COMENIUS UNIVERSITY IN BRATISLAVA
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OPTIMIZATION IN FINANCIAL MATHEMATICS

Dissertation thesis

submitted in fulfilment of the degree **Philosophiae Doctor (PhD.)**

in the doctoral branch of study **1114 Applied Mathematics**

Supervisor: **doc. Ing. Aleš Černý, PhD.**

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UNIVERZITA KOMENSKÉHO V BRATISLAVE
Fakulta matematiky, fyziky a informatiky
Katedra aplikovanej matematiky a štatistiky

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OPTIMALIZÁCIA VO FINANČNEJ MATEMATIKE

Dizertačná práca

na získanie akademického titulu **Philosophiae Doctor (PhD.)**

v odbore doktorandského štúdia **1114 Aplikovaná matematika**

Školiteľ: **doc. Ing. Aleš Černý, PhD.**

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THESIS ASSIGNMENT

Name and Surname:	Mgr. Ján Komadel
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Title: Optimization in financial mathematics

Annotation: Different areas of financial mathematics often rely on optimizing certain objectives. This dissertation thesis focuses on two such areas: optimal liquidation of a trading position and option hedging. We aim for comprehensible explanations of both topics for a reader without extensive background in the field. In the optimal liquidation part we formulate the problem with the pressure to liquidate given endogenously, which is rare in literature, and we propose and carefully study a method of solving the severely singular and numerically unstable ordinary differential equation arising from the optimization. In the option hedging part we propose an approximation for the mean squared hedging error of a discretely applied delta hedging strategy in case of an Asian option. We then develop a method of evaluating the approximation by means of solving a system of two partial differential equations and we analyze it numerically.

Aim: The thesis aims to study two applications of optimization in financial mathematics with focus on computational aspects. The first area to examine is the problem of optimal trade execution when the traded quantity adversely affects the price. The second application concerns hedging Asian options in discrete setting.

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ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Mgr. Ján Komadel
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Študijný odbor: aplikovaná matematika
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Názov: Optimization in financial mathematics
Optimalizácia vo finančnej matematike

Anotácia: Rôzne oblasti finančnej matematiky sa často spoliehajú na optimalizáciu určitých cieľov. Táto dizertačná práca je zameraná na dve také oblasti: optimálnu likvidáciu obchodnej pozície a zaisťovanie opcií. Našou snahou je vysvetliť obidve témy čitateľovi, ktorý nemá rozsiahle znalosti v danej oblasti. V časti o optimálnej likvidácii sformulujeme problém tak, že dôvod na likvidáciu je endogénny, čo je v literatúre zriedkavé, a navrhujeme a pozorne študujeme metódu riešenia ťažko singulárnej a numericky nestabilnej obyčajnej diferenciálnej rovnice, ktorá vzniká z optimalizácie. V časti o zaistení opcií navrhujeme aproximáciu očakávanej kvadratickej chyby zaistenia pre diskrétne aplikované delta zaistenie v prípade ázijskej opcie. Následne odvodíme metódu výpočtu tejto aproximácie pomocou riešenia systému dvoch parciálnych diferenciálnych rovníc a numericky ho analyzujeme.

Cieľ: Cieľom dizertačnej práce je študovať dve použitia optimalizácie vo finančnej matematike s dôrazom na výpočtové aspekty. Prvou skúmanou oblasťou bude problém optimálneho obchodovania, keď obchodované množstvo negatívne ovplyvňuje cenu. Druhá aplikácia bude pri diskrétnom zaisťovaní ázijských opcií

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Abstract

Mgr. Ján Komadel: *Optimization in financial mathematics* [Dissertation thesis] Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics, supervisor: doc. Ing. Aleš Černý, PhD., Bratislava, 2018

This dissertation thesis focuses on two applications of optimization in financial mathematics. The first application is in optimal liquidation in presence of an adverse temporary price impact. The novel aspect of our formulation of the problem is that we give the pressure to liquidate endogenously and we use a stochastic time horizon. This leads to a severely singular initial value problem for which standard numerical methods fail. We propose a procedure to overcome the singularity by solving related finite horizon boundary value problems obtained by introducing a time dimension into the time-homogenous problem. The convergence of the solutions of the finite horizon problems to the solution of the original problem is analyzed analytically and, subsequently, confirmed numerically. We find that the model is consistent with the square root law known from empirical literature.

The second examined application of optimization in financial mathematics is quadratic hedging of options. We focus on studying the mean squared hedging error (MSHE) of a discretely implemented delta hedging strategy for an arithmetic Asian option. We heuristically derive an approximation of the MSHE which is consistent with known approximations for European options. We propose a method of evaluating the approximation by solving a system of two partial differential equations and use this method to numerically confirm that the approximation produces reasonable estimates of the MSHE.

Keywords: optimal liquidation, singular boundary value problem, quadratic hedging, delta hedging, Asian options

Abstrakt

Mgr. Ján Komadel: *Optimalizácia vo finančnej matematike* [Dizertačná práca] Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky, školiteľ: doc. Ing. Aleš Černý, PhD., Bratislava, 2018

Táto dizertačná práca je zameraná na dve aplikácie optimalizácie vo finančnej matematike. Prvá aplikácia je pri optimálnej likvidácii s nepriaznivým dočasným dopadom na cenu. Novinkou v našej formulácii je, že tlak na predaj zadávame endogénne a používame stochastický časový horizont. To vedie k prudko singulárnej počiatočnej úlohe, pre ktorú štandardné numerické metódy zlyhávajú. Navrhujeme postup, ako si s touto singularitou poradiť riešením príbuzných okrajových úloh na konečnom časovom horizonte, ktoré získame zavedením časového rozmeru do inak časovo-homogénneho problému. Konvergenciu riešení úloh na konečnom horizonte k riešeniu pôvodnej úlohy skúmame analyticky a následne ju potvrdíme numericky. Zistíme, že náš model je konzistentný s odmocninovým zákonom známym z empirickej literatúry.

Druhou skúmanou aplikáciou optimalizácie vo finančnej matematike je kvadratické zaisťovanie opcií. Zameriame sa na očakávanú kvadratickú chybu zaistenia pre diskrétne uplatnené delta zaistenie v prípade aritmetickej ázijskej opcie. Heuristicky odvodíme pre túto očakávanú kvadratickú chybu aproximáciu, ktorá je konzistentná so známymi aproximáciami pre európske opcie. Navrhneme metódu vyčíslenia našej aproximácie riešením systému dvoch parciálnych diferenciálnych rovníc a následne túto metódu použijeme pri numerickom potvrdení toho, že naša aproximácia dáva rozumné odhady kvadratickej zaisťovacej chyby.

Kľúčové slová: optimálna likvidácia, singulárna okrajová úloha, kvadratické zaistenie, delta zaistenie, ázijské opcie

Contents

Introduction

In finance, it is often one's aim to find an optimal strategy of execution. Common objectives include maximizing revenue, profit, or utility in a broader sense, and minimizing risk. Naturally, mathematical optimization is a key instrument in finding the optimal strategy. We study two applications of optimization in finance. The first one is the optimal liquidation problem where the investor aims to maximize the expected revenue from selling a certain amount of an asset while the received price at each time is adversely affected by the amount sold at this time. The other application is option hedging in an incomplete market where the mean squared hedging error is analyzed.

This dissertation thesis is divided into eight chapters. The first chapter provides an introduction to the theory of stochastic calculus. We define a stochastic process and we mention important classes such as càdlàg, predictable, optional, and finite variation processes as well as martingales. Then we introduce Itō's integral and stochastic differential equations and we explain the fundamentals of Itō calculus. Finally, we formulate Girsanov's theorem which is used to change measure.

The remainder of the work consists of two parts, Part I containing Chapters 2-4 and being devoted to optimal liquidation and Part II containing Chapters 5-8 and dealing with quadratic hedging. Each of the two parts includes a more detailed introduction so we only mention the contents briefly at this point.

In Chapter 2 we present the studied optimal liquidation problem and the corresponding severely singular ordinary differential equation. Then we propose a method of solving the severely singular ODE and, as we show, thus also the optimal liquidation problem. The method involves solving a related boundary value problem for a parabolic partial differential equation. In Chapter 3 we solve the optimal liquidation problem numerically and we analyze the results. In particular, we show that our proposed method is capable of solving the severely singular boundary value problem for which standard methods, such as the Matlab routine `bvp5c`, fail. Moreover, we find that our results agree with the square root law known from empirical studies. Chapter 4 concludes Part I.

Chapter 5 introduces the reader to options and their pricing using partial differential equations. We derive a PDE for the arithmetic Asian option price and we reduce the dimension in line with [54] in the next chapter. In Chapter 6 give a brief overview of relevant results from quadratic hedging literature and we focus on the mean squared hedging error of a discretely implemented strategy. We heuristically derive an approximation of this error which we apply to discrete delta hedging of Asian options. Then we propose a method of estimating this approximation by solving a system of two partial differential equations which we numerically solve in Chapter 7. We also verify the obtained MSHE approximation by comparing it to simulated actual MSHE. Chapter 8 concludes Part II.

Chapter 1

Stochastic calculus

In this chapter we provide a short overview of the stochastic calculus theory. We start by defining basic terms and introducing martingales. Then we present Itō calculus and we conclude this chapter with change of measure and Girsanov's theorem. For more detailed information we refer the reader to the monographs by Øksendal [39], Revuz and Yor [44], Shreve [47], Protter [42], He, Wang and Yan [23], or Mikosch [38].

We start by defining a *stochastic process*, the central concept of stochastic calculus, as it is defined in [39, Definition 2.1.4].

Definition 1.1 (Stochastic process)**.** *A stochastic process is a parametrized collection of random variables*

$$\{X_t\}_{t\geq 0}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ *and assuming values in* $\mathbb{R}$.

The index t denotes time and we assume it to take values in $\mathbb{R}_+$. Alternatively, X_t is sometimes defined on an interval $t \in [a, b]$, $a < b \in \mathbb{R}$, or for t from a discrete set of indices. The process $X_t(\omega)$ can be thought of as a function of two variables (t, ω) from $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}$. For any fixed time $t \geq 0$ one has a random variable mapping Ω into $\mathbb{R}$

$$\omega \mapsto X_t(\omega)$$

and for a fixed $\omega \in \Omega$ one has a function from $\mathbb{R}_+$ into $\mathbb{R}$

$$t \mapsto X_t(\omega)$$

which is called a *path* of the process X_t.

Definition 1.2 (Filtration). *A filtration on $(\Omega, \mathcal{F})$ is a family $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ of nested σ-algebras $\mathcal{F}_t \subseteq \mathcal{F}$, i.e.*

$$0 \leq s \leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t.$$

Every σ-algebra $\mathcal{F}_t$ of the filtration represents information available at time t. The non-decreasingness of the filtration ensures that no information is lost over time.

We say that a process $\{X_t\}_{t\geq 0}$ is *adapted* to the filtration $\{\mathcal{F}_t\}$ if for all t the random variable X_t is $\mathcal{F}_t$-measurable.

We will work with the *natural filtration* (cf. [42, p. 16]) generated by a standard Brownian motion W (see Definition 1.3 below). It is denoted by $\{\mathcal{F}_t^W\}$ and defined by

$$\mathcal{F}_t^W = \sigma(W_s : s \leq t)$$

so at time t it only contains the information about the path of the Brownian motion up to time t. The natural filtration $\{\mathcal{F}_t^W\}$ is the smallest filtration with respect to which W is adapted.

We assume that the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfies the *usual conditions* or the *usual hypotheses* (cf. [23, Definition 2.63] or [42, p. 3]), i.e. it is *complete* ($\mathcal{F}_0$ contains all zero probability sets with all their subsets, which are also null sets) and right-continuous ($\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$). Note that the non-decreasingness of filtration implies that every $\mathcal{F}_t$ contains all P-null sets as well. Throughout this work, we will understand natural filtration satisfying the usual conditions when talking about filtration or under $\{\mathcal{F}_t\}$.

Process X_t is called *càdlàg* if all its paths are right-continuous and with left limits. Right-continuity means that for every $t \geq 0$ one has $\lim_{s\searrow t} X_s = X_t$. Left limits mean that for every $t > 0$ the limit $X_{t-} := \lim_{u\nearrow t} X_u$ exists. The word càdlàg is an acronym of the French expression for "right-continuous, left limits," *continue à droite, limite à gauche*.

Denote $\mathcal{P}$ the σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by all left-continuous adapted processes. $\mathcal{P}$ is called the *predictable σ-algebra*. We say that process X is *predictable* if it is $\mathcal{P}$-measurable as a mapping from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}$.

Similarly, the σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by all càdlàg adapted processes is called the *optional σ-algebra* and denoted by $\mathcal{O}$. A stochastic process is *optional* if it is $\mathcal{O}$-measurable.

For a càdlàg adapted process X, the left limit process $X_- := \{X_{t-}\}$ is predictable (cf. [23, p. 87]).

Process X is said to be measurable, if $X_t(\omega)$, as a function of (t, ω), is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable, where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel σ-algebra on $\mathbb{R}_+$. The product σ-algebra $\mathcal{M} = \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ on $\mathbb{R}_+ \times \Omega$ is called the *measurable σ-algebra*.

A stochastic process X is called *progressively measurable* or *progressive* if, for every $t \geq 0$, X restricted on $[0, t] \times \Omega$ is measurable with respect to $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$. The smallest σ-algebra on $\mathbb{R}_+ \times \Omega$ which makes all progressive processes measurable is the *progressive σ-algebra* $\mathcal{A}$.

Every progressively measurable process is measurable and adapted. Conversely, every adapted process with right- or left-continuous paths is progressively measurable (cf. [44, Proposition 4.8]). In general, one has (cf. [42, p. 103])

$$\mathcal{P} \subset \mathcal{O} \subset \mathcal{A} \subset \mathcal{M}.$$

Processes X and Y are *modifications* or *versions* of each other if for every $t \geq 0$, $X_t = Y_t$ a.s.. They are *indistinguishable* if almost all their sample paths agree, i.e. if for almost all $\omega \in \Omega$ one has $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$.

If X and Y are modifications of each other, then for each $t \geq 0$ there is a null set N_t such that for all $\omega \notin N_t$ on has $X_t(\omega) = Y_t(\omega)$. These null sets depend on t and the union $\cup_{t \geq 0} N_t$ does not need to be a null set. If X and Y are indistinguishable, however, there is a single null set N such that if $\omega \notin N$, then $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$. Clearly, indistinguishability is a stronger concept and indistinguishable processes are also modifications of each other. The opposite is not true in general but if X and Y are modifications of each other and their paths are a.s. right-continuous, then X and Y are indistinguishable (cf. [42, Theorem I.2]).

If all paths of a process X are nonnegative, increasing, right-continuous functions, we say that X is *increasing*. If a process is the difference of two increasing processes, it is called a process with *finite variation* (or, shortly, an FV process).

Next, we define a very important process in stochastic calculus – standard Brownian motion, in line with [32, Definition II.1.1].

Definition 1.3 (Standard Brownian motion)**.** *Standard Brownian motion is a continuous, adapted stochastic process $\{W_t\}_{t\geq 0}$ with the properties that $W_0 = 0$ a.s. and for $0 \leq s < t$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean zero and variance $t - s$.*

Brownian motion is named after Robert Brown, a 19th century Scottish botanist, who first observed and described the chaotic movement of pollen grains in water. Standard Brownian motion is also called the *Wiener process* after the American mathematician Norbert Wiener who came up with a mathematical model describing it.

We will use the name *Brownian motion* for a process X of the form

$$X_t = \mu t + \sigma W_t \tag{1.1}$$

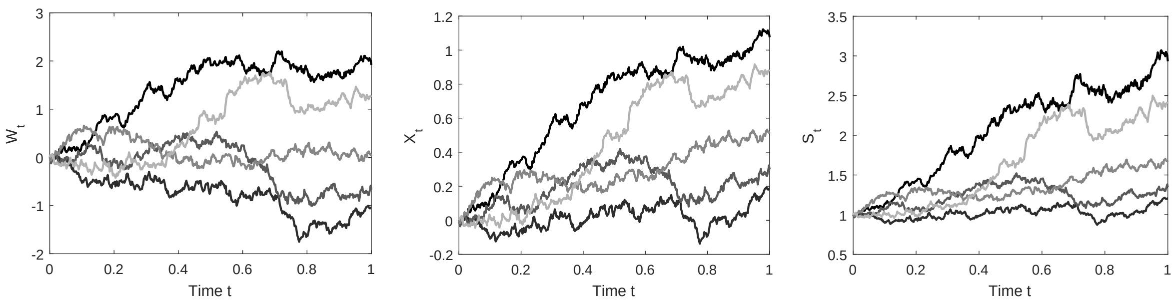
whose distribution is $X_t \sim \mathcal{N}(\mu t, \sigma^2 t)$. A *geometric Brownian motion* is defined as

$$S_t = S_0 e^{X_t} = S_0 e^{\mu t + \sigma W_t}, \tag{1.2}$$

where X is a Brownian motion. Since the logarithm of S is a Brownian motion, which has normal distribution, geometric Brownian motion is said to have *lognormal distribution*. Sometimes, authors use the name arithmetic Brownian motion for Brownian motion X_t to distinguish it from geometric Brownian motion S_t.

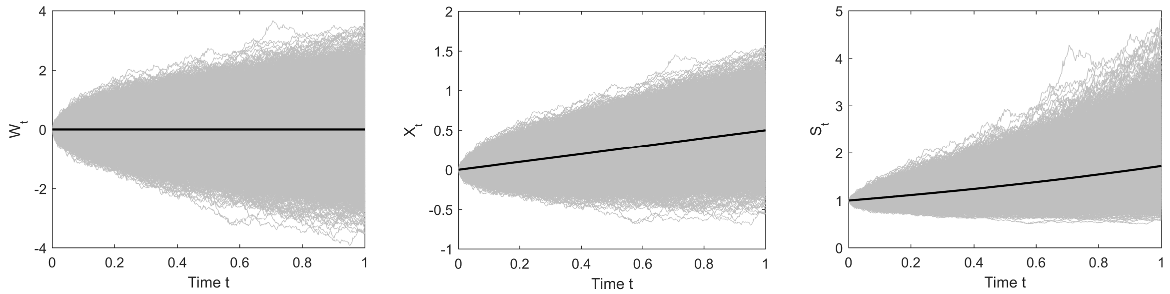
Panel (a) of Figure 1.1 shows five randomly generated sample paths of a standard Brownian motion W_t. The corresponding paths of the (arithmetic) Brownian motion $X_t = 0.5t + 0.3W_t$ and the geometric Brownian motion $S_t = e^{X_t}$, respectively, are shown in panels (b) and (c) of the same figure.

Figure 1.2 shows 10 000 randomly generated paths of the same processes together with their means. Observe the zero mean of the standard Brownian motion $E[W_t] = 0$, the linear mean of the (arithmetic) Brownian motion $E[X_t] = \mu t = 0.5t$ and the exponential mean of the geometric Brownian motion $E[S_t] = e^{\left(\mu + \frac{1}{2}\sigma^2\right)t} = e^{\left(0.5 + \frac{1}{2}0.3^2\right)t}$.



(a) Standard BM. **(b)** Arithmetic BM. **(c)** Geometric BM.

Figure 1.1: Five randomly generated paths of a standard Brownian motion and the corresponding paths of arithmetic and geometric Brownian motions.



(a) Standard BM. **(b)** Arithmetic BM. **(c)** Geometric BM.

Figure 1.2: 10 000 randomly generated paths of a standard Brownian motion and the corresponding paths of arithmetic and geometric Brownian motions. The solid black lines show the means.

1.1 Martingale theory

The following definition of a martingale is adapted form [39, Definition 3.2.2].

Definition 1.4 (Martingale). *A process $\{M_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if*

(a) M_t is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$,

(b) $E\left(|M_t|\right) < \infty$ for all t,

(c) $E_s(M_t) = M_s$ for all $s \leq t$.

The notation $E_s(M_t)$ in part (c) of Definition 1.4 means the expected value conditional on the information available at time s, i.e. $E_s(M_t) := E(M_t|\mathcal{F}_s)$. The martingale property says the expectation of any future value M_t is the current value M_s at any time s before time t. In other words, the process M does not change on average.

If we change the (c) property in Definition 1.4 to the inequality $E_s(M_t) \geq M_s$, we get a definition of a *submartingale*. Similarly, a *supermartingale* is defined by changing the (c) property in Definition 1.4 to the inequality $E_s(M_t) \leq M_s$. A submartingale may have a tendency to rise on average while a supermartingale may have a tendency to fall. A martingale is both a submartingale and a supermartingale.

If X is a supermartingale, then the function $t \mapsto E(X_t)$ is right-continuous if and only if X has a càdlàg modification (cf. [42, Theorem I.9]). As mention earlier, modifications which are right-continuous are indistinguishable so the càdlàg modification is unique up to indistinguishability. If X is a martingale, $t \mapsto E(X_t)$ is constant and it follows that every martingale has a càdlàg modification. For this reason martingales can be assumed to be càdlàg, meaning that one always works with the càdlàg version.

A *stopping time* is a random variable τ for which the event $\tau \leq t$ is $\mathcal{F}_t$-measurable for every t. In other words, we can tell, based on the information available at time t, whether τ has already occurred or not. For an adapted process X and a stopping time τ we define the *stopped process* X^τ as

$$X_t^\tau := X_{t\wedge\tau},$$

where $t \wedge \tau = \min\{t, \tau\}$.

A collection of random variables $\mathcal{H}$ is *uniformly integrable* if for each $\varepsilon > 0$ there is a constant $K \in \mathbb{R}$ such that

$$\sup_{V\in\mathcal{H}} E\big(|V|\mathbf{1}_{|V|>K}\big) < \varepsilon.$$

Similarly, a process $\{X_t\}_{t\geq 0}$ is *uniformly integrable* if for each $\varepsilon > 0$ there is a constant $K \in \mathbb{R}$ such that for all $t \geq 0$

$$E\big(|X_t|\mathbf{1}_{|X_t|>K}\big) < \varepsilon.$$

Definition 1.5 (Local martingale). *An adapted, càdlàg process $\{X_t\}_{t\geq 0}$ is a local martingale if there is an increasing sequence of stopping times $\{\tau_n\}_{n=1}^\infty$ such that*

$$\lim_{n\to\infty} \tau_n = \infty$$

and for each n the stopped process X^{τ_n} is a uniformly integrable martingale.

Every martingale is a local martingale but, in general, not every local martingale is a martingale (cf. [42, p. 37]).

Definition 1.6 (Semimartingale). *A process $\{X_t\}_{t\geq 0}$ is called a semimartingale if it can be decomposed as*

$$X_t = M_t + A_t,$$

where M_t is a local martingale and A_t is a càdlàg adapted process with finite variation.

A semimartingale is an adapted càdlàg process and the class of semimartingales includes Brownian motion as well as all submartingales and supermartingales.

If X is a bounded, measurable process, there exists a unique optional process ${}^oX$, also bounded, such that for any stopping time τ one has

$$E\big[X_\tau \mathbf{1}_{\tau<\infty}\big] = E\big[\,{}^oX_\tau \mathbf{1}_{\tau<\infty}\big].$$

The process ${}^oX$ is called the optional projection of X (cf. [42, p. 367]).

A measurable process X is said to be of *class (D)* if the set of random variables $\{X_\tau : \tau < \infty$ is a stopping time $\}$ is uniformly integrable.

Theorem 1.7 (Doob-Meyer decomposition). *Let X be a right-continuous supermartingale of class (D). Then X can be uniquely decomposed as $X = M - A$, where M is a uniformly integrable martingale and A is a predictable, integrable, increasing process with $A_0 = 0$.*

This theorem can be found in [23, Theorem 5.48]. The Doob-Meyer decomposition in Theorem 1.7 may seem similar to the Definition 1.6 of a semimartingale but note that the process M is only a local martingale in Definition 1.6 while it is a uniformly integrable martingale in the Doob-Meyer decomposition. Moreover, the process A is a càdlàg adapted process with finite variation in Definition 1.6 while it is a predictable, integrable, increasing process in Theorem 1.7.

1.2 Itō calculus

The integral with respect to a smooth function f is defined as

$$\int_a^b g(u)df(u) = \int_a^b g(u)f'(u)du.$$

In finance, one often encounters integrals where the integrator is not smooth. In particular, integrals with respect to standard Brownian motion W

$$\int_0^t \varphi_u dW_u \tag{1.3}$$

are common. The problem is that the paths of W are not differentiable with probability 1 and so the classical approach fails and one has to proceed differently.

The integral (1.3) is in fact called an *Itō integral* and it is defined as the limit

$$\int_0^t \varphi_u dW_u = \lim_{\|\Pi\|\to 0} \sum_{k=1}^{n} \varphi_{t_{k-1}} \left(W_{t_k} - W_{t_{k-1}}\right), \tag{1.4}$$

where $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ is a partition of the interval $[0, t]$ and $\|\Pi\| = \max_{j=1,\ldots,n} t_j - t_{j-1}$. The *Itō integral* is properly constructed for example in Shreve [47, Chapter 4] or in Mikosch [38, Chapter 2].

The Itō integral (1.4) is defined for adapted processes φ such that

$$E\left(\int_0^t \varphi_u^2 du\right) < \infty.$$

The next theorem adopted from Shreve [47, Theorem 4.3.1] lists properties of the Itō integral.

Theorem 1.8 (Properties of the Itō integral)**.** *The Itō integral $\int_0^t \varphi_u dW_u$ defined by (1.4) has following properties.*

(a) ***(Continuity)*** *As a function of the upper limit t, the paths of $\int_0^t \varphi_u dW_u$ are continuous.*

(b) ***(Adaptivity)*** *For each t, $\int_0^t \varphi_u dW_u$ is $\mathcal{F}_t$-measurable.*

(c) ***(Linearity)*** *For every $a, b \in \mathbb{R}$, one has*

$$\int_0^t a\,\varphi_u dW_u + \int_0^t b\,\psi_u dW_u = \int_0^t a\varphi_u + b\psi_u dW_u.$$

(d) ***(Martingale)*** *$\int_0^t \varphi_u dW_u$ is a martingale.*

(e) ***(Itō isometry)***

$$E\left[\left(\int_0^t \varphi_u dW_u\right)^2\right] = E\left[\int_0^t \varphi_u^2 du\right].$$

We define an *Itō process* X as an adapted process that can be written as

$$X_t = X_0 + \int_0^t \mu(u, X_u)du + \int_0^t \sigma(u, X_u)dW_u. \tag{1.5}$$

An Itō process is the sum of a nonrandom initial value X_0, a Riemann integral, and an Itō stochastic integral. We make the following assumptions about μ and σ

$$\int_0^t |\mu(u, X_u)|du < \infty, \qquad E\left[\int_0^t \sigma^2(u, X_u)du\right] < \infty$$

for all $t \geq 0$ or that these at least hold with probability 1 (cf. [47, Definition 4.4.3] or [39, Definition 4.1.1]). The function $\mu(t, X_t)$ is called the *drift* of X and $\sigma(t, X_t)$ is its *volatility*.

The Itō process X (1.5) is often written in the differential form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{1.6}$$

We call (1.6) a *stochastic differential equation* (SDE).

A Brownian motion X from (1.1) has the SDE

$$dX_t = \mu dt + \sigma dW_t$$

corresponding to both $\mu(t, X_t) \equiv \mu$ and $\sigma(t, X_t) \equiv \sigma$ being constant. To see this, we may write the integral form

$$X_t = X_0 + \int_0^t \mu du + \int_0^t \sigma dW_u = \mu t + \sigma W_t,$$

which is how we defined Brownian motion in (1.1).

The definition of stochastic integral can be extended to Itō process integrators given by (1.6). The integral of an adapted process φ with respect to an Itō process X is defined as

$$\int_0^t \varphi_u dX_u = \int_0^t \varphi_u \mu(u, X_u) du + \int_0^t \varphi_u \sigma(u, X_u) dW_u,$$

assuming $E\left[\int_0^t \varphi_u^2 \sigma^2(u, X_u) du\right]$ and $\int_0^t |\varphi_u \mu(u, X_u)| \, du$ are finite for all $t > 0$ (cf. [47, Definition 4.4.5]).

1.2.1 Quadratic variation

Next, we define *quadratic variation* of a function, inspired by [47, Definition 3.4.1]. This concept is special to stochastic calculus because it is zero for all continuous, piecewise differentiable functions and so it is not considered in standard calculus.

Definition 1.9 (Quadratic variation of a function). *Let* $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ *be a partition of* $[0, t]$ *and* $f(u)$ *be a function defined on* $[0, t]$*. The quadratic variation of* f *up to time* t *is defined as*

$$[f](t) = \lim_{\|\Pi\| \to 0} \sum_{k=1}^{n} \Big(f(t_k) - f(t_{k-1}) \Big)^2 .$$

In case of a function f with a continuous derivative, one has

$$\sum_{k=1}^{n} \Big(f(t_k) - f(t_{k-1}) \Big)^2 = \sum_{k=1}^{n} \Big(f'(t_k^*) \Big)^2 (t_k - t_{k-1})^2 \leq \|\Pi\| \sum_{k=1}^{n} \Big(f'(t_k^*) \Big)^2 (t_k - t_{k-1})$$

and thus

$$[f](t) = \lim_{\|\Pi\| \to 0} \sum_{k=1}^{n} \Big(f(t_k) - f(t_{k-1}) \Big)^2 \leq \lim_{\|\Pi\| \to 0} \|\Pi\| \int_0^t \Big(f'(u) \Big)^2 du = 0.$$

A similar argument can be made for piecewise continuous functions which justifies why quadratic variation is not considered in ordinary calculus.

Protter defines quadratic variation of a semimartingale and *quadratic covariation* of two semimartingales in [42, p. 66] in the following way.

Definition 1.10 (Quadratic variation and quadratic covariation of semimartingales)**.** *Let X and Y be semimartingales. The quadratic variation process of X, denoted $[X]_t = [X, X]_t$ is defined by*

$$[X]_t = X_t^2 - X_0^2 - 2\int_0^t X_{u-} dX_u.$$

The quadratic covariation of X and Y, also called the bracket process or the sharp bracket process, is defined by

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{u-} dY_u - \int_0^t Y_{u-} dX_u.$$

There is an alternative process to quadratic variation, called the *predictable quadratic variation* or the angle bracket process, which is defined for a square integrable martingale M as the unique, predictable, integrable, increasing process $\langle M \rangle$ such that $M^2 - \langle M \rangle$ is a martingale (cf. [23, Definition 6.24]). Note that the existence of $\langle M \rangle$ follows from the Doob-Meyer decomposition. For continuous martingales one has $\langle M \rangle = [M]$ (cf. [31]).

It can be shown (cf. [47, Theorem 3.4.3]) that the (predictable) quadratic variation of a standard Brownian motion is

$$[W]_t = t$$

which is often written in differential form as

$$d[W]_t = (dW_t)^2 = dt. \tag{1.7}$$

For an Itō process X, given by (1.5), the quadratic variation is, according to [49, Theorem 8.6], given by

$$[X]_t = \int_0^t \sigma^2(u, X_u) du.$$

1.2.2 Itō's lemma

Now we formulate Itō's lemma which is a central tool in stochastic calculus. We start with the 1-dimensional case adapted from [39, Theorem 4.1.2].

Theorem 1.11 (1-dimensional Itō's lemma)**.** *Let X be an Itō process given by*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

and let $f(t,x) \in C^2\big([0,\infty] \times \mathbb{R}\big)$. Then $Y_t = f(t, X_t)$ is again an Itō process with

$$dY_t = df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2} f_{xx}(t, X_t)d[X]_t. \tag{1.8}$$

Note that (1.8) is equivalent to

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \left(f_t(u, X_u) + \mu_u f_x(u, X_u) + \frac{1}{2}\sigma_u^2 f_{xx}(u, X_u) \right) du \\ &+ \int_0^t \sigma_u f_x(u, X_u) dW_u, \end{aligned}$$

where $\mu_u = \mu(u, X_u)$ and $\sigma_u = \sigma(u, X_u)$.

When computing the term $d[X]_t = (dX_t)^2$ in (1.8), the rules

$$(dW_t)^2 = dt, \quad dW_t dt = dt dW_t = (dt)^2 = 0$$

apply (cf. [39, Eq. (4.1.8)]). The first one is the differential of quadratic variation of a standard Brownian motion (1.7) and the others follow from quadratic covariation being zero when at least one of the processes is nonrandom.

An important stochastic process in finance is the before mentioned geometric Brownian motion defined by (1.2). Let us illustrate the use of Itō's lemma to find the stochastic differential equation for the geometric Brownian motion S. We define $f(t,x) = e^x$, i.e. f is only a function of one variable, and S can be written as $S_t = f(X_t) = S_0 e^{X_t}$ where X is the Brownian motion

$$dX_t = \mu dt + \sigma dW_t.$$

Itō's lemma yields

$$\begin{aligned} dS_t &= f_x(t, X_t)dX_t + \frac{1}{2} f_{xx}(t, X_t)d[X]_t = S_t\big(\mu dt + \sigma dW_t\big) + \frac{1}{2} S_t \big(\mu dt + \sigma dW_t\big)^2 \\ &= \left(\mu + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dW_t = \tilde{\mu} S_t dt + \sigma S_t dW_t, \end{aligned}$$

where $\tilde{\mu} = \mu + \frac{\sigma^2}{2}$. A Geometric Brownian motion is usually written as

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

or equivalently as

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dW_u.$$

It is often used in finance to model the price of an asset. As illustrated in Figure 1.1, its main advantages over a Brownian motion are that it only takes positive values (provided that S_0 is positive) and the fact that relative returns do not depend on the price.

Itō's lemma can be generalized to the case when X is multivariate (cf. [39, Theorem 4.2.1]). Let X be a d-dimensional Itō process, i.e. $X = (X^1, X^2, \ldots, X^d)$ and X^k are Itō processes. Let Y be given by $Y = f(x)$, where f is a twice continuously differentiable function from $\mathbb{R}^d$ to $\mathbb{R}$. Then one has

$$dY_t = df(X_t) = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_t) dX_t^i + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 f}{\partial x_i x_j}(X_t) d[X^i, X^j]_t.$$

1.2.3 Integration by parts

The integration by parts rule known from ordinary calculus needs to be adjusted when working with stochastic processes. For a pair of semimartingales X, Y the integration by parts formula follows directly from Definition 1.10 an it reads

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{u-} dY_u + \int_0^t Y_{u-} dX_u + [X, Y]_t,$$

Equivalently, one may write

$$d(X_t Y_t) = X_{t-} dY_t + Y_{t-} dX_t + d[X, Y]_t. \tag{1.9}$$

As opposed to the standard integration by parts formula, there is an extra term representing quadratic covariation. In case that X and Y are continuous, the left limits X_{t-}, Y_{t-} can by replaced with X_t, Y_t. If in addition, one of the processes is deterministic, the quadratic covariation is zero and (1.9) reduces to the standard formula

$$d\left(X_tY_t\right) = X_t dY_t + Y_t dX_t.$$

1.3 Change of measure

In finance, it is often more convenient to work with a different probability measure than the objective one which corresponds to the objective probability distributions of price movements. Under this other measure, the *risk-neutral probability measure*, every investment has the same mean return. This implies that, in presence of a risk-free asset, no-arbitrage prices of all assets denominated in terms of the risk-free asset, i.e. discounted by the risk-free return, need to be martingales under the risk-neutral probability.

We say that probability measures P and Q on $(\Omega, \mathcal{F})$ are *equivalent*, and write $P \sim Q$, if for every set A

$$P(A) = 0 \Longleftrightarrow Q(A) = 0,$$

i.e. P is absolutely continuous w.r.t Q and Q is absolutely continuous w.r.t P (cf. [27, Definition 28.1]).

We now define the density process of the change of measure and then we formulate the Girsanov's theorem which is proven in Protter [42, Theorem III.35]. If $P \sim Q$, then there exists a dP-integrable random variable Z_T, such that $dQ/dP = Z_T$ and $E^P(Z_T) = 1$, where E^P is expectation under P. Z_T is called the *Radon-Nikodým derivative* of Q with respect to P. Define

$$Z_t = E_t^P\left(\frac{dQ}{dP}\right) \tag{1.10}$$

as the right-continuous version. Then Z is a uniformly integrable martingale and it is called the Radon-Nikodým derivative process or the *density process* of the change of measure.

Consider times t and s such that $0 \leq s \leq t \leq T$. The time s conditional expectation under Q of an $\mathcal{F}_t$-measurable X can be calculated as

$$E_s^Q(X) = \frac{1}{Z_s} E_s^P(XZ_t).$$

Theorem 1.12 (Girsanov's theorem)**.** *Let P and Q be equivalent. Let X be a semimartingale under P with decomposition $X = M^P + A^P$, where, under P, M^P is a local martingale and A^P is an FV process. Then X is also a semimartingale under Q and has a decomposition $X = M^Q + A^Q$, where*

$$M_t^Q = M_t^P - \int_0^t \frac{1}{Z_u} d\left[Z, M^P\right]_u$$

is a local martingale under Q, with Z being the density process from (1.10) and

$$A_t^Q = X_t - M_t^Q = A_t^P + \int_0^t \frac{1}{Z_u} d\left[Z, M^P\right]_u$$

is an FV process under Q.

Girsanov's theorem allows us to change from the objective probability measure P to the risk-neutral Q. Under Q, the martingale property of discounted prices can be used for asset pricing.

More generally, Girsanov's theorem can be used to determine how the dynamics of a process change when measure is changed. We demonstrate this use on the so-called stochastic logarithm of the stock price (cf. [28, p. 134]).

Definition 1.13 (Stochastic logarithm)**.** *For a real-valued semimartingale X, such that the two processes X_t and X_{t-} do not vanish, the stochastic logarithm $\mathcal{L}(X)$ is defined as $\mathcal{L}(X)_t = X_t/X_{t-}$ and its dynamics read*

$$d\mathcal{L}(X)_t = \frac{dX_t}{X_{t-}}.$$

Note that the dynamics of stochastic logarithm is analogous to the derivative of natural logarithm of a differentiable function $x(t)$ in standard calculus

$$\frac{d\ln\big(x(t)\big)}{dt} = \frac{1}{x(t)}\frac{dx(t)}{dt}.$$

Stochastic logarithm is often used to express the dynamics of processes such as geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dW_t$ which can be written as

$$d\mathcal{L}(S)_t = \frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

From Girsanov's theorem we can tell how these dynamics change when measure is changed and we formulate it in form in the following lemma.

Lemma 1.14. *Let the P-dynamics of $\mathcal{L}(S)_t$ be given by*

$$d\mathcal{L}(S)_t = b_P^{\mathcal{L}(S)} dt + \sqrt{c_P^{\mathcal{L}(S)}} dW_t.$$

Consider a change of measure to a new measure Q with the density Z_t given by (1.10). *The drift of $\mathcal{L}(S)_t$ under Q is then given by*

$$b_Q^{\mathcal{L}(S)} = b_P^{\mathcal{L}(S)} + \frac{1}{dt} d\big[\mathcal{L}(S), \mathcal{L}(Z)\big]$$

and its volatility remains unchanged, i.e. $c_Q^{\mathcal{L}(S)} = c_P^{\mathcal{L}(S)}$.

Proof. The lemma follows directly from Girsanov's theorem with S playing the role of X and taking $A_t^P = b_P^{\mathcal{L}(S)} t$ and $A_t^P = \sqrt{c_P^{\mathcal{L}(S)}} W_t$. □

Part I

Optimal liquidation

Introduction to Part I

The first part of this thesis is devoted to the problem of optimal liquidation of a trading position in presence of an adverse temporary price impact and it consists of Chapters 2 - 4.

Chapter 2 introduces the problem in the context of known optimal liquidation literature and presents the approach to its solution as proposed in our article Brunovský, Černý and Komadel [8]. We point out that the used formulation differs from most of optimal liquidation literature in giving the pressure to liquidate endogenously and using a stochastic time horizon. After introducing the problem and the initial value problem for a severely singular ODE to which the optimization problem leads, we formulate a series of related boundary value problems. The last of these BVPs, which we denote $\mathrm{BVP}^t_{[0,L]}$, is suitable for numerical treatment and we link its solutions $w(t,x)$ to the solutions $u_\infty(x)$ of the original IVP.

In Chapter 3 we propose a numerical scheme for $\mathrm{BVP}^t_{[0,L]}$ as well as a procedure of using this scheme to obtain a sufficiently accurate approximation of $u_\infty(x)$. Furthermore, we investigate the implementation shortfall and the time to liquidation, determined by the optimal liquidation strategy, for three sets of parameters corresponding to three market situations of interest. We find that our results agree with the square root law, known from empirical studies, which says that the price impact is proportional to the square root of the total trade size. Finally, we combine the numerical solutions with Monte Carlo simulations to get better insight into the dynamics of optimal liquidation. Chapter 4 concludes this part of the thesis.

Chapter 2

The optimal liquidation problem

The topic of optimal liquidation or optimal trade execution addresses the question of how to sell a given amount of an asset, maximizing the investor's utility from the sale, when the execution price is adversely affected by the sale. Alternatively, the theory can also be applied to the optimal purchase of an asset rather than its sale. The seminal papers in this field are by Bertsimas and Lo [4] and Almgren and Chriss [1]. In these works the asset price is assumed to be a martingale and the pressure to liquidate is exogenous. In particular, the authors use a fixed date by which the whole position must be liquidated.

We study the problem of optimal liquidation as it was first formulated by Černý [9] where the pressure to liquidate is given endogenously. It may be due to the asset price falling on average or due to time discounting and the setting rules out short sales which proves to be important. The combination of a bearish market and allowed short sales lead to a surprising result found in Schied [46], where the investor can gain from short selling the asset near the end of the time horizon and buying it later at a lower price. While this looks promising in theory, it may be problematic to find a counterparty for the transaction in practice and thus we find it important to examine the situation that does not allow such short sales.

Černý [9] showed that the problem reduces to solving a severely singular ordinary differential equation which was later treated analytically in Brunovský, Černý and Winkler [7] and in Quittner [43]. In the master's thesis [34] we developed a numerical procedure which appeared to converge to a solution of the original problem but we did not provide proofs that the found solution actually solves the ODE from [9] and [7] and, more importantly, that the solution of the ODE indeed is the value function of the liquidation problem. These questions are addressed in our paper Brunovský, Černý and Komadel [8]. We summarize the most important theoretical results in this chapter and we present numerical analysis and results in Chapter 3.

2.1 Formulation of the problem

We consider a problem of maximizing the expected revenue from liquidating a position $Z_0 = z > 0$ of an asset whose price is adversely affected by the amount being sold. The so called *unaffected price* process, i.e. the price prevailing in the market in absence of our trading, is given by the geometric Brownian motion

$$dS_t = \lambda S_t dt + \sigma S_t dW_t, \tag{2.1}$$

with an initial price $S_0 = s > 0$, and the amount of the asset yet to be sold Z_t has the dynamics

$$dZ_t = \big(rZ_t - v_t\big)dt, \tag{2.2}$$

where r is the growth rate of the asset and v_t is the selling rate. The objective function, which is to be maximized over selling strategies v, is

$$E\left(\int_0^{T(Z=0)} e^{-\rho t} v_t \big(S_t - \eta v_t\big)\, dt\right),$$

where ρ is the discount rate and $T(Z = 0)$ denotes the first time t such that $Z_t = 0$, i.e. the time when the whole amount is sold. The use of the stopping time $T(Z = 0)$ is a novel feature in optimal liquidation where the time horizon is typically given exogenously. Our approach rules out short sales once the inventory is disposed of but it leaves open the possibility of intermediate purchases. However, these turn out to be never optimal, as we show later.

Note that the price received by the investor is negatively affected by the amount being sold. This is represented by the term ηv_t by which the unaffected price S_t is reduced. This represents the instantaneous effect, or *temporary impact*, of selling on the price and it creates an incentive to sell at a low rate as opposed to the pressure to liquidate. Some authors (cf. [1, 4, 17, 20, 46]) also include a *permanent* *impact*, where not just the current selling rate but also the total amount sold upto current time deteriorates the selling price. The permanent impact is assumed to be zero in our model, which makes the model simpler, and it can be argued that its presence would not change the optimal strategy dramatically (cf. [8]).

We say that v is an admissible control, and write $v \in \mathcal{A}$, if process v is predictable,

$$E\left[\int_0^t |v_s|^m\, ds\right] < \infty \text{ for all } t > 0 \text{ and } m = 1, 2, \ldots, \tag{2.3}$$

and

$$E\left(\int_0^{T(Z=0)} e^{-\rho t} \big| v_t \big(S_t - \eta v_t\big) \big| dt\right) < \infty. \tag{2.4}$$

As shown in [9] or [34], the Hamilton-Jacobi-Bellman equation for the value function

$$V(s,z) = \sup_{v \in \mathcal{A}} E\left(\int_0^{T(Z=0)} e^{-\rho t} v_t \big(S_t - \eta v_t\big) dt\right) \tag{2.5}$$

subject to the dynamics (2.1), (2.2) is

$$\sup_{v \in \mathcal{A}} \left\{ v(s - \eta v) + \frac{1}{2} s^2 \sigma^2 V_{ss} + \lambda s V_s + (rz - v) V_z - \rho V \right\} = 0.$$

The formal optimal control

$$v^* = \frac{s - V_z}{2\eta}$$

leads to the partial differential equation

$$\frac{1}{2} s^2 \sigma^2 V_{ss} + \lambda s V_s + rz V_z - \rho V + \frac{(s - V_z)^2}{4\eta} = 0 \tag{2.6}$$

for $s > 0$, $z > 0$, with the condition $V(s, 0) = 0$ corresponding to the revenue being zero whenever the amount of the asset is $z = 0$, regardless of the price.

Employing the scaling

$$V(s,z) = \frac{s^2}{\eta \sigma^2} u(x), \quad x = \eta \sigma^2 \frac{z}{s} \tag{2.7}$$

the PDE (2.6) can be reduced to the ordinary differential equation for $x > 0$

$$x^2 u'' = a x u' + b u - \frac{1}{2}(u' - 1)^2, \tag{2.8}$$

$$u(0) = 0, \tag{2.9}$$

where we define

$$\begin{aligned} a &= \frac{2}{\sigma^2}\left(\lambda - r + \sigma^2\right), \\ b &= -\frac{2}{\sigma^2}\left(2\lambda - \rho + \sigma^2\right). \end{aligned} \tag{2.10}$$

We make the assumption throughout that $a + b > 0$ which translates to

$$\rho > \lambda + r, \tag{2.11}$$

creating the endogenous pressure to liquidate. The interpretation is that the discount rate ρ is higher than the average growth of the capital invested in the asset, $\lambda + r$, so that the investor has an incentive to sell the asset. Were it not satisfied, the investor could profit from postponing the sale infinitely. Note that the assumption is satisfied in two important cases where the pressure to liquidate is solely due to:

(i) the asset price falling on average ($\lambda < 0$, $\rho = 0$, $r = 0$);

(ii) time discounting ($\lambda = 0$, $\rho > 0$, $r = 0$).

2.2 Optimality

In this section we address the question of whether the solution u of the initial value problem (2.8), (2.9) indeed defines the value function V of the optimal liquidation problem (2.5) since the Hamilton-Jacobi-Bellman equation is in general only a necessary condition. In addition we examine the optimal liquidation strategy which we compare to a myopic strategy. We finish the section by studying the resulting implementation shortfall which we connect to the square root law observed in empirical literature.

2.2.1 Initial value problem IVP$_0$

We denote the initial value problem (2.8), (2.9)

$$\begin{aligned} &x^2 u'' = axu' + bu - \frac{1}{2}(u' - 1)^2, \\ &u(0) = 0, \end{aligned} \tag{IVP$_0$}$$

by IVP_0, the 0 signifying that the initial condition is set at $x = 0$. This problem is studied in Brunovský, Černý and Winkler [7] and in Quittner [43] where its severe singularity is shown.

For $a + b > 0$ IVP_0 has infinitely many solutions with identical asymptotics near 0 given by the formal power series

$$h_n(x) = \sum_{i=0}^{n} k_i x^{1+i/2},\ n \in \mathbb{N}, \tag{2.12}$$

with $k_0 = 1$, $k_1 = -\frac{2}{3}\sqrt{2(a+b)}$ and the other k_i obtained recursively by

$$k_{n+1} = \frac{1}{3\,(n+3)\,k_1}\left[k_n\Big(\,(n+2)\,(2a-n)+4b\Big) - \frac{1}{2}\sum_{j=1}^{n-1}(3+j)(n-j+3)k_{j+1}k_{n-j+1}\right].$$

The highly degenerate nature of IVP_0 does not stem from the singularity of the linear terms in the ODE, which is well known and rather innocuous in the context of the Black-Scholes model, but from the singularity of the non-linear term. Liang [36] studies singular IVPs of the form $u'' = \frac{1}{x}f(x,u,u')$ where f is continuous on $[0,\infty) \times \mathbb{R} \times \mathbb{R}$. In case of IVP_0 function f is given by

$$f(x,u,u') = au' + \frac{2bu - (u'-1)^2}{2x}$$

which disqualifies IVP_0 from Liang's category.

Brunovský, Černý and Winkler [7, Proposition 5.1] show that IVP_0 does, however, have a unique solution under additional conditions and they also prove certain characteristics of the solution. We summarize these results in the following proposition [8, Proposition 5.1].

Proposition 2.1. *Under the assumption $a+b > 0$ there is a unique solution of IVP_0 denoted by u_∞ satisfying $u_\infty \in \mathcal{C}^0[0,\infty) \times \mathcal{C}^2(0,\infty)$ and*

$$0 \leq u_\infty(x) \leq x \textit{ for } x > 0.$$

The solution u_∞ further satisfies $u_\infty'(0) = 1$, $u_\infty'(x) > 0, u_\infty''(x) < 0$, $u_\infty'''(x) > 0$ for all

$x > 0$ *as well as* $u'_\infty(x) \searrow 0$ *for* $x \to \infty$.

2.2.2 Boundary value problem $\mathbf{BVP}_{[0,\infty)}$

Inspired by Proposition 2.1 we add to IVP_0 the boundary condition

$$u'(\infty) = 0 \tag{2.13}$$

where we write $u'(\infty) = \lim_{x\to\infty} u'(x)$ whenever the limit on the right-hand side exists. This defines the boundary value problem given by (2.8), (2.9) and (2.13)

$$\begin{aligned} &x^2u'' = axu' + bu - \frac{1}{2}(u'-1)^2, \\ &u(0) = 0, \qquad u'(\infty) = 0, \end{aligned} \tag{$\mathrm{BVP}_{[0,\infty)}$}$$

which we denote by $\mathrm{BVP}_{[0,\infty)}$. In [8, Proposition 6.1] we show that the addition of the boundary condition (2.13) has the effect of uniquely determining the solution of IVP_0 established in Proposition 2.1, i.e. u_∞ from Proposition 2.1 is the unique solution $\mathrm{BVP}_{[0,\infty)}$.

The next theorem, proven in [8, Theorem 7.2], confirms that by solving $\mathrm{BVP}_{[0,\infty)}$ and subsequent use of scaling (2.7) we indeed find the value function of the optimal liquidation problem. Furthermore, the theorem characterizes the optimal strategy v^* which turns out to be nonnegative due to $0 \le u'_\infty(x) \le 1$. This means that it is never optimal to acquire more of the asset even though our setting only rules out short sales and intermediate purchases are admissible.

Theorem 2.2. *Assume* (2.11). *Let* u_∞ *be the unique solution of* $BVP_{[0,\infty)}$, *with* a, b *given by* (2.10). *Then the function* $V(s,z) = \frac{s^2}{\eta\sigma^2} u_\infty\left(\eta\sigma^2 \frac{z}{s}\right) \le sz$ *is the value function of the optimization* (2.5) *and*

$$v_t^* = \frac{1}{2\eta}\Big(S_t - V_z\left(S_t, Z_t^*\right)\Big) = \frac{S_t}{2\eta}\left(1 - u'_\infty\left(\eta\sigma^2\frac{Z_t^*}{S_t}\right)\right) \ge 0 \tag{2.14}$$

is the optimal control among all admissible controls $\mathcal{A}$ *defined in* (2.3), (2.4).

Note that one can compare the optimal strategy v^* in (2.14) to a myopic strategy $v^{myopic}(t) = S_t/2\eta$ which maximizes the integrand $e^{-\rho t}v_t\big(S_t - \eta v_t\big)$ in the objective function (2.5). Unlike v^{myopic}, the optimal strategy v^* accounts for the fact that the current selling rate affects future inventory and thus also the revenue from its sale. In particular one has

$$v_t^* = v_t^{myopic} - \frac{1}{2\eta} V_z\left(S_t, Z_t^*\right)$$

where $V_z\left(S_t, Z_t^*\right)$ is the marginal value of the optimal revenue with respect to the size of the remaining inventory.

Furthermore, by Proposition 2.1 $u'_\infty(x)$ decreases from 1 to 0 as increases from 0 to infinity. Using this in (2.14), one can conclude that for large values of the inventory Z_t^*, v^* is close to v^{myopic}. On the other hand, as Z_t^* approaches zero, the optimal liquidation rate is significantly lower than the myopic one. Using the asymptotic expansion (2.12) one can express an expansion for the derivative

$$u'_\infty(x) = 1 - \sqrt{2(a+b)x} + \mathcal{O}(x)$$

which, combined with (2.14), yields that the optimal strategy v^* is roughly proportional to $\sqrt{Z_t^*}$.

Another observation from Theorem 2.2 is that for the value function one has $V(s,z) \leq sz$. The value function describes the optimal expected revenue from the liquidation of z units of the asset when the initial price is s. The expression sz thus gives the revenue the investor would receive if it were possible to sell the whole inventory immediately without any effect on the price. The difference $sz - V(s,z)$ is thus the shortfall resulting from the fact that the selling rate negatively affects the selling price and it is called *implementation shortfall* in literature (cf. [41]). According to Theorem 2.2 the implementation shortfall is nonnegative in our setting which rules out short sales. This agrees with intuition and differs from results of [46] where the investor can benefit from short sales in a bearish market (see the beginning of Chapter 2).

We rewrite the asymptotic expansion (2.12) as

$$u_\infty(x) = x - \frac{2}{3}\sqrt{2(a+b)}x^{3/2} + \mathcal{O}(x^2)$$

to express the asymptotic relative implementation shortfall for small values of the inventory z

$$I(s,z) = \frac{sz - V(s,z)}{sz} = \frac{4}{3}\sqrt{\eta(\rho - \lambda - r)\frac{z}{s}} + \mathcal{O}(z). \tag{2.15}$$

Since sz is the revenue from an immediate sale of the whole inventory z at the current price s,

$$I(s,z) = \frac{s - \frac{V(s,z)}{z}}{s}$$

can also be interpreted as the relative difference between the initial price s and the average realized price $V(s,z)/z$. For this reason $I(s,z)$ is often referred to in empirical literature as the *price impact*. The result (2.15) agrees with the square root law, observed in empirical studies such as [15, 16, 52], which says that the price impact is proportional to the square root of the total trade size z.

2.3 Computation of the solution

By adding a second boundary condition for $x \to \infty$ to IVP$_0$ we formed problem BVP$_{[0,\infty)}$ and we have shown that its solution $u_\infty(x)$ defines the value function $V(s,z)$ of the optimization (2.5) as well as the optimal liquidation strategy v_t^*. However, we still need to address the question of how $u_\infty(x)$ can be calculated. In this section we describe a procedure which enables us to do so.

2.3.1 Truncated problem BVP$_{[0,L]}$

The first step is to truncate x to a finite interval $[0,L]$, $L<\infty$, defining the boundary value problem BVP$_{[0,L]}$ which consists of IVP$_0$ accompanied by the condition $u'(L) = 0$, i.e.

$$\begin{aligned} x^2u'' &= axu' + bu - \frac{1}{2}(u'-1)^2, \\ u(0) &= 0, \qquad u'(L) = 0. \end{aligned} \tag{BVP$_{[0,L]}$}$$

One can view BVP$_{[0,L]}$ as a modification of BVP$_{[0,\infty)}$ where we still impose a zero derivative condition at the end of the spatial interval but the interval is now finite.

We formulate a theorem from [8, Theorem 8.1] where it is shown that BVP$_{[0,L]}$ has a unique solution $u_L(x)$ which tends pointwisely to $u_\infty(x)$, the solution of BVP$_{[0,\infty)}$, as $L \to \infty$. This justifies looking for $u_\infty(x)$ by solving the truncated problem BVP$_{[0,L]}$ with L sufficiently large.

Theorem 2.3. *Let $a+b>0$. For given $L>0$, BVP$_{[0,L]}$ has a unique solution $u_L \in C^2\big((0,L]\big) \cap C^0\big([0,L]\big)$ such that $0 \le u_L(x) \le x$ for all $x \in [0,L]$. The solution*

u_L is strictly increasing, concave and satisfies $u_{L_1}(x) \leq u_{L_2}(x)$ for $L_1 \leq L_2,\ 0 \leq x \leq L_1$. Furthermore, $\lim_{L\to\infty} u_L(x) = u_\infty(x)$ for $0 \leq x < \infty$, where u_∞ is the unique solution of BVP$_{[0,\infty)}$.

Literature studies numerical methods for boundary value problems of similar type as BVP$_{[0,L]}$ (cf. [29, 33, 55, 56]). The closest formulation to BVP$_{[0,L]}$ we were able to find was in [56] which studies problems with singular coefficients of the form

$$u''(x) = \frac{A_1}{x}u'(x) + \frac{A_0}{x^2}u(x) + f\big(x, u(x), u'(x)\big),$$

where one of the boundaries is at $x = 0$ and f is assumed to be continuous at this point. However, in case of BVP$_{[0,L]}$ the nonlinear term is given by $f(x, u, u') = \frac{(u'-1)^2}{2x^2}$ so the assumption of continuity is not met. The singularity of BVP$_{[0,L]}$, caused by the nonlinear term, is more severe than singularities typically considered and thus standard methods for numerical treatment of BVPs fail in this case.

As suggested in [2, Eq. (1.3)], second-order problems of the type of BVP$_{[0,L]}$ can be transformed to two-dimensional first-order problems. In particular, ODE (2.8) can be written in the form

$$xy'(x) = My(x) - \tilde{f}\big(x, y(x)\big), \tag{2.16}$$

where

$$y(x) = \begin{pmatrix} u(x) \\ xu'(x) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ b & 1+a \end{pmatrix}, \quad \tilde{f}\big(x, y(x)\big) = \begin{pmatrix} 0 \\ \frac{1}{2}\left(\frac{y_2(x)}{x} - 1\right)^2 \end{pmatrix}$$

and $y_2(x) = xu'(x)$ is the second element of $y(x)$. Using the form (2.16) for the ODE, one can use the Matlab function bvp5c to compute numerical solutions of an augmented version of BVP$_{[0,L]}$ which starts at $x = \varepsilon > 0$

$$\begin{aligned} x^2u'' &= axu' + bu - \frac{1}{2}(u'-1)^2, \\ u(\varepsilon) &= 0, \qquad u'(L) = 0. \end{aligned} \tag{BVP$_{[\varepsilon,L]}$}$$

Setting ε far enough from zero, the solutions found by bvp5c are stable but they do not agree with the asymptotics (2.12) near zero, as we show in Figure 3.3 in the following chapter.

To overcome the problem with the severe singularity it turns out to be advantageous to introduce a time dimension into the optimal liquidation problem (2.5) which corresponds to setting a finite time horizon. This results in finite horizon problems $\mathrm{BVP}^t_{[0,L]}$ which are examined in the follwoing subsection.

2.3.2 Finite horizon problems $\mathrm{BVP}^t_{[0,L]}$

By introducing a time variable into the optimal liquidation problem one obtains instead of the ODE (2.8) a parabolic partial differential equation

$$w_t = x^2 w_{xx} - axw_x - bw + \frac{1}{2}(w_x - 1)^2 \tag{2.17}$$

for the function $w(t,x)$. For the PDE (2.17) we define on $[0,\infty) \times [0,L]$ the boundary value problem $\underline{\mathrm{BVP}}^t_{[0,L]}$ by

$$\begin{gathered} w_t = x^2 w_{xx} - axw_x - bw + \frac{1}{2}(w_x - 1)^2, \\ w(t,0) = 0, \qquad w_x(t,L) = 0, \qquad w(0,x) = 0. \end{gathered} \tag{$\underline{\mathrm{BVP}}^t_{[0,L]}$}$$

In addition, we define a second problem, $\overline{\mathrm{BVP}}^t_{[0,L]}$, by replacing the zero initial condition with $w(0,x) = x$

$$\begin{gathered} w_t = x^2 w_{xx} - axw_x - bw + \frac{1}{2}(w_x - 1)^2, \\ w(t,0) = 0, \qquad w_x(t,L) = 0, \qquad w(0,x) = x. \end{gathered} \tag{$\overline{\mathrm{BVP}}^t_{[0,L]}$}$$

By $\mathrm{BVP}^t_{[0,L]}$ we refer to either of the problems $\underline{\mathrm{BVP}}^t_{[0,L]}$ and $\overline{\mathrm{BVP}}^t_{[0,L]}$.

Because of the spatial variable x being limited to a finite interval, problems $\underline{\mathrm{BVP}}^t_{[0,L]}$ and $\overline{\mathrm{BVP}}^t_{[0,L]}$ do not correspond to any optimal control problems. In [8, Theorem 8.2], which we formulate in the following theorem, we show, however, that as $t \to \infty$ the solutions of $\underline{\mathrm{BVP}}^t_{[0,L]}$ and $\overline{\mathrm{BVP}}^t_{[0,L]}$ tend to the solution of the time homogenous problem $\mathrm{BVP}_{[0,L]}$ monotonically from below, resp. from above, see Figure 3.2 in the following chapter.

Theorem 2.4. *For given L the problems $\underline{\mathrm{BVP}}^t_{[0,L]}$ and $\overline{\mathrm{BVP}}^t_{[0,L]}$ have a unique solution in $\mathcal{C}^{1,2}\big((0,\infty)\times(0,L]\big)\cap\mathcal{C}\big([0,\infty)\times[0,L]\big)$. These solutions, denoted by $\underline{w}$ and $\overline{w}$ respectively, satisfy*

Table 2.1: Computation of the value function from the finite horizon problems.

$\underline{\mathrm{BVP}}^t_{[0,L]}$ or $\overline{\mathrm{BVP}}^t_{[0,L]}$	$\xrightarrow[t\to\infty]{}$	$\mathrm{BVP}_{[0,L]}$	$\xrightarrow[L\to\infty]{}$	$\mathrm{BVP}_{[0,\infty)}$	$\xrightarrow[(2.7)]{}$	(2.5)
$\underline{w}(t,x)$ or $\overline{w}(t,x)$	$\xrightarrow[t\to\infty]{}$	$u_L(x)$	$\xrightarrow[L\to\infty]{}$	$u_\infty(x)$	$\xrightarrow[(2.7)]{}$	$V(s,z)$

$$\begin{array}{ccccc} 0 \leq \underline{w}(t,x) & \leq & u_L(x) & \leq & \overline{w}(t,x) \leq x \\ \dfrac{\partial \overline{w}(t,x)}{\partial t} & \leq & 0 & \leq & \dfrac{\partial \underline{w}(t,x)}{\partial t} \end{array}$$

and $\lim_{t\to\infty} \overline{w}(t,x) = \lim_{t\to\infty} \underline{w}(t,x) = u_L(x)$.

2.3.3 Summary of the computational procedure

The finite horizon problems $\underline{\mathrm{BVP}}^t_{[0,L]}$ and $\overline{\mathrm{BVP}}^t_{[0,L]}$ are numerically well behaved and the singularity at $x = 0$ no longer causes problems, as we demonstrate in the next chapter. We already observed this in the master's thesis [34] where we formulated the finite horizon problems, solved them numerically and showed that their solutions $\underline{w}(t,x)$ and $\overline{w}(t,x)$ appear to converge to the same limit. Now we have proven analytically how these finite horizon problems relate to the original optimal liquidation problem and the connection can be summarized in the following way:

1. Based on Theorem 2.4 the solutions of the finite horizon problems $\mathrm{BVP}^t_{[0,L]}$ get arbitrarily close to the solution of the truncated problem $\mathrm{BVP}_{[0,L]}$ if t is large enough.

2. Based on Theorem 2.3 the solution of the truncated problem $\mathrm{BVP}_{[0,L]}$ gets arbitrarily close to the solution of the solution of $\mathrm{BVP}_{[0,\infty)}$ if L is large enough.

3. Based on Theorem 2.2 the solution of $\mathrm{BVP}_{[0,\infty)}$ determines the value function $V(s,z)$ of the optimization (2.5) through the scaling (2.7) and the optimal liquidation strategy v^*_t through (2.14).

This procedure, which connects the numerically amenable finite horizon problems to value function (2.5) and which will be used in numerical computations, is schematically written in Table 2.1.

Chapter 3

Numerical results

In his chapter we describe the numerical procedure which used to solve the optimal liquidation problem described in Chapter 2. First, we propose a numerical scheme for the finite horizon problems $\mathrm{BVP}^t_{[0,L]}$. Then we describe a procedure used to obtain a sufficiently precise approximation of u_∞ from $\underline{w}(t,x)$ or $\overline{w}(t,x)$, in line with Table 2.1. In the last section of this chapter we use the numerical approximation of u_∞ to investigate the optimal liquidation problem by studying the relative implementation shortfall and the time to liquidation.

3.1 Solving $\mathrm{BVP}^t_{[0,L]}$

Recall that the finite horizon problems $\underline{\mathrm{BVP}}^t_{[0,L]}$ and $\overline{\mathrm{BVP}}^t_{[0,L]}$ are given by the PDE (2.17)

$$w_t = x^2 w_{xx} - axw_x - bw + \frac{1}{2}(w_x - 1)^2$$

which we now treat numerically. For the spatial variable $x \in [0, L]$ we employ a non-equidistant partition $0 = x_0 < x_1 < \cdots < x_N = L$, where the partition points are defined as

$$x_j = e^{\xi_j} - 1 - \xi_j + \xi_j^{3/2}, \qquad j = 0, 1, \ldots, N \tag{3.1}$$

with $\{\xi_j\}_{j=0}^N$ being equidistant, i.e. $\xi_j = jL^*/N$ with L^* such that $x_N = e^{L^*} - 1 - L^* + (L^*)^{3/2} = L$. Figure 3.1 illustrates how the used non-equidistant partition (3.1) is finer for small values of x and coarser for larger values. This is advantageous in capturing the development of function w (and consequently u) near the singularity at $x = 0$ while saving memory by not using unnecessarily many partition points further away from the singularity.

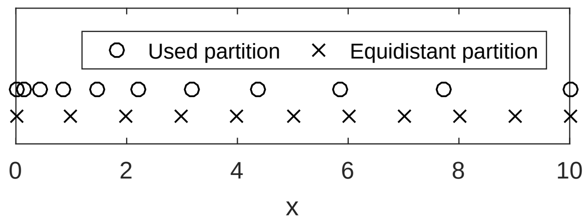


Figure 3.1: Comparison of partition (3.1) to an equidistant partition of the interval $[0, 10]$ with $N + 1 = 11$ partition points.

For the time variable $t \in [0, T]$ we use an equidistant partition with the time step $h = T/M$ so the partition points are $t_i = ih$, $i = 0, 1, \dots, M$. Denoting by w^i_j the numerical approximation of $w(t_i, x_j)$ we approximate for $i = 0, 1, \dots, M$ and $j = 1, 2, \dots, (N-1)$ the first spatial derivative of w by the central difference

$$\frac{\partial w(t_i, x_j)}{\partial x} \approx \frac{w^i_{j+1} - w^i_{j-1}}{x_{j+1} - x_{j-1}},$$

and the second spatial derivative by the difference

$$\frac{\partial^2 w(t_i, x_j)}{\partial x^2} \approx \frac{\frac{w^i_{j+1} - w^i_j}{x_{j+1} - x_j} - \frac{w^i_j - w^i_{j-1}}{x_j - x_{j-1}}}{\frac{x_{j+1} - x_{j-1}}{2}}.$$

The time derivative is approximated by the forward difference

$$\frac{\partial w(t_i, x_j)}{\partial x} \approx \frac{w^{i+1}_j - w^i_j}{h}$$

for $i = 0, 1, \dots, (M-1)$ and $j = 0, 1, \dots, N$. In this notation the explicit Euler scheme for PDE (2.17) reads

$$\begin{aligned} w^{i+1}_j = w^i_j + h \Bigg[& \frac{2\, x_j^2}{x_{j+1} - x_{j-1}} \left(\frac{w^i_{j+1} - w^i_j}{x_{j+1} - x_j} - \frac{w^i_j - w^i_{j-1}}{x_j - x_{j-1}} \right) \\ & - a\, x_j \frac{w^i_{j+1} - w^i_{j-1}}{x_{j+1} - x_{j-1}} - b\, w^i_j + \frac{1}{2} \left(\frac{w^i_{j+1} - w^i_{j-1}}{x_{j+1} - x_{j-1}} - 1 \right)^2 \Bigg] \end{aligned} \tag{3.2}$$

for $i = 0, 1, \dots, (M-1)$ and $j = 1, 2, \dots, (N-1)$.

The boundary conditions, given for both $\underline{\mathrm{BVP}}^t_{[0,L]}$ and $\overline{\mathrm{BVP}}^t_{[0,L]}$ by $w(t, 0) = 0$ and

$w_x(t, L) = 0$, dictate

$$w_0^i = 0, \qquad w_N^i = w_{N-1}^i, \qquad i = 0, 1, \ldots, M, \tag{3.3}$$

while $\underline{\text{BVP}}^t_{[0,L]}$ has the zero initial condition

$$w_j^0 = 0, \qquad j = 0, 1, \ldots, N \tag{3.4}$$

and $\overline{\text{BVP}}^t_{[0,L]}$ has

$$w_j^0 = x_j, \qquad j = 0, 1, \ldots, N. \tag{3.5}$$

Denote by $W^i = \left(w_1^i, w_2^1, \ldots, w_{N-1}^i\right)^T$ the vector of interior points at the i-th time layer and by $\tilde{W}^i = (w_0^i, w_1^i, \ldots, w_N^i)^T$ the complete vector of all points at time layer t_i. The explicit scheme (3.2) can then be written as

$$W^{i+1} = W^i + h\left[A\tilde{W}^i + F\left(\tilde{W}^i\right)\right], \tag{3.6}$$

where matrix $A \in \mathbb{R}^{(N-1)\times(N+1)}$ is tridiagonal with non-zero elements given by

$$\begin{aligned}
A_{j,j-1} &= \frac{x_j}{x_{j+1} - x_{j-1}}\left(\frac{2\,x_j}{x_j - x_{j-1}} + a\right), \\
A_{j,j} &= -\frac{2\,x_j^2}{x_{j+1} - x_{j-1}}\left(\frac{1}{x_{j+1} - x_j} + \frac{1}{x_j - x_{j-1}}\right) - b, \\
A_{j,j+1} &= \frac{x_j}{x_{j+1} - x_{j-1}}\left(\frac{2\,x_j}{x_j - x_{j-1}} - a\right),
\end{aligned}$$

for $j = 1, 2, \ldots, (N-1)$, and the non-linear term F is given by

$$F\left(\tilde{W}^i\right) = \frac{1}{2}\left[\left(\frac{w_2^i - w_0^i}{x_2 - x_0} - 1\right)^2, \ldots, \left(\frac{w_{j+1}^i - w_{j-1}^i}{x_{j+1} - x_{j-1}} - 1\right)^2, \ldots, \left(\frac{w_N^i - w_{N-2}^i}{x_N - x_{N-2}} - 1\right)^2\right]^T.$$

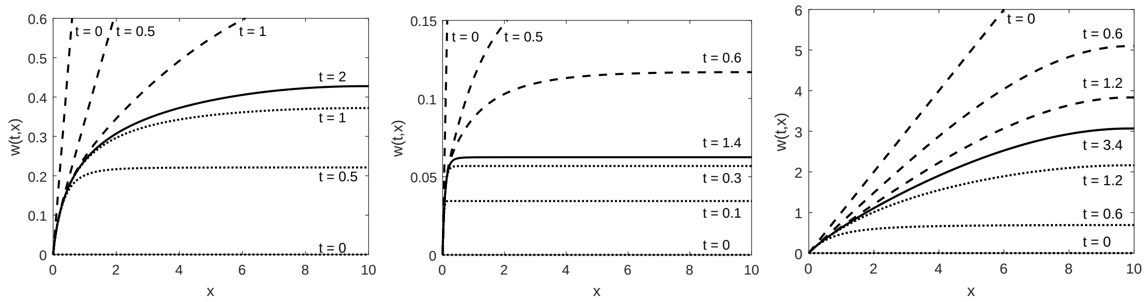
Starting from the initial time layer given by the initial condition (3.4) (or (3.5) in case of $\overline{\text{BVP}}^t_{[0,L]}$) and given L, N and time step h we are able to use (3.6), (3.3) to calculate an approximation of a new time layer $w(t_{i+1}, x)$ from the last known time layer $w(t_i, x)$.

As claimed by Theorem 2.4 the solutions of $\underline{\text{BVP}}^t_{[0,L]}$ and $\overline{\text{BVP}}^t_{[0,L]}$ should monotonically converge to u_L, the solution of $\text{BVP}_{[0,L]}$, from below and above, respectively, as t increases. This can be observed in Figure 3.2 for the three sets of parameters listed

Table 3.1: Parameter values used in numerical examples.

	σ	λ	r	ρ	a	b
Parametrization 1	0.2	0	0	0.05	2	0.5
Parametrization 2	0.2	-0.1	0	0	-3	8
Parametrization 3	0.2	0.03	0.01	0.05	3	-2.5

in Table 3.1.



(a) Parametrization 1. **(b)** Parametrization 2. **(c)** Parametrization 3.

Figure 3.2: Solutions of $\underline{\mathrm{BVP}}^t_{[0,L]}$ (dotted) and $\overline{\mathrm{BVP}}^t_{[0,L]}$ (dashed) for $L = 10$ and different values of t. The solid lines represent the solution of BVP$_{[0,L]}$.

The parameter sets used numerical examples and listed in Table 3.1 were chosen to describe three interesting cases from financial perspective. The first parametrization uses $\lambda = r = 0$ and the pressure to liquidate is solely due to time discounting by the factor $\rho = 0.05$. In parametrization 2 we set $r = \rho = 0$ and the pressure to liquidate is solely due to the unaffected asset price drifting downwards by $\lambda = -0.1$. In the third parametrization we demonstrate a situation where all parameters are positive. Note that in all three case the assumption (2.11) creating the endogenous pressure to liquidate is satisfied. Furthermore, the three parameter sets cover the three possible sign combinations of parameters a and b which allow for $a + b > 0$ to hold.

3.2 Solving BVP$_{[0,\infty)}$

In this section we describe how we use the numerical method of solving the finite horizon problems BVP$^t_{[0,L]}$, described in the previous section, to obtain a sufficiently precise approximation of u_∞, the solution of BVP$_{[0,\infty)}$. As shown in Table 2.1, this can be achieved by increasing t and L.

Our goal is to achieve sufficient precision of $u_\infty(x)$ on the interval $[0, 1]$. The procedure uses four nested loops and we describe them from the innermost one moving outwards.

Determining T For a given time step h, length of the spatial interval L and number of x-partition points N we find new time layers by using (3.6), (3.3). The number of time steps M (and thus also the time horizon $T = Mh$) is determined by considering two time layers for $T_1 < T_2$ for $i = 1, 2$ and using stopping criteria based on changes in the relative implementation shortfall. Denote the numerical solutions corresponding to the two time layers by $u_i(x) = w(T_i, x)$, $i = 1, 2$.

Recall that we defined relative implementation shortfall or the price impact in (2.15) as $I(s, z) = \big(sz - V(s, z)\big)/sz$ which can be rewritten using the scaling (2.7) as

$$I(s,z) = \frac{sz - \frac{s^2}{\eta\sigma^2} u\left(\eta\sigma^2 \frac{z}{s}\right)}{sz} = 1 - \frac{u(x)}{x}. \tag{3.7}$$

We use this to express the relative implementation shortfall $f_i(x) = 1 - u_i(x)/x$ for the two time layers and we distinguish between two regions for x: $\mathcal{X} = \big\{x > 0 : f_2(x) \leq 0.01\big\}$ and its complement in $[0, 10]$ denoted by $\mathcal{X}^c$.

For small x we aim for a low relative difference in the shortfall

$$\sup_{x\in\mathcal{X}} \left|1 - \frac{f_2(x)}{f_1(x)}\right| \leq 0.1 \tag{3.8}$$

while for larger x we aim for low absolute difference

$$\sup_{x\in\mathcal{X}^c} |f_2(x) - f_1(x)| \leq 10^{-4}. \tag{3.9}$$

In the innermost loop we start with $T_1 = 0.1$, $T_2 = 0.2$ and increase T_i by 0.1 until conditions (3.8) and (3.9) are satisfied.

Determining N One level up, for given h and L, we determine the number of partition points for the spatial interval N. We start with $N_1 = 10$, $N_2 = 20$ and denote the corresponding solutions from the innermost loop by u_1 and u_2. Then we increase N_i by 10 until conditions (3.8) and (3.9) are satisfied. Thus, we obtain an approximation of $u_L(x)$ for a given L.

When moving to a larger N, i.e. moving to a finer partition, we use the previously calculated solution, interpolated by cubic spline to match the finer grid, as an initial condition to improve computational efficiency.

Determining L Two levels up, for a given h we determine the length of the spatial interval L. We start with $L_1 = 1$, $L_2 = 1.1$ and denote the corresponding solutions from the previous loop by u_1 and u_2. Then we increase L_i by 0.1 until conditions (3.8) and (3.9) are satisfied.

When moving from a smaller L_1 to a larger L_2 we again improve computational efficiency by using a previously calculated solution u_{L_1}, extended constantly to $[0, L_2]$, as the initial condition.

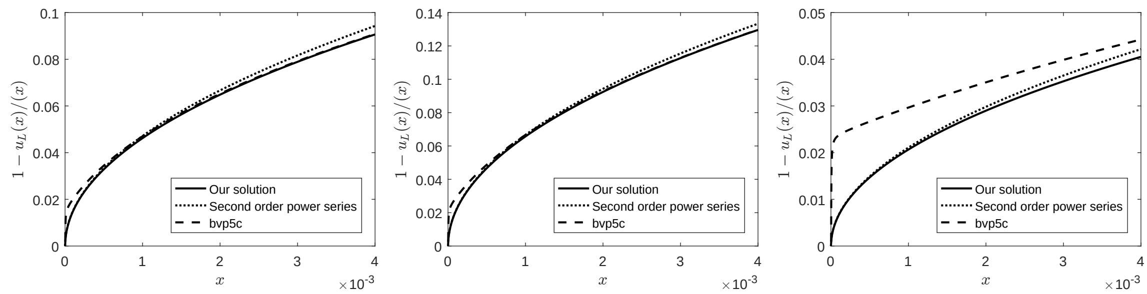
Determining h In the outermost loop we determine the time step h so that it is small enough not to affect the solution. We start by setting $h_1 = 10^{-5}$, $h_2 = 0.5 \times 10^{-5}$ and denote the corresponding solutions from the previous loop by u_1 and u_2. Then we halve h_i until conditions (3.8) and (3.9) are satisfied.

This procedure, excluding the loop for L, was also used to determine the solutions u_L shown in Figure 3.2. We observe that the convergence occurs for different values of T for the three parameter sets. Moreover, while for parametrizations 1 and 3 the convergence of $\underline{w}(t, x)$ and $\overline{w}(t, x)$ to $u_L(x)$ is similar, for parametrization 2 the convergence occurs sooner for $\underline{w}(t, x)$ than for $\overline{w}(t, x)$. Conditions (3.8) and (3.9) were satisfied for $T = 0.9$ in case of $\underline{w}(t, x)$ while $T = 1.4$ was need in case of $\overline{w}(t, x)$.

Figure 3.3 shows the approximate relative implementation shortfall $I(s, z) = 1 - u_L(x)/(x)$ given by the solutions u_L of $\mathrm{BVP}_{[0,L]}$ in close neighborhood of zero for the three sets of parameters listed in Table 3.1. Our solutions are compared to the second-order asymptotic power series $h_2(x) = x - \frac{2}{3}\sqrt{2(a+b)}x^{3/2}$ defined in (2.12) and to the solution obtained from the Matlab routine `bvp5c`. For all three cases our method outperforms `bvp5c` in capturing the dynamics of the solution in proximity of the singularity at $x = 0$. This is most apparent for the third parametrization.

3.3 Results for the optimal liquidation problem

Having established a procedure to calculate the solution of $\mathrm{BVP}_{[0,\infty)}$ with sufficient accuracy, we can examine the implications for the optimal liquidation problem. Re-



(a) Parametrization 1. **(b)** Parametrization 2. **(c)** Parametrization 3.

Figure 3.3: Comparison of $\mathrm{BVP}_{[0,L]}$ solution to solution from Matlab routine bvp5c near the singularity at $x = 0$. The displayed quantity $1 - u_L(x)/(x)$ represents approximate implementation shortfall.

call that u_∞ defines the value function $V(s, z)$ through (2.7) as

$$V(s,z) = \frac{s^2}{\eta\sigma^2} u_\infty(x) = sz\frac{u_\infty(x)}{x},$$
$$x = \eta\sigma^2 \frac{z}{s}.$$

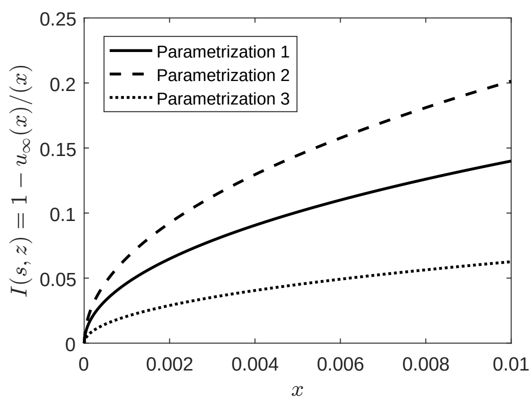
In panel (a) of Figure 3.4 we plot the relative implementation shortfall or the price impact $I(s, z)$ which we introduced in (2.15) and from (3.7) it can be written in terms of u_∞ as

$$I(s,z) = 1 - \frac{u_\infty(x)}{x}.$$

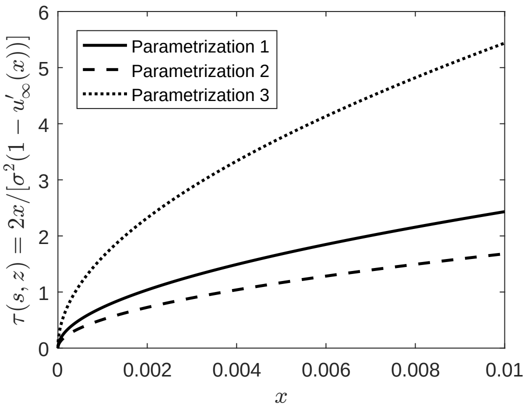
For a given asset, i.e. for fixed initial price s and volatility σ, and a fixed value of the temporary impact parameter η, the reduced variable x is proportional to the order size z. From Figure 3.4 (a) $I(s, z)$ is increasing for all three parameter sets which confirms the intuition that selling a larger amount z of the asset leads to a larger drop in the average per-share price received, relative to the initial price s.

The value of η, which measures the strength of the temporary impact of liquidation on the selling price, can be estimated from from empirical studies such as [6] or [25]. The authors in [6] study the sale of 1 000 shares in a 5-minute window and estimate the price impact at around 0.18 % of the unaffected price. If we set the initial price to $s = 100$, interpret the variable z, measuring the inventory, in thousands and assume that there are $250 \times 8 \times 60 = 120\,000$ trading minutes in a year, their result implies the value

$$\eta = 0.0018 \times s \times \frac{5}{250 \times 8 \times 60} = 7.5 \times 10^{-6}.$$



(a) $I(s,z)$.



(b) $\tau(s,z)$.

Figure 3.4: Relative implementation shortfall $I(s,z)$ and time to liquidation assuming constant liquidation speed and no accruing interest $\tau(s,z)$, for three parametrizations in Table 3.1.

Using this value of η, the initial asset price $s = 100$ and volatility of $\sigma = 0.2$, which we assume in all our parametrizations in Table 3.1, the range of x shown in Figure 3.4 of $[0, 0.01]$ translates to a range of z from 0 to approximately $\frac{s \times 0.01}{\eta\sigma^2} = 3 \times 10^6$ thousands of units of the asset.

Panel (b) of Figure 3.4 shows, for the three sets of parameters listed in Table 3.1, the quantity

$$\tau(s,z) = \frac{z}{v(s,z)} = \frac{2\,x}{\sigma^2\big(1 - u'_\infty(x)\big)}$$

which is related to the optimal liquidation strategy given by (2.14) as

$$v(s,z) = \frac{1}{2\eta}\Big(S_t - V_z(s,z)\Big) = \frac{S_t}{2\eta}\Big(1 - u'_\infty(x)\Big).$$

Since $\tau(s,z)$ is defined as the size of the inventory z divided by the optimal liquidation rate v at that moment, it expresses the time the liquidation of z would take if the the selling rate stayed constant at $v(s,z)$. Figure 3.4 (b) shows $\tau(s,z)$ to be increasing for all three parameter sets which means that the approximate time to liquidation is longer for larger amounts to be sold.

However, the actual liquidation rate is not constant. By (2.14), it is nonnegative and it decreases with decreasing inventory size z, meaning that the liquidation slows down as the inventory is being sold (this can be observed in the third column of Figure 3.6). Thus, the actual time to liquidation is longer than $\tau(s,z)$, as can be seen in Figure 3.5 where we compare, for the three parametrizations, $\tau(s,z)$ to estimations of the actual time to liquidation $T(Z = 0)$ obtained from 10 000 simulations. The initial inventory was set to $z = 100$ thousands of units of the asset. The time to liquidation increases with increasing strength of the temporary price impact η which agrees with the intuition that with stronger temporary impact, the investor will choose to sell at a lower speed.

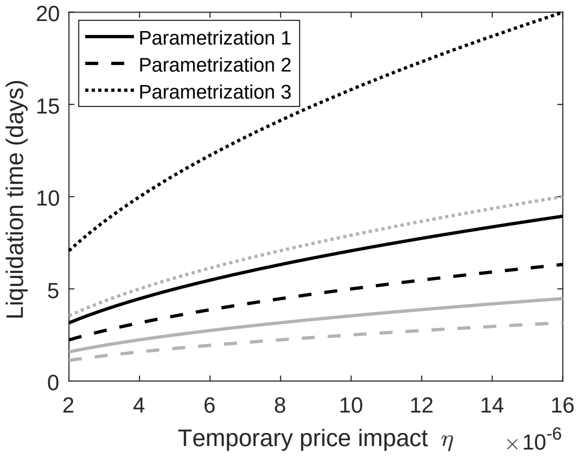


Figure 3.5: Actual average time to liquidation, $T(Z = 0)$, based on 10 000 simulations (black) and approximate time to liquidation, assuming constant liquidation speed, $\tau(z, s)$, (grey), for three parametrizations in Table 3.1 and changing values of the temporary price impact parameter η.

The simulations were performed by generating 10 000 realizations of the unaffected asset price process S_t which is given by the geometric Brownian motion (2.1) starting at $s = 100$. The time step was set to one trading minute, i.e. $\delta t = 1/(250 \times 8 \times 60)$ years. In each time step, the numerical approximation of u_∞ was used to calculate the current optimal liquidation rate $v^*(S_t, Z_t)$ from (2.14). Subsequently, the inventory level was updated by the discrete version of (2.2)

$$Z^*_{t+\delta t} = (1 + r\,\delta t)Z^*_t - v^*(S_t, Z_t)\,\delta t.$$

Once Z^*_t hit zero, the simulation was terminated and the t was taken for the stochastic liquidation time $T(Z = 0)$.

Figure 3.6 shows 10 000 simulations of the liquidation with both the initial price s and the initial inventory z set to 100 and $\eta = 7.5 \times 10^{-6}$. All lines are shown until the (stochastic) time of liquidation $T(Z = 0)$ is reached. In the first column, we observe that, with each of the parameter sets, the execution time increases when the asset price is falling. On average, the liquidation takes 6.17, 4.36 and 13.80 days for the three parametrizations in Table 3.1, respectively.

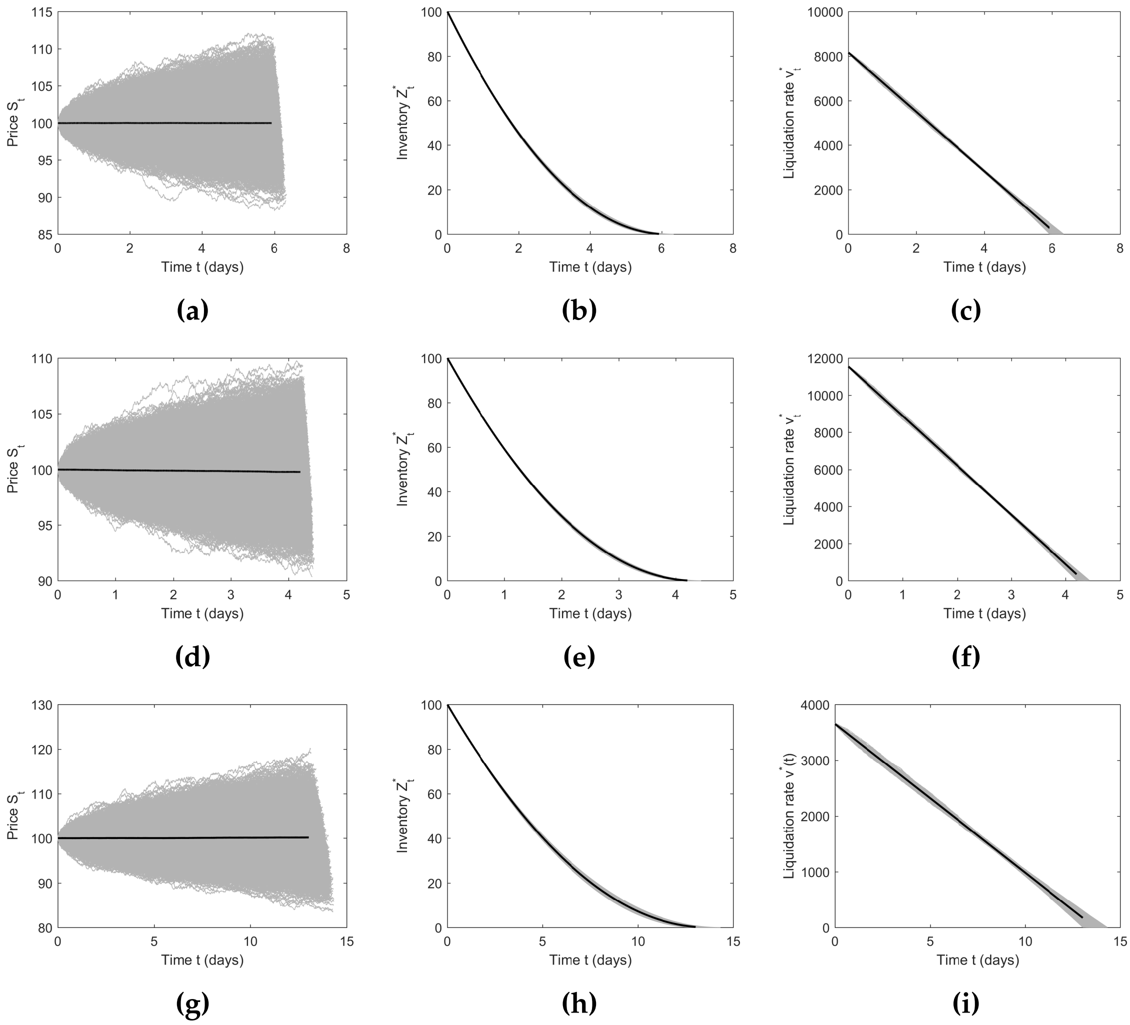


Figure 3.6: Each row shows 10 000 simulations of the unaffected price S_t (first column), inventory Z_t^* (second column) and the optimal strategy v_t^* (third column), for one of the three parametrizations in Table 3.1. The solid black lines show the averages of the simulations.

Chapter 4

Conclusion of Part I

In Part I of this thesis we dealt with the problem of optimal liquidation. In Chapter 2 we presented our formulation of the problem, which first appeared in [9], in the context of optimal liquidation literature. The main feature that differentiates this formulation from the rest of the papers is that in our case the pressure to liquidate is given endogenously which results in a stochastic liquidation horizon, given as a part of the optimal strategy. Moreover, our formulation rules out short sales and we find that intermediate purchases turn out to be never optimal, even though they are permitted.

The optimal liquidation problem leads to a severely singular initial value problem IVP_0 which has been studied in [7] and [43] and for which standard numerical methods fail. We presented a method of overcoming the singularity and solving IVP_0 which consists of solving related boundary value problems $\mathrm{BVP}^t_{[0,L]}$ and stretching the finite time horizon t and the length of the spatial interval L.

In Chapter 3 we described a numerical scheme which produces stable solutions of $\mathrm{BVP}^t_{[0,L]}$ and we proposed a procedure how this scheme can be used to obtain solutions of IVP_0 with sufficient precision. We demonstrated numerically the convergence of the solutions of $\mathrm{BVP}^t_{[0,L]}$ described theoretically in Chapter 2. Numerical approximations of the solutions of IVP_0 were then used to examine the relative implementation shortfall, or the realized price impact, resulting from liquidation. This realized price impact was found to be consistent with the square root law known from empirical literature.

Furthermore, we examined the stochastic time to liquidation resulting from the optimal execution strategy, comparing the approximate time to liquidation, assuming constant liquidation speed, to estimates of the actual time obtained by simulations and we studied the process of optimal liquidation by simulating the liquidation process for three parameter settings corresponding to three different incentives to liquidate – pure time discounting, falling price and a combination of different factors.

The research in the field of optimal liquidation presented in this part of the thesis could be extended in the future by including permanent price impact in the model. While we believe that its presence should not significantly affect the optimal strategy, it is nevertheless interesting to examine its exact effect as well as its effect on the implementation shortfall and the liquidation times. Another interesting extension would be to consider nonlinear utility functions in the optimization.

Part II

Quadratic hedging

Introduction to Part II

The second part of this thesis is devoted to quadratic hedging with application to Asian options and it consists of Chapters 5 - 8.

In Chapter 5 we introduce the reader to options and we begin with the simplest type – European options. We present the theory of Black and Scholes which played a crucial role in the history of financial mathematics and derivative pricing in particular. We develop the concept of a self-financing replicating portfolio and we use it to derive the Black-Scholes partial differential equation. Then we introduce Asian options and we derive a PDE for their price. This PDE has an extra dimension, corresponding to the additional state variable, compared to the Black-Scholes PDE for a European option and we reduce the dimension later in Section 6.5.

Chapter 6 deals with the theory of quadratic hedging which is motivated by the presence of unattainable contingent claims in an incomplete market. We focus on studying the mean squared hedging error of a discretely applied delta hedging strategy. It is well-known that the delta hedging strategy provides perfect replication if implemented continuously. In the more likely case, when the replicating portfolio is rebalanced at discrete times, the replication is no longer perfect and a tracking error is introduced. The asymptotics of the mean squared hedging error with respect to the length of the rehedging interval have been studied in literature for European options as well as for some other option types. We extend these results for the case of Asian options and we propose a method how the MSHE approximation can be calculated by solving a PDE which contains the second derivative of the option price with respect to the underlying stock price, i.e. the option gamma. We conclude the chapter by reducing the dimension for the option pricing PDE, recovering the reduced equation of Večeř [54], and for the PDE describing the MSHE.

Chapter 7 describes the numerical schemes used and presents results. First, we deal with finding the Asian option price where we use the reduced PDE proposed by Večeř [54] and we contrast the results with Monte Carlo estimates to verify their validity. Then we use the results to solve the PDE describing the MSHE where we again use a dimension reduction described in Chapter 6. Finally, we use the PDE solution to evaluate an approximation of the MSHE and compare this to Monte Carlo estimates to find that our proposed approximation agrees reasonably well with the simulated MSHE. Chapter 8 concludes this part of the thesis.

Chapter 5

Introduction to options

A financial derivative is an instrument whose value depends on the value of another instrument, the *underlying asset*. Over the past decades, derivatives have become a major part of financial markets. One of most widely used types of derivatives are *options*. While the underlying instrument may be of different types, including exchange rates or other derivatives, we focus on stock options.

In this chapter we introduce the reader to options, mainly following the ideas presented in Shreve [47]. We start with basic European options for which we develop the concept of a replicating portfolio and use it to derive the Black-Scholes equation for the option price. Then we introduce Asian options which are more complex than European options because their value depends not only on time and the current price of the underlying stock but it depends on the average stock price over some time period as well. We derive a PDE for the Asian option price which has an additional dimension corresponding to the average stock price being a new state variable.

5.1 European options

A basic *European option* gives the holder the right, but not the obligation, to either buy or to sell the underlying stock for a given *strike price* at a given date of *maturity* or *expiration*. If the right is to buy, we speak of a *call option*, and if it is to sell, it is a *put option*.

At time of maturity, $t = T$, a European call option with strike price K generates a payoff of

$$h(S_T) = (S_T - K)^+ := \max\{S_T - K, 0\},$$

where S_T is the price of the underlying stock at $t = T$. The reasoning behind this is that the holder of a call option can exercise the option, buying the stock for K while its market price is S_T. Of course, he will only choose to do so when the option is *in the money*, i.e. when $S_T > K$. In that case he may sell the stock for its market price, leaving him with $S_T - K > 0$. If the option is *at the money* ($S_T = K$) or *out of the money* ($S_T < K$), exercising it brings no benefit to the holder and it has zero value. Similarly, a European put option generates a payoff of

$$h(S_T) = (K - S_T)^+ := \max\{K - S_T, 0\}.$$

The seminal works in option pricing are by Black and Scholes [5] and Merton [37] where the price of the underlying stock S_t is assumed to be governed by geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t W_t.$$

Further assumptions include existence of a constant, risk-free interest rate r in the market, no dividend being paid, and a frictionless market (i.e. no transaction costs). Lastly, trading is assumed to take place continuously in time and there are no restrictions to the amounts being bought and sold (including fractions and short positions). Based on these assumptions the authors derive the well known pricing formulas for the European call and put options

$$\begin{aligned} C_t &= S_t \Phi\left(d_1\right) - K e^{-r(T-t)} \Phi\left(d_2\right), \\ P_t &= K e^{-r(T-t)} \Phi\left(-d_2\right) - S_t \Phi\left(-d_1\right), \end{aligned}$$

where Φ is the cumulative distribution function of standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

and d_1 and d_2 are given by

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{5.1}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{5.2}$$

The option price is a function of

S_t	the price of the underlying stock,
K	the option's strike price,
r	the market risk-free interest rate,
$T-t$	time to maturity,
σ	the volatility of the underlying stock's return.

5.2 Replicating portfolio

Consider a portfolio consisting of the stock and a risk-free bond. The stock price S is governed by the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t W_t, \tag{5.3}$$

and the value of the bond is $B_t = e^{rt}$, with r being the constant risk-free return. We are assuming here for simplicity that the initial bond price is one. Another interpretation would be that B_t represents the value of one dollar deposited in a bank account. The portfolio's value is given by

$$V_t = \varphi_t S_t + \psi_t B_t.$$

where φ_t is the number of shares in the portfolio at time t, ψ_t is the number of bonds. We require the portfolio to be to be *self-financing*, i.e. no funds are added or withdrawn after the initial time. This translates to the immediate change in the portfolio's value at time t being

$$dV_t = \varphi_t dS_t + \psi_t dB_t,$$

where $dB_t = rB_t dt$, i.e. the structure of the portfolio does not change immediately and the change in value arises only from the changes in prices, dS_t and dB_t.

We define the discounted stock price process

$$\tilde{S}_t = e^{-rt} S_t = \frac{S_t}{B_t}$$

whose differential is

$$\begin{aligned}
d\tilde{S}_t &= d(e^{-rt}S_t) = -re^{-rt}S_t dt + e^{-rt}dS_t \\
&= -re^{-rt}S_t dt + e^{-rt}\big(\mu S_t dt + \sigma S_t dW_t\big) \\
&= \sigma \tilde{S}_t \left(\frac{\mu - r}{\sigma}dt + dW_t\right) = \sigma \tilde{S}_t d\tilde{W}_t,
\end{aligned}$$

where $\tilde{W}_t = \frac{\mu-r}{\sigma}t + W_t$ is the standard Brownian motion under Q from Girsanov's theorem. The discounted price $\tilde{S}$ is hence a martingale under Q.

Let us now calculate the differential of the discounted value of the portfolio

$$\begin{aligned}
d(e^{-rt}V_t) &= -re^{-rt}V_t dt + e^{-rt}dV_t \\
&= -re^{-rt}\big(\varphi_t S_t + \psi_t B_t\big)dt + e^{-rt}\big(\varphi_t dS_t + \psi_t dB_t\big) \\
&= -re^{-rt}\varphi_t S_t dt + e^{-rt}\varphi_t dS_t = \varphi_t d\tilde{S}_t,
\end{aligned}$$

which yields the expression for the value of the portfolio

$$e^{-rt}V_t = V_0 + \int_0^t \varphi_\tau d\tilde{S}_\tau = V_0 + \sigma \int_0^t \varphi_\tau \tilde{S}_\tau d\tilde{W}_\tau.$$

This also proves that $e^{-rt}V_t$ is a martingale under Q, since it is an Itō integral.

Taking B_t as numéraire, i.e. expressing prices in terms of the money market instead of in monetary units, we can switch to the risk-neutral probability measure Q and to new variables $\tilde{S}_t$, $\tilde{W}_t$ and $\tilde{V}_t = e^{-rt}V_t$ instead of S_t, W_t and V_t and simplify (5.3) to

$$d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{W}_t.$$

Furthermore, we obtain the time t value of a portfolio created by the strategy φ as

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \varphi_\tau d\tilde{S}_\tau. \tag{5.4}$$

In order for the self-financing portfolio to be a *replicating portfolio* of an option, we require that its terminal value is equal to the option payoff H, i.e. $V_T = H$. In the absence of arbitrage, the price of the option must be equal to the value of its replicating portfolio at every time $0 \leq t \leq T$. Thus, the option price at time t is given by $V_t = \varphi_t S_t + \psi_t B_t$, which, after the change of numéraire, is equivalently written as (5.4).

5.3 The Black-Scholes PDE

Let V_t be the value of a European option at time t. Black and Scholes [5] and Merton [37] argued that V_t should be a function of two variables, the price of the underlying stock S_t and the time t, i.e.

$$V_t = u(t, S_t) \tag{5.5}$$

for a smooth function $u(t, S)$ defined on $[0, T] \times [0, \infty)$. It also depends on the option's strike price K, the risk-free interest rate r, and the volatility of the underlying stock's return σ, but these are assumed to be constant. As a matter of fact, there must be a function $u(t, S_t)$ such that (5.5) holds because V_t is given as

$$V_t = E_t^Q \left[e^{-r(T-t)} V_T \right]$$

and this is a conditional expectation of a deterministic function $V_T = h(S_T)$ of the Markov process S. Thus the conditional expectation only depends on the time t, when it is computed, and on the value of S_t.

We consider a replicating portfolio and we set

$$u(t, S_t) = \varphi_t S_t + \psi_t B_t. \tag{5.6}$$

Appying Itō's formula to $V_t = u(t, S_t)$, one obtains

$$\begin{aligned} dV_t &= u_t(t, S_t)dt + u_S(t, S_t)dS_t + \frac{1}{2}u_{SS}(t, S_t)d[S]_t \\ &= \left(u_t(t, S_t) + u_S(t, S_t)\mu S_t + \frac{1}{2}u_{SS}(t, S_t)\sigma^2 S_t^2 \right) dt + u_S(t, S_t)\sigma S_t dW_t. \end{aligned} \tag{5.7}$$

On the other hand, the portfolio is self-financing so

$$dV_t = \varphi_t dS_t + \psi_t r B_t dt. \tag{5.8}$$

Combining this with (5.6), we obtain

$$dV_t = \varphi_t dS_t + \frac{u(t, S_t) - \varphi_t S_t}{B_t} r B_t dt$$

$$= \big[(\mu - r)\varphi_t S_t + ru(t, S_t)\big]\,dt + \varphi_t \sigma S_t dW_t. \tag{5.9}$$

Expressions (5.9) and (5.7) express dV. Comparing the terms containing dW, we conclude that the number of shares in the replicating portfolio is given by

$$\varphi_t = u_S(t, S_t).$$

This called *delta hedging* because the derivative of the option price w.r.t. the stock price is called the option's delta. Comparing the terms containing dt, we obtain

$$u_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 u_{SS}(t, S_t) = -rS_t u_S(t, S_t) + ru(t, S_t).$$

Since this has to hold for any positive value of S, the function $u(t, S)$ needs to satisfy

$$u_t(t, S) + rSu_S(t, S) + \frac{1}{2}\sigma^2 S^2 u_{SS}(t, S) - ru(t, S) = 0 \tag{5.10}$$

for $t \in [0, T]$ and $S > 0$. Equation (5.10) is called the *Black-Scholes partial differential equation*.

Note that we have not used the fact that we are pricing a European option. The Black-Scholes PDE (5.10) holds for any derivative whose value depends only on time and the stock price.

To price a European call option, we impose a terminal condition on u, setting it equal to the option payoff

$$u(T, S_T) = (S_T - K)^+ . \tag{5.11}$$

The solution of (5.10) subject to the terminal condition (5.11) yields the famous Black-Scholes pricing formula

$$V_t = S_t \Phi\left(d_1\right) - Ke^{-r(T-t)}\Phi\left(d_2\right)$$

with d_1 and d_2 as in (5.1), (5.2).

Black-Scholes PDE and the change of numéraire

Note that we could change to the risk-neutral measure Q and change numéraire to the money market $B_t = e^{rt}$. This results in discounted prices being used instead of the original prices. Thus, one works with $d\tilde{S}_t = \sigma\tilde{S}_t d\tilde{W}_t$ instead of the original dynamics (5.3) and

$$d\tilde{V}_t = \varphi_t d\tilde{S}_t \tag{5.12}$$

instead of the self financing condition (5.8). This is compared to the Itō formula applied to $\tilde{V}_t = u(t, \tilde{S}_t)$, which yields

$$\begin{aligned} d\tilde{V}_t &= u_t(t,\tilde{S}_t)dt + u_S(t,\tilde{S}_t)d\tilde{S}_t + \frac{1}{2}u_{SS}(t,\tilde{S}_t)d\big[\tilde{S}\big]_t \\ &= \left(u_t(t,\tilde{S}_t) + \frac{1}{2}u_{SS}(t,\tilde{S}_t)\sigma^2\tilde{S}_t^2\right)dt + u_S(t,\tilde{S}_t)\sigma S_t d\tilde{W}_t. \end{aligned} \tag{5.13}$$

Comparing (5.13) to (5.12), we obtain that the delta hedging strategy remains unchanged after the change of numéraire

$$\varphi_t = u_S(t, \tilde{S}_t),$$

which is known as numéraire invariance, but u solves a reduced Black-Scholes PDE

$$u_t(t,S) + \frac{1}{2}\sigma^2 S^2 u_{SS}(t,S) = 0$$

for $t \in [0,T]$ and $S > 0$, with the terminal condition

$$u(T,\tilde{S}_T) = e^{-rT}h(e^{rT}\tilde{S}_T).$$

In case of a European call option, the terminal condition is

$$u(T,\tilde{S}_T) = e^{-rT}h(e^{rT}\tilde{S}_T) = e^{-rT}\left(e^{rT}\tilde{S}_T - K\right)^+ = e^{-rT}\left(S_T - K\right)^+,$$

which is the discounted payoff.

5.4 Asian options

Options with more complex dynamics, whose value depends on other variables in addition to time and the price of the underlying stock, are referred to as *exotic options*. *Asian options*, a common type of exotic options, are derivatives whose payoff depends on the average of the underlying stock price over a given period of time.

5.4.1 Arithmetic and geometric average

Based on the type of averaging used, we distinguish between *arithmetic average* Asian options and *geometric average* Asian options. The arithmetic average process is given by

$$A_t = \frac{1}{T} \int_0^t S_\tau d\tau,$$

whose differential is

$$dA_t = \frac{1}{T} S_t dt. \tag{5.14}$$

The geometric average of numbers $x_1, \ldots, x_n$ is defined as

$$\sqrt[n]{x_1 \times x_2 \times \cdots \times x_n} = \exp\left(\frac{\ln x_1 + \cdots + \ln x_n}{n}\right)$$

and so we define the geometric average process as

$$A_t = \exp\left(\frac{1}{T} \int_0^t \ln S_\tau d\tau\right).$$

To express the differential, we first calculate

$$d \ln A_t = d\left(\frac{1}{T} \int_0^t \ln S_\tau d\tau\right) = \frac{1}{T} \ln S_t dt$$

and hence

$$dA_t = d\left(e^{\ln A_t}\right) = A_t d \ln A_t + \frac{1}{2} A_t \big(d \ln A_t\big)^2 = \frac{1}{T} A_t \ln S_t dt \tag{5.15}$$

because $\big(d \ln A_t\big)^2 = 0$.

Note that if we define the average processes as $A_t = \frac{1}{t} \int_0^t S_\tau d\tau$, resp. $A_t =$

$\exp\left(\frac{1}{t}\int_0^t \ln S_\tau d\tau\right)$ for the arithmetic and geometric options respectively, it allows us to unify the expressions for the differential of A as

$$dA_t = A_t f\left(t, \frac{S}{A}\right) \tag{5.16}$$

with $f(t,x) = \frac{x-1}{t}$ for the arithmetic average option and $f(t,x) = \frac{\ln x}{t}$ for the geometric average option.

In this thesis we focus solely on arithmetic average Asian options which are more common.

5.4.2 Fixed strike and floating strike

If the average price comes into place of the spot price, we say the option is a *fixed strike* Asian option. Its payoff is

$$h(S_T, A_T) = (A_T - K)^+$$

in the case of a call option and

$$h(S_T, A_T) = (K - A_T)^+$$

in the case of a put option. The other type is a *floating strike* Asian option where the strike price is not known in advance and it is determined by A_T. The payoff of a floating strike call option is given by

$$h(S_T, A_T) = (S_T - A_T)^+$$

and the payoff of a floating strike put option by

$$h(S_T, A_T) = (A_T - S_T)^+ .$$

5.5 PDE for Asian options

Since the payoff of Asian options depends an the average stock price as well as on time and the stock price at maturity, it is not possible to write the value of an Asian option as a function of only t and S_t. We need to add a third variable A_t and instead of (5.5) we assume

$$V_t = u(t, S_t, A_t) \tag{5.17}$$

for a smooth function $u(t, S, A)$ defined on $[0, T] \times [0, \infty) \times [0, \infty)$. The pair (S, A) is given by a pair of stochastic differential equations so it is a two-dimensional Markov process and there is a function u such that (5.17) holds. Using Itō's lemma, we express

$$\begin{aligned} dV_t &= u_t dt + u_S dS_t + u_A dA_t + \frac{1}{2} u_{SS} d[S]_t \\ &= \left(u_t + \mu S_t u_S + \frac{dA_t}{dt} u_A + \frac{1}{2}\sigma^2 S_t^2 u_{SS} \right) dt + \sigma S_t u_S dW_t, \end{aligned} \tag{5.18}$$

where we dropped the arguments (t, S_t, A_t) of the derivatives of u. The value of $\frac{dA_t}{dt}$ in (5.18) depends on the type of averaging. It is given by (5.14) and (5.15) for an arithmetic and a geometric average Asian option respectively. Expression (5.18) can again be compared to the self-financing replicating portfolio with dV expressed in (5.9) as

$$dV_t = \left[(\mu - r)\varphi_t S_t + r u(t, S_t, A_t) \right] dt + \varphi_t \sigma S_t dW_t. \tag{5.19}$$

Expressions (5.18), (5.19) yield that the replicating portfolio is again achieved by delta hedging

$$\varphi_t = u_S(t, S_t, A_t) \tag{5.20}$$

and $u(t, S, A)$ solves the PDE

$$u_t + rSu_S + \frac{dA_t}{dt} u_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0 \tag{5.21}$$

for $t \in [0, T]$, $S > 0$ and $A > 0$. The terminal condition is

$$u(T, S_T, A_T) = h(S_T, A_T),$$

with h giving the payoff for the particular type of Asian option.

In case of an arithmetic option, the PDE (5.21) becomes

$$u_t + rSu_S + \frac{1}{T} S u_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0 \tag{5.22}$$

where we used (5.14), and for a geometric option we use (5.15) to obtain the PDE

$$u_t + rSu_S + \frac{1}{T}A\ln S\,u_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0. \tag{5.23}$$

Compared with the Black-Scholes PDE for European options (5.10), the PDE for Asian options (5.21) has an extra dimension and it contains the additional term $\frac{dA_t}{dt}u_A(t,S,A)$.

The dimension of equation (5.21) can be reduced from two space variables to a single one, as we show later in Section 6.5 and use in numerical solutions in Section 7.1. In Appendix A we show also other dimension reductions.

Chapter 6

Quadratic hedging theory

In this chapter we present selected results from the theory of quadratic hedging, relying mainly on Pham [40] and Schweizer [48] both of which provide an overview of results and developments in this field. Other notable sources include the early papers by Föllmer and Sondermann [19] and Föllmer and Schweizer [18] or more recent papers which extend the results for more general settings. Černý and Kallsen [12] study quadratic hedging in a general setting of a semimartingale market. Denkl et al. [14] study delta hedging strategies in exponential Lévy models which is followed by Černý et al. [11] where the authors find that the presence of jumps introduces a comparable hedging error as discrete implementation of a continuous strategy in the Black-Scholes model.

We begin this chapter by describing the simple motivation for quadratic hedging – in an incomplete market there are unattainable claims and thus a hedging error is introduced which is then studied and possibly minimized. Then we describe discrete delta hedging strategies which have been studied for European as well as for certain other types of options. We develop an approximation of the mean squared hedging error for a discrete hedging strategy which agrees with known approximations for discrete delta hedging strategies from literature. Finally, we apply this approximation to Asian options and propose a method of evaluating the hedging error approximation by use of partial differential equations.

6.1 Motivation of quadratic hedging

A simple financial market can be described on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P\big)$ by the processes B and S giving the prices of basic assets – a risk-free bond and a stock, respectively. It is convenient to take the bond as numéraire, making the bond price 1 at all times, and work only with the stock price $\tilde{S} = S/B$ expressed in terms of the bond rather than with the original price S expressed in monetary terms.

A *portfolio strategy* is a pair (φ, ψ), where φ_t is the number of stocks in the portfolio at time t and ψ_t is the amount invested in the bond. Since the bond's price is 1 at any time, ψ_t is also the number of bonds held in the portfolio. The process φ is assumed to be predictable and ψ is adapted. The value of portfolio (φ, ψ) at time t is given by

$$V_t = \varphi_t \tilde{S}_t + \psi_t.$$

A portfolio is self-financing (see Section 5.2) if

$$dV_t = \varphi_t d\tilde{S}_t$$

and in that case the portfolio's value can be written as

$$V_t = V_0 + \int_0^t \varphi_\tau d\tilde{S}_\tau, \tag{6.1}$$

where V_0 is the initial investment needed to start the strategy. Note that ψ can be written as

$$\psi_t = V_t - \varphi_t \tilde{S}_t = V_0 + \int_0^t \varphi_\tau d\tilde{S}_\tau - \varphi_t \tilde{S}_t$$

so a self-financing strategy (φ, ψ) is fully described by the pair (V_0, φ).

A contingent claim H is *attainable* if there exists a self-financing strategy with $V_T = H$. If the market allows no arbitrage, the price of H at any time must be given by V_t. In particular, the initial price must be $H_0 = V_0$. An attainable claim H can be written as

$$H_t = H_0 + \int_0^t \varphi_\tau^H d\tilde{S}_\tau$$

where (H_0, φ^H) is a hedging strategy against H. A market is said to be *complete*, if any $\mathcal{F}_T$-measurable claim H is attainable. Otherwise, the market is *incomplete*.

Recall Section 5.3 where we showed that in the Black-Scholes model, the hedging strategy φ for a European option is given by delta hedging, i.e.

$$\varphi_t = u_S(t, \tilde{S}_t),$$

where $u(t, \tilde{S}_t)$ denotes the option price at time t.

In reality, however, one rarely encounters a complete market. For this reason we consider an incomplete market where non-attainable contingent claims exist. These can no longer be hedged perfectly and so it is sensible to look for the best possible hedge according to some criterion. One of the most widely used criteria is based on keeping the self-financing condition but relaxing the replication condition $V_T = H$. Thus, a hedging error, also referred to as *tracking error* in literature (cf. [3]), is introduced

$$H - V_T(V_0, \varphi) = H - V_0 - \int_0^T \varphi_\tau d\tilde{S}_\tau.$$

The aim of *quadratic hedging* or *mean-variance hedging* is to look for a strategy (V_0, φ) which minimizes the *mean squared hedging error* (MSHE)

$$\varepsilon^2(V_0, \varphi) = E\left[\left(H - V_0 - \int_0^T \varphi_\tau d\tilde{S}_\tau\right)^2\right]. \tag{6.2}$$

over all $V_0 \in \mathbb{R}$ and $\tilde{S}$-integrable processes φ. The use of a quadratic criterion means that positive and negative values of the hedging error are penalized equally. This might seem as a drawback, but it makes the resulting strategy applicable to both buyers and sellers of options and it leads to close replication rather than profit.

The strategy which minimizes the mean squared hedging error error ε^2, defined in (6.2), among all admissible strategies is referred to as the *globally optimal strategy*, dynamically optimal strategy or the mean-variance hedge.

In a complete market any risk connected with investment in any claim can by offset by investing in the replicating portfolio. If the market is incomplete, however, it is not possible to eliminate completely the risk connected with investing in an unattainable claim. The globally optimal strategy minimizes the total variance of the hedge and thus investing in this strategy is also known as global risk minimization.

6.2 Discrete delta hedging for a European call

An important aspect of hedging strategies is that they often require the portfolio to be continuously dynamically adjusted. This is also the case of delta hedging in the Black-Scholes model, where the strategy dictates that the portfolio contains $\varphi_t = u_S(t, S_t)$ units of the underlying stock at each time t. In reality, this is not plausible mainly because of transaction costs which make continuous trading infinitely expensive.

When a continuous replication strategy is applied discretely, it no longer provides perfect replication and the resulting hedging error has been studied by numerous authors. The hedging error of discretely rebalanced European option hedges in the presence of transaction cost was first studied in Leland [35] and the results were later extended in Toft [51]. Both papers show evidence that the option cash gamma $S^2 u_{SS}(t, S)$ plays an important role in hedging error approximations. More recent results on quadratic hedging in presence of transaction costs can be found in Kallsen and Muhle-Karbe [30].

However, the majority of quadratic hedging literature assumes no transaction costs and so do we in this thesis. We mention two such papers by Bertsimas, Kogan and Lo [3] and by Gobet and Temam [22] which study the hedging error of discrete delta hedging strategies and are of importance for our work. Bertsimas, Kogan and Lo [3] study the discretely implemented delta hedging for European options in the Black-Scholes model. They derive an approximation for the mean squared hedging error and show that it is of the order $O(\Delta t)$, where Δt is the length of the rehedging interval. The same approximation was derived independently in Zhang [57]. The results of [3] have been extended to stochastic volatility models by [24], exponential Lévy models by [14] and to general Itō processes with jumps by [50]. Gobet and Temam [22] show for a wider class of options that the rate of convergence is between $\sqrt{\Delta t}$ and Δt, depending on smoothness of the payoff. Their research was followed by [21] where hedging errors of delta-gamma strategies are studied.

Consider a delta hedging strategy implemented at discrete times during $[0, T]$. Let the trading times t_k be given by $t_k = k\Delta t$ for $k = 0, \ldots, n$ with $\Delta t = T/n$. At each t_k the investor rebalances the portfolio so that it contains $\varphi_{t_k} = u_S(t_k, \tilde{S}_{t_k})$ units of the stock. This amount is then held until the next trading time t_{k+1}. A discretely implemented delta hedging strategy is then given by

$$\varphi_t^{\Delta} = \varphi_{\theta(t)} = u_S\big(\theta(t), \tilde{S}_{\theta(t)}\big),$$

where $\theta(t) = \sup\{t_k : t_k \leq t\} = \lfloor \frac{t}{\Delta t} \rfloor \Delta t$ denotes the last trading time up to time t.

The hedging error of this strategy is given by

$$H - V_T(V_0, \varphi) = \int_0^T \left(\varphi_\tau - \varphi_\tau^{\Delta}\right) d\tilde{S}_\tau = \int_0^T \left[u_S(\tau, \tilde{S}_\tau) - u_S\big(\theta(\tau), \tilde{S}_{\theta(\tau)}\big)\right] d\tilde{S}_\tau$$

and the mean squared hedging error is

$$\varepsilon^2(V_0, \varphi^{\Delta}) = E\left[\left(\int_0^T \left[u_S(\tau, \tilde{S}_\tau) - u_S\big(\theta(\tau), \tilde{S}_{\theta(\tau)}\big)\right] d\tilde{S}_\tau\right)^2\right]. \tag{6.3}$$

Bertsimas, Kogan and Lo [3] derived the approximation of (6.3)

$$\varepsilon^2(V_0, \varphi^{\Delta}) = \frac{\Delta t}{2} E\left[\int_0^T \Big(\sigma^2(\tau, S_\tau) S_\tau^2 u_{SS}(\tau, S_\tau)\Big)^2 d\tau\right] + o(\Delta t) \tag{6.4}$$

under the assumption that the option is of European type and the stock price follows the generalized Brownian motion

$$dS_t = \mu(t, S_t) S_t dt + \sigma(t, S_t) S_t dW_t,$$

where the coefficients $\mu(t, S)$ and $\sigma(t, S)$ are such that $\sigma(t, S)$ is bounded from below by a positive constant σ_0 and there exists a constant $K_1 > 0$ such that

$$\left|\frac{\partial^{\beta+\gamma}}{\partial t^\beta \partial S^\gamma} \mu(t, S)\right| + \left|\frac{\partial^{\beta+\gamma}}{\partial t^\beta \partial S^\gamma} \sigma(t, S)\right| + \left|\frac{\partial^{\alpha}}{\partial S^\alpha}\Big(S\sigma(t, S)\Big)\right| \leq K_1$$

where $(t, S) \in [0, T] \times [0, \infty)$, $1 \leq \alpha \leq 6$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 3$, and all partial derivatives are continuous. Furthermore, $S^2 \frac{\partial^\alpha \sigma(t,S)}{\partial S^\alpha}$ must be bounded for all $2 \leq \alpha \leq 6$. Alternatively, the approximation (6.4) is also shown to be valid without the condition on boundedness of $S^2 \frac{\partial^\alpha \sigma(t,S)}{\partial S^\alpha}$ in case that the derivative payoff is six time continuously differentiable with bounded derivatives which is not satisfied by simple European options.

The square root of the coefficient of the $\mathcal{O}(\Delta t)$ term in the approximation (6.4)

$$\sqrt{\frac{1}{2}E\left[\int_0^T \left(\sigma^2(\tau,S_\tau)S_\tau^2 u_{SS}(\tau,S_\tau)\right)^2 d\tau\right]}$$

is called *temporal granularity* by Bertsimas, Kogan and Lo [3] because it measures how granular time is, i.e. how much hedging error is introduced by implementing a continuous strategy discretely. The square root is used because the authors work with the root mean squared error instead of the mean squared error.

Connection to optimal hedging

In addition to the globally optimal hedge, which minimizes the MSHE (6.2), quadratic hedging literature also works with the so called *locally optimal hedge* defined for discretely applied strategies to minimize the conditional expected one-step squared hedging error. More precisely, the locally optimal strategy is defined at each trading time t_k, $k = 0, \ldots, (n-1)$, to minimize with respect to $\mathcal{F}_{t_k}$-measurable V_{t_k} and φ_{t_k} and subject to $V_T = H$ the expectation

$$E_{t_k}\left[\left(V_{t_k} + \varphi_{t_k}\Delta\tilde{S}_{t_{k+1}} - V_{t_{k+1}}\right)^2\right], \tag{6.5}$$

where $\Delta\tilde{S}_{t_{k+1}} = \tilde{S}_{t_{k+1}} - \tilde{S}_{t_k}$. The process V_t is called the mean-value process and it can be expressed as

$$V_{t_k} = E^*_{t_k}\left[V_{t_{k+1}}\right] = E^*_{t_k}\left[E^*_{t_{k+1}}\left[V_{t_{k+2}}\right]\right] = \cdots = E^*_{t_k}\left[H\right],$$

where the expectations are taken under the so-called minimal martingale measure (cf. [13]).

In contrast to the locally optimal strategy which minimizes (6.5) with respect to both φ_{t_k} and V_{t_k}, the globally optimal strategy acknowledges that at time t_k the value of the self-financing portfolio is given by (6.1) and cannot be chosen arbitrarily. While the locally optimal strategy assumes that the value V_t of the hedging portfolio is always at its optimum, the globally optimal strategy adjust for this not always being true. The two strategies are well explained and compared to the delta hedging strategy on a specific example in [10, Chapter 13].

The locally optimal strategy is according to [13, eqn. (4.6)] given by

$$\xi_{t_k} = \frac{Cov_{t_k}\left(V_{t_{k+1}}, \Delta\tilde{S}_{t_{k+1}}\right)}{Var_{t_k}\left(\Delta\tilde{S}_{t_{k+1}}\right)}.$$

It is suboptimal in terms of minimizing the unconditional MSHE (6.2) which is for the locally optimal strategy (cf. [13, eqn. (4.10)]) given as the sum of the one-period expected squared hedging errors plus the square of the difference between the initial endowment v and the value of V_0

$$\varepsilon^2(v,\xi) = (v - V_0)^2 + \sum_{k=0}^{n-1} E\left(E_{t_k}\left[\left(V_{t_k} + \varphi_{t_k}\Delta\tilde{S}_{t_{k+1}} - V_{t_{k+1}}\right)^2\right]\right).$$

Literature finds that the delta hedging strategy performs very similarly to the locally optimal strategy and in case of independent and identically distributed returns it is also very closely related to the globally optimal strategy (cf. [10, Chapter 13]). Moreover, in the Black-Scholes setting, where the option price is driven by a geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dW_t$ with nonnegative drift μ, the delta hedging strategy coincides with the globally optimal strategy (cf. [14, Remark 3.5]). Thus, studying the performance of a delta hedging strategy gives valuable insight into locally and globally optimal hedging.

6.3 Mean squared hedging error approximation

In this section we provide a heuristic derivation of an approximation of the mean squared hedging error (6.3) for the discretely applied delta hedging strategy. Assume the discounted stock price dynamics under the risk-neutral measure are

$$d\tilde{S}_t = \sigma(\tilde{S}_t)d\tilde{W}_t$$

so that one has

$$d[\tilde{S}]_t = \left(d\tilde{S}_t\right)^2 = \sigma^2(\tilde{S}_t)dt.$$

The integral in the expression (6.3) for the MSHE of a discrete delta hedging strategy can be written as the sum of integrals

$$\int_0^T \left(\varphi_\tau - \varphi_\tau^\Delta\right) d\tilde{S}_\tau = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(\varphi_\tau - \varphi_\tau^\Delta\right) d\tilde{S}_\tau$$

so the total hedging error is the sum of hedging errors in the individual rehedging periods.

Consider the one-period mean squared hedging error and for simplicity consider the first interval between $t_0 = 0$ and $t_1 = \Delta t$

$$E\left[\left(\int_0^{\Delta t} \left(\varphi_\tau - \varphi_\tau^\Delta\right) d\tilde{S}_\tau\right)^2\right] = E\left[M_{\Delta t}^2\right], \tag{6.6}$$

where we write the one-period MSHE in terms of the martingale

$$M_{\Delta t} = \int_0^{\Delta t} \left(\varphi_\tau - \varphi_\tau^\Delta\right) d\tilde{S}_\tau = \int_0^{\Delta t} \left(\varphi_\tau - \varphi_0\right) d\tilde{S}_\tau,$$

whose differential and quadratic variation are

$$\begin{aligned} dM_\tau &= \left(\varphi_\tau - \varphi_0\right) d\tilde{S}_\tau, \\ d[M]_\tau &= \left(dM_\tau\right)^2 = \left(\varphi_\tau - \varphi_0\right)^2 d[\tilde{S}]_\tau = \left(\varphi_\tau - \varphi_0\right)^2 \sigma^2(\tilde{S}_\tau) d\tau. \end{aligned}$$

Note that $M_0 = 0$ so for M^2 one has

$$M_{\Delta t}^2 = 2\int_0^{\Delta t} M_\tau dM_\tau + [M]_{\Delta t}. \tag{6.7}$$

Using the martingale property of M, $E\left(\int_0^{\Delta t} M_\tau dM_\tau\right) = 0$, (6.6) becomes

$$\begin{aligned} E\left[M_{\Delta t}^2\right] &= E\left[\int_0^{\Delta t} d[M]_\tau\right] = E\left[\int_0^{\Delta t} \left(\varphi_\tau - \varphi_0\right)^2 \sigma^2(\tilde{S}_\tau) d\tau\right] \\ &= \int_0^{\Delta t} E\left[\left(\varphi_\tau - \varphi_0\right)^2 \sigma^2(\tilde{S}_\tau)\right] d\tau = \int_0^{\Delta t} E\left[\tilde{M}_\tau^2\right] d\tau, \end{aligned} \tag{6.8}$$

where we define $\tilde{M}_\tau = \left(\varphi_\tau - \varphi_0\right) \sigma(\tilde{S}_\tau)$. Expression (6.8) contains the expectation $E\left[\tilde{M}_\tau^2\right]$ which resembles the term $E\left[M_{\Delta t}^2\right]$ of (6.6). $\tilde{M}$ starts at zero so (6.7) holds also for $\tilde{M}$ but, unlike M, $\tilde{M}$ is not a martingale so $E\left(\int_0^{\Delta t} \tilde{M}_\tau d\tilde{M}_\tau\right)$ will no longer be zero and thus one has

$$E\left[\tilde{M}_\tau^2\right] = 2E\left[\int_0^\tau \tilde{M}_t d\tilde{M}_t\right] + E\left[\int_0^\tau d[\tilde{M}]_t\right].$$

Assume for now, that the first term is negligible compared to the second one, i.e.

$$E\left[\tilde{M}_u^2\right] \approx E\left[\int_0^u d[\tilde{M}]_t\right].$$

Using Itō's lemma, we compute

$$\begin{aligned} d\tilde{M}_t &= (\varphi_t - \varphi_0)\, d\big(\sigma(\tilde{S}_t)\big) + \sigma(\tilde{S}_t) d\varphi_t + d\big[\varphi, \sigma(\tilde{S})\big]_t, \\ d[\tilde{M}]_t &= \sigma^2(\tilde{S}_t) d[\varphi]_t + (\varphi_t - \varphi_0)^2\, d\big[\sigma(\tilde{S})\big]_t + 2\,(\varphi_t - \varphi_0)\, \sigma(\tilde{S}_t) d\big[\varphi, \sigma(\tilde{S})\big]_t. \end{aligned}$$

Now, as $\Delta t \to 0$, the terms containing $(\varphi_t - \varphi_0)$ for $t \in [0, \Delta t]$ will be relatively small and for this reason they should be dominated by the first term, i.e.

$$d[\tilde{M}]_t \approx \sigma^2(\tilde{S}_t) d[\varphi]_t.$$

Thus we obtain the approximation

$$E\left[M_{\Delta t}^2\right] = \int_0^{\Delta t} E\left[\tilde{M}_\tau^2\right] d\tau \approx \int_0^{\Delta t} E\left[\int_0^\tau \sigma^2(\tilde{S}_t) d[\varphi]_s\right] d\tau.$$

Recall that the delta hedging strategy is given by $\varphi_t = u_S(t, \tilde{S}_t)$. Itō's lemma yields

$$d\varphi_t = u_{tS}(t, \tilde{S}_t) dt + u_{SS}(t, \tilde{S}_t) d\tilde{S}_t + \frac{1}{2} u_{SSS}(t, \tilde{S}_t) d[\tilde{S}]_t, \tag{6.9}$$

$$d[\varphi]_t = (d\varphi_t)^2 = \Big(u_{SS}(t, \tilde{S}_t)\Big)^2 d[\tilde{S}]_t = \Big(\sigma(\tilde{S}_t) u_{SS}(t, \tilde{S}_t)\Big)^2 dt. \tag{6.10}$$

For $t \in [0, \tau]$ we make the approximation

$$\int_0^\tau \sigma^2 \tilde{S}_t^2 d[\varphi]_t = \int_0^\tau \sigma^4(\tilde{S}_t) \Gamma_t^2 dt = \sigma^4\left(\tilde{S}_0\right) \Gamma_0^2 \tau + \mathcal{O}(\tau^2),$$

where $\Gamma_t = u_{SS}(t, \tilde{S}_t)$ is the option's gamma at time t, and we obtain

$$\int_0^{\Delta t} E\left[\int_0^{\tau} \sigma^2(\tilde{S}_t) d[\varphi]_t\right] d\tau \approx \int_0^{\Delta t} \sigma^4(\tilde{S}_0)\Gamma_0^2\, \tau\, d\tau = \frac{(\Delta t)^2}{2}\sigma^4\left(\tilde{S}_0\right)\Gamma_0^2.$$

The one-period mean squared hedging error approximation for the first interval is thus

$$E\left[\left(\int_0^{\Delta t}\left(\varphi_\tau - \varphi_\tau^\Delta\right) d\tilde{S}_\tau\right)^2\right] \approx \frac{(\Delta t)^2}{2}\sigma^4\left(\tilde{S}_0\right)\Gamma_0^2$$

which translates to the general one-period MSHE approximation

$$E_{t_{k-1}}\left[\left(\int_{t_{k-1}}^{t_k}\left(\varphi_\tau - \varphi_\tau^\Delta\right) d\tilde{S}_\tau\right)^2\right] \approx \frac{(\Delta t)^2}{2}\sigma^4\left(\tilde{S}_{t_{k-1}}\right)\Gamma_{t_{k-1}}^2. \tag{6.11}$$

The total mean squared hedging error (6.3) for the discrete delta hedging strategy can be evaluated as

$$\begin{aligned}
\varepsilon^2(V_0, \varphi^\Delta) &= E\left[\sum_{k=1}^n\left(\int_{t_{k-1}}^{t_k}\left(\varphi_\tau - \varphi_\tau^\Delta\right) d\tilde{S}_\tau\right)^2\right] \\
&= E\left[\sum_{k=1}^n E_{t_{k-1}}\left\{\left(\int_{t_{k-1}}^{t_k}\left(\varphi_\tau - \varphi_\tau^\Delta\right) d\tilde{S}_\tau\right)^2\right\}\right] \\
&\approx E\left[\sum_{k=1}^n \frac{(\Delta t)^2}{2}\sigma^4\left(\tilde{S}_{t_{k-1}}\right)\Gamma_{t_{k-1}}^2\right] \\
&= \frac{\Delta t}{2} E\left[\sum_{k=1}^n \sigma^4\left(\tilde{S}_{t_{k-1}}\right)\Gamma_{t_{k-1}}^2 \Delta t\right].
\end{aligned}$$

For $\Delta t \to 0$ the sum can be replaced by an integral leading to the MSHE approximation

$$\varepsilon^2(V_0, \varphi^\Delta) \approx \frac{\Delta t}{2} E\left[\int_0^T \sigma^4\left(\tilde{S}_t\right)\Gamma_t^2 dt\right]. \tag{6.12}$$

For the setting considered by Bertsimas, Kogan and Lo, i.e. $\sigma(\tilde{S}) = \sigma(t, \tilde{S})\tilde{S}$, this approximation coincides with their expression (6.4).

The full expression for the one-period mean squared hedging error (6.11) is

$$E_{t_{k-1}}\left[\left(\int_{t_{k-1}}^{t_k}\left(\varphi_\tau-\varphi_\tau^\Delta\right)d\tilde{S}_\tau\right)^2\right]=\frac{(\Delta t)^2}{2}\sigma^4\left(\tilde{S}_{t_{k-1}}\right)\Gamma_{t_{k-1}}^2$$
$$+\int_{t_{k-1}}^{t_k}\big(R_1+R_2+R_3+R_4\big)d\tau, \qquad (6.13)$$

where R_1-R_4 represent the remainders

$$R_1=E_{t_{k-1}}\left[2\int_{t_{k-1}}^{\tau}\tilde{M}_t d\tilde{M}_t\right], \qquad (6.14)$$
$$R_2=E_{t_{k-1}}\left[\int_{t_{k-1}}^{\tau}\left(\varphi_t-\varphi_{t_{k-1}}\right)^2 d\big[\sigma(\tilde{S})\big]_t\right], \qquad (6.15)$$
$$R_3=E_{t_{k-1}}\left[\int_{t_{k-1}}^{\tau}2\left(\varphi_t-\varphi_{t_{k-1}}\right)\sigma(\tilde{S}_t)d\big[\varphi,\sigma(\tilde{S})\big]_t\right], \qquad (6.16)$$
$$R_4=E_{t_{k-1}}\left[\int_{t_{k-1}}^{\tau}\left(\sigma^4(\tilde{S}_t)\Gamma_t^2-\sigma^4(\tilde{S}_{t_{k-1}})\Gamma_{t_{k-1}}^2\right)dt\right], \qquad (6.17)$$

with $\tilde{M}$ is defined by (6.8) as $\tilde{M}_t=\left(\varphi_t-\varphi_{t_{k-1}}\right)\sigma(\tilde{S}_t)$. It remains yet to be shown in future work that these remainder expressions R_1-R_4 are of order $o(\Delta t)$ so that they are dominated by the first term of (6.13) and the one-period approximation (6.11) is valid and then also the total mean squared hedging error approximation (6.12). However, numerical experiments for Asian options in next chapter support the validity of the approximation.

Mean squared hedging error for a general strategy

In case of a general strategy φ we no longer have explicit expressions for its dynamics of the form (6.9), (6.10) but we have general expressions

$$d\varphi_t=b_t^\varphi dt+\sqrt{c_t^\varphi}d\tilde{W}_t, \qquad (6.18)$$
$$d[\varphi]_t=c_t^\varphi dt, \qquad (6.19)$$

with b^φ and $\sqrt{c^\varphi}$ being the drift and volatility processes of the strategy φ.

This reflects in the expression for the one-period mean squared hedging error

(6.13) of discretely exercised strategy φ in the following way

$$E_{t_{k-1}}\left[\left(\int_{t_{k-1}}^{t_k}\left(\varphi_\tau-\varphi_\tau^\Delta\right)d\tilde{S}_\tau\right)^2\right]=\frac{(\Delta t)^2}{2}\sigma^2(\tilde{S}_{t_{k-1}})c_{t_{k-1}}^\varphi$$
$$+\int_{t_{k-1}}^{t_k}\left(R_1+R_2+R_3+\tilde{R}_4\right)d\tau, \qquad (6.20)$$

where R_1-R_3 are the same as in (6.14)-(6.16) and $\tilde{R}_4$ is given as

$$\tilde{R}_4=E_{t_{k-1}}\left[\int_{t_{k-1}}^{\tau}\left(\sigma^2(\tilde{S}_t)c_t^\varphi-\sigma^2(\tilde{S}_{t_{k-1}})c_{t_{k-1}}^\varphi\right)dt\right]. \qquad (6.21)$$

Again, as in the case of the discrete delta hedging strategy, if the remainder terms R_1-R_3 and $\tilde{R}_4$ are of order $o(\Delta t)$, then we obtain the mean squared hedging error approximation for a general strategy

$$\varepsilon^2(V_0,\varphi^\Delta)\approx\frac{\Delta t}{2}E\left[\int_0^T\sigma^2(\tilde{S}_t)c_t^\varphi dt\right]. \qquad (6.22)$$

6.4 Mean squared hedging error for Asian options

As we discussed in Section 5.4, in case of Asian options one has to consider an additional state variable A_t, describing the average price of the underlying stock, in addition to time t and the stock price S_t.

We are interested in approximating the mean squared hedging error for a discretely implemented strategy φ for an Asian option. The general expression is given by (6.22) and one needs to evaluate the expectation

$$E\left[\int_0^T\sigma^2(\tilde{S}_\tau)c_\tau^\varphi d\tau\right]$$

in order to obtain an estimate of the MSHE.

Define martingale M given as the conditional expectation

$$M_t=E_t\left[\int_0^T\sigma^2(\tilde{S}_\tau)c_\tau^\varphi d\tau\right]=\int_0^t\sigma^2(\tilde{S}_\tau)c_\tau^\varphi d\tau+E_t\left[\int_t^T\sigma^2(\tilde{S}_\tau)c_\tau^\varphi d\tau\right]$$

and write the last conditional expectation as a function v of the state variables t, S

and A, i.e.

$$v(t, S_t, A_t) = E_t \left[\int_t^T \sigma^2(\tilde{S}_\tau) c_\tau^\varphi d\tau \right]. \tag{6.23}$$

The MSHE approximation (6.22) can then be expressed in terms of v as

$$\varepsilon^2(V_0, \varphi^\Delta) \approx \frac{\Delta t}{2} v(0, S_0, A_0).$$

Consider the stock price dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and the arithmetic average process

$$dA_t = \frac{1}{T} S_t dt.$$

Itō's lemma for M yields

$$\begin{aligned} dM_t &= \sigma^2 S_t^2 c_t^\varphi dt + v_t(t, S_t, A_t)dt + v_S(t, S_t, A_t)dS_t + v_A(t, S_t, A_t)dA_t \\ &\quad + \frac{1}{2} v_{SS}(t, S_t, A_t)(dS_t)^2 \\ &= \left[\sigma^2 S_t^2 c_t^\varphi + v_t + \mu S_t v_S + \frac{1}{T} S_t v_A + \frac{1}{2}\sigma^2 S_t^2 v_{SS} \right] dt + \sigma S_t v_S dW_t. \end{aligned} \tag{6.24}$$

Since M is a martingale, its drift must be zero and we obtain the PDE for $v(t, S, A)$

$$\sigma^2 S^2 c_t^\varphi + v_t(t, S, A) + \mu S v_S(t, S, A) + \frac{1}{T} S v_A(t, S, A) + \frac{1}{2}\sigma^2 S^2 v_{SS}(t, S, A) = 0 \tag{6.25}$$

for $t \in [0, T]$, $S > 0$ and $A > 0$.

The terminal condition is

$$v(T, S, A) = 0 \tag{6.26}$$

as at $t = T$, the integral in the definition (6.23) of v runs from T to T.

The delta hedging strategy for an Asian option is given by (5.20) as $\varphi_t = u_S(t, S_t, A_t)$, where $u(t, S_t, A_t)$ is the option price with $u(t, S, A)$ being given as the solution of the PDE (5.22)

$$u_t + rSu_S + \frac{1}{T} S u_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0$$

for $t \in [0, T]$, $S > 0$ and $A > 0$, together with the terminal condition

$$u(T, S_T, A_T) = h(S_T, A_T), \tag{6.27}$$

with h giving the payoff for the particular type of Asian option.

For $\varphi_t = u_S(t, S_t, A_t)$ Itō's lemma yields

$$\begin{aligned} d\varphi_t &= u_{tS}dt + u_{SS}dS_t + u_{SA}dA_t + \frac{1}{2}u_{SSS}(t, S_t)d[S]_t, \\ d[\varphi]_t = (d\varphi_t)^2 &= \Big(u_{SS}(t, S_t, A_t)\Big)^2 d[S]_t = \Big(\sigma S_t u_{SS}(t, S_t, A_t)\Big)^2 dt \end{aligned}$$

so $c_t^\varphi = \big(\sigma S_t u_{SS}(t, S_t, A_t)\big)^2$ in the PDE for v (6.25).

The general MSHE approximation (6.22) thus becomes in this case

$$\varepsilon^2(V_0, \varphi^\Delta) \approx \frac{\Delta t}{2} E\left[\int_0^T \sigma^2(\tilde{S}_t)c_t^\varphi dt\right] = \frac{\Delta t}{2} E\left[\int_0^T \sigma^4 S_t^4 u_{SS}^2(t, S_t, A_t)dt\right]. \tag{6.28}$$

The Asian option price and the mean squared hedging error of the discretely implemented delta hedging strategy can be found by solving the system of PDEs

$$u_t + rSu_S + \frac{1}{T}Su_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0 \tag{6.29}$$

$$\Big(\sigma^2 S^2 u_{SS}\Big)^2 + v_t + \mu S v_S + \frac{1}{T}Sv_A + \frac{1}{2}\sigma^2 S^2 v_{SS} = 0 \tag{6.30}$$

with terminal conditions (6.27), (6.26).

6.5 Dimension reduction for the Asian MSHE

The dimension of the PDE (6.29), which we derived in Section 5.5 and which describes the Asian option price, can be reduced from two state variables, S and A, to a single state variable. In Appendix A we survey four dimension reductions in the order they were proposed by Ingersoll [26], Rogers and Shi [45], Večeř (2001) [53] and by Večeř (2002) [54]. The last mentioned reduction of Večeř (2002) [54] applies to both fixed and floating strike options and leads to the simplest reduced PDE which is numerically well-behaved for a wide range of parameter values.

In this section we derive the reduced equation of Večeř (2002) [54] and we reduce the dimension of the MSHE equation (6.30) in a similar manner, leading to a reduced version of the system (6.29), (6.30) describing the mean squared hedging error of a discretely applied delta hedging strategy for an Asian option. This dimension reduction enables us to formulate the model in a way which resembles models for European options from literature and we compare our model to these models.

Consider a risk-free bond that pays the value of 1 at maturity T, whose price is

$$P(t,T) = e^{-r(T-t)},$$

and a stock, with a dividend yield $\hat{\delta}$, whose price S_t is governed under the physical measure P by the geometric Brownian motion

$$d\mathcal{L}(S)_t = \mu dt + \sigma dW_t.$$

In addition, consider a forward fund containing one stock at time T, whose value is

$$F_t = S_t e^{-\hat{\delta}(T-t)}.$$

Finally, define fund $\tilde{F}$ whose value at T equals $-K + \frac{1}{T}\int_0^T S_\tau d\tau$. The value of this fund at times $t \le T$ is

$$\tilde{F}_t = e^{-r(T-t)} \left(-K + \frac{1}{T}\int_0^t S_\tau d\tau + S_t \frac{e^{(r-\hat{\delta})(T-t)} - 1}{(r-\hat{\delta})T} \right),$$

so, denominating the value in terms of the risk-free bond, we may write

$$\frac{\tilde{F}_t}{P(t,T)} = -K + \frac{1}{T}\int_0^t S_\tau d\tau + \frac{F_t}{P(t,T)} \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}$$

and

$$d\left(\frac{\tilde{F}_t}{P(t,T)}\right) = \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T} d\left(\frac{F_t}{P(t,T)}\right).$$

We intend to use fund F as numéraire and, in particular, we are interested in the dynamics of $\tilde{F}/F$, i.e. the dynamics of fund $\tilde{F}$ denominated in units of fund F, which we express in (6.34) below.

Using the dynamics of the risk-free bond, $dP(t,T) = rP(t,T)dt$, the Itō quotient rule yields

$$d\frac{\tilde{F}}{P} = \frac{\tilde{F}}{P}\left(\frac{d\tilde{F}}{\tilde{F}} - \frac{dP}{P} - \frac{d\tilde{F}}{\tilde{F}}\frac{dP}{P} + \left(\frac{dP}{P}\right)^2\right) = \frac{d\tilde{F}}{P} - \frac{\tilde{F}}{P}\frac{dP}{P}$$

and an analogous formula holds also for F/P instead of $\tilde{F}/P$. Thus, we may express

$$\begin{aligned} d\tilde{F} = Pd\frac{\tilde{F}}{P} + \tilde{F}\frac{dP}{P} &= \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}Pd\frac{F}{P} + \tilde{F}\frac{dP}{P} \\ &= \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\left(dF - F\frac{dP}{P}\right) + \tilde{F}\frac{dP}{P} \\ &= \left(\tilde{F} - \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}F\right)\frac{dP}{P} + \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}dF. \end{aligned} \tag{6.31}$$

Taking F as numéraire and using numéraire invariance, (6.31) yields

$$d\chi = \left(\tilde{F} - \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}F\right)\frac{1}{P}d\frac{P}{F}, \tag{6.32}$$

where $d(P/F)$ can be calculated by the Itō quotient rule

$$d\frac{P}{F} = \frac{P}{F}\left(\frac{dP}{P} - \frac{dF}{F} - \frac{dP}{P}\frac{dF}{F} + \left(\frac{dF}{F}\right)^2\right) = \frac{P}{F}\left(rdt - d\mathcal{L}(S) - \hat{\delta}dt + \sigma^2 dt\right).$$

Finally, we denote $\tilde{F}/F$ by χ, introducing the new, reduced variable

$$\chi_t = \frac{\tilde{F}_t}{F_t} = e^{-(r-\hat{\delta})(T-t)}\frac{\frac{1}{T}\int_0^t S_\tau d\tau - K}{S_t} + \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T} \tag{6.33}$$

and write its dynamics (6.32) as

$$d\chi = \left(\chi - \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)\left((r-\hat{\delta}+\sigma^2)dt - d\mathcal{L}(S)\right). \tag{6.34}$$

6.5.1 Option price

Note that since $\tilde{F}_T = -K + \frac{1}{T}\int_0^T S_\tau d\tau$ we may write the price of a fixed strike Asian call option to be the discounted expectation of the payoff under the risk-neutral measure Q

$$C_t = e^{-r(T-t)} E_t^Q \left[\tilde{F}_T^+ \right] = e^{-r(T-t)} E_t^Q \left[F_T \right] E_t^\bullet \left[\frac{\tilde{F}_T^+}{F_T} \right] = F_t E_t^\bullet \left[\chi_T^+ \right]. \tag{6.35}$$

Under Q, the dynamics of S read

$$d\mathcal{L}(S)_t = \left(r - \hat{\delta} \right) dt + \sigma dW_t^Q,$$

where W^Q is a standard Brownian motion under Q. The Radon-Nikodým derivative of the change of measure from Q to $P^\bullet$ is

$$Z_T = \frac{dP^\bullet}{dQ} = \frac{F_T}{E^Q[F_T]}.$$

The density of this change of measure is

$$Z_t = E_t^Q \left[Z_T \right] = \frac{E_t^Q[F_T]}{E^Q[F_T]} = \frac{E_t^Q[S_T]}{E^Q[S_T]} = \frac{S_t e^{\left(r-\hat{\delta}\right)(T-t)}}{S_0 e^{\left(r-\hat{\delta}\right)T}} = \frac{S_t}{S_0} e^{-\left(r-\hat{\delta}\right)t}$$

and Itō's lemma yields

$$dZ_t = \frac{Z_t}{S_t} dS_t - \left(r - \hat{\delta} \right) Z_t dt = \sigma Z_t dW_t^Q$$

$P^\bullet$ is a forward measure associated with F which means that prices denominated in units of F are $P^\bullet$-martingales (cf. [47, Section 9.4]). Indeed, form (6.35) it is readily seen that C/F, the option price denominated in F, is a $P^\bullet$-martingale because $C_T = \tilde{F}_T^+$ so (6.35) can be written as

$$\frac{C_t}{F_t} = E_t^\bullet \left[\frac{C_T}{F_T} \right].$$

Now we confirm that $\tilde{F}/F$ is also a $P^\bullet$-martingale by calculating its drift. Lemma 1.14 yields the $P^\bullet$-drift of $\mathcal{L}(S)$

$$b_\bullet^{\mathcal{L}(S)} = b_Q^{\mathcal{L}(S)} + \frac{d\mathcal{L}(S)d\mathcal{L}(Z)}{dt} = \left(r - \hat{\delta} \right) + \frac{1}{dt} \sigma^2 \left(dW^Q \right)^2 = r - \hat{\delta} + \sigma^2$$

and from (6.34) the drift and volatility of χ under $P^\bullet$ are

$$b_\bullet^\chi = \left(\frac{\tilde{F}}{F} - \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T} \right) \left(r - \hat{\delta} + \sigma^2 - b_\bullet^{\mathcal{L}(S)} \right) = 0, \tag{6.36}$$

$$\sqrt{c_\bullet^\chi} = \left(\frac{\tilde{F}}{F} - \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T} \right) \sigma. \tag{6.37}$$

The $P^\bullet$ expectation in (6.35) can be written as a function of the state variables

$$f(t, \chi_t) = E_t^\bullet \left[(\chi_T)^+ \right].$$

where $f(t, \chi)$ satisfies

$$f_t + b_\bullet^\chi f_\chi + \frac{1}{2} c_\bullet^\chi f_{\chi\chi} = 0. \tag{6.38}$$

due to the martingale property of conditional expectation. The terminal condition $C_T = \tilde{F}_T^+$ translates to $f(T, \chi) = \chi^+$.

Substituting for the drift and volatility from (6.36) and (6.37), the PDE (6.38) for describing the option price becomes

$$f_t + \frac{\sigma^2}{2} \left(\chi - \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T} \right)^2 f_{\chi\chi} = 0, \tag{6.39}$$
$$f(T, \chi) = \chi^+.$$

Having solved (6.39), the option price can be calculated from (6.35) as

$$C_t = F_t E_t^\bullet \left[\chi_T^+ \right] = F_t f(t, \chi_t).$$

For a stock without dividends, i.e. with $\hat{\delta} = 0$, equation (6.39) coincides with the PDE of Večeř (2002) [54] which we present in Appendix A. The reduced PDE (6.39) can also be derived from the original PDE for an arithmetic Asian option (5.22) by use of the scaling

$$u(t, S_t, A_t) = e^{-\hat{\delta}(T-t)} S_t f(t, \chi_t), \qquad \chi_t = e^{-(r-\hat{\delta})(T-t)} \frac{A_t - K}{S_t} + \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}. \tag{6.40}$$

In case of a floating strike call the price can be calculated by solving the same PDE (A.21) but the terminal condition changes to $f(T, \chi) = (1 - \chi)^+$ to describe the floating strike payoff. The parameter K denoting the fixed strike is in this case set to zero.

6.5.2 Mean squared hedging error

Now we describe the mean squared hedging error of a self-financing strategy consisting of $\tilde{\varphi}$ units of fond $\tilde{F}$ and φ units of fond F. The value of this portfolio is given by

$$V_t = V_0 + \int_0^t \tilde{\varphi}_\tau d\tilde{F}_\tau + \int_0^t \varphi_\tau dF_\tau$$

which becomes

$$\frac{V_t}{F_t} = \frac{V_0}{F_0} + \int_0^t \tilde{\varphi}_\tau d\chi_\tau,$$

when F is taken as numéraire. Note that $\tilde{\varphi}$ is the same as before due to numéraire invariance.

The mean squared hedging error for strategy $\tilde{\varphi}$ is given by

$$\varepsilon^2 = E\left[\left(\tilde{F}_T^+ - V_T\right)^2\right] = E\left[F_T^2\right] E^\star\left[\left(\frac{\tilde{F}_T^+}{F_T} - \frac{V_T}{F_T}\right)^2\right] \tag{6.41}$$

with the Radon-Nikodým derivative of the change of measure from the physical measure P to $P^\star$

$$Z_T = \frac{dP^\star}{dP} = \frac{F_T^2}{E[F_T^2]}.$$

The density of the change of measure is

$$Z_t = E_t\left[Z_T\right] = \frac{E_t[F_T^2]}{E[F_T^2]} = \frac{E_t[S_T^2]}{E[S_T^2]} = \frac{S_t^2 e^{\left(2\mu+\sigma^2\right)(T-t)}}{S_0^2 e^{(2\mu+\sigma^2)T}} = \left(\frac{S_t}{S_0}\right)^2 e^{-\left(2\mu+\sigma^2\right)t},$$

where we used the fact that

$$S_T^2 = \left(S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T}\right)^2 = S_0^2 e^{\left(2\mu-\sigma^2\right)T+2\sigma W_T}$$

which is, under the physical measure P, lognormally distributed with $E_t\left[S_T^2\right] = S_t^2 e^{\left(2\mu+\sigma^2\right)(T-t)}$. Itō's lemma for Z yields

$$\frac{dZ}{Z} = \frac{1}{S^2} d\left(S^2\right) - \left(2\mu+\sigma^2\right) dt = \left(2\frac{dS}{S} + \left(\frac{dS}{S}\right)^2 - \left(2\mu+\sigma^2\right) dt\right) = \sigma dW.$$

By Lemma 1.14, the $P^\star$-drift of $\mathcal{L}(S)$ is

$$b_\star^{\mathcal{L}(S)} = b^{\mathcal{L}(S)} + \frac{d\mathcal{L}(S)d\mathcal{L}(Z)}{dt} = \mu + \sigma^2$$

and hence, from (6.34), the drift and volatility of χ under $P^\star$ are

$$b_\star^\chi = b\,(t,\chi) = \left(\chi - \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)(r - \hat{\delta} - \mu), \tag{6.42}$$

$$\sqrt{c_\star^\chi} = \sqrt{c\,(t,\chi)} = \left(\chi - \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)\sigma. \tag{6.43}$$

Note from (6.43) and (6.37) that $c_\star^\chi = c_\bullet^\chi = c\,(t,\chi)$ since the volatility of a process is not affected by a change of measure.

6.5.3 Reduced model compared to literature

We have reduced the dimension of the problem by using the variable $\chi_t = \tilde{F}_t/F_t$. The fixed strike Asian option payoff is given by $h(\chi) = \chi^+$ and the mean squared hedging error (6.41) is proportional to

$$E^\star\left[\left(\chi_T^+ - \frac{V_0}{F_0} - \int_0^T \tilde{\varphi}_\tau d\chi_\tau\right)^2\right]. \tag{6.44}$$

This setting resembles those used by Bertsimas, Kogan and Lo [3] and related literature which studies hedging errors for European options. In [3] generalized geometric Brownian motion

$$dS_t = \mu(t,S_t)S_t dt + \sigma(t,S_t)S_t dW_t$$

is used for the stock price dynamics. The authors assume that $\sigma(t,S)$ is bounded from below by a positive constant and partial derivatives of $\mu(t,S)$, $\sigma(t,S)$ and $S\sigma(t,S)$ are bounded (see Section 6.2 for a more detailed description of the assumptions) to derive the MSHE approximation (6.4).

Now we want to verify whether these assumptions are fulfilled by our reduced model where χ takes the place of the stock price S. One difference is that χ takes negative as well as positive values ($\tilde{F}_0$ may well be negative) while S is positive.

The dynamics of χ read

$$d\chi_t = \left(\chi_t - \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)\left((r - \hat{\delta} - \mu)dt + \sigma dW_t^\star\right),$$

where $W^\star$ is a standard Brownian motion under $P^\star$, which can be written in the form

$$d\chi_t = \tilde{\mu}(t, \chi_t)\chi_t dt + \tilde{\sigma}(t, \chi_t)\chi_t dW_t^\star \tag{6.45}$$

where we denote

$$\tilde{\mu}(t, \chi_t) = \left(1 - \frac{1}{\chi_t}\frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)(r - \hat{\delta} - \mu),$$
$$\tilde{\sigma}(t, \chi_t) = \left(1 - \frac{1}{\chi_t}\frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)\sigma.$$

Dynamics (6.45) do resemble the generalized geometric Brownian motion but $\tilde{\mu}(t, \chi_t)$ and $\tilde{\sigma}(t, \chi_t)$ do not fulfill the conditions imposed in [3]. The volatility process $\tilde{\sigma}(t, \chi_t)$ can get arbitrarily close to zero when

$$\chi_t \approx \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}$$

and it may even be negative at times. Furthermore, partial derivatives of $\tilde{\mu}(t, \chi_t)$ and $\tilde{\sigma}(t, \chi_t)$ all contain the term $1/\chi^\alpha$ with $\alpha \geq 1$ so they are are not bounded for $\chi_t \approx 0$ like required by [3].

Other papers like Zhang [57] or Gobet and Temam [22] make similar assumptions about the coefficients having bounded derivatives so their results do not directly apply to our reduced model.

6.5.4 Mean squared hedging error approximation

To evaluate the mean squared hedging error approximation (6.28) for a discretely implemented delta hedging strategy for an Asian option we need to evaluate the expectation

$$E\left[\int_0^T \left(\sigma^2 S_\tau^2 u_{SS}(\tau, S_\tau, A_\tau)\right)^2 d\tau\right]. \tag{6.46}$$

From (6.40) with $\hat{\delta} = 0$, $u(t, S, A)$ can be found by transform $u(t, S_t, A_t) = S_t f(t, \chi_t)$ where

$$\chi_t = e^{-r(T-t)} \frac{A_t - K}{S_t} + \frac{1 - e^{-r(T-t)}}{rT},$$

and $f(t, \chi)$ solves the reduced PDE (6.39). We express u_{SS} in terms of f

$$u_{SS}(t, S, A) = e^{-2r(T-t)} \frac{(A-K)^2}{S^3} f_{\chi\chi} = \frac{1}{S} \left(\frac{1 - e^{-r(T-t)}}{rT} - \chi \right)^2 f_{\chi\chi}$$

to write the expectation (6.46) as

$$\begin{aligned}
E&\left[\int_0^T \left(\sigma^2 S_\tau^2 u_{SS}(\tau, S_\tau, A_\tau) \right)^2 d\tau \right] \\
&= E\left[\int_0^T \left(\sigma^2 S_\tau \left(\frac{1 - e^{-r(T-\tau)}}{rT} - \chi_\tau \right)^2 f_{\chi\chi}(\tau, \chi_\tau) \right)^2 d\tau \right] \\
&= E\left[F_T^2\right] E^\star \left[\int_0^T \left(\sigma^2 \left(\frac{1 - e^{-r(T-\tau)}}{rT} - \chi_\tau \right)^2 f_{\chi\chi}(\tau, \chi_\tau) \right)^2 d\tau \right] \\
&= E\left[F_T^2\right] E^\star \left[\int_0^T \Big(c\left(t, \chi_\tau\right) f_{\chi\chi}\left(\tau, \chi_\tau\right) \Big)^2 d\tau \right],
\end{aligned} \tag{6.47}$$

where we used the same change of measure as in (6.41) with the density

$$Z_t = \frac{E_t[F_T^2]}{E[F_T^2]} = \frac{S_t}{S_0} e^{-\left(2\mu + \sigma^2\right)t}$$

and $c(t, \chi)$ is the variance process given by (6.43).

To evaluate the $P^\star$-expectation in (6.47) we define the $P^\star$-martingale

$$\begin{aligned}
M_t &= E_t^\star \left[\int_0^T \Big(c\left(t, \chi_\tau\right) f_{\chi\chi}\left(\tau, \chi_\tau\right) \Big)^2 d\tau \right] \\
&= \int_0^t \Big(c\left(t, \chi_\tau\right) f_{\chi\chi}\left(\tau, \chi_\tau\right) \Big)^2 d\tau + E_t^\star \left[\int_t^T \Big(c\left(t, \chi_\tau\right) f_{\chi\chi}\left(\tau, \chi_\tau\right) \Big)^2 d\tau \right]
\end{aligned}$$

and write the last expectation as a function of the state variables t and χ

$$g(t, \chi_t) = E_t^\star \left[\int_t^T \Big(c\left(t, \chi_\tau\right) f_{\chi\chi}\left(\tau, \chi_\tau\right) \Big)^2 d\tau \right].$$

For $t = T$ the integral runs from T to T so one has $g(T, \chi) = 0$. Itō's lemma for M_t

together with the martingale zero drift condition yield the PDE for $g(t,\chi)$

$$\Big(c\,(t,\chi)\,f_{\chi\chi}\Big)^2 + g_t + b(t,\chi)g_\chi + \frac{1}{2}c\,(t,\chi)\,g_{\chi\chi} = 0,$$

where $b(t,\chi)$ and $c(t,\chi)$ are given by (6.42) and (6.43), respectively. Function $f(t,\chi)$ solves (6.39), from where $c\,(t,\chi)\,f_{\chi\chi} = 2f_t$, which can be substituted into the PDE.

The mean squared hedging error for a fixed strike Asian call option is thus described by the system

$$f_t(t,\chi) + \frac{1}{2}c\,(t,\chi)\,f_{\chi\chi}(t,\chi) = 0, \tag{6.48}$$

$$4\Big(f_t(t,\chi)\Big)^2 + g_t(t,\chi) + b\,(t,\chi)\,g_\chi(t,\chi) + \frac{1}{2}c\,(t,\chi)\,g_{\chi\chi}(t,\chi) = 0. \tag{6.49}$$

with terminal conditions $f(T,\chi) = \chi^+$ and $g(T,\chi) = 0$. In case of a floating strike call, only the terminal condition for f changes to $f(T,\chi) = (1-\chi)^+$ as argued earlier in this section. System (6.48), (6.49) for the reduced functions $f(t,\chi)$, $g(t,\chi)$ can be compared to the system (6.29), (6.30) for the original functions $u(t,S,A)$, $v(t,S,A)$ so that one can see the simpler form of the reduced system.

Having solved the system (6.48), (6.49) the MSHE approximation (6.22) can be evaluated using (6.47) as

$$\varepsilon^2 \approx \frac{\Delta t}{2}E\left[\int_0^T \left(\sigma^2 S_\tau^2 u_{SS}(\tau,S_\tau,A_\tau)\right)^2 d\tau\right] = \frac{\Delta t}{2}S_0^2 e^{\left(2\mu+\sigma^2\right)T} g(0,\chi_0). \tag{6.50}$$

Chapter 7

Numerical results

In this chapter we describe the numerical procedure used to calculate the the approximation (6.50) of the mean squared hedging error for discretely applied delta hedging for an Asian option. As shown in Section 6.5 the approximation can be evaluated by solving the system of PDEs (6.48), (6.49) which describes the Asian option price and the MSHE. We first present solutions of the option price equation (6.48) which is independent of the other PDE and can thus be solved on its own. We then use the found solution $f(t, \chi)$ to solve the MSHE equation (6.49) which enables us to evaluate the MSHE approximation (6.50). Finally, we compare these approximations to actual MSHE values obtained from simulations.

7.1 Computing the option price

We start by solving the PDE describing the price of an Asian option (5.22)

$$u_t + rSu_S + \frac{1}{T}Su_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0$$

which was derived in Section 5.5. As proposed by Večeř [54] and shown in Section 6.5, the dimension of this equation can be reduced and the problem translates to solving (6.48) (which was also labeled (6.39) earlier in Section 6.5)

$$f_t + \frac{\sigma^2}{2}\left(\chi - \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)^2 f_{\chi\chi} = 0$$

with the terminal condition $f(T, \chi) = \chi^+$ in case of a fixed strike call option and $f(T, \chi) = (1 - \chi)^+$ in case of a floating strike call option.

The option price at time t is then given as

$$C_t = u(t, S_t, A_t) = S_t e^{-\hat{\delta}(T-t)} f\left(t, e^{-(r-\hat{\delta})(T-t)} \frac{A_t - K}{S_t} + \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r - \hat{\delta})T}\right) \tag{7.1}$$

with the fixed strike K set to zero in case of a floating strike option.

For the purpose of numerical treatment we limit the considered range for $\chi \in \mathbb{R}$ to $\chi \in [-\chi_{max}, \chi_{max}]$ and we define the discretization for the time variable t as $t_i = i \times \delta t$, $i = 0, \ldots, n_t$, with $\delta t = T/n_t$ and the discretization for the spatial variable χ as $\chi_j = -\chi_{max} + j \times \delta\chi$, $j = 0, \ldots, n_\chi$, with $\delta\chi = 2\chi_{max}/n_\chi$. Furthermore we denote by f_j^i the approximation of $f(t_i, \chi_j)$ and by c_j^i the value of the coefficient which multiplies $\frac{1}{2} f_{\chi\chi}$ in equation (6.39), i.e.

$$c_j^i = c_\star^\chi(t_i, \chi_j) = \sigma^2 \left(\chi_j - \frac{1 - e^{-(r-\hat{\delta})(T-t_i)}}{(r - \hat{\delta})T}\right)^2 \tag{7.2}$$

with $c(t, \chi)$ defined in (6.43).

The Crank-Nicolson method for PDE (6.48) reads

$$\begin{aligned} -\frac{1}{4} c_j^i \frac{\delta t}{(\delta\chi)^2} f_{j-1}^i + \left(1 + \frac{c_j^i}{2} \frac{\delta t}{(\delta\chi)^2}\right) f_j^i - \frac{1}{4} c_j^i \frac{\delta t}{(\delta\chi)^2} f_{j+1}^i \\ = \frac{1}{4} c_j^{i+1} \frac{\delta t}{(\delta\chi)^2} f_{j-1}^{i+1} + \left(1 - \frac{c_j^{i+1}}{2} \frac{\delta t}{(\delta\chi)^2}\right) f_j^{i+1} + \frac{1}{4} c_j^{i+1} \frac{\delta t}{(\delta\chi)^2} f_{j+1}^{i+1} \end{aligned} \tag{7.3}$$

for $i = 0, \ldots, (n_t - 1)$ and $j = 1, \ldots, (n_\chi - 1)$, which applies to both fixed and floating strike options. What differs are the terminal and boundary conditions described in the following subsections.

7.1.1 Fixed strike call

For the fixed strike call option the terminal condition for equation (7.3), established in Section 6.5,

$$f_j^{n_t} = \max(0, \chi_j), \qquad j = 0, \ldots, (n_\chi - 1) \tag{7.4}$$

is accompanied by the boundary conditions

$$f_0^i = 0, \qquad f_{n_\chi}^i = \chi_{max}, \qquad i = 0, \dots, (n_t - 1), \tag{7.5}$$

which correspond to $\lim_{\chi \to -\infty} f(t,\chi) = 0$ and $\lim_{\chi \to +\infty} f(t,\chi) = \chi_{max}$, respectively. Recall (6.40)

$$\chi_t = e^{-(r-\hat{\delta})(T-t)} \frac{A_t - K}{S_t} + \frac{1 - e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}$$

so $\chi \to -\infty$ refers to situations where S and A are small. In that case the fixed strike call with the payoff $(A_T - K)^+$ is unlikely to end up in the money, justifying the former condition in (7.5). On the other hand, $\chi \to +\infty$ refers to situations where A is large (and S possibly small). In that case the fixed strike call is very likely to end up in the money, justifying the latter condition in (7.5).

Denoting F^i the column vector of interior point values in the i-th time layer, i.e. $F^i = \left(f_1^i, \dots, f_{n_\chi - 1}^i\right)^T$, we can rewrite (7.3) as

$$M_1^i F^i = M_2^{i+1} F^{i+1} + B^{i+1}, \qquad i = 0, \dots, (n_t - 1), \tag{7.6}$$

where matrices $M_1^i, M_2^i \in \mathbb{R}^{(n_\chi - 1)\times(n_\chi - 1)}$ are tridiagonal with non-zero elements given by

$$\begin{aligned}
M_1^i(j, j-1) &= -\frac{1}{4} c_j^i \frac{\delta t}{(\delta\chi)^2}, & j &= 2, \dots, (n_\chi - 1), \\
M_1^i(j, j) &= 1 + \frac{c_j^i}{2} \frac{\delta t}{(\delta\chi)^2}, & j &= 1, \dots, (n_\chi - 1), \\
M_1^i(j, j+1) &= -\frac{1}{4} c_j^i \frac{\delta t}{(\delta\chi)^2}, & j &= 1, \dots, (n_\chi - 2)
\end{aligned}$$

and

$$\begin{aligned}
M_2^i(j, j-1) &= \frac{1}{4} c_j^i \frac{\delta t}{(\delta\chi)^2}, & j &= 2, \dots, (n_\chi - 1), \\
M_2^i(j, j) &= 1 - \frac{c_j^i}{2} \frac{\delta t}{(\delta\chi)^2}, & j &= 1, \dots, (n_\chi - 1), \\
M_2^i(j, j+1) &= \frac{1}{4} c_j^i \frac{\delta t}{(\delta\chi)^2}, & j &= 1, \dots, (n_\chi - 2),
\end{aligned}$$

and vector $B^i \in \mathbb{R}^{(n_\chi - 1)\times 1}$ compensates for the boundary conditions (7.5). The elements of B^i are all zeros except the last one

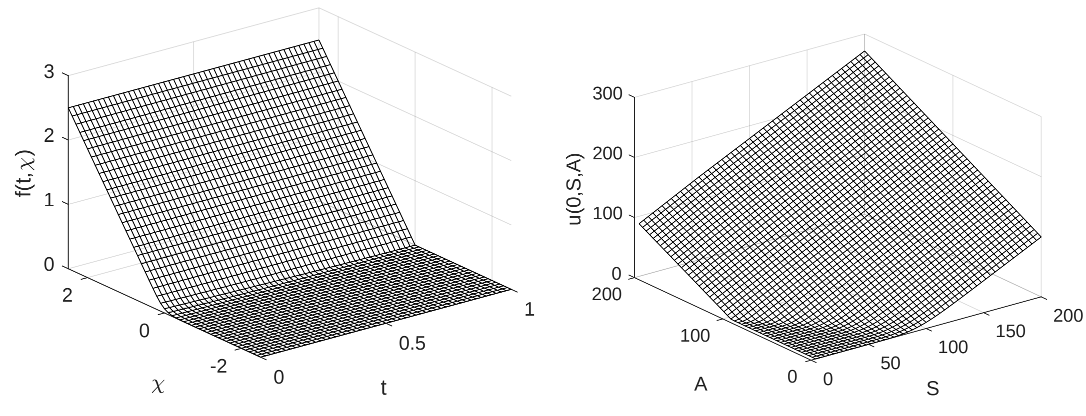
$$B^i(n_\chi - 1) = \frac{1}{2} c^i_{n_\chi - 1} \frac{\delta t}{(\delta \chi)^2} \chi_{max}, \tag{7.7}$$

which stems from the second of the boundary conditions (7.5) and equation (7.3) with $j = n_\chi - 1$.

Starting with the terminal condition (7.4) and solving the system (7.6) we can calculate the internal points of the next time layer $F^i = \left(f^i_1, \ldots, f^i_{n_z - 1}\right)^T$ from the last known time layer $F^{i+1} = \left(f^{i+1}_1, \ldots, f^{i+1}_{n_z - 1}\right)^T$. The boundary points are then given by the boundary conditions (7.5).

Having calculated the numerical approximations of $f(t, \chi)$, the fixed strike call option price at any time can be obtained using the transform (7.1).

The left panel of Figure 7.1 shows the solution $f(t, \chi)$ of (6.48) for a fixed strike Asian call option with parameters set to $r = 0.15$, $\sigma = 0.30$, $\hat{\delta} = 0$, $T = 1$. The right panel of the same figure shows the solution $u(t, S, A)$ to the Asian option pricing equation (5.22) at time $t = 0$. This was calculated from $f(t, \chi)$ by the transform (7.1). The fixed strike was set to $K = 100$. We observe that the option price rises with increasing S and A which can be expected from the fixed strike call payoff $(A_T - K)^+$.



(a) $f(t, \chi)$. **(b)** $u(0, S, A)$.

Figure 7.1: Fixed strike call: (a) Soluiton $f(t, \chi)$ of the reduced equation (6.48) and (b) the option price $C_0 = u(0, S, A)$ calculated by (7.1).

To verify whether the prices obtained by the numerical procedure are sensible, we calculated Monte Carlo estimates of option prices with the same parameters and the current stock price set to $S_0 = 90, 100$ and 110. We used the Matlab function *asianbyls*

Table 7.1: Comparison of Asian option prices calculated by Monte Carlo simulations and by the described PDE approach.

	Fixed strike		Floating strike	
S_0	MC Price	PDE Price	MC Price	PDE Price
90	4.8031	4.8096	9.6634	9.6719
100	10.2131	10.2101	10.7439	10.7465
110	17.4551	17.4606	11.7948	11.8108

with variance reduced by the antithetic variate method and we set the number of simulations to 1 000 000. The resulting MC prices are shown in Table 7.1 together with prices calculated by (7.1) which confirms that the fixed strike call option prices obtained by the two methods well agree.

7.1.2 Floating strike call

For the floating strike call option the terminal condition for equation (7.3), established in Section 6.5,

$$f_j^{n_t} = \max(0, 1 - \chi_j), \qquad j = 0, \ldots, (n_\chi - 1) \tag{7.8}$$

is accompanied by the boundary conditions

$$f_0^i = 1 + \chi_{max}, \qquad f_{n_\chi}^i = 0, \qquad i = 0, \ldots, (n_t - 1), \tag{7.9}$$

which can be justified by similar reasoning as (7.5) in Section 7.1.1 using the floating strike call with the payoff $(S_T - A_T)^+$.

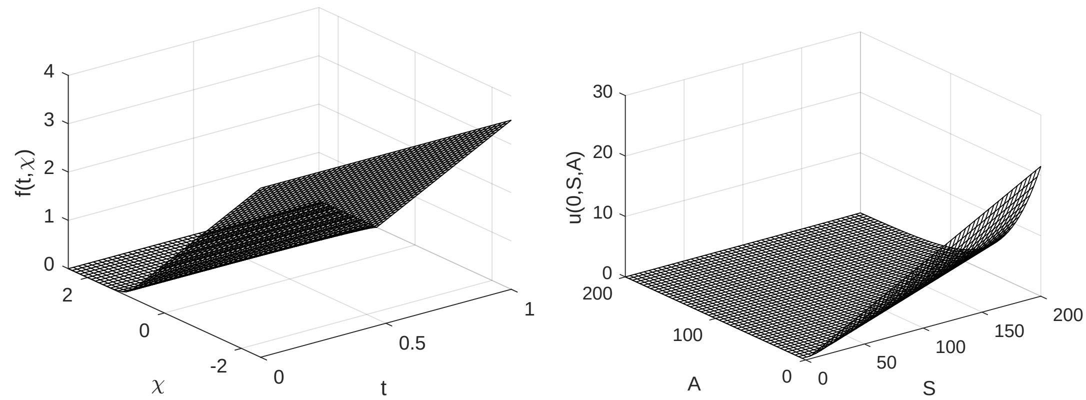
The vector form (7.6) of the numerical scheme still applies with the same M_1^i and M_2^i but the vector B^i, corresponding to the boundary conditions, differs. Instead of (7.7), we define $B^i \in \mathbb{R}^{(n_\chi - 1)\times 1}$ as a vector consisting of all zeros except the first element which is given as

$$B^i(1) = \frac{1}{2} c_1^i \frac{\delta t}{(\delta\chi)^2} \left(1 + \chi_{max}\right), \tag{7.10}$$

obtained from the boundary conditions (7.9) and equation (7.3) with $j = 1$.

With these adjustments, the procedure described in Section 7.1.1 can be used to find the price of a floating strike call.

Figure 7.2 shows the solution $f(t,\chi)$ of (6.48) for a floating strike Asian call option and solution $u(t,S,A)$ of (5.22) at time $t=0$. The same parameters $r=0.15$, $\sigma=0.30$, $\hat{\delta}=0$, $T=1$ were used as in Figure 7.1 in case of the fixed strike call. We observe that the option price $u(t,S,A)$, shown in panel (b) of Figure 7.2, rises with increasing S and decreasing A which is consistent with the floating strike call payoff $(S_T-A_T)^+$.



(a) $f(t,\chi)$. **(b)** $u(0,S,A)$.

Figure 7.2: Floating strike call: (a) Soluiton $f(t,\chi)$ of the reduced equation (6.48) and (b) the option price $C_0=u(0,S,A)$ calculated by (7.1).

Once again, we confronted the prices found by the numerical procedure to Monte Carlo estimated prices of floating strike Asian call options. The same settings were used for the Matlab function *asianbyls* as in Section 7.1.1 with 1 000 000 simulations resulting with prices shown in Table 7.1. These can be compared to prices calculated by solving (7.6) and using the transform (7.1). The floating strike call prices also agree with the MC estimates.

7.2 Solving the MSHE equation

Having found a numerical solution $f(t,\chi)$ to the reduced option price equation (6.48), which was independent of $g(t,\chi)$, we can approach to solving the other equation of our interest (6.49)

$$4\Big(f_t(t,\chi)\Big)^2+g_t(t,\chi)+b_\star^\chi g_\chi(t,\chi)+\frac{1}{2}c_\star^\chi g_{\chi\chi}(t,\chi)=0$$

which characterizes the mean squared hedging error. We use the same discretizations t_i and χ_j as in the previous section for the time and spatial variable, respectively. We denote by g_j^i the approximation of $g(t_i,\chi_j)$ and by b_j^i the coefficient next to g_x in (6.49), i.e. the drift of χ under $P^\star$ defined in (6.42), as

$$b_j^i = b_\star^\chi(t_i, \chi_j) = (r - \hat{\delta} - \mu)\left(\chi_j - \frac{1 - e^{-(r-\hat{\delta})(T-t_i)}}{(r-\hat{\delta})T}\right). \tag{7.11}$$

Furthermore, we use again c_j^i defined in (7.2) and express the Crank-Nicolson scheme for PDE (6.49)

$$\begin{aligned} -\frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c_j^i}{\delta\chi} - b_j^i\right)g_{j-1}^i + \left(1 + \frac{c_j^i}{2}\frac{\delta t}{(\delta\chi)^2}\right)g_j^i - \frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c_j^i}{\delta\chi} + b_j^i\right)g_{j+1}^i \\ = \frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c_j^{i+1}}{\delta\chi} - b_j^{i+1}\right)g_{j-1}^{i+1} + \left(1 - \frac{c_j^{i+1}}{2}\frac{\delta t}{(\delta\chi)^2}\right)g_j^{i+1} \\ + \frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c_j^{i+1}}{\delta\chi} + b_j^{i+1}\right)g_{j+1}^{i+1} + \frac{4}{\delta t}\left(f_j^{i+1} - f_j^i\right)^2. \end{aligned} \tag{7.12}$$

for $i = 0, \ldots, (n_t - 1)$ and $j = 1, \ldots, (n_\chi - 1)$. The zero terminal condition

$$g_j^{n_t} = 0, \qquad j = 0, \ldots, (n_\chi - 1) \tag{7.13}$$

is accompanied by zero boundary conditions

$$g_0^i = 0, \qquad g_{n_\chi}^i = 0, \qquad i = 0, \ldots, (n_t - 1). \tag{7.14}$$

System (7.12) can be written in vector notation

$$M_3^i G^i = M_4^{i+1} G^{i+1} + C^i, \qquad i = 0, \ldots, (n_t - 1), \tag{7.15}$$

where matrices $M_3^i, M_4^i \in \mathbb{R}^{(n_\chi - 1)\times(n_\chi - 1)}$ are tridiagonal with non-zero elements given by

$$\begin{aligned} M_3^i(j, j-1) &= -\frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c_j^i}{\delta\chi} - b_j^i\right), && j = 2, \ldots, (n_\chi - 1), \\ M_3^i(j, j) &= 1 + \frac{c_j^i}{2}\frac{\delta t}{(\delta\chi)^2}, && j = 1, \ldots, (n_\chi - 1), \\ M_3^i(j, j+1) &= -\frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c_j^i}{\delta\chi} + b_j^i\right), && j = 1, \ldots, (n_\chi - 2) \end{aligned}$$

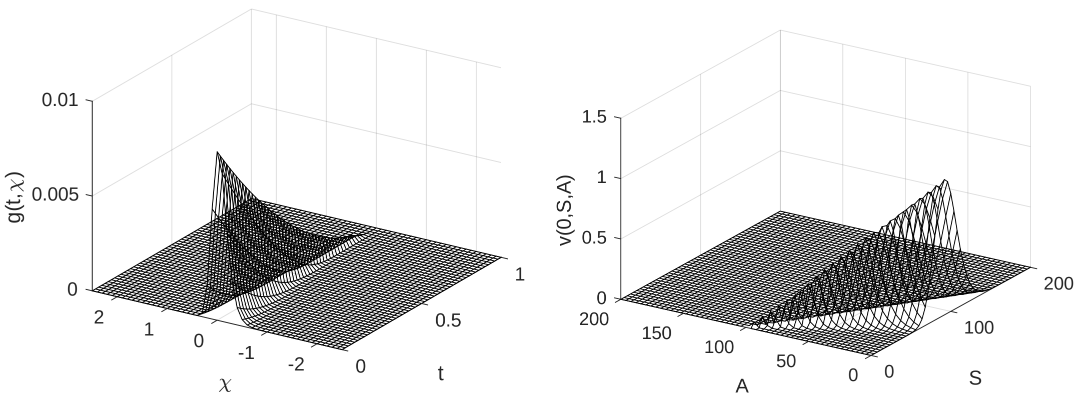
and

$$
\begin{aligned}
M_4^i(j,j-1) &= \frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c_j^i}{\delta\chi} - b_j^i\right), && j = 2,\dots,(n_\chi - 1),\\
M_4^i(j,j) &= 1 - \frac{c_j^i}{2}\frac{\delta t}{(\delta\chi)^2}, && j = 1,\dots,(n_\chi - 1),\\
M_4^i(j,j+1) &= \frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c_j^i}{\delta\chi} + b_j^i\right), && j = 1,\dots,(n_\chi - 2)
\end{aligned}
$$

and $C^i \in \mathbb{R}^{(n_\chi-1)\times 1}$ is a vector containing the numerical approximation of the term containing time derivative of f in (6.48)

$$
C^i(j) = \frac{4}{\delta t}\left(f_j^{i+1} - f_j^i\right)^2, \qquad j = 1,\dots,(n_\chi - 1). \tag{7.16}
$$

Starting from the terminal condition (7.13) and solving the system (7.15) one can calculate the internal points of the next time layer $G^i = \left(g_1^i,\dots,g_{n_z-1}^i\right)^T$ from the last known time layer $G^{i+1} = \left(g_1^{i+1},\dots,g_{n_z-1}^{i+1}\right)^T$. The boundary points are then given by the boundary conditions (7.14).

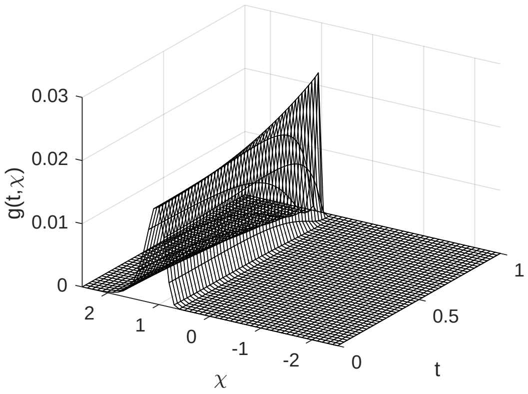


(a) $g(t,\chi)$. **(b)** $v(0,S,A)$.

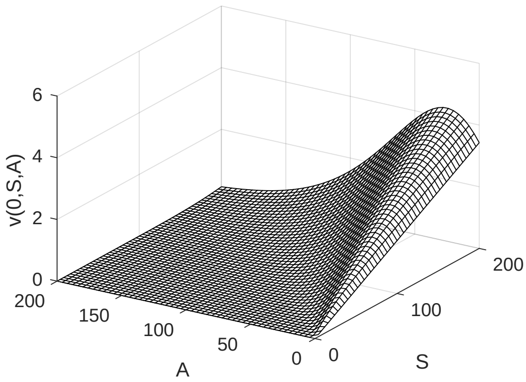
Figure 7.3: Fixed strike call: (a) Soluiton $g(t,\chi)$ of the reduced equation (6.49) and (b) function $v(0,S,A)$ describing the MSHE.

Left panels of Figures 7.3 and 7.4 show for the fixed strike call and floating strike call, respectively, the solution $g(t,\chi)$ of (6.49), the reduced version of (6.30). The same parametrization was used as in Figures 7.1 and 7.2, i.e. $r = 0.15$, $\sigma = 0.30$, $\hat{\delta} = 0$, $T = 1$. In addition, the drift of the stock price was set to $\mu = 0.2$ and the fixed strike was set to $K = 100$. The right panels of the same figures show $v(t,S,A)$, the solutions of (6.30), at time $t = 0$ which were calculated from $g(t, \chi)$ by the transform

$$v(0, S, A) = S_0^{(2\mu+\sigma^2)T} g\left(0, e^{-(r-\hat{\delta})T}\frac{A-K}{S} + \frac{1-e^{-(r-\hat{\delta})T}}{(r-\hat{\delta})T}\right).$$



(a) $g(t, \chi)$.



(b) $v(0, S, A)$.

Figure 7.4: Floating strike call: (a) Soluiton $g(t, \chi)$ of the reduced equation (6.49) and (b) function $v(0, S, A)$ describing the MSHE.

7.3 Mean squared hedging error approximation

In this section we verify for an Asian option the mean squared hedging error approximation (6.28)

$$e_n^2 = \frac{\Delta t}{2} E\left[\int_0^T \sigma^4 S_t^4 u_{SS}^2(t, S_t, A_t)dt\right] = \frac{\Delta t}{2} S_0^{(2\mu+\sigma^2)T} g(0, \chi_0) \tag{7.17}$$

by comparing it to the actual MSHE

$$\begin{aligned}\varepsilon_n^2 &= E\left[\left(\int_0^T \Big[u_S(t, S_t, A_t) - u_S\big(\theta(t), S_{\theta(t)}, A_{\theta(t)}\big)\Big] dS_t\right)^2\right] \\ &= E\left[\left(\sum_{k=1}^n \left[\int_{t_{k-1}}^{t_k} \Big(u_S(t, S_t, A_t) - u_S\big(t_{k-1}, S_{t_{k-1}}, A_{t_{k-1}}\big)\Big) dS_t\right]\right)^2\right].\end{aligned} \tag{7.18}$$

Recall that $\Delta t = t_k - t_{k-1} = T/n$ so both (7.17) and (7.18) depend on n and we denote this dependence by the subscript in e_n^2 and ε_n^2.

The actual MSHE ε_n^2 is found by 50 000 simulations of the stock price S_t and subsequent use of the numerical approximation of $f(t, \chi)$ to calculate the option delta as

$$u_S(t, S_t, A_t) = e^{-\hat{\delta}(T-t)} f(t, \chi) - e^{-r(T-t)} \frac{A-K}{S} f_\chi(t, \chi)$$

at 10 001 different times τ_i ranging from 0 to $T = 1$. These values of delta are in turn used to numerically approximate the integrals in (7.18) as

$$\begin{aligned}\int_{t_{k-1}}^{t_k} & \Big(u_S(t, S_t, A_t) - u_S\big(t_{k-1}, S_{t_{k-1}}, A_{t_{k-1}}\big)\Big) dS_t \\ & \approx \sum_{i=0}^{n_k - 1} \Big(u_S(\tau_i, S_{\tau_i}, A_{\tau_i}) - u_S\big(t_{k-1}, S_{t_{k-1}}, A_{t_{k-1}}\big)\Big)\big(S_{\tau_{i+1}} - S_{\tau_i}\big)\end{aligned}$$

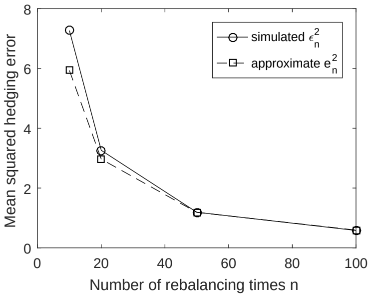
where $t_{k-1} = \tau_0 < \tau_1 < \cdots < \tau_{n_k} = t_k$.

Figures 7.5 and 7.6 show, respectively, for the fixed strike and floating strike call the development of the MSHE approximation e_n^2 (7.17) together with the simulated MSHE ε_n^2 (7.18) as the number of trading times n increases. We considered $n = 10$, 20, 50 and 100. The same parametrization $r = 0.15$, $\sigma = 0.30$, $\hat{\delta} = 0$, $T = 1$ was used as in previous sections and we considered three different values for the initial stock price S_0: 90, 100 and 110. The first rows of Figures 7.5 and 7.6 (panels (a), (b) and (c)) show that the mean squared hedging error tends to zero as trading becomes more frequent and also that the approximate MSHE e_n^2 agrees with the simulated MSHE ε_n^2 for both the fixed and floating strike call.

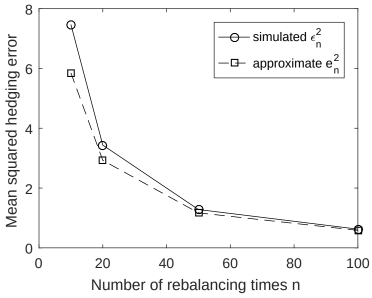
Panels (d) of Figures 7.5 and 7.6 show the development of the absolute difference between the approximate and simulated MSHE, i.e. $|\varepsilon_n^2 - e_n^2|$, as n increases. Panels (e) of the same figures show the development of relative differences

$$\left| \frac{\varepsilon_n^2 - e_n^2}{\varepsilon_n^2} \right|.$$

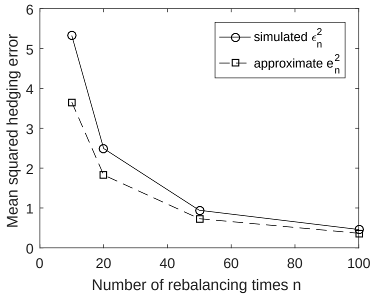
The absolute differences of ε_n^2 and e_n^2 get close to zero with more frequent rebalancing and the relative differences get to around 5% for all cases except the in-the-money fixed strike call with $S_0 = 110$ where the difference is 20.5%. However, the difference does have a decreasing tendency as can be seen in panels (c) and (e) of Figure 7.5.



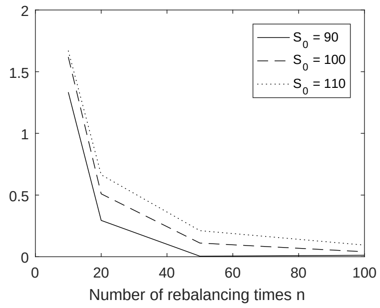
(a) $S_0 = 90$.



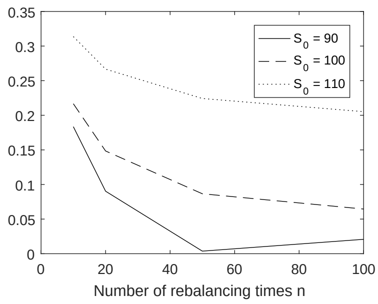
(b) $S_0 = 100$.



(c) $S_0 = 110$.

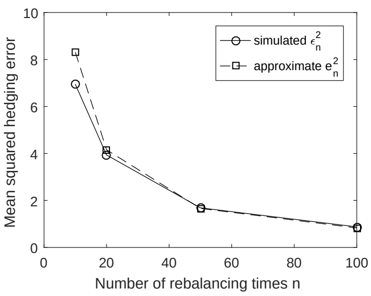


(d) $\left|\varepsilon_n^2 - e_n^2\right|$.

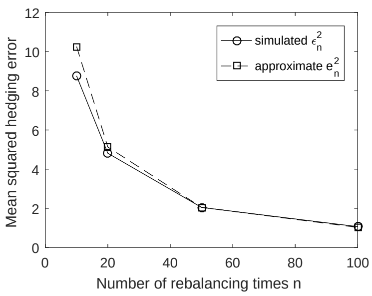


(e) $\left|\varepsilon_n^2 - e_n^2\right| / \varepsilon_n^2$.

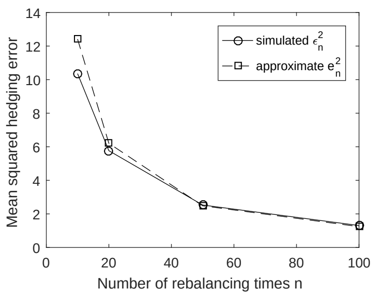
Figure 7.5: Fixed strike call: Top row shows the development of ε_n^2 and e_n^2 for increasing number of trading times and different S_0. Bottom row shows the (d) absolute and (e) relative differences between the simulated and approximate MSHE.



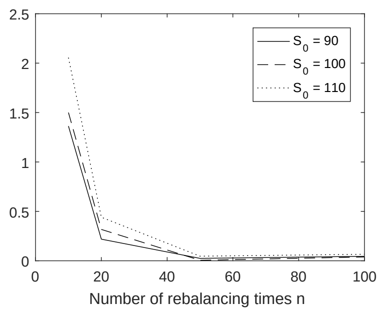
(a) $S_0 = 90$.



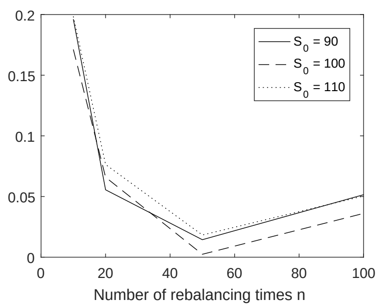
(b) $S_0 = 100$.



(c) $S_0 = 110$.



(d) $\left|\varepsilon_n^2 - e_n^2\right|$.



(e) $\left|\varepsilon_n^2 - e_n^2\right| / \varepsilon_n^2$.

Figure 7.6: Floating strike call: Top row shows the development of ε_n^2 and e_n^2 for increasing number of trading times and different S_0. Bottom row shows the (d) absolute and (e) relative differences between the simulated and approximate MSHE.

Chapter 8

Conclusion of Part II

In Part II of this thesis we dealt with quadratic hedging, focusing on the mean squared hedging error of a discretely implemented delta hedging strategy for arithmetic Asian options.

In Chapter 5 we introduced basic concepts from tie field of option pricing to the reader. We used plain European options to explain the derivation of the Black-Scholes partial differential equation for the option price, using a self-financing replicating portfolio. Then we introduced Asian options and derived a PDE for their price which has an additional dimension compared to the European option price PDE. The dimension was then later reduced in Section 6.5 in line with [54].

Chapter 6 focused on quadratic hedging. After giving a brief overview of relevant results from literature and establishing the presence of hedging errors in an incomplete market, we moved our attention to the mean squared hedging error of discretely implemented delta hedging strategies. In Section 6.3 we heuristically derived the approximation (6.12) of the MSHE of a discretely implemented strategy in a more general setting than the ones used in literature. In the following section we applied this approximation to discrete delta hedging of Asian options and we derived the PDE (6.30) for function v describing the MSHE based on our approximation (6.12). In Section 6.5 we reduced the dimension of PDE (6.30), recovering the reduced equation for the Asian option price of [54] in the process. We explained how the MSHE approximation (6.12) can be evaluated using the solution g of the reduced MSHE equation (6.49).

In Chapter 7 we described numerical methods used and we presented results. In Section 7.1 we solved PDE (6.48) for the price of a fixed and floating strike Asian call option. We then used this solution in the following section in solving PDE (6.49) describing the MSHE. Finally, in Section 7.3 we verified the MSHE approximation (6.12), evaluated by means of the PDE solutions from previous sections, by comparing it to simulated actual MSHE and found that the approximation fits reasonably well with the actual mean squared hedging error.

An obvious extension of the research presented in Part II of this thesis is proving rigorously that the MSHE approximation (6.12), heuristically derived in Section 6.3, indeed holds. Another possible extension would be studying hedging errors for Asian options under generalized Brownian motion or in Lévy models. This could be motivated by the findings of [11] where the authors show that in case of European options in a Lévy model the mean squared hedging error is related to the MSHE of a discretely implemented strategy in the Black-Scholes model.

Conclusion

In this dissertation thesis we examined two applications of optimization in financial mathematics. Part I, consisting of Chapters 2-4 dealt with optimal liquidation or optimal trade execution when the selling price is adversely affected by the current liquidation rate. Our formulation differs from most of optimal liquidation literature in giving the pressure to liquidate endogenously and using a stochastic time horizon which is determined as a part of the optimal strategy. The endogenous pressure to liquidate in our model may be due to several reasons including the asset price falling on average or time discounting. We found that liquidation in presence of a temporary price impact inevitably leads to an implementation shortfall and this shortfall is consistent with the square root law, known from empirical studies, which says that the resulting relative implementation shortfall is proportional to the square root of the initial size of the inventory.

We presented the optimal liquidation problem in Chapter 2 where we also showed that it leads to a severely singular ordinary differential equation. In the same chapter we described a procedure which transforms the original problem IVP_0 by adding the boundary condition $u'(\infty) = 0$, truncating the spatial interval to $[0, L]$ and introducing a time variable to solving the boundary value problem $\mathrm{BVP}^t_{[0,L]}$. The solutions of $\mathrm{BVP}^t_{[0,L]}$ are shown to be unique and converging to the solutions of IVP_0. This knowledge was then used in Chapter 3 to solve the optimal liquidation numerically. The findings of Part I were in more detail concluded in Chapter 4 where we also mention possible ideas of future research in the field of optimal liquidation.

Part II, consisting of Chapters 5-8 dealt with quadratic hedging. In particular, we focused on approximating the mean squared hedging error of a discretely implemented delta hedging strategy for arithmetic Asian options. In Chapter 5 we introduced options and we showed a derivation of a partial differential equation for the price of an Asian option.

In Chapter 6 we heuristically derived the approximation (6.12) of the mean squared hedging error of a discretely implemented hedging strategy. This approximation agrees with the MSHE approximation found in [22] and [57] for a discretely applied delta hedging strategy in case of a European option in the Black-Scholes setting. We then applied the MSHE approximation (6.12) to the case of a discretely implemented delta hedging strategy for an arithmetic Asian option and we proposed that the approximation can be evaluated by solving the system of partial differential equations (6.48), (6.49). In Chapter 7 we solved system system (6.48), (6.49) numerically, solving first (6.48) which describes the option price and which is independent of the other PDE. Subsequently, we used the solution f of (6.48) to solve (6.49). Finally, we used the solution g of (6.49) to evaluate the MSHE approximation (6.12) which we verified by comparing it to simulated actual MSHE. A more detailed conclusion of the results of Part II was made in Chapter 8 where we also suggested possible future extensions of the research of Asian option hedging.

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Appendix

A Dimension reduction for the Asian option PDE

In Section 5.5 we derived the partial differential equation (5.22)

$$u_t + rSu_S + \frac{1}{T}Su_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0$$

for the price of an arithmetic average Asian option which is a function $u(t, S_t, A_t)$ of time t, the current stock price S_t and the average stock price A_t. In this appendix we survey four different dimension reductions used in literature. We present them in the order they appeared and we use the same notation χ for the reduced variable, which replaces the original variables S and A, and $f(t, \chi_t)$ for the reduced function which replaces the original $u(t, S_t, A_t)$. The definitions of χ and $f(t, \chi)$ differ for each of the dimension reductions.

A.1 Reduction by Ingersoll

Ingersoll [26] reduces the dimension of the PDE (5.22) in case of a floating strike option. He uses the self-similarity of the payoff

$$h(S_T, A_T) = (S_T - A_T)^+ = A_T \left(\frac{S_T}{A_T} - 1\right)^+ = A_T h\left(\frac{S_T}{A_T}, 1\right),$$

introduces a new variable $\chi_t = S_t/A_t$ and expresses the option price as

$$u(t, S_t, A_t) = A_t f(t, \chi_t), \tag{A.1}$$

where $f(t,\chi)$ is a function of only two variables. The partial derivatives of $u(t,S,A)$ can be expressed as

$$\begin{aligned}
u_t(t,S,A) &= Af_t(t,\chi)\\
u_S(t,S,A) &= Af_\chi(t,\chi)\frac{1}{A} = f_\chi(t,\chi)\\
u_{SS}(t,S,A) &= \frac{1}{A}f_{\chi\chi}(t,\chi)\\
u_A(t,S,A) &= f(t,\chi) + Af_\chi(t,\chi)\left(-\frac{S}{A^2}\right) = f(t,\chi) - \chi f_\chi(t,\chi)
\end{aligned}$$

which, when substituted into equation (5.22), reduce the PDE to

$$f_t + \left(r - \frac{1}{T}\chi\right)\chi f_\chi + \frac{1}{2}\sigma^2\chi^2 f_{\chi\chi} - \left(r - \frac{1}{T}\chi\right) f = 0 \tag{A.2}$$

and the terminal condition (6.27) becomes $f(T,\chi_T) = (\chi_T - 1)^+$. In addition, there are the boundary conditions $f(t,0) = 0$, $\lim_{\chi\to\infty} f_\chi(t,\chi) = 1$.

A.2 Reduction by Rogers and Shi

Rogers and Shi [45] propose a dimension reduction which works for both fixed and floating strike Asian options, unlike the previously mentioned method by Ingersoll. Instead of the average price process $A_t = \frac{1}{T}\int_0^t S_t dt$, they work with a shifted variable

$$I_t = A_t - K = -K + \frac{1}{T}\int_0^t S_u du, \tag{A.3}$$

where K is the strike price in case of a fixed strike option. In case of a floating strike option it is defined as $K = 0$ so I coincides with A.

Under the the risk-neutral measure the dynamics of the variables are

$$\begin{aligned}
dS &= rSdt + \sigma SdW,\\
dI &= \frac{1}{T}Sdt.
\end{aligned}$$

The option price C_t at time t can be calculated as the discounted expected value under the risk-neutral probability of the option payoff. For a fixed strike call option this is

$$C_t = e^{-r(T-t)}E_t[I_T^+]. \tag{A.4}$$

A new variable $\chi_t = I_t/S_t$ is introduced and its dynamics read

$$\begin{aligned} d\chi = d\left(\frac{I}{S}\right) &= \frac{I}{S}\left(\frac{dI}{I} - \frac{dS}{S} - \frac{dI}{I}\frac{dS}{S} + \left(\frac{dS}{S}\right)^2\right) \\ &= \frac{dI}{S} - \chi\frac{dS}{S} - \frac{dI}{S}\frac{dS}{S} + \chi\left(\frac{dS}{S}\right)^2 \\ &= \frac{1}{T}dt - \chi\Big(d\mathcal{L}(S) - \sigma^2 dt\Big). \end{aligned} \tag{A.5}$$

Then the measure is changed to $P^\bullet$ defined by $dP^\bullet/dP = Z_T/Z_0$ with

$$Z_t = e^{r(T-t)}S_t \tag{A.6}$$

being the forward price of the stock.

Under the new measure $P^\bullet$ the option pricing formula (A.4) becomes

$$C_t = e^{-r(T-t)}E_t[I_T^+] = e^{-r(T-t)}Z_t E_t^\bullet\left[\frac{I_T^+}{Z_T}\right] = S_t E_t^\bullet\left[\chi_T^+\right], \tag{A.7}$$

where we used that $Z_T = S_T$ and $\frac{I_T^+}{S_T} = \left(\frac{I_T}{S_T}\right)^+$. We need the $P^\bullet$-dynamics of χ to evaluate the expectation in (A.7). Lemma 1.14 yields the $P^\bullet$-drift of $\mathcal{L}(S)$

$$b_\bullet^{\mathcal{L}(S)} = b^{\mathcal{L}(S)} + \frac{d\mathcal{L}(S)d\mathcal{L}(Z)}{dt} = b^{\mathcal{L}(S)} + c^{\mathcal{L}(S)} = r + \sigma^2 \tag{A.8}$$

and this combined with (A.5) allows us to express the drift and volatility of χ under measure $P^\bullet$

$$\begin{aligned} b_\bullet^\chi &= \frac{1}{T} - \chi\left(b_\bullet^{\mathcal{L}(S)} - \sigma^2\right) = \frac{1}{T} - r\chi \\ \sqrt{c_\bullet^\chi} &= \sigma\chi. \end{aligned}$$

In other words, the dynamics of χ can be written as

$$d\chi = \left(\frac{1}{T} - r\chi\right)dt + \sigma\chi dW_t^\bullet,$$

where $W_t^\bullet$ is a standard Brownian motion under $P^\bullet$. Thus, χ is Markov, the conditional expectation in (A.7) can be written as a function $f(t,\chi) = E_t^\bullet\left[\chi_T^+\right]$ and it is a $P^\bullet$-martingale.

Using Itō's lemma, we express

$$\begin{aligned} df(t,\chi) &= \left(f_t + b^{\chi}_{\bullet} f_{\chi} + \frac{1}{2} c^{\chi}_{\bullet} f_{\chi\chi}\right) dt + \sqrt{c^{\chi}_{\bullet}} f_{\chi} dW^{\bullet}_t \\ &= \left(f_t + \left(\frac{1}{T} - r\chi\right) f_{\chi} + \frac{\sigma^2}{2}\chi^2 f_{\chi\chi}\right) dt + \sigma\chi f_{\chi} dW^{\bullet}_t. \end{aligned} \tag{A.9}$$

The martingale property of f yields the PDE

$$f_t + \left(\frac{1}{T} - r\chi\right) f_{\chi} + \frac{\sigma^2}{2}\chi^2 f_{\chi\chi} = 0 \tag{A.10}$$

and from (A.7) the terminal condition $f(T,\chi) = \chi^+$.

In case of a floating strike option, equation (A.4) is replaced by

$$C_t = e^{-r(T-t)} E_t[(S_T - I_T)^+],$$

which leads to the same PDE (A.10) for $f(t,\chi)$, defined as $f(t,\chi) = E^{\bullet}_t[(1-\chi_T)^+]$, but with a different terminal condition $f(T,\chi) = (1-\chi)^+$.

Note that the reduced PDE (A.10) of Rogers and Shi can also be derived from the original PDE for an arithmetic Asian option (5.22) by use of the scaling

$$u(t,S_t,A_t) = S_t f(t,\chi_t), \qquad \chi_t = \frac{A_t - K}{S_t}.$$

A.3 Reduction by Večeř (2001)

Večeř (2001) [53] uses a different dimension reduction than Rogers and Shi which also works for both floating and fixed strike options and can also be applied to discretely sampled Asian options. In comparison to the approach of Rogers and Shi, Večeř's equation is well-behaved also for low volatilities and short maturities.

Instead of variable I_t, defined in (A.3) as $I_t = -K + \frac{1}{T}\int_0^t S_u du$, Večeř uses variable J_t defined by (A.11) below. He expresses

$$d\left(\frac{t}{T} S_t\right) = S_t \frac{dt}{T} + \frac{t}{T} dS_t,$$

which yields

$$\frac{1}{T}\int_0^t S_u du = S_T - \frac{1}{T}\int_0^t u dS_u.$$

He then defines

$$J_t = -K + S_t - \frac{1}{t}\int_0^t u dS_u \tag{A.11}$$

so that $J_T = I_T$ and the differential of J is

$$dJ_t = \left(1 - \frac{t}{T}\right) dS_t.$$

The reduced variable is $\chi_t = J_t/S_t$ so one has

$$\begin{aligned} d\chi = d\left(\frac{J}{S}\right) &= \frac{dJ}{S} - \chi\frac{dS}{S} - \frac{dJ}{S}\frac{dS}{S} + \chi\left(\frac{dS}{S}\right)^2 \\ &= \left(1 - \frac{t}{T} - \chi\right)\frac{dS}{S} - \left(1 - \frac{t}{T} - \chi\right)\left(\frac{dS}{S}\right)^2 \\ &= \left(1 - \frac{t}{T} - \chi\right)\left(d\mathcal{L}(S) - \sigma^2 dt\right). \end{aligned} \tag{A.12}$$

Then the same change of numéraire/change of measure as in the approach of Rogers and Shi is used with $dP^\bullet/dP = Z_T/Z_0$ and Z_t given by (A.6). Since the same change of measure is used, the $P^\bullet$-drift of $\mathcal{L}(S)$ is again given by (A.8) and from (A.12) the drift and volatility of χ under $P^\bullet$ are

$$\begin{aligned} b_\bullet^\chi &= \left(1 - \frac{t}{T} - \chi\right) r\left(b_\bullet^{\mathcal{L}(S)} - \sigma^2\right) = \left(1 - \frac{t}{T} - \chi\right) r \\ \sqrt{c_\bullet^\chi} &= \left(1 - \frac{t}{T} - \chi\right)\sigma. \end{aligned} \tag{A.13}$$

The fixed strike Asian option price is again expressed by means of a conditional expectation of a deterministic function of the Markov process χ

$$C_t = e^{-r(T-t)}E_t[J_T^+] = e^{-r(T-t)}Z_t E_t^\bullet\left[\frac{J_T^+}{Z_T}\right] = S_t E_t^\bullet\left[\chi_T^+\right]. \tag{A.14}$$

The conditional expectation is a martingale under the new measure and it can be written as a function of t and χ_t, $f(t,\chi) = E_t^\bullet\left[\chi_T^+\right]$. Itō's lemma reads

$$df(t,\chi) = \left(f_t + b_\bullet^\chi f_\chi + \frac{1}{2}c_\bullet^\chi f_{\chi\chi}\right)dt + \sqrt{c_\bullet^\chi} f_\chi dW_t^\bullet$$

which, combined with the zero drift condition and substituting for $b^{\chi}_{\bullet}$ and $c^{\chi}_{\bullet}$ from (A.13), yields the PDE

$$f_t + r\left(1 - \frac{t}{T} - \chi\right) f_\chi + \frac{\sigma^2}{2}\left(1 - \frac{t}{T} - \chi\right)^2 f_{\chi\chi} = 0. \tag{A.15}$$

From (A.14) the terminal condition is again $f(T,\chi) = \chi^+$.

The reduced equation (A.15) can also be derived from the original PDE for an arithmetic Asian option (5.22) by use of the scaling

$$u(t, S_t, A_t) = S_t f(t, \chi_t), \qquad \chi_t = \frac{A_t - K}{S_t} + 1 - \frac{t}{T}.$$

In case of a floating strike option the scaling

$$u(t, S_t, A_t) = S_t f(t, \chi_t), \qquad \chi_t = \frac{t}{T} - \frac{A_t}{S_t}.$$

yields the reduced equation

$$f_t + r\left(\frac{t}{T} - \chi\right) f_\chi + \frac{1}{2}\sigma^2\left(\frac{t}{T} - \chi\right)^2 f_{\chi\chi} = 0 \tag{A.16}$$

with the same terminal condition

$$f(T,\chi) = \frac{1}{S_t} u(T, S_t, A_t) = \frac{1}{S_t}\left(S_t - A_t\right)^+ = \left(\frac{T}{T} - \frac{A_t}{S_t}\right)^+ = \chi^+.$$

Note that the reduced equations (A.15), (A.16) for the fixed and floating strike, respectively, have the same form

$$f_t + r\left(q_t - \chi\right) f_\chi + \frac{1}{2}\sigma^2\left(q_t - \chi\right)^2 f_{\chi\chi} = 0, \tag{A.17}$$

where q_t represents the position in the underlying stock at time t in the option-on-a-traded-account interpretation of Večeř [53]. In particular, the stock position in case of fixed strike is given by $q_t = 1 - \frac{t}{T}$ and in case of floating strike it is $q_t = \frac{t}{T}$.

Alternatively, the floating strike option price can also be found by solving the same PDE (A.15) as for the fixed strike option but the terminal condition changes to $f(T,\chi) = (1-\chi)^+$. This follows directly from our derivation of the PDE (A.15) where the payoff in (A.14) needs to be changed to

$$C_t = e^{-r(T-t)} E_t[(S_T - J_T)^+] = S_t E_t^{\bullet}\left[(1-\chi_T)^+\right]$$

for the floating strike option. It is readily seen that PDE (A.15)

$$f_t + r\left(1 - \frac{t}{T} - \chi\right) f_\chi + \frac{\sigma^2}{2}\left(1 - \frac{t}{T} - \chi\right)^2 f_{\chi\chi} = 0$$
$$f(T,\chi) = (1-\chi)^+$$

and PDE (A.16)

$$f_t + r\left(\frac{t}{T} - \chi\right) f_\chi + \frac{1}{2}\sigma^2\left(\frac{t}{T} - \chi\right)^2 f_{\chi\chi} = 0$$
$$f(T,\chi) = \chi^+$$

describe the same function with the simple change of variable $\tilde{\chi} = 1 - \chi$.

A.4 Reduction by Večeř (2002)

A year after having published [53] with the previous dimension reduction, Večeř (2002) [54] proposed a different reduction which leads to a driftless variable χ and produces a simpler PDE containing only two terms.

A new variable J_t is defined by means of I_t, which was defined in (A.3), as

$$\begin{aligned} J_t = E_t\left[I_T\right] &= E_t\left[-K + \frac{1}{T}\int_0^T S_u du\right] \\ &= E_t\left[-K + \frac{1}{T}\int_0^t S_u du + \frac{1}{T}\int_t^T S_u du\right] = I_t + E_t\left[\frac{1}{T}\int_t^T S_u du\right]. \end{aligned}$$

For $u \geq t$, the expectation of the stock price is $E_t[S_u] = e^{r(u-t)} S_t$ so we obtain

$$J_t = I_t + \frac{1}{T} S_t e^{-rt} \int_t^T e^{ru} du = I_t + S_t \frac{e^{r(T-t)} - 1}{rT}.$$

The dynamics of J read

$$dJ_t = dI_t + \frac{e^{r(T-t)} - 1}{rT} dS_t - S_t d\left(\frac{e^{r(T-t)} - 1}{rT}\right)$$

$$= \frac{1}{T} S_t dt + \frac{e^{r(T-t)} - 1}{rT} dS_t - \frac{e^{r(T-t)}}{T} S_t dt = \frac{e^{r(T-t)} - 1}{rT} \big(dS_t - r S_t dt \big).$$

The same change of measure is used again with $dP^{\bullet}/dP = Z_T/Z_0$ and Z_t given by (A.6). A new variable χ is defined as

$$\chi_t = \frac{J_t}{Z_t} = e^{-r(T-t)} \frac{I_t}{S_t} + \frac{1 - e^{-r(T-t)}}{rT}.$$

Using the differential of the change of measure density

$$dZ_t = e^{r(T-t)} dS_t - r Z_t dt$$

we express the dynamics of χ

$$\begin{aligned} d\chi = d\left(\frac{J}{Z}\right) &= \frac{dJ}{Z} - \chi \frac{dZ}{Z} - \frac{dJ}{Z}\frac{dZ}{Z} + \chi \left(\frac{dZ}{Z}\right)^2 \\ &= \left(\frac{1 - e^{-r(T-t)}}{rT} - \chi\right) \left(\frac{dS}{S} - rdt\right) - \left(\frac{1 - e^{-r(T-t)}}{rT} - \chi\right) \left(\frac{dS}{S} - rdt\right)^2 \\ &= \left(\frac{1 - e^{-r(T-t)}}{rT} - \chi\right) \Big(d\mathcal{L}(S) - rdt - \sigma^2 dt \Big), \end{aligned} \tag{A.18}$$

where we used $\left(\frac{dS}{S} - rdt\right)^2 = \left(\frac{dS}{S}\right)^2 = \sigma^2 dt$ in the last step.

The $P^{\bullet}$-drift of $\mathcal{L}(S)$ is again given by (A.8) and from (A.18) the drift and volatility of χ under $P^{\bullet}$ are

$$\begin{aligned} b^{\chi}_{\bullet} &= \left(\frac{1 - e^{-r(T-t)}}{rT} - \chi\right) \left(b^{\mathcal{L}(S)}_{\bullet} - r - \sigma^2\right) = 0, \\ \sqrt{c^{\chi}_{\bullet}} &= \left(\frac{1 - e^{-r(T-t)}}{rT} - \chi\right) \sigma. \end{aligned} \tag{A.19}$$

Note that χ is a $P^{\bullet}$-martingale by construction so $b^{\chi}_{\bullet} = 0$ only confirms this.

We again arrived at the option price being expressed as a conditional expectation of a deterministic function of a Markov process

$$C_t = S_t E^{\bullet}_t \left[\chi^+_T\right]. \tag{A.20}$$

This expectation is a $P^{\bullet}$-martingale and it can be written as $f(t, \chi) = E^{\bullet}_t \left[\chi^+_T\right]$ which yields

$$f_t + b_\bullet^\chi f_\chi + \frac{1}{2} c_\bullet^\chi f_{\chi\chi} = 0.$$

Substituting for $b_\bullet^\chi$ and $c_\bullet^\chi$ from (A.19) we obtain the PDE

$$f_t + \frac{\sigma^2}{2} \left(\frac{1 - e^{-r(T-t)}}{rT} - \chi \right)^2 f_{\chi\chi} = 0, \tag{A.21}$$

which has the simplest form out of all the reduced equations presented because of the eliminated drift of the reduced variable χ. The terminal condition is again given by $f(T, \chi) = \chi^+$.

The reduced PDE (A.21) can also be derived from the original PDE for an arithmetic Asian option (5.22) by use of the scaling

$$u(t, S_t, A_t) = S_t f(t, \chi_t), \qquad \chi_t = e^{-r(T-t)} \frac{A_t - K}{S_t} + \frac{1 - e^{-r(T-t)}}{rT}.$$

In case of a floating strike call the price can be calculated by solving the same PDE (A.21) but the terminal condition changes to $f(T, \chi) = (1 - \chi)^+$ to describe the floating strike payoff.