*: AP Pension, [62].

life annuities to term annuities, phased withdrawals and lump sum payments. Most plans offer this choice of products. Plans differ in the degree of flexibility and choice they allow to their members.

Personal pension plans constitute the third pillar of the Danish pension system. They are offered by banking, insurance and pension institutions and are established on a voluntary basis for persons who are not covered by occupational pension schemes or wish to obtain additional coverage.

One component of the system suitable for our models is the occupational schemes, which offer numerous products, such as the possibility to invest in various funds with different risk profiles. As an example we can mention a product of AP Pension ([62]), offering the following four funds: AP Profil 35, 25, 15, 7, with risk profiles given in Table 2.3.

The information on the Danish pension system presented here was obtained from [4]. We refer readers to this source for a more detailed information.

2.4 The United States of America

The pension system of the United States of America is rather complicated. Therefore, we pay attention only to one particular part of the overall pension system: TIAA-CREF (Teachers Insurance and Annuity Association - College Retirement Equities, founded 1918). It is a nonprofit organization serving employees of educational and research institutions. Today 8,700 college, universities, and institutions, and 2 million individuals are part of the TIAA-CREF pension system.

The retirement program administered by TIAA-CREF is a popular benefit. Since contributions may be made on a tax-deferred basis, many faculty and staff members use the pension to lower current taxes. Salary reduction agreements can be changed 4 times per calendar year ([63]). The contribution levels are restricted depending on the particular college or university. For example, Dixie State College in Utah contributes an amount equal to 14.20% of the employee's annual salary to their TIAA-CREF retirement Plan. No individual contributions are required.

TIAA-CREF offers nine accounts, one with a guarantee and eight that are variable, or nonguaranteed. *The TIAA-CREF Traditional Annuity* guarantees the principal as well as a specified interest rate, plus provision of an opportunity for additional growth through dividends. *The TIAA-CREF Real Estate Account* invests the majority of its assets in a portfolio of income producing commercial and residential properties. The remainder is kept in more liquid investments. *The CREF Stock Account* invests a major portion of its assets in a portfolio that tracks the performance of the US stock market as a whole. Other segments consist of foreign and domestic stocks selected for their above-average investment potential. *The CREF Money Market Account* invests in short-term interest-earning securities. *The CREF Bond Market Account* invests in a portfolio of medium- to long-term US government bonds, corporate bonds, and asset-backed securities. *The CREF Social Choice Account* invests in a diversified portfolio of stocks, bonds, and money market instruments of companies that follow certain standards for social responsibility. *The CREF Global Equities Account* invests in a portfolio of stocks from around the world, including the US. *The CREF Equity index Account* invests in a diversified portfolio that tracks the overall US stock market, as represented by a broad market index. *The CREF Growth Account* invests in a portfolio of stocks selected for exceptional growth potential. Like returns from all variable annuities, returns from the TIAA-CREF variable accounts will fluctuate and principal is not guaranteed.

The employees can allocate their contributions among the TIAA-CREF accounts in any whole-number percentage, including full allocation to any option. Once participation begins they can change their allocation of future premiums or transfer existing accumulations. The employees can also make supplemental tax-deferred contributions through TIAA and CREF Supplemental Retirement Annuities (SRA's) or Retirement Annuities. A more detailed description on TIAA-CREF can be found in [63], [64], [65].

Chapter 3

Conceptual model and goals of dissertation

The second pillar of some of the countries mentioned in Table 2.1, or a part of it, can be generally defined as the following problem.

Assume that the worker's expected retirement time is in T years and they save for their pension in a pension fund management institution offering investment in funds labeled by $1, ..., J$. Next we assume that the saver is allowed to decide which fund they want to invest the savings into and to revise this decision in later times if they wish so. We can assume, without loss of generality, that they revise their decision every twelve months. We can formulate the problem:

For each time $t \in \{0, 1, ..., T-1\}$, determine the fund $j_t \in \{1, ..., J\}$ so that we obtain the best possible outcome at time T.

Since the pension funds invest in financial markets, i.e. in financial instruments with more or less volatile returns, the outcome of pension saving in pension funds is stochastic. It is therefore necessary to introduce a measure that gives us means for comparing two random outcomes and determining the better one. We use two approaches for this purpose: the expected utility and the risk measures concept. Hence, we approach the above formulated problem in two different ways:

I. at a given level of saver's risk aversion or risk tolerance, the expected utility from the saved amount at time T is maximized;

II. at a given target terminal value of savings at time T, the insecureness (riskiness) of achieving it is minimized.

In approach (I), we assume to know the saver's utility function representing their preferences. We deal with the notions of a utility function and risk aversion in Chapter 4. Let us denote by d_T a random variable representing the saved amount at time T. Let U be the saver's utility function and R the corresponding risk aversion coefficient fixed at the value $\bar{R} \in \mathbb{R}$. Next, let $\mathcal{J}$ be a set representing eventual restrictions on the fund selection imposed by government and other constraints that may come into consideration. Let us denote by d_t the state variables representing the saved amount at time t and $\mathcal{X}$ the set representing the constraints on d_t. Problem (I) can then be formulated as the following optimization problem:

$$\max_{j_t, t \in \{0,\dots,T-1\}} \mathbb{E}(U(d_T))$$

subject to

$$\begin{aligned} R &= \bar{R}\,, \\ d_t &\in \mathcal{X}\,, \\ j_t &\in \mathcal{J}\,. \end{aligned} \tag{3.1}$$

We specify this problem in Chapter 6. We show that it leads to a stochastic dynamic programming problem.

The second approach (II) uses the notion of risk measures which are usually statistical tools suitable for quantifying the insecureness (risk) of a future outcome. Basically, we distinguish two types of risk measures, depending on whether we measure a future outcome after one single period, or during several periods of time: we speak of the so called *static (single-period)* or *dynamic (multi-period) risk measures*. If M is the used risk measure (single-period or multi-period), and μ the target terminal amount, we are interested in solving the problem

$$\min_{j_t, t \in \{0,\dots,T-1\}} M(d_T)$$

subject to

$$\begin{aligned} \mathbb{E}(d_T) &\geq \mu\,, \\ d_t &\in \mathcal{X}\,, \\ j_t &\in \mathcal{J}\,. \end{aligned} \tag{3.2}$$

In the case of a multi-period risk measure, M is a function of the state variable d_t in all considered time periods; that is, $M = M(d_0, ..., d_T)$. We apply the *average value-at-risk deviation* as the risk measure M in Chapter 6. We show that the problem above may be rewritten to a large-scale linear program. Before, we introduce the theoretical framework of risk measures with emphasis on the average value-at-risk deviation in Chapter 5.

The following questions are important for a saver:

- What is my risk tolerance or risk aversion?
- Which sum do I want to achieve?

The answers to these questions specify the utility function U in (3.1) and the risk measure M and the target wealth μ in (3.2).

Goals of the thesis

In this thesis we aim to achieve the following goals:

- ♢ To develop a mathematical model for the utility function approach (I).
- ♢ To develop a mathematical model for the risk measures approach (II).
- ♢ To propose numerical schemes and implement both models for the example of Slovak Republic.
- ♢ To study the sensitivity of the results to varying parameters of the models.
- ♢ To make conclusions and to summarize rules for the fund selection in pension saving which depend on selected parameters.

Chapter 4

Utility functions in decision problems

In economic analysis, preferences of individuals about having n goods in quantities $x_1, ..., x_n$ are often represented by a utility function $U(x_1, ..., x_n)$. In this chapter, we introduce basic properties of utility functions and the notion of expected utility in a short extent that is sufficient for our purposes in this thesis. We refer to books [17], [21] and [58] for a detailed theory of consumer's preferences, utility functions and expected utility.

In the case of a single good, utility function $U : \mathbb{R} \to \mathbb{R}$ usually has the following specific properties:

1. $U(x)$ is increasing in x on $(0, \infty)$, i.e. $U'(x) > 0$. That is, more is always better. The function U' is referred to as a marginal utility, so this criterion says that the marginal utility is always positive.

2. $U(x)$ is concave in x, i.e. $U''(x) < 0$. This property is referred to as a risk aversion. It implies that the certainty of an expected value of outcomes is preferred to an uncertain situation. It also means that the marginal utility $U'(x)$ is a decreasing function of wealth.

The only relevant feature of a utility function is its ordinal character, not its absolute values. If $U(x)$ is a utility function representing one's preferences and $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, then $f(U(x))$ represents exactly the same preferences since $f(U(x)) \geq f(U(y))$ if and only if $U(x) \geq U(y)$.

Investors often face the necessity of making decisions about investments, the efficiency or return of which depend on unknown future behavior of some stochastic environment. Their behavior can be described by the notion of expected utility. The expected utility theory states that the decision maker chooses between risky or uncertain prospects by comparing their expected utility values. If X is a random variable, then the expected utility associated with X is $\mathbb{E}(U(X))$ where $\mathbb{E}$ is the expectation operator.

In the context of the problem of pension saving, let us denote d the random variable representing future pensioner's saved amount in their pension account. If the random wealth d depends among other stochastic or deterministic factors also upon a decision variable j and $\mathcal{J}$ is the set of all feasible decisions j, the future pensioners solve the problem

$$\max_{j \in \mathcal{J}} \mathbb{E}(U(d(j)))\,.$$

The most crucial thing here is the right choice of the utility function and its parameters, reflecting in particular investors' attitude to risk. Usually, the parameters entering the utility functions are estimated using some statistical methods or psychological experiments.

Based on the attitude to risk, we distinguish *risk averse*, *risk neutral*, and *risk loving* investors. Their utility functions are concave, affine, and convex, correspondingly. Most investors are assumed to be risk averse and it is often convenient to have a measure of risk aversion. We discuss various measures of risk aversion in the next section.

4.1 Risk aversion

A risk aversion coefficient is a special measure reflecting the character and degree of investor's risk aversion. Intuitively, the more concave the expected utility function, the more risk averse the investor. We could measure risk aversion by the second derivative of the utility function. However, this definition is sensitive to changes in the utility function: if we consider any positive multiple of the utility function, the second derivative changes but the consumer's behavior does not. If we normalize the second derivative by dividing by the first, we get a reasonable measure known as the *Arrow-Pratt absolute risk aversion coefficient* ([50]). The next most common measure is the *risk aversion coefficient, relative*.

Definition 4.1.1. *The absolute risk aversion coefficient at a point x pertaining to a utility function U is defined as*

$$\lambda_A(x) = -\frac{U''(x)}{U'(x)}\,. \tag{4.1}$$

Utility functions with a constant absolute risk aversion coefficient are called CARA utility functions.

A utility function U exhibits constant absolute risk aversion (CARA) if the absolute risk aversion coefficient does not depend on the wealth or $\lambda'_A(x) = 0$. U exhibits decreasing absolute risk aversion (DARA) if richer people are less absolutely risk averse than poorer ones or $\lambda'_A(x) < 0$. U exhibits increasing absolute risk aversion (IARA) if $\lambda'_A(x) > 0$. We notice that there is a natural assumption that most investors have decreasing absolute risk aversion.

Definition 4.1.2. *The relative risk aversion coefficient at a point x pertaining to a utility function U is defined as*

$$\lambda_R(x) = -x\frac{U''(x)}{U'(x)}\,. \tag{4.2}$$

Utility functions with a constant relative risk aversion coefficient are called CRRA utility functions.

Most often investors are assumed to have constant relative risk aversion.

4.2 Examples of utility functions

There are several classes of utility functions suitable for describing various types of investors' or consumers' economic behavior. We look at examples of the well known classes: the quadratic, exponential and power-like utility functions.

A quadratic utility function

Definition 4.2.1. *A quadratic utility function is of the form*

$$U(x) = ax - bx^2\,.$$

Its Arrow-Pratt absolute risk aversion coefficient is

$$\lambda_A(x) = \frac{2b}{a-2bx}$$

and the Arrow-Pratt relative risk aversion coefficient $\lambda_R(x) = x\lambda_A$. Since the derivatives of both λ_A and λ_R with respect to x are positive, the absolute and relative risk aversion coefficients of a quadratic utility function are increasing in x. Since $U''' \equiv 0$ there is no motive for precautionary saving which is understood as additional saving resulting from the knowledge that the future is uncertain. Additional saving can be achieved either by consuming less or by working more. For some discussion on precautionary saving we refer the reader to e.g. [37] or [45].

A quadratic utility function is mainly used in the context of permanent income and life cycle hypotheses ([9]).

An exponential utility function

Definition 4.2.2. *A negative exponential utility function is of the form*

$$U(x) = -e^{-ax}\,.$$

The absolute risk aversion coefficient of negative exponential utility function is $\lambda_A = a$ and it is constant in x. Hence, the negative exponential utility function is CARA. The relative risk aversion coefficient has the value $\lambda_R = ax$, that is, it is increasing in x. The CARA function implies a positive motive for precautionary saving.

A power-like utility function

Definition 4.2.3. *A power-like utility function is of the form*

$$U(x) = \frac{x^{1-a}}{1-a}\,.$$

The ratio $1/a$ is the intertemporal substitution elasticity between consumption in any two periods, i.e., it measures the willingness to substitute consumption between different periods. The smaller the value of a (the larger $1/a$), the more willing the household is to substitute consumption over time. Note also that a is the coefficient of relative risk aversion defined by (4.2). Since the coefficient of relative risk aversion is constant, this utility function is a CRRA (or *isoelastic*) utility function.

There are three other important properties. First, the expression x^{1-a} is increasing in x if $a < 1$ but decreasing if $a > 1$. Therefore, dividing by $1 - a$ ensures that the marginal utility is positive for all values of a. Second, if $a \to 1$, the utility function converges to $\ln a$. Third, $U'''(x) > 0$, implying a positive motive for precautionary saving. Therefore, one often uses this utility function when studying savings behavior (see [9]). We will use a power-like utility function in expected utility maximization based models for pension saving in Chapter 6.

Chapter 5

Risk measures in decision problems

Measures of risk or *risk measures* are functions that describe risk and give the manager or decision maker a quantitative tool to compare different insecure alternatives. In the context of static financial positions, economically meaningful axioms for risk measures were proposed by Artzner et al. in [6]. Well known static risk measures are value-at-risk ([20]), coherent risk measures ([6]), sublinear ([26]) and convex risk measures ([23], [24], [27]). Furthermore, a large part of literature is concerned with quantile-based alternatives to value at risk. For excellent overviews on static risk measures, we refer to Föllmer and Schied ([25]), Delbaen ([18]) and Scandolo ([55]).

For the case of multi-period decision problems, a concept of dynamic risk measures was developed. The basic idea of risk measures in a dynamic setting was presented in the papers of Cvitanic & Karatzas [16] and Wang [59]. Recent approaches to this subject can be found in the papers by Artzner et al. [7], Pflug & Ruszczynski [48] and Riedel [51].

5.1 Measuring single-period risk

In this section, we define the basic notions used in the theory of risk measures. Although there is a plenty of various risk measures, we focus only on the so called *value-at-risk deviation* and the *average value-at-risk deviation*. We use the latter one in our pension planning models in the next chapter.

There are three types of functionals connected to the theory of risk measures. Functionals describing preferences in the sense that higher values of the functional mean higher preference are called acceptability-type functionals. Among them, we call *accept-*

ability functionals $\mathcal{A}$ those, which have some important additional properties: translation equivariance, concavity and monotonicity. Another group of functionals is formed by the translation-invariant ones. We call them deviation-type functionals. Within these, we identify a sub-group of functionals satisfying the property of convexity and monotonicity. We call them *deviation risk functionals* and denote usually by $\mathcal{D}$. We recall that $\mathcal{D}$ is a deviation risk functional if and only if $\mathcal{A}(Y) = \mathbb{E}(Y) - \mathcal{D}(Y)$ is an acceptability functional. The third type of functional connected to risk measures theory is called *risk (capital) functional* which can be viewed as a mirror image of the concept of acceptability functionals. We refer the reader to the book of Pflug [47] for proper definitions of acceptability, deviation and risk capital functionals. Rockafellar et al. in [53] use the notions *sureness valuation, expectation-bounded risk measures*, and *general deviation measures* instead of acceptability functionals, risk capital functionals and deviation risk functionals, respectively.

A classical example of a deviation risk functional is the standard deviation $\mathcal{D}(Y) = \sigma(Y)$. However, the disadvantage of the standard deviation in measuring risk is that it treats the negative and the positive deviations from the mean in the same way. Already Markowitz realized this feature and proposed other measures to be used, e.g. the semi-variance.

In the following two sections we define the value-at-risk deviation and the average-value-at-risk deviation. They are often used risk measures and have better properties than the standard deviation.

5.1.1 Value-at-risk

Given a probability distribution of future wealth of a financial institution or an investor, the value-at-risk at the level α of the future wealth random variable is a maximum wealth exceeded with probability $1-\alpha$ where α is a given confidence level. In practice, the level α is quite low, typically $0.5\%, 1\%$ or 5%. When this risk measure is used, we accept positions as safe if in less than $\alpha\%$ of the cases we experience difficulties.

Although the value-at-risk has poor mathematical properties (e.g. it is not convex), it is very relevant in many decision models, see e.g. Duffie & Pan [20] and Gourieroux et al. [29].

Definition 5.1.1. *[47, Section 2.2] The value-at-risk $VaR_\alpha(Y)$ of a profit random variable Y with a distribution function F at a confidence level α, $0 < \alpha < 1$, is defined as the α-quantile $F^{-1}(\alpha)$, i.e.*

$$VaR_\alpha(Y) = F^{-1}(\alpha) = \inf\{u : F(u) \geq \alpha\}, \quad 0 < \alpha < 1 .$$

The value-at-risk deviation of a profit random variable Y at a confidence level α is defined by

$$VaRD_\alpha(Y) = \mathbb{E}(Y) - VaR_\alpha(Y), \quad 0 < \alpha < 1 .$$

Notice that $VaRD_\alpha$ may also take negative values.

Since a distribution function F of a random variable Y, defined by $F(u) = P(Y \leq u)$, is continuous from the right, the infimum in the above definition of the value-at-risk is in fact a minimum.

Please note that the nomenclature is inconsistent in the literature. Some authors call VaR_α the value-at-risk of level $1-\alpha$. Some other authors take the negative value $-F^{-1}(\alpha)$ as the value-at-risk.

5.1.2 The average value-at-risk

The average value-at-risk of a loss random variable was generally defined by Uryasev in [57]. We modify this definition slightly for profit random variables.

Definition 5.1.2. *Let Y be a profit random variable with a distribution function F, possibly not continuous. Let F_α be the lower α-tail distribution, which equals to 1 for profits exceeding VaR_α, and equals to $\frac{F}{\alpha}$ for profits below or equal to VaR_α. The average value-at-risk of Y at the level α is defined as the mean of the α-tail distribution F_α.*

Acerbi [1] gave a representation in terms of an average over α of the VaR_α values.

Definition 5.1.3. *[1] Let Y be a continuous random variable. The average value-at-risk of Y at level α, $0 < \alpha \leq 1$, is defined as*

$$AVaR_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha F^{-1}(u)du \tag{5.1}$$

where F is the distribution of Y. The average value-at-risk deviation is defined by

$$AVaRD_\alpha(Y) = \mathbb{E}(Y) - AVaR_\alpha(Y)\,. \tag{5.2}$$

The average value-at-risk ($AVaR$) is also known under the names of *conditional value-at-risk* ($CVaR$, see e.g. [52]), *tail value-at-risk* ($TVaR$), *mean shortfall*, or *expected shortfall*. Defining the value-at-risk as the quantile $F^{-1}(u)$ (see Definition 5.1.1), the $AVaR_\alpha$ is the average of these values, averaged over $u \in [0, \alpha]$, and this justifies the name.

There are many alternative ways of representing $AVaR$. The following one, proposed by Uryasev ([52]), says that the $AVaR$ may be expressed by a maximization formula. Since the statement is not trivial, we present a proof of it in Appendix A for reader's convenience. For other representations of the $AVaR$ see [47, Section 2.2.3].

Theorem 5.1.1. *[52] The average value-at-risk of a random variable Y at the level α may be represented as the optimal value of the following optimization problem:*

$$AVaR_\alpha(Y) = \max_{x\in\mathbb{R}}\{x - \frac{1}{\alpha}\mathbb{E}([Y-x]^-)\} \tag{5.3}$$

where $[g]^- = \max\{-g, 0\}$ is the negative part of g. The maximum in (5.3) is attained.

For detailed discussions of the properties of $AVaR$ see Rockafellar and Uryasev [52], Acerbi [1] and Pflug [47]. We present some of them in the following proposition.

Proposition 5.1.1. *Let $Y, Y^{(1)}, Y^{(2)}$ be random variables. The average value-at-risk $AVaR_\alpha$, $0 < \alpha \leq 1$, is*

(i) translation equivariant:

$$AVaR_\alpha(Y + c) = AVaR_\alpha(Y)$$

for all $c \in \mathbb{R}$,

(ii) concave:

$$AVaR_\alpha(\lambda Y^{(1)} + (1-\lambda)Y^{(2)}) \geq \lambda AVaR_\alpha(Y^{(1)}) + (1-\lambda)AVaR_\alpha(Y^{(2)})$$

for $0 \leq \lambda \leq 1$,

(iii) positively homogeneous:

$$AVaR_\alpha(\lambda Y) = \lambda AVaR_\alpha(Y)$$

for any $\lambda > 0$,

(iv) strict

$$AVaR_\alpha(Y) \leq \mathbb{E}(Y)\,.$$

Proof. The properties (i) and (iii) follow directly from the Definition 5.1.3. Properties (ii) and (iv) follow from the dual representation of $AVaR$ which we omitted in this thesis but it can be found in [47, Section 2.2.3]. □

Property (iv) verifies the following statement.

Corollary 5.1.1. *The average value-at-risk deviation AVaRD is non-negative.*

The $VaR_\alpha = F^{-1}(\alpha)$ defined in Definition 5.1.1 is related to the average value-at-risk by the following two relationships:

$$F^{-1}(\alpha) \in \text{ argmax } \{x - \frac{1}{\alpha}\mathbb{E}([Y-x]^-) : x \in \mathbb{R}\} \tag{5.4}$$

which is the relationship (8.2) from Appendix A, and

$$VaR_\alpha(Y) = F^{-1}(\alpha) \geq \frac{1}{\alpha}\int_{-\infty}^{\alpha} F^{-1}(p)dp = AVaR_\alpha(Y) \tag{5.5}$$

for all $\alpha \in (0, 1)$.

5.2 Measuring multi-period risk

So far, we considered economic activities that resulted in just one random income or one random change in wealth at a fixed time. In this section, we generalize this concept by considering activities which result in an insecure cash-flow stream during a longer period.

Denote by $Y = (Y_1, ..., Y_T)$ a stochastic cash-flow process defined on some probability space $(\Omega, \mathcal{F}, P)$ to which we wish to assign an acceptability value $\mathcal{A}$ or a risk value $\mathcal{D}$. In the multi-period situation, there is typically also other information than just the observation of the income values Y_t, which is available and which is relevant to the quantification of risk. The standard way of dealing with information in probability is done by introducing filtrations. We recall that a filtration $\mathcal{F} = (\mathcal{F}_1, ..., \mathcal{F}_T)$ is an increasing sequence of σ-algebras, i.e. $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$. The cash-flow process $Y = (Y_1, ..., Y_T)$ is *adapted* to $\mathcal{F}$, if Y_t is $\mathcal{F}_t$-measurable for $t = 1, ..., T$. A filtration may be specified in a tree process. A good insight into filtrations and tree processes is given in [42] or [47, Chapter 5.1].

Similarly as in the single-period case, we may define acceptability and deviation multi-period functionals. We refer to Appendix B where we recall their definitions.

5.2.1 Multi-period average value-at-risk

The multi-period average value-at-risk and multi-period average value-at-risk deviation are defined as follows.

Definition 5.2.1. *[47, Section 3.3.3] Let $Y = (Y_1, ..., Y_T)$ be an integrable stochastic process. For a given sequence of constants $c = (c_1, ..., c_T)$, probabilities $\alpha = (\alpha_1, ..., \alpha_T)$, and a filtration $\mathcal{F} = (\mathcal{F}_0, ..., \mathcal{F}_T)$, the multi-period average value-at-risk is defined as*

$$AVaR_{\alpha,c}(Y;\mathcal{F}) = \sum_{t=1}^{T} c_t \mathbb{E}[AVaR_{\alpha_t}(Y_t|\mathcal{F}_{t-1})]\,.$$

Definition 5.2.2. *[47, Section 3.3.3] Let $Y = (Y_1, ..., Y_T)$ be an integrable stochastic process. For a given sequence of constants $c = (c_1, ..., c_T)$, probabilities $\alpha = (\alpha_1, ..., \alpha_T)$, and a filtration $\mathcal{F} = (\mathcal{F}_0, ..., \mathcal{F}_T)$, the multi-period average value-at-risk deviation is defined as*

$$AVaRD_{\alpha,c}(Y;\mathcal{F}) = \sum_{t=1}^{T} c_t \mathbb{E}[AVaRD_{\alpha_t}(Y_t|\mathcal{F}_{t-1})]\,.$$

The multi-period average value-at-risk is concave and monotone in $\mathbf{Y}$. The proof of these properties together with other properties of the multi-period AVaR may be found in [47, Section 3.3.3].

5.3 Risk measures and decision problems

When investors want to construct a portfolio from certain assets, they aim to maximize the portfolio return. Risk averse investors minimize the risk associated with the investment as well. This problem has not a unique solution in general. One has to find a compromise between return and risk. The curve comprising all optimal solutions, i.e. portfolios with maximal return and minimal risk, is called the *efficient frontier* and is a well known issue in financial mathematics. Markowitz in his theory of portfolio [40] constructed an efficient frontier which consisted of a relationship between portfolio return and its variance. Of course, one can construct the efficient frontier for arbitrary risk measure (deviation functional) $\mathcal{D}$.

In the pension planning models introduced in Chapter 6, the random future outcome is the amount d_T of money saved at the terminal year T of pension saving. The saved amount is influenced by the following factors: the stochastic fund returns, the saver's decision abound the fund selection, wage growth. If we denote these factors symbolically by $\mathbf{x}$, then we may write $d_T = d_T(\mathbf{x})$.

The standard decision problem is to maximize the acceptability of the outcome over all feasible decisions $\mathbf{x} \in \mathbb{X}$. Thus, the optimization problem is

$$\begin{aligned} &\max \mathcal{A}(d_T(\mathbf{x})) \\ &\mathbf{x} \in \mathbb{X} \end{aligned} \tag{5.6}$$

and after taking $\mathcal{A} = \mathbb{E} - \mathcal{D}$ it can be rewritten to a form

$$\begin{aligned} &\max \mathbb{E}(d_T(\mathbf{x})) - \mathcal{D}(d_T(\mathbf{x})) \\ &\mathbf{x} \in \mathbb{X}\,. \end{aligned} \tag{5.7}$$

The family of problems (5.7) is closely related to the following family of problems

$$\begin{aligned} &\min \mathcal{D}(d_T(\mathbf{x})) \\ &\mathbb{E}(d_T(\mathbf{x})) \geq \mu \\ &\mathbf{x} \in \mathbb{X} \end{aligned} \tag{5.8}$$

with μ as a parameter. Solving the problem (5.8) for an appropriate range of μ leads to the efficient frontier function

$$\mu \mapsto F(\mu) = \min\{\mathcal{D}(d_T(\mathbf{x})) : \mathbb{E}(d_T(\mathbf{x})) \geq \mu, \mathbf{x} \in \mathbb{X}\} \tag{5.9}$$

pertaining to the deviation functional $\mathcal{D}$, which can be either a static risk measure or a dynamic one. In particular, we will use the single-period and multi-period average value-at-risk deviation $AVaRD(d_T)$ in models in Chapter 6.

Chapter 6

Mathematical models for pension planning

This chapter is dedicated to construction of models suitable for solving the problem defined in Chapter 3. We recall that the problem is to find an optimal switching strategy between several pension funds with different risk profiles in a time horizon of T (typically $T = 40$) years. Since the pension funds invest in financial markets, the saver bears the risk of asset returns during the saving phase. They may influence the exposition to risk but also the return by balancing between the pension funds.

In pension saving, one should also take into account the future contributions. If a series of contributions throughout a lifespan is made, a fall in the assets value early in life does not affect the future contributions, i.e. only part of one's future pension wealth is affected. On the other hand, if it occurs close to retirement it affects all past accumulated contributions and returns on them, i.e. most of one's pension wealth. Therefore, it is reasonable that the investment decision depends on the time to the maturity of saving. Since conventional wisdom, evidential in historical data, confirms that stock returns outperform bond ones in the long run, it is reasonable to assume that investors with a long time horizon prefer stocks to bonds.

There are several models that help, but do not ensure the saver to reach a target level of pension savings. The well known Markowitz portfolio selection model [40] relates the return and risk of efficient portfolios in the so called efficient frontier. Bodie et al. in [11] developed a model for lifetime consumption-portfolio choice with a labor/leisure decision. The authors concluded that pension saving becomes more conservative as retirement approaches. In [10], Bodie suggested a model to guarantee a minimum living standard in retirement.

In this chapter, we propose two types of models for the problem of optimal fund selection in pension planning:

Expected utility maximization models:

Ia: the Dynamic Accumulation Model (DAM),

Ib: the Proportional Investment Allocation Model (PIAM).

Risk minimizing models:

IIa: the Terminal Risk Minimizing Model (TRMM), in which the terminal risk is measured by the single-period average value-at-risk deviation,

IIb: the Multi-period Risk Minimizing Model (MRMM), in which the multi-period risk is measured by the multi-period average value-at-risk deviation.

The models determine an optimal strategy of the fund selection. However, since a retiring person strives to maintain their living standard at the same level as their last preretirement income, the wealth at year t is measured by multiples of the t-year's salary instead of the absolute value of saved money.

Models DAM and PIAM assume a given utility function and thereby also the saver's risk attitude. Then, the expected utility of the saved amount is maximized. The models lead to a Bellman equation of stochastic dynamic programming. Moreover, we also derive a partial differential equation determining the optimal strategy for the PIAM model.

Models TRMM and MRMM are based on an opposite approach. The target amount to be saved is determined first and then the riskiness of the investment is minimized. In the TRMM model, we consider a future pensioner who is interested in their terminal wealth at time T of retirement only, that is, they do not care about the evolution of their account in intermediate times. Using a static risk measure we minimize the uncertainty of achieving the target wealth. The MRMM model considers a saver who is interested in saving throughout their whole period of saving. This can be argued by the fact that, in the case of early death, the savings become a subject of heritage. We use a dynamic risk measure to measure the overall insecureness of the savings and minimize it taking the requirement on the target terminal amount into account. Both TRMM and MRMM models lead to large-scale linear programs with sparse and block matrix representation of the constraints.

The dynamic accumulation model has been presented by the author et al. in [34] and [35], the risk minimizing models in [36].

6.1 Dynamic accumulation model

The dynamic accumulation model (DAM) is the first of the models we introduce in this thesis for solving the problem defined in Chapter 3. We assume that a future pensioner is given a possibility to choose one (and only one) from a finite number of pension funds with different risk profiles, and may change the decision in certain periods. Without loss of generality, we assume yearly rebalancing. Next, we assume regular yearly contributions transferred from saver's salary to their pension account. We also assume that the saver's utility function U is known. Therefore their attitude to risk represented by the risk aversion coefficient is also known. We maximize the expected utility from the terminal wealth.

6.1.1 Problem formulation and assumptions

Before proceeding to the problem formulation, we list and clarify the notation that will be used in the DAM model:

T	expected retirement time,
J	number of funds,
r_t^j	return of fund j at time t,
u_t	accumulated sum at time t,
w_t	gross salary at time t,
β_t	wage growth at time t defined by $w_{t+1} = w_t(1+\beta_t)$,
d_t	ratio of accumulated sum u_t to the salary w_t,
τ	rate of regular yearly contribution as a part of gross salary.

In the sequel, we embed the above listed variables and parameters to the context of our problem. Suppose that a future pensioner with the expected retirement time in T years deposits once a year a τ-part of their yearly salary w_t at year t to a fund $j \in \{1, 2, \dots, J\}$. Since the funds invest in financial markets, their returns r_t^j are assumed to be stochastic.

Denote by u_t, $t = 0, 1, \dots, T$, the accumulated sum at time t. The startup value u_0 is equal to the very first contribution. At each next decision time $t = 1, ..., T-1$, the amount u_t is appreciated by a return corresponding to the chosen fund j at the previous time stage $t-1$, and a new contribution is added to the account. Under the assumption of constant contribution rate τ, the equations describing the time evolution of the account are

$$\begin{aligned} u_0 &= w_0\tau\,, \\ u_{t+1} &= u_t(1+r_t^j) + w_{t+1}\tau\,,\ t = 0, 1, \dots, T-1\,. \end{aligned} \tag{6.1}$$

At the time of retirement T, the pensioner will strive to maintain their living standard at the level of the last salary. From this point of view, the saved sum u_T at time T is not precisely what the future pensioner cares about. The ratio of the cumulative sum u_T and the yearly salary w_T, i.e. $d_T = u_T/w_T$, is more important. Using the quantity $d_t = u_t/w_t$, one can reformulate the budget-constraint equation (6.1):

$$\begin{aligned} d_0 &= \tau\,, \\ d_{t+1} &= F_t(d_t, j)\,,\ t = 0, 1, \dots, T-1\,, \end{aligned} \tag{6.2}$$

where

$$F_t(d,j) = d\,\frac{1+r_t^j}{1+\beta_t} + \tau\,,\ t = 0,1,\dots,T-1\,, \tag{6.3}$$

and β_t denotes the wage growth defined by the equation

$$w_{t+1} = w_t(1+\beta_t)\,, \quad t = 0,1,...,T-1\,.$$

The saver's decision about the fund selection at the time t is based on their information at that time. That is, if I_t denotes the information consisting of the history of returns $r_{t'}^j$, $t' = 0,1,\dots,t-1$, $j \in \{1,2,\dots,J\}$, and the wage growth $\beta_{t'}$, $t' = 0,1,\dots,t-1$, until the time t, then we have $j = j(t, I_t)$.

At this point, we make two assumptions for the DAM model:

Assumption A1. The fund returns r_t^j for all funds $j \in \{1,...,J\}$ and all time stages $t \in \{0,1,...,T\}$ are stochastic and mutually independent for fixed j.

Assumption A2. The wage growth rates β_t, $t = 0,1,...,T-1$, are deterministic and prescribed.

Assumptions A1 and A2 imply that the quantity d_t is the only relevant information from I_t. Hence, $j(t,I_t) \equiv j(t,d_t)$. In order to maximize the saver's utility from the terminal wealth, we can formulate a problem of stochastic dynamic programming

$$\max_{\mathcal{J}} E(U(d_T)) \tag{6.4}$$

with the following recurrent budget constraint:

$$\begin{aligned} d_0 &= \tau\,, \\ d_{t+1} &= F_t(d_t, j(t,d_t))\,,\ t = 0,1,\dots,T-1\,, \end{aligned} \tag{6.5}$$

where the maximum in (6.4) is taken over all non-anticipative strategies $\mathcal{J} = \{j(t,d_t) : t = 0,1,...,T-1\}$. Here U denotes a preferred utility function of a saver. We discussed the theory of utility functions in Chapter 4.

We recall a fact from the theory of conditional expectations that a sequence of nondecreasing information $\{I_t, t = 0,1,...,T\}$ may be considered as a sequence of nondecreasing σ-algebras. The so called tower-law holds for conditional expectation.

Theorem 6.1.1. *[8, pg. 34],[42] Tower law for conditional expectations.*

Let X be a random variable on a probabilistic space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}(|X|) < \infty$. Let $\mathcal{G}, \mathcal{H}$ be σ-algebras with $\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$. Then

$$\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G})\,.$$

The following theorem states that the optimal decision strategy for (6.4)–(6.5) can be found as a solution to a Bellman equation.

Theorem 6.1.2. *The optimal strategy of the problem (6.4)-(6.5) is the solution of the Bellman equation*

$$V_t(d) = \max_{j\in\{1,2,\dots,J\}} \mathbb{E}[V_{t+1}(F_t(d,j,r_t^j))] = \mathbb{E}[V_{t+1}(F_t(d,j(t,d),r_t^j))]\,, \tag{6.6}$$

for $t = 0,1,\dots,T-1$, *where* $V_T(d) = U(d)$. *The optimal feedback strategy* $j(t,d_t)$ *of (6.6) can be found backwards.*

Proof. Since the sequence of information $\{I_t, t = 0,1,...,T\}$ can be considered as a sequence of nondecreasing $\sigma-$algebras with the trivial σ-algebra $I_0 = \{\emptyset,\Omega\}$ with no information, we can apply the tower law of conditional expectations and obtain

$$\mathbb{E}(U(d_T)) = \mathbb{E}(U(d_T)|I_0) = \mathbb{E}(\mathbb{E}(U(d_T)|I_t)|I_0) = \mathbb{E}(\mathbb{E}(U(d_T)|d_t))\,.$$

We conclude that maximizing $\mathbb{E}(U(d_T))$ is the same as maximizing $\mathbb{E}(U(d_T)|d_t)$ for arbitrary t. Moreover, using the tower law again, we have

$$\mathbb{E}(U(d_T)|d_t) = \mathbb{E}(\mathbb{E}(U(d_T)|d_{t+1})|d_t)\,. \tag{6.7}$$

Let us now denote $\mathcal{J}_t = \{j(\tau,d_\tau) : \tau = t,...,T\}$ and

$$V_t(d) = \max_{\mathcal{J}_t} \mathbb{E}(U(d_T)|d_t = d)\,. \tag{6.8}$$

Then, using (6.7), we obtain

$$\begin{aligned} V_t(d) &= \max_{\mathcal{J}_t} \mathbb{E}(\mathbb{E}(U(d_T)|d_{t+1})|d_t = d) \\ &= \max_{j_t} \max_{\mathcal{J}_{t+1}} \mathbb{E}(\mathbb{E}(U(d_T)|d_{t+1})|d_t = d) \end{aligned} \tag{6.9}$$

At this place, a further discussion is needed. If $\mathcal{J}_{t+1}$ is a strategy, then we denote by $\mathcal{R}_{t+1}(\mathcal{J}_{t+1})$ the sequence of fund returns determined by the strategy $\mathcal{J}_{t+1}$, i.e.

$$\mathcal{R}_{t+1}(\mathcal{J}_{t+1}) = \{r_\tau^{j_\tau}; \tau = t+1,...,T, \text{ and } j_\tau \text{ given in } \mathcal{J}_{t+1}\}\,.$$

It is important to notice that the inner mean value in (6.9) has the following property: it is a function

$$\mathbb{E}(U(d_T)|d_{t+1}) \equiv h(d_{t+1}, \mathcal{J}_{t+1}, \mathcal{R}_{t+1}(\mathcal{J}_{t+1}))\,,$$

depending only on the d_{t+1} variable and the decision variables $j_{t+1},...,j_T$ and the fund returns determined by them. When proceeding backwards to the next time stage t, the only variable that depends on the control variable j_t or the stochastic variable $r_t^{j_t}$, is $d_{t+1} = d_{t+1}(d_t, j_t, r_t^{j_t})$. The variables $\mathcal{J}_{t+1}, \mathcal{R}_{t+1}$ are not affected. Thus, in (6.9) we are considering

$$\max_{j_t} \max_{\mathcal{J}_{t+1}} \mathbb{E}(h(d_{t+1}(d_t, j_t, r_t^{j_t}), \mathcal{J}_{t+1}, \mathcal{R}_{t+1}(\mathcal{J}_{t+1}))|d_t = d)\,.$$

To obtain the relationship (6.6), it is desirable to shift the maximum over $\mathcal{J}_{t+1}$ operator inside the mean value. Clearly, we have

$$h(d_{t+1}(d_t, j_t, r_t^{j_t}), \mathcal{J}_{t+1}, \mathcal{R}_{t+1}(\mathcal{J}_{t+1})) \le \max_{\mathcal{J}_{t+1}} h(d_{t+1}(d_t, j_t, r_t^{j_t}), \mathcal{J}_{t+1}, \mathcal{R}_{t+1}(\mathcal{J}_{t+1}))$$

for all $\mathcal{J}_{t+1}$, implying

$$\max_{\mathcal{J}_{t+1}} \mathbb{E}(h(d_{t+1}(d_t, j_t, r), \mathcal{J}_{t+1}, \mathcal{R}_{t+1}))$$
$$\leq \mathbb{E}(\max_{\mathcal{J}_{t+1}} h(d_{t+1}(d_t, j_t, r), \mathcal{J}_{t+1}, \mathcal{R}_{t+1}))$$

where the mean value is with respect to the random variable r. The opposite inequality follows from the fact that

$$\max_{\mathcal{J}_{t+1}} \mathbb{E}(h(d_{t+1}(d_t, j_t, r), \mathcal{J}_{t+1}, \mathcal{R}_{t+1}))$$
$$\geq \mathbb{E}(h(d_{t+1}(d_t, j_t, r), \mathcal{J}_{t+1}, \mathcal{R}_{t+1}))\,,$$

which holds for all $\mathcal{J}_{t+1}$, and, in particular for $\hat{\mathcal{J}}_{t+1} = \operatorname{argmax}_{\mathcal{J}_{t+1}} h(d_{t+1}, \mathcal{J}_{t+1}, \mathcal{R}_{t+1})$.

That is, we may continue in (6.9) as follows:

$$\begin{aligned} V_t(d) &= \max_{j_t} \max_{\mathcal{J}_{t+1}} \mathbb{E}(\mathbb{E}(U(d_T)|d_{t+1})|d_t = d) && (6.10)\\ &= \max_{j_t} \mathbb{E}(\max_{\mathcal{J}_{t+1}} \mathbb{E}(U(d_T)|d_{t+1})|d_t = d) && (6.11)\\ &= \max_{j_t} \mathbb{E}(V_{t+1}(F_t(d, j_t, r_t^{j_t})))\\ &= \mathbb{E}(V_{t+1}(F_t(d, j_t(d), r_t^{j_t(d)})))\,. \end{aligned}$$

Starting from $V_T(d) = U(d)$ and proceeding backwards from $t = T-1$ down to $t = 0$, one can calculate the optimal feedback strategy $j_t(d_t)$. Referring to Theorem C.1 in Appendix C, the solution to the Bellman equation can be found backwards. □

The optimal feedback strategy of (6.6) gives the saver the information about the optimal fund selection for each time t in dependency on the value of savings d_t. Now, suppose that the stochastic fund returns r_t^j are represented by their densities f_t^j. Moreover, let us assume that there are some govermental restrictions imposed on the fund selection. We denote by $\Delta_t \subset \{1, ..., J\}$ the set of all funds that may be chosen by a saver at time t. We present the exact form of the barrier set Δ_t in implementation of the model in Section 7.1.1. Equation (6.6) can be rewritten in the form

$$\begin{aligned} V_t(d) &= \max_{j \in \Delta_t} \mathbb{E}[V_{t+1}(F_t(d, j, r_t^j))]\\ &= \max_{j \in \Delta_t} \int_{\mathbb{R}} V_{t+1}\left(d\frac{1+r}{1+\beta_t} + \tau\right) f_t^j(r)dr\\ &= \max_{j \in \Delta_t} \int_{\mathbb{R}} V_{t+1}(y) f_t^j\left((y-\tau)\frac{1+\beta_t}{d} - 1\right)\frac{1+\beta_t}{d}dy\\ &= \int_{\mathbb{R}} V_{t+1}(y) f_t^{j(t,d)}\left((y-\tau)\frac{1+\beta_t}{d} - 1\right)\frac{1+\beta_t}{d}dy && (6.12) \end{aligned}$$

where the substitution $y = d(1+r)(1+\beta_t)^{-1} + \tau$ has been used and $\mathbb{R}$ denotes the set of real numbers. Thus, the optimal j at time t and for a given d may be found as the argument of maximum of J one-dimensional integrals, in which the value function V_{t+1} from time $t+1$ appears.

6.1.2 Proportional investment allocation model

Let us make an assumption that all funds $j \in \{1, ..., J\}$ invest to the same set of assets or financial instruments $i \in \{1, ..., I\}$. The funds differ in the weights of assets in their investment allocations. This assumption is not restrictive nor generalizing, because if an asset is not comprised in the investment strategy of a fund, the corresponding weight can be considered zero. As a special and simplified case, let us assume $I = 2$ and that the funds invest either to stocks, represented for example by a stock index, or bonds. Thus, a fund j is specified by the weight θ_t^j of stocks and $(1 - \theta_t^j)$ of bonds at time t in its investment strategy.

Example 6.1.1. *In Chapter 2 we summarized that in the case of Slovak Republic each pension fund management institution is obliged to offer the possibility of investing to three funds: The Growth Fund, The Balanced Fund, and The Conservative Fund. The definition of these funds is based on the ratios that can be invested to risky assets (e.g. stocks) and to secure assets (e.g. bonds). In particular, The Growth fund can invest up to* 80% *to risky assets, The Balanced Fund up to* 50%*, and The Conservative Fund cannot invest to risky assets at all. That is, if we denote the three funds correspondingly by* $i = 1, 2, 3$*, the theta parameters are* $\theta^1 = 0.8$*,* $\theta^2 = 0.5$*, and* $\theta^3 = 0$ *and they are constant over time.*

As a slight modification of the problem of choosing one among J given funds studied in the DAM model, we can ask a similar question: what is the optimal weight $\theta_t \in [0, 1]$ of stocks for the investment at each particular time t? Hence, the weight θ_t of stocks in the fund investment is a new control variable. The saver can subsequently choose a fund which is most similar to their investment decision θ_t. This version of the model was also studied in [44]. We call this model the proportional investment allocation model (PIAM).

We assume that the fund return R_t is normally distributed, i.e.

$$R_t \sim N(\mu(\theta_t), \sigma^2(\theta_t)) .$$

R_t has the density function

$$g_t(R) = \frac{1}{\sigma(\theta_t)\sqrt{2\pi}} \exp\left\{-\frac{(R - \mu(\theta_t))^2}{2\sigma^2(\theta_t)}\right\} .$$

Its mean value and volatility depend on the mean values and volatilities of stocks and bonds, and on parameter θ_t in the following way: if $R_t^{(s)}$ and $R_t^{(b)}$ denote the return of stocks and bonds, with mean values $\mu^{(s)}$ and $\mu^{(b)}$ and volatilities $\sigma^{(s)}$ and $\sigma^{(b)}$, respectively, then the mean value of a fund return is given by

$$\mu(\theta_t) = \theta_t \mu_t^{(s)} + (1 - \theta_t)\mu_t^{(b)} . \tag{6.13}$$

Its volatility is given by

$$\sigma^2(\theta_t) = \theta_t^2(\sigma_t^{(s)})^2 + 2\theta_t(1 - \theta_t)\sigma_t^{(s)}\sigma_t^{(b)} corr(R_t^{(s)}, R_t^{(b)}) + (1 - \theta_t)^2(\sigma_t^{(b)})^2 \tag{6.14}$$

where $corr$ is the corelation coefficient of $R_t^{(s)}$ and $R_t^{(b)}$.

For the sake of better computations, we now use exponential discounting. That is the reason why we now use capital R to denote the returns, as is common in the theory of financial mathematics. The evolution equation for the variable d_t becomes $d_{t+1} = F_t(d, \theta_t, y)$ where

$$F_t(d, \theta_t, y) = d_t \exp\left\{\mu(\theta_t) - \frac{1}{2}\sigma^2(\theta_t) - \beta_t + \sigma(\theta_t)y\right\} + \tau\,, \quad t = 1, ..., T\,, \tag{6.15}$$

starting from $d_0 = \tau$. Here y is a realization of the random variable $\Psi \sim N(0,1)$. The appearance of the expression $-\frac{1}{2}\sigma^2(\theta_t)$ in (6.15) follows from the well known Itô lemma ([39], Appendix E) implying that if μ is the mean of a stochastic process $\ln(S_t), t = 1, 2, ...$, satisfying the stochastic differential equation $d(\ln S_t) = \frac{dS_t}{S_t} = \mu dt + \sigma dW_t$, then the solution of this differential equation is $S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)$.

Let Δ_t^θ denote the barrier set reflecting the eventual governmental restrictions on the fund selection, expressed in terms of the proportion of stocks in the fund investment. By (6.6), the optimal strategy is a solution to

$$V_t(d) = \max_{\theta_t \in \Delta_t^\theta} \int_{\mathbb{R}} V_{t+1}\left(d \exp\left\{\mu(\theta_t) - \frac{1}{2}\sigma^2(\theta) - \beta_t + \sigma(\theta_t)y\right\} + \tau\right) f(y)dy\,, \tag{6.16}$$

for $t = T-1, ..., 1$, where $f(y) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}$ is the density function of standard normal distribution.

Theorem 6.1.3. *If $U(d)$ is continuous and increasing in d, then also $V_t(d)$ is continuous and increasing in d for all t.*

Proof. We prove all properties by mathematical induction.

Continuity: for $t = T$, the function $V_T(d) = U(d)$ is continuous in d. We assume that $V_{t+1}(d)$ is continuous in d and show that so is $V_t(d)$. We denote

$$\phi(d, \theta) = \int_{\mathbb{R}} V_{t+1}(F_t(d, \theta, y))f(y)dy\,. \tag{6.17}$$

Clearly, the integrand function is continuous in d. Hence, $\phi(d, \theta)$ is continuous in d. Finally, the maximum $V_{t+1} = \max_\theta \phi(d, \theta)$ is continuous in d.

Monotonicity: for $t = T$, the function $V_T(d) = U(d)$ is increasing in d. We assume that $V_{t+1}(d)$ is increasing in d and show that so is $V_t(d)$. Let $d_1 < d_2$. Then $F_t(d_1, \theta_t, y) < F_t(d_2, \theta_t, y)$ for all θ_t and also $V_{t+1}(F_t(d_1, \theta_t, y)) < V_{t+1}(F_t(d_2, \theta_t, y))$. Thus,

$$\int_{\mathbb{R}} V_{t+1}(F_t(d_1, \theta_t, y)f(y)dy < \int_{\mathbb{R}} V_{t+1}(F_t(d_2, \theta_t, y)f(y)dy$$

holds for all θ_t. Finally, the inequality holds also for maximum over $\theta_t \in \Delta_t^\theta$ on both sides. $\square$

6.1.3 PDE for the value function V

In this section, we derive a partial differential equation for the function V satisfying (6.16) when considering infinitely small time change $\epsilon \to 0^+$. The discrete model is the time-discretization of the obtained PDE with the time step 1.

In a short time interval $[t, t+\epsilon], \epsilon > 0$ small, (6.15) and (6.16) change to

$$F_t^\epsilon(d) = d \exp\left\{[\mu(\theta_t) - \frac{1}{2}\sigma^2(\theta_t) - \beta_t]\epsilon + \sigma(\theta_t) y \sqrt{\epsilon}\right\} + \tau\epsilon$$

and

$$V_t(d) = \max_{\theta_t \in \Delta_t^\theta} \int_{\mathbb{R}} V_{t+\epsilon}(F_t^\epsilon(d, \theta_t, y)) f(y) dy \,. \tag{6.18}$$

For $\epsilon = 0$ we have $F_t^0 = d$. The function F_t^ϵ has the following Taylor expansion about $\epsilon = 0$:

$$F_t^\epsilon = d + d\sigma(\theta_t) y \sqrt{\epsilon} + \left(d\left((\mu(\theta_t) - \frac{\sigma^2(\theta_t)}{2} - \beta_t) + \frac{\sigma^2(\theta_t)}{2} y^2\right) + \tau\right)\epsilon + O(\epsilon^{3/2}) \,. \tag{6.19}$$

The partial derivation of F_t^ϵ with respect to $\sqrt{\epsilon}$ reads

$$\frac{\partial F_t^\epsilon}{\partial \sqrt{\epsilon}} = d\sigma(\theta_t) y + 2\sqrt{\epsilon}\left(d\left(\mu(\theta_t) - \frac{\sigma^2(\theta_t)}{2} - \beta_t + \frac{\sigma^2(\theta_t)}{2} y^2\right) + \tau\right) + O(\epsilon) \,. \tag{6.20}$$

The Taylor expansion of the $V_{t+\epsilon}(F_t^\epsilon)$ function of two variables about (t, d) reads as follows:

$$\begin{aligned} V_{t+\epsilon}(F_t^\epsilon) &= V_t(d) + \frac{\partial V_t(d)}{\partial t}\epsilon + \frac{\partial V_t(d)}{\partial d}(F_t^\epsilon - F_t^0) \\ &+ \frac{1}{2}\left(\frac{\partial^2 V_t(d)}{\partial t^2}\epsilon^2 + \frac{\partial^2 V_t(d)}{\partial t \partial d}\epsilon(F_t^\epsilon - F_t^0) + \frac{\partial^2 V_t(d)}{\partial d^2}(F_t^\epsilon - F_t^0)^2\right) \\ &+ \text{h.o.t.} \end{aligned}$$

where

$$F_t^\epsilon - F_t^0 = A\sqrt{\epsilon} + B\epsilon \,,$$

with

$$A = d\sigma(\theta_t) y, \qquad B = d\left(\mu(\theta_t) - \frac{\sigma^2(\theta_t)}{2} - \beta_t + \frac{\sigma^2(\theta_t)}{2} y^2\right) + \tau \,.$$

Dividing the above Taylor expansion of $V_{t+\epsilon}(F_t^\epsilon)$ by $\epsilon > 0$ we obtain

$$\frac{V_{t+\epsilon}(F_t^\epsilon) - V_t(F_t^0)}{\epsilon} = \frac{\partial V_t}{\partial t} + \frac{\partial V_{t+\epsilon}}{\partial d}(\frac{A}{\sqrt{\epsilon}} + B) + \frac{1}{2}\frac{\partial^2 V_{t+\epsilon}}{\partial d^2} A^2 + O(\sqrt{\epsilon}) \,.$$

Equation (6.18) implies

$$0 = \max_{\theta_t \in \Delta_t^\theta} \int_{\mathbb{R}} \frac{V_{t+\epsilon}(F_t^\epsilon) - V_t(d)}{\epsilon} f(y) dy$$

and therefore

$$0 = \max_{\theta_t \in \Delta_t^\theta} \int_{\mathbb{R}} \left(\frac{\partial V_t}{\partial t} + \frac{\partial V_{t+\epsilon}}{\partial d} \frac{A}{\sqrt{\epsilon}} + \frac{\partial V_{t+\epsilon}}{\partial d} B + \frac{1}{2} \frac{\partial^2 V_{t+\epsilon}}{\partial d^2} A^2 + O(\sqrt{\epsilon}) \right) f(y) dy \,. \qquad (6.21)$$

If we denote $\varphi(\sqrt{\epsilon}, y) = \frac{\partial}{\partial d} V_{t+\epsilon}(F_t^\epsilon)$, then the Taylor expansion of the function $\varphi(\sqrt{\epsilon}, y)$ about $\sqrt{\epsilon} = 0$ is

$$\varphi(\sqrt{\epsilon}, y) = \varphi_0 + \sqrt{\epsilon}\varphi_1 + \epsilon\varphi_2 + \text{h.o.t.}$$

where

$$\begin{aligned}
\varphi_0 &= \frac{\partial V_t(F_t^\epsilon)}{\partial d}\Big|_{\epsilon=0}, \\
\varphi_1 &= \frac{\partial^2 V_t(F_t^\epsilon)}{\partial d^2} \frac{\partial F_t^\epsilon}{\partial \sqrt{\epsilon}}\Big|_{\epsilon=0}, \\
\varphi_2 &= \frac{\partial^3 V_t(F_t^\epsilon)}{\partial d^3} \left(\frac{\partial F_t^\epsilon}{\partial \sqrt{\epsilon}} \right)^2 \Big|_{\epsilon=0} + \frac{\partial^2 V_t(F_t^\epsilon)}{\partial d^2} \frac{\partial^2 F_t^\epsilon}{(\partial \sqrt{\epsilon})^2}\Big|_{\epsilon=0}.
\end{aligned}$$

Then we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{1}{\sqrt{\epsilon}} \varphi(\sqrt{\epsilon}, y) y f(y) dy = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \varphi_1 y f(y) dy \,.$$

Using (6.20) and the fact that $\int_{\mathbb{R}} y^2 f(y) dy = 1$ we obtain

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{\partial}{\partial d} V_{t+\epsilon}(F_t^\epsilon) y \frac{1}{\sqrt{\epsilon}} f(y) dy = \frac{\partial^2 V_t}{\partial d^2} d\sigma \,. \qquad (6.22)$$

That is, in the limit case $\lim_{\epsilon \to 0^+}$ in (6.21) we obtain the following theorem.

Theorem 6.1.4. *The function V satisfies the following fully nonlinear partial differential equation:*

$$\frac{\partial V}{\partial t} + \max_{\theta_t \in \Delta_t^\theta} \left\{ \left(d \left(\mu(\theta_t) - \beta \right) + \tau \right) \frac{\partial V}{\partial d} + \frac{1}{2} d^2 \sigma^2(\theta_t) \frac{\partial^2 V}{\partial d^2} \right\} = 0 \qquad (6.23)$$

with the terminal condition $V_T(d) = U(d)$ for $d > 0$.

If V is concave in d, i.e. $\frac{\partial^2 V}{\partial d^2} < 0$, then from the first order necessary condition for maximum of $\left(d \left(\mu(\theta_t) - \beta \right) + \tau \right) \frac{\partial V}{\partial d} + \frac{1}{2} d^2 \sigma^2(\theta_t) \frac{\partial^2 V}{\partial d^2}$ in (6.23) we obtain

$$\hat{\theta}_t(d) = \max \left(\min(\hat{\theta}_t^{int}, u_t), l_t \right) \qquad (6.24)$$

where l_t and u_t are the bounds of the interval $\Delta_t^\theta = [l_t, u_t]$ and

$$\hat{\theta}_t^{\,int} = - \frac{(\mu^{(s)} - \mu^{(b)}) \frac{\partial V}{\partial d} + dL \frac{\partial^2 V}{\partial d^2}}{dK \frac{\partial^2 V}{\partial d^2}} \,. \qquad (6.25)$$

Here

$$K = (\sigma^{(s)})^2 - 2\sigma^{(s)}\sigma^{(b)} corr(R_t^{(s)}, R_t^{(b)}) + (\sigma^{(b)})^2$$

and

$$L = \sigma^{(s)}\sigma^{(b)} corr(R_t^{(s)}, R_t^{(b)}) - (\sigma^{(b)})^2 \,.$$

6.2 Model minimizing the terminal risk

The risk minimizing models for solving the problem of pension planning described in Chapter 3 are based on minimizing the uncertainty of the savings, whereas the target terminal amount is self given. The terminal wealth is a random variable, as it depends on the random returns of pension funds that invest into various financial instruments. We look for an optimal selection of pension funds so that the target terminal wealth is achieved (in the sense of the average value) but the uncertainty of the savings is minimized.

In the Terminal Risk Minimizing Model (TRMM), we consider a future pensioner who is interested in their wealth at time T of retirement only, that is, they do not care about the evolution of their account in intermediate times. Using a static risk measure we minimize the uncertainty of achieving the target wealth.

At the very beginning, we have to point out one important change in comparison to the DAM model: we allow a continuous fund selection, that is, the saver can split their money into more than just one fund at the same time. There are two reasons for making this assumption: first, the possibility of splitting the saved amount into several funds can be viewed as a possible form of the legislative change in the future; second, the current legislature allowing the choice of one fund only would lead to mixed-integer programming, which is in general a NP-hard problem.

6.2.1 Linear constraints

We start with a description the natural constraints that have to be taken into account. In the DAM model they were given in (6.2). They express the appreciation of savings between two time stages and the regular contributions to the account.

We recall that $t \in \{0, ..., T-1\}$ denotes the time at which the saver makes a decision about the fund selection, and $j \in \{1, ..., J\}$ denotes different funds with returns r_t^j at time t, for which the Assumption A1 from Section 6.1.1 holds). In the DAM model, the variable d_t denotes the ratio of the accumulated amount at time t to the salary at time t. We introduce new variables for model TRMM (and also MRMM developed later):

- $y_t^1, ..., y_t^J$ denote the amounts invested at time t in funds $1, ..., J$, respectively. Of course, there is a natural constraint on nonnegativity of these variables, that is, $y_t^j \geq 0$ for all t, j.

- $\mathbf{y}_t$ denotes the vector $[y_t^1, ..., y_t^J]^\top$.
  Remark: the relationship between the variable d_t from the DAM model and the variable $\mathbf{y}_t$ from models TRMM and MRMM is $d_t = \mathbf{y}_t^\top \mathbf{1} = \sum_{j=1}^J y_t^j$ where $\mathbf{1}$ is a vector with all elements equal to 1.

- $s_t^j = \frac{1+r_t^j}{1+\beta_t}$ denotes the return of the fund j in the time interval $[t-1, t]$ adjusted by the wage growth rate β_t, which, like in the DAM model, is defined by the relationship $w_{t+1} = w_t(1+\beta_t)$. It is assumed to be deterministic and prescribed, see Assumption A2, Section 6.1.1.

- $\mathbf{s}_t$ denotes the vector $[s_t^1, ..., s_t^J]^\top$.

We notice that variables r_t^j, β_t, τ stand for the same parameters as in the DAM model, i.e. r_t^j is the return of fund j at time t, β_t is the wage growth at time t, and τ is the rate of regular contribution to the pension account. The problem is to find optimal y_t^j for all t, j, that is, the optimal distribution of the savings into the funds $j \in \{1, ..., J\}$ at each time t, so that the target wealth is achieved and the uncertainty of it is minimized.

The equations describing the time evolution of savings dependent on balancing between funds are

$$\begin{aligned} \mathbf{y}_0^\top \mathbf{1} &= \tau\,, && (6.26)\\ \mathbf{y}_t^\top \mathbf{1} &= \mathbf{y}_{t-1}^\top \mathbf{s}_t + \tau \quad \text{for all } t \in \{1, ..., T-1\}, && (6.27)\\ \mathbf{y}_T^\top \mathbf{1} &= \mathbf{y}_{T-1}^\top \mathbf{s}_T\,, && (6.28)\\ \mathbf{y}_t &\geq 0 \quad \text{for all } t \in \{1, ..., T\}\,. && (6.29) \end{aligned}$$

The meaning of the equations is as follows. The initial saved money at $t = 0$, i.e. the value $\mathbf{y}_0^\top \mathbf{1}$, is equal to the first contribution τ. At each next time stage $t = 1, ..., T - 1$, the amounts y_{t-1}^j from the previous time stage are appreciated by a corresponding random adjusted fund return s_t^j and a new contribution τ is added. That is, the overall accumulated amount at time t is $\mathbf{y}_{t-1}^\top \mathbf{s}_t + \tau$. Again, the saver has to distribute this amount into the funds $j \in \{1, ..., J\}$ in parts y_t^j for which (6.27) must hold. At the end of saving, at the terminal time T, no contribution τ is added.

The next constraint appearing in the problem is the requirement on the minimal target amount μ, in terms of the yearly salary. If $\mathbf{y}_T^\top \mathbf{1}$ (=d_T) denotes the *terminal wealth* random variable, then the following condition must be satisfied:

$$\mathbb{E}(\mathbf{y}_T^\top \mathbf{1}) \geq \mu\,. \qquad (6.30)$$

All constraints (6.26) - (6.30) are linear in the y_t^j variables.

6.2.2 The objective function

The TRMM model is based on minimizing the uncertainty of the terminal outcome $d_T = \mathbf{y}_T^\top \mathbf{1}$ under conditions (6.26) - (6.30). This can be measured by a static (one-period) risk measure. In our model, we use the average value-at-risk deviation. The main advantage of using it is that the problem becomes a linear program, as we will see in the following section. We gave a brief overview on the average value-at-risk measure in Chapter 5.

Hence, the objective function of the optimization problem is

$$g(\mathbf{y}) = \mathbb{E}(\mathbf{y}_T^\top \mathbf{1}) - AVaR_\alpha(\mathbf{y}_T^\top \mathbf{1}) \qquad (6.31)$$

and it is minimized with respect to $y_t^j, t \in \{0, ..., T-1\}, j \in \{1, ..., J\}$, under constraints (6.26)–(6.30).

6.2.3 Tree representation

Adjusted returns $\mathbf{s}_t$ form a stochastic process in a discrete time. It can be approximated by a scenario tree ([30], [42], [47, Chapter 5.1]). Each node at a stage t of the scenario tree represents one possible state of the random vector $\mathbf{s}_t$ at the (future) time t. Each path in the tree, starting from the root, represents one scenario of evolution of the random process $\mathbf{s}_t$ over time. We denote

0	the root of the tree,
$\mathcal{N} = \{0, 1, ..., N\}$	the set of all nodes in the tree,
S	the number of terminal nodes in the tree,
$\mathcal{T} = \{N - S + 1, ..., N\}$	the set of all S terminal nodes in the tree,
$\mathcal{N}_0 = \{1, ..., N - S\}$	the set of "inner" nodes,
n_-	the unique predecessor of the node $n \in \mathcal{N} \setminus \{0\}$,
$\{n\}^+$	the set of successors of the node $n \in \mathcal{N} \setminus \mathcal{T}$,
$\xi(n) \in \{0, ..., T\}$	time stage of the node $n \in \mathcal{N}$.

Each node n, except the root $n = 0$, has exactly one predecessor n_-. However, each node $n \in \mathcal{N} \setminus \mathcal{T}$ has a set of successors $\{n\}^+$. We illustrate this notation in Figure 6.1.

Now, we have to slightly adjust the notation from Section 6.2.1 to the tree representation, in the sense that the lower index of variables y_t^j and r_t^j, s_t^j will denote the particular node n in the time stage t. It means, we use the notation y_n^j and r_n^j, s_n^j. We clarify that the variable r_n^j represents the return of the fund j valid in the period (part of the scenario path) from the node n_- to n. Similarly to the older t-indexed notation, we denote $\mathbf{y}_n = [y_n^1, ..., y_n^J]^\top$. Let $d_n = \mathbf{y}_n^\top \mathbf{1}$ and $y_n^j \geq 0$ for all n, j. The notation of the wage growth rate changes from β_t to $\beta_{\xi(n)}$ to denote the wage growth rate corresponding to the time stage $\xi(n)$ of the node n. That is, the adjusted returns in the tree notation are defined by

$$s_n^j = \frac{1 + r_n^j}{1 + \beta_{\xi(n)}} \tag{6.32}$$

for all $n \in \mathcal{N} \setminus \{0\}, j \in \{1, ..., J\}$. We put $\mathbf{s}_n = [s_n^1, ..., s_n^J]^\top$. Again, the adjusted return s_n^j of fund j is valid in the period $\xi(n_-)$ to $\xi(n)$ with the corresponding scenario path between nodes n_- and n. The terminal wealth random variable d_T is represented by a vector of discrete values $d_m = \mathbf{y}_m^\top \mathbf{1}$, $m \in \mathcal{T}$, with corresponding scenario probabilities $p_m > 0$, $\sum_{m \in \mathcal{T}} p_m = 1$. The sum of node probabilities $p_n > 0$ in every time stage t of the tree is equal to one, i.e. $\sum_{n:\xi(n)=t} p_n = 1$ for all $t \in \{0, ..., T\}$.

Using Uryasev's representation (5.3) of $AVaR$, the minimized objective function (6.31) is expressed in the tree notation as

$$\min_{\mathbf{y}} \left\{ \sum_{m \in \mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) - \max_a \{a - \frac{1}{\alpha} \sum_{m \in \mathcal{T}} p_m [\mathbf{y}_m^\top \mathbf{1} - a]^-\} \right\} . \tag{6.33}$$

The constraints (6.26)-(6.30) can be rewritten as

$$\sum_{m \in \mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) \geq \mu , \tag{6.34}$$

$$\mathbf{y}_0^\top \mathbf{1} = \tau , \tag{6.35}$$

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau \quad \text{for all } n \in \mathcal{N}_0, \tag{6.36}$$

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n \quad \text{for all } n \in \mathcal{T}, \tag{6.37}$$

$$\mathbf{y}_n \geq 0 \quad \text{for all } n \in \mathcal{N} . \tag{6.38}$$

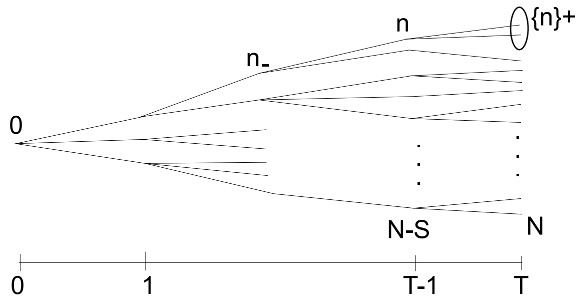


Figure 6.1: An example of a scenario tree. The bottom line represents the time line.

Next, we show that the optimization problem (6.33)–(6.38) can be rewritten to a linear program.

Proposition 6.2.1. *The optimization problem (6.33)–(6.38) is equivalent to the following linear problem:*

$$\min_{a,\mathbf{z},\mathbf{y}} \left(\sum_{m\in\mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) - a + \frac{1}{\alpha} \sum_{m\in\mathcal{T}} p_m z_{m-N+S} \right) \tag{6.39}$$

subject to

$$\begin{aligned}
-a + \mathbf{y}_m^\top \mathbf{1} + z_{m-N+S} &\geq 0, \quad z_{m-N+S} \geq 0, \quad \textit{for all } m \in \mathcal{T}, && (6.40)\\
\sum_{m\in\mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) &\geq \mu\,, && (6.41)\\
\mathbf{y}_0^\top \mathbf{1} &= \tau\,, && (6.42)\\
\mathbf{y}_n^\top \mathbf{1} &= \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau \quad \textit{for all } n \in \mathcal{N}_0, && (6.43)\\
\mathbf{y}_n^\top \mathbf{1} &= \mathbf{y}_{n-}^\top \mathbf{s}_n \quad \textit{for all } n \in \mathcal{T}, && (6.44)\\
\mathbf{y}_n &\geq 0 \quad \textit{for all } n \in \mathcal{N}\,. && (6.45)
\end{aligned}$$

It means that (i) every optimal solution to (6.39)–(6.45) is optimal for (6.33)–(6.38), and (ii) for every optimal solution to (6.33)–(6.38) there exists a unique corresponding optimal solution to (6.39)–(6.45).

Proof. First, let us notice that (6.40) implies $z_{m-N+S} \geq [\mathbf{y}_m^\top \mathbf{1} - a]^- = \max\{0, a - \mathbf{y}_m^\top \mathbf{1}\}$ for all $m \in \mathcal{T}$.

To prove (i), we will show that for an optimal solution to (6.39)–(6.45) we have $z_{m-N+S} = [\mathbf{y}_m^\top \mathbf{1} - a]^-$, and therefore, the optimal solution to (6.39)–(6.45) is the $AVaRD_\alpha$ minimizer subject to (6.34)–(6.38).

Let $\hat{a}, \hat{\mathbf{z}}, \hat{\mathbf{y}}$ be optimal for (6.39)–(6.45). Denote $h(\hat{a}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ the value of the objective function at the optimal solution. Assume that $m^\star \in \mathcal{T}$ exists such that $\hat{z}_{m^\star-N+S} > [\hat{\mathbf{y}}_{m^\star}^\top \mathbf{1} - \hat{a}]^-$. Then, choosing $\tilde{\mathbf{z}}$ with $\tilde{z}_{m-N+S} = \hat{z}_{m-N+S}$ for all $m \neq m^\star$ and $\hat{z}_{m^\star-N+S} > \tilde{z}_{m^\star-N+S} \geq [\mathbf{y}_{m^\star}^\top \mathbf{1} - a]^-$ we obtain $h(\hat{a}, \hat{\mathbf{z}}, \hat{\mathbf{y}}) > h(\hat{\mathbf{a}}, \tilde{\mathbf{z}}, \hat{\mathbf{y}})$, which is a contradiction to the optimality of $\hat{a}, \hat{\mathbf{z}}, \hat{\mathbf{y}}$.

To prove (ii), let us assume that $\bar{a}, \bar{\mathbf{y}}$ are optimal for (6.33)–(6.38). We put $\bar{z}_{m-N+S} = [\bar{\mathbf{y}}_m^\top \mathbf{1} - \bar{a}]^-$ for all $m \in \mathcal{T}$, satisfying the set of inequalities (6.40). It follows from the optimality of $\bar{a}, \bar{\mathbf{y}}$ for (6.33)–(6.38) that $\bar{a}, \bar{\mathbf{z}}, \bar{\mathbf{y}}$ are optimal for (6.39)–(6.45). □

Problem (6.39)–(6.45) is a linear program that can be symbolically written as

$$\min_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} \tag{6.46}$$

subject to

$$\begin{aligned} \mathbf{A}_{ineq}\mathbf{x} &\leq \mathbf{b}_{ineq}\,, &(6.47)\\ \mathbf{A}_{eq}\mathbf{x} &= \mathbf{b}_{eq}\,, &(6.48)\\ \mathbf{y}_n &\geq 0 \text{ for all } n \in \mathcal{N}, &(6.49)\\ z_m &\geq 0 \text{ for all } m \in \{1,...,S\}. &(6.50)\end{aligned}$$

The vector of variables $\mathbf{x} = (a, \mathbf{z}, \mathbf{y})$ has the length $vars = 1+S+J(1+N)$. The matrix $\mathbf{A}_{ineq}$ is of type $(1+S) \times vars$ and it is sparse with $(2J+2)S$ nonzero elements. The sparse $(1+N) \times vars$ matrix $\mathbf{A}_{eq}$ has $(1+2N)J$ nonzero elements. For reader's convenience, we present the exact form of matrices $\mathbf{A}_{ineq}$ and $\mathbf{A}_{eq}$ in Appendix D.

6.2.4 Existence of a solution

Let us now investigate the feasibility and optimality of the problem (6.46)–(6.50). First, we notice that the constraints

$$\mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq} \tag{6.51}$$

have no real impact on feasibility of the problem; they just describe explicitly the wealth evolution along the scenario paths. Similarly, the constraints

$$\mathbf{A}_{ineq}\mathbf{x} \leq \mathbf{b}_{ineq} \tag{6.52}$$

do not influence the feasibility of (6.46)–(6.50), with exception of one equality. Indeed, given any $\mathbf{y}_m, m \in \mathcal{T}$, we may put $z_{m-N+S} \geq 0$ arbitrary, $a \leq \mathbf{y}_m^\top \mathbf{1} + z_{m-N+S}$. Hence, given any $\mathbf{y}$, we may easily find feasible values of the variables a and $\mathbf{z}$. However, since we may treat the scenarios of s_n^j as fixed and given, the constraint

$$-\sum_{m \in \mathcal{T}} p_m \mathbf{y}_m^\top \mathbf{1} \leq -\mu$$

is crucial in determining whether the problem is feasible or not. It is clear that, given fixed scenarios s_n^j and following (6.51), it is not possible to achieve an arbitrary value of $\mathbb{E}(\mathbf{y}_T^\top \mathbf{1})$. Clearly, it is not possible to achieve a too high terminal outcome μ when the fund returns simulated in scenarios are low. There exists a μ_{max} such that if $\mu > \mu_{max}$, the problem (6.46)–(6.50) is infeasible. The value of this bound is determined by the particular scenarios of s_n^j and the rate of regular contribution τ by the folowing relationship:

$$\mu_{max} = \max_{\mathbf{y}} \sum_{m \in \mathcal{T}} p_m \mathbf{y}_m^\top \mathbf{1} \tag{6.53}$$

subject to constraints (6.48) and (6.49), i.e. subject to (6.42)–(6.45).

We summarize the above considerations in the following proposition.

Proposition 6.2.2. *Let μ_{max} be given by (6.53). If $\mu \leq \mu_{max}$, then the optimization problem (6.46)–(6.50) is feasible.*

Before we proceed to a statement about optimality of the above problem, we discuss the boundedness of the $\mathbf{y}$ variable in the following proposition.

Proposition 6.2.3. *The variable $\mathbf{y}$ from (6.47)–(6.50) is bounded.*

Proof. We denote

$$\Theta := \max_{n,j} |s_n^j| \,. \tag{6.54}$$

Then it is easy to show that

$$0 \leq y_n^j \leq \tau \sum_{i=0}^{\xi(n)} \Theta^i \,. \tag{6.55}$$

□

Theorem 6.2.1. *If the problem (6.46)–(6.50) is feasible then it attains an optimum.*

Proof. If a problem of linear programming is feasible, then it may attain an optimum or it may be unbounded ([49]). We show that the unboundedness is not the case in (6.46)–(6.50).

We recall that the objective function has the form

$$\min_{a,\mathbf{z},\mathbf{y}} \sum_{m\in\mathcal{T}} \left(p_m \mathbf{y}_m^\top \mathbf{1} - a + \frac{1}{\alpha} \sum_{m\in\mathcal{T}} p_m z_{m-N+S} \right) ,$$

or equivalently,

$$\min_{\mathbf{z},\mathbf{y}} \sum_{m\in\mathcal{T}} \left(p_m \mathbf{y}_m^\top \mathbf{1} - \max_a \{ a + \frac{1}{\alpha} \sum_{m\in\mathcal{T}} p_m z_{m-N+S} \} \right)$$

with $z_{m-N+S} = [\mathbf{y}_m^\top \mathbf{1} - a]^-$ in the optimum (see the proof of Proposition 6.2.1). We showed in Proposition 6.2.3 that the $\mathbf{y}$ variable is bounded. Therefore, the unboundedness of the objective function may be caused only by the unboundedness of variables a or $\mathbf{z}$. However, $\max_a\{a - \frac{1}{\alpha}\sum_{m\in\mathcal{T}} p_m z_{m-N+S}\} = AVaR_\alpha(\mathbf{y}_m^\top \mathbf{1})$. It is finite, hence bounded, because $\mathbf{y}$ is finite and bounded.

Another argument for excluding the unboundedness of (6.46)–(6.50) is that the objective function stands for the $AVaRD_\alpha$ and Corollary 5.1.1 states that it is nonnegative, hence bounded below. □

The next statement follows from the well known fact that, in linear programming, if the primal (dual) problem attains an optimum, so does the dual (primal) one ([49]).

Corollary 6.2.1. *For $\mu \leq \mu_{max}$, the optimization problem (6.46)–(6.50) attains an optimum and so does its dual.*

6.2.5 A nonlinear constraint

The size (number of nodes) of a tree is determined by the degree of the nonterminal nodes and the depth of the tree (number of time stages). If b_n are the degrees of the nodes $n \in \mathcal{N} \setminus \mathcal{T}$, then the size of the tree is $1 + \sum_{n \in \mathcal{N} \setminus \mathcal{T}} b_n$. In particular, if $b_n \equiv b$ for all $n \in \mathcal{N} \setminus \mathcal{T}$, the size of the tree is $\sum_{i=0}^{T} b^i$, that is, it depends polynomially on the degree b of nonterminal nodes and exponentially on the depth of the tree.

A typical length of the active life of a person during which they are working and saving for pension is, say, 40 years. The size of the simplest – binomial – tree is of order $2 * 10^{12}$. The number $vars$ of variables in our linear programs is then of the order $7 * 10^{12}$. If one would like to consider a tree with more scenarios, e.g. for a better approximation of the underlying stochastic process, the size of the tree would grow even more.

To lower the number of variables, let us assume that the savers do not make decisions about the fund selection every year but only in years $0 = t_0 < t_1 < \ldots < t_\omega$ represented by the nonterminal tree stages $0, 1, \ldots, \omega$; see Figure 6.2. We notice that the last time stage of the tree corresponds to real time T but this time is the time of retirement where no more decisions about investments are to be made, and therefore the last decision time is $t_\omega < T$. Hence, the depth of the tree is $\omega + 1$.

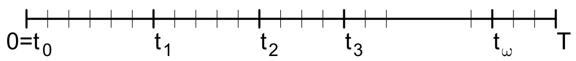


Figure 6.2: Decision periods.

Let $l_k = t_k - t_{k-1}$ denote the length of the period $[t_{k-1}, t_k], k \in \{1, \ldots, \omega + 1\}$. The basic problem (6.33)–(6.38) and its equivalent linear counterpart (6.39)–(6.45) are based on the assumption that $l_k = 1$ for all k. We now assume $l_k > 1$ for all or at least some k. That means, the account is appreciated l_k-times during the period $[t_{k-1}, t_k]$. The regular contribution τ is transferred to the account l_k-times too. We assume that it is distributed each time between funds $j \in \{1, \ldots, J\}$, maintaining their weights from the previous decision time t_{k-1}. If $\tau_n = [\tau_n^1, \ldots, \tau_n^J]^\top$ is the vector of yearly contributions transferred $l_{\xi(n)+1}$ times to funds $\{1, \ldots, J\}$ during the period $[t_{\xi(n)}, t_{\xi(n)+1}]$, from the node n to any of its successors from the set $\{n\}^+$, $n \in \mathcal{N} \setminus \mathcal{T}$, then

$$\frac{\tau_n^j}{\tau} = \frac{y_n^j}{\mathbf{y}_n^\top \mathbf{1}}. \tag{6.56}$$

Of course, $\tau_n^\top \mathbf{1} = \tau$ for all $n \in \mathcal{N} \setminus \mathcal{T}$.

Constraints (6.36) and (6.43), concerning the appreciation of the wealth in nonterminal stages, have to be modified in the following way:

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n_-}^\top \mathbf{s}_n + \tau_{n_-}^\top \sum_{i=0}^{l_{\xi(n)}-1} (\mathbf{s}_n)^{i/l_{\xi(n)}} \quad \text{for all } n \in \mathcal{N}_0 \tag{6.57}$$

where the components of the vector τ_{n_-} are given by (6.56). The power $(\mathbf{s}_n)^{i/l_{\xi(n)}}$ of the vector of yearly adjusted returns is considered componentwise, i.e. $(\mathbf{s}_n)^{i/l_{\xi(n)}} = [(s_n^1)^{i/l_{\xi(n)}}, ..., (s_n^J)^{i/l_{\xi(n)}}]^\top$. We obtain an optimization problem with one nonlinear constraint (6.61):

$$\min_{\mathbf{y}} \left\{ \sum_{m\in\mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) - \max_a \{a - \frac{1}{\alpha}\sum_{m\in\mathcal{T}} p_m[\mathbf{y}_m^\top \mathbf{1} - a]^-\} \right\} \tag{6.58}$$

subject to

$$\begin{aligned}
\sum_{m\in\mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) &\geq \mu\,, & (6.59)\\
\mathbf{y}_0^\top \mathbf{1} &= \tau\,, & (6.60)\\
\frac{\tau_n^j}{\tau} &= \frac{y_n^j}{\mathbf{y}_n^\top \mathbf{1}} \quad \text{for all } j\in\{1,...,J\}, n\in\mathcal{N}\setminus\mathcal{T}\,, & (6.61)\\
\mathbf{y}_n^\top \mathbf{1} &= \mathbf{y}_{n_-}^\top \mathbf{s}_n + \tau_{n_-}^\top \sum_{i=0}^{l_{\xi(n)}-1} (\mathbf{s}_n)^{i/l_{\xi(n)}} \quad \text{for all } n\in\mathcal{N}_0\,, & (6.62)\\
\mathbf{y}_n^\top \mathbf{1} &= \mathbf{y}_{n-}^\top \mathbf{s}_n \quad \text{for all } n\in\mathcal{T}, & (6.63)\\
\mathbf{y}_n &\geq 0 \quad \text{for all } n\in\mathcal{N}\,. & (6.64)
\end{aligned}$$

Remark. Similarly as in the proof of Proposition 6.2.1, the above problem is equivalent to a problem of minimizing the linear objective function (6.39) subject to (6.59)–(6.64) and the additional constraints (6.40). Next, we notice that (6.61) and (6.62) can be joined to one nonlinear constraint by expressing $\tau_{n_-}^j$ from (6.61).

6.3 Model minimizing the multi-period risk

The multi-period risk minimizing model (MRMM) for solving the pension planning problem defined in Chapter 3 is suitable for savers who are interested in the value of their account at all intermediate times rather than just at the terminal time. The reason for it could be the possibility of heritage of the saved amount in the case of early death. We use a multi-period risk measure to measure the uncertainty of all intermediate savings. In particular, we use the multi-period average value-at-risk deviation.

We recall from Section 5.2.1 that the multi-period average value-at-risk deviation is defined by

$$\begin{aligned}
AVaRD_{\alpha,c}(Y,\mathcal{F}) &= \sum_{t=1}^{T} c_t \mathbb{E}[AVaRD_{\alpha_t}(Y_t|\mathcal{F}_{t-1})]\\
&= \sum_{t=1}^{T} c_t \mathbb{E}[\mathbb{E}(Y_t|\mathcal{F}_{t-1}) - AVaR_{\alpha_t}(Y_t|\mathcal{F}_{t-1})]
\end{aligned}$$

where c_t are real constants representing a discount factor or weights of importance of conditional AVaRD-s at the time t. We notice that the confidence level α_t may be different for different times t. However, we consider $\alpha_t \equiv \alpha$ but the generalization of the upcoming equations and formulations to nonconstant confidence levels α_t is straightforward.

If we denote the conditional probabilities of the nodes $k \in \mathcal{N} \setminus \{0\}$ by

$$pc(k) = \frac{p_k}{\sum_{l \in \{n\}^+} p_l}, \quad k \in \{n\}^+, n \in \mathcal{N} \setminus \mathcal{T},$$

then the multi-period average value-at-risk deviation in the tree notation reads

$$\sum_{n \in \mathcal{N} \setminus \mathcal{T}} c_{\xi(n)} p_n \left(\sum_{k \in \{n\}^+} \left(pc(k) \mathbf{y}_k^\top \mathbf{1} \right) - \max_{a_n} \{ a_n - \frac{1}{\alpha} \sum_{k \in \{n\}^+} pc(k) [\mathbf{y}_k \mathbf{1} - a_n]^- \} \right). \tag{6.65}$$

The objective function (6.65) is minimized subject to the same constraints like in the one-period case, i.e. (6.34)–(6.38).

Using the same principle as in Proposition 6.2.1, we conclude that the problem of minimizing the multi-period risk, measured by the dynamic average value-at-risk deviation (6.65), can be rewritten to a linear program:

$$\min_{a,z,y} \sum_{n \in \mathcal{N} \setminus \mathcal{T}} c_{\xi(n)} p_n \left(\sum_{k \in \{n\}^+} \left(pc(k) \mathbf{y}_k^\top \mathbf{1} \right) - a_n + \frac{1}{\alpha} \sum_{k \in \{n\}^+} pc(k) z_{kn} \right) \tag{6.66}$$

subject to

$$\begin{aligned}
-a_n + \mathbf{y}_k^\top \mathbf{1} + z_{kn} &\geq 0, \; z_{kn} \geq 0, \quad \text{for all } k \in \{n\}^+, n \in \mathcal{N} \setminus \mathcal{T}, && (6.67)\\
\sum_{m \in \mathcal{T}} p_m \mathbf{y}_m^\top \mathbf{1} &\geq \mu, && (6.68)\\
\mathbf{y}_0^\top \mathbf{1} &= \tau, && (6.69)\\
\mathbf{y}_n^\top \mathbf{1} &= \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau \quad \text{for all } n \in \mathcal{N}_0, && (6.70)\\
\mathbf{y}_n^\top \mathbf{1} &= \mathbf{y}_{n-}^\top \mathbf{s}_n \quad \text{for all } n \in \mathcal{T}, && (6.71)\\
\mathbf{y}_n &\geq 0 \quad \text{for all } n \in \mathcal{N}. && (6.72)
\end{aligned}$$

The above optimization problem can be represented as a linear program having the matrix form (6.46)–(6.50) with the vector of variables $\mathbf{x} = (\mathbf{a}, \mathbf{z}, \mathbf{y})$ having the length $vars = 1 + N - S + N + J(1 + N)$. Both matrices A_{ineq} and A_{eq} have a sparse structure with size $(1 + N) \times vars$ and they have $JS + (J + 2)N$ and $(1 + 2N)J$ nonzero elements.

As in the TRMM, it is clear that, for the above problem to be feasible, the target wealth μ must be in a favorable relationship with the scenarios $\mathbf{s}_n$. This is due to the fact that it is not possible to achieve a too high outcome with low fund returns. Analogously to the TRMM model, the following theorem holds. Its proof is similar to that of Theorem 6.2.1 and therefore we will omit it here.

Theorem 6.3.1. *There exists a constant μ_{max} such that for $\mu \leq \mu_{max}$ the problem of minimizing (6.65) subject to (6.34)–(6.38) and its equivalent problem (6.66)–(6.72) are feasible and attain an optimum.*

The number of variables in the MRMM model is bigger than that of TRMM, because the variables $\mathbf{a}, \mathbf{z}$ are of a higher dimension. It means that, during implementation of the problem, one may expect difficulties with memory requirements. For this reason, we consider, again like in Section 6.2.5, a case when the savers do not rebalance their account yearly but only once during a period of several years. We obtain the following optimization problem with one nonlinear constraint (6.77):

$$\min_{a,z,y} \sum_{n \in \mathcal{N} \setminus \mathcal{T}} c_{\xi(n)} p_n \left(\sum_{k \in \{n\}^+} \left(pc(k) \mathbf{y}_k^\top \mathbf{1} \right) - a_n + \frac{1}{\alpha} \sum_{k \in \{n\}^+} pc(k) z_{kn} \right) \tag{6.73}$$

subject to

$$\begin{aligned}
-a_n + \mathbf{y}_k^\top \mathbf{1} + z_{kn} &\geq 0, \quad z_{kn} \geq 0, \quad \text{for all } k \in \{n\}^+, n \in \mathcal{N} \setminus \mathcal{T}, && (6.74)\\
\sum_{m \in \mathcal{T}} p_m (\mathbf{y}_m^\top \mathbf{1}) &\geq \mu\,, && (6.75)\\
\mathbf{y}_0^\top \mathbf{1} &= \tau\,, && (6.76)\\
\frac{\tau_n^j}{\tau} &= \frac{y_n^j}{\mathbf{y}_n^\top \mathbf{1}} \quad \text{for all } j \in \{1, ..., J\}, n \in \mathcal{N} \setminus \mathcal{T}\,, && (6.77)\\
\mathbf{y}_n^\top \mathbf{1} &= \mathbf{y}_{n_-}^\top \mathbf{s}_n + \tau_{n_-}^\top \sum_{i=0}^{l_{\xi(n)}-1} (\mathbf{s}_n)^{i/l_{\xi(n)}} \quad \text{for all } n \in \mathcal{N}_0\,, && (6.78)\\
\mathbf{y}_n^\top \mathbf{1} &= \mathbf{y}_{n-}^\top \mathbf{s}_n \quad \text{for all } n \in \mathcal{T}, && (6.79)\\
\mathbf{y}_n &\geq 0 \quad \text{for all } n \in \mathcal{N}\,. && (6.80)
\end{aligned}$$

We notice that constraints (6.77) and (6.78) can be joined to one nonlinear constraint

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n_-}^\top \mathbf{s}_n + \tau \frac{\mathbf{y}_{n_-}^\top \mathbf{q}_n}{\mathbf{y}_{n_-}^\top \mathbf{1}} \tag{6.81}$$

where $\mathbf{q}_n = \sum_{i=0}^{l_{\xi(n)}-1} (\mathbf{s}_n)^{i/l_{\xi(n)}}$.

Chapter 7

Implementation of the models and sensitivity analysis

In this chapter, we implement the DAM, PIAM, TRMM and MRMM models for the case of the pension system of Slovak Republic. We start with description of the system in more detail and description of the data used for computations and simulations. For each model we introduce a numerical approximation scheme and present results afterwards. The results concerning the dynamic strategies will be summarized in graphical plots as well as several tables displaying computed results of optimization. In the DAM model, we also investigate the sensitivity of the results with respect to the change of the risk aversion parameter, stocks and bonds returns and wage growth rate. In the TRMM model, we investigate the sensitivity of the results with respect to the α parameter specifying the average value-at-risk measure. As we will see, the results from all models confirm a property that the proportion of risky assets in the optimal investment strategy is decreasing over time.

7.1 Description of the system and data used

Let us now review the characteristics of the Slovak pension system described in Chapter 2, going into more detail. We begin with specifying the barrier function representing the governmental limitations imposed on the fund selection. Next, we make an assumption about the portfolio composition of particular funds. Finally, we specify the values of parameters entering the models.

7.1.1 Barrier function

In Slovakia, there are several commercial pension fund management institutions managing savers' pension savings within the 2^{nd} pillar of the pension system. Each of them is obliged to manage three funds with different limits for investment (see Tab. 2.2). As was already mentioned in Chapter 2, instant savers may hold assets at a particular time in one fund only. The decision about the fund selection is up to the savers. We recall the governmental restrictions that apply: the savers may not hold assets in the Growth Fund up to 15 years before retirement; moreover, all assets should be held in the Conservative Fund up to 7 years before retirement. The intention of these restrictions and governmental regulations was to lower the risk of the value of savings falling substantially, shortly before retirement. Intuitively it is clear that e.g. a 20% fall of asset values after the first year of saving affects the level of future pension less than a 20% fall in the last year of saving leading to a 20% fall of the pension benefits. In [35] we give more precise mathematical arguments to justify the idea of gradual switching to the less risky funds as the retirement age approaches.

In mathematical terms, the governmental limitations may be expressed by imposing a barrier on the fund selection. We denote the corresponding barrier function by Δ_t which is a function depending on the time parameter. The corresponding barrier function is defined by

$$\Delta_t = \begin{cases} \{1,2,3\} & \text{if } t \leq 25\,, \\ \{2,3\} & \text{if } 25 \leq t \leq 32\,, \\ \{3\} & \text{if } t \geq 33\,. \end{cases} \tag{7.1}$$

We can impose a constraint on the fund selection:

$$j(t,d) \in \Delta_t \text{ for all } t = 1, ..., T\,,$$

where we refer to the Growth, Balanced, and Conservative fund as 1, 2 and 3, correspondingly.

7.1.2 Portfolio composition

Let us now present a more detailed characterization of the three funds. The funds typically invest to stocks (European, American, Asian), bonds (typically governmental bonds) and other money market instruments. A typical portfolio composition is depicted in Figure 7.1 as an average portfolio of several pension fund management institutions obtained from their monthly reviews from October 2007. Geographically, most investments are allocated in European financial instruments. This is the reason why we decided, for simplicity, to use a European stock index to model the stock investments, and a European bond index to represent bond and money market investments.

We have chosen the Standard & Poor's Europe 350 Index to characterize the stock part of investments. S&P Europe 350 combines the benefits of representation with investability for the Europe region, spanning 17 exchanges. S&P Europe 350 is the foundation of the European index series. It is also the Europe component of S&P Global 1200. Index constituents exhibit the following characteristics. Market coverage of the index is over 70%

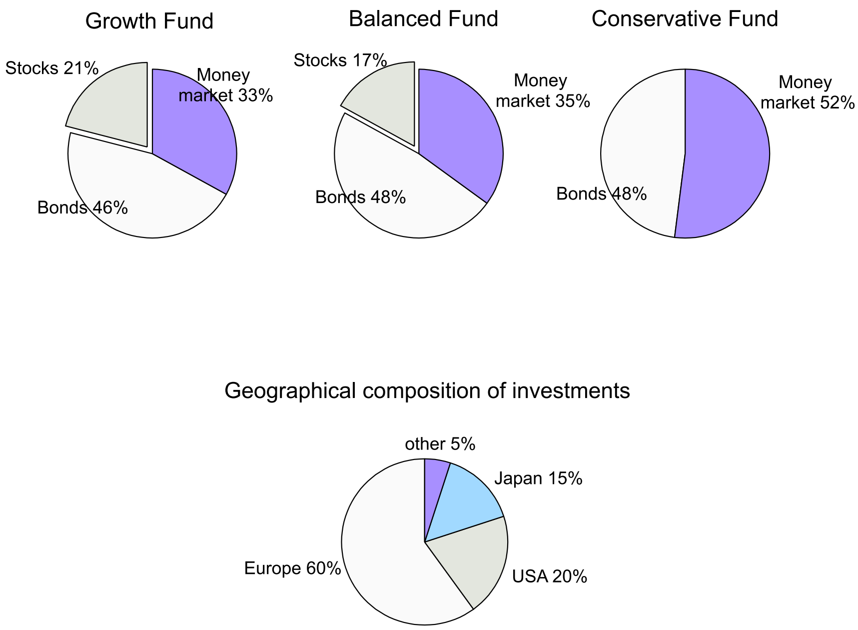


Figure 7.1: Top: Typical allocation of investments in particular funds. Computed as an average from several pension fund management institutions from their monthly reviews, obtained from their official webpages. Bottom: geographical diversification of stock investments, approximately.

of the Europe equities market. The S&P Europe 350 is a market capitalization weighted index. More information on the index is to be found at the official website of Standard and Poor's ([66]). For the historical evolution of S&P Europe 350 see Figure 7.2. To represent the bond part of the investment, we will use governmental bonds of European countries. The MSCI EMU Sovereign Debt Index contains the local currency government debt of 11 EMU member states. It is denominated in the EUR currency. For the historical evolution of this index see Figure 7.2. The index was incepted on December 31, 1993. The average yearly returns and standard deviations implied by historical data of the two indices are presented in Table 7.1. The correlation coefficient of S&P Europe 350 and MSCI EMU Sovereign Debt Index in the considered period is $corr = -0.07943$.

Figure 7.1 indicates that the proportion of stocks in the investment strategy of the funds is currently much smaller than the barrier restrictions imposed by governmental prohibitions allow. For example, the Growth fund invests approximately only 21% to stocks instead of the possible 80%. The Balanced fund invests approximately 17% to stocks instead of the possible 50%. In our computations, we consider two situations. First, we assume that the funds fulfill the barrier to a maximal possible extend. Second, we take into consideration the fact that the real ratio of stocks in the portfolio is relatively low, and we assume that the pension fund management institutions will raise this ratio over time.

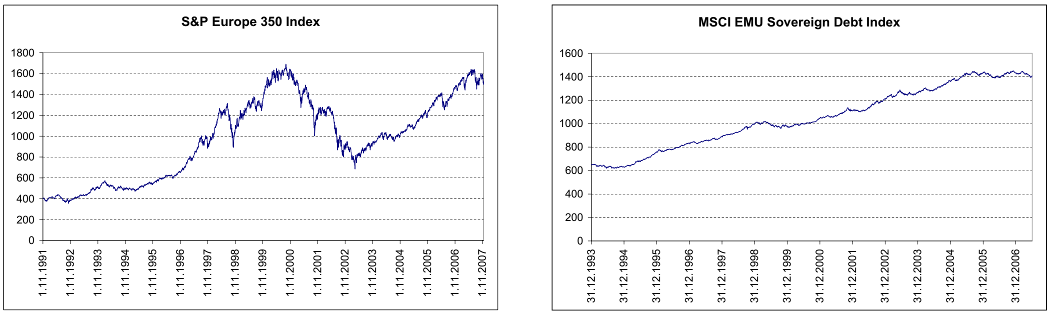


Figure 7.2: History of The Standard and Poor's Europe 350 Index (left) in the period 1991–2007, and The MSCI EMU Sovereign Debt Index (right) in the period 1994–2007. *Source*: Bloomberg.

	Average Return	StdDev
S&P Europe 350	$\bar{r}^{(s)} = 0.09185$	$\sigma^{(s)} = 0.17259$
MSCI EMU Sovereign Debt Index	$\bar{r}^{(b)} = 0.05594$	$\sigma^{(b)} = 0.03340$

Table 7.1: The average yearly returns and standard deviations of S&P Europe 350 and MSCI EMU Sovereign Debt Index based on historical data Jan 1994–June 2007. The correlation coefficient between S&P Europe 350 and MSCI EMU Sovereign Debt Index is $corr = -0.07943$. *Source*: Bloomberg.

7.1.3 Model parameters

Before presenting the results of simulations we have to discuss the input data. As mentioned in Chapter 2, there are three types of pension funds in the case of Slovak Republic, i.e. we put $J = 3$. As we discussed in the previous section, we suppose that the funds are constructed from stocks (S) and secure bonds (B) where stocks are represented by S&P Poor's Europe 350 Index, whereas the secure bonds by MSCI EMU Sovereign Debt Index. The average returns, standard deviations and correlation may be found in Table 7.1. According to Table 2.2 and under assumption that the pension fund management institutions use the investment restrictions to the maximal possible extent, we may express the value of funds symbolically in the following way:

$$\begin{aligned} \mathrm{F1} &= 0.8 \times \mathrm{S} + 0.2 \times \mathrm{B}\,, \\ \mathrm{F2} &= 0.5 \times \mathrm{S} + 0.5 \times \mathrm{B}\,, \\ \mathrm{F3} &= \mathrm{B}\,. \end{aligned} \tag{7.2}$$

By a simple computation we obtain average returns and standard deviations of the funds as in Table 7.2.

According to the Slovak legislature, the percentage of salary transferred each year to the pension account is $\tau = 9\%$. We assume that the period of saving is $T = 40$ years. The

Fund	Average Return	StdDev
F1 (Growth)	$\bar{r}_1 = 8.47\%$	$\sigma_1 = 13.80\%$
F2 (Balanced)	$\bar{r}_2 = 7.39\%$	$\sigma_2 = 8.73\%$
F3 (Conservative)	$\bar{r}_3 = 5.59\%$	$\sigma_3 = 3.40\%$

Table 7.2: Average return and its standard deviation for the Growth, Balanced and Conservative Fund.

Period	2008–10	2011–15	2016–21	2022–24	2025–50
t	1–4	5–10	11–16	17–19	20–60
wage growth $(1+\beta_t)$	1.07	1.071	1.065	1.060	1.050

Table 7.3: Predictions of the wage growth rate in Slovak Republic or years 2008–2050. *Source*: [38].

prediction of the wage growth rate β_t is summarized in Table 7.3.

7.2 The DAM and PIAM models

In implementation of the DAM and PIAM models, we consider two variants:

A: we do not assume any restrictions on the decision variables j or θ;

B: we do take the governmental regulatory restrictions on the fund selection given by the barrier function (7.1) into account.

We use the DAM model for studying the sensitivity of the optimal choice $j = j(t, d)$ with respect to varying parameters.

In the PIAM model, we discretize the decision variable θ by a discrete vector of values from $[0, 1]$. Both models DAM and PIAM share the same numerical approximation scheme presented in the following section.

7.2.1 Numerical approximation scheme

We start by recalling the mathematical formulation of the optimization problem in DAM. Our aim is to determine optimal j for the problem (6.12), i.e.

$$V_t(d) = \max_{j\in\Delta_t} \int_{\mathbb{R}} V_{t+1}(y) f_t^j \left((y-\tau)\frac{1+\rho_t}{d} - 1 \right) \frac{1+\rho_t}{d} dy \tag{7.3}$$

where $V_T(d) = U(d)$ and f_t^j is the density function of normally distributed fund returns r_t^j. We use the constant relative risk aversion utility function of the form

$$U(d) = -d^{1-a}\,, \quad d > 0\,, \tag{7.4}$$

where $a > 1$ is the constant coefficient of relative risk aversion. We note that the function $U(d)$ defined by (7.4) is a smooth, increasing and strictly concave function for $d > 0$. The coefficient of relative risk aversion a is commonly suggested to be less than 10 ([41]).

The principal difficulty in computing the integrals in (7.3) is due to significant oscillations in the integrand function. More precisely, it may attain both large values as well as low values of the order one. Therefore a scaling technique is needed when computing the integral (7.3). The idea of scaling is rather standard and is widely used in similar circumstances.

Let $H_t(d)$ be any bounded positive function for $t = 1, 2, ..., T$. We scale the function V_t by H_t, i.e. we define a new auxiliary function

$$W_t(d) = H_t(d)V_t(d)\,.$$

Clearly, the original function $V_t(d)$ can be easily calculated from $W_t(d)$ as follows: $V_t(d) = W_t(d)/H_t(d)$. Then, for each time step t from $t = T$ down to $t = 1$ we have

$$W_T(d) = H_T(d)V_T(d)$$

and

$$\begin{aligned} W_{t-1}(d) &= H_{t-1}(d)V_{t-1}(d) \\ &= \max_{j\in\Delta_t} \int_{\mathbb{R}} H_{t-1}(d)V_t\left(\frac{d}{1+\rho_t}(1+r)+\tau\right) f_t^j(r)dr \\ &= \max_{j\in\Delta_t} \int_{\mathbb{R}} \frac{H_{t-1}(d)W_t\left(\frac{d}{1+\rho_t}(1+r)+\tau\right)}{H_t\left(\frac{d}{1+\rho_t}(1+r)+\tau\right)} f_t^j(r)dr \\ &= \max_{j\in\Delta_t} \int_{\mathbb{R}} \frac{H_{t-1}(d)W_t(y)}{H_t(y)} f_t^j\left((y-\tau)\frac{1+\rho_t}{d}-1\right)\frac{1+\rho_t}{d}dy\,. \end{aligned} \tag{7.5}$$

It is worthwhile to note that any choice of the family $H_t, t = 0, ..., T$, of positive bounded scaling functions does not change the result. It may however significantly improve the stability of numerical computation.

In order to capture both large and small values of V_t we recursively define the scaling functions $H_t, t = T, T-1, ..., 1, 0$, depending on the previously computed solution V_{t+1} as follows:

$$H_T = \frac{1}{\sqrt{1+V_T^2}}, \text{ and } H_t = \frac{1}{\sqrt{1+V_{t+1}^2}} \text{ for } t = T-1, ..., 0\,. \tag{7.6}$$

In our algorithm we compute values of the function $W_t = W_t(d)$ for discrete values of d from the time dependent interval $d \in (d_{min}, t/2)$, where we use $d_{min} = d_0 = 0.09$. The upper bound $t/2$ has been chosen with respect to maximal expected values of the savings-to-yearly-salary ratio d. In each time level $t = T$ down to $t = 1$ we choose a uniform spatial discretization of the interval $(d_{min}, t/2)$ consisting of $k = 500$ mesh points. Stochastic fund returns r_t^j were assumed to have normal distributions with densities f_t^j having constant in-time means $\bar{r}^j$ and standard deviations $\sigma^j, j = 1, ..., J$. In order to compute numerically

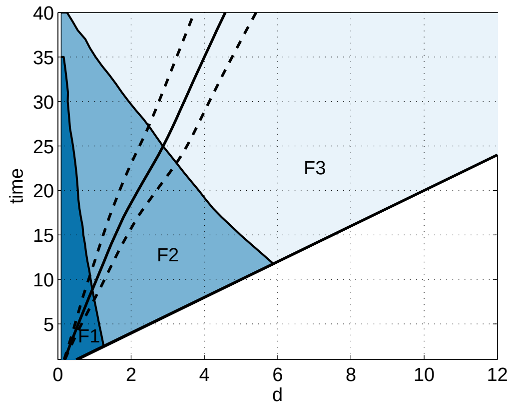


Figure 7.3: Regions of optimal choice, the path $\mathbb{E}(d_t)$ of average saved money to wage ratio (solid line) and the paths $\mathbb{E}(d_t) \pm \sigma(d_t)$ (dashed lines).

the Bellman type integral with normal distribution densities f_t^j we used the Simpson rule with 11 equidistant grid points covering the essential interval $(\bar{r}^j - 3\sigma^j, \bar{r}^j + 3\sigma^j)$. The numerical calculations were realized in Matlab 7 environment.

7.2.2 The DAM model: results and simulations

We present a typical result of our analysis for the DAM model, Variant A, in Figure 7.3. It contains three distinct regions in the (d, t) plane determining the optimal choice $j = j(d, t)$ of a fund depending on time $t \in [0, T-1]$ and the average saved money to wage ratio $d \in [d_{min}, t/2]$. The curvilinear solid line represents the path of the averaged wealth $\mathbb{E}(d_t)$, obtained by 50000 Monte Carlo simulations. Notice that, for $t > 0$, the ratio d_t is a random variable depending on (in our case normally distributed) random returns of the funds and on the computed optimal fund choice matrix $j(d, t'), t' < t$. The dashed curvilinear lines correspond to $\mathbb{E}(d_t) \pm \sigma_t$ intervals where σ_t is the standard deviation of the random variable d_t. One can observe that the optimal strategy is to choose the most risky fund in the first years of saving, and to switch to less risky funds in later times. In particular, if we view the funds as representations of the stock part θ_t (see Example 6.1.1), the numerical evidence says that the optimal weight of stocks θ_t in investment is nonincreasing over time t.

mean $\mathbb{E}(d_T)$	switch F1–F2	switch F2–F3
4.57	9 (8–11)	25 (23–27)

Table 7.4: Summary of computation of the averaged saved money to wage ratio d_T and switching times.

a	mean $E(d_T)$	switch F1–F2	switch F2–F3
5	5.81	15 (13–17)	never
7	5.09	11 (10–14)	33 (32–35)
9	4.57	9 (8–11)	25 (23–27)
11	4.36	8 (7–9)	21 (19–23)

Table 7.5: Results for fixed wage growths, fixed returns and standard deviations (see values in Tab. 7.2), various risk aversion parameter a.

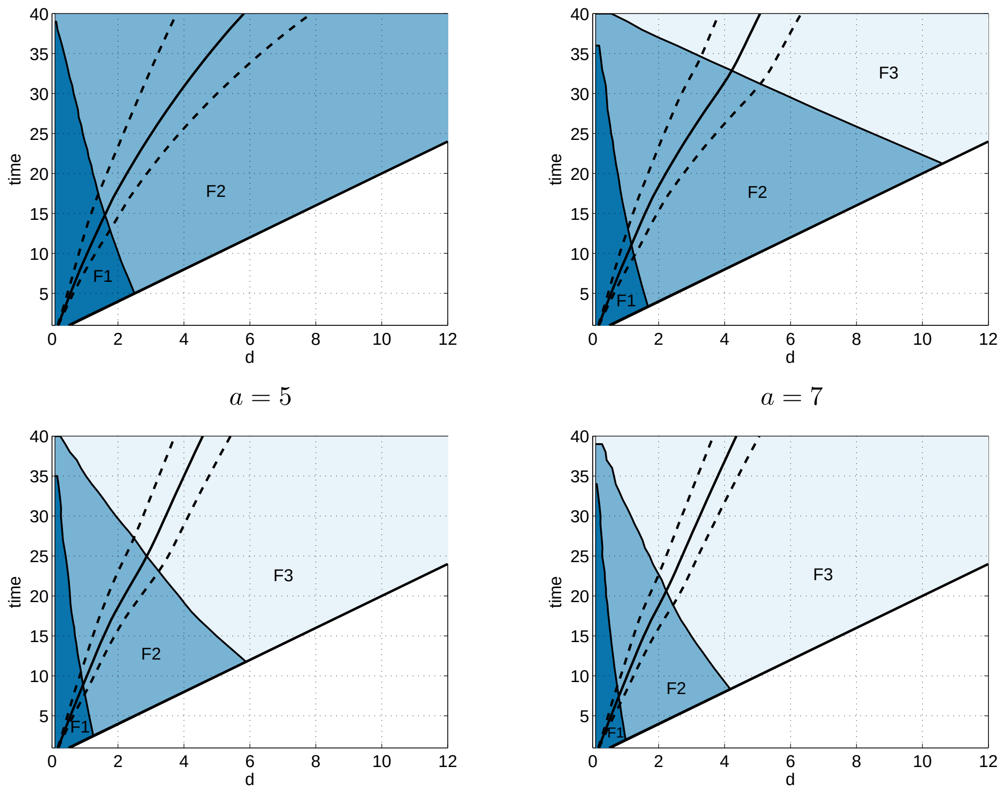
In Table 7.4 we present the averaged final wealth $\mathbb{E}(d_T)$ as well as the so-called *switching-times* for averaged path $\mathbb{E}(d_t), t \in [0, T]$, and the intervals (in brackets) of switching times for one standard deviation of the averaged path, i.e. for the paths $\mathbb{E}(d_t) \pm \sigma(d_t)$.

In the next subsections we pay our attention to sensitivity analysis of the results when some parameters are changing. In particular, we study the behavior of the optimal strategy j and the saved amount $\mathbb{E}(d_T)$ when the risk aversion parameter entering the utility function changes. Next, we study the sensitivity of the results with respect to varying stocks and bond returns and finally with respect to the wage growth rate.

Sensitivity analysis

Varying risk aversions.
Let us consider various values of the risk aversion parameter a in the utility function U given in (7.4): $a = 5, 7, 9, 11$. It is intuitive to expect that increasing risk aversion leads to preferring less risky funds. Indeed, as illustrated in Figure 7.4 and Table 7.5, one can observe that increasing a (i.e. increasing aversion to risk) causes that the switching-times between funds move to earlier times; that is, the saver changes from fund F1 to fund F2 earlier, as well as from fund F2 to fund F3. One may also expect that for higher values of the risk aversion parameter a one typically obtains lower levels of the final wealth and this is confirmed in the first column of Table 7.5; although, in general, higher exposure to risky assets for low risk aversion a may result in low outcome in the case of unfavorable behavior of stock market.

The efficient frontier relating the average terminal value of savings $\mathbb{E}(d_T)$ and the volatility $\sigma(d_T)$ to the risk aversion parameter a is depicted in Figure 7.5. Three different parts can be observed on the curves' shape: the first one (left from $a = 2$) corresponds to those a where the optimal strategy consists of investing in the most risky fund F1 only and the saver stays in this fund for the whole period of saving; the second part (between $a = 2$ and $a = 6$), the optimal strategy contains investing to the Growth Fund F1 in the first and Balanced Fund F2 in later years, but there is no switch to Conservative Fund F3. Finally, the part after $a = 6$ represents cases when all three funds are contained in the optimal strategy. The figure documents the expectation that with lowering risk aversion, one obtains a higher outcome on average.



$a = 5$ $a = 7$

$a = 9$ $a = 11$

Figure 7.4: The DAM model, Variant A. Sensitivity of regions of optimal choice with respect to various values of the risk aversion parameter $a = 5, 7, 9, 11$ and the average paths $\mathbb{E}(d_T)$, $\mathbb{E}(d_T) \pm \sigma(d_T)$.

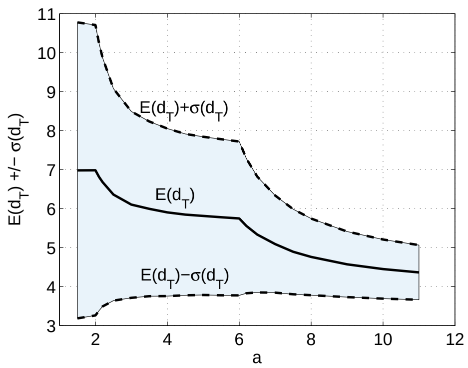
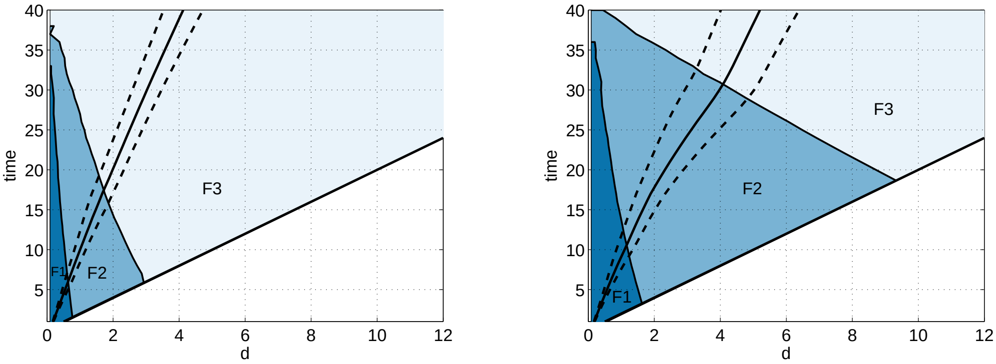
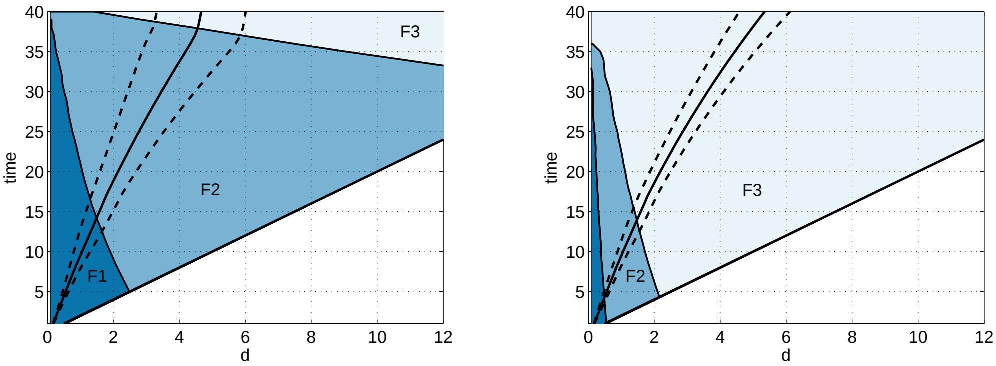


Figure 7.5: The DAM model, Variant A. The averaged saved wealth $\mathbb{E}(d_T)$ plus/minus simulated standard deviation $\sigma(d_T)$ in relationship to the risk aversion parameter a.



a) lower stock return $r^{(s)} = 0.08$ b) higher stock return $r^{(s)} = 0.10$
and fixed bond return $r^{(b)} = 0.05594$



c) lower bond return $r^{(b)} = 0.04$ d) higher bond return $r^{(b)} = 0.07$
and fixed stock return $r^{(s)} = 0.09185$

Figure 7.6: The DAM model, Variant A. Sensitivity of regions of optimal choice with respect to various expected values of asset and bond returns.

Varying stock and bond returns.
Now, let us examine the impact of a change in stock or bond returns on the optimal strategy and savings. For example, if the return of stocks increases (at a fixed volatility), one can expect that it will be more favorable to stay in funds with a higher proportion of stocks investment for a longer time than before. In our experiments, we first fix the bond return and increase/decrease the stock return, whereas $a = 9$ and other parameters including volatilities of stocks and bonds are fixed. The change of stock returns is reflected in the change of funds' F1 and F2 returns. The obtained results confirm the intuitive expectation that a higher return of stocks implies a later switch from more risky to less risky funds. Understandably, the average wealth $\mathbb{E}(d_T)$ in the terminal year of saving is higher. Secondly, we fix the stock return and increase/decrease the bond return. A higher return of bonds implies an earlier switch from more risky to less risky funds. The results of experiments are summarized in Figure 7.6 and Table 7.6.

Stock & Bond returns	Fund returns	mean $\mathbb{E}(d_T)$	switch F1–F2	switch F2–F3
$\bar{r}^{(s)} = 0.09185$ $\bar{r}^{(b)} = 0.05594$	$\bar{r}_1 = 0.08466$ $\bar{r}_2 = 0.07389$ $\bar{r}_3 = 0.05594$	4.57	9 (8–11)	25 (24–28)
$\bar{r}^{(s)} = 0.08$ $\bar{r}^{(b)} = 0.05594$	$\bar{r}_1 = 0.07519$ $\bar{r}_2 = 0.06797$ $\bar{r}_3 = 0.05594$	4.12	6 (6–7)	18 (16–20)
$\bar{r}^{(s)} = 0.10$ $\bar{r}^{(b)} = 0.05594$	$\bar{r}_1 = 0.09119$ $\bar{r}_2 = 0.07797$ $\bar{r}_3 = 0.05594$	5.20	11 (10–13)	31 (29–33)
$\bar{r}^{(s)} = 0.09185$ $\bar{r}^{(b)} = 0.04$	$\bar{r}_1 = 0.08148$ $\bar{r}_2 = 0.06592$ $\bar{r}_3 = 0.04$	4.66	15 (13–17)	38 (37–39)
$\bar{r}^{(s)} = 0.09185$ $\bar{r}^{(b)} = 0.07$	$\bar{r}_1 = 0.08748$ $\bar{r}_2 = 0.08092$ $\bar{r}_3 = 0.07$	5.35	4 (4–5)	14 (13–16)

Table 7.6: The DAM model, Variant A. Results for fixed wage growths, fixed $a = 9$, fixed standard deviations of fund returns $\sigma_1 = 0.1380, \sigma_2 = 0.0873, \sigma_3 = 0.0340$, and various bond and stock returns $\bar{r}^{(b)}$ and $\bar{r}^{(s)}$, hence fund returns $\bar{r}_1, \bar{r}_2, \bar{r}_3$.

Interest rate targeting.
Based on the calibration of Cox-Ingersoll-Ross interest rate model it was shown in [4] that it is reasonable to expect that the bond return will be approaching the value of approximately 2% in the time horizon of 30–40 years. Therefore, as a special case in studying the sensitivity of the results with respect to varying bond returns, we investigate how the optimal strategy changes when bond returns are decreasing monotonically to a certain target level.

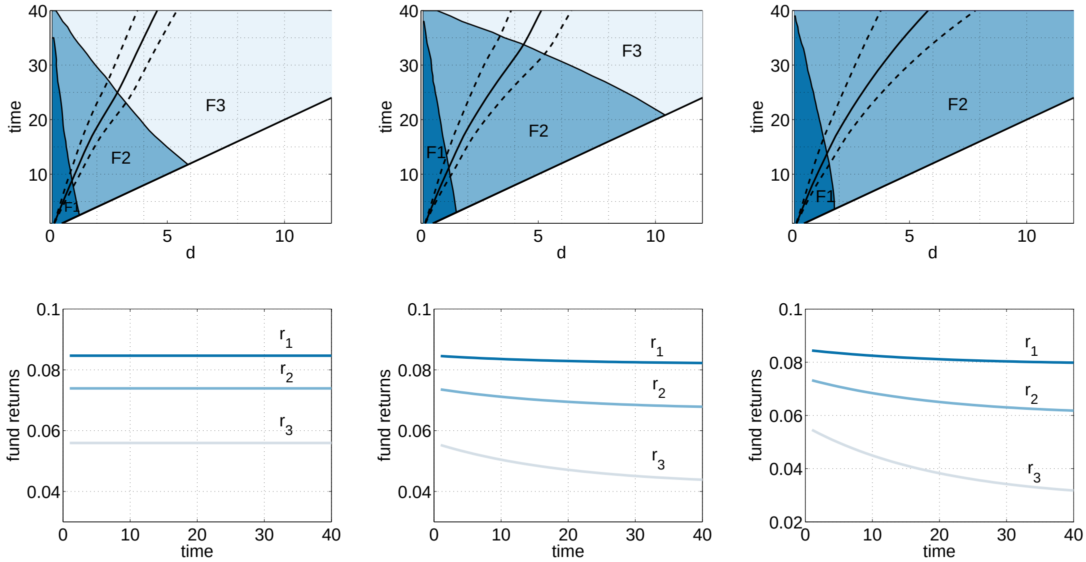
Let us assume that the bond return decreases exponentially from the starting value $\bar{r}_0^{(b)} = 0.05594$ given in Table 7.1 to a value $\bar{r}_\infty^{(b)}$. More precisely, we suppose that the rates in years $t = 1, 2, ..., T$, are given by formula

$$\bar{r}_t^{(b)} = \bar{r}_\infty^{(b)} + (\bar{r}_0^{(b)} - \bar{r}_\infty^{(b)})e^{-Kt/T}$$

for some constant K. We notice that decreasing bond returns affect the returns of all funds, F3, F2, as well as F1.

Figure 7.7 presents the results obtained for $K = 2$ and $\bar{r}_\infty^{(b)} = 0.75\bar{r}_0^{(b)}$ and $\bar{r}_\infty^{(b)} = 0.5\bar{r}_0^{(b)}$ in comparison to constant bond return $\bar{r}_0^{(b)}$. As expected, decreasing bond returns imply enlargement of the optimality regions of funds with low proportion of bonds in their portfolio. At a very strongly decreasing bond return (towards $\bar{r}_\infty^{(b)} = 0.5\bar{r}_0^{(b)}$), the Conservative Fund F3 is not selected at all throughout the whole period of saving.

A similar experiment was also presented in author's paper [34] in which the returns of all funds were considered to be exponentially decreasing to their half values.



a) $\bar{r}_{\infty}^{(b)} = \bar{r}_0^{(b)}$ b) $\bar{r}_{\infty}^{(b)} = 0.75\bar{r}_0^{(b)}$ c) $\bar{r}_{\infty}^{(b)} = 0.5\bar{r}_0^{(b)}$

Figure 7.7: Sensitivity of regions of optimal choice for exponentially decreasing bond returns.

wage growth	mean $\mathbb{E}(d_T)$	switch F1–F2	switch F2–F3
$\beta^{(-1\%)}$	5.53	8 (7–10)	23 (22–26)
β	4.57	9 (8–11)	25 (24–28)
$\beta^{(+1\%)}$	3.82	11 (10–13)	27 (25–29)

Table 7.7: Results for fixed returns and standard deviations (see values in Tab.7.2), fixed $a = 9$, and various wage growth rates.

Varying wages.
Finally, let us investigate the impact of various wage growth rates on the optimal strategy. Intuition says that one can expect a lower saved money to wage ratio d_t for a higher wage growth β; however, of course, this does not mean a lower absolute value of money. To examine this, we consider the wage growth being raised (lowered) uniformly for all time periods by 1%. We denote by $\beta^{(+1\%)}$ ($\beta^{(-1\%)}$) the wage growth predictions derived from Table 7.3 by increasing (decreasing) the β_t values by 1%. As we can see in Figure 7.8 and Table 7.7, a higher wage growth leads to a lower wealth $\mathbb{E}(d_T)$, guided by a shift of switching-times to later years.

Variant B of the DAM reflects the governmental limitations on fund selection, as described in Section 7.1.1. We present two illustrative results in Figure 7.9. One may observe the horizontal parts of the switching-borders at corresponding years where limitations apply.

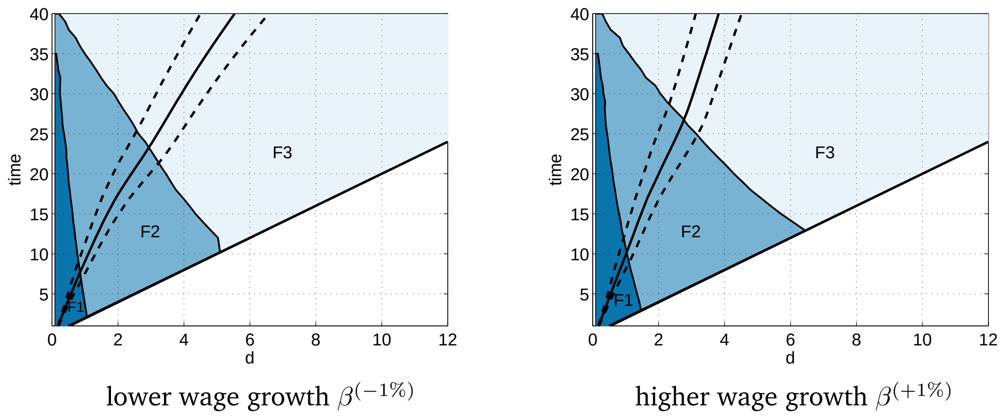


Figure 7.8: The DAM model, Variant A. Sensitivity of regions of optimal choice with respect to various wage growth β scenarios.

7.2.3 The PIAM model: results and simulations

In the PIAM model, the weight θ_t of stocks in investment is the decision variable. In our experiments, we discretize $\theta_t \in [0, 1]$ by 11 equidistant points. At each time stage t, an optimal $\theta_t \in \{1, 0.9, 0.8, ..., 0.1, 0\}$ was calculated. Figure 7.10 depicts a decreasing trend of the proportion of stocks in the optimal saving strategy, similarly as in Variant A and B of the DAM model.

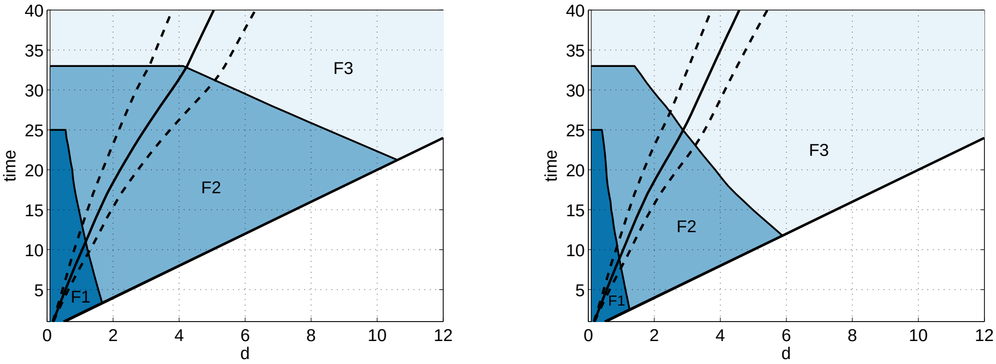
7.2.4 A case study

As the last example of results, we consider again the DAM model and its Variant B with portfolio composition representing the current situation in pension funds of Slovak Republic. It means, we take the governmental limitations on investments into account but, in contrary to all simulations in previous sections, we consider the current portfolio composition of funds presented in Figure 7.1. For illustrative purposes, we use a slightly more "optimistic" portfolio composition, as presented in Table 7.9. All other parameters are used as in the default case studied in the previous sections: risk aversion parameter $a = 9$, regular contribution $\tau = 9\%$ of gross salary, stock and bond returns as in Table 7.1, wage growth rates as in Table 7.3. The results obtained with these parameters are depicted in Figure 7.11 a).

A first look at the left-hand side plot of Figure 7.11 a) invokes an impression that the low weight of stocks in funds' investment has no impact on the amount of savings, because

govt. limits	no	yes
mean $\mathbb{E}(d_T)$	4.83	4.60

Table 7.8: The effect on the amount of savings with or without governmental limitations.



$a = 7$ $a = 9$

Figure 7.9: The DAM model, Variant B. Regions of optimal selection under governmental limitations, $a = 7, a = 9$.

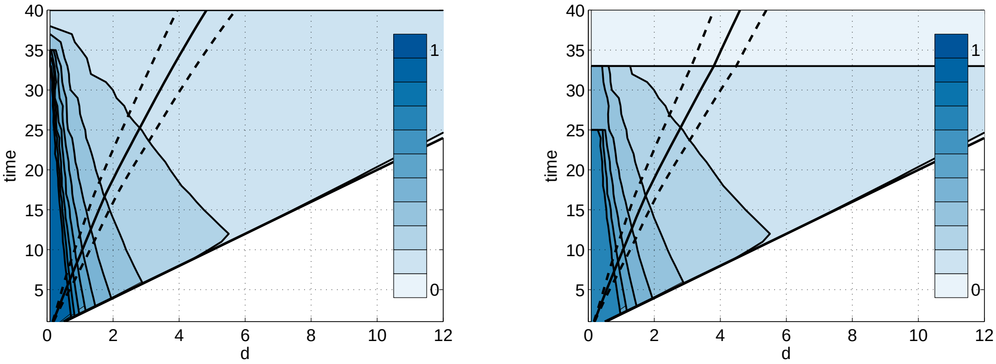
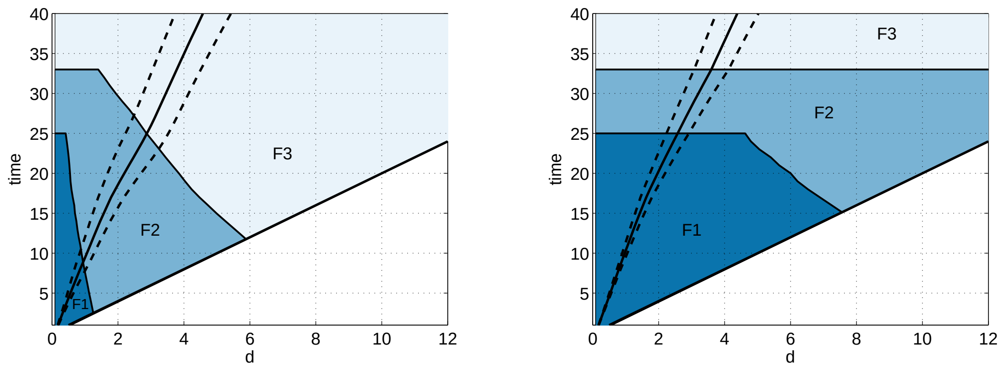


Figure 7.10: The PIAM model. Regions of optimal selection without (left) and with (right) governmental limitations, $a = 9$, continuous $\theta_t \in [0, 1]$ discretized by $\theta_t \in \{0, 0.1, 0.2, ..., 1\}$.

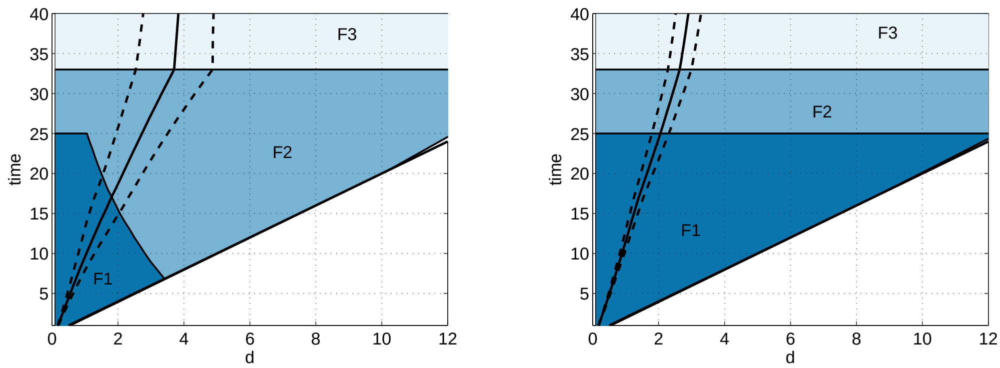
Fund	Stocks weight	Average return	StdDev
F1 (Growth)	0.25	6.49 %	4.92 %
F2 (Balanced)	0.2	5.31 %	4.31 %
F3 (Conservative)	0	5.59 %	3.40 %

Table 7.9: Current portfolio composition of pension funds in Slovak Republic.

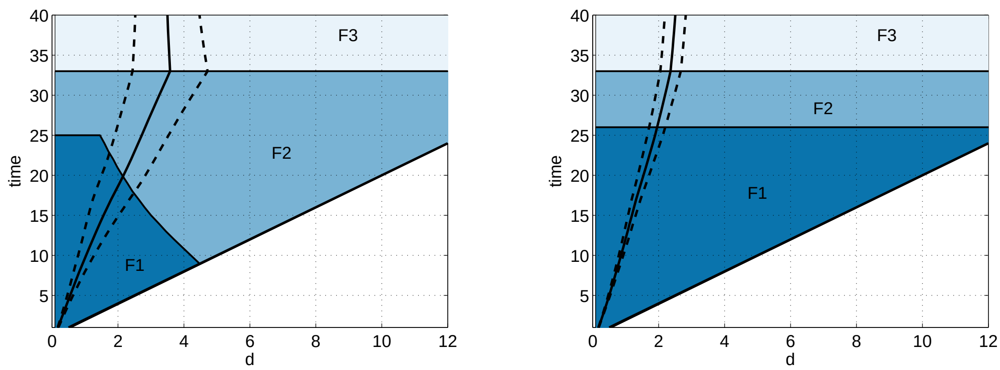
the average terminal wealth $\mathbb{E}(d_T)$ is close to that one of the default case in the right-hand side plot of Figure 7.11 a). However, it is important to notice that the time spent in risky funds is much larger than in the default case. One could argue that the volatility of funds F1 and F2 is lowered by lower proportion of stocks, see Table 7.9, and that even if the savers stay in the Growth fund for, say, 25 years, the risk they face is much smaller than



a) $r^{(b)} = 5.59\%$



b) $r^{(b)} = 3\%$



c) $r^{(b)} = 2\%$

Figure 7.11: The DAM model, a case study. Sensitivity of regions of optimal fund selection for fund investment composition with $\theta_1 = 0.8, \theta_2 = 0.5, \theta_3 = 0$ (left) vs. $\theta_1 = 0.25, \theta_2 = 0.2, \theta_3 = 0$ (right), and for $\bar{r}^{(b)} = 5.59\%$ (a), $\bar{r}^{(b)} = 3\%$ (b), and $\bar{r}^{(b)} = 2\%$ (c).

the one in the default case, where they stayed in fund F1 for 10 years only but with almost four times higher volatility.

In order to keep from making conclusions based on the impression that the proportion

Fund	$\bar{r}^{(b)} = 3\%$	$\bar{r}^{(b)} = 2\%$
F1 (Growth)	4.55 %	3.80 %
F2 (Balanced)	4.24 %	3.44 %
F3 (Conservative)	3 %	2 %

Table 7.10: The DAM model, a case study. Average fund returns for two pessimistic scenarios of bond return $\bar{r}^{(b)}$.

	$\bar{r}^{(b)} = 5.59\%$	$\bar{r}^{(b)} = 3\%$	$\bar{r}^{(b)} = 2\%$
$\theta_1 = 0.8, \theta_2 = 0.5$	$\mathbb{E}(d_T) = 4.57$ $\sigma(d_T) = 0.8479$	$\mathbb{E}(d_T) = 3.83$ $\sigma(d_T) = 1.0654$	$\mathbb{E}(d_T) = 3.50$ $\sigma(d_T) = 0.9731$
$\theta_1 = 0.25, \theta_2 = 0.2$	$\mathbb{E}(d_T) = 4.39$ $\sigma(d_T) = 0.6398$	$\mathbb{E}(d_T) = 2.90$ $\sigma(d_T) = 0.3848$	$\mathbb{E}(d_T) = 2.51$ $\sigma(d_T) = 0.3170$

Table 7.11: The DAM model, a case study. Results for a default bond return $\bar{r}^{(b)} = 5.59\%$ and for two pessimistic scenarios of the bond return $\bar{r}^{(b)} = 3\%$ and $\bar{r}^{(b)} = 2\%$.

of stocks in the funds does not play any important role in affecting the amount of savings, we do two more experiments. We show that if the bond return is lower, the overall savings may be much lower too. In fact, the results in Figure 7.11 a) are more or less a good occasional interaction of input parameters of the model, but in general the average value of savings depends strongly on the portfolio composition. Let us consider two pessimistic scenarios for the bond return $r^{(b)}$ as in Table 7.10. The results of the experiments are depicted in Figure 7.11 b) and c). In the optimal strategy, savers stay in the most risky funds as long as possible and switch to less risky funds only due to governmental limitations. The amount of terminal savings is legibly lower than in the case of $\theta_1 = 0.8, \theta_2 = 0.5$. However, the the standard deviation is much lower, see Table 7.11.

Hence, we may summarize that, on one hand, a very conservative policy of pension funds leads in average to low amount of savings, but on the other hand, the volatility of them is much lower also and thus it is worth consideration to decide which is better. Risk neutral managers would prefer the case with a higher average amount of savings, strongly risk averse managers would prefer the case with a lower volatility.

7.2.5 Summary

The results and simulations based on the DAM model give us an experience about the qualitative character of optimal fund selection strategies. They also illustrate how the optimal strategy and the amount of savings change under a change in the values of some parameters. We notice that the quantitative and also qualitative properties depend on the average return and volatility of stock and bond returns that enter the model. In our experiments, the average return of stocks was higher than the average return of bonds, and the volatility of the stock return was higher than the volatility of the bond return. We may conclude that, under this assumption on average returns and volatilities of stock and bond returns, the optimal weight of stocks in the optimal pension saving strategy is:

1. decreasing in time t and also decreasing in d_t,
2. decreasing in risk aversion parameter a,
3. increasing in average stock return $r^{(s)}$,
4. decreasing in average bond return $r^{(b)}$,
5. increasing in wage growth rate β_t,

The average amount of savings is:

1. increasing in time t
2. decreasing in risk aversion parameter a,
3. increasing in average stock return $r^{(s)}$,
4. increasing in average bond return $r^{(b)}$,
5. decreasing in wage growth rate β_t,

7.3 The TRMM and MRMM models

In this section we implement the TRMM model minimizing (6.39) subject to (6.59)–(6.64) and the MRMM model (6.73)–(6.80) for the case of the Slovak pension system with three funds. We suggest the use of an iterative algorithm for solving the problems in order to cope with the nonlinearity comprised in constraints (6.61) and (6.77). We discuss the data used for computation and describe the scenario tree generation. We implement the problem in MATLAB and use the MATLAB built-in *linprog* function and also MOSEK *mosekopt* function that can be implemented into a MATLAB code for solving sparse large-scale linear optimization problems. The two optimization softwares use different methods for linear programming: MATLAB uses the modified Mehrotra predictor-corrector primal-dual infeasible interior point method ([61]), MOSEK uses the homogeneous interior point method ([3]). We compare the results obtained with both of them.

7.3.1 An iterative algorithm

We suggest to cope with the nonlinearity in both models TRMM and MRMM by solving the problems iteratively in the following way:

1. fix the starting point $\tau_n^j = \tau/J$ for all $n \in \mathcal{N} \setminus \mathcal{T}, j \in \{1, ..., J\}$;
2. solve the linearized problems (6.39) subject to (6.59)–(6.64) and (6.73)–(6.80) with fixed parameters τ_n^j. Obtain optimal y_n^j for all n, j.
3. Compute new τ_n^j for all n, j using (6.56);
4. repeat steps 2, 3, 4 until prescribed accuracy in the norm of the difference of two successive iterates is attained.

If we denote the solution of the k-th iterate by $\mathbf{x}^{(k)} = (\mathbf{a}^{(k)}, \mathbf{z}^{(k)}, \mathbf{y}^{(k)})$, then the optimization problem solved by the proposed iterative algorithm is:

$$\min_{\mathbf{x}^{(k+1)}} \mathbf{c}^\top \mathbf{x}^{(k+1)} \tag{7.7}$$

subject to

$$\begin{aligned} \mathbf{A}_{ineq}\mathbf{x}^{(k+1)} &\leq \mathbf{b}_{ineq}\,, \\ \mathbf{A}_{eq}\mathbf{x}^{(k+1)} &= \mathbf{b}_{eq}(\mathbf{x}^{(k)})\,, \\ \mathbf{z}^{(k+1)}, \mathbf{y}^{(k+1)} &\geq 0\,. \end{aligned} \tag{7.8}$$

where the right-hand side vector from the equality constraints has elements

$$[\mathbf{b}_{eq}(\mathbf{x}^{(k)})]_i = [\mathbf{b}_{eq}(\mathbf{y}_{n_-}^{(k)})]_i = \begin{cases} \tau & i = 1\,, \\ \tau \frac{(\mathbf{y}_{n_-}^{(k)})^\top \mathbf{q}_n}{(\mathbf{y}_{n_-}^{(k)})^\top \mathbf{1}} & i = 2, ..., N-S+1, \text{ where } n = i-1\,, \\ 0 & i = N-S+2, ..., N+1\,, \end{cases}$$

where $\mathbf{q}_n = \sum_{i=0}^{l_{\xi(n)}-1}(\mathbf{s}_n)^{i/l_{\xi(n)}}$. According to point 1 from the description of the iterative algorithm and using (6.81), the elements of the right hand side vector, depending on the old iterates, are in the initial iterate chosen as

$$[\mathbf{b}_{eq}(\mathbf{y}^{(0)})]_i = \tau \frac{(\mathbf{y}_{n_-}^{(0)})^\top \mathbf{q}_n}{(\mathbf{y}_{n_-}^{(0)})^\top \mathbf{1}} = \frac{\tau}{J}\mathbf{1}^\top \mathbf{q}_n, \quad i = 2, ..., N-S+1; n = i-1\,.$$

They may be chosen in other ways also. We discuss the convergence of this algorithm in Section 7.4.

As a stopping criterion (step 4) we consider the difference between the optimal values of the objective function in two successive iterates k and $k+1$: if $\mathbf{x}^{(k)}$ and $\mathbf{x}^{(k+1)}$ are solutions of the k-th and $(k+1)$-th iterate and if $\epsilon > 0$ is fixed and small, then the stopping criterion is $|\mathbf{c}^\top \mathbf{x}^{(k+1)} - \mathbf{c}^\top \mathbf{x}^{(k)}| \leq \epsilon$. In our computations, the number of iterates for $\epsilon = 0.001$ is typically 5–6.

7.3.2 Data discussion and variants

The data set used in implementation of the TRMM and MRMM models is the same as in the DAM model. Table 2.2 explains the structure of the three funds of the Slovak pension system according to governmental regulation. The funds differ in various proportions of investing to stocks and to bonds. Table 7.1 contains historical returns and standard deviations for stocks represented by the S&P Europe 350 Index and for bonds represented by MSCI EMU Sovereign Debt Index. Data for the expected wage growth rate β_t are presented in Table 7.3. The regular contribution to the account is set by the law at the level of $\tau = 9\%$ of the gross salary. In the objective function of the MRMM model, we take the discount factors to be $c_n = 1$ for all n, and the average value-at-risk parameters $\alpha_n \equiv \alpha = 0.05$.

At this place it is important to recall that there are additional governmental restrictions on the fund selection. They define a barrier function Δ_t for each t, see (7.1). Therefore, like in the DAM model, we consider two variants of the TRMM and MRMM:

A: Governmental limitations are not considered; it means, the savers may choose any fund at any time. We split the saving period of 40 years into periods of years 1–10, 11–18, 19–25, 26–33, 33–40, with lengths $[l_1, ..., l_5] = [10, 8, 7, 8, 7]$. We put $T = 5$ and the target returns $\mu = 5.5$ and $\mu = 6$ at the year 40.

B: Governmental limitations are considered. We split the saving period of 40 years into periods of years 1–10, 11–18, 19–25, 26–29, 30–33, 33–40, having the lengths $[l_1, ..., l_6] = [10, 8, 7, 4, 4, 7]$. The time division is finer in later times because a higher amount of money is more sensitive to changes of fund returns and therefore a more frequent balancing is important. Since the fund selection is prescribed for the last seven years of saving, the last period is omitted from the optimization. We implement the regulations for periods 4 and 5 by adding a simple constraint $y_n^1 = 0$ for all nodes n belonging to the corresponding time stages. Since we consider only the first 33 years of saving in the optimization, we set $T = 5$, and consider target returns

$\mu = 4$ and 4.25 to be the target wealth at the end of the year 33. The wealth at the time of retirement is then given simply by the bond return (divided by the wage growth) in years 33–40.

The saver makes decisions at the beginning of each period. These decisions are followed by regular contributions distributed into the funds for the rest of the particular period.

7.3.3 Scenario tree generation

The values at the scenario tree nodes $n \in \mathcal{N} \setminus \{0\}$ are triple vectors $[s_n^1, s_n^2, s_n^3]$ representing the adjusted fund returns during the period $[t_{\xi(n)-1}, t_{\xi(n)}]$ from node n_- to n. They are used in appreciation equations for $\mathbf{y}_n$. Since the periods represent several years, the triples $[s_n^1, s_n^2, s_n^3]$ are returns for the whole length $l_{\xi(n)}$ of the period, not just for one year. The yearly returns needed in the vector $\mathbf{q}_n$ from (6.81) for the appreciation of τ_{n_-} are then obtained as the $l_{\xi(n)}$-th root of the components of $[s_n^1, s_n^2, s_n^3]$.

The values of s_n^j are calculated according to their definition in Section 6.2.3, modified to

$$s_n^j = \frac{1 + r_n^j}{1 + \beta_n^{avg}}$$

where we take $\beta_n^{avg} = \mathbb{E}([\beta_{l_{\xi(n)-1}+1}, ..., \beta_{l_{\xi}(n)}])$ in order to consider the average wage growth rate in the given period. The fund returns r_n^j for the period $[t_{\xi(n)-1}, t_{\xi(n)}]$ from node n_- to n are calculated from scenarios of stock and bond returns $r_n^{(s)}$ and $r_n^{(b)}$ for the given period and using the corresponding weights given in Table 2.2, i.e. $r_n^1 = 0.8r_n^{(s)} + 0.2r_n^{(b)}, r_n^2 = 0.5r_n^{(s)} + 0.5r_n^{(b)}, r_n^3 = r_n^{(b)}$. Hence, scenarios of s_n^j are determined by scenarios of the stock and bond returns.

We generate three scenarios for both stock and bond returns, in order to simulate an increment, decrement and no change of them. The scenarios for $r_n^{(s)}$ of stock return from node n_- to node n are generated in a standard way based on the assumption that the stock prices S_t follow the geometrical Brownian motion (see Appendix E):

$$dS_t = \bar{r}^{(s)} S_t dt + \sigma S_t dW_t$$

where $\bar{r}^{(s)}$ is the average return of stocks, σ is its standard deviation (see Table 7.1), and W_t is a standard Brownian motion. Using Itô lemma ([39], Appendix E) we obtain that

$$S_{t+l} = S_t \exp\left((\bar{r}^{(s)} - 0.5(\sigma^{(s)})^2)l + \sigma(W_{t+l} - W_t)\right)$$

for some time interval of length l. Then the returns are given by

$$1 + r_{\{n\}^+}^{(s)} = \exp\left((\bar{r}^{(s)} - 0.5(\sigma^{(s)})^2)l_{\xi(n)+1} + \sigma\sqrt{l_{\xi(n)+1}}Z_{\xi(n)+1}\right)$$

where $Z_{\xi(n)+1}$ are $N(0,1)$ distributed and independent for non-intersecting time intervals $[t_{\xi(n)}, t_{\xi(n)+1})$. The discrete scenarios are generated using a 3-point discretization for the standard normal distribution: point masses are concentrated at $(-\sqrt{2}, 0, \sqrt{2})$ with probabilities $(1/4, 1/2, 1/4)$. This discretization coincides in the first two moments with the standard normal distribution $N(0,1)$. Hence, we obtain $K_s = 3$ scenarios for stock returns and in the same way we generate $K_b = 3$ scenarios for bond returns. Combinations of them determine fund returns $r^j_{\{n^+\}}$ and thereby also adjusted returns $s^j_{\{n^+\}}$ in the $K_s K_b = 9$ successors of each node $n \in \mathcal{N} \setminus \mathcal{T}$.

Remark. A correlated bivariate Brownian motion should be used to generate the scenarios of stock and bond returns. However, since the correlation coefficient between them is very small ($corr \sim -0.08$), we use two independent univariate Brownian motions for simplicity.

7.3.4 Results

The output of the Matlab program is the set of triples $\mathbf{y}_n = [y^1_n, y^2_n, y^3_n]^\top$, $n \in \mathcal{N}$. We normalize the values y^j_n to weights w^j_n as follows:

$$w^j_n = \frac{y^j_n}{\sum_{i=1}^3 y^i_n}, \quad j \in \{1,2,3\},$$

such that $w^j_n \geq 0$. If we denote $\mathbf{w}_n = [w^1_n, ..., w^3_n]^\top$, then $\mathbf{w}_n^\top \mathbf{1} = 1$. We transform the amounts to weights in order to represent the distribution of the investment to funds as a percentage. There is no need to know the weights in the terminal time stage, because the savings are not distributed to different funds anymore. Therefore, the weights are calculated for nonterminal nodes $n \in \mathcal{N} \setminus \mathcal{T}$ only. The results have the form of a matrix of size J funds times the number of nonterminal nodes including the root, i.e. $J \times 1 + N - S$.

Since it is impractical to present the results for all $1 + N - S = \sum_{i=0}^4 9^i = 7381$ nonterminal nodes at this point, we investigate the average optimal strategy in each time stage. We denote $\bar{\mathbf{w}}_t = [\bar{w}^1_t, \bar{w}^2_t, \bar{w}^3_t]^\top$ the vector of average optimal weights of particular funds in the optimal strategy at time $t \in \{0, ..., \omega\}$, i.e.

$$\bar{\mathbf{w}}_t = \sum_{n:\xi(n)=t} p_n \mathbf{w}_n \,. \tag{7.9}$$

Averaging takes place over all nodes n in the same time stage $\xi(n) = t$ with probabilities p_n of the corresponding nodes. The overall share of stocks in the investment can be calculated from the funds' composition and the funds' weights $\bar{\mathbf{w}}_t$ as

$$\bar{w}^{(s)}_t = 0.8\bar{w}^1_t + 0.5\bar{w}^2_t \,. \tag{7.10}$$

In the forthcoming figures, we depict the average fund weights (7.9) in two left-hand side plots and the average stock weights (7.10) in the right-hand side plot.

Variant A

We present the results obtained for the Variant A (without considering governmental limitations) of models TRMM and MRMM in Figures 7.12 and 7.13. In Table 7.12 we present

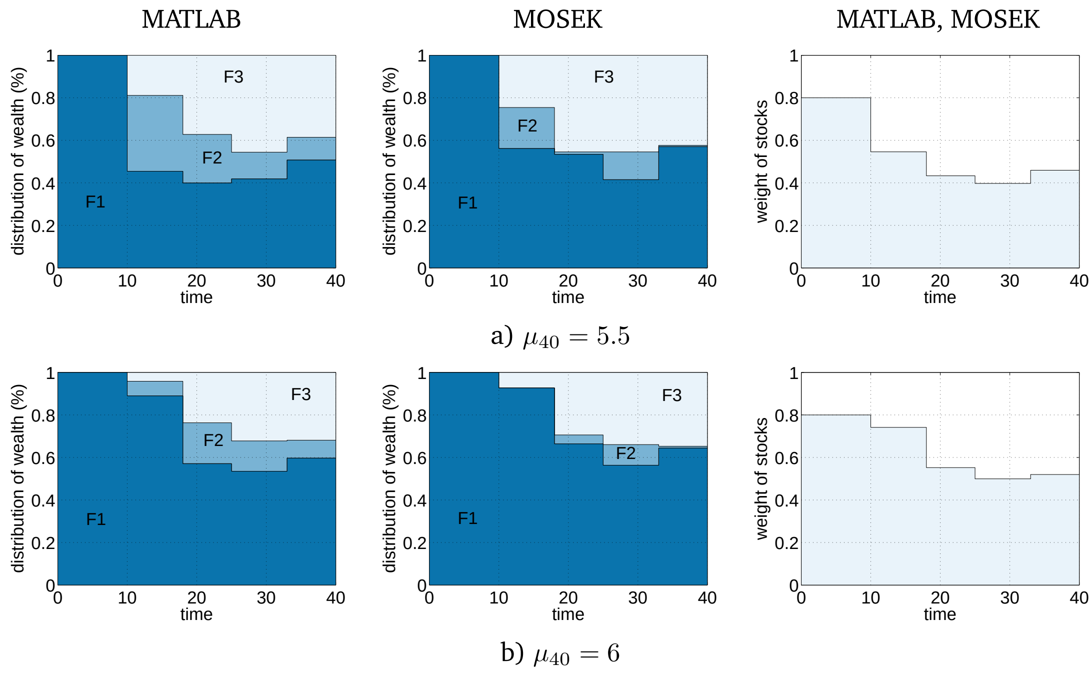


Figure 7.12: TRMM, Variant A without governmental limitations, MATLAB and MOSEK optimization results. Average optimal percentual allocation of the saved amount into the funds (left two columns) and average weights of stocks in investment (right) for target wealth after 40 years equal to $\mu_{40} = 5.5$ (top) and $\mu_{40} = 6$ (bottom).

Model		$\mu_{40} = 5.5$	$\mu_{40} = 6$
TRMM	single-risk 40	2.8525	3.4506
	multi-risk 1–40	5.4236	6.4024
	number of iterates	5	4
MRMM	single-risk 40	3.6853	4.1865
	multi-risk 1–40	3.4511	4.9677
	number of iterates	5	3

Table 7.12: TRMM and MRMM, Variant A without governmental limitations. Results for target amounts $\mu_{40} = 5.5$ and $\mu_{40} = 6$ after 40 years.

the single-period $AVaRD_{0.05}$ from the *wealth at year 40* random variable (it is minimal in the TRMM model but is not minimal in MRMM), the multi-period $AVaRD_{0.05}$ (which is not minimal in the TRMM model but is minimal in MRMM) and the number of iterates needed to achieve the accuracy $\epsilon \leq 0.001$ in the sense of Section 7.3.1.

The meaning of the sigle-period risk from the terminal wealth random variable (Table 7.12, row *single-risk 40*) is that the saver following the optimal strategy and having the target terminal wealth e.g. $\mu_{40} = 5.5$ will with probability $\alpha = 5\%$ save less than $5.5 - 2.8525 = 2.6475$ in terms of the multiples of the terminal salary level.

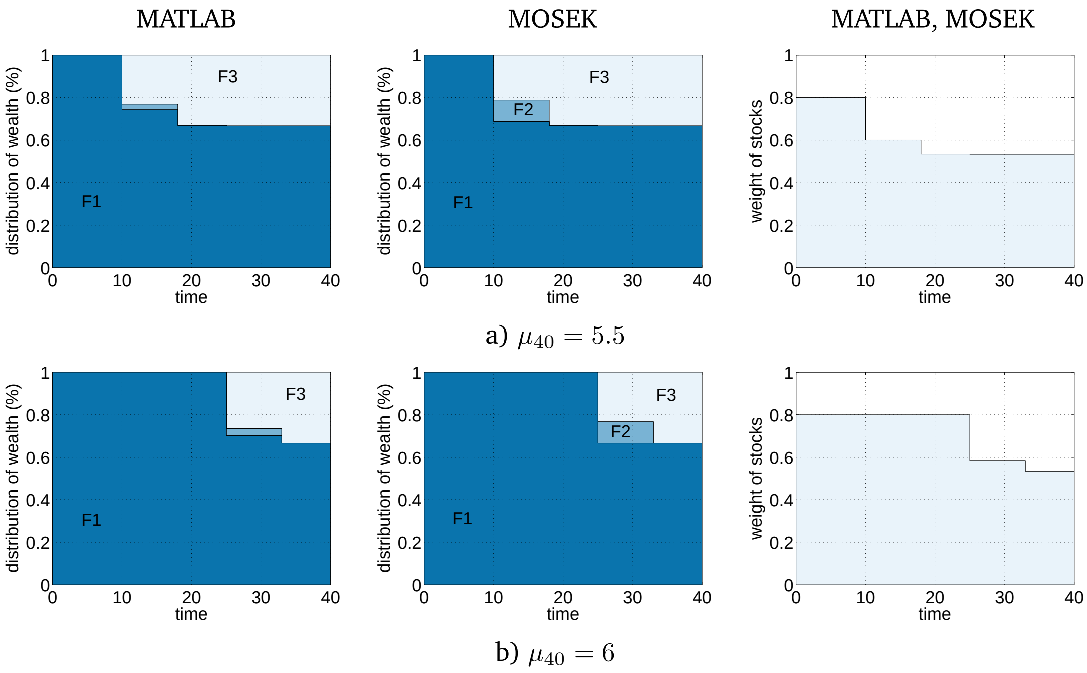


Figure 7.13: MRMM, Variant A without governmental limitations, MATLAB and MOSEK optimization results. Average optimal percentual allocation of the saved amount into the funds (left two columns) and average weights of stocks in investment (right) for target wealth after 40 years $\mu_{40} = 5.5$ (top) and $\mu_{40} = 6$ (bottom).

One can observe that, over time, the most risky (Growth) fund has a decreasing character whereas there is evidence of an increasing weight for the less risky funds. The share of stocks decreasing in time is in accordance to expectation because a higher amount of money is more sensitive to a change in the level of fund returns so the trend is to lower the risk in the future. It is also in accordance to the results obtained for the DAM and PIAM models where this feature was confirmed as well.

The next property of the optimal solution that appears in the empirical results is that the share of stocks in the investment increases when the target wealth μ increases. Of course, this is true under the assumption that stocks have a higher average return than bonds and that the returns have appropriate volatility. For example, if the volatility of stock returns was significantly lower than the volatility of bond returns, we could eventually expect a trend of lowering the weight of bonds and raising the weight of stocks over time.

Remark. Comparing numerical implementation results obtained by MATLAB and MOSEK one can observe a difference in the optimal solutions (see Figures 7.12 and 7.13). The optimal value obtained by both software packages was the same. There is empirical as well as evidence that the optimal solution is not unique. The figures also illustrate that although the optimal strategies differ under the two different interior point methods, the weights of stocks in the investment are the same. This leads us to a conclusion that the weight of stocks in investment is what is really relevant in the optimal strategy, and it

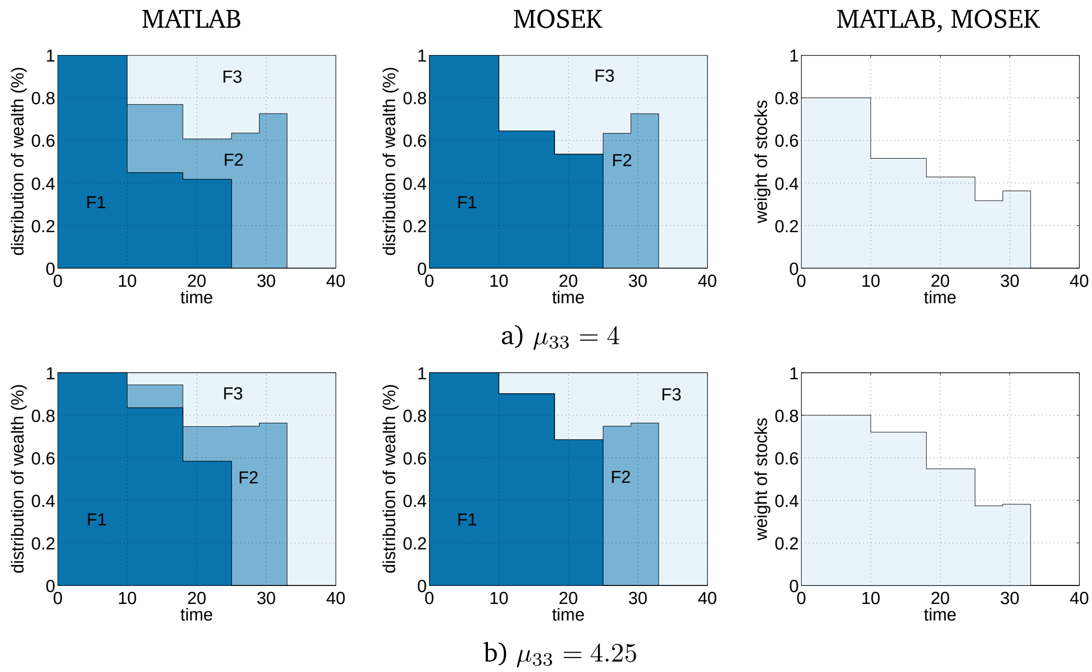


Figure 7.14: TRMM, Variant B with governmental limitations, MATLAB and MOSEK optimization results. Average optimal percentual allocation of the saved amount into the funds (left two columns) and average weights of stocks in investment (right) for target wealth after 33 years $\mu_{33} = 4$ (top) and $\mu_{33} = 4.25$ (bottom).

does not really matter by which combination of funds it is achieved. Indeed, the value of savings $\mathbf{y}_n^\top \mathbf{1}$ in each node n may be expressed as a sum of stocks value S_n and bonds value B_n:

$$\mathbf{y}_n^\top \mathbf{1} = S_n + B_n$$

where $S_n = 0.8y_n^1 + 0.5y_n^2$ and $B_n = 0.2y_n^1 + 0.5y_n^2 + y_n^3$. Similarly, the appreciation of the $y_{n\_}^j$ amounts in particular fonds, $\mathbf{y}_{n\_}^\top \mathbf{s}_n$, may be expressed as appreciation of the stocks and bonds part with using a suitable vector $\nu = (\nu_n^S, \nu_n^B)$ of returns:

$$\mathbf{y}_{n\_}^\top \mathbf{s}_n = S_{n\_}\nu_n^S + B_{n\_}\nu_n^B$$

where $\nu_n^S = 2s_n^2 - s_n^3$ and $\nu_n^B = s_n^3$. That means, the optimal triples $[y_n^1, y_n^2, y_n^3]$ of the optimization problems in all iterates with fixed right-hand side vectors may be expressed by pairs $[S_n, B_n]$. Therefore, there may be many possibilities how to combine the values $[y_n^1, y_n^2, y_n^3]$ in order to get the values $[S_n, B_n]$. The difference in the optimal vector solutions obtained by MATLAB and MOSEK is caused by different optimization algorithms.

Variant B

We present the results obtained for the version B of the models TRMM and MRMM in Figures 7.14 and 7.15 and in Table 7.13.

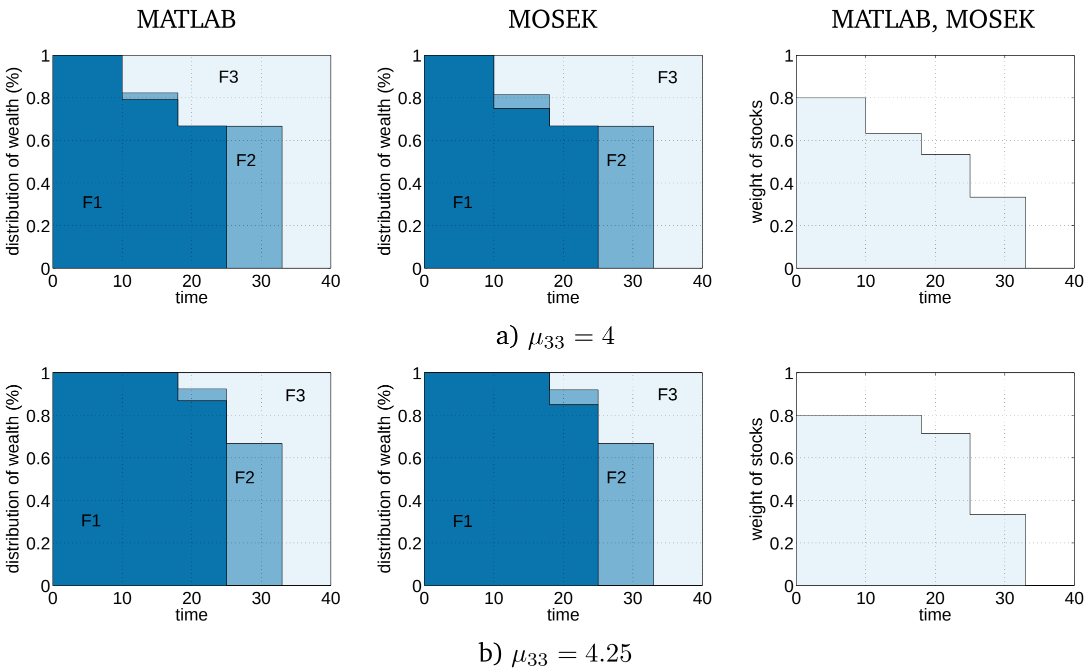
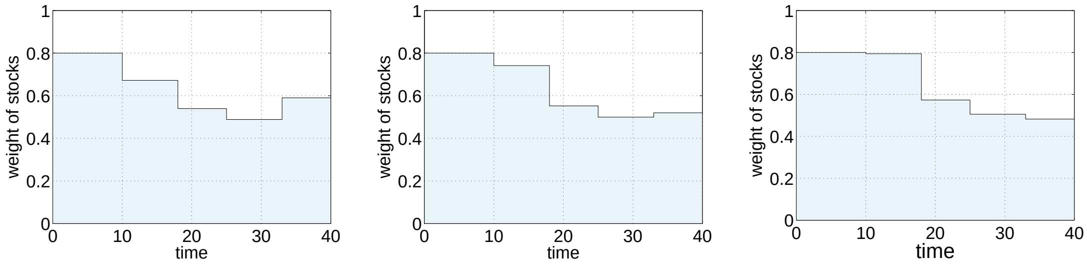


Figure 7.15: MRMM, Variant B with governmental limitations, MATLAB and MOSEK optimization results. Average optimal percentual allocation of the saved amount into the funds (left two columns) and average weights of stocks in investment (right) for target wealth after 33 years $\mu_{33} = 4$ (top) and $\mu_{33} = 4.25$ (bottom).

Model		$\mu_{33} = 4$	$\mu_{33} = 4.25$
TRMM	single-risk 33	1.7301	2.0757
	multi-risk 1–33	3.5582	4.1055
	single-risk 40	1.9975	2.3454
	mean 40	4.7586	5.0214
	number of iterates	6	5
MRMM	single-risk 33	2.1768	2.4495
	multi-risk 1–33	2.3394	3.2230
	single-risk 40	2.3586	2.6438
	mean 40	4.7586	5.0214
	number of iterates	6	10

Table 7.13: Optimization results for the TRMM and MRMM models, Variant B.

From Section 7.3.2 we recall that the period considered in the optimization is 33 years and therefore also the parameter μ entering the problem represents the average target terminal wealth after 33 years of saving. For clarity, we use subindex 33 at the μ variable, μ_{33}. The wealth achieved after 40 years of saving is calculated simply using 3 scenarios of bond returns from each node $n \in \mathcal{T}$. Table 7.13 summarizes the results from optimization



a) $\alpha = 0.01$ b) $\alpha = 0.05$ c) $\alpha = 0.1$

Figure 7.16: TRMM, Variant A without governmental limitations. Sensitivity of the results to varying α parameter. Average proportion of stocks in investment for target wealth after 40 years $\mu_{40} = 6$ and for a) $\alpha = 0.01$, b) $\alpha = 0.05$, c) $\alpha = 0.1$.

alpha	μ_{40}	$AVaRD40_\alpha$
$\alpha = 0.01$	6	3.6366
$\alpha = 0.05$	6	3.4506
$\alpha = 0.1$	6	3.3165

Table 7.14: TRMM, Variant A without governmental limitations. Sensitivity of the results to varying α parameter.

for both models TRMM and MRMM taking the governmental limitations into consideration. In the row "single-risk 33" we present the single-period $AVaRD_{0.05}$ from the wealth in the year 33 random variable (it is minimal in the TRMM model but is not minimal in MRMM). Next we present the multi-period $AVaRD_{0.05}$ (which is not minimal in the TRMM model but is minimal in MRMM), the single-period risk calculated additionally for the year 40, the mean value of the wealth at year 40, and the number of iterates needed to achieve the accuracy $\epsilon = 0.001$ in the sense of Section 7.3.1. The results confirm the natural expectation that a higher target return is accompanied by a higher risk.

Similarly as in Version A, the trend of the weight of stocks in the investment is decreasing over time. Again, different optimal solutions obtained by the two different interior point methods in MATLAB and MOSEK indicate that the solution is not unique, but the weight of stocks in the optimal strategies is the same and therefore of crucial importance.

Sensitivity with respect to the α parameter

Finally, we investigate experimentally the sensitivity of the results with respect to varying parameter α. We use the TRMM model, Variant A. We put $\mu = 6$ and assume $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.1$. We present the results in Figure 7.16 and Table 7.14. It is clear from the definition of the value-at-risk VaR_α that a higher α implies a higher VaR_α, thereby also a higher average value-at-risk $AVaR_\alpha$ and hence a lower $AVaRD_\alpha = \mathbb{E} - AVaR_\alpha$. The results in Table 7.14 confirm this feature. Figure 7.16 indicates that a higher α leads to an investment strategy with a higher proportion of stocks on average.

7.4 Notes on convergence of the iterative scheme for the risk models

It is important to verify that the proposed iterative algorithm for coping with the nonlinearity in the constraints indeed converges to its solution. We consider the MRMM model, since the TRMM model is its simplified version. For purposes of this section, we denote $\mathbf{x}^{(k)} = (\mathbf{a}_+^{(k)}, \mathbf{a}_-^{(k)}, \mathbf{z}^{(k)}, \mathbf{y}^{(k)}, \mathbf{u}^{(k)}) \geq \mathbf{0}$ the solutions to the k-th iterate, with $\mathbf{a}_+^{(k)}, \mathbf{a}_-^{(k)}$ being such that $\mathbf{a}^{(k)} = \mathbf{a}_+^{(k)} - \mathbf{a}_-^{(k)}$ and $\mathbf{u}$ the slack variables for the inequality constraints. The optimization problem (6.73)–(6.80) solved by the iterative algorithm proposed in Section 7.3.1 can be symbolically rewritten in the standard form as

$$\min_{\mathbf{x}} \mathbf{c}^\top \mathbf{x}^{(k)} \tag{7.11}$$

subject to

$$\begin{aligned} \mathbf{A}\mathbf{x}^{(k)} &= \mathbf{b}(\mathbf{x}^{(k-1)})\,, && (7.12)\\ \mathbf{x}^{(k)} &\geq 0\,, && (7.13) \end{aligned}$$

where $k = 1, 2, ...$ For the matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$ we have

$$\mathbf{A} = \begin{pmatrix} \tilde{\mathbf{A}}_{ineq} & \mathbf{I} \\ \tilde{\mathbf{A}}_{eq} & \mathbf{0} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \mathbf{b}_{ineq} \\ \mathbf{b}_{eq}(\mathbf{x}^{(k-1)}) \end{pmatrix}$$

where $\tilde{\mathbf{A}}_{ineq}$ and $\tilde{\mathbf{A}}_{eq}$ are the matrices $\mathbf{A}_{ineq}$ and $\mathbf{A}_{eq}$ having adjusted first columns in order to represent the coefficients at variables $a_n \in \mathbb{R}$ split to a_n^+, $a_n^- > 0$. The matrix $\mathbf{I}$ is an $1 + N \times 1 + N$ identity matrix of coefficients at the slack variables $\mathbf{u}$. We slightly abuse the notation and use $\mathbf{c}$ again to denote the vector of coefficients of the objective function that is adjusted in the first elements pertaining to the original $\mathbf{a}$ variables so that they represent the coefficients at $\mathbf{a}_+$ and $\mathbf{a}_-$. Moreover, it is enlarged at its end by zero elements pertaining to the slack variables $\mathbf{u}$. We notice that the number of variables of the above standard form problem is $vars = 2(1 + N - S) + N + J(1 + N) + 1 + N$.

The elements of the right-hand side vector $\mathbf{b}$ are:

$$b_i = \begin{cases} 0 & i = 1, ..., N\,, \\ -\mu & i = N+1\,, \\ \tau & i = N+2\,, \\ \tau \frac{\mathbf{y}_{n\_}^\top \mathbf{q}_n}{\mathbf{y}_{n\_}^\top \mathbf{1}} & i = N+3, ..., 2N+2-S \quad \text{where } n = i - (N+2)\,, \\ 0 & i = 2N+2-S+1, ..., 2N+2\,, \end{cases}$$

where $\mathbf{q}_n = \sum_{i=0}^{l_{\xi(n)}-1} (\mathbf{s}_n)^{i/l_{\xi(n)}}$.

First, we make clear that the varying elements of the right-hand side vector $\mathbf{b}$ attain their values from a compact set and we estimate the bounds on some parameters. We denote

$$\Theta := \max_{n,j} |s_n^j|\,, \quad \Xi := \min_{n,j} |s_n^j|\,.$$

Then it is easy to show that

$$Q_n^l \mathbf{1} := \frac{1-\Xi}{1-\Xi^{1/l_{\xi(n)}}}\mathbf{1} \leq \mathbf{q}_n \leq \frac{1-\Theta}{1-\Theta^{1/l_{\xi(n)}}}\mathbf{1} =: Q_n^u \mathbf{1} \tag{7.14}$$

where $Q_n^l = \frac{1-\Xi}{1-\Xi^{1/l_{\xi(n)}}}$ and $Q_n^u = \frac{1-\Theta}{1-\Theta^{1/l_{\xi(n)}}}$. Next,

$$0 \leq b_i = \tau \frac{\mathbf{y}_{n_-}^\top \mathbf{q}_n}{\mathbf{y}_{n_-}^\top \mathbf{1}} \leq \tau Q_n^u \leq \tau Q^u \,, \quad n = i-(N+2)\,, \tag{7.15}$$

where $Q^u := \max_n Q_n^u$. Then

$$0 \leq y_n^j \leq \mathbf{y}_n^\top \mathbf{1} \leq \tau(\Theta)^{\xi(n)} + \tau Q^u \frac{1-\Theta^{\xi(n)}}{1-\Theta} =: \tau Y_n^u \tag{7.16}$$

and

$$\mathbf{y}_{n_-}^\top \mathbf{q}_n \leq \mathbf{y}_{n_-}^\top \mathbf{1} Q_n^u \leq \tau Y_{n_-}^u Q_n^u \,. \tag{7.17}$$

Moreover,

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n_-}^\top \mathbf{s}_n + b_i(\mathbf{y}_{n_-}) \geq \mathbf{y}_{n_-}^\top \mathbf{s}_n > \tau \Xi^{\xi(n)} =: \tau Y_n^l \,, \quad \text{for all } n \geq 1. \tag{7.18}$$

Hence, we emphasise that the right-hand side vectors $\mathbf{b}$ appearing in the iterates belong to a compact set $\mathcal{K} \subset \mathbb{R}^{2N+2}$. The lower and upper bounds on its elements are determined by the estimates in (7.15). The $\mathbf{y}$ also belongs to a compact set which we denote $\mathcal{Y} \subset \mathbb{R}^{J(1+N)}$ and its bounds are given by (7.16). From now on, we will denote by M the set

$$M := \{\mathbf{x} = (\mathbf{a}_+, \mathbf{a}_-, \mathbf{z}, \mathbf{y}, \mathbf{u}); \mathbf{a}_+, \mathbf{a}_- \in \mathbb{R}^{1+N-S}, \mathbf{z} \in \mathbb{R}_+^N, \mathbf{y} \in \mathcal{Y} \subset \mathbb{R}^{J(1+N)}, \mathbf{u} \in \mathbb{R}_+^{1+N}\} \tag{7.19}$$

where $\mathbb{R}_+^N$ denotes the nonnegative orthant in $\mathbb{R}^N$.

The feasibility of the iterates follows from the same argument as in Proposition 6.2.2. We summarize that for each iterate k there exists a bound $\mu_{max}^{(k)}$ given by

$$\mu_{max}^{(k)} := \max_{\mathbf{y}} \sum_{m \in \mathcal{T}} p_m (\mathbf{y}_m^{(k)})^\top \mathbf{1} \tag{7.20}$$

where the maximum is subject to $[\widetilde{\mathbf{A}}_{eq} \quad \mathbf{0}]\, \mathbf{x}^{(k)} = \mathbf{b}_{eq}(\mathbf{x}^{(k-1)})$ and $\mathbf{x}^{(k)} \geq 0$. We may conclude that if

$$\mu \leq \min_{k \geq 1} \mu_{max}^{(k)} \,, \tag{7.21}$$

then all optimization problems in iterative algorithm (7.11)–(7.13) are feasible. It is not possible to give an explicit formula for any of $\mu_{max}^{(k)}$, because they are specific for scenarios and model parameters.

We summarize the above considerations in the following proposition.

Proposition 7.4.1. *If $\mu \leq \min_k \mu_{max}^{(k)}$, then the optimization problems in all iterates in (7.11)–(7.13) are feasible.*

Next, we discuss the optimality of the iterates. The proof of the following theorem is a straightforward extension of the proof of Theorem 6.2.1.

Theorem 7.4.1. *If the iterates in (7.11)–(7.13) are feasible then they attain their optimum.*

Proof. If a problem of linear programming is feasible, then it may attain an optimum or it may be unbounded ([49]). We show that the unboundedness is not the case in (7.11)–(7.13). We drop the numbering of iterates k for simplification of notation.

We recall that the objective function has the form

$$\min_{a,z,y} \sum_{n\in\mathcal{N}\setminus\mathcal{T}} c_{\xi(n)} p_n \left(\sum_{k\in\{n\}^+} \left(pc(k)\mathbf{y}_k^\top \mathbf{1} \right) - a_n + \frac{1}{\alpha} \sum_{k\in\{n\}^+} pc(k) z_{kn} \right),$$

or equivalently,

$$\min_{z,y} \sum_{n\in\mathcal{N}\setminus\mathcal{T}} c_{\xi(n)} p_n \left(\sum_{k\in\{n\}^+} \left(pc(k)\mathbf{y}_k^\top \mathbf{1} \right) - \max_{a_n} \{ a_n - \frac{1}{\alpha} \sum_{k\in\{n\}^+} pc(k) z_{kn} \} \right)$$

with $z_{kn} = [\mathbf{y}_k^\top \mathbf{1} - a_n]^-$ in the optimum (see the proof of Proposition 6.2.1). We showed in (7.16) that the $\mathbf{y}$ variable is bounded. Therefore, the unboundedness of the objective function may be caused only by the unboundedness of variables $\mathbf{a}$ or $\mathbf{z}$. However, $\max_{a_n}\{a_n - \frac{1}{\alpha}\sum_{k\in\{n\}^+} pc(k)z_{kn}\} = AVaR_\alpha(\mathbf{y}_{\{n\}^+}^\top \mathbf{1})$ which is finite, hence bounded, because $\mathbf{y}$ is finite and bounded.

Another argument for optimality follows from the fact that the average value-at-risk deviation is bounded from below by 0 and hence the unboundedness is not the case. $\square$

The next statement follows from the well known fact that, in linear programming, if the primal (dual) problem attains an optimum, so does the dual (primal) one ([49]).

Corollary 7.4.1. *For $\mu \leq \min_k \mu_{max}^{(k)}$, the optimization problems in all iterates in (7.11)–(7.13) attain their optimum and so do their duals.*

Finally, we investigate the convergence of the iterative algorithm (7.11)–(7.13). The solutions $\mathbf{x}^{(k+1)}(\mathbf{b}(\mathbf{x}^{(k)})) \equiv \mathbf{T}(\mathbf{x}^{(k)})$ of this iterative scheme converge to a solution $\mathbf{x}^* = \lim_{k\to\infty} \mathbf{T}(\mathbf{x}^{(k)})$ if the mapping $\mathbf{T}$ is contractive. We show that there exists a positive constant $\kappa < 1$ such that the inequality

$$||\mathbf{T}'(\mathbf{x})|| \leq ||\mathbf{x}'(\mathbf{b})||\;||\mathbf{b}'(\mathbf{x})|| \leq \kappa \tag{7.22}$$

holds for all $\mathbf{x} \in M$.

Proposition 7.4.2. *For the Mehrotra primal-dual infeasible interior point method, implemented in Matlab, there exists a constant $C > 0$ such that $||\mathbf{x}'(\mathbf{b})|| \leq C$ for any $\mathbf{b}$ from the compact set $\mathcal{K} \in \mathbb{R}^{2N+2}$ determined by (7.15).*

Proof. Primal-dual infeasible interior point methods ([60, Chapter 6]) find a solution of a standard linear program (7.11)–(7.13) with a constant right-hand side vector $\mathbf{b}$ by solving the system of linear equations

$$\begin{aligned} \mathbf{A}^\top\lambda + \mathbf{s} &= \mathbf{c} + \gamma\Delta\mathbf{c}\,, && (7.23)\\ \mathbf{A}\mathbf{x} &= \mathbf{b} + \gamma\Delta\mathbf{b}\,, && (7.24)\\ x_i s_i &= \gamma \quad \text{for all } i = 1, ..., \dim(\mathbf{x})\,, && (7.25)\\ (\mathbf{x}, \mathbf{s}) &\geq 0\,, && (7.26) \end{aligned}$$

where $\gamma > 0$ and the solution to the linear program is obtained in the limit $\gamma \to 0$. The initial point is set as arbitrary $\gamma_0, \mathbf{x}_0, \mathbf{s}_0, \lambda_0$ all strictly positive and

$$\Delta b_i = \frac{(\mathbf{A}\mathbf{x}_0)_i - b_i}{\gamma_0}\,, \quad i = 1, ..., \dim(\mathbf{b})\,,$$

and

$$\Delta c_i = \frac{(\mathbf{A}^\top\lambda_0)_i + s_i - c_i}{\gamma_0}\,, \quad i = 1, ..., \dim(\mathbf{c})\,.$$

Using this initial point, satisfying (7.23)–(7.26), and keeping γ positive ensures that the solutions $\mathbf{x}, \mathbf{s}$ in each iterate will be infeasible but positive. As $\gamma \to 0^+$, the solution of (7.23)–(7.26) converges to an optimal solution lying on the boundary of the set of feasible solutions, if both primal and dual attain their optimum. We discussed this issue above in Theorem 7.4.1.

For simplicity and clarity, let us forget the real dimensions and their notation used in previous sections. In this short section we will assume the following dimensions of matrices and vectors appearing in (7.23)–(7.26): $\mathbf{A}$ is of type $m \times n$ with $m < n$ and full rank, vectors $\mathbf{x}, \mathbf{c}, \Delta\mathbf{c}, \mathbf{s} \in \mathbb{R}^n$ and $\mathbf{b}, \Delta\mathbf{b}, \lambda \in \mathbb{R}^m$. The dimensions of the derivatives are then $n \times m$ for $\mathbf{x}'(\mathbf{b})$, $m \times m$ for $\lambda'(\mathbf{b})$, and $n \times m$ for $\mathbf{s}'(\mathbf{b})$. Differentiating (7.23)–(7.25) with respect to the vector $\mathbf{b}$ we obtain

$$\begin{aligned} \mathbf{A}^\top\lambda'(\mathbf{b}) + \mathbf{s}'(\mathbf{b}) &= \mathbf{0}\,, && (7.27)\\ \mathbf{A}\mathbf{x}'(\mathbf{b}) &= (1 - \frac{\gamma}{\gamma_0})\mathbf{I}\,, && (7.28)\\ \mathbf{X}\mathbf{s}'(\mathbf{b}) + \mathbf{S}\mathbf{x}'(\mathbf{b}) &= 0\,, && (7.29) \end{aligned}$$

where $\mathbf{X} = diag(x_1, ..., x_n)$, $\mathbf{S} = diag(s_1, ..., s_n)$. Equation (7.29) yields

$$\mathbf{x}'(\mathbf{b}) = -\mathbf{S}^{-1}\mathbf{X}\mathbf{s}'(\mathbf{b})\,. \tag{7.30}$$

By multiplying from left by $\mathbf{A}$ and using (7.27) and (7.28), we obtain

$$\lambda'(\mathbf{b}) = (1 - \frac{\gamma}{\gamma_0})(\mathbf{A}\mathbf{S}^{-1}\mathbf{X}\mathbf{A}^\top)^{-1}$$

where we notice that the matrix $\mathbf{A}\mathbf{S}^{-1}\mathbf{X}\mathbf{A}^\top$ is of type $m \times m$ and has a full rank, thus it is invertible. Realizing from (7.25) that $\mathbf{S}^{-1} = \frac{1}{\gamma}\mathbf{X}$, we get

$$\lambda'(\mathbf{b}) = \gamma(1 - \frac{\gamma}{\gamma_0})(\mathbf{A}\mathbf{X}^2\mathbf{A}^\top)^{-1}\,.$$

Then (7.27) implies

$$\mathbf{s}'(\mathbf{b}) = -\gamma(1 - \frac{\gamma}{\gamma_0})\mathbf{A}^\top(\mathbf{A}\mathbf{X}^2\mathbf{A}^\top)^{-1}$$

and subsequently, from (7.30),

$$\mathbf{x}'(\mathbf{b}) = (1 - \frac{\gamma}{\gamma_0})\mathbf{X}^2\mathbf{A}^\top(\mathbf{A}\mathbf{X}^2\mathbf{A}^\top)^{-1}\,. \tag{7.31}$$

We notice that $\mathbf{X} = \mathbf{X}(\mathbf{b})$. We recall that the infeasible interior point algorithms keep the solutions strictly positive for each $\gamma > 0$. Therefore the inversion in (7.31) is well defined. The derivative in (7.31) is well defined for arbitrary $\mathbf{b}$, and thus also for all $\mathbf{b} \in \mathcal{K}$. That is, $\mathbf{x}(\mathbf{b})$ is continuous in each $\mathbf{b} \in \mathcal{K}$ and hence continuous on $\mathcal{K}$. The image of the compact set $\mathcal{K}$ under the continuous mapping $\mathbf{x}$, i.e. $\mathbf{x}(\mathcal{K})$, is a again a compact set. In addition, since $\mathbf{b} \mapsto \mathbf{x}(\mathbf{b})$ is continuous so is the right hand side of (7.31). Hence $\mathbf{x}'$ is continuous on the compact $\mathcal{K}$. Thus $\mathbf{x}'(\mathcal{K})$ is a compact set too. Finally, a continuous function $\mathbf{x}'$ attains on a compact set its maximum and minimum, and therefore there exists a positive constant C such that for all $\mathbf{b} \in \mathcal{K}$ we have $\|\mathbf{x}'(\mathbf{b})\| \leq C$. □

Proposition 7.4.3. *For any $\epsilon > 0$ there exists a $\delta > 0$ such that for all n the following holds: if $\|\mathbf{s}_n - \mathbf{1}\| \leq \delta$ then $\|\mathbf{b}'(\mathbf{x})\| \leq \epsilon$ for all $\mathbf{x} \in M$.*

Proof. It is clear that the derivative of the constant elements of the right-hand side vector $\mathbf{b}$ is zero. Therefore, we consider only nonconstant elements of it.

We recall that the directional derivative of the function $\mathbf{b}(\mathbf{x})$ in a direction ξ is defined by

$$\mathbf{b}'(\mathbf{x})\xi = \lim_{t\to 0}\frac{\mathbf{b}(\mathbf{x}+t\xi) - \mathbf{b}(\mathbf{x})}{t} = \frac{\partial}{\partial t}\mathbf{b}(\mathbf{x}+t\xi)\bigg|_{t=0}\,.$$

We also recall that $\mathbf{x} = (\mathbf{a}^+, \mathbf{a}^-, \mathbf{z}, \mathbf{y}, \mathbf{u})$ and that $b_i(\mathbf{x}) = \tau\frac{\mathbf{y}_{n\_}^\top\mathbf{q}_n}{\mathbf{y}_{n\_}^\top\mathbf{1}}$ for $i = N{+}3, ..., 2N{+}2{-}S$ and $n = i - (N+2)$. Hence, we may write $b(\mathbf{x}) \equiv b(\mathbf{y})$ and in particular $b_i(\mathbf{x}) \equiv b_i(\mathbf{y}_{n\_})$. Let us consider the mapping

$$t \mapsto b_i(\mathbf{y}_{n\_} + t\xi) = \tau\frac{\mathbf{y}_{n\_}^\top\mathbf{q}_n + t\xi^\top\mathbf{q}_n}{\mathbf{y}_{n\_}^\top\mathbf{1} + t\xi^\top\mathbf{1}}\,.$$

For its derivative with respect to t we have

$$\frac{\partial}{\partial t}b_i(\mathbf{y}_{n\_} + t\xi) = \tau\frac{\xi^\top\mathbf{q}_n(\mathbf{y}_{n\_}^\top\mathbf{1} + t\xi^\top\mathbf{1}) - (\mathbf{y}_{n\_}^\top\mathbf{q}_n + t\xi^\top\mathbf{q}_n)\xi^\top\mathbf{1}}{(\mathbf{y}_{n\_}^\top\mathbf{1} + t\xi^\top\mathbf{1})^2}$$

and then we subsequently obtain

$$\mathbf{b}'(\mathbf{x})\xi = \tau\frac{\xi^\top\mathbf{q}_n}{\mathbf{y}_{n\_}^\top\mathbf{1}} - \frac{\mathbf{y}_{n\_}^\top\mathbf{q}_n}{(\mathbf{y}_{n\_}^\top\mathbf{1})^2}\xi^\top\mathbf{1}\,.$$

From the assumption $\|\mathbf{s}_n - \mathbf{1}\| \leq \delta$ it follows that

$$\mathbf{q}_n = l_{\xi(n)}\mathbf{1} + \eta_n$$

such that $\eta_n \to 0$ as $\mathbf{s}_n \to \mathbf{1}$. Therefore,

$$\mathbf{b}'(\mathbf{x})\xi = \tau\left(\frac{\xi^\top \mathbf{1}}{\mathbf{y}_{n_-}^\top \mathbf{1}} l_{\xi(n)} + \frac{\xi^\top \eta_n}{\mathbf{y}_{n_-}^\top \mathbf{1}} - \frac{\xi^\top \mathbf{1}}{\mathbf{y}_{n_-}^\top \mathbf{1}} l_{\xi(n)} - \frac{\mathbf{y}_{n_-}^\top \eta_n}{(\mathbf{y}_{n_-}^\top \mathbf{1})^2}\xi^\top \mathbf{1}\right) \tag{7.32}$$

$$= \tau\left(\frac{\xi^\top \eta_n}{\mathbf{y}_{n_-}^\top \mathbf{1}} - \frac{\mathbf{y}_{n_-}^\top \eta_n}{(\mathbf{y}_{n_-}^\top \mathbf{1})^2}\xi^\top \mathbf{1}\right). \tag{7.33}$$

Since $\mathbf{y}_{n_-}^\top \mathbf{1} \geq \tau Y_{n_-}^l > 0$ and $y_{n_-}^j \leq \tau Y_{n_-}^u$ we infer the existence of a constant $\widetilde{C} > 0$ such that

$$\|\mathbf{b}'(\mathbf{x})\xi\| \leq \widetilde{C}\|\eta_n\|\|\xi\|.$$

Hence $\|\mathbf{b}'(\mathbf{x})\| \leq \widetilde{C}\|\eta_n\|$. If $\delta > 0$ is sufficiently small and $\|\mathbf{s}_n - \mathbf{1}\| \leq \delta$ then $\widetilde{C}\|\eta_n\| \leq \epsilon$. Therefore $\|\mathbf{b}'(\mathbf{x})\| \leq \epsilon$. □

Theorem 7.4.2. *The iterative scheme (7.11)–(7.13) converges on M to a solution if $\|\mathbf{s}_n - \mathbf{1}\|$ is small enough.*

Proof. Based on previous propositions, for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all n we have

$$\|\mathbf{T}'(\mathbf{x})\| \leq \|\mathbf{x}'(\mathbf{b})\|\,\|\mathbf{b}'(\mathbf{x})\| \leq C\epsilon. \tag{7.34}$$

Putting $\epsilon < \frac{1}{C}$ and $\kappa = C\epsilon$, there exists a $\delta > 0$ such that if $\|\mathbf{s}_n - \mathbf{1}\| \leq \delta$, we have $\|\mathbf{T}'(\mathbf{x})\| \leq \kappa < 1$. The rest of the proof is a consequence of the Banach fixed point argument. □

Chapter 8

Conclusions

In this thesis, we proposed two types of models for the problem of optimal fund selection in funded schemes of pension planning:

Expected utility maximization models:

Ia: the Dynamic Accumulation Model (DAM);

Ib: the Proportional Investment Allocation Model (PIAM).

Risk minimizing models:

IIa: the Terminal Risk Minimizing Model (TRMM), in which the terminal risk is measured by the single-period average value-at-risk deviation;

IIb: the Multi-period Risk Minimizing Model (MRMM), in which the multi-period risk is measured by the multi-period average value-at-risk deviation.

We investigated their solvability and qualitative and quantitative properties. We showed that a solution to the DAM model can be found as a solution to a Bellman equation in Theorem 6.1.2. We proposed a generalization of the DAM model in the so-called Proportional Investment Allocation Model (PIAM) assuming that the funds proportionally invest to stocks and bonds with weights θ_t and $1 - \theta_t$, respectively. The PIAM model finds an optimal weight of stocks in the investment strategy over time. We also derived a nonlinear partial differential equation for the value function V. Next, we expressed explicitly the optimal solution θ_t under the assumption that the value function V is concave.

The Terminal Risk Minimizing Model TRMM is formulated on a scenario tree with adjusted fund returns as the underlying process. In Proposition 6.2.1 we showed that the model can be rewritten to a linear program. Theorem 6.2.1 states that it attains an optimum for feasible values of the μ parameter. The TRMM model is generalized to a case where the savers do not make decisions about the fund selection every year but only once during a period of several years. The resulting model becomes nonlinear, making thus its analysis and numerical approximation more complex.

The Multi-period Risk Minimizing Model (MRMM) uses the multi-period average value-at-risk as the objective function and it is also defined on a scenario tree. Similarly to the TRMM model, the MRMM model is generalized to a case where the savers do not make decisions about the fund selection every year but only once in several years. Again, the model becomes nonlinear.

The numerical schemes for solving all models are proposed in Chapter 7. An iterative algorithm is proposed for coping with the nonlinearity in models TRMM and MRMM. We investigated its convergence in Section 7.4.

All models are implemented in Chapter 7 for the case of Slovak Republic. We implemented them in two basic variants:

A: without considering any governmental limitations on the fund selection;

B: taking the governmental restrictions imposed on the fund selection into account.

In the DAM model we also included a case study considering the real portfolio composition of the pension funds that reflects the real situation in the current pension market in Slovak Republic.

The results obtained from all models exhibit the following common feature:

The proportion of risky assets in the optimal strategy decreases over time.

We conclude that this property is intuitively understandable, as a higher amount of saved money is more sensitive to changes in the fund returns. Hence, secure funds are preferred to funds with high volatility of returns in later times. We investigated the sensitivity of the results with respect to varying parameters. Based on our extensive experiments, done under the assumption that the average stock return is higher than the average bond return, and that the volatility of the stock return is higher than the volatility of the bond return, we conclude that

- the optimal weight of stocks is *decreasing* in time t, saved amount d_t, bond return $r^{(b)}$ and it is *increasing* in stock return $r^{(s)}$ and the wage growth rate β_t;
- the average amount of savings is *decreasing* in the risk aversion parameter a and the wage growth rate β_t and it is *increasing* in the time t, stock return $r^{(s)}$ and bond return $r^{(b)}$;
- in the TRMM and MRMM models, the weight of stocks in the optimal strategy is *increasing* in the α parameter specifying the confidence level of the average value-at-risk measure.

Appendix A

Proof of Theorem 5.1.1

as in [47].

Proof. We introduce the following quantities, which may be different, if the distribution F of Y has jumps at $F^{-1}(\alpha)$:

$$\begin{aligned} \alpha_F^+ &= \inf\{F(u) : F(u) \geq \alpha\} = F(F^{-1}(\alpha)), \\ \alpha_F^- &= \sup\{F(u) : F(u) < \alpha\} = F(F^{-1}(\alpha)-) = \lim_{h\downarrow 0} F(F^{-1}(\alpha) - h). \end{aligned}$$

It holds that $\alpha_F^- \leq \alpha \leq \alpha_F^+$, see Figure 8.1. Notice that

$$P\{Y \leq F^{-1}(\alpha)\} = \alpha_F^+, \quad P\{Y < F^{-1}(\alpha)\} = \alpha_F^-. \tag{8.1}$$

Next, let us denote $F(x-)$ the left-sided limit $F(x-) = \lim_{u\uparrow x}$ and $\mathbf{1}_A$ the characteristic function of a set A, that is,

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

We first prove the following fact:

$$F^{-1}(\alpha) \in \text{ argmax } \{x - \frac{1}{\alpha}\mathbb{E}([Y-x]^-) : x \in \mathbb{R}\}. \tag{8.2}$$

To this end we notice that for all x

$$\begin{aligned} \alpha x - \mathbb{E}([Y-x]^-) &= \alpha x - \mathbb{E}((x-Y)\mathbf{1}_{\{Y<x\}}) \\ &= \alpha x - xF(x-) + \mathbb{E}(Y\mathbf{1}_{\{Y<x\}}) \end{aligned} \tag{8.3}$$

and also

$$\begin{aligned} \alpha x - \mathbb{E}([Y-x]^-) &= \alpha x - \mathbb{E}((x-Y)\mathbf{1}_{\{Y\leq x\}}) \\ &= \alpha x - xF(x-) + \mathbb{E}(Y\mathbf{1}_{\{Y\leq x\}}). \end{aligned} \tag{8.4}$$

Let $b = F^{-1}(\alpha)$. Then

$$F(b-) \leq \alpha \leq F(b). \tag{8.5}$$

Suppose that $x \leq b$. Then using (8.5)

$$\begin{aligned} \mathbb{E}(Y\mathbf{1}_{\{Y<b\}}) - \mathbb{E}(Y\mathbf{1}_{\{Y<x\}}) &= \mathbb{E}(Y\mathbf{1}_{\{x\leq Y<b\}}) \\ &\geq x[F(b-) - F(x-)] \\ &\geq x[F(b-) - F(x-)] - (b-x)[\alpha - F(b-)] \\ &= b[F(b-) - \alpha] - x[F(x-) - \alpha] \end{aligned}$$

or equivalently

$$b[\alpha - F(b-)] + \mathbb{E}(Y\mathbf{1}_{\{Y<b\}}) \geq x[\alpha - F(x-)] + \mathbb{E}(Y\mathbf{1}_{\{Y<x\}}),$$

which using (8.4) can be rewritten as

$$\alpha b - \mathbb{E}([Y-b]^-) \geq \alpha x - \mathbb{E}([Y-x]^-)\,. \tag{8.6}$$

Similarly, for $x \geq b$, we have

$$\begin{aligned}
\mathbb{E}(Y\mathbf{1}_{\{Y\leq x\}}) - \mathbb{E}(Y\mathbf{1}_{\{Y\leq b\}}) &= \mathbb{E}(Y\mathbf{1}_{\{b<Y\leq x\}}) \\
&\geq x[F(x)-F(b)] \\
&\geq x[F(x)-F(b)] - (x-b)[F(b)-\alpha] \\
&= x[F(x)-\alpha] - b[F(b)-\alpha]
\end{aligned}$$

or equivalently

$$b[\alpha - F(b)] + \mathbb{E}(Y\mathbf{1}_{\{Y\leq b\}}) \geq x[\alpha - F(x)] + \mathbb{E}(Y\mathbf{1}_{\{Y\leq x\}})\,,$$

which thanks to (8.5) leads also to (8.6). After dividing (8.6) by α one gets the assertion (8.3), i.e.:

$$\max\{x - \frac{1}{\alpha}\mathbb{E}([Y-x]^-) : x \in \mathbb{R}\} = F^{-1}(\alpha) - \frac{1}{\alpha}\mathbb{E}([Y-F^{-1}(\alpha)]^-)\,.$$

By partial Stieltjes integral

$$\begin{aligned}
& F^{-1}(\alpha) - \frac{1}{\alpha}\mathbb{E}([Y-F^{-1}(\alpha)]^-) \\
= \;& F^{-1}(\alpha) - \frac{1}{\alpha}\int_{-\infty}^{F^{-1}(\alpha)} (F^{-1}(\alpha)-u)F(u) \\
= \;& F^{-1}(\alpha) - \frac{1}{\alpha}\int_{(-\infty,F^{-1}(\alpha))} F^{-1}(\alpha)dF(u) + \frac{1}{\alpha}\int_{(-\infty,F^{-1}(\alpha))} udF(u) \\
= \;& F^{-1}(\alpha) - \frac{\alpha_F^-}{\alpha}F^{-1}(\alpha) + \frac{1}{\alpha}\int_0^{\alpha_F^-} F^{-1}(u)du \\
= \;& \frac{\alpha-\alpha_F^-}{\alpha}F^{-1}(\alpha) + \frac{1}{\alpha}\int_0^{\alpha} F^{-1}(u)du - \frac{1}{\alpha}\int_{\alpha_F^-}^{\alpha_F} F^{-1}(u)du \\
= \;& \frac{\alpha-\alpha_F^-}{\alpha}F^{-1}(\alpha) + \frac{1}{\alpha}\int_0^{\alpha} F^{-1}(u)du - \frac{\alpha-\alpha_F^-}{\alpha}F^{-1}(\alpha) \\
= \;& \frac{1}{\alpha}\int_0^{\alpha} F^{-1}(u)du = AVaR_\alpha(Y)\,.
\end{aligned}$$

□

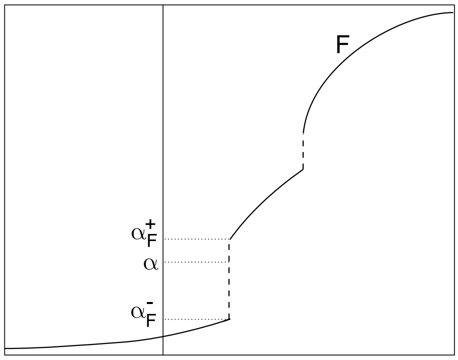


Figure 8.1: $\alpha_F^- \leq \alpha \leq \alpha_F^+$

Appendix B

Definition of multi-period acceptability and deviation risk functionals

Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T)$ be a filtration where $\mathcal{F}_0 = \{\Omega, \emptyset\}$ is the trivial σ-algebra with no information. Let $\mathcal{Y}$ be a linear space of income process $Y = (Y_1, ..., Y_T)$, which is adapted to the filtration $\mathcal{F}$. An acceptability functional assigns real value to the combination of a process and a filtration

$$\mathcal{A} = \mathcal{A}(Y; \mathcal{F}) = \mathcal{A}(Y_1, ..., Y_T; \mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_{T-1}) .$$

The deviation functional assigns the process and filtration a value of risk,

$$\mathcal{D} = \mathcal{D}(Y; \mathcal{F}) = \mathcal{D}(Y_1, ..., Y_T; \mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_{T-1}) .$$

Let us now define acceptability functionals as functionals satisfying certain properties.

Definition B.1. *[47, Chapters 2, 3] A multi-period functional $\mathcal{A}(Y; \mathcal{F})$ is called multi-period acceptability functional, if it is proper (i.e. $\mathcal{A}(Y) < +\infty$ for all Y and $\mathcal{A}(Y) > -\infty$ for some Y) and satisfies the following properties:*

(MA0) Information monotonicity. If $\mathcal{F}_t \subseteq \mathcal{F}'_t$ for all t, then

$$\mathcal{A}(Y_1, ..., Y_T; \mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_{T-1}) \leq \mathcal{A}(Y_1, ..., Y_T; \mathcal{F}'_0, \mathcal{F}'_1, ..., \mathcal{F}'_{T-1}) .$$

(MA1) Predictable translation equivariance.

$$\mathcal{A}(Y_1, ..., Y_t + C_t, ..., Y_T; \mathcal{F}) = \mathbb{E}(C_t) + \mathcal{A}(Y_1, ..., Y_T; \mathcal{F})$$

for all $\mathcal{F}_{t-1}$ measurable functions C_t.

(MA2) Concavity.

$$(Y_1, ..., Y_T) \mapsto \mathcal{A}(Y_1, ..., Y_T; \mathcal{F})$$

is concave.

(MA3) Monotonicity.

$$Y_t \leq \tilde{Y}_t \text{ a. s. for all } t \text{ implies } \mathcal{A}(Y_1, ..., Y_T; \mathcal{F}) \leq \mathcal{A}(\tilde{Y}_1, ..., \tilde{Y}_T; \mathcal{F})\,.$$

The equivariance condition (MA1) is relatively strong. Some multi-period functionals do not fulfill this condition, but a weaker one, which is called the weak translation equivariance.

(MA1') Weak translation equivariance.

$$\mathcal{A}(Y_1, ..., Y_t + c_t, ..., Y_T; \mathcal{F}) = c_t + \mathcal{A}(Y_1, ..., Y_T; \mathcal{F})$$

for all constants c_t.

A multi-period acceptability functional is positively homogeneous, if it satisfies

(MA4) Positive homogeneity.

$$\mathcal{A}(\lambda Y_1, ..., \lambda Y_T; \mathcal{F}) = \lambda \mathcal{A}(Y_1, ..., Y_T; \mathcal{F})$$

for $\lambda > 0$.

A multi-period acceptability functional is strict, if it satisfies

(MA5) Strictness.

$$\mathcal{A}(Y_1, ..., Y_T; \mathcal{F}) \leq \sum_{t=1}^{T} \mathbb{E}(Y_t)\,.$$

For any multi-period acceptability functional $\mathcal{A}$, the functional $\varrho = -\mathcal{A}$ is called a *multi-period risk (capital) functional*.

Definition B.2. *[47, Chapters 2, 3] A multi-period functional $\mathcal{D}(Y; \mathcal{F})$ is called multi-period deviation risk functional, if it is proper (i.e. $\mathcal{D}(Y) > -\infty$ for all Y and $\mathcal{D}(Y) < +\infty$ for some Y) and satisfies the following properties:*

(MD0) Information monotonicity. If $\mathcal{F}_t \subseteq \mathcal{F}'_t$ for all t, then

$$\mathcal{D}(Y_1, ..., Y_T; \mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_{T-1}) \geq \mathcal{D}(Y_1, ..., Y_T; \mathcal{F}'_0, \mathcal{F}'_1, ..., \mathcal{F}'_{T-1})\,.$$

(MD1) Predictable translation invariance.

$$\mathcal{D}(Y_1, ..., Y_t + C_t, ..., Y_T; \mathcal{F}) = \mathcal{D}(Y_1, ..., Y_T; \mathcal{F})$$

for all $\mathcal{F}_{t-1}$ measurable functions C_t.

(MD2) Convexity.

$$(Y_1, ..., Y_T) \mapsto \mathcal{D}(Y_1, ..., Y_T; \mathcal{F})$$

is convex.

The condition (MD1) has a weaker version, called weak translation invariance.

(MD1') Weak translation invariance.

$$\mathcal{D}(Y_1, ..., Y_t + c_t, ..., Y_T; \mathcal{F}) = \mathcal{D}(Y_1, ..., Y_T; \mathcal{F})$$

for all constants c_t.

A multi-period risk functional is positively homogeneous, if it satisfies

(MD4) Positive homogeneity.

$$\mathcal{D}(\lambda Y_1, ..., \lambda Y_T; \mathcal{F}) = \lambda \mathcal{D}(Y_1, ..., Y_T; \mathcal{F})$$

for $\lambda > 0$.

A multi-period risk functional is strict, if it satisfies

(MD5) Strictness.

$$\mathcal{D}(Y_1, ..., Y_T; \mathcal{F}) \geq 0\,.$$

Appendix C

Introduction to stochastic dynamic programming

We review the stochastic dynamic programming framework, as it is done in [12] and [31]. We often face problems in which some random effects influence the state of the system. Let us assume that we are given an object, the current state of which is described by a point $x \in X$ and its immediate input by $u \in U$. We observe the behavior of the object in time $i \in \{0, ..., k-1\}$. That is, the values of x, u, as well as those of X, U, become dependent on time. The behavior of the system is described by equations

$$x_{i+1} = F_i(x_i, u_i, z_i), \quad i = 0, ..., k-1, \quad x_0 = \hat{x}_0, x_i \in X_i, u_i \in U_i \tag{8.7}$$

where x_i is called a *state variable*, u_i a *control* and z_i are independent random variables. A *feasible* control u_i satisfies constraints

$$u_i \in U_i, \quad i = 0, ..., k-1\,. \tag{8.8}$$

State variable constraints are given by

$$x_i \in X_i, \quad i = 0, ..., k-1\,. \tag{8.9}$$

We can exclude the initial and terminal constraints from the overall state variable constraints, and denote $P \subset X_0$ the set of possible states at the beginning of the process in time $t = 0$, and $C \subset X_k$ the set of possible terminal states at the end of the process at time k. That is,

$$x_0 \in P, x_k \in C. \tag{8.10}$$

The quality of the process can be measured by an objective function

$$J(\mathcal{U},\mathcal{Z}) = \sum_{i=1}^{k-1} f_i^0(x_i,u_i,z_i) \tag{8.11}$$

where f_i^0 are given real functions and

$$\mathcal{U} = \{u_0,...,u_{k-1}\}, \quad \mathcal{Z} = \{z_0,...,z_{k-1}\}\,.$$

It is reasonable to define an *optimal control* as minimizing the mean value of the objective function subject to the multi-dimensional random variable $\mathcal{Z} = \{z_0,...z_{k-1}\}$. That is, we look for such $\hat{\mathcal{U}}$ for which

$$\mathbb{E}(J(\hat{\mathcal{U}},\mathcal{Z})) \le \mathbb{E}(J(\mathcal{U},\mathcal{Z}))$$

holds for every feasible control $\mathcal{U}$. For arbitrary $\mathcal{U}$, the statistical mean $\mathbb{E}(J)$ is a number, and thus the definition is correct.

Let us define a control in the form of a *strategy* $\mathcal{V} = \{v_o,...,v_{k-1}\}$ as a sequence of functions $v_i : X_i \to U_i$, with $X_0 = \hat{x}_0$. It is clear that, for a given strategy and given realization of the random process $\mathcal{Z} = \{z_0,...,z_{k-1}\}$, the feedback $\mathcal{X} = \{x_0,...,x_{k-1}\}$ is uniquely determined by a recurrent formula

$$x = \hat{x}_0, \quad x_{i+1} = F_i(x_i, v_i(x_i), z_i)$$

and thereby also the value of the objective function

$$J(\mathcal{V},\mathcal{Z}) = \sum_{i=0}^{k-1} f_i^0(x_i, v_i(x_i), z_i)\,.$$

We denote this problem, depending on the initial state x_0, by $D_0(x_0)$, and call it the *problem of optimal transition from point x_0 to the set C in the interval* $[0,k]$. We recall that, if the feedback $\mathcal{X} = \{x_0,...,x_k\}$ to a given control $\mathcal{U} = \{u_0,...,u_{k-1}\}$ satisfies all constraints, i.e. $x_i \in X_i$ for all $i = 0,...,k-1$ and $x_k \in C$, then the control $\mathcal{U}$ is called *feasible*. We define a system of problems $\mathcal{D} = \{D_j(x); j \in \{0,k-1\}, x \in X_j\}$, where $D_j(x)$ is a problem of optimal transition from the point x to the set C in the interval $[j,k]$.

It is meaningfull to define an optimal strategy as the one minimizing $\mathbb{E}(J(\mathcal{V},\mathcal{Z}))$ under given constraints. Let us denote, for each $j \in \{0,...,k-1\}$, the sets $\mathcal{Z}_j = \{z_j,...,z_{k-1}\}$, $\mathcal{V}_j = \{v_j,...,v_{k-1}\}$ and the function $V_j(x) = \min_{\mathcal{V}_j} \mathbb{E}(J_j(x,\mathcal{V}_j,\mathcal{Z}_j))$ where $J_j(\mathcal{V},\mathcal{Z}) = \sum_{i=j}^{k-1} f_i^0(x_i,v_i(x_i),z_i)$. We call the function V_j the *value function*. Next, for all $j \in [0,k-1]$ and $x \in X_j$, we define $\Gamma_j(x)$ as a set of such $u \in U_j$ for which there exists an optimal control $\hat{\mathcal{U}} = \{\hat{u}_j,...\}$ for a problem $D_j(x)$ such that $\hat{u}_j = u$. Notice that, for some j,x, the set $\Gamma_j(x)$ can be empty.

The following theorem compares an optimal strategy and an optimal control. It shows that it is possible to calculate an optimal strategy recurrently by means of dynamic programming.

Theorem C.1.*[12], [31]*

1. *It holds:* $\min_{\mathcal{V}} \mathbb{E}(J(\mathcal{V}, \mathcal{Z})) \leq \min_{\mathcal{U}} \mathbb{E}(J(\mathcal{U}, \mathcal{Z}))$.

2. *Let* $\hat{\mathcal{V}} = \{\hat{v}_0, ..., \hat{v}_{k-1}\}$ *be an optimal strategy and* V_j *the value function. Then the functions* $V_j, \hat{v}_j$ *satisfy the dynamic programming equation:*

$$\begin{aligned} V_j(x) &= \mathbb{E}(f_j^0(x, \hat{v}_j(x), z_j) + V_{j+1}(F_j(x, \hat{v}_j(x), z_j))) \\ &= \min_{u \in \Gamma(x)} \mathbb{E}(f_j^0(x, u, z_j) + V_{j+1}(F_j(x, u, z_j))) \end{aligned} \tag{8.12}$$

for $j = 0, ..., k-1$*, where*

$$V_k(x) = 0 \text{ for all } x \in C\,. \tag{8.13}$$

Conversely, if $V_j, \hat{v}_j$ *satisfy (8.12) and (8.13), then* V_j *is a value function and* $\hat{v}_j$ *an optimal strategy.*

Finally, we notice that the Bellman equation holds for minimization as well as for maximization.

Appendix D

The form of matrices $\mathbf{A}_{ineq}$ and $\mathbf{A}_{eq}$ from the TRMM is as follows:

0						$-p_1$	$-p_1$	$-p_1$	$-p_2$	$-p_2$	$-p_2$	$\cdots$	$-p_S$	$-p_S$	$-p_S$
1	-1					-1	-1	-1							
1		-1							-1	-1	-1				
$\vdots$			$\ddots$									$\ddots$			
1				-1									-1	-1	-1

Table 8.1: Inequality constraints matrix A_{ineq}

		1	1	1									
		$-s_1^1$	$-s_1^2$	$-s_1^3$	1	1	1						
								$-s_{n-}^1$	$-s_{n-}^2$	$-s_{n-}^3$	1	1	1

Table 8.2: Equality constraints matrix A_{eq}

Appendix E

Brownian motion

Suppose we are given a set Ω. We denote by $(\Omega, \mathcal{F}, P)$ a probability space, where $\mathcal{F}$ is a σ-algebra of measurable sets and P is a probability measure on Ω.

Definition E.1. *[42] A stochastic process is a set of random variables $X = \{X_t; 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$. For all t,*

$$\omega \to X_t(\omega); \omega \in \Omega$$

is a random variable. For fixed $\omega \in \Omega$, the function

$$t \to X_t(\omega); 0 \leq t < \infty$$

is called a trajectory of X corresponding to ω.

Sometimes it is useful to consider the variable t as time and each ω as a particle or an experiment. Then, $X_t(\omega)$ can be considered as a position of the particle (result of the experiment) ω at the time t. Sometimes it is more suitable to write $X(t, \omega)$ instead of $X_t(\omega)$. Then, a stochastic process can be viewed as a function of two variables:

$$(t, \omega) \to X(t, \omega)$$

from $[0, \infty) \times \Omega$ to $\mathbb{R}^d$.

Since each ω corresponds to a function $t \to X_t(\omega)$ from $[0, \infty)$ to $\mathbb{R}^d$, the set Ω can be considered as a subset of the space $\tilde{\Omega} = (\mathbb{R}^d)^{[0,\infty)}$ of all functions from $[0, \infty)$ to $\mathbb{R}^d$. In this sense, $(\Omega, \mathcal{F}, P)$ is a probabilistic description of the set of trajectories. The σ-algebra $\mathcal{F}$ represents the set of possible random events.

Definition E.2. *[56] Brownian motion $\{X_t; t \geq 0\}$ is a t-parametric system of random variables for which*

(i) all increments $X(t + \Delta) - X(t)$ have normal distribution with mean $\mu\Delta$ and variance $\sigma^2\Delta$,

(ii) the increments $W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_k} - W_{t_{k-1}}$ are independent random variables for all $0 \leq t_1 < ... < t_k$, with parameters according to (i),

(iii) $W_0 = 0$ and the trajectories $W_t(\omega)$ are continuous with probability 1.

Definition E.3. *[56] Brownian motion with $\mu = 0$ and $\sigma^2 = 1$ is called the Wiener process.*

Remark. If $\{w_t; t \geq 0\}$ is a Wiener process, then $w_t \sim N(0, t)$.

It is possible to analyze the Brownian motion also by the means of its increments $dX(t) = X(t+dt) - X(t)$. According to (i) from Definition E.2., for their mean value and variance, $\mathbb{E}(dX(t)) = \mu dt$ and $var(dX(t)) = \sigma dt = \sigma var(dw(t))$ must hold, correspondingly. That is, the Brownian motion may be characterized by its deterministic and stochastic part and the increments $dX(t)$ may be written in the form of the total differential

$$dX(t) = \mu dt + \sigma dw(t) \tag{8.14}$$

where $\{w(t); t \geq 0\}$ is a Wiener process. Equation (8.14) is called a *stochastic differential equation*.

Definition E.4. *[56] If $\{X(t); t \geq 0\}$ is a Brownian motion with parameters μ, σ and if $y_0 \in \mathbb{R}$ then the system of random variables $\{Y(t); t \geq 0\}$ defined by*

$$Y(t) = y_0 e^{X(t)}\,, \quad t \geq 0$$

is called a geometrical Brownian motion.

We will denote a Wiener process by $\{w(t); t \geq 0\}$ and its increments in a short time span dt by dw, that is, $dw(t) = w(t+dt) - w(t)$. Based on the definition of Wiener process, the increments $dw(t)$ are not correlated in time t. Their mean value is zero, i.e. $\mathbb{E}(dw(t)) = 0$, and for variance we have $var(dw(t)) = dt$. Thus, we may write

$$dw = \Phi\sqrt{dt}$$

where $\Phi \sim N(0,1)$ is a random variable with standard normal distribution.

The Itô lemma

The Itô lemma gives us an answer to the question, what is the stochastic differential equation for arbitrary smooth function $f(x,t)$ if the variable x self is a solution to a given stochastic differential equation.

Lemma E.1. *[39], [56] Let $f(x,t)$ be a smooth function of two variables, with x being the solution to stochastic differential equation $dx = \mu(x,t)dt + \sigma(x,t)dw$ where w is a Wiener process. Then the first differential of the function f is given by*

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt\,,$$

that is, the function f satisfies the stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma(x,t)\frac{\partial f}{\partial x}dw\,. \tag{8.15}$$

The intuitive proof of this lemma is based on Taylor expansion of the order 2 of the function $f = f(x,t)$.

Bibliography

[1] Acerbi, C. (2002): Spectral Measures of Risk: a Coherent Representation of Subjective Risk Aversion. Journal of Banking and Finance, Vol. 26, 1505–1518.

[2] Acuña, R. R. and Iglesias, A. P. (2001): Chile's Pension Reform After 20 Years. Social Protection Discussion Paper Series, No. 0129, The World Bank.

[3] Andersen, E. D. and Andersen, K. D. (2000): The MOSEK interior point optimizer for linear programming: an implementation of the homogeneous algorithm. In: H. Frenk, K. Roos, T. Terlaky, and S. Zhang, Eds., High Performance Optimization, pp. 197–232. Kluwer Academic Publishers, 2000.

[4] Andersen, C. and Skjodt, P. (2007): Pension Institutions and Annuities in Denmark. World Bank Policy Research Working Paper No. 4437, December 1, 2007.

[5] Anusic, Z., O'Keefe, P. and Madzarevic-Sujster, S. (2003): Pension Reform in Croatia. Social Protection Discussion Paper Series, No. 0304, The World Bank.

[6] Artzner, P., Delbaen, F., Eber, J. M. and Heath, D. (1999): Coherent measures of risk. Mathematical Finance, Vol. 9, No. 3, pp. 203–228.

[7] Artzner, P., Delbaen, F., Eber, J. M., Heath, D. and Ku, H. (2007): Coherent multiperiod risk adjusted values and Bellman's principle. Annals of Operations Research, Vol. 152, No. 1, pp. 5–22.

[8] Baxter, M. and Rennie, A.: Financial Calculus: An Introduction to Derivative Pricing. Cambridge University Press, 1996.

[9] Bergman, U. M. (2005): Lecture Notes: Dynamic optimization and utility functions. Institute of Economics, University of Copenhagen, Denmark.

[10] Bodie, Z. (2001): Retirement Investing: A New Approach. Pension Research Council Working Paper 2001-8.

[11] Bodie, Z., Merton, R. and Samuelson, W. F. (1992): Labor Supply Flexibility and Portfolio Choice in a Life- Cycle Model. Journal of Economic Dynamics & Control, Vol. 16, pp. 427–449.

[12] Brunovský, P.: Matematická teória optimálneho riadenia. Alfa, Bratislava, 1980.

[13] Carone, G. and Costello, D. (2006): Can Europe Afford to Grow Old? Finance and Development, A Quarterly Magazine of the IMF, Vol. 43, No 3, September 2006.

[14] Chan-Lau, J. A. (2004): Pension Funds and Emerging Markets. International Monetary Fund Working Paper WP/04/181, September 2004.

[15] Chlon, A. (2000): Pension Reform and Public Information in Poland. Social Protection Discussion Paper Series, No. 0019, The World Bank.

[16] Cvitanic, J. and Karatzas, I. (1999): On Dynamic Measures of Risk. Finance and Stochastics, Vol. 3, Iss. 4, August 1999, pp. 451–482.

[17] Davis, J., Hands, W. and Maki, U. (Eds): Handbook of Economic Methodology. Edward Elgar, 1998.

[18] Delbaen, F. (2000): Coherent risk measures. Lecture Notes, Scuola Normale Superiore di Pisa.

[19] Devesa-Carpio, J. E. and Vidal-Meliá, C. (2002): The Reformed Pension Systems in Latin America. Social Protection Discussion Paper Series, No. 0209, The World Bank.

[20] Duffie, D., and Pan, J. (1997): An Overview of Value at Risk. The Journal of Derivatives. Vol. 4, Spring 1997, pp. 7–49.

[21] Fishburn, P. C.: Utility Theory for Decision Making. John Wiley & Sons, Inc., 1970.

[22] Fox, L., and Palmer, E. (1999): New approaches to multi-pillar pension systems: What in the world is going on?. Social Protection – Labor Markets, Pensions, Social Assistance. The World Bank.

[23] Föllmer, H. and Schied, A. (2002a): Convex measures of risk and trading constraints. Finance and Stochastics, Vol. 6, No. 4, pp. 429–448.

[24] Föllmer, H. and Schied, A. (2002b): Robust preferences and convex measures of risk. Advances in Finance and Stochastics, 39–56, Springer-Verlag, 2002.

[25] Föllmer, H. and Schied, A. (2002c): Stochastic Finance – An Introduction in Discrete Time. Walter de Gruyter, Berlin, 2002.

[26] Fritelli, M. (2000): Representing sublinear risk measures and Pricing rules. Working paper no. 10, Universita di Milano Bicocca, Italy.

[27] Fritelli, M. and Rosazza, G. E. (2004): Dynamic Convex Risk Measures. In: Risk Measures for the 21st Century, Chapter 12, Wiley Finance, 2004.

[28] Goliaš, P. (2003): Pension Calculations for the PAYG and the Funded Pension System in Slovakia. Academia Istropolitana Nova, Professional Programme in Applied Economics and Finance, August 2003.

[29] Gourieroux, C., Laurent, J. P. and Scaillet, O. (2000): Sensitivity analysis of values at risk. Journal of Empirical Finance, Vol. 7, pp. 225–245.

[30] Halada, L., Lucká, M. and Melicherčík, I. (2005): Optimal Multistage Portfolio Management Using a Parallel Interior Point Method. Algoritmy 2005: Proceedings of contributed papers and posters, Bratislava, Slovak University of Technology, pp. 359–368.

[31] Halická, M.: Lecture notes on Dynamic Programming.

[32] Holzmann, R., Orenstein, M. and Rutkowski, M. (Eds.) (2003): Pension Reform in Europe: Process and Progress. The World Bank, Washington, DC, 2003.

[33] Holzmann, R. and Pallarès-Miralles, M. (2005): The Role, Limits of, and Alternatives to Financial Education in Support of Retirement Saving in the OECD, Eastern Europe and beyond. The World Bank, Draft/Work in Progress as of October 12, 2005.

[34] Kilianová, S. (2006): The Second Pillar of the Slovak Pension System – Interest Rate Targeting. Journal of Electrical Engineering, Vol. 57, No. 12/s.

[35] Kilianová, S., Melicherčík, I. and Ševčovič, D. (2006): Dynamic Accumulation Model for the Second Pillar of the Slovak Pension System. Finance a úvěr – Czech Journal of Economics and Finance, Vol. 56, No. 11–12, 506–521.

[36] Kilianová, S. and Pflug, G.: Optimal pension fund management under multi-period risk minimization. To appear in Annals of Operations Research.

[37] Kimball, M. S. (1990): Precautionary Saving in the Small and in the Large. Econometrica, Vol. 58, pp. 53–73, 1990.

[38] Kvetan, V., Mlýnek, M., Páleník, V. and Radvanský M. (2007): Starnutie, zdravotný stav a determinanty výdavkov na zdravie v podmienkach Slovenska. Working paper, Ekonomický ústav SAV, Bratislava 2007.

[39] Kwok, Y. K. (1998): Mathematical Models of Financial Derivatives. Springer-Verlag Singapore Pte. Ltd., 1998.

[40] Markowitz, H. (1952): Portfolio Selection. The Journal of Finance, Vol. 7, No. 1, pp. 77–91.

[41] Mehra R, Prescott E. (1985): The Equity Premium: a Puzzle. Journal of Monetary Economics, Vol. 15, pp. 145–161.

[42] Melicherčík, I., Olšarová, L. and Úradníček, V. (2005): Kapitoly z finančnej matematiky. EPOS, 2005.

[43] Melicherčík, I. and Ungvarský, C. (2004): Pension Reform in Slovakia: Perspectives of the Fiscal Debt and Pension Level. Finance a úvěr - Czech Journal of Economics and Finance, Vol. 54, No. 9–10, 391–404.

[44] Melicherčík, I. and Ševčovič, D.: Dynamic Stochastic Accumulation Model with Application to Pension Savings Management, submitted.

[45] Menegatti, M. (2006): A Note on a New Interpretation for the Precautionary Saving Motive. Dipartimento di Economia, Università di Parma, Italy, WP 5/2006, Serie: Economia e Politica Economica.

[46] Natali, D. (2004): {Denmark. The Pension System.; Hungary. The Reformed Pension System.; Latvia. The Reformed Pension System.; Lithuania. The Reformed Pension System.; Poland. The Reformed Pension System.; Slovak Republic. The Reformed Pension System.; Sweden. The Reformed Pension System.; The Netherlands. The Pension System.; United Kingdom. The Pension System.} Observatoire social européen, Research Project "La methode ouverte de coordination (MOC) en matiére des pensions et de l'integration européenne", 2004. Available at: http://www.ose.be/

[47] Pflug, G. and Römisch, W. (2007): Modeling, Measuring and Managing Risk. World Scientific Publishing Company, 2007.

[48] Pflug, G. and Ruszczynski, A. (2004): Measuring Risk for Income Streams. In: Risk Measures for the 21st Century, Chapter 13, Wiley Finance, 2004.

[49] Plesník, J., Dupačová, J. and Vlach, M. (1990): Lineárne programovanie. Alfa, Bratislava 1990.

[50] Pratt, J.W. (1964): Risk Aversion in the Small and in the Large. Econometrica, Vol. 32, No. 1–2, pp. 122–136.

[51] Riedel, F. (2004): Dynamic coherent risk measures. Stochastic Processes and their Applications, Vol. 112, No. 2, pp. 185–200.

[52] Rockafellar, R. T. and Uryasev, S. (2000): Optimization of Conditional Value-at-Risk. The Journal of Risk, Vol. 2, No. 3, pp. 21–41.

[53] Rockafellar, R. T., Uryasev, S. and Zabarankin, M. (2002): Deviation measures in risk analysis and optimization. University of Florida, Department of Industrial & Systems Engineering Working Paper No. 2002-7.

[54] Rofman, R. (2000): The Pension System in Argentina: Six Years After the Reform. Social Protection Discussion Paper Series, No. 0015, The World Bank.

[55] Scandolo, G. (2003): Risk measures in a dynamic setting. Ph.D. thesis, Università degli Studi Milano & Università di Firenze.

[56] Ševčovič, D.: Analytické a numerické metódy oceňovania finančných derivátov. Textbook, Comenius University, Bratislava.

[57] Uryasev, S. (2000): Conditional Value-at-Risk: Optimization Algorithms and Applications. Financial Engineering News, Vol. 14, February 2000.

[58] Varian, H. R.: Microeconomic Analysis, Third edition. New York, W.W. Norton and comp., Inc. 1992.

[59] Wang, T. (2002): A class of dynamic risk measures. Working Paper, University of British Columbia, Vancouver, BC, Canada.

[60] Wright, S. J.: Primal-dual interior point methods. SIAM, Philadelphia, 1997.

[61] Zhang, Y. (1995): Solving Large-Scale Linear Programs by Interior-Point Methods Under the MATLAB Environment. Optimization Methods and Software, Vol. 10, pp. 1–31, 1998.

[62] https://www.apnetlink.dk/Demo/Default.asp, APPension Denmark.

[63] http://www.canisius.edu/hr/tiaacref.asp

[64] http://www.tiaa-cref.org

[65] http://www.dixie.edu/humanres/tiaa.html

[66] http://www2.standardandpoors.com

[67] http://www.worldbank.org/pensions

Index

Resumé

V tejto dizertačnej práci sme sa zaoberali témou dôchodkového sporenia v rámci kapitalizačného druhého piliera dôchodkového systému. Cieľom bolo vybudovať matematicko-štatistický aparát uľahčujúci sporiteľom rozhodovanie o výbere typu dôchodkového fondu a simulovať výšku úspor. K problému sme pristupovali dvoma rôznymi spôsobmi:

I maximalizuje sa očakávaná užitočnosť sporiteľa z nasporenej sumy po skončení sporenia v okamihu odchodu do dôchodku,

II minimalizuje sa rizikovosť investícií, teda neistota dosiahnutia želanej sumy.

Predpokladom prvého prístupu je, že úžitková funkcia sporiteľa je známa. Špecifikovaním úžitkovej funkcie sa špecifikuje aj sporiteľov postoj k riziku, ktorý je vyjadrený koeficientom averzie k riziku. Predpokladom druhého prístupu je, že sporiteľ má stanovenú cieľovú sumu, ktorú by chcel sporením dosiahnuť. Pritom platí pravidlo, že čím je vyššia želaná suma, tým vyššie je riziko spojené s investovaním.

Prvá kapitola dizertačnej práce uvádza čitateľa do problematiky dôchodkového sporenia v kapitalizačnom pilieri. Druhá kapitola popisuje konkrétne príklady krajín, vrátane Slovenskej republiky, v ktorých je prítomný viac-pilierový dôchodkový systém a v ktorých naše modely môžu byť aplikované. Samotnému budovaniu modelov predchádzajú aj dve krátke kapitoly so zhrnutím základných poznatkov potrebných pre oba vyššie spomínané prístupy. Venujeme sa konceptu úžitkových funkcií a konceptu mier rizika v rozhodovacích úlohách. V kapitole 6 napokon pristupujeme k odvodeniu jednotlivých modelov.

Dynamický akumulačný model (DAM) je založený na maximalizácii očakávanej užitočnosti z nasporenej sumy. Predpokladom modelu je, že sporitelia si vyberajú v každom čase jediný z troch fondov. Ukázali sme, že optimálne riešenie modelu je riešením Bellmanovej rovnice stochastického dynamického programovania. Modifikáciou modelu DAM je Model proporcionálnych investícií (Proportional Investment Allocation Model, PIAM). Rozhodovacou premennou už nie je jednotlivý typ fondu, ale váha akciovej zložky v dôchodkovom portfóliu sporiteľa. Pre riešenie PIAM modelu sme odvodili plne nelineárnu parciálnu diferenciálnu rovnicu. Pritom, model DAM je diskretizáciou modelu PIAM.

Druhým typom modelov vybudovaných v tejto práci sú modely založené na minimalizácii rizika spojeného s investovaním v dôchodkových fondoch. Model TRMM je model minimalizujúci neistotu výšky úspor v záverečnom čase sporenia (Terminal Risk Minimizing Model). Model MRMM minimalizuje rizikovosť úspor počas celej doby sporenia, čo môže byť odôvodnené možnosťou dedenia priebežných úspor v prípade predčasného úmrtia sporiteľa. Pri oboch modeloch sme použili mieru rizika *average value-at-risk deviation*, v TRMM jej statickú verziu, v MRMM dynamickú. Za predpokladu, že

sporitelia zvažujú zmenu výberu fondu jedenkrát ročne, vedú oba modely vedú na úlohu vysokorozmerného lineárneho programovania s riedkou maticou. Špecifikovali sme podmienky, za ktorých existuje optimálne riešenie daných optimalizačných úloh. Z dôvodu vysokej pamäťovej náročnosti pri implementácii sme modely zovšeobecnili pre prípad, kedy rozhodovacie okamžiky nastávajú iba jedenkrát počas niekoľkých rokov. Touto modifikáciou sa modely stávajú nelineárnymi úlohami.

V ďalšej kapitole sa venujeme navrhnutiu numerických schém pre oba typy modelov a ich následnej numerickej implementácii. Pre modely DAM a PIAM navrhujeme škálovaciu techniku, ktorá umožní lepšiu numerickú stabilitu úloh pri použití mocninovej úžitkovej funkcie. Pre modely TRMM a MRMM navrhujeme iteračnú schému, ktorou sa nelineárna úloha v každej iterácii opäť linearizuje. Konvergencia navrhnutej iteračnej schémy je študovaná v jednej z podkapitol. Pri numerickej implementácii modelov sme vychádzali z predpokladov o rozložení investícií dôchodkových fondov. Numerické výsledky sme prezentovali grafickou i tabuľkovou formou. Použijúc model DAM sme skúmali citlivosť výsledkov na meniace sa hodnoty rôznych parametrov, ako sú miera averzie k riziku, výška výnosov akcií a dlhopisov, či miera rastu miezd. V modeli TRMM sme skúmali citlivosť výsledkov na meniacu sa hodnotu parametra α vystupujúceho v špecifikácii miery average value-at-risk. Výsledky získané zo simulácií vykazujú pri všetkých modeloch spoločnú vlastnosť:

Zastúpenie rizikových aktív v optimálnej stratégii výberu fondov má klesajúci charakter v čase.

Táto vlastnosť je v súlade intuícou, nakoľko vyššia suma úspor je citlivejšia na zmeny vo výnosoch fondov. Ostaté kvalitatívne vlastnosti výsledkov sme zosumarizovali v prehľadných bodoch. Záverečnou kapitolou je ôsma kapitola so závermi a zhrnutím celej práce.