FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY
UNIVERZITY KOMENSKÉHO
BRATISLAVA

Ústav aplikovanej matematiky

Uniqueness Results for Some Parabolic Systems

Dizertačná práca

Bratislava
2004

Predkladateľ RNDr. Matej Kordoš
Školiteľ Doc. RNDr. Marek Fila, DrSc.

Prehlasujem, že som dizertačnú prácu
vypracoval samostatne a uviedol som
všetku použitú literatúru.

Matej Kordoš

Bratislava, máj 2004

Matej Kordoš

Ďakujem Marekovi Filovi za vedenie
a neoceniteľné rady pri vypracovaní
dizertačnej práce.

UNIQUENESS RESULTS FOR SOME PARABOLIC SYSTEMS

Matej Kordoš

Ústav aplikovanej matematiky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského
Mlynská dolina
842 48 Bratislava
Slovakia
E-mail: kordos@fmph.uniba.sk

30.05.2004

Contents

1 Introduction

1.1 Formulation and motivation

This thesis solves several uniqueness questions for nonnegative solutions of the following parabolic problems on the halfspace $\mathbb{R}^N_+ = \{(x_1, x') : x' \in \mathbb{R}^{N-1}, x_1 > 0\}$, $N \geqslant 1$. The first problem is to find u_i, $i = 1, 2, \ldots, n$, $n > 1$ such that

$$\text{(P)} \qquad \begin{aligned} &\frac{\partial u_i}{\partial t} = \triangle u_i, && x \in \mathbb{R}^N_+, && t > 0, && \\ &-\frac{\partial u_i}{\partial x_1} = u_{i+1}^{p_i}, && x_1 = 0, && t > 0, && u_{n+1} = u_1, \\ &u_i(x,0) = v_i(x), && x \in \mathbb{R}^N_+, && && \end{aligned}$$

where p_i are positive numbers and v_i are nonnegative, smooth, and bounded functions satisfying the compatibility condition. The second problem is a non-symmetric semilinear system with two equations

$$\text{(FL)} \qquad \begin{aligned} &u_t = \triangle u + v^p, && v_t = \triangle v, && x \in \mathbb{R}^N_+, && t > 0, \\ &-\frac{\partial u}{\partial x_1} = 0, && -\frac{\partial v}{\partial x_1} = u^q, && x_1 = 0, && t > 0, \\ &u(x,0) = u_0(x), && v(x,0) = v_0(x), && x \in \mathbb{R}^N_+, && \end{aligned}$$

where p, q are positive numbers, and u_0, v_0 are nonnegative, smooth, and bounded functions satisfying the compatibility condition.

In order to motivate the results of this thesis, we briefly discuss the uniqueness of the following system's nonnegative solutions

$$\begin{aligned} &u' = v^p, && v' = u^q && \text{for } t > 0, \\ &u(0) = 0, && v(0) = 0, && \end{aligned}$$

where p, q are positive numbers. The system may be transformed into an ordinary differential equation for u as follows. We differentiate the first equation and substitute for both v and v'

$$u'' = pv^{p-1}v' = p(u')^{\frac{p-1}{p}}u^q.$$

An elementary algebra yields

$$\begin{aligned} (u')^{\frac{1}{p}}u'' &= pu^q u', \\ \left((u')^{\frac{1+p}{p}}\right)' &= \frac{1+p}{1+q}\left(u^{q+1}\right)'. \end{aligned}$$

We integrate the last equation using the initial conditions for u, u' and arrive at

$$u' = \left(\frac{1+p}{1+q}\right)^{\frac{p}{1+p}} u^{\frac{pq+p}{1+p}}.$$

It is known that the equation $u' = cu^\gamma$ with the zero initial condition has only the trivial solution if $\gamma \geqslant 1$, while there exists a set of nontrivial solutions if $\gamma \in (0,1)$. See Figure 1 for an example. Adapting this result to the system under discussion, we see that its solution is unique if and only if $pq \geqslant 1$. The question is whether a similar claim is true for some parabolic systems having this kind of nonlinearity as well.

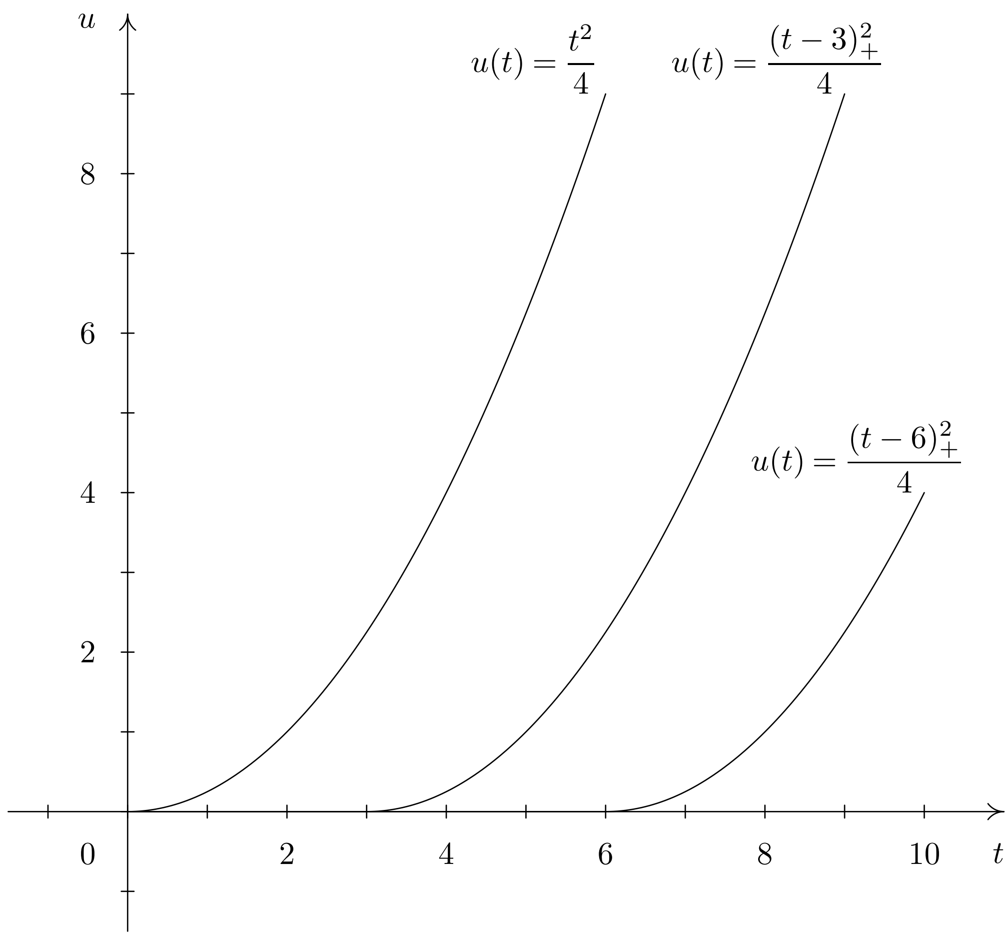

Figure 1. Some solutions of $u' = u^{\frac{1}{2}}$, $u(0) = 0$; here $(r)_+^2 = \max^2\{r, 0\}$.

1.2 Some blow-up results

Let us first look at some results concerning the nonlinear evolution equations, especially with polynomial nonlinearities. Naturally, the results solve the questions of existence and uniqueness of the classical solutions. The first type of results specifies the conditions for the global (in time) existence, since the local one can be established with standard arguments. However, a solution does not have to exist globally in the classical sense. A situation, when a solution becomes unbounded in a finite time, is called "blow-up". Deeper results of this type discuss either the large time behaviour of a global solution or the blow-up rate of a nonglobal solution. The authors also study so called weak continuation of a nonglobal solution beyond the blow-up time, when it exists, as well as the blow-up set (single point, whole domain). The second type of the results answers the questions of uniqueness of the classical solutions. The main results of this thesis belong to the second type.

At the beginning of our brief and by far not complete results review, we recall a classical result of Fujita [F] for the problem

$$\text{(F)} \qquad \begin{aligned} u_t &= \triangle u + u^p, && x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) &= u_0(x), && x \in \mathbb{R}^N, \end{aligned}$$

with nonnegative initial data u_0. He showed that (i) if $1 < p < 1 + 2/N$, then (F) possesses no global nonnegative solutions while (ii) if $p > 1 + 2/N$, both global and nonglobal nonnegative solutions exist. The number $1+2/N$ is called the critical exponent which turns out to belong to the case (i). See [W] for an elegant proof by Weissler as well as references to earlier proofs of this result. Over the past years there have been many extensions of Fujita's result in various directions.

In 1991, Escobedo and Herrero investigated the initial value problem for a weakly coupled system

$$\text{(EH1)} \qquad \begin{aligned} u_t &= \triangle u + v^p, & v_t &= \triangle v + u^q, & x &\in \mathbb{R}^N, \quad t > 0, \\ u(x,0) &= u_0(x) \geqslant 0, & v(x,0) &= v_0(x) \geqslant 0, & x &\in \mathbb{R}^N. \end{aligned}$$

Set, when $pq \neq 1$,

$$\alpha_1 = \frac{p+1}{pq-1}, \qquad \beta_1 = \frac{q+1}{pq-1}.$$

The results of [EH2] for (EH1) take the following form. If $\max(\alpha_1, \beta_1) \geqslant N/2$ then all nontrivial solutions are nonglobal. If $0 < \max(\alpha_1, \beta_1) < N/2$ then there are global and nonglobal solutions. When $\max(\alpha_1, \beta_1)$ is negative or not defined, all solutions with L^∞ initial values are global.

Galaktionov and Levine considered in [GL] the boundary value problem

$$\text{(GL)} \qquad \begin{aligned} u_t &= u_{xx}, && x > 0, \quad t > 0, \\ -u_x &= u^p, && x = 0, \quad t > 0, \\ u(x,0) &= u_0(x) \geqslant 0, && x > 0, \\ -u_0'(0) &= u_0^p(0). && \end{aligned}$$

They showed that if $1 < p \leqslant 2$, then $u(x,t)$ blows up in a finite time for all nontrivial u_0; whereas if $p > 2$, then $u(x,t)$ becomes unbounded in a finite time for large u_0 and $u(x,t)$ exists globally for small initial data.

Deng, Fila, and Levine extended later their result to the problem

$$\text{(DFL)} \quad \begin{aligned} u_t &= \triangle u, & v_t &= \triangle v, && x \in \mathbb{R}_+^N, && t > 0, \\ -\frac{\partial u}{\partial x_1} &= v^p, & -\frac{\partial v}{\partial x_1} &= u^q, && x_1 = 0, && t > 0, \\ u(x,0) &= u_0(x) \geqslant 0, & v(x,0) &= v_0(x) \geqslant 0, && x \in \mathbb{R}_+^N, && \end{aligned}$$

where $\mathbb{R}_+^N = \left\{(x_1, x') : x' \in \mathbb{R}^{N-1}, x_1 > 0\right\}, N \geqslant 1$, p, q are positive numbers, and u_0, v_0 are nonnegative, smooth, and bounded functions satisfying the compatibility condition. It was shown in [DFL] that the result for (DFL) takes the form as in [EH2] for (EH1) if we replace α_1, β_1 by $\alpha_2 = \alpha_1/2$, $\beta_2 = \beta_1/2$.

Fila and Levine studied the problem (FL) which is "intermediate" between the problems (EH1) and (DFL). They proved in [FL] the same result for "intermediate" powers

$$\alpha_3 = \frac{p+2}{2(pq-1)}, \qquad \beta_3 = \frac{2q+1}{2(pq-1)}.$$

Obviously,

$$\alpha_1 > \alpha_3 > \alpha_2, \qquad \beta_1 > \beta_3 > \beta_2.$$

Chlebík and Fila derived estimates of blow-up rates for the systems (EH1), (DFL), and (FL) in [ChF]. They showed that there is a constant $C > 0$ such that

$$\text{(ChF)} \qquad u(x,t) \leqslant C(T-t)^{-\alpha}, \quad v(x,t) \leqslant C(T-t)^{-\beta}$$

hold true in $\Omega \times (0,T)$ for every positive solution (u,v) of (EH1), (DFL) if $pq > 1$ and $\max\{\alpha, \beta\} \geqslant N/2$, where $T < \infty$ is the blow-up time and

(i) $\alpha = \alpha_1$, $\beta = \beta_1$, $\Omega = \mathbb{R}^N$ for (EH1), (ii) $\alpha = \alpha_2$, $\beta = \beta_2$, $\Omega = \mathbb{R}^N_+$ for (DFL). The estimates (ChF) with $\alpha = \alpha_3$, $\beta = \beta_3$, $\Omega = \mathbb{R}^N_+$ hold true also for every positive solution (u, v) of (FL) satisfying u_{x_1}, $v_{x_1} \leqslant 0$, if $pq > 1$ and either $\max\{\alpha, \beta\} > N/2$ or $\max\{\alpha, \beta\} = N/2$, p, $q \geqslant 1$.

Rencławowicz [R] extended some results of [EH2] to the parabolic system of three equations

$$\text{(R)} \qquad \begin{aligned} u_t - \triangle u &= v^p, \\ v_t - \triangle v &= w^q, \qquad x \in \mathbb{R}^N, \quad t > 0, \\ w_t - \triangle w &= u^r, \end{aligned}$$

with p, q, r positive numbers, $N \geqslant 1$, and nonnegative, bounded, continuous initial values. Set, when $pqr > 1$,

$$\alpha = \frac{1 + p + pq}{pqr - 1}, \qquad \beta = \frac{1 + q + qr}{pqr - 1}, \qquad \gamma = \frac{1 + r + rp}{pqr - 1}.$$

The results of [R] take the following form. If $pqr \leqslant 1$ then every solution is global. If $pqr > 1$ and $\max(\alpha, \beta, \gamma) \geqslant N/2$ then (R) never has nontrivial global solutions.

1.3 Known uniqueness results

Before introducing the uniqueness results for systems with polynomial coupling, we recall a result by Fujita and Watanabe [FW] for the Cauchy-Dirichlet problem

$$\text{(FW)} \qquad \begin{aligned} u_t - \triangle u &= u^p, & x &\in \Omega, \quad t > 0, \\ u(x, 0) &= u_0(x), & x &\in \Omega, \\ u(x, t) &= 0, & x &\in \partial\Omega, \quad t \geqslant 0, \end{aligned}$$

where $p > 0$, u_0 is a continuous, nonnegative and bounded real function, and Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$. They showed that uniqueness fails when $p < 1$.

All known results for systems with polynomial coupling follow the same pattern. It is not necessary for each of the exponents from the coupling to be at least 1, i.e., Lipschitz continuous, to secure the uniqueness of a solution. The uniqueness holds in the case of nonzero initial data, whereas it depends on the product of the exponents from the coupling in the case of zero initial data.

First of all we mention a paper by Escobedo and Herrero. In [EH1] they proved a uniqueness result of the above mentioned type for an initial value problem on the whole space. The original formulation follows.

> Let us consider the problem (EH1) with $N \geqslant 1$, $p > 0$, $q > 0$, and where u_0 and v_0 are nonnegative, continuous, and bounded real functions.
>
> [EH1, Theorem]. *Assume that p and q are different from zero and $p < 1$ or $q < 1$. Then*
>
> (a) *If $0 < pq < 1$ and $(u_0, v_0) \not\equiv (0,0)$, problem* (EH1) *has a unique solution.*
>
> (b) *If $0 < pq < 1$ and $(u_0, v_0) \equiv (0,0)$, the set of nontrivial nonnegative solutions of* (EH1) *is given by*
>
> $$u(x,t;s) = c_1 (t-s)_+^{\alpha}, \qquad v(x,t;s) = c_2 (t-s)_+^{\beta},$$
>
> *where $(r)_+ = \max\{r, 0\}$, s is any nonnegative real constant, and*
>
> $$\alpha = \frac{p+1}{1-pq}, \qquad c_1^{1-pq} = (1-pq)^{p+1}(p+1)^{-1}(q+1)^{-p},$$
> $$\beta = \frac{q+1}{1-pq}, \qquad c_2 \beta = c_1^q.$$
>
> (c) *If $pq \geqslant 1$, there is a unique solution of* (EH1).

The results formulated and proven later in this thesis are inspired by the paper [EH1] and the proofs richly use the ideas and tricks from there.

The bounded domain version of the result for the problem (EH1) was presented in [EH3]. It was formulated as follows.

> We shall consider the following Cauchy-Dirichlet problem
>
> (EH2.1a) $u_t - \triangle u = v^p$ when $x \in \Omega$, $t > 0$,
>
> (EH2.1b) $v_t - \triangle v = u^q$ when $x \in \Omega$, $t > 0$,
>
> (EH2.2) $u = v = 0$ if $x \in \partial\Omega$, $t \geqslant 0$,
>
> (EH2.3) $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ if $x \in \Omega$,

where

(EH2.4) $p > 0$, $q > 0$ and $u_0(x)$, $v_0(x)$ are continuous, nonnegative and bounded real functions.

[EH3, Theorem 1]. *Assume that* (EH2.4) *holds. We then have*

a) *If one of the initial values* $u_0(x)$, $v_0(x)$ *is different from zero, or if* $pq \geqslant 1$, *there exists a unique solution of* (EH2.1)-(EH2.3) *which is defined in some time interval* $(0, T)$ *with* $T \leqslant +\infty$.

b) *If* $pq < 1$ *and* $u_0(x) = v_0(x) \equiv 0$, *the set of solutions of* (EH2.1)-(EH2.3) *consists of*

  b1) *The trivial solution* $u(x,t) = v(x,t) \equiv 0$,

  b2) *A solution* $U(x,t), V(x,t)$ *such that* $U(x,t) > 0$ *and* $V(x,t) > 0$ *for any* $x \in \Omega$ *and* $t > 0$,

  b3) *A monoparametric family* $U_\mu(x,t), V_\mu(x,t)$ *where* μ *is any positive number,* $U_\mu(x,t) = U(x,(t-\mu)_+)$, $V_\mu(x,t) = V(x,(t-\mu)_+)$ *and* $\xi_+ = \max\{\xi, 0\}$.

A nonuniqueness result is obtained by Deng, Fila, and Levine in the work [DFL] where they studied the system (DFL), i.e., (P) with $n = 2$. Their results solve the questions of global existence mainly, nevertheless in the dimension $N = 1$ they constructed a nontrivial solution with zero initial data if $pq < 1$. Their formulation is the following.

Let us consider the one dimensional problem

$$\text{(DFL1)}\quad \begin{array}{llll} u_t = u_{xx}, & v_t = v_{xx}, & x > 0, & t > 0, \\ -u_x = v^p, & -v_x = u^q, & x = 0, & t > 0, \\ u(x,0) = u_0 \geqslant 0, & v(x,0) = v_0 \geqslant 0, & x > 0. & \end{array}$$

[DFL, Theorem 3.5]. *If* $pq < 1$ *then problem* (DFL1) *with* $u_0 \equiv v_0 \equiv 0$ *has a nontrivial, nonnegative solution.*

The one dimensional solution was explicitly constructed in the proof and it can be easily generalized for higher-dimension and more-equation problems.

The bounded domain version of the problem (DFL) was discussed in a work by Cortazar, Elgueta, and Rossi. The result of their paper [CER] reads as follows.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary and let p and q be two positive real numbers. Consider the system

$$\text{(CER)}\quad \begin{array}{lll} u_t = \triangle u, & v_t = \triangle v & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = v^p, & \dfrac{\partial v}{\partial \nu} = u^q & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x) & \text{in } \Omega \end{array}$$

with smooth initial data $u_0 \geqslant 0$ and $v_0 \geqslant 0$, and ν being the outer normal to $\partial\Omega$.

[CER, Theorem 1]**.** *Assume* $(u_0, v_0) \equiv (0,0)$. *Then*

a) *If* $pq \geqslant 1$, *then the unique solution of problem* (CER) *is* $(u,v) \equiv (0,0)$.
b) *If* $pq < 1$, *then there exists exactly one solution* $(\tilde{u}, \tilde{v})$ *of problem* (CER) *such that both* $\tilde{u}$ *and* $\tilde{v}$ *are strictly positive for every positive time. Moreover, any other nonnegative nontrivial solution of* (CER) *is of the form*

$$(\tilde{u}((t-\tau)_+), \tilde{v}((t-\tau)_+))$$

for some fixed $\tau > 0$. *Here* $r_+ = \max(r, 0)$.

[CER, Theorem 2]**.** *If* $(u_0, v_0) \not\equiv (0,0)$, *then the solution of* (CER) *is unique.*

Wang, Xie, and Wang showed in [WXW] besides the blow-up estimates also the uniqueness for (DFL) with zero initial data in the case $pq \geqslant 1$. The result was formulated in the following way.

[WXW, Theorem 3]**.** *Assume that* $pq \geqslant 1$. *Then the only solution of the problem* (DFL) *with* $(u_0, v_0) \equiv (0,0)$ *is the trivial one, i.e.,* $(u,v) \equiv (0,0)$.

The corresponding result has been recently proven for the problem (P) by Lin.

[L, Theorem 4.1]. *Assume that* $\sum_{i=1}^{n} p_i \geqslant 1$. *Then the only solution of the problem* (P) *with vanishing initial values is the trivial one, i.e.,* $u_i \equiv 0$, $i = 1, \ldots, n$.

Later on we present also its proof, since it concerns the uniqueness of one of the problems under discussion.

1.4 Notation, solution formulae, and preliminaries

Similarly as in [FL], we denote

$$\begin{aligned}
G_N(x,y;t) &= (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right), \\
H_N(x,y;t) &= G_N(x,y;t) + G_N(x,-y;t), \\
H_1(x_1,y_1;t) &= \frac{1}{2}(\pi t)^{-\frac{1}{2}} \left(\exp\left(-\frac{|x_1-y_1|^2}{4t}\right) + \exp\left(-\frac{|x_1+y_1|^2}{4t}\right)\right), \\
R(x_1,t) &= H_1(x_1,0;t) = (\pi t)^{-\frac{1}{2}} \exp\left(-\frac{x_1^2}{4t}\right)
\end{aligned}$$

for $t > 0$, x, $y \in \mathbb{R}^N$, x_1, $y_1 \in \mathbb{R}$, x', $y' \in \mathbb{R}^{N-1}$, and $x = (x_1, x')$, $y = (y_1, y')$. We use these functions to define several operators for $w \in L^1_{loc}(\mathbb{R}^N_+)$, namely

$$\begin{aligned}
\mathcal{S}_N(t)w(x) &= \int_{\mathbb{R}^N} G_N(x,y;t)w(y)dy, \\
\mathcal{S}_{N-1}(t)w(x_1,x') &= \int_{\mathbb{R}^{N-1}} G_{N-1}(x',y';t)w(x_1,y')dy', \\
\mathcal{T}(t)w(x) &= \int_{\mathbb{R}_+} H_1(x_1,y_1;t)w(y_1,x')dy_1, \\
\mathcal{R}(t)w(x) &= R(x_1,t)\mathcal{S}_{N-1}(t)w(0,x').
\end{aligned}$$

These integral operators allow us to write the variation of constants formulae for solutions of both systems. For (P) we have

$$u_i(x,t) = \mathcal{T}(t)\mathcal{S}_{N-1}(t)\upsilon_i(x) + \int_0^t \mathcal{R}(t-\eta)u_{i+1}^{p_i}(x,\eta)d\eta, \tag{1.1}$$

and for (FL)

$$
\begin{aligned}
u(x,t) &= \mathcal{T}(t)\mathcal{S}_{N-1}(t)u_0(x) + \int_0^t \mathcal{T}(t-\eta)\mathcal{S}_{N-1}(t-\eta)v^p(x,\eta)d\eta, \\
v(x,t) &= \mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x) + \int_0^t \mathcal{R}(t-\eta)u^q(x,\eta)d\eta.
\end{aligned} \tag{1.2}
$$

It is possible to prove the local existence of the solution in time using (1.1) or (1.2), and the contraction mapping principle (cf. proof of Theorem 1.1). Since in some cases the solutions do not have to exist globally in the classical sense, we define a strip $S_T = \mathbb{R}^N_+ \times (0,T)$ for any $0 < T \leqslant \infty$. See [DFL] and [FL] for more detailed results on the global existence.

We use the following notation for $i = 1, 2, \ldots, n$ when dealing with the problem (P)

$$
\begin{aligned}
&p_{i+n} = p_i, \qquad u_{i+n} = u_i \qquad \text{for } i = 1, 2, \ldots, n, \\
&p = \prod_{i=1}^{n} p_i, \qquad (u_i) = (u_1, u_2, \ldots, u_n), \quad (v_i) = (v_1, v_2, \ldots, v_n), \quad \text{etc.}
\end{aligned}
$$

We also set for $i = 1, 2, \ldots, n$ and $k = 0, 1, \ldots, n$

$$
\begin{aligned}
\pi_k^{(i)} &= \prod_{j=1}^{k} p_{i+j-1} = p_i p_{i+1} \cdots p_{i+k-1}, \qquad \text{i.e.,} \quad \pi_0^{(i)} = 1, \quad \pi_n^{(i)} = p, \\
\alpha_i &= \frac{1}{2(1-p)} \sum_{k=0}^{n-1} \pi_k^{(i)}, \quad \alpha_{i+n} = \alpha_i \qquad \text{for } p < 1, \\
B(\gamma) &= \int_0^1 (1-t)^{-\frac{1}{2}} t^{\gamma - \frac{1}{2}} dt = \beta\left(\frac{1}{2} + \gamma, \frac{1}{2}\right) = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+\gamma)}{\Gamma(1+\gamma)}
\end{aligned}
$$

for convenience. Observe that $\frac{1}{2} + p_i \alpha_{i+1} = \alpha_i$ for $i = 1, 2, \ldots, n$.

We point out several useful relationships. One can easily check that for $w \in L^1_{loc}(\mathbb{R}^N_+)$, $s, t > 0$, the equalities

$$
\begin{aligned}
\mathcal{T}(t)\mathcal{S}_{N-1}(t)w &= \mathcal{S}_{N-1}(t)\mathcal{T}(t)w, \\
\mathcal{S}_{N-1}(t)\mathcal{S}_{N-1}(s)w &= \mathcal{S}_{N-1}(t+s)w
\end{aligned}
$$

hold. We use them later without referring to them. We also often use Jensen's inequality without referring, mainly in two following forms:

$$
\begin{aligned}
&\text{if} \quad r \geqslant 1 \quad \text{then} \quad \left(\int_0^t f(s)ds\right)^r \leqslant t^{r-1} \int_0^t f^r(s)ds, \\
&\text{if} \quad r \leqslant 1 \quad \text{then} \quad \int_0^t f^r(s)ds \leqslant t^{1-r} \left(\int_0^t f(s)ds\right)^r.
\end{aligned}
$$

1.5 Main results and methods

The thesis shows the conjectured results for both problems (P) and (FL). The solutions of the systems are unique if the product of the exponents in the nonlinearities is at least 1. On the other hand, there are nontrivial solutions of the systems with trivial initial data if the product is less than 1.

For the problem (P), we answer several finer uniqueness questions as well. The class of nontrivial solutions for zero initial data and $p < 1$ is fully classified. The solution is unique for the nonzero initial data even if all of the exponents are less than 1. The uniqueness questions are fully solved for the problem (DFL), i.e., (P) with $n = 2$.

We prove the following results. Theorems 1.1 through 1.4 deal with the problem (P) and they are proved in Chapters 2 through 5. Chapter 2 also contains the proof of [L, Theorem 4.1]. Theorems 1.5 and 1.6 deal with the system (FL) and they are proved in Chapters 6 and 7. Theorem 1.7 completes the result for the system (DFL) and it is proved in Chapter 8. The main tool in the proofs are fine estimates of the solutions gained by iterating their integral representations.

Theorem 1.1. *If* (u_i) *and* $(\bar{u}_i)$ *are solutions of the problem* (P) *with* $p_i \geqslant 1$, $i = 1, 2, \dots, n$, *in some strip* S_T, *then* $(u_i) = (\bar{u}_i)$ *in* S_T.

We look for the solution of the problem (P) with $p < 1$ and trivial initial data in the selfsimilar form

$$u_i(x,t) = t^{\alpha_i} f_i(y) \qquad \text{for } y = \frac{x_1}{\sqrt{t}}, \quad t > 0, \quad i = 1, 2, \dots, n.$$

The problem (P) transforms into following ordinary initial value problem for f_i, $i = 1, 2, \dots, n$

$$\begin{aligned} f_i''(y) + \frac{y}{2} f_i'(y) - \alpha_i f_i(y) &= 0, & y &> 0, \\ -f_i'(0) &= f_{i+1}^{p_i}(0), & f_{n+1} &= f_1, \end{aligned} \tag{1.3}$$

where $(f_i) \to 0$ as $y \to \infty$.

Theorem 1.2. *Let* $s \geqslant 0$. *The functions*

$$\Upsilon_i(x,t;s) = (t-s)_+^{\alpha_i} f_i(y) \quad \textit{for } y = \begin{cases} \dfrac{x_1}{\sqrt{t-s}} & \textit{when } t > s, \\ 0 & \textit{otherwise,} \end{cases} \tag{1.4}$$

$i = 1, 2, \dots, n$, *solve the problem* (P) *with* $p < 1$ *and trivial initial data* $(v_i) \equiv 0$. *Here* $r_+ = \max\{r, 0\}$ *and* f_i *solve ordinary boundary value problem* (1.3).

Theorem 1.3. *For every nontrivial nonnegative solution* (u_i) *of the problem* (P) *with* $p < 1$ *and trivial initial data* $(v_i) \equiv 0$ *there exists* $s \geqslant 0$ *such that* $(u_i(x,t)) = (\Upsilon_i(x,t;s))$*, where* Υ_i *are given in* (1.4).

Theorem 1.4. *If* (u_i) *and* $(\bar{u}_i)$ *are solutions of the problem* (P) *with* $p_i < 1$, $i = 1, 2, \ldots, n$ *and nontrivial initial condition* $(v_i) \not\equiv 0$*, then* $(u_i) = (\bar{u}_i)$.

Theorem 1.5. *If* (u, v) *and* $(\bar{u}, \bar{v})$ *are solutions of the problem* (FL) *with* $pq \geqslant 1$ *in some strip* S_T*, then* $(u, v) = (\bar{u}, \bar{v})$ *in* S_T.

Theorem 1.6. *There exists a nontrivial nonnegative solution* (u, v) *of the problem* (FL) *with* $pq < 1$ *and trivial initial data.*

Theorem 1.7. *If* (u, v) *and* $(\bar{u}, \bar{v})$ *solve the problem* (DFL) *with nontrivial initial data* $(u_0, v_0) \not\equiv 0$*, then* $(u, v) = (\bar{u}, \bar{v})$.

The uniqueness results for the problem (DFL), i.e., Theorems 1.2, 1.3 for $n = 2$, and Theorem 1.7, are presented in [K1]. There remain open uniqueness questions for both problems (P) with $n > 2$ and (FL), although we expect that the corresponding results are true for them as well.

In the problem (P), the difficulties lie in the need to discuss different possibilities of ordering the exponents p_i and the number 1. In each of them, different estimates for the solutions obtained from their integral representations (1.1) and mean value theorems may lead to the desired result (cf. proof of Theorem 1.7).

In the problem (FL), the main complications lie in the nonsymmetry of the system and also of the integral representation (1.2), and in the dependence of the solution formulae not only on the initial condition and values on the boundary, but also on the values in the whole domain. The results of Theorems 1.5 and 1.6 are presented in [K2].

2 Uniqueness in the Case $p \geqslant 1$ for (P)

We start this chapter with the elegant proof of [L, Theorem 4.1] (see the Section 1.3) by Zhigui Lin presented in [L]. However, the trivial initial condition is necessary in the argument.

Proof of [L, Theorem 4.1]. The representation formula (1.1) gives

$$\begin{aligned}\sup_{x\in\mathbb{R}^N_+} u_i(x,t) &\leqslant \pi^{-\frac{1}{2}}\int_0^t (t-\eta)^{-\frac{1}{2}} \sup_{y'\in\mathbb{R}^{N-1}} u_{i+1}^{p_i}(0,y';\eta)d\eta\\ &\leqslant \frac{2\sqrt{t}}{\sqrt{\pi}} \max_{0\leqslant\eta\leqslant t}\sup_{x\in\mathbb{R}^N_+} u_{i+1}^{p_i}(x,\eta), \qquad i=1,2,\dots,n.\end{aligned}$$

Setting $F_i(t) = \max_{0\leqslant\eta\leqslant t}\sup_{x\in\mathbb{R}^N_+} u_i(x,\eta)$ we get

$$F_i(t) \leqslant \frac{2\sqrt{t}}{\sqrt{\pi}} F_{i+1}^{p_i}(t), \qquad t\geqslant 0, \quad i=1,2,\dots,n.$$

Therefore

$$\begin{aligned}F_1(t) &\leqslant \frac{2\sqrt{t}}{\sqrt{\pi}} F_2^{p_1}(t)\\ &\leqslant \left(\frac{2\sqrt{t}}{\sqrt{\pi}}\right)^{1+p_1} F_3^{p_1p_2}(t)\\ &\dots\\ &\leqslant \left(\frac{4}{\pi}\right)^{(1-p)\alpha_1} t^{(1-p)\alpha_1} F_1^p(t).\end{aligned}$$

Since $F_1(0)=0$ and $p\geqslant 1$, we have $F_1(t)\equiv 0$, i.e., $u_1(x,t)\equiv 0$. Similarly we get $u_i(x,t)\equiv 0$ for $i=2,3,\dots,n$. $\square$

The following proof of Theorem 1.1 uses the standard argument to show the uniqueness of the solution (as well as the existence) - the fixed point theorem.

Proof of Theorem 1.1. Consider the integral system associated to the solution formulae (1.1)

$$\begin{aligned}u_i(x,t) &= \Phi_i(u_{i+1})(x,t)\\ &= \mathcal{T}(t)\mathcal{S}_{N-1}(t)v_i(x) + \int_0^t \mathcal{R}(t-\eta)|u_{i+1}(x,\eta)|^{p_i-1}u_{i+1}(x,\eta)d\eta.\end{aligned} \tag{2.1}$$

For arbitrary fixed $0<\tau<T$ we set

$$E_\tau = \left\{(u_i)\in\left(L^\infty\left(\mathbb{R}^N_+\times[0,\tau]\right)\right)^n : \|(u_i)\|_E < \infty\right\},$$

where

$$\|(u_i)\|_E = \sup_{0 \leqslant t \leqslant \tau} \sum_{i=1}^{n} \|u_i(\cdot, t)\|_\infty.$$

Clearly, E_τ is a Banach space, and $P_\tau = \{(u_i) \in E_\tau : u_i \geqslant 0, i = 1, 2, \ldots, n\}$ is its closed subset. Further let $B_R = \{(u_i) \in P_\tau : \|(u_i)\|_E < R\}$. We can easily see that $\Psi((u_i)) = ((\Phi_i(u_{i+1})))$ is a strict contraction of B_R into itself if $R > 0$ is large enough and $\tau > 0$ is sufficiently small, which gives the result. $\square$

3 Nonuniqueness in the Case $p < 1$ for (P) with Zero Initial Data

In this chapter we consider the problem (P) with $p < 1$ and trivial initial condition. We find a class of nontrivial solutions and extend so the result of [DFL, Theorem 3.5].

Before we find the solution of (1.3), we introduce following notation. We define

$$U(a,b,r) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-rt}\, t^{a-1}(1+t)^{b-a-1}dt \tag{3.1}$$

and

$$\begin{aligned} D_i &= \pi^{-\frac{1}{2}} \left(\frac{\Gamma(\frac{1}{2}+\alpha_i)}{\Gamma^p(1+\alpha_i)} \right)^{\frac{1}{1-p}} \prod_{k=1}^{n-1} \left(\frac{\Gamma(\frac{1}{2}+\alpha_{i+k})}{\Gamma(1+\alpha_{i+k})} \right)^{\frac{\pi_k^{(i)}}{1-p}} \\ &= \pi^{-\frac{1}{2}-\alpha_i}\Gamma(1+\alpha_i) \prod_{k=0}^{n-1} B^{\frac{\pi_k^{(i)}}{1-p}}(\alpha_{i+k}), \qquad i = 1, 2, \dots, n, \\ D_{n+1} &= D_1. \end{aligned} \tag{3.2}$$

Lemma 3.1. *The function U fulfills the following relations.*

$$\begin{aligned} &\text{(i)} && U_r(a,b,r) = -aU(a+1,b+1,r) \\ &\text{(ii)} && U\left(a, \frac{1}{2}, 0\right) = \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+a)} \\ &\text{(iii)} && U\left(a, \frac{3}{2}, r\right) = \frac{\pi^{\frac{1}{2}}}{\Gamma(a)} r^{-\frac{1}{2}} + O(1) \qquad \textit{for } r \to 0 \\ &\text{(iv)} && U(a,b,r) = r^{-a}(1+O(r^{-1})) \qquad \textit{for } r \to \infty \end{aligned} \tag{3.3}$$

Proof. The relations (i), (iii), and (iv) can be found in [AS 13.4.21, 13.5.8, 13.1.8] respectively, (ii) can be obtained directly from (3.1). $\square$

Lemma 3.2. *The constants D_i fulfill the recurrent relation*

$$D_{i+1}^{p_i} = \pi^{\frac{1-p_i}{2}} \frac{\Gamma^{p_i}(1+\alpha_{i+1})}{\Gamma(\frac{1}{2}+\alpha_i)} D_i, \qquad i = 1, 2, \dots, n. \tag{3.4}$$

Proof. We verify the relation (3.4) using the definition of D_i in (3.2) directly. Recalling the simple facts $p_i\pi_k^{(i+1)} = \pi_{k+1}^{(i)}$ and $1 + p_i\alpha_{i+1} = \frac{1}{2} + \alpha_i$, we write

$$\begin{aligned}
D_{i+1}^{p_i} &= \pi^{-p_i(\frac{1}{2}+\alpha_{i+1})}\Gamma^{p_i}(1+\alpha_{i+1}) \prod_{k=0}^{n-1} B^{\frac{p_i\pi_k^{(i+1)}}{1-p}}(\alpha_{i+k+1}) \\
&= \pi^{-p_i(\frac{1}{2}+\alpha_{i+1})}\Gamma^{p_i}(1+\alpha_{i+1}) B^{-1}(\alpha_i) B^{\frac{1}{1-p}}(\alpha_i) \prod_{k=0}^{n-2} B^{\frac{\pi_{k+1}^{(i)}}{1-p}}(\alpha_{i+k+1}) \\
&= \pi^{\frac{1-p_i}{2}-(1+p_i\alpha_{i+1})} \frac{\Gamma^{p_i}(1+\alpha_{i+1})}{\Gamma(\frac{1}{2}+\alpha_i)} \Gamma(1+\alpha_i) \prod_{k=0}^{n-1} B^{\frac{\pi_k^{(i)}}{1-p}}(\alpha_{i+k}) \\
&= \pi^{\frac{1-p_i}{2}} \frac{\Gamma^{p_i}(1+\alpha_{i+1})}{\Gamma(\frac{1}{2}+\alpha_i)} D_i.
\end{aligned}$$

□

Lemma 3.3. *The functions*

$$f_i(y) = D_i\, \mathrm{e}^{-\frac{y^2}{4}}\, U\left(\frac{1}{2}+\alpha_i, \frac{1}{2}, \frac{y^2}{4}\right), \qquad i = 1, 2, \ldots, n \tag{3.5}$$

where U is given in (3.1) *and D_i are given in* (3.2), *solve the problem* (1.3).

Proof. We show that the functions f_i defined in (3.5) solve the equations from (1.3) arguing similarly as in the proof of [FQ, Lemma 3.1] (see also [DFL, Lemma 3.2]). Each of the equations from (1.3) is a generalized Whittaker's equation and can be written in the form (see [AS, 13.1.35] for $A = 0$)

$$\begin{aligned}
w'' &+ \left(2f' + \frac{bh'}{h} - h' - \frac{h''}{h'}\right) w' \\
&+ \left(\left(\frac{bh'}{h} - h' - \frac{h''}{h'}\right) f' + f'' + (f')^2 - \frac{a(h')^2}{h}\right) w = 0
\end{aligned} \tag{3.6}$$

with $a = \frac{1}{2} + \alpha_i$, $b = \frac{1}{2}$, $f(y) = h(y) = \frac{y^2}{4}$. The equation (3.6) can be solved explicitly. One part of its general solution is given by (cf. [AS, 13.1.37])

$$w_1(y) = \mathrm{e}^{-f(y)}\, U(a, b, h(y)). \tag{3.7}$$

Obviously, the functions f_i defined in (3.5) solve the equations from (1.3).

Now we use the relations from Lemma 3.1 and Lemma 3.2 to verify the boundary condition from (1.3)

$$\begin{aligned}
-f_i'(0) = {} & D_i \frac{y}{2}\, \mathrm{e}^{\frac{y^2}{4}} \left(U\left(\frac{1}{2}+\alpha_i, \frac{1}{2}, \frac{y^2}{4}\right) + \left(\frac{1}{2}+\alpha_i\right) U\left(\frac{3}{2}+\alpha_i, \frac{3}{2}, \frac{y^2}{4}\right)\right)\Bigg|_{y\to 0} \\
&= \frac{\pi^{\frac{1}{2}} D_i}{\Gamma(\frac{1}{2}+\alpha_i)} = \frac{D_{i+1}^{p_i}}{\pi^{\frac{p_i}{2}}\Gamma(1+\alpha_{i+1})} = f_{i+1}^{p_i}(0).
\end{aligned}$$

The fact $(f_i) \to 0$ as $y \to \infty$ follows directly from the definition (3.5) and the relation (3.3.iv). $\square$

Proof of Theorem 1.2. Obviously, the functions Υ_i fulfill (P) when $t \neq s$. We need only to show

$$\lim_{t\to s^+} \frac{\partial \Upsilon_i}{\partial t}(x,t;s) = 0 \qquad \text{for} \quad x \in \mathbb{R}^N_+, \quad s \geqslant 0 \tag{3.8}$$

to claim that (Υ_i) converges uniformly to 0 as $t \to s^+$. We use the relations from Lemma 3.1 to write

$$\begin{aligned}
\frac{\partial \Upsilon_i}{\partial t}(x,t;s) &= D_i\alpha_i\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}}\,(t-s)^{\alpha_i-1} U\left(\frac{1}{2}+\alpha_i, \frac{1}{2}, \frac{x_1^2}{4(t-s)}\right) \\
&\quad + D_i\frac{x_1^2}{4}\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}}\,(t-s)^{\alpha_i-2} U\left(\frac{1}{2}+\alpha_i, \frac{1}{2}, \frac{x_1^2}{4(t-s)}\right) \\
&\quad + D_i\left(\frac{1}{2}+\alpha_i\right)\frac{x_1^2}{4}\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}}\,(t-s)^{\alpha_i-2} U\left(\frac{3}{2}+\alpha_i, \frac{3}{2}, \frac{x_1^2}{4(t-s)}\right) \\
&\underset{t\to s^+}{=} D_i\alpha_i\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}}\,(t-s)^{2\alpha_i-\frac{1}{2}}\left(\frac{x_1}{2}\right)^{-\alpha_i-1}(1+O(t-s)) \\
&\quad + D_i\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}}\,(t-s)^{2\alpha_i-\frac{3}{2}}\left(\frac{x_1}{2}\right)^{1-2\alpha_i}(1+O(t-s)) \\
&\quad + D_i\left(\frac{1}{2}+\alpha_i\right)\frac{x_1^2}{4}\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}}\,(t-s)^{2\alpha_i-\frac{3}{2}}\left(\frac{x_1}{2}\right)^{-1-2\alpha_i}(1+O(t-s)) \\
&= \mathrm{e}^{-\frac{x_1^2}{4(t-s)}}\,(t-s)^{2\alpha_i-\frac{3}{2}}\left(\varphi_i(x_1)+O(t-s)\right).
\end{aligned}$$

Thus (3.8) holds and the proof is finished. $\square$

4 Uniqueness of the Nontrivial Solution in the Case $p < 1$ for (P) with Zero Initial Data

As in the previous chapter, we consider the problem (P) with $p < 1$ and trivial initial condition. We prove that the class of functions given in (1.4) contains all nontrivial nonnegative solutions of problem (P) with $p < 1$ and trivial initial data. Every nontrivial nonnegative solution is then uniquely determined by time $s \geqslant 0$ when it branches off the trivial zero solution ($s = \inf\{t > 0 : u_i(x,t) > 0, \quad x \in \mathbb{R}_+^N\}$).

Let us introduce further notation for $i = 1, 2, \ldots, n$

$$\begin{aligned} C_i &= \pi^{\frac{1}{2}} \Gamma^{-1}(1+\alpha_i) D_i = \prod_{k=0}^{n-1} \left(\frac{\Gamma(\frac{1}{2} + \alpha_{i+k})}{\Gamma(1+\alpha_{i+k})} \right)^{\frac{\pi_k^{(i)}}{1-p}} \\ &= \pi^{-\alpha_i} \prod_{k=0}^{n-1} B^{\frac{\pi_k^{(i)}}{1-p}}(\alpha_{i+k}), \\ \beta_i &= 2(1-p)\alpha_i, \\ \rho_i &= \sum_{k=0}^{n-i} \pi_k^{(i)} \qquad (\rho_n = 1, \rho_{n-1} = 1 + p_{n-1}, \ldots, \rho_1 = \beta_1) \end{aligned} \tag{4.1}$$

Lemma 4.1. *Let $p_k > 0$, $k = 1, 2, \ldots, n$, $\prod_{k=1}^{n} p_k = p < 1$, $p_{i+n} = p_i$, and let*

$$\begin{aligned} \alpha_n^{(0)} &= \frac{1}{2}, \\ \alpha_k^{(i)} &= \frac{1}{2} + p_k \alpha_{k+1}^{(i)}, \qquad k = 1, 2, \ldots, n-1, \\ \alpha_n^{(i+1)} &= \frac{1}{2} + p_n \alpha_1^{(i)}. \end{aligned} \tag{4.2}$$

Then $\alpha_k^{(i)} = \alpha_k(1-p^i) + \dfrac{p^i \rho_k}{2}$ for all $i \in \mathbb{N}_0, k = 1, 2, \cdots, n$. The constants ρ_k are given in (4.1).

Proof. The verification is simpler in terms of βs. Set $\beta_k^{(i)} = 2(1-p)\alpha_k^{(i)}$. The recurrent relations corresponding to (4.2) are then

$$\begin{aligned} \beta_n^{(0)} &= 1 - p, \\ \beta_k^{(i)} &= 1 - p + p_k \beta_{k+1}^{(i)}, \qquad k = 1, 2, \ldots, n-1, \\ \beta_n^{(i+1)} &= 1 - p + p_n \beta_1^{(i)} \end{aligned}$$

and it can be easily checked that $\beta_k^{(i)} = \beta_k(1-p^i) + (1-p)p^i \rho_k$. $\square$

Remark 4.2. Notice that $\alpha_k^{(i)}$ increases to α_k as $i \to \infty$ for all $k = 1, 2, \ldots, n$. Namely

$$\alpha_k^{(i)} = \alpha_k - p^i \frac{2\alpha_k - \rho_k}{2}$$

and

$$2\alpha_k > \beta_k \geqslant \rho_k$$

hold.

Lemma 4.3. *If $t > 0$ and (u_i) is a solution of the system* (P) *with an initial condition fulfilling $\upsilon_i \not\equiv 0$, then we can find $\gamma, \sigma > 0$ such that*

$$u_i(x,t) \geqslant \gamma\, \mathrm{e}^{-\sigma|x|^2}, \qquad x \in \mathbb{R}^N_+. \tag{4.3}$$

Proof. Since $\upsilon_i \not\equiv 0$, we can find $\Omega \subset \mathbb{R}^N_+$ such that $\delta = \inf\{\upsilon_i(x) : x \in \Omega\} > 0$. Now

$$\begin{aligned} u_i(x,t) &\geqslant \mathcal{T}(t)\mathcal{S}_{N-1}(t)\upsilon_i(x) \\ &= \int_{\mathbb{R}_+} \frac{1}{2}(\pi t)^{-\frac{1}{2}} \left(\mathrm{e}^{-\frac{|x_1-y_1|^2}{4t}} + \mathrm{e}^{-\frac{|x_1+y_1|^2}{4t}} \right) \\ &\quad \times \int_{\mathbb{R}^{N-1}} (4\pi t)^{-\frac{N-1}{2}}\, \mathrm{e}^{-\frac{|x'-y'|^2}{4t}}\, \upsilon_i(y_1, y')dy'dy_1 \\ &\geqslant \delta \int_\Omega (4\pi t)^{-\frac{N}{2}}\, \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, dy \geqslant \delta (4\pi t)^{-\frac{N}{2}}\, \mathrm{e}^{-\frac{|x|^2}{2t}} \int_\Omega \mathrm{e}^{-\frac{|y|^2}{2t}}\, dy, \end{aligned}$$

which proves the desired estimate with

$$\gamma(t) = \delta(4\pi t)^{-\frac{N}{2}} \int_\Omega \mathrm{e}^{-\frac{|y|^2}{2t}}\, dy \quad \text{and} \quad \sigma(t) = \frac{1}{2t}.$$

□

Lemma 4.4. *For $t > 0,\ \sigma > 0,\ x \in \mathbb{R}^M,\ M \in \mathbb{N}$,*

$$\mathcal{S}_M(t)\, \mathrm{e}^{-\sigma|x|^2} = (1+4\sigma t)^{-\frac{M}{2}}\, \mathrm{e}^{-\frac{\sigma}{1+4\sigma t}|x|^2}$$

holds.

Proof. The verification is not very difficult. We write

$$\begin{aligned} &(4\pi t)^{-\frac{M}{2}} \int_{\mathbb{R}^M} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, \mathrm{e}^{-\sigma|y|^2}\, dy \\ &\qquad = (4\pi t)^{-\frac{M}{2}}\, \mathrm{e}^{-\frac{|x|^2 - \frac{|x|^2}{1+4\sigma t}}{4t}} \int_{\mathbb{R}^M} \mathrm{e}^{-\frac{\left|\frac{x}{\sqrt{1+4\sigma t}} - \sqrt{1+4\sigma t}\, y\right|^2}{4t}}\, dy \\ &\qquad = (4\pi t)^{-\frac{M}{2}}\, \mathrm{e}^{-\frac{\sigma|x|^2}{1+4\sigma t}} \int_{\mathbb{R}^M} \mathrm{e}^{-\frac{1+4\sigma t}{4t}|y|^2}\, dy = (1+4\sigma t)^{-\frac{M}{2}}\, \mathrm{e}^{-\frac{\sigma}{1+4\sigma t}|x|^2}. \end{aligned}$$

□

Remark 4.5. For $0 \leqslant \eta \leqslant t$, we have

$$\mathcal{S}_{N-1}(t-\eta)\,\mathrm{e}^{-\sigma|x'|^2} \geqslant (1+4\sigma t)^{-\frac{N-1}{2}}\,\mathrm{e}^{-\sigma|x'|^2}. \tag{4.4}$$

Proposition 4.6. *If* (u_i) *is a solution of the system* (P) *with nontrivial initial condition* $(v_i) \not\equiv 0$ *and* $p < 1$, *then*

$$u_i(0,x';t) \geqslant C_i t^{\alpha_i}, \qquad i = 1,2,\dots,n, \tag{4.5}$$

where C_i *are given in* (4.1).

Proof. We use the ideas from the proof of [EH1, Lemma 2]. We first prove the result assuming that $v_1(x) \geqslant \gamma\,\mathrm{e}^{-\sigma|x|^2}$ for some $\gamma, \sigma > 0$. Lemma 4.4 then yields

$$u_1(0,x';t) \geqslant \mathcal{T}(t)\mathcal{S}_{N-1}(t)v_1(0,x') \geqslant \gamma(1+4\sigma t)^{-\frac{N-1}{2}}\,\mathrm{e}^{-\sigma|x'|^2}$$

and from (4.4) follows

(4.6.n.0)

$$\begin{aligned}
u_n(0,x';t) &\geqslant \int_0^t \mathcal{R}(t-\eta)u_1^{p_n}(x,\eta)d\eta \\
&\geqslant \pi^{-\frac{1}{2}}\gamma^{p_n}(1+4\sigma t)^{-\frac{N-1}{2}p_n}\int_0^t (t-\eta)^{-\frac{1}{2}}\mathcal{S}_{N-1}(t-\eta)\,\mathrm{e}^{-\sigma p_n|x'|^2}\,d\eta \\
&\geqslant \pi^{-\frac{1}{2}}\gamma^{p_n}(1+4\sigma t)^{-\frac{N-1}{2}p_n}(1+4\sigma p_n t)^{-\frac{N-1}{2}}\,\mathrm{e}^{-\sigma p_n|x'|^2}\int_0^t (t-\eta)^{-\frac{1}{2}}d\eta \\
&= 2\pi^{-\frac{1}{2}}\gamma^{p_n}(1+4\sigma t)^{-\frac{N-1}{2}p_n}(1+4\sigma p_n t)^{-\frac{N-1}{2}}\,\mathrm{e}^{-\sigma p_n|x'|^2}\,t^{\frac{1}{2}}
\end{aligned}$$

as an initial inequality for an iterating argument. Namely, using (4.4) we can derive the following implication

$$\text{if} \quad u_{i+1}(0,x';t) \geqslant c\prod_{j=1}^{m}(1+a_j t)^{b_j}\,\mathrm{e}^{-\sigma|x'|^2}\,t^{\gamma}$$

$$\text{where} \quad \gamma,\sigma > 0, \quad m \in \mathbb{N}, \quad a_j > 0, \quad b_j < 0,$$

$$\begin{aligned}
\text{then} \quad u_i(0,x';t) &\geqslant \int_0^t \mathcal{R}(t-\eta)u_{i+1}^{p_i}(x,\eta)d\eta \\
(b_j<0) \quad &\geqslant \pi^{-\frac{1}{2}}c^{p_i}\prod_{j=1}^{m}(1+a_j t)^{p_i b_j} \\
&\quad\times\int_0^t (t-\eta)^{-\frac{1}{2}}\eta^{p_i\gamma}\mathcal{S}_{N-1}(t-\eta)\left(e^{-p_i\sigma|x'|^2}\right)d\eta \\
(4.4) \quad &\geqslant \pi^{-\frac{1}{2}}c^{p_i}\prod_{j=1}^{m}(1+a_j t)^{p_i b_j}(1+4p_i\sigma t)^{-\frac{N-1}{2}} \\
&\quad\times\mathrm{e}^{-p_i\sigma|x'|^2}\,B\left(\frac{1}{2}+p_i\gamma\right)t^{\frac{1}{2}+p_i\gamma}
\end{aligned} \tag{4.7}$$

for $i=1,2,\dots,n$ (setting $u_{n+1}=u_1$). Now it is only a matter of a patient calculation to prove that for any $k\in\mathbb{N}_0$

(4.6.1.k)

$$\begin{aligned} u_1(0,x';t) \geqslant{}& 2^{\pi_{n-1}^{(1)}p^k}\pi^{-\alpha_1^{(k)}}\gamma^{p^{k+1}} \\ &\times \prod_{i=0}^{k}\prod_{m=1}^{n}\left(1+4\sigma t p^i \textstyle\prod_{j=n-m+1}^{n} p_j\right)^{-\frac{N-1}{2}p^{k-i}\prod_{j=1}^{n-m}p_j} \\ &\times (1+4\sigma t)^{p^{k+1}}\prod_{i=1}^{k} B^{\pi_{n-1}^{(1)}p^{k-i}}\left(\alpha_n^{(i)}\right) \\ &\times \prod_{m=1}^{n-1}\prod_{i=0}^{k} B^{\pi_{m-1}^{(1)}p^{k-i}}\left(\alpha_m^{(i)}\right)\mathrm{e}^{-\sigma p^{k+1}|x'|^2}\, t^{\alpha_1^{(k)}} \end{aligned}$$

holds.

Now let us see what happens when $k\to\infty$. Clearly, $2^{\pi_{n-1}^{(1)}p^k}\xrightarrow[k\to\infty]{}1$, $\gamma^{p^{k+1}}\xrightarrow[k\to\infty]{}1$, $\mathrm{e}^{-\sigma p^{k+1}|x'|^2}\xrightarrow[k\to\infty]{}1$, $(1+4\sigma t)^{p^{k+1}}\xrightarrow[k\to\infty]{}1$. Moreover, since $\alpha_m^{(k)}$ increases to α_m and B is decreasing in γ, we have

$$\prod_{i=1}^{k} B^{\pi_{n-1}^{(1)}p^{k-i}}\left(\alpha_n^{(i)}\right)\prod_{m=1}^{n-1}\prod_{i=0}^{k} B^{\pi_{m-1}^{(1)}p^{k-i}}\left(\alpha_m^{(i)}\right) \geqslant \prod_{m=1}^{n} B^{\frac{\pi_{m-1}^{(1)}}{1-p}}(\alpha_m).$$

And finally setting $a_m = 4\sigma t\displaystyle\prod_{j=n-m+1}^{n} p_j > 0$ and $b_m = -\dfrac{N-1}{2}\displaystyle\prod_{j=1}^{n-m} p_j < 0$, we may write

$$\prod_{i=0}^{k}\prod_{m=1}^{n}\left(1+a_m p^i\right)^{b_m p^{k-i}} \xrightarrow[k\to\infty]{} 1$$

since for $m=1,2,\dots,n$

$$\begin{aligned} 0 \geqslant \ln\prod_{i=0}^{k}\left(1+a_m p^i\right)^{b_m p^{k-i}} &= b_m\sum_{i=0}^{k} p^{k-i}\ln\left(1+a_m p^i\right) \\ &\geqslant a_m b_m \sum_{i=0}^{k} p^k = (k+1)a_m b_m p^k \xrightarrow[k\to\infty]{} 0 \end{aligned}$$

holds. Letting $k\to\infty$ in (4.6.1.k) therefore gives

$$u_1(0,x';t) \geqslant C_1 t^{\alpha_1}$$

which is (4.5) for $i=1$.

Considering the implication ($u_{n+1}=u_1$, $C_{n+1}=C_1$, $\frac{1}{2}+p_i\alpha_{i+1}=\alpha_i=\alpha_{i+n}$)

$$\begin{aligned} &\text{if} \quad u_{i+1}(0,x';t) \geqslant C_{i+1}t^{\alpha_{i+1}} \\ &\text{then} \quad u_i(0,x';t) \geqslant \int_0^t \mathcal{R}(t-\eta)u_{i+1}^{p_i}(x,\eta)d\eta \\ &\qquad = \pi^{-\frac{1}{2}}C_{i+1}^{p_i}\int_0^t (t-\eta)^{-\frac{1}{2}}\eta^{p_i\alpha_{i+1}} \\ &\qquad = \pi^{-\alpha_i}\prod_{m=1}^{n-1} B^{\frac{\pi_m^{(i)}}{1-p}}(\alpha_{i+m})\cdot B^{\frac{p}{1-p}}(\alpha_i)\,B(\alpha_i)\,t^{\alpha_i} \\ &\qquad = \pi^{-\alpha_i}\prod_{m=0}^{n-1} B^{\frac{\pi_m^{(i)}}{1-p}}(\alpha_{i+m})\,t^{\alpha_i} = C_i t^{\alpha_i} \end{aligned} \tag{4.8}$$

for $i=1,2,\dots,n$, we obtain (4.5) under current assumption on the initial data.

Now we generalize the estimate for any nontrivial initial data $\upsilon_1\not\equiv 0$ using Lemma 4.3. We take arbitrary positive ε and set $\tilde{u}_i(\cdot,t;\varepsilon)=u_i(\cdot,t+\varepsilon)$. The autonomous nature of the system (P) implies

$$\tilde{u}_i(\cdot,t;\varepsilon)=\mathcal{T}(t)\mathcal{S}_{N-1}(t)\tilde{u}_i(\cdot,0;\varepsilon)+\int_0^t \mathcal{R}(t-\eta)\tilde{u}_{i+1}^{p_i}(\cdot,\eta;\varepsilon)d\eta,$$

where $\tilde{u}_1(0,x';0;\varepsilon) > \gamma\,\mathrm{e}^{-\sigma|x'|^2}$ for some positive numbers γ and σ. Therefore $\tilde{u}_i(0,x';t;\varepsilon)\geqslant C_i t^{\alpha_i}$, and accordingly

$$u_i(0,x';t)\geqslant C_i(t-\varepsilon)^{\alpha_i}.$$

Thus (4.5) holds for any $\upsilon_1\not\equiv 0$, since ε is arbitrary. Obviously, the assumption $\upsilon_1\not\equiv 0$ is made without loss of generality. □

Proof of Theorem 1.3. We apply the idea from the proof of [EH1, Lemma 4]. Let (u_i) be a nontrivial nonnegative solution of the problem (P) with $p<1$ and trivial initial data $(\upsilon_i)\equiv 0$. Without loss of generality, we assume that there are $x\in\mathbb{R}^N_+$ and $t>0$ such that $u_n(x,t)=\int_0^t \mathcal{R}(t-\eta)u_1^{p_n}(x,\eta)d\eta>0$. Set

$$\tau=\inf\{t>0: u_1(0,x';t)>0, x'\in\mathbb{R}^{N-1}\}.$$

By standard results, $u_i(x,t)>0$ for any $x\in\mathbb{R}^N_+$ and $t>\tau$, $i=1,2,\dots,n$. Now we take $\bar{t}>\tau$ and set $\bar{u}_i(x,t)=u_i(x,\bar{t}+t)$. Obviously, $(\bar{u}_i)$ solves (P) and $\bar{\upsilon}_i=\bar{u}_i(\cdot,0)>0$, $i=1,2,\dots,n$, and according to Proposition 4.6

$$u_i(0,x';\bar{t}+t)\geqslant C_i t^{\alpha_i}$$

for any $x' \in \mathbb{R}^{N-1}$ and $t \geqslant 0$. This implies

$$u_i(0,x';t) \geqslant C_i(t-\tau)_+^{\alpha_i}, \qquad x' \in \mathbb{R}^{N-1}, \quad t \geqslant 0, \quad i=1,2,\dots,n. \tag{4.9}$$

We use another slight modification of the often used iteration argument to obtain the corresponding upper estimate for $u_i(0,x';t)$. Let $T>0$ be arbitrary and $M>0$ such that

$$\|u_1(0,\cdot;\eta)\|_\infty \leqslant M\|u_1(0,\cdot;t)\|_\infty \qquad \text{for} \quad 0 \leqslant \eta \leqslant t \leqslant T.$$

Since the initial condition is trivial, we can write

$$\begin{aligned} u_n(0,x';t) &\leqslant \int_0^t \mathcal{R}(t-\eta)\mathcal{S}_{N-1}(t-\eta)\|u_1(\eta)\|_\infty^{p_n}\,d\eta \\ &= \pi^{-\frac12}\int_0^t \|u_1(0,\cdot;\eta)\|_\infty^{p_n}\,d\eta \leqslant \pi^{-\frac12}M^{p_n}\|u_1(0,\cdot;t)\|_\infty^{p_n}2t^{\frac12}. \end{aligned}$$

The induction implication reads as follows

$$\begin{aligned} \text{if} \quad & u_{i+1}(0,x';t) \leqslant ct^\gamma \\ \text{then} \quad & u_i(0,x';t) \leqslant \int_0^t \mathcal{R}(t-\eta)\|u_{i+1}(0,\cdot;\eta)\|_\infty^{p_i}\,d\eta \\ & \qquad \leqslant \pi^{-\frac12}c^{p_i}B\left(\frac12+p_i\gamma\right)t^{\frac12+p_i\gamma} \end{aligned}$$

for $i=1,2,\dots,n$, so that

$$\|u_1(0,\cdot;t)\|_\infty \leqslant \|u_1(0,\cdot;t)\|_\infty^p 2^{\pi_{n-1}^{(1)}}\pi^{-(1-p)\alpha_1}M^p\prod_{k=1}^{n-1}B^{\pi_{n-1}^{(1)}}\left(\frac{\rho_k}{2}\right)t^{(1-p)\alpha_1},$$

$$\|u_1(0,\cdot;t)\|_\infty \leqslant 2^{\frac{\pi_{n-1}^{(1)}}{1-p}}\pi^{-\alpha_1}M^{\frac{p}{1-p}}\prod_{k=1}^{n-1}B^{\frac{\pi_{n-1}^{(1)}}{1-p}}\left(\frac{\rho_k}{2}\right)t^{\alpha_1} = Pt^{\alpha_1}.$$

Substituting this inequality into the solution formulae (1.1) we get

$$\begin{aligned} \|u_1(0,x';t)\|_\infty &\leqslant \int_0^t \pi^{-\frac12}(t-t_1)^{-\frac12} \\ &\times\left(\int_0^{t_1}\cdots\left(\int_0^{t_{n-1}}\pi^{-\frac12}(t_{n-1}-t_n)^{-\frac12}P^{p_n}t_n^{p_n\alpha_1}dt_n\right)^{p_{n-1}}\cdots dt_2\right)^{p_1}dt_1 \\ &= P^p\pi^{-(1-p)\alpha_1}\prod_{i=1}^n B^{\pi_{i-1}^{(1)}}(\alpha_i)t^{\alpha_1}, \end{aligned} \tag{4.10.1.1}$$

and iterating this step yields

$$\|u_1(0,x';t)\|_\infty \leqslant P^{p^k}\pi^{-(1-p^k)\alpha_1}\prod_{i=1}^{n} B^{\pi^{(1)}_{i-1}\frac{1-p^k}{1-p}}(\alpha_i)t^{\alpha_1}. \tag{4.10.1.k}$$

Letting $k\to\infty$ implies

$$\|u_1(0,x';t)\|_\infty \leqslant C_1 t^{\alpha_1} \tag{4.10.1}$$

and as in (4.8) we obtain

$$u_i(0,x';t) \leqslant \|u_i(0,\cdot;t)\|_\infty \leqslant C_i t^{\alpha_i} \tag{4.10}$$

for $x'\in\mathbb{R}^{N-1}$, $t\geqslant 0$, $i=1,2,\dots,n$. When $\tau>0$, we take $0<\underline{t}\leqslant\tau$ and define $\underline{u}_i(x,t)=u_i(x,\underline{t}+t)$. The definition of τ and a simple contradiction argument imply that $(u_i(\underline{t}))\equiv 0$, and therefore $(\underline{u}_i)$ solves (P) with trivial initial data. From (4.10) we obtain $u_i(0,x';\underline{t}+t)\leqslant C_i t^{\alpha_i}$ for any $x'\in\mathbb{R}^{N-1}$ and $t\geqslant 0$. This implies

$$u_i(0,x';t)\leqslant C_i(t-\tau)_+^{\alpha_i}, \qquad x'\in\mathbb{R}^{N-1}, \quad t\geqslant 0, \quad i=1,2,\dots,n, \tag{4.11}$$

and, by (4.10), it is valid for $\tau=0$ as well.

We conclude from (4.9,11) that

$$u_i(0,x';t)=C_i(t-\tau)_+^{\alpha_i}=\Upsilon_i(0,x';t;\tau), \quad x'\in\mathbb{R}^{N-1}, \quad t\geqslant 0, \quad i=1,2,\dots,n.$$

The maximum principle implies that $(u_i(x,t))=(\Upsilon_i(x,t;s))$ for $s=\tau$. $\square$

5 Uniqueness in the Case $p<1$ for (P) with Nonzero Initial Data

In this chapter we consider the problem (P) with nontrivial initial condition. We prove that the solution is unique if $p_i < 1$, $i=1,2,\dots,n$. We use the notation from (4.1) and the estimate from Proposition 4.6.

Lemma 5.1. *The constants C_i from* (4.1) *fulfill the equality*

$$\prod_{i=1}^{n} C_i^{p_{i-1}-1} = \pi^{\frac{n}{2}} \prod_{i=1}^{n} B^{-1}(\alpha_i). \tag{5.1}$$

Proof. First notice that $p_{i-1}\beta_i = \beta_{i-1} + p - 1$, $i=1,2,\cdots,n$, and therefore

$$\sum_{i=1}^{n} \alpha_i(p_{i-1}-1) = \frac{\beta_n + p - 1 - \beta_1}{2(1-p)} + \sum_{i=2}^{n} \frac{\beta_{i-1}+p-1-\beta_i}{2(1-p)} = -\frac{n}{2}.$$

Furthermore, $p_{i+j-1}\pi_{n-j}^{(i+j)} = \pi_{n-j+1}^{(i+j-1)}$, $i=1,2,\dots,n$, $j=1,2,\dots,n-1$. So we can write

$$\begin{aligned}
\prod_{i=1}^{n} C_i^{p_{i-1}-1} &= \prod_{i=1}^{n}\left(\pi^{-\alpha_i(p_{i-1}-1)} \prod_{j=0}^{n-1} B^{\frac{p_{i-1}-1}{1-p}\pi_j^{(i)}}(\alpha_{i+j})\right)\\
&= \pi^{\frac{n}{2}} \prod_{i=1}^{n}\left(B^{\frac{p_{n+i-1}-1}{1-p}}(\alpha_i) \prod_{j=1}^{n-1} B^{\frac{p_{i+j-1}-1}{1-p}\pi_{n-j}^{(i+j)}}(\alpha_i)\right)\\
&= \pi^{\frac{n}{2}} \prod_{i=1}^{n} B^{\frac{p_{n+i-1}-1}{1-p}+\frac{1}{1-p}\sum_{j=1}^{n-1}(\pi_{n-j+1}^{(i+j-1)}-\pi_{n-j}^{(i+j)})}(\alpha_i)\\
&= \pi^{\frac{n}{2}} \prod_{i=1}^{n} B^{\frac{p_{n+i-1}-1+\pi_n^{(i)}-\pi_1^{n+i-1}}{1-p}}(\alpha_i) = \pi^{\frac{n}{2}} \prod_{i=1}^{n} B^{-1}(\alpha_i).
\end{aligned}$$

□

Proof of Theorem 1.4. Let us introduce the notation $\|f(t)\| = \sup\{f_+(0,x';t) : x' \in \mathbb{R}^{N-1}\}$ and $f_+ = \max\{f,0\}$. We use the contradiction argument from the proof of [EH1, Lemma 3].

Suppose that $(u_i) \neq (\bar{u}_i)$. Then we can find $t>0$ such that without loss of generality we may assume $\|(u_1-\bar{u}_1)(\eta)\| \leqslant \|(u_1-\bar{u}_1)(t)\| > 0$ for $0 \leqslant \eta \leqslant t$. Since $p_i < 1$, we have $|a^{p_i} - b^{p_i}| \leqslant |a-b|^{p_i}$ for $a,b>0$. The solution formulae (1.1) imply

$$(u_i - \bar{u}_i)_+(0,x';t) \leqslant \int_0^t \mathcal{R}(t-\eta)(u_{i+1}^{p_i} - \bar{u}_{i+1}^{p_i})_+(0,x';\eta)d\eta$$

for $i=1,2,\dots,n$. We use these facts n times to show for some positive constant P that $\|(u_1-\bar{u}_1)(t)\|\leqslant Pt^{\alpha_1}$. First we write

$$\begin{aligned}(u_n-\bar{u}_n)_+(0,x';\tau)&\leqslant\int_0^\tau\pi^{-\frac{1}{2}}(\tau-\eta)^{-\frac{1}{2}}\|(u_1-\bar{u}_1)(\eta)\|^{p_n}d\eta\\&\leqslant 2\pi^{-\frac{1}{2}}\|(u_1-\bar{u}_1)(t)\|^{p_n}\tau^{\frac{1}{2}}\end{aligned}$$

for $0\leqslant\tau\leqslant t$. The inequality $(u_{i+1}-\bar{u}_{i+1})_+(0,x';\tau)\leqslant c\tau^\gamma$ implies

$$(u_i-\bar{u}_i)_+(0,x';\tau)\leqslant\pi^{-\frac{1}{2}}c^{p_i}B\left(\frac{1}{2}+p_i\gamma\right)\tau^{\frac{1}{2}+p_i\gamma}$$

for $i=n-1,n-2,\dots,1$. So by induction we obtain

$$\|(u_1-\bar{u}_1)(t)\|\leqslant\|(u_1-\bar{u}_1)(t)\|^p2^{\pi^{(1)}_{n-1}}\pi^{-(1-p)\alpha_1}\prod_{k=1}^{n-1}B^{\pi^{(1)}_{k-1}}\left(\frac{\rho_k}{2}\right)t^{(1-p)\alpha_1}$$

with ρ_k defined in (4.1), and therefore

$$\|(u_1-\bar{u}_1)(t)\|\leqslant Pt^{\alpha_1},\qquad P=2^{\frac{\pi^{(1)}_{n-1}}{1-p}}\pi^{-\alpha_1}\prod_{k=1}^{n-1}B^{\pi^{(1)}_{k-1}/(1-p)}\left(\frac{\rho_k}{2}\right)\tag{5.2}$$

holds.

The mean value theorem for $f(\xi)=\xi^{p_i}$ gives

$$(u^{p_i}_{i+1}-\bar{u}^{p_i}_{i+1})(0,x';t)=p_iw^{p_i-1}_{i+1}(0,x';t)(u_{i+1}-\bar{u}_{i+1})(0,x';t),\quad i=1,2,\dots,n,$$

for some w_{i+1} between u_{i+1} and $\bar{u}_{i+1}$. Now all p_i are less than 1, so by Proposition 4.6 we have

$$w^{p_i-1}_{i+1}(0,x';t)\leqslant C^{p_i-1}_{i+1}t^{(p_i-1)\alpha_i},\qquad i=1,2,\dots,n.$$

By the solution formulae (1.1), inequalities

$$\begin{aligned}(u_i-\bar{u}_i)_+(0,x';t)\leqslant&\int_0^t\mathcal{R}(t-\eta)(u^{p_i}_{i+1}-\bar{u}^{p_i}_{i+1})_+(0,x';\eta)d\eta\\&\leqslant\pi^{-\frac{1}{2}}p_iC^{p_i-1}_{i+1}\int_0^t(t-\eta)^{-\frac{1}{2}}\eta^{(p_i-1)\alpha_{i+1}}\|(u_{i+1}-\bar{u}_{i+1})(\eta)\|d\eta\end{aligned}$$

hold for $i = 1, 2, \dots, n$, so that we can claim

(5.3)
$$\begin{aligned}
\|(u_1 - \bar{u}_1)(t)\| \leqslant \pi^{-\frac{n}{2}} p \prod_{i=1}^{n} C_{i+1}^{p_i - 1} \int_0^t (t - t_1)^{-\frac{1}{2}} t_1^{(p_1-1)\alpha_2} \\
\times \int_0^{t_1} (t_1 - t_2)^{-\frac{1}{2}} t_2^{(p_2-1)\alpha_3} \int_0^{t_2} \cdots \int_0^{t_{n-1}} (t_{n-1} - t_n)^{-\frac{1}{2}} \\
\times t_n^{(p_n-1)\alpha_1} \|(u_1 - \bar{u}_1)(t_n)\| dt_n \cdots dt_3 dt_2 dt_1 \\
= p \prod_{i=1}^{n} B^{-1}(\alpha_i) \int_0^t (t - t_1)^{-\frac{1}{2}} t_1^{(p_1-1)\alpha_2} \\
\times \int_0^{t_1} (t_1 - t_2)^{-\frac{1}{2}} t_2^{(p_2-1)\alpha_3} \int_0^{t_2} \cdots \int_0^{t_{n-1}} (t_{n-1} - t_n)^{-\frac{1}{2}} \\
\times t_n^{(p_n-1)\alpha_1} \|(u_1 - \bar{u}_1)(t_n)\| dt_n \cdots dt_3 dt_2 dt_1
\end{aligned}$$

by Lemma 5.1. By (5.2), we see that righthand side of (5.3) is integrable. Moreover combining with (5.2) yields

$$\begin{aligned}
\int_0^{t_{n-1}} (t_{n-1} - t_n)^{-\frac{1}{2}} t_n^{(p_n-1)\alpha_1} \|(u_1 - \bar{u}_1)(t_n)\| dt_n \\
\leqslant P \int_0^{t_{n-1}} (t_{n-1} - t_n)^{-\frac{1}{2}} t_n^{p_n \alpha_1} dt_n = P B(\alpha_n) t_{n-1}^{\alpha_n}.
\end{aligned}$$

Similarly we get

$$\|(u_1 - \bar{u}_1)(t)\| \leqslant p P t^{\alpha_1},$$

and obviously

$$\|(u_1 - \bar{u}_1)(t)\| \leqslant p^k P t^{\alpha_1}, \qquad k \in \mathbb{N}, \tag{5.4}$$

as well. Letting $k \to \infty$ implies $u_1(\cdot, t) = \bar{u}_1(\cdot, t)$ on the boundary $x_1 = 0$, and the contradiction argument is finished. $\square$

6 Uniqueness for the Nonsymmetric Problem (FL)

This chapter proves uniqueness for the problem (FL) with $pq \geqslant 1$.

Proof of Theorem 1.5. We omit the standard argument when both nonlinearities are Lipschitz continuous, i.e., $p, q \geqslant 1$ (cf. proof of Theorem 1.1). However, we have to discuss both cases $p < 1$, $q > 1$ and $p > 1$, $q < 1$ ($pq \geqslant 1$) since the system (FL) is not symmetric in the sense of interchanging p and q.

(a) We start with the case $p < 1$. Let $\tau \in (0, T)$ be an arbitrary time and let $0 \leqslant s \leqslant \eta \leqslant t \leqslant \tau$ be always ordered this way in further discussion. We fix $(x, \eta) \in S_\tau$ and define a functional $g(\cdot)(x, \eta) : L^\infty(S_\tau) \to \mathbb{R}$

$$
\begin{aligned}
g(w)(x,\eta) &= \mathcal{T}(\eta)\mathcal{S}_{N-1}(\eta)v_0(x) + \int_0^\eta \mathcal{R}(\eta - s) w^q(x,s)ds, \\
f(\xi) &= \xi^p, \qquad \xi > 0,
\end{aligned}
$$

so that we obtain by the mean value theorem for $f \circ g$

$$
\begin{aligned}
V(x,\eta) &= \left(v^p - \bar{v}^p\right)(x,\eta) = \left(g(u)(x,\eta)\right)^p - \left(g(\bar{u})(x,\eta)\right)^p \\
&= pq\left(g(w)(x,\eta)\right)^{p-1} \int_0^\eta \mathcal{R}(\eta - s)\left(w^{q-1}(u - \bar{u})\right)(x,s)ds
\end{aligned}
\tag{6.1}
$$

for some w between u and $\bar{u}$. More precisely we write

$$
w(\cdot, s) = \rho(x,\eta)u(\cdot, s) + (1 - \rho(x,\eta))\bar{u}(\cdot, s)
$$

where $0 < \rho(x, \eta) < 1$. We also define $F(t) = \sup\{\|(u - \bar{u})(\cdot, \eta)\|_\infty : 0 \leqslant \eta \leqslant t\}$, and by Hölder's inequality we derive (since $\frac{1}{q} \leqslant p < 1$)

$$
\begin{aligned}
|V(x,\eta)| &\leqslant pqF(\eta)\left(\int_0^\eta \mathcal{R}(\eta - s)w^q(x,s)ds\right)^{p-1} \\
&\qquad \times \int_0^\eta \mathcal{R}(\eta - s)w^{q-1}(x,s)ds \\
&\leqslant pqF(\eta)\left(2^{\frac{1}{q}}\pi^{-\frac{1}{2q}}\eta^{\frac{1}{2q}}\right)\left(\int_0^\eta \mathcal{R}(\eta - s)w^q(x,s)ds\right)^{p-1+1-\frac{1}{q}} \\
&\leqslant pq2^p\pi^{-\frac{p}{2}}U^{pq-1}F(\eta)\eta^{\frac{p}{2}},
\end{aligned}
\tag{6.2}
$$

where U is the upper bound of w in $\mathbb{R}^N_+ \times [0, \tau]$. Hence, applying the solution formulae (1.2), we obtain for any $x \in \mathbb{R}^N_+$, $\eta \in [0, \tau]$

$$
\begin{aligned}
|u - \bar{u}|(x,\eta) &\leqslant \int_0^\eta \mathcal{T}(\eta - s)\mathcal{S}_{N-1}(\eta - s)|V(x,s)|ds \\
&\leqslant pq2^p\pi^{-\frac{p}{2}}U^{pq-1}F(\eta)\int_0^\eta s^{\frac{p}{2}}ds \leqslant Kt^{\frac{2+p}{2}}F(t),
\end{aligned}
\tag{6.3}
$$

where the constant K depends on p, q, and on the bounds of u and $\bar{u}$ in $\mathbb{R}^N_+ \times [0,\tau]$. The supremum property implies $F(t) \leqslant Kt^{\frac{2+p}{2}} F(t)$ on $[0,\tau]$, and thus $F(t) = 0$ for $t \in (0, K^{-\frac{2}{2+p}})$. Since the system is autonomous, finite iterating of the argument yields $u = \bar{u}$ in $\mathbb{R}^N_+ \times [0,\tau]$. The equality $v = \bar{v}$ follows consequently from (1.2).

(b) The case $q < 1$ is dealt with in a slightly different manner. We set

$$\begin{aligned} F(t) &= \sup\{\|(v-\bar{v})(\cdot,\eta)\|_\infty : 0 \leqslant \eta \leqslant t\}, \qquad t \geqslant 0,\\ g(z)(x,\eta) &= \mathcal{T}(\eta)\mathcal{S}_{N-1}(\eta)u_0(x) + \int_0^\eta \mathcal{T}(\eta-s)\mathcal{S}_{N-1}(\eta-s)z^p(x,s)ds\\ &\qquad \text{for } z \in L^\infty(S_\tau),\\ f(\xi) &= \xi^q, \qquad \xi > 0, \end{aligned}$$

so that we arrive by the mean value theorem and Hölder's inequality (since $\frac{1}{p} \leqslant q < 1$) at

$$\begin{aligned} |U(x,\eta)| &= |(g(v)(x,\eta))^q - (g(\bar{v})(x,\eta))^q|\\ &\leqslant pqF(\eta)\,(g(z)(x,\eta))^{q-1}\left(\int_0^\eta \mathcal{T}(\eta-s)\mathcal{S}_{N-1}(\eta-s)z^{p-1}(x,s)ds\right)\\ &\leqslant pqV^{pq-1}F(\eta)\eta^q, \end{aligned} \tag{6.4}$$

where $z(\cdot,s) = \rho(x,\eta)v(\cdot,s) + (1-\rho(x,\eta))\bar{v}(\cdot,s)$, $0 < \rho(x,\eta) < 1$ and V is its upper bound on $\mathbb{R}^N_+ \times [0,\tau]$. Thus the difference of solutions in the v component is bounded above by

$$\begin{aligned} |v-\bar{v}|(x,\eta) &\leqslant \int_0^\eta \mathcal{R}(\eta-s)|U(x,s)|ds\\ &\leqslant pqV^{pq-1}F(\eta)\int_0^\eta R(\eta-s)s^q ds \leqslant Lt^{\frac{1}{2}+q}F(t) \end{aligned} \tag{6.5}$$

for all $x \in \mathbb{R}^N_+$, $\eta \in [0,\tau]$. We complete the proof by the same final argument as earlier. $\square$

7 Nonuniqueness for the Nonsymmetric Problem (FL)

Let us first study the one dimensional problem:

$$
\text{(FL1)}\qquad
\begin{array}{llll}
u_t = u_{xx} + v^p, & v_t = v_{xx}, & x > 0, & t > 0, \\
-u_x = 0, & -v_x = u^q, & x = 0, & t > 0, \\
u(x,0) = u_0 \geqslant 0, & v(x,0) = v_0 \geqslant 0, & x > 0. &
\end{array}
$$

We want to find a nonnegative nontrivial solution starting from the zero initial condition if $pq < 1$. In this chapter, we set

$$
\alpha = \frac{2+p}{2(1-pq)} > 1, \qquad \beta = \frac{1+2q}{2(1-pq)} > \frac{1}{2},
$$

and as in [DFL], we look for a self-similar solution of the form

$$
u(x,t) = t^{\alpha} f(y), \qquad v(x,t) = t^{\beta} g(y) \qquad \text{for } y = \frac{x}{\sqrt{t}}, \quad t > 0,
$$

where (f,g) is a positive solution of the problem

$$
\begin{aligned}
f''(y) + \frac{y}{2} f'(y) - \alpha f(y) + g^p(y) &= 0, \\
g''(y) + \frac{y}{2} g'(y) - \beta g(y) &= 0 \qquad \text{for } y > 0, \\
f'(0) &= 0, \\
g'(0) &= -f^q(0),
\end{aligned}
\tag{7.1}
$$

and where $(t^{\alpha} f(y), t^{\beta} g(y))$ converges to $(0,0)$ as $t \to 0^+$, i.e. $y \to \infty$. This transformation can be easily verified.

Remark 7.1. In order to prove that the blow-up rate estimates (ChF) for (FL) are optimal (see Section 1.2), backward self-similar solutions of (FL1) are constructed in [ChF]. They are of the form

$$
\tilde{u}(x,t) = (T-t)^{\alpha} \tilde{f}(y), \qquad \tilde{v}(x,t) = (T-t)^{\beta} \tilde{g}(y), \qquad y = \frac{x}{\sqrt{T-t}},
$$

where positive bounded $(\tilde{f}, \tilde{g})$ satisfies

$$
\begin{aligned}
\tilde{f}''(y) - \frac{y}{2} \tilde{f}'(y) + \alpha \tilde{f}(y) + \tilde{g}^p(y) &= 0, \\
\tilde{g}''(y) - \frac{y}{2} \tilde{g}'(y) + \beta \tilde{g}(y) &= 0 \qquad \text{for } y > 0, \\
\tilde{f}'(0) &= 0, \\
\tilde{g}'(0) &= -\tilde{f}^q(0).
\end{aligned}
$$

Recall that in this case $pq > 1$, i.e., α, $\beta < 0$, and $T < \infty$ is the blow-up time.

As in the proof of Lemma 3.3, the linear equations

$$\begin{aligned} &f''(y) + \frac{y}{2} f'(y) - \alpha f(y) = 0, \\ &g''(y) + \frac{y}{2} g'(y) - \beta g(y) = 0 \qquad \text{for } y > 0, \end{aligned}$$

are generalized Whittaker's equations (see (3.6)). Their solutions are given by the formulae (cf. [AS, 13.1.36,37])

$$\begin{aligned} f_1(y) &= \mathrm{e}^{-\frac{y^2}{4}}\, U\left(\frac{1}{2} + \alpha, \frac{1}{2}, \frac{y^2}{4}\right), & f_2(y) &= \mathrm{e}^{-\frac{y^2}{4}}\, M\left(\frac{1}{2} + \alpha, \frac{1}{2}, \frac{y^2}{4}\right), \\ g_1(y) &= \mathrm{e}^{-\frac{y^2}{4}}\, U\left(\frac{1}{2} + \beta, \frac{1}{2}, \frac{y^2}{4}\right), & g_2(y) &= \mathrm{e}^{-\frac{y^2}{4}}\, M\left(\frac{1}{2} + \beta, \frac{1}{2}, \frac{y^2}{4}\right), \end{aligned}$$

where $M(a, b, r) = 1 + \frac{a}{b} r + \ldots + \frac{a(a+1)\ldots(a+n-1)}{b(b+1)\ldots(b+n-1)} r^n + \ldots$ and U is given in (3.1).

Lemma 7.2. *The functions M, U fulfill the following relations.*

$$\begin{aligned} &\text{(i)} && M_r(a, b, r) = \frac{a}{b} M(a+1, b+1, r) \\ &\text{(ii)} && M(a, b, r) \to 1 \qquad \textit{for } r \to 0, \quad b \notin \mathbb{N} \\ &\text{(iii)} && U\left(a, \frac{1}{2}, r\right) = \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + a\right)} + O(r^{\frac{1}{2}}) \qquad \textit{for } r \to 0 \\ &\text{(iv)} && M(a, b, r) = \frac{\Gamma(b)}{\Gamma(a)}\, \mathrm{e}^r\, r^{a-b}(1 + O(r^{-1})) \qquad \textit{for } r \to \infty \end{aligned} \tag{7.2}$$

Proof. The relations (i), (iii), and (iv) can be found in [AS 13.4.8, 13.5.10, 13.1.4] respectively, (ii) can be obtained directly from the definition of M. $\square$

Remark 7.3. We will need also the properties of U given in (3.3) from Lemma 3.1.

We look for the solution of (7.1) which fulfills the simplified initial conditions with two positive parameters F and G

$$\begin{aligned} f(0) &= F, & f'(0) &= 0, \\ g(0) &= G, & g'(0) &= -F^q. \end{aligned}$$

The solution of the ordinary initial value problem

$$g''(y) + \frac{y}{2} g(y) - \beta g(y) = 0 \text{ for } y > 0, \qquad g(0) = G, \qquad g'(0) = -F^q$$

is obviously given by

$$g(y) = F^q \frac{\Gamma(\frac{1}{2}+\beta)}{\sqrt{\pi}} g_1(y) + \left(G - F^q \frac{\Gamma(\frac{1}{2}+\beta)}{\Gamma(1+\beta)} \right) g_2(y).$$

Now we need to solve the problem

$$f''(y) + \frac{y}{2} g(y) - \alpha f(y) = -\, g^p(y) \text{ for } y > 0, \qquad f(0) = F, \qquad f'(0) = 0.$$

We look for the solution of the equation in the form

$$f(y) = d(y) f_1(y),$$

which transforms it into

$$f_1(y) d''(y) + \left(2f'(y) + \frac{y}{2} f_1(y) \right) d'(y) = -\, g^p(y), \qquad y > 0$$

or

$$\varphi'(y) + \left(2\frac{f_1'(y)}{f_1(y)} + \frac{y}{2} \right) \varphi(y) = -\, \frac{g^p(y)}{f_1(y)}, \qquad y > 0$$

for $\varphi(y) = d'(y)$. This equation can be solved explicitly. The solution is given by

$$\varphi(y) = \frac{C_1 - \int_0^y \mathrm{e}^{\frac{s^2}{4}} f_1(s) g^p(s) ds}{\mathrm{e}^{\frac{y^2}{4}} f_1^2(y)}$$

and we have

$$f(y) = \left(\int_0^y \varphi(s) ds + C_2 \right) f_1(y).$$

Since $f(0) = C_2 f_1(0) = C_2 \pi^{\frac{1}{2}} \Gamma^{-1}(1+\alpha)$, we have to set

$$C_2 = F\Gamma(1+\alpha)\pi^{-\frac{1}{2}}$$

in order to satisfy $f(0) = F$. Similarly

$$\begin{aligned} f'(0) = {} & \varphi(y) f_1(y) + \left(\int_0^y \varphi(s) ds + C_2 \right) f_1(y) f_1'(y) \bigg|_{y \to 0} \\ & = \frac{C_1 \Gamma(1+\alpha)}{\pi^{\frac{1}{2}}} - \frac{F\Gamma(1+\alpha)}{\Gamma(\frac{1}{2}+\alpha)}, \end{aligned}$$

so we set

$$C_1 = \frac{F\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+\alpha)}.$$

Thus we found the following solution of (7.1)

$$\begin{aligned} f(y) &= \left(\int_0^y \varphi(s)ds + C_2\right) f_1(y), \\ g(y) &= F^q \frac{\Gamma(\frac{1}{2}+\beta)}{\sqrt{\pi}} g_1(y) + \left(G - F^q \frac{\Gamma(\frac{1}{2}+\beta)}{\Gamma(1+\beta)}\right) g_2(y), \end{aligned} \tag{7.3}$$

where

$$\begin{aligned} \varphi(y) &= \frac{C_1 - \int_0^y \mathrm{e}^{\frac{s^2}{4}} f_1(s) g^p(s) ds}{\mathrm{e}^{\frac{y^2}{4}} f_1^2(y)}, \\ C_1 &= \frac{F\sqrt{\pi}}{\Gamma(\frac{1}{2}+\alpha)} = -Ff_1'(0), \\ C_2 &= \frac{F\Gamma(1+\alpha)}{\sqrt{\pi}} = Ff_1^{-1}(0). \end{aligned}$$

Now we need to choose F and G so that the solution from (7.3) is positive and has the required growth for $y \to \infty$.

Lemma 7.2 (iv) implies that $g_2(y)$ grows as $y^{2\beta}$, i.e., $t^\beta g_2(y)$ does not converge to 0 for $t = x^2 y^{-2} \to 0^+$. We have to set

$$G = F^q \frac{\Gamma(\frac{1}{2}+\beta)}{\Gamma(1+\beta)}$$

in order to cancel the g_2 component of the solution g. For convenience we denote $\psi(s) = \mathrm{e}^{\frac{s^2}{4}} f_1(s) g^p(s)$. Obviously, $\left(C_1 - \int_0^y \psi(s)ds\right)$ must stay positive so that f is positive as well. Furthermore, it has to converge to 0 for $y \to \infty$ so that f has the desired growth. The integral

$$\begin{aligned} &\int_o^\infty \psi(s)ds \\ &\quad = F^{pq} \frac{\Gamma^p(\frac{1}{2}+\beta)}{\sqrt{\pi^p}} \int_o^\infty \mathrm{e}^{-p\frac{s^2}{4}} U\left(\frac{1}{2}+\alpha, \frac{1}{2}, \frac{s^2}{4}\right) U^p\left(\frac{1}{2}+\beta, \frac{1}{2}, \frac{s^2}{4}\right) ds \end{aligned}$$

is convergent and we can choose F so that $C_1 = \int_0^\infty \psi(s)ds$. Now since

$$C_1 - \int_0^y \psi(s)ds = \varphi(y)\,\mathrm{e}^{\frac{y^2}{4}} f_1^2(y) = \frac{\varphi(y)}{\mathrm{e}^{\frac{y^2}{4}}} U^2\left(\frac{1}{2}+\alpha, \frac{1}{2}, \frac{y^2}{4}\right) \xrightarrow[y\to\infty]{} 0,$$

we have

$$\frac{\varphi(y)}{\mathrm{e}^{\frac{y^2}{4}}\, y^{2+4\alpha}} \xrightarrow[y\to\infty]{} 0$$

and obviously also

$$\frac{\varphi(y)}{e^{\frac{y^2}{4}}(\frac{1}{2}y^{2+4\alpha}+(1+4\alpha)y^{4\alpha})}\xrightarrow[y\to\infty]{}0.$$

We apply L'Hospital's rule to get

$$\frac{\int_0^y\varphi(s)ds}{e^{\frac{y^2}{4}}\,y^{1+4\alpha}}\xrightarrow[y\to\infty]{}0,$$

which finally implies

$$\frac{f(y)}{y^{2\alpha}}\xrightarrow[y\to\infty]{}0.$$

The functions f and g defined in (7.3) with chosen F and G solve (7.1) and $(t^\alpha f(y), t^\beta g(y))$ converges to $(0,0)$ as $t\to 0^+$. Finally, the functions $u(x,t)=t^\alpha f(y)$, $v(x,t)=t^\beta g(y)$ solve the problem (FL1) with $pq<1$ and zero initial condition.

Proof of Theorem 1.6. The generalization of the one-space dimensional solution for $N>1$ is simple. The nontrivial solution of the problem (FL) with $pq<1$ and trivial initial data is, as in the Theorem 1.2, spatially homogenous except in the x_1 direction:

$$u(x,t)=t^\alpha f\left(\frac{x_1}{\sqrt{t}}\right),\quad v(x,t)=t^\beta g\left(\frac{x_1}{\sqrt{t}}\right),\qquad x=(x_1,x')\in\mathbb{R}^N_+,\quad t>0,$$

where f and g are given in (7.3). $\square$

8 Uniqueness for (DFL) with Nonzero Initial Data

The uniqueness result is complete for the system (DFL), i.e., (P) with two equations. In addition to the u, v, p, q-notation, we also set $C = C_1$, $D = C_2$, $\alpha = \alpha_1$, $\beta = \alpha_2$ in this chapter.

Proof of Theorem 1.7. We start with the case $pq \geqslant 1$, in which the assumption on initial condition is not necessary ([L, Theorem 4.1], [WXW, Theorem 3] deal with zero initial data only). Theorem 1.1 proves uniqueness when both $p,\ q \geqslant 1$. Since the system (DFL) is symmetric in the sense of interchanging p and q, it is sufficient to prove the result for $p < 1$, $q > 1$.

We obtain (6.1,2) as in the proof of Theorem 1.5. The solution formulae (1.1) and (1.2) differ in the u component, thus we derive

$$
\begin{aligned}
|u-\bar{u}|(x,\eta) &\leqslant \int_0^\eta \mathcal{R}(\eta-s)|V(x,s)|ds \\
&\leqslant pq2^p\pi^{-\frac{p}{2}}U^{pq-1}F(\eta)\int_0^\eta \mathcal{R}(\eta-s)s^{\frac{p}{2}}ds \\
&\leqslant K\eta^{\frac{1+p}{2}}F(\eta) \leqslant Kt^{\frac{1+p}{2}}F(t)
\end{aligned}
\tag{8.1}
$$

for all $x \in \mathbb{R}^N_+$ and $0 \leqslant \eta \leqslant t \leqslant \tau$ instead of (6.3). However, the same final argument as in the proof of Theorem 1.5 is applicable. The constant K depends only on p, q, and the bounds of u and $\bar{u}$ in $\mathbb{R}^N_+ \times [0,\tau]$ again. We have $F(t) \leqslant Kt^{\frac{1+p}{2}}F(t)$ on $[0,\tau]$, i.e., $F(t) = 0$ for $t \in (0, K^{-\frac{2}{1+p}})$. Finite iterating of the argument yields $u = \bar{u}$ in $\mathbb{R}^N_+ \times [0,\tau]$, and from the solution formulae (1.1) we get $v = \bar{v}$ as well.

Now we discuss the case $pq < 1$. The symmetric case $p, q < 1$ is shown in Theorem 1.4. In the nonsymmetric case we can assume $p < 1$, $q \geqslant 1$ without loss of generality. We introduce following notations $f_+ = \max\{f, 0\}$ and $\|f(t)\| = \sup\{|f_+(0,x';t)| : x' \in \mathbb{R}^{N-1}\}$, and we use the contradiction argument from the proof of Theorem 1.4 again. We assume $\|(u-\bar{u})(t)\| > 0$ and $\|(u-\bar{u})(\eta)\| \leqslant \|(u-\bar{u})(t)\|$ for $0 \leqslant \eta \leqslant t$.

We apply the ideas from the proof of [EH1, Lemma 3] to get the estimate (5.4) in this case as well. For arbitrary $\theta \in (0,1)$, using the inequalities $u \leqslant \bar{u} + (u-\bar{u})_+$ and $u^\theta \leqslant \bar{u}^\theta + (u-\bar{u})_+^\theta$, we obtain

$$
\begin{aligned}
v(x,t) = &\mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x)\\
&+\int_0^t\int_{\mathbb{R}_{N-1}}\left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{q-\theta}{q}}u^{q-\theta}(x_1,y';\eta)\\
&\qquad\left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{\theta}{q}}u^{\theta}(x_1,y';\eta)dy'd\eta\\
\leqslant&\mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x)\\
&+\int_0^t\int_{\mathbb{R}_{N-1}}\left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{q-\theta}{q}}u^{q-\theta}(x_1,y';\eta)\\
&\qquad\left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{\theta}{q}}\bar{u}^{\theta}(x_1,y';\eta)dy'd\eta\\
&+\int_0^t\int_{\mathbb{R}_{N-1}}\left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{q-\theta}{q}}u^{q-\theta}(x_1,y';\eta)\\
&\qquad\left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{\theta}{q}}(u-\bar{u})_+^{\theta}(x_1,y';\eta)dy'd\eta.
\end{aligned}
$$

We apply Hölder's inequality twice to get

$$
\begin{aligned}
v(x,t)\leqslant&\mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x)\\
&+\int_0^t\left(\mathcal{R}(t-\eta)u^q(x,\eta)\right)^{\frac{q-\theta}{q}}\left(\mathcal{R}(t-\eta)\bar{u}^q(x,\eta)\right)^{\frac{\theta}{q}}d\eta\\
&+\int_0^t\left(\mathcal{R}(t-\eta)u^q(x,\eta)\right)^{\frac{q-\theta}{q}}\left(\mathcal{R}(t-\eta)(u-\bar{u})_+^q(x,\eta)\right)^{\frac{\theta}{q}}d\eta\\
\leqslant&\mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x)\\
&+\left(\int_0^t\mathcal{R}(t-\eta)u^q(x,\eta)d\eta\right)^{\frac{q-\theta}{q}}\left(\int_0^t\mathcal{R}(t-\eta)\bar{u}^q(x,\eta)d\eta\right)^{\frac{\theta}{q}}\\
&+\left(\int_0^t\mathcal{R}(t-\eta)u^q(x,\eta)d\eta\right)^{\frac{q-\theta}{q}}\left(\int_0^t\mathcal{R}(t-\eta)(u-\bar{u})_+^q(x,\eta)d\eta\right)^{\frac{\theta}{q}}.
\end{aligned}
$$

The last inequality yields

$$
v(x,t)\leqslant v^{\frac{q-\theta}{q}}(x,t)\bar{v}^{\frac{\theta}{q}}+v^{\frac{q-\theta}{q}}(x,t)\left(\int_0^t\mathcal{R}(t-\eta)(u-\bar{u})_+^q(x,\eta)d\eta\right)^{\frac{\theta}{q}}
$$

using $a+b^{1-\gamma}c^{\gamma}\leqslant(a+b)^{1-\gamma}(a+c)^{\gamma}$ for any nonnegative a,b,c, and any $\gamma\in(0,1)$. Setting $\theta=pq$ we obtain

$$
(v^p-\bar{v}^p)(x,t)\leqslant\left(\int_0^t\mathcal{R}(t-\eta)(u-\bar{u})_+^q(x,\eta)d\eta\right)^{p}, \tag{8.2}
$$

that we use to get (5.2) in the same way as in the symmetric case. Using the solution formulae (1.1) and recalling the assumption $\|(u-\bar{u})(\eta)\|\leqslant\|(u-\bar{u})(t)\|$ for $0\leqslant\eta\leqslant t$, we write

$$
\begin{aligned}
(u-\bar u)_+(0,x';t) &\leqslant \int_0^t \mathcal{R}(t-\eta)(v^p-\bar v^p)_+(0,x';\eta)d\eta \\
&\leqslant \int_0^t \mathcal{R}(t-\eta)\left(\mathcal{R}(\eta-s)\|(u-\bar u)(s)\|^q ds\right)^p d\eta \\
&\leqslant 2^p\pi^{\frac{1+p}{2}}B\left(\frac{1+p}{2}\right)t^{\frac{1+p}{2}}\|(u-\bar u)(t)\|^{pq},
\end{aligned}
$$

i.e.,

$$
\|(u-\bar u)(t)\|\leqslant Pt^\alpha, \qquad P=2^{\frac{p}{1-pq}}\pi^{-\alpha}B^{\frac{1}{1-pq}}\left(\frac{1+p}{2}\right),
$$

which is exactly (5.2).

Now we need an inequality like (5.3), such that its combining with (5.2) implies (5.4). We set $g(w)(x,t)=\mathcal{T}(t)\mathcal{S}_{N-1}v_0(x)+\int_0^t\mathcal{R}(t-\eta)w^q(x,\eta)d\eta$, $f(\xi)=\xi^p$, and by the mean value theorem for $f\circ g$ we write (using the assumption $p<1$ as well)

$$
\begin{aligned}
(u-\bar u)(x,t) \leqslant pq\int_0^t \mathcal{R}(t-\eta)&\left(\int_0^\eta \mathcal{R}(\eta-s)w^q(x,s)ds\right)^{p-1} \\
&\times\left(\int_0^\eta \mathcal{R}(\eta-s)(w^{q-1}(u-\bar u))(x,s)ds\right)d\eta
\end{aligned}
\tag{8.3}
$$

for some $w(x,t)=(1-\rho(t))u(x,t)+\rho(t)\bar u(x,t)$, $0<\rho(t)<1$. We also have by Hölder's inequality

$$
\begin{aligned}
&\int_0^\eta \mathcal{R}(\eta-s)(w^{q-1}(u-\bar u))(x,s)ds \\
&\leqslant \left(\int_0^\eta \mathcal{R}(\eta-s)w^q(x,s)ds\right)^{\frac{q-1}{q}}\left(\int_0^\eta \mathcal{R}(\eta-s)|u-\bar u|^q(x,s)ds\right)^{\frac{1}{q}},
\end{aligned}
\tag{8.4}
$$

and since $w^q(0,x';s)\geqslant C^q s^{\alpha q}$, $pq-1<0$, we derive from inequalities (8.3,4) that

$$
\begin{aligned}
\|(u-\bar u)(t)\| &\leqslant pq\int_0^t(\pi(t-\eta))^{-\frac12}\left(\int_0^\eta \mathcal{R}(\eta-s)C^q s^{\alpha q}ds\right)^{\frac{pq-1}{q}} \\
&\qquad\times\left(\int_0^\eta(\pi(\eta-s))^{-\frac12}\|(u-\bar u)(s)\|^q ds\right)^{\frac1q}d\eta \\
&= pqB^{-1}(\alpha)B^{-\frac1q}(\beta)\int_0^t(t-\eta)^{-\frac12}\eta^{-\frac{1+q}{2q}} \\
&\qquad\times\left(\int_0^\eta(\eta-s)^{-\frac12}\|(u-\bar u)(s)\|^q ds\right)^{\frac1q}d\eta.
\end{aligned}
\tag{8.5}
$$

It takes the role of (5.3) in the iterating, because combining (5.2,8.5) yields

$$\begin{aligned}\|(u-\bar{u})(t)\| &\leqslant pqB^{-1}(\alpha)\,B^{-\frac{1}{q}}(\beta)\,PB^{\frac{1}{q}}(\beta)\int_0^t (t-\eta)^{-\frac{1}{2}}\eta^{-\frac{1+q}{2q}}\eta^{\frac{1+q}{2(1-pq)q}}\,d\eta\\ &= pqPt^{\alpha},\end{aligned}$$

hence (5.4) does hold in this case as well.

The final steps are the same as in the proof of Theorem 1.4. Letting $k\to\infty$ in (5.4) we have $u(\cdot,t)=\bar{u}(\cdot,t)$ on the boundary $x_1=0$ which contradicts the assumption $\|(u-\bar{u})(t)\|>0$. $\square$

9 References

[AS] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, DC, 1964.

[ChF] M. Chlebík, M. Fila, *From critical exponents to blow-up rates for parabolic systems*, Rend. Mat. Appl. (7) **19** (1999), 449–470.

[CER] C. Cortazar, M. Elgueta, J. D. Rossi, *Uniqueness and non-uniqueness for a system of heat equations with non-linear coupling at the boundary*, Nonlinear Anal. **37** (1999), 257–267.

[DFL] K. Deng, M. Fila, H. A. Levine, *On critical exponents for a system of heat equations coupled in the boundary conditions*, Acta Math. Univ. Comenian. (N.S.) **63** (1994), 169–192.

[EH1] M. Escobedo, M. A. Herrero, *A uniqueness result for a semilinear reaction diffusion system*, Proc. Amer. Math. Soc. **112** (1991), 175–186.

[EH2] ———, *Boundedness and blow up for a semilinear reaction diffusion system*, J. Diff. Eq. **89** (1991), 176–202.

[EH3] ———, *A semilinear parabolic system in a bounded domain*, Ann. Mat. Pura Appl. (4) **165** (1993), 317–336.

[FL] M. Fila, H. A. Levine, *On critical exponents for a semilinear parabolic system coupled in an equation and in a boundary condition*, J. Math. Anal. Appl. **204** (1996), 494–521.

[FQ] M. Fila, P. Quittner, *The blow-up rate for the heat equation with a non-linear boundary condition*, Math. Methods Appl. Sci. **14** (1991), 197–205.

[F] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t - \triangle u = u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math **13** (1966), 109–124.

[FW] H. Fujita, S. Watanabe, *On the uniqueness and non-uniqueness of solutions of initial value problems for some quasi-linear parabolic equations*, Comm. Pure Appl. Math. **21** (1968), 631–652.

[GL] V. Galaktionov, H. A. Levine, *On critical Fujita exponents for heat equations with a nonlinear flux condition on the boundary*, Israel J. Math. **94** (1996), 125–146.

[K1] M. Kordoš, *On uniqueness for a system of heat equations coupled in the boundary conditions*, Acta Math. Univ. Comenian. (N.S.), to appear.

[K2] ———, *On uniqueness for a semilinear parabolic system coupled in an equation and a boundary condition*, preprint.

[L] Z. Lin, *Blowup behaviors for diffusion system coupled through nonlinear boundary conditions in a half space*, Sci. China Ser. A **47** (2004), 72–82.

[R] J. Rencławowicz, *Global existence and blow up of solutions for a completely coupled Fujita type system of reaction-diffusion equations*, Appl. Math. (Warsaw) **25** (1998), 313–326.

[WXW] S. Wang, C. Xie, M. Wang, *Note on critical exponents for a system of heat equations coupled in the boundary conditions*, J. Math. Anal. Appl. **218** (1998), 313–324.

[W] F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981), 29–40.