Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Properties of Partial-Sum Discrete Probability Distributions

Dissertation thesis

2020 Mgr. Michaela Koščová

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics
Department of Applied Mathematics and Statistics

Properties of Partial-Sum Discrete Probability Distributions

Dissertation thesis

Study programme:	Applied Mathematics
Field of Study:	1114 Applied Mathematics
Department:	Department of Applied Mathematics and Statistics
Supervisor:	doc. Mgr. Ján Mačutek, PhD.

Bratislava 2020

Mgr. Michaela Koščová

51798459

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname:	Mgr. Michaela Koščová
Study programme:	Applied Mathematics (Single degree study, Ph.D. III. deg., full time form)
Field of Study:	Applied Mathematics
Type of Thesis:	Dissertation thesis
Language of Thesis:	English
Secondary language:	Slovak

Title: Properties of partial-sum discrete probability distributions

Annotation: The dissertation thesis will focus mainly on two topics:
1) Partial summations can be applied iteratively. The question then is whether there exists a limit distribution for the iterated summations. So far, the only case where the limit distribution (geometric) is known is the most simple partial summation in which the summed probabilities are multiplied by a constant. This result will be generalized for other types of partial summations.
2) For each discrete probability distribution there exists one and only one summation under which the distribution is invariant (the parent distribution is the same as its descendant). If the summations are parametrized, the parent distribution remains invariant in some cases, but sometimes we obtain another distribution as the descendant. The two families of partial summations will be characterized in the thesis.

Tutor:	doc. Mgr. Ján Mačutek, PhD.
Department:	FMFI.KAMŠ - Department of Applied Mathematics and Statistics
Head of department:	prof. RNDr. Marek Fila, DrSc.
Assigned:	21.02.2014
Approved:	21.02.2014

prof. RNDr. Marek Fila, DrSc.
Guarantor of Study Programme

....

Student Tutor

51798459

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Mgr. Michaela Koščová
Študijný program: aplikovaná matematika (Jednoodborové štúdium, doktorandské III. st., denná forma)
Študijný odbor: aplikovaná matematika
Typ záverečnej práce: dizertačná
Jazyk záverečnej práce: anglický
Sekundárny jazyk: slovenský

Názov: Properties of partial-sum discrete probability distributions
Vlastnosti diskrétnych rozdelení pravdepodobnosti vyjadrených parciálnymi sumáciami

Anotácia: Dizertačná práca sa bude venovať primárne dvom témam:
1) Parciálne sumácie je možné opakovať. Otázna je existencia limitného rozdelenia pre takéto opakované sumácie. Zatiaľ je limitné rozdelenie (geometrické) známe len pre najjednoduchší prípad, keď sú sumované pravdepodobnosti prenásobené konštantou. V práci bude tento výsledok rozšírený aj pre ďalšie typy sumácií.
2) Pre každé diskrétne rozdelenie existuje práve jedna sumácia, vzhľadom na ktorú je toto rozdelenie invariantné (rodič sa zhoduje s potomkom). Ak ale sumácie parametrizujeme, v niektorých prípadoch zostáva rozdelenie invariatné, ale niekedy dostávame ako potomka iné rozdelenie pravdepodobnosti. V práci budú odvodené charakterizácie týchto dvoch tried sumácií.

Školiteľ: doc. Mgr. Ján Mačutek, PhD.
Katedra: FMFI.KAMŠ - Katedra aplikovanej matematiky a štatistiky
Vedúci katedry: prof. RNDr. Marek Fila, DrSc.

Spôsob sprístupnenia elektronickej verzie práce:
bez obmedzenia

Dátum zadania: 21.02.2014

Dátum schválenia: 21.02.2014

prof. RNDr. Marek Fila, DrSc.
garant študijného programu

....

študent školiteľ

Acknowledgement

I would like to express my deep gratitude to my supervisor doc. Mgr. Ján Mačutek, PhD. for the opportunity to participate on this research, for his very useful guidance, insightful comments and considerable encouragements to complete this thesis. I also send my appreciation to all the members of the department for their unceasing helpfulness and willingness shown from the very beginning of my PhD. studies. Last but not least, I would like to thank my closest family who have given me constant support and love during the completion of the thesis.

Abstract

The dissertation thesis focuses on two main topics, namely invariance of partial summation and iterated partial summation. 1) For each discrete probability distribution there exists one and only one summation under which the distribution is invariant. If the summations are parametrized, the parent distribution remains invariant in some cases, but sometimes we obtain another distribution as the descendant. We suggest a division of discrete distributions into two families which reflect the behaviour with respect to the parametrization. Some examples of distributions from both families are shown, and the necessary and sufficient condition for a distribution to belong to one of the two families is proven. 2) Partial summations can be applied iteratively. It is shown that the limit distribution exists for a wide spectrum of partial summations when applied to a parent with a finite support. The existence of such limit is proved by the power method which was originally developed in matrix theory to find eigenvalues and eigenvectors.

Keywords: discrete probability distributions, partial summation, partial summation invariance, iterated partial summation.

Abstrakt

Dizertačná práca sa venuje primárne dvom témam, konkrétne invariancii vzhľadom na parciálnu sumáciu a iterovanej parciálnej sumácii. 1) Pre každé diskrétne rozdelenie existuje práve jedna sumácia, vzhľadom na ktorú je toto rozdelenie invariantné. Ak ale sumácie parametrizujeme, v niektorých prípadoch zostáva rozdelenie invariantné, no niekedy dostávame ako potomka iné rozdelenie pravdepodobnosti. Táto vlastnosť vytvára dve triedy diskrétnych rozdelení, ktoré reflektujú správanie parametra parciálnej sumácie. V práci sú uvedené príklady z obidvoch tried a tiež je odvodená nutná a postačujúca podmienka na to, aby rozdelenie patrilo do jednej z týchto dvoch tried. 2) Parciálne sumácie je možné opakovať. V práci je dokázané, že limitné rozdelenie existuje pre široké spektrum parciálnych sumácií aplikovaných na rodiča s konečným nosičom. Existencia tohoto limitného rozdelenia vyplýva z tzv. mocninovej metódy, ktorá bola pôvodne navrhnutá pre potreby teórie matíc.

Kľúčové slová: diskrétne rozdelenia pravdepodobnosti, parciálna sumácia, invariancia vzhľadom na parciálnu sumáciu, iterovaná parciálna sumácia.

Contents

List of Figures

List of Tables

Notation

$x_{(r)}$	r-th descending factorial of a real number x
$x^{(r)}$	r-th ascending factorial of a real number x
$\Gamma(a)$	gamma function
${}_2F_1(a,b;c;t)$	hypergeometric function
$\Vert U\Vert_1$	L1-norm of a vector U
$\Vert U\Vert_2$	L2-norm of a vector U
$\{P_j^*\}_{j=0}^{\infty}$	parent (discrete probability distribution)
$\{P_j\}_{j=0}^{\infty}$	descendant (discrete probability distribution)
$\{P_j^{(k)}\}_{j=0}^{\infty}$	k-th descendant (discrete probability distribution)
μ^*	mean of the parent
μ	mean of the descendant
$\mu^*_{(r)}$	r-th descending factorial moment of the parent
$\mu_{(r)}$	r-th descending factorial moment of the descendant
$G^*(t)$	probability generating function of the parent
$G(t)$	probability generating function of the descendant
$G_k(t)$	probability generating function of the k-th descendant

Introduction

The dissertation thesis focuses on partial-sum discrete probability distributions. In this thesis we recapitulate results achieved in the field of partial-sum distributions, we summarize our own results and we present several possibilities for further research in this area.

Chapter 1 contains a detailed overview of the relevant literature. An insight into two basic types of partial summation is provided, together with relations between some properties of parent and descendant distributions (probability generating functions, moments). The concept of invariance under partial summation is defined and the conditions of invariance are listed. Iterated partial summations are introduced, with examples of their limit behaviour. Some of the applications of partial summations in reliability analysis or economical modelling, among others, are mentioned.

Chapters 2 and 3 bring new results regarding partial summations that arose during our research. The topics of these two chapters both concern partial summations, but they are independent of each other.

In Chapter 2 we classify discrete probability distributions into two families. The two families reflect the behaviour of a parameter of partial summation. Some examples of distributions from both families are shown, and the necessary and sufficient condition for a distribution to belong to one of the two families is proven.

Chapter 3 focuses on the limit distributions obtained by iterated applications of partial summations. In order to gain a preliminary insight in this field, a computational study using the R software was performed. We studied the limit distributions with the use of three different partial summations. Our findings were substantiated by further research, where we found the connection between partial summations and the power method, usually used as a computational method for finding eigenvalues and eigenvectors, from the matrix theory. We managed to reformulate the iterated partial summation as repeated matrix multiplication, which provides an effective tool to prove the existence of the limit distribution, which is then expressed as a solution of a system of linear equations. We present the result of repeated application of partial summation to a wide class of discrete distributions with a finite support and we show some specific examples.

The probability mass functions, parametric spaces and other properties of the discussed discrete distributions can be found in Wimmer and Altmann (1999). All computations were run in statistical software R[1].

[1] www.r-project.org

1 Partial summations

Let $\{P_j^*\}_{j=0}^{\infty}$ and $\{P_j\}_{j=0}^{\infty}$ be discrete probability distributions defined on the set of non-negative integers. The distribution $\{P_j\}_{j=0}^{\infty}$ is the result of a partial summation applied to $\{P_j^*\}_{j=0}^{\infty}$ if

$$P_x = \sum_{j=x}^{\infty} u(x,j) P_j^*, \qquad x = 0, 1, 2, \dots\ , \tag{1.1}$$

where $u(\cdot,\cdot)$ is a real function. The distributions $\{P_j^*\}_{j=0}^{\infty}$ and $\{P_j\}_{j=0}^{\infty}$ are called the parent and the descendant, respectively. Different types of partial summation can be obtained by different choices of $u(\cdot,\cdot)$[1] in (1.1). A brief overview of partial-sum distributions can be found in Johnson et al. (2005, pp. 508-512). Summation (1.1) was previously used mainly as a tool for creating new discrete distributions, see, e.g., Wimmer and Altmann (2001a), where four particular types of partial summations are proposed:

$$P_x = \frac{1}{\mu} \sum_{j=x}^{\infty} P_j^*, \qquad x = 1, 2, \dots,$$

$$P_x = \sum_{j=x}^{\infty} \frac{P_j^*}{j}, \qquad x = 1, 2, \dots,$$

$$P_x = \frac{1}{\mu - 1} \sum_{j=x+1}^{\infty} P_j^*, \qquad x = 1, 2, \dots,$$

$$P_x = \frac{1}{1 - P_1^*} \sum_{j=x+1}^{\infty} \frac{P_j^*}{j-1}, \qquad x = 1, 2, \dots,$$

μ being the mean of the parent. Naranan and Balasubrahmanyan (2005) suggest a new discrete distribution created by partial summation to model phoneme frequencies.

[1] We remind that the function $u(\cdot,\cdot)$ does not necessarily have to depend on both of its arguments.

In the field of economic modelling, Li and Garrido (2002) focus on the discrete time risk model. They set $u(\cdot,\cdot)$ from 1.1 to the non-negative values of a penalty function. Such partial summation is used to determine a recursive expression for the expected discounted penalty due at ruin. Also in the recent research by Baker (2019), the partial summations are studied as a method for obtaining new discrete distributions. Baker (2019) also proposes integrating probability density functions by parts for providing new continuous probability distributions. However, other applications of partial summations are available in literature, as we describe in this chapter.

Mačutek (2003) proposed two types of partial summation that are special cases of (1.1), namely

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0, 1, 2, \ldots \tag{1.2}$$

and

$$P_x = h(x) \sum_{j=x}^{\infty} P_j^*, \qquad x = 0, 1, 2, \ldots\ , \tag{1.3}$$

where $g(\cdot)$ and $h(\cdot)$ are real functions, $g(\cdot)$ depends solely on j, and $h(\cdot)$ is a constant with respect to the index of summation j. There is a one-to-one correspondence between discrete distributions and partial summations (1.2). For every distribution, there exists one and only one partial summation defined by (1.2), which results in the descendant identical with its parent, i.e.,

$$P_x = P_x^*, \qquad x = 0, 1, 2, \ldots\ .$$

The same is true for summation (1.3). Then we say that the distribution $\{P_j^*\}_{j=0}^{\infty}$ is invariant with respect to summation (1.2), or (1.3). Hence, the functions $g(\cdot)$ and $h(\cdot)$ can be considered new characteristics of discrete distributions. Mačutek (2003) derives these characteristics for a wide class of discrete distributions. Let the parent distribution satisfy the recurrence relation

$$P_{x+1}^* = f(x+1) P_x^* \qquad x = 0, 1, 2, \ldots\ .$$

The distribution is invariant with respect to the summation (1.2) if and only if

$$g(x) = 1 - f(x+1), \qquad x = 0, 1, 2, \ldots\ . \tag{1.4}$$

On the other hand, the distribution is invariant with respect to the summation (1.3) if and only if

$$h(x) = \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} f(x+k)\right)^{-1}, \qquad x = 0, 1, 2, \ldots\ .$$

Table 1.1: *Characteristics $g(\cdot)$ and $h(\cdot)$ for some distributions*

distribution (*parameters*)	$g(x)$	$h(x)$
geometric (p)	p	p
Poisson (λ)	$\frac{x-\lambda+1}{x+1}$	$\frac{1}{{}_1F_1(1;x+1;\lambda)}$
negative binomial (k, p)	$\frac{1-k+p(k+x)}{x+1}$	$\frac{1}{{}_2F_1(x+k,1;x+1;1-p)}$
1-shifted logarithmic (θ)	$\frac{x(1-\theta)-\theta+2}{j+2}$	$\frac{1}{{}_2F_1(x+1,1;x+2;\theta)}$
2-shifted Flory	$\frac{x}{2(x+1)}$	$\frac{1}{{}_2F_1(x+2,1;x+1;\frac{1}{2})}$
Salvia-Bolinger (α)	$\frac{\alpha+1}{x+2}$	$\frac{\alpha}{x+1}$
Yule (b)	$\frac{b+1}{x+b+2}$	$\frac{b}{x+b+1}$
family (*parameters*)	$g(x)$	$h(x)$
power series (a_x, θ, $f(\cdot)$)	$\frac{a_x-\theta a_{x+1}}{a_x}$	$\frac{a_x}{\sum_{i=0}^{\infty} a_{x+i}\theta^i}$
Katz (α, β), $\beta \neq 0$	$\frac{1-\alpha+(1-\beta)x}{x+1}$	$\frac{1}{{}_2F_1(x+\frac{\alpha}{\beta},1;x+1;\beta)}$
Irwin (a_x, θ, $\psi(\cdot)$)	$-\frac{(x+\theta+1)a_x-a_{x+1}}{(x+\theta+1)a_x}$	$\frac{a_x}{\sum_{i=0}^{\infty} \frac{a_{x+i}}{(x+\theta+1)^{(j)}}}$

We provide several examples taken from Mačutek (2003), see Table 1.1. The mentioned distributions and distribution families can be found in Wimmer and Altmann (1999). Also the relations between probability generating functions $G^*(\cdot)$ and $G(\cdot)$ of the parent and descendant distributions, respectively, for arbitrary meaningful function $g(\cdot)$ in the summation 1.2 are presented by Mačutek (2003), with

$$G(t) = \frac{1 - tG^*(t) - \sum_{x=0}^{\infty} P_x^* f(x+1)(1-t^{x+1})}{(1-t)\left(1 + \mu^* - \sum_{x=0}^{\infty}(x+1)f(x+1)P_x^*\right)},$$

μ^* being finite mean of the parent. Special cases of this relation between probability generating functions can be found in Mačutek (2001), Mačutek (2002) and Wimmer and Altmann (2000).

Wimmer and Kalas (1999) had derived the characteristics $g(\cdot)$ and $h(\cdot)$ for the geometric distribution before Mačutek (2003) presented the general formula. The geometric distribution is invariant with respect to both of summations (1.2) and (1.3) if and only if the function $g(\cdot)$ (as well as $h(\cdot)$) is equal to p, which is the parameter of the geometric distribution, i.e., if

$$P_x = p\sum_{j=x}^{\infty} P_j^*, \qquad x = 0, 1, 2, \ldots \quad . \tag{1.5}$$

The same is shown by Li and Garrido (2002) as a by-product of their time-to-ruin analysis. The partial summation (1.5) can be also called the geometric partial summation (similarly, the partial summation, with respect to which the Poisson distribution is invariant, is called the Poisson partial summation, etc.). The geometric partial summation is widely used in reliability analysis, mostly to determine the residual life of a component in a system, see e.g. Unnikrishnan Nair and Hitha (1989) or Unnikrishnan Nair et al. (2012). An application of this type of partial summation in risk models in insurance is presented by Willmot (1986). The geometric distribution is somewhat special among discrete distributions, being invariant to the partial summation defined by $g(j) = h(x) = p$. In general, the functions $g(\cdot)$ and $h(\cdot)$ which determine partial summations under which a distribution is invariant differ from each other.

A special case of (1.2),

$$P_x = c \sum_{j=x+1}^{\infty} \frac{1}{j} P_j^*, \qquad x = 0, 1, 2, \dots \ , \tag{1.6}$$

where c is a proper normalization constant, also known as the STER summation (Sums successively Truncated from the Expectation of the Reciprocal of a random variable having the parent distribution), can be found in Bissinger (1965). According to Xekalaki (1983), the Yule distribution is the only one for which the parent and the descendant are identical under (1.6).

A generalization of (1.6) is proposed by Wimmer and Altmann (2000) and another one by Mačutek (2002). In Mačutek (2002) the generalization is

$$P_x(k, l) = A(k, l) \sum_{j=x}^{\infty} \frac{1}{j+l} P_{j+k}^*, \qquad x = 0, 1, 2, \dots \ , \tag{1.7}$$

$k \geq 0$ and $l \geq 0$ are integers, $A(k, l)$ is a normalization constant. For $k = l = 1$ it is the usual STER summation. The author presents recursive formulas to compute the descendant probabilities,

$$P_{x+1}(k, l) = P_x(k, l) - \frac{A(k, l)}{x+l} P_{x+k}^*,$$

with

$$A(k, l) = \left(1 - \sum_{i=0}^{k-1} P_i - (l-1) \sum_{i=k}^{\infty} \frac{1}{i-k+l} P_i \right)^{-1}$$

and provides relations between the probability generating functions $G^*(\cdot)$ and $G(\cdot)$ of the parent and the descendant under summation (1.7), respectively, which is

$$G(t) = \frac{A(k, l)}{1-t} \int_t^1 \left(G^*(z) - \sum_{i=0}^{k-1} P_i z^i - (l-1) \sum_{i=k}^{\infty} \frac{1}{i-k+l} P_i z^i \right) z^{-k} dz$$

and also the relation between the r-th descending factorial moments. Let $\mu^*_{(r)}$ denote the r-th descending factorial moment of parent and $\mu_{(r)}$ denote the r-th descending factorial moment of descendant under summation (1.7). If $\mu^*_{(r)}$ and $\mu_{(r)}$ exist for all $r \geq 1$ then

$$\mu_{(r)} = \frac{A(k,l)}{r+1} \left(\sum_{i=0}^{r} \binom{r}{i} (-k)_{(i)} \mu^*_{(r-i)} - \sum_{i=0}^{k-1} (i-k)_{(r)} P_i - (l-1) \sum_{i=k}^{\infty} \frac{(i-k)_{(r)}}{i-k+l} P_i \right),$$

where $x_{(r)} = x(x-1)\dots(x-r+1)$ denotes the r-th descending factorial of a real number x, $r \geq 0$ is an integer and $x_{(0)} = 1$. The $(k-1)$-displaced Yule distribution, $k \geq 1$, is invariant with respect to the generalized STER summation (1.7), which is in accordance with the result of Xekalaki (1983) for $k = l = 1$.

The generalization of the STER summation presented by Wimmer and Altmann (2000) is simpler than the previously discussed one,

$$P_x = \frac{1}{\sum\limits_{s=k}^{\infty} P^*_s} \sum_{j=x+k}^{\infty} \frac{1}{j-k+1} P^*_j, \qquad x = 0,1,2,\dots\ ,$$

$k \geq 0$ is integer. It is the same as (1.7) for $l = 1$. Many pairs of the parent and the descendant distributions from the generalized hypergeometric family are presented.

Another special case of (1.2), namely

$$P_x = c \sum_{j=x}^{\infty} r^{j-x} P^*_j, \qquad x = 0,1,2,\dots\ , \tag{1.8}$$

$r \in \mathbb{R}$, c is a proper constant, finds its application in economic modelling, see e.g. Lin and Willmot (1999), or Li (2005) and Li et al. (2009). In the latter two works, the summation (1.8) is used to characterize a class of renewal risk models. The partial summation is understood more generally there, as an operator on any real-valued function.

Mačutek (2001) focuses on the partial summation (1.2) with such function $g(\cdot)$ that leaves the Katz family of discrete distributions unaltered, i.e.,

$$g(j) = 1 - \frac{\alpha + \beta j}{1+j},$$

$\alpha \geq 0$, $\beta > 1$ being the parameters of distributions from the Katz family. The Katz partial summation can be written as

$$P_x = c \sum_{j=x}^{\infty} \left(1 - \frac{\alpha + \beta j}{1+j}\right) P^*_j, \qquad x = 0,1,2,\dots,$$

where c is a proper constant and $\{P_j^*\}_{j=0}^{\infty}$ is a proper probability mass function with finite mean μ^*. The relation between the probability generating functions $G^*(\cdot)$, $G(\cdot)$ of the parent and descendant distributions, respectively, is provided,

$$G(t) = c\frac{(1-\beta)(1-tG^*(t)) - (\alpha-\beta)\int_t^1 G^*(z)dz}{1-t},$$

where

$$c = \frac{1}{1-\alpha+(1-\beta)\mu^*}.$$

The relation between the r-th descending factorial moments, denoted $\mu^*_{(r)}$ for parent and $\mu_{(r)}$ for descendant according to Mačutek (2001) is

$$\mu_{(r)} = \frac{c}{r+1}\left((1-\beta)\mu^*_{(r)}(1-\alpha-\beta r + r\mu^*_{(r)})\right),$$

supposing the generating functions of the descending factorial moments of parent and descendant distributions exist.

Wimmer and Altmann (2001b) inspect the condition of invariance for summation

$$P_x = \frac{k}{x}\sum_{j=x}^{\infty} P_j^* \qquad x = 1, 2, 3, \ldots$$

which is a special case of (1.3) for

$$h(x) = \frac{k}{x},$$

k is a proper normalization constant. This type of partial summation arose from generalizing the theoretical explorations of the Bradford law by Brookes (1977) and from searching for a law-like hypotheses in quantitative linguistics and musicology. It turns out that the 1-displaced Salvia-Bolinger distribution (see e.g. Salvia and Bolinger, 1982) is the invariant distribution for such choice of the function $h(\cdot)$. The relation between probability generating functions $G^*(\cdot)$ and $G(\cdot)$ of parent and descendant, respectively, connected by this partial summation is provided,

$$G(t) = k\int_0^t \frac{1-G^*(z)}{1-z}dz,$$

as well as the relation between the r-th descending factorial moments $\mu^*_{(r)}$ and $\mu_{(r)}$ of the parent and descendant, respectively,

$$\mu_{(r)} = \frac{k}{r}\mu^*_{(r)}.$$

A method for finding continuous analogues of discrete distributions (and vice versa) based on the partial summation (1.2) is provided by Mačutek and Altmann (2007). The authors consider an integral transformation

$$\varphi(x) = \int_x^\infty g(t)\varphi^*(t)dt, \tag{1.9}$$

where $\varphi(\cdot)$ and $\varphi^*(\cdot)$ are probability density functions and the function $g(\cdot)$ satisfies (1.4) for a certain discrete distribution. It is easy to see the analogy between summation (1.2) and integration (1.9). Supposing $\varphi(\cdot)$ is differentiable on the interval $(0;\infty)$, $\varphi(x) = 0$ for $x < 0$ and $\varphi(x) > 0$ for $x > 0$, the function $\varphi(\cdot)$ satisfying

$$\varphi(x) = \int_x^\infty g(t)\varphi(t)dt$$

is probability density function of the continuous analogue for the considered discrete distribution. In the mentioned work there are shown continuous analogues for several discrete distributions, see Table 1.2. The discrete distributions listed in the Table 1.2 can be found in Wimmer and Altmann (1999) while for the continuous distributions see Johnson and Kotz (1970). Mačutek and Altmann (2007) also provide a backward procedure, which finds its use in the field of quantitative linguistics.

It is possible to apply partial summation repeatedly as follows:

$$P_x^{(1)} = c_1 \sum_{j=x}^\infty u(x,j)P_j^*, \qquad x = 0, 1, 2, \dots\ ,$$

$$P_x^{(2)} = c_2 \sum_{j=x}^\infty u(x,j)P_j^{(1)}, \qquad x = 0, 1, 2, \dots\ ,$$

$$\vdots$$

$$P_x^{(n)} = c_n \sum_{j=x}^\infty u(x,j)P_j^{(n-1)}, \qquad x = 0, 1, 2, \dots\ ,$$

$$\vdots$$

where c_i is a normalization constant which ensures that $\{P_x^{(i)}\}_{x=0}^\infty$ is a proper distribution for $i = 1, 2, 3, \dots$.

Mačutek (2006) focuses on partial summation with

$$u(x,j) = p,$$

Table 1.2: Continuous analogues to some discrete distributions

discrete distribution P_x	continuous distribution $\varphi(x)$
geometric $p(1-p)^x$	exponential Ce^{-px}
negative binomial $\binom{k+x-1}{x}p^k(1-p)^x$	Pearson type III $Ce^{-px}(x+1)^{(k-1)(1-p)}$
Poisson $\frac{e^{-a}a^x}{x!}$	Pearson type III $Ce^{-x}(x+1)^a$
hyperpoisson $\frac{a^x}{{}_1F_1(1;b;a)b^{(x)}}$	Pearson type III $Ce^{-px}(x+b)^a$
Yule $\frac{bx!}{(b+1)^{(x+1)}}$	Pearson type VIII/Zipf-Mandelbrot $C(x+b+2)^{-b-1}$
Waring $\frac{bn^{(x)}}{(b+n)(b+n+1)^{(x)}}$	Pearson type VIII/Zipf-Mandelbrot $C(x+b+n+1)^{-b-1}$
Simon $\frac{1}{(x+1)(x+2)}$	Lotka $C(x+3)^{-2}$
Johnson-Kotz $\frac{a}{(a+x)(a+x+1)}$	generalized Lotka $C(x+a+2)^{-2}$
hyperpascal $\frac{\binom{k+x-1}{x}}{\binom{m+x-1}{x}}q^xP_0$	Pearson type III $Ce^{-(1-q)x}(x+m)^{q(k-m)}$

$p \in (0,1)$. We remind that the geometric distribution is invariant with respect to this partial summation, see Wimmer and Kalas (1999). Mačutek (2006) shows that the sequence of probability mass functions is in this case

$$P_x^{(n)} = \prod_{s=1}^{n} c_s \sum_{j=0}^{\infty} \binom{n+j-1}{j} P^*_{x+j}, \qquad k = 0,1,2,\ldots, \qquad n = 1,2,\ldots$$

and proves, using the convergence of the probability generating functions

$$G_n(t) = \sum_{j=0}^{n-1} \frac{(-1)^j t^j \prod_{s=0}^{j} c_{n-s}}{(1-t)^{j+1}} + \frac{(-1)^n t^n \prod_{r=1}^{n} c_r}{(1-t)^n} G^*(t), \qquad n = 1,2,\ldots,$$

that if the parent distribution $\{P^*_j\}_{j=0}^{\infty}$ satisfies

$$p = \lim_{j\to\infty} \frac{P^*_{j+1}}{P^*_j}, \tag{1.10}$$

then the limit distribution is the geometric distribution with the parameter value equal to p in (1.10), i.e.,

$$\lim_{i\to\infty} P_x^{(i)} = p(1-p)^x, \qquad x = 0, 1, 2, \ldots \ .$$

However, if the parent distribution has a finite support, the limit distribution is deterministic.

Generally, in the partial summation (1.1) there are three elements:

(1) the parent distribution $\{P_j^*\}_{j=0}^{\infty}$,

(2) the descendant distribution $\{P_j\}_{j=0}^{\infty}$, and

(3) the function $u(\cdot,\cdot)$ (or in special cases (1.2) and (1.3) the functions $g(\cdot)$ and $h(\cdot)$, respectively) - the link between the parent and the descendant.

Wimmer and Altmann (2000) show that for two different parent distributions (they call them parent and godparent) it is possible to obtain the same descendant if different type of partial summation is applied (they work with the generalized STER summation (1.6) and the geometric summation (1.5)). However, if we fix two of the three elements, the remaining one is almost[2] uniquely given by (1.1). What is more, Wimmer and Mačutek (2012) prove that any two discrete distributions defined on the same support with only non-zero probabilities are connected by a partial summation for some choice of the function $u(\cdot,\cdot)$.

The result given by Wimmer and Mačutek (2012) has its consequences. The so-called Wimmer-Altmann family of distributions was suggested as a general model in quantitative linguistics (see Wimmer and Altmann, 2005). Under this assumption the question arises what to do with models which appear in linguistics but do not belong to the suggested family of distributions. Mačutek (2005) attempts to answer the question by enlarging the above mentioned family, namely by presenting the "problematic" models as partial-sums distributions with a parent from the Wimmer-Altmann family of distributions. However, due to Wimmer and Mačutek (2012) this attempt fails, since all discrete distributions with non-zero probabilities can be presented in this way.

The work of Kotz and Johnson (1991), besides a little criticism towards Unnikrishnan Nair and Hitha (1989) due to some inconsistencies, suggests a bivariate extension of partial sum distributions. The idea of bivariate partial-sums distributions is further discussed by Leššová and Mačutek (2020), who define the bivariate partial summation

[2] The non-uniqueness concerns only parent distributions for some modifications of partial summation (1.3) and is caused by the normalization constant.

as follows. Let $\{P^*_{x,y}\}_{x,y=0}^{\infty}$ and $\{P_{x,y}\}_{x,y=0}^{\infty}$ be bivariate discrete distributions and let $g(x, y)$ be a real function. Then $\{P_{x,y}\}_{x,y=0}^{\infty}$ is the descendant of the parent $\{P^*_{x,y}\}_{x,y=0}^{\infty}$ if

$$P_{x,y} = c \sum_{i=x}^{\infty} \sum_{j=y}^{\infty} g(i, j) P^*_{i,j}, \qquad x = 0, 1, 2, \dots \quad y = 0, 1, 2, \dots \; .$$

The authors continue with finding the limit distribution of iterated bivariate partial summation for parent distribution of a finite support. Similarly as Koščová et al. (2020), they use the power method. They discover that there are sequences of descendant distributions which do not converge, but oscillate instead. For example, when the parent is alternative distribution with the parameter $\frac{1}{2}$ and the type of repeated partial summation is given by $g(j) = 1 - \frac{\frac{4}{3}}{j+1}$, which corresponds to Poisson distribution with the parameter $\frac{4}{3}$, it holds

$$\{P^*_j\}_{j=0}^{1} = \{P^*_0, P^*_1\} = \left\{\frac{1}{2}, \frac{1}{2}\right\},$$

$$\left\{P_j^{(1)}\right\}_{j=0}^{1} = \left\{P_0^{(1)}, P_1^{(1)}\right\} = \{0, 1\},$$

$$\left\{P_j^{(2)}\right\}_{j=0}^{1} = \left\{P_0^{(2)}, P_1^{(2)}\right\} = \left\{\frac{1}{2}, \frac{1}{2}\right\},$$

$$\left\{P_j^{(3)}\right\}_{j=0}^{1} = \left\{P_0^{(3)}, P_1^{(3)}\right\} = \{0, 1\},$$

etc. For a detailed research on oscillation of univariate partial-sums discrete probability distributions of the support size 2 and 3, see Leššová (2019).

2 Parametrization of partial summations

In this chapter we consider the partial summation (1.2)

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0, 1, 2, \dots \ . \tag{2.1}$$

Let $f(x)$ be a function given by

$$P_{x+1}^* = f(x+1) P_x^*, \qquad x = 0, 1, 2, \dots \ . \tag{2.2}$$

Then (according to Mačutek, 2003) the function $g(\cdot)$, which leaves the parent distribution unaltered under summation (2.1), is

$$g(x) = 1 - f(x+1), \qquad x = 0, 1, 2, \dots \ . \tag{2.3}$$

For the sake of simplicity, in the following considerations we limit ourselves to discrete distributions with one parameter only.

Denote a the parameter of the distribution $\{P_j^*\}_{j=0}^{\infty}$. In order to emphasize the role of the parameter, we can use the notation $\{P_j^*(a)\}_{j=0}^{\infty}$ and condition of invariance (2.3) can be rewritten as

$$g(x; a) = 1 - f(x+1; a), \qquad x = 0, 1, 2, \dots \ . \tag{2.4}$$

The descendant distribution is uniquely given by $\{P_j^*(a)\}_{j=0}^{\infty}$ and $g(x; a),\ x = 0, 1, \dots$, i.e., it also depends solely on the parameter a. Hence the distribution resulting from (2.1) is in fact $\{P_j(a)\}_{j=0}^{\infty}$:

$$P_x(a) = \sum_{j=x}^{\infty} g(j; a) P_j^*(a), \qquad x = 0, 1, 2, \dots \ . \tag{2.5}$$

2.1 Change of parameter in partial summation

Let us consider a modification of summation (2.5), namely,

$$P_x = c\sum_{j=x}^{\infty} g(j;\lambda)P_j^*(a), \qquad x = 0,1,2,\dots\ , \tag{2.6}$$

with

$$g(x;\lambda) = 1 - f(x+1;\lambda), \qquad x = 0,1,2,\dots\ , \tag{2.7}$$

where the formula for the function from (2.4) is kept, but parameter a was replaced with λ; c is a proper constant (with respect to x) which should ensure that $\{P_j(a)\}_{j=0}^{\infty}$ sums to 1. Our aim is to investigate consequences of the above mentioned change of the parameter value.

2.2 Examples

2.2.1 Geometric distribution

Let $\{P_j^*(a)\}_{j=0}^{\infty}$ be the geometric distribution with parameter $a \in (0;1\rangle$, i.e.,

$$P_j^*(a) = a(1-a)^j, \qquad j = 0,1,2,\dots\ .$$

Summation (2.6) yields

$$P_x = c\sum_{j=x}^{\infty} g(j;\lambda)a(1-a)^j, \qquad x = 0,1,2,\dots\ ,$$

and function $g(\cdot)$ is determined by (2.7) as

$$\begin{aligned} g(x;\lambda) =& 1 - f(x+1;\lambda) = 1 - \frac{P_{x+1}^*(\lambda)}{P_x^*(\lambda)} = 1 - \frac{\lambda(1-\lambda)^{x+1}}{\lambda(1-\lambda)^x} = \\ =& 1-(1-\lambda) = \lambda, \qquad x = 0,1,2,\dots\ . \end{aligned}$$

To find the descendant distribution $\{P_j\}_{j=0}^{\infty}$ it is necessary to find the normalization constant c,

$$c = \left[\sum_{x=0}^{\infty}\sum_{j=x}^{\infty} g(j;\lambda)P_j^*(a)\right]^{-1} = \left[\lambda a \sum_{x=0}^{\infty}\sum_{j=x}^{\infty}(1-a)^j\right]^{-1} =$$

$$= \left[\lambda a \sum_{x=0}^{\infty}\frac{(1-a)^x}{a}\right]^{-1} = \left[\lambda\frac{1}{a}\right]^{-1} = \frac{a}{\lambda}.$$

Hence the descendant is

$$P_x = \frac{a}{\lambda}\sum_{x=j}^{\infty}\lambda a(1-a)^j = a^2\sum_{j=x}^{\infty}(1-a)^j = a(1-a)^x, \qquad x = 0, 1, 2, \dots \ .$$

For the geometric distribution, the change of the parameter value from a to λ in summation (2.6) does not affect the resulting descendant distribution, as the new parameter value is eliminated by the constant c. The descendant distribution is identical with its parent distribution.

2.2.2 Poisson distribution

If $\{P_j^*(a)\}_{j=0}^{\infty}$ is the Poisson distribution with parameter $a \geq 0$, i.e.,

$$P_j^*(a) = \frac{e^{-a}a^j}{j!}, \qquad j = 0, 1, 2, \dots \ ,$$

the summation (2.6) has the form

$$P_x = c\sum_{j=x}^{\infty} g(j;\lambda)\frac{e^{-a}a^j}{j!}, \qquad x = 0, 1, 2, \dots \ .$$

The function $g(\cdot)$ is determined by (2.7), i.e.,

$$g(x;\lambda) = 1 - f(x+1;\lambda) = 1 - \frac{P_{x+1}^*(\lambda)}{P_x^*(\lambda)} = 1 - \frac{\dfrac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}}{\dfrac{e^{-\lambda}\lambda^x}{x!}} =$$

$$= 1 - \frac{\lambda}{x+1}, \qquad x = 0, 1, 2, \dots \ .$$

The normalization constant is

$$c = \left[\sum_{x=0}^{\infty}\sum_{j=x}^{\infty} g(j;\lambda)P_j^*(a)\right]^{-1} = \left[\sum_{x=0}^{\infty}\sum_{j=x}^{\infty}\left(1-\frac{\lambda}{j+1}\right)\frac{e^{-a}a^j}{j!}\right]^{-1} =$$

$$= \left[\sum_{j=0}^{\infty}\sum_{x=0}^{j}\left(1-\frac{\lambda}{j+1}\right)\frac{e^{-a}a^j}{j!}\right]^{-1} =$$

$$= \left[\sum_{j=0}^{\infty}(j+1)\left(1-\frac{\lambda}{j+1}\right)\frac{e^{-a}a^j}{j!}\right]^{-1} = \left[\sum_{j=0}^{\infty}(j+1-\lambda)\frac{e^{-a}a^j}{j!}\right]^{-1} =$$

$$= \left[1-\lambda+\sum_{j=1}^{\infty} j\frac{e^{-a}a^j}{j!}\right]^{-1} = \left[1-\lambda+a\sum_{j=1}^{\infty}\frac{e^{-a}a^{j-1}}{(j-1)!}\right]^{-1} = \frac{1}{1-\lambda+a}.$$

Therefore, the descendant distribution is given by

$$P_x = \frac{1}{1-\lambda+a}\sum_{x=j}^{\infty}\left(1-\frac{\lambda}{j+1}\right)\frac{e^{-a}a^j}{j!}, \qquad x = 0, 1, 2, \dots \ .$$

The resulting descendant distribution for the Poisson parent is a two-parameter distribution (i.e. it differs from its parent, which has only one parameter). As both parameters a and λ are parameters of certain Poisson distributions, they must not be negative. Moreover, it must hold $a + 1 > \lambda$, because the normalization constant c is supposed to be positive.

2.2.3 Logarithmic distribution

We consider the logarithmic distribution in the form

$$P_j^*(a) = -\frac{a^{j+1}}{(j+1)\log(1-a)}, \qquad j = 0, 1, 2, \dots \ ,$$

$a \in \langle 0; 1)$ is parameter, to be the parent distribution. We shifted the classical logarithmic distribution to the left by 1 so that the parent distribution is defined on all non-negative integers (like all other examples). The descendant distribution with respect to summation (2.6) is

$$P_x = -c\sum_{j=x}^{\infty} g(j;\lambda)\frac{a^{j+1}}{(j+1)\log(1-a)}, \qquad x = 0, 1, 2, \dots \ ,$$

where function $g(\cdot)$ defined by (2.7) is

$$g(x;\lambda) = 1 - f(x+1;\lambda) = 1 - \frac{P_{x+1}^*(\lambda)}{P_x^*(\lambda)} = 1 - \frac{\dfrac{\lambda^{x+2}}{(x+2)\log(1-\lambda)}}{\dfrac{\lambda^{x+1}}{(x+1)\log(1-\lambda)}} =$$

$$= 1 - \lambda\frac{x+1}{x+2}, \qquad x = 0, 1, 2, \dots \ .$$

The normalization constant c is

$$c = \left[\sum_{x=0}^{\infty}\sum_{j=x}^{\infty} g(j;\lambda)P_j^*(a)\right]^{-1} =$$

$$= \left[-\sum_{x=0}^{\infty} \sum_{j=x}^{\infty} \left(1 - \lambda \frac{j+1}{j+2} \right) \frac{a^{j+1}}{(j+1)\log(1-a)} \right]^{-1} =$$

$$= \left[-\sum_{j=0}^{\infty} \sum_{x=0}^{j} \left(1 - \lambda \frac{j+1}{j+2} \right) \frac{a^{j+1}}{(j+1)\log(1-a)} \right]^{-1} =$$

$$= \left[-\sum_{j=0}^{\infty} \left(1 - \lambda \frac{j+1}{j+2} \right) \frac{a^{j+1}}{\log(1-a)} \right]^{-1} =$$

$$= \left[-\sum_{j=0}^{\infty} (1-\lambda) \frac{a^{j+1}}{\log(1-a)} - \sum_{j=0}^{\infty} \frac{\lambda}{j+2} \frac{a^{j+1}}{\log(1-a)} \right]^{-1} =$$

$$= \left[-\frac{1-\lambda}{\log(1-a)} \frac{a}{1-a} - \frac{\lambda}{a\log(1-a)} \sum_{j=0}^{\infty} \frac{a^{j+2}}{j+2} \right]^{-1} =$$

$$= \left[-\frac{1-\lambda}{\log(1-a)} \frac{a}{1-a} - \frac{\lambda}{a\log(1-a)} \int \sum_{j=0}^{\infty} a^{j+1} da \right]^{-1} =$$

$$= \left[-\frac{1-\lambda}{\log(1-a)} \frac{a}{1-a} - \frac{\lambda}{a\log(1-a)} \int \frac{a}{1-a} da \right]^{-1} =$$

$$= \left[-\frac{1-\lambda}{\log(1-a)} \frac{a}{1-a} + \frac{\lambda(a + \log(1-a))}{a\log(1-a)} \right]^{-1} =$$

$$= \frac{a(1-a)\log(1-a)}{(\lambda-1)a^2 + \lambda(a-1)(a+\log(1-a))}.$$

The descendant distribution turns out to be

$$P_x = -\frac{a(1-a)\log(1-a) \sum_{x=j}^{\infty} \left(1 - \lambda \frac{j+1}{j+2} \right) \frac{a^{j+1}}{(j+1)\log(1-a)}}{(\lambda-1)a^2 + \lambda(a-1)(a+\log(1-a))}, \qquad x = 0, 1, 2, \dots ,$$

which is a rather complicated distribution with two parameters λ and a.

2.2.4 Salvia-Bolinger distribution

Let the parent be the Salvia-Bolinger distribution with parameter $a \in (0; 1\rangle$, which is defined as

$$P_0^*(a) = a,$$

$$P_j^*(a) = \frac{a\prod_{k=1}^{j}(k-a)}{(j+1)!}, \qquad j = 1, 2, 3, \dots \; .$$

To find the descendant distribution we need the function $g(\cdot)$ from (2.7), which can be expressed as

$$\begin{aligned} g(x;\lambda) =& 1 - f(x+1;\lambda) = 1 - \frac{P_{x+1}^*(\lambda)}{P_x^*(\lambda)} = 1 - \frac{x+1-\lambda}{x+2} \\ =& \frac{\lambda+1}{x+2}, \qquad x = 0, 1, 2, \dots \; . \end{aligned}$$

The normalization constant c in this case evaluates to

$$\begin{aligned} c =& \left[\sum_{x=0}^{\infty}\sum_{j=x}^{\infty} g(j;\lambda)P_j^*(a)\right]^{-1} = \left[\sum_{j=0}^{\infty}(j+1)g(j,\lambda)P_j^*(a)\right]^{-1} = \\ =& \left[\sum_{j=0}^{\infty}(j+1)\frac{\lambda+1}{j+2}P_j^*(a)\right]^{-1} = \\ =& \left[(\lambda+1)\left(\sum_{j=0}^{\infty}P_j^*(a) - \sum_{j=0}^{\infty}\frac{1}{j+2}P_j^*(a)\right)\right]^{-1} = \\ =& \left[(\lambda+1)\left(1 - \frac{a}{2} - \sum_{j=1}^{\infty}\frac{a(1-a)(2-a)\dots\;(j-a)}{(j+2)!}\right)\right]^{-1} = \\ =& \left\{(\lambda+1)\left[1 - \frac{a}{2}\left(1 + \frac{1-a}{3} + \frac{(1-a)(2-a)}{3\cdot 4} + \dots\right)\right]\right\}^{-1} = \\ =& \left[(\lambda+1)\left(1 - \frac{a}{2}{}_2F_1(1-a,1;3;1)\right)\right]^{-1} = \\ =& \left[(\lambda+1)\left(1 - \frac{a}{2}\frac{\Gamma(a+1)\Gamma(3)}{\Gamma(2)\Gamma(a+2)})\right)\right]^{-1} = \\ =& \left[(\lambda+1)\left(1 - \frac{a}{2}\frac{2}{a+1})\right)\right]^{-1} = \frac{a+1}{\lambda+1}, \end{aligned}$$

where ${}_2F_1(a,b;c;t)$ is the hypergeometric function defined as

$${}_2F_1(a,b;c;t) = \sum_{j=0}^{\infty}\frac{a^{(j)}b^{(j)}t^j}{j!c^{(j)}} = 1 + \frac{abt}{1!c} + \frac{a(a+1)b(b+1)t^2}{2!c(c+1)} + \dots$$

(for more detail see e.g. Abramowitz and Stegun, 1972). In the derivation of the constant c above we applied the Gauss summation theorem, i.e.

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, $$

for $c-a-b>0$. Then the descendant distribution is

$$ P_x = \frac{a+1}{\lambda+1}\sum_{j=x}^{\infty}\frac{\lambda+1}{j+2}P_j^*(a) = \sum_{j=x}^{\infty}\frac{a+1}{j+2}P_j^*(a), \qquad x=0,1,2,\dots\ . $$

The new parameter of summation, i.e., λ is eliminated by the normalization constant c and the resulting partial summation is according to (2.4) the Salvia-Bolinger partial summation. Therefore,

$$ P_x = P_x^*(a). $$

The Salvia-Bolinger distribution is, like the geometric distribution, resistant to the change of partial summation parameter.

2.3 Two families of discrete distributions

We use the examples to demonstrate that the descendant distribution of summation (2.6) is either a one-parameter distribution identical with its parent or a two-parameter distribution. This behaviour defines two families of discrete distributions. The first family consists of those distributions that yield a two-parameter descendant under summation (2.6), e.g., the Poisson and the logarithmic distribution (see Section 2.2.2 and Section 2.2.3). In the following, this family of distributions is denoted by the term family of sensitive distributions (or the sensitive family). The name of this family is chosen to emphasize the sensitivity of the parent to the change of summation parameter value. On the other hand, the group of distributions that remain unaltered under summation (2.6) is henceforward denoted as the family of resistant distributions (or the resistant family). The resistant family contains, e.g., the geometric and the Salvia-Bolinger distribution (as is shown in Section 2.2.1 and in Section 2.2.4). The members of the resistant family are, in fact, resistant to the change of summation parameter value, when summation (2.6) is applied to them. In the following part of this chapter we provide the theoretical background.

Lemma 1. *The normalization constant c can be expressed as*

$$ c = \frac{1}{1 - f(1;\lambda)P_0^*(a) + \sum_{s=1}^{\infty} P_s^*(a)[s-(s+1)f(s+1;\lambda)]}, $$

and if the distribution $\{P_j^*(a)\}_{j=0}^{\infty}$ *has a finite mean* μ*, then it holds*

$$c = \frac{1}{\mu + 1 - \sum_{s=1}^{\infty} s f(s;\lambda) P_{s-1}^*(a)}.$$

Proof. It is possible to rewrite summation (2.6) as a series of equations

$$\begin{aligned}
P_0 =& c[g(0;\lambda)P_0^*(a) + g(1;\lambda)P_1^*(a) + g(2;\lambda)P_2^*(a) + \dots],\\
P_1 =& c[\qquad\qquad\qquad g(1;\lambda)P_1^*(a) + g(2;\lambda)P_2^*(a) + \dots],\\
P_2 =& c[\qquad\qquad\qquad\qquad\qquad\qquad g(2;\lambda)P_2^*(a) + \dots],\\
& \vdots
\end{aligned}$$

or

$$\begin{aligned}
P_0 =& c\{[1-f(1;\lambda)]P_0^*(a) + [1-f(2;\lambda)]P_1^*(a) + [1-f(3;\lambda)]P_2^*(a) + \dots\},\\
P_1 =& c\{\qquad\qquad\qquad [1-f(2;\lambda)]P_1^*(a) + [1-f(3;\lambda)]P_2^*(a) + \dots\},\\
P_2 =& c\{\qquad\qquad\qquad\qquad\qquad\qquad [1-f(3;\lambda)]P_2^*(a) + \dots\},\\
& \vdots
\end{aligned}$$

which is the same as

$$\begin{aligned}
P_0 =& c\{P_0^*(a) + P_1^*(a) + P_2^*(a) + \dots - [f(1;\lambda)P_0^*(a) + f(2;\lambda)P_1^*(a) + \dots]\},\\
P_1 =& c\{\qquad\quad P_1^*(a) + P_2^*(a) + \dots - [\qquad\qquad f(2;\lambda)P_1^*(a) + \dots]\},\\
& \vdots
\end{aligned}$$

After summing both sides of these equations we have

$$1 = c\left\{[P_0^*(a) + 2P_1^*(a) + \dots] - \sum_{s=1}^{\infty} s f(s;\lambda) P_{s-1}^*(a)\right\},$$

which leads to

$$c = \frac{1}{1 + \sum_{k=1}^{\infty} k P_k^*(a) - \sum_{s=1}^{\infty} s f(s;\lambda) P_{s-1}^*(a)} =$$

$$= \frac{1}{1 + \sum_{k=2}^{\infty}(k-1)P_{k-1}^*(a) - \sum_{s=2}^{\infty} sf(s;\lambda)P_{s-1}^*(a) - f(1;\lambda)P_0^*(a)}$$

and, consequently,

$$c = \frac{1}{1 - f(1;\lambda)P_0^*(a) + \sum_{s=1}^{\infty}[s - (s+1)f(s+1;\lambda)]P_s^*(a)}. \tag{2.8}$$

If the distribution $\{P_j^*(a)\}_{j=0}^{\infty}$ has the finite mean μ, the constant c is obviously

$$c = \frac{1}{1 + \mu - \sum_{s=1}^{\infty} sf(s;\lambda)P_{s-1}^*(a)}. \tag{2.9}$$

□

Theorem 1. *A one-parameter discrete distribution* $\{P_j^*(a)\}_{j=0}^{\infty}$ *belongs to the resistant family if and only if*

$$\frac{1 - f(j+1;\lambda)}{1 - f(1;\lambda)P_0^*(a) + \sum_{s=1}^{\infty}[s - (s+1)f(s+1;\lambda)]P_s^*(a)} = 1 - f(j+1;a), \tag{2.10}$$

$$j = 0, 1, 2, \dots .$$

Proof. Using the invariance condition (2.4), which is

$$g(x;a) = 1 - f(x+1;a), \qquad x = 0, 1, 2, \dots ,$$

the parent distribution $\{P_j^*(a)\}_{j=0}^{\infty}$ is not changed by summation (2.6) if and only if

$$P_x(a) = \sum_{j=x}^{\infty}(1 - f(j+1;a))P_j^*(a), \qquad x = 0, 1, 2, \dots . \tag{2.11}$$

Next, using the result of Lemma 1 it holds

$$P_x(a;\lambda) = \sum_{j=x}^{\infty} \frac{g(j;\lambda)P_j^*(a)}{1 - f(1;\lambda)P_0^*(a) + \sum_{s=1}^{\infty}[s - (s+1)f(s+1;\lambda)]P_s^*(a)}, \tag{2.12}$$

$$x = 0, 1, 2, \dots .$$

Then combining (2.11) and (2.12) we obtain

$$\frac{g(j;\lambda)}{1-f(1;\lambda)P_0^*(a)+\sum_{s=1}^{\infty}[s-(s+1)f(s+1;\lambda)]P_s^*(a)}=1-f(j+1;a),$$

$$j=0,1,2,\ldots\ ,$$

which is the same as

$$\frac{1-f(j+1;\lambda)}{1-f(1;\lambda)P_0^*(a)+\sum_{s=1}^{\infty}[s-(s+1)f(s+1;\lambda)]P_s^*(a)}=1-f(j+1;a),$$

$$j=0,1,2,\ldots\ .$$

□

Theorem 1 offers a rule which strictly distinguishes between the family of resistant distributions and the family of sensitive distributions. For distributions with the finite mean μ, the rule (2.10) can be expressed analogously to (2.9) as

$$\frac{1-f(j+1;\lambda)}{1+\mu-\sum_{s=1}^{\infty}sf(s;\lambda)P_{s-1}^*(a)}=1-f(j+1;a).$$

Theorem 2. *If the distribution $\{P_j^*\}$ belongs to the resistant family then the ratio*

$$\frac{1-\dfrac{P_{j+1}^*(\lambda)}{P_j^*(\lambda)}}{1-\dfrac{P_{j+1}^*(a)}{P_j^*(a)}} \tag{2.13}$$

does not depend on j.

Proof. The sufficient and necessary condition (2.10) for a distribution to belong to the resistant family can be reformulated as

$$1-f(1;\lambda)P_0^*(a)+\sum_{s=1}^{\infty}[s-(s+1)f(s+1;\lambda)]P_s^*(a)=\frac{1-f(j+1;\lambda)}{1-f(j+1;a)}.$$

Using (2.2) we obtain

$$1-f(1;\lambda)P_0^*(a)+\sum_{s=1}^{\infty}[s-(s+1)f(s+1;\lambda)]P_s^*(a)=\frac{1-\dfrac{P_{j+1}^*(\lambda)}{P_j^*(\lambda)}}{1-\dfrac{P_{j+1}^*(a)}{P_j^*(a)}}. \tag{2.14}$$

It is easy to see that the left side of (2.14) does not depend on j, therefore the right side necessarily must not depend on j too. $\square$

Theorem 2 presents a useful necessary condition which can be used to classify distributions to the sensitive family. Discrete distributions with one parameter for which (2.13) is dependent on j are certainly from the family of sensitive distributions. Those for which (2.13) is independent of j are candidates for the resistant family and validity of the condition (2.10) comes into question. In addition, if the necessary condition is satisfied, combining (2.8) with (2.14) we have

$$c = \frac{1 - \dfrac{P^*_{j+1}(a)}{P^*_j(a)}}{1 - \dfrac{P^*_{j+1}(\lambda)}{P^*_j(\lambda)}}.$$

For the Poisson distribution it holds

$$\frac{1 - \dfrac{P^*_{j+1}(\lambda)}{P^*_j(\lambda)}}{1 - \dfrac{P^*_{j+1}(a)}{P^*_j(a)}} = \frac{1 - \dfrac{\frac{e^{-\lambda}\lambda^{j+1}}{(j+1)!}}{\frac{e^{-\lambda}\lambda^j}{j!}}}{1 - \dfrac{\frac{e^{-a}a^{j+1}}{(j+1)!}}{\frac{e^{-a}a^j}{j!}}} = \frac{1 - \dfrac{\lambda}{j+1}}{1 - \dfrac{a}{j+1}} = \frac{j+1-\lambda}{j+1-a},$$

hence the ratio (2.13) is dependent on j. Thus, the Poisson distribution belongs to the family of sensitive distributions, as was shown above in Section 2.2.2. Similarly for logarithmic distribution the ratio (2.13) leads to

$$\frac{1 - \dfrac{P^*_{j+1}(\lambda)}{P^*_j(\lambda)}}{1 - \dfrac{P^*_{j+1}(a)}{P^*_j(a)}} = \frac{1 - \dfrac{-\frac{\lambda^{j+2}}{(j+2)\log(1-\lambda)}}{-\frac{\lambda^{j+1}}{(j+1)\log(1-\lambda)}}}{1 - \dfrac{-\frac{a^{j+2}}{(j+2)\log(1-a)}}{-\frac{a^{j+1}}{(j+1)\log(1-a)}}} = \frac{1 - \dfrac{\lambda(j+1)}{j+2}}{1 - \dfrac{a(j+1)}{j+2}} = \frac{j+2-\lambda(j+1)}{j+2-a(j+1)},$$

which is also dependent on j. The logarithmic distribution also belongs to the sensitive family. However, for the geometric distribution it holds

$$\frac{1 - \dfrac{P^*_{j+1}(\lambda)}{P^*_j(\lambda)}}{1 - \dfrac{P^*_{j+1}(a)}{P^*_j(a)}} = \frac{1 - \dfrac{\lambda(1-\lambda)^{j+1}}{\lambda(1-\lambda)^j}}{1 - \dfrac{a(1-a)^{j+1}}{a(1-a)^j}} = \frac{1-(1-\lambda)}{1-(1-a)} = \frac{\lambda}{a} \tag{2.15}$$

and for the Salvia-Bollinger distribution,

$$\frac{1-\dfrac{P^*_{j+1}(\lambda)}{P^*_j(\lambda)}}{1-\dfrac{P^*_{j+1}(a)}{P^*_j(a)}} = \frac{1-\dfrac{j+1-\lambda}{j+2}}{1-\dfrac{j+1-a}{j+2}} = \frac{1+\lambda}{1+a}. \tag{2.16}$$

Both (2.15) and (2.16) do not depend on j.

2.4 Other distributions in the family of resistant distributions?

The family of resistant distributions seems to be significantly smaller than the family of sensitive distributions. So far only the geometric and the Salvia-Bolinger distribution are known to be from that family. We perform a search for other members of the resistant family.

The Kemp-Dacey-hypergeometric family will be scrutinized. The probability mass function of this family of distributions is

$$P_x = \frac{C a_1^{(x)} a_2^{(x)} \dots\ a_k^{(x)} \theta^x}{x! b_1^{(x)} b_2^{(x)} \dots\ b_r^{(x)}}, \qquad x \in T,$$

where T is a subset of $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $k, r \in \mathbb{N}_0$, $(a_1, a_2, \dots, a_k, b_1, b_2, \dots\ b_r, \theta)$ are parameters and $p^{(x)}$ is the ascending factorial function, i.e.

$$p^{(x)} = p(p+1)(p+2)\dots\ (p+x-1), \qquad p \in \mathbb{R}, \quad x \in \mathbb{N},$$

$$p^{(0)} = 1, \qquad p \in \mathbb{R}.$$

The constant C is a normalization constant, which can be expressed using the generalized Gauss hypergeometric series ${}_kF_r(a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_r; \theta)$ (see Abramowitz and Stegun, 1972) as

$$C = \left[\sum_{x\in T} P_x\right]^{-1} = [{}_kF_r(a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_r; \theta)]^{-1}.$$

The Kemp-Dacey-hypergeometric family is a very wide family of discrete distributions. Hence we restrict the search for distributions belonging to the family of resistant distributions by fixing $k = 2$ and $r = 1$. This restriction implies

$$P_x = \frac{C a_1^{(x)} a_2^{(x)} \theta^x}{x! b_1^{(x)}}, \qquad x \in T,$$

and consequently,

$$\frac{1-\dfrac{P^*_{j+1}(a_1,a_2;b_1;\theta)}{P^*_j(a_1,a_2;b_1;\theta)}}{1-\dfrac{P^*_{j+1}(\alpha_1,\alpha_2;\beta_1;\eta)}{P^*_j(a)}}=\frac{1-\dfrac{(a_1+j)(a_2+j)\theta}{(1+j)(b_1+j)}}{1-\dfrac{(\alpha_1+j)(\alpha_2+j)\eta}{(1+j)(\beta_1+j)}}. \tag{2.17}$$

In order to get rid of j in (2.17), we propose two different sets of parameter restrictions. These parameter restrictions also reduce the dimension of parametric space from four to one, hence the previous results become applicable. The first set of restrictions, $\theta = 1$, $a_1 = b_1$, eliminates j from (2.17) but they do no yield a proper distribution[1].

The other set of restrictions is:

$$\theta = 1, \qquad a_2 = 1, \qquad b_1 = B, \tag{2.18}$$

B is a fixed constant from parametric space of parameter b_1. The only remaining free parameter a_1 is further denoted as a. Then the probability mass function is

$$P_x(a) = \frac{Ca^{(x)}1^{(x)}}{x!B^{(x)}} = \frac{Ca^{(x)}}{B^{(x)}}, \qquad x = 0, 1, 2, \dots\ , \tag{2.19}$$

where C is a normalization constant. We consider $a > 0$ and $B > a + 2$ to ensure non-negative values of the probability mass function and to secure the validity of the Gauss summation theorem, which is widely used in the computations below. The restrictions (2.18) result in

$$\frac{1-\dfrac{P_{j+1}(a)}{P_j(a)}}{1-\dfrac{P_{j+1}(\lambda)}{P_j(\lambda)}}\frac{1-\dfrac{a+j}{B+j}}{1-\dfrac{\lambda+j}{B+j}} = \frac{B+j-a-j}{B+j-\lambda-j} = \frac{B-a}{B-\lambda},$$

which is obviously independent of j. The necessary condition, according to Theorem 2, is fulfilled. Therefore, the distribution (2.19) might belong to the family of resistant distributions. Validity of the condition (2.10) must be investigated.

To evaluate the veracity of the condition (2.10) it is necessary to find C:

$$C = \left[\sum_{x=0}^{\infty} \frac{a^{(x)}}{B^{(x)}}\right]^{-1} = \left[{}_2F_1(a,1;B;1)\right]^{-1} = \left[\frac{\Gamma(B)\Gamma(B-a-1)}{\Gamma(B-a)\Gamma(B-1)}\right]^{-1} = \frac{B-a-1}{B-1},$$

The Gauss summation theorem and a property of the Gamma function ($\Gamma(t+1) = t\Gamma(t)$) were used. The probability $P_0(a)$ takes then the value

$$P_0(a) = \frac{B-a-1}{B-1},$$

[1]In the derivation of the constant C, a condition $a_2 < 0$ arises. However, it is easy to see that $P_0 = C$ and $P_1 = Ca_2$, which fails to ensure the non-negative probabilities.

and for the function $f(s+1;\lambda)$ it holds

$$f(s+1;\lambda) = \frac{P_{s+1}(\lambda)}{P_s(\lambda)} = \frac{\lambda+s}{B+s}.$$

Combining last two equations with (2.10), the aim is to verify whether

$$1 - \frac{\lambda}{B}\frac{B-a-1}{B-1} + \sum_{s=1}^{\infty}\left(s - (s+1)\frac{\lambda+s}{B+s}\right)P_s(a) = \frac{B-\lambda}{B-a} \tag{2.20}$$

is true. In order to evaluate the sum in the equation above, we split it in four parts,

$$\sum_{s=1}^{\infty} sP_s(a) - \lambda\sum_{s=1}^{\infty}\frac{P_s(a)}{B+s} - (\lambda+1)\sum_{s=1}^{\infty}\frac{sP_s(a)}{B+s} - \sum_{s=1}^{\infty}\frac{s^2P_s(a)}{B+s}. \tag{2.21}$$

The first part is of value

$$\begin{aligned}
\sum_{s=1}^{\infty} sP_s(a) =& C\sum_{s=1}^{\infty} s\frac{a^{(s)}}{B^{(s)}} = C\sum_{s=1}^{\infty}\frac{s!a^{(s)}}{(s-1)!B^{(s)}} = C\sum_{s=0}^{\infty}\frac{(s+1)!a^{(s+1)}}{s!B^{(s+1)}} = \\
=& C\frac{a}{B}\sum_{s=0}^{\infty}\frac{2^{(s)}(a+1)^{(s)}}{s!(B+1)^{(s)}} = C\frac{a}{B}{}_2F_1(a+1,2;B+1;1) = \\
=& C\frac{a}{B}\frac{\Gamma(B+1)\Gamma(B-a-2)}{\Gamma(B-a)\Gamma(B-1)} \\
=& C\frac{a}{B}\frac{B\Gamma(B)\Gamma(B-a-2)}{(B-a-1)\Gamma(B-a-1)\Gamma(B-1)} = \\
=& Ca\frac{B-1}{(B-a-1)(B-a-2)} = \frac{a}{B-a-2},
\end{aligned}$$

while the second part is (not considering the constant $-\lambda$ in front of the sum)

$$\begin{aligned}
\sum_{s=1}^{\infty}\frac{P_s(a)}{B+s} =& C\sum_{s=1}^{\infty}\frac{a^{(s)}}{B^{(s+1)}} = C\sum_{s=0}^{\infty}\frac{a^{(s+1)}}{B^{(s+2)}} = C\sum_{s=0}^{\infty}\frac{a(a+1)^{(s)}}{B(B+1)(B+2)^{(s)}} = \\
=& \frac{Ca}{B(B+1)}\sum_{s=0}^{\infty}\frac{(a+1)^{(s)}1^{(s)}}{(B+2)^{(s)}s!} = \frac{Ca}{B(B+1)}{}_2F_1(a+1,1;B+2;1) = \\
=& \frac{Ca}{B(B+1)}\frac{\Gamma(B+2)\Gamma(B-a)}{\Gamma(B-a+1)\Gamma(B+1)} = \frac{Ca}{B(B+1)}\frac{B+1}{B-a} = \\
=& \frac{a(B-a-1)}{B(B-1)(B-a)}.
\end{aligned}$$

For the third part (again without the constant $-(\lambda+1)$) it holds

$$\sum_{s=1}^{\infty}\frac{sP_s(a)}{B+s}=C\sum_{s=1}^{\infty}\frac{sa^{(s)}}{B^{(s+1)}}=C\sum_{s=1}^{\infty}\frac{s!a^{(s)}}{(s-1)!B^{(s+1)}}=C\sum_{s=0}^{\infty}\frac{(s+1)!a^{(s+1)}}{s!B^{(s+2)}}=$$

$$=C\sum_{s=0}^{\infty}\frac{2^{(s)}a(a+1)^{(s)}}{s!B(B+1)(B+2)^{(s)}}=\frac{Ca}{B(B+1)}{}_2F_1(a+1,2;B+2;1)=$$

$$=\frac{Ca}{B(B+1)}\frac{\Gamma(B+2)\Gamma(B-a-1)}{\Gamma(B-a+1)\Gamma(B)}=$$

$$=\frac{Ca}{B(B+1)}\frac{(B+1)\Gamma(B+1)\Gamma(B-a-1)}{(B-a)\Gamma(B-a)\Gamma(B)}=$$

$$=\frac{Ca}{B}\frac{B}{(B-a)(B-a-1)}=\frac{a}{(B-1)(B-a)},$$

and the fourth part yields

$$\sum_{s=1}^{\infty}\frac{s^2P_s(a)}{B+s}=C\sum_{s=1}^{\infty}\frac{s^2a^{(s)}}{B^{(s+1)}}=\frac{C}{B}\sum_{s=1}^{\infty}\frac{s!s^2a^{(s)}}{s!(B+1)^{(s)}}=$$

$$=\frac{C}{B}\left(\sum_{s=1}^{\infty}\frac{(s+2)!a^{(s)}}{s!(B+1)^{(s)}}-3\sum_{s=1}^{\infty}\frac{(s+1)!a^{(s)}}{s!(B+1)^{(s)}}+\sum_{s=1}^{\infty}\frac{s!a^{(s)}}{s!(B+1)^{(s)}}\right)=$$

$$=\frac{C}{B}\left(2\sum_{s=1}^{\infty}\frac{3^{(s)}a^{(s)}}{s!(B+1)^{(s)}}-3\sum_{s=1}^{\infty}\frac{2^{(s)}a^{(s)}}{s!(B+1)^{(s)}}+\sum_{s=1}^{\infty}\frac{1^{(s)}!a^{(s)}}{s!(B+1)^{(s)}}\right)=$$

$$=\frac{C}{B}[2({}_2F_1(a,3;B+1;1)-1)-3({}_2F_1(a,2;B+1;1)-1)+$$

$$+({}_2F_1(a,1;B+1;1)-1)]=$$

$$=\frac{2C}{B}\left(\frac{\Gamma(B+1)\Gamma(B-a-2)}{\Gamma(B-a+1)\Gamma(B-2)}-1\right)-$$

$$-\frac{3C}{B}\left(\frac{\Gamma(B+1)\Gamma(B-a-1)}{\Gamma(B-a+1)\Gamma(B-1)}-1\right)+$$

$$+\frac{C}{B}\left(\frac{\Gamma(B+1)\Gamma(B-a)}{\Gamma(B-a+1)\Gamma(B)}-1\right)=$$

$$=\frac{2C}{B}\left(\frac{B(B-1)\Gamma(B-1)\Gamma(B-a-2)}{(B-a)(B-a-1)\Gamma(B-a-1)\Gamma(B-2)}-1\right)-$$

$$-\frac{3C}{B}\left(\frac{B\Gamma(B)\Gamma(B-a-1)}{(B-a)\Gamma(B-a)\Gamma(B-1)}-1\right)+\frac{C}{B}\left(\frac{B}{B-a}-1\right)=$$

$$=\frac{2C}{B}\left(\frac{B(B-1)(B-2)}{(B-a)(B-a-1)(B-a-2)}-1\right)-$$

$$-\frac{3C}{B}\left(\frac{B(B-1)}{(B-a)(B-a-1)}-1\right)+\frac{a(B-a-1)}{B(B-1)(B-a)}=$$

$$=\frac{2}{B-a}-\frac{2(B-a-1)}{B(B-1)}-\frac{3}{B-a}-\frac{3(B-a-1)}{B(B-1)}+$$

$$+\frac{a(B-a-1)}{B(B-1)(B-a)}=$$

$$=\frac{B-a-1}{B(B-1)}-\frac{1}{B-a}+\frac{a(B-a-1)}{B(B-1)(B-a)},$$

where we used a factorial equality

$$s!s^2=(s+2)!-3(s+1)!+s!.$$

Having all four parts of (2.21), we can write (2.20) as:

$$1-\frac{\lambda(B-a-1)}{B(B-1)}+\frac{a}{B-a-2}-\frac{(\lambda+1)a}{(B-a)(B-1)}-$$

$$-\frac{\lambda a(B-a-1)}{B(B-1)(B-a)}-\frac{B-a-1}{B(B-1)}+\frac{1}{B-a}-\frac{a(B-a-1)}{B(B-1)(B-a)}=\frac{B-\lambda}{B-a},$$

which can be simplified to

$$B-a-\frac{\lambda(B-a)(B-a-1)}{B(B-1)}+\frac{a(B-a)}{B-a-2}-\frac{(\lambda+1)aB}{B(B-1)}-$$

$$-\frac{\lambda a(B-a-1)}{B(B-1)}-\frac{(B-a)(B-a-1)}{B(B-1)}+1-\frac{a(B-a-1)}{B(B-1)}=B-\lambda,$$

and further to

$$B-a-\lambda+\frac{a(B-a)}{B-a-2}-\frac{a}{B-1}-\frac{B-a-1}{B-1}+1=B-\lambda,$$

which is the same as

$$a(B-1)=0.$$

To conclude, a distribution with the probability mass function (2.19) belongs to the family of resistant distributions if and only if $a = 0$ or $B = 1$. However, neither of these conditions can be fulfilled as it was stated before, right above (2.19). The candidate for the family of resistant distributions discussed in this section turned out to belong to the sensitive family.

For the time being, it remains an open question whether the family of resistant distributions consists of only two distributions (the geometric distribution and the Salvia-Bolinger distribution) or if there are more of them.

3 Iterated partial summations

In this chapter we propose some solutions regarding iterated partial summations. It is possible to apply partial summation (1.2) repeatedly, i.e.,

$$
\begin{aligned}
P_x^{(1)} =& C_1 \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0, 1, 2, \dots\ , \\
P_x^{(2)} =& C_2 \sum_{j=x}^{\infty} g(j) P_j^{(1)}, \qquad x = 0, 1, 2 \dots\ , \\
&\vdots \\
P_x^{(n)} =& C_n \sum_{j=x}^{\infty} g(j) P_j^{(n-1)}, \qquad x = 0, 1, 2 \dots\ , \\
&\vdots
\end{aligned}
$$

C_i is a normalization constant which ensures that $\{P_x^{(i)}\}_{x=0}^{\infty}$, also called the i-th descendant, is a proper distribution for $i = 1, 2, 3, \dots$. The distribution $\{P_x^*\}_{x=0}^{\infty}$ is hereafter called the original parent. As it was mentioned in Chapter 1, Mačutek (2006) proves that if there exists

$$p = \lim_{j \to \infty} \frac{P_{j+1}^*}{P_j^*},$$

the iterated geometric partial summation yields the geometric distribution with the parameter p as the limit distribution. One of the aims of our research is to examine the behaviour of other types of iterated partial summations. At first we perform a computational study for iterated partial summations with finite-support original parents. Then we show the relation between the iterated partial summations and the power method, which is an apparatus from the matrix theory designed to find dominant eigenvalues and eigenvectors of matrices. The power method is applied to several particular partial summations.

3.1 Computational study

The presented computational study inquires about the limit distribution

$$\lim_{i\to\infty} P_x^{(i)}, \qquad x = 0, 1, 2, \dots .$$

In order to obtain some preliminary insight in this field, a computational study using the R software was performed. Basically, in the R software it is possible to work only with objects of finite length. Therefore, the computational study is restricted by the choice of parent distribution. Only parent distributions with finite supports are considered, which ensures that the ith descendant has also finite support, $i = 1, 2, 3, \dots$.

We show the results of the computational study with the binomial distribution playing the role of the original parent. We use two different types of partial summation, the Salvia-Bolinger and the Poisson partial summation.

3.1.1 Salvia-Bolinger partial summation

The probability mass function of the Salvia-Bolinger distribution is

$$Q_0 = a,$$

$$Q_x = \frac{a\prod_{k=1}^{x}(k-a)}{(x+1)!}, \qquad 0 < a \le 1, \quad x = 1, 2, 3, \dots .$$

This distribution is invariant with respect to the partial summation (1.2) for

$$g(x) = 1 - \frac{Q_{x+1}}{Q_x} = 1 - \frac{x+1-a}{x+2} = \frac{a+1}{x+2}, \qquad x = 0, 1, 2, \dots .$$

Having binomial distribution with parameters n, p as the original parent,

$$P_x^* = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0, 1, \dots, n,$$

the descendant distributions are computed iteratively for several selected parameter values using the R software:

1. $\displaystyle\sum_{j=x}^{n} \left(\frac{a+1}{j+2}\right) \binom{n}{j} p^j (1-p)^{n-j}$ is computed for each $x = 0, 1, 2, \dots, n$,
2. C_1 is determined as $\displaystyle\left[\sum_{x=0}^{n}\sum_{j=x}^{n} \left(\frac{a+1}{j+2}\right) \binom{n}{j} p^j (1-p)^{n-j}\right]^{-1}$,

3. $P_x^{(1)}$, $x = 0, 1, 2, \dots, n$ are obtained by combining the results from previous two steps;

4. $\displaystyle\sum_{j=x}^{n} \left(\frac{a+1}{j+2}\right) P_j^{(1)}$ is computed for each $x = 0, 1, 2, \dots, n$,

5. C_2 is determined as $\displaystyle\left[\sum_{x=0}^{n}\sum_{j=x}^{n} \left(\frac{a+1}{j+2}\right) P_j^{(1)}\right]^{-1}$,

6. $P_x^{(2)}$, $x = 0, 1, 2, \dots, n$ are obtained combinming the results from previous two steps;

7. etc.

The steps 1.-3. describe the first iteration, the steps 4.-6. stand for the second iteration, etc. The procedure is run for several combinations of values of the parameters n, p, a. The number of iterations is set to 10000. Hence, for each combination of parameters we obtain a sequence of 10000 partial-sum distributions $\{P_x^{(1)}\}_{x=0}^n$, $\{P_x^{(2)}\}_{x=0}^n$, $\dots$, $\{P_x^{(10000)}\}_{x=0}^n$.

The script for the R software to perform the procedure for the choice of parameters $n = 6$, $p = 0.2$ and $a = 0.5$ is following,

```
n<-6; p<-0.2; a<-0.5
desc<-matrix(0,ncol=n+1,nrow=10001)
for(i in 1:ncol(desc))
  desc[1,i]<-choose(n,i-1)*p^(i-1)*(1-p)^(n-i+1)
for(r in 2:nrow(desc)){
  for(x in 0:n)
    for(j in x:n)
      desc[r,x+1]<-desc[r,x+1]+((a+1)/(j+2))*desc[r-1,j+1]
  desc[r,]<-desc[r,]/sum(desc[r,])}
```

yielding a matrix `desc` as a result. The ith row of the matrix `desc` contains the probabilities of the $(i-1)$th descendant distribution. The study reveals that the 41th descendant is the deterministic distribution ($P_0^{(41)} = 1$ and all other probabilities are zero), which is necessarily the limit distribution. Some of the descendants are depicted in the Figure 3.1, the darker the green colour, the latter the descendant. The limit distribution, deterministic in this case, is illustrated by the black colour and the original parent, the binomial distribution, is of the blue colour. The dots, representing

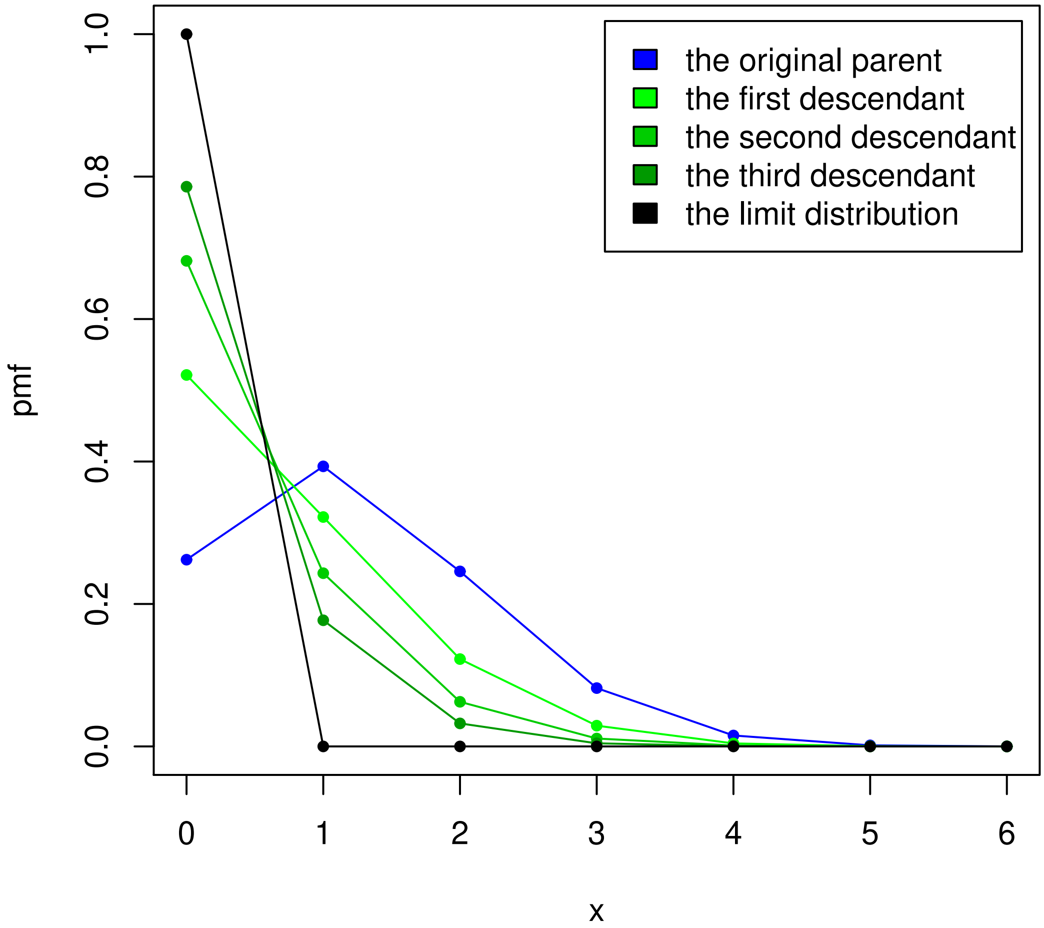

Figure 3.1: *Descendants of the binomial distribution under the iterated Salvia-Bolinger partial summations*

probability mass functions, are connected for a better orientation. We note that the deterministic distribution is a special case of the Salvia-Bolinger distribution, for $a = 1$.

Results of iterated application of Salvia-Bolinger partial summation with binomial distribution as the original parent, for different combinations of parameters n, p, a, are shown in Appendix A. The appendix also contains results for different choices of the original parent distribution. All of the examined combinations of parameters yield the deterministic limit distribution.

3.1.2 Poisson partial summation

The Poisson distribution has the probability mass function

$$Q_x = \frac{e^{-a}a^x}{x!}, \qquad a \geq 0, \quad x = 0, 1, 2, \dots ,$$

a is the parameter and it is invariant with respect to the summation (1.2) for

$$g(x) = 1 - \frac{Q_{x+1}}{Q_x} = 1 - \frac{a}{x+1}, \qquad x = 0, 1, 2, \dots .$$

The procedure of the computational study in this case is analogical to the previous one. The script for the R software is provided here:

```
n<-6; p<-0.2; a<-0.5
desc<-matrix(0,ncol=n+1,nrow=10001)
for(i in 1:ncol(desc))
  desc[1,i]<-choose(n,i-1)*p^(i-1)*(1-p)^(n-i+1)
for(r in 2:nrow(desc)){
  for(x in 0:n)
    for(j in x:n)
      desc[r,x+1]<-desc[r,x+1]+(1-a/(j+1))*desc[r-1,j+1]
  desc[r,]<-desc[r,]/sum(desc[r,])}
```

with the choice of the parameters $n = 6$, $p = 0.2$ and $a = 0.5$, the same as in the previous example. When a closer look at the matrix `desc` is taken, one can notice that the descendant $\{P_x^{(1265)}\}_{x=0}^{6}$ is equal to all of the following descendants, hence, we consider the 1265th descendant to be the limit distribution. Its probabilities are

$$\{P_x^{(1265)}\}_{x=0}^{6} =$$

$$= (0.6410500, 0.2958692, 0.0568979, 0.0058357, 0.0003367, 0.0000104, 0.0000001). \tag{3.1}$$

The situation is depicted in Figure 3.2. The blue colour is again reserved for the original parent - the binomial distribution, green colours represent descendant distributions and black is for the limit distribution.

For a few other choices of parameter values or the original parent see Appendix A. The limit distributions are not deterministic, in contrast to the Salvia-Bolinger partial summation.

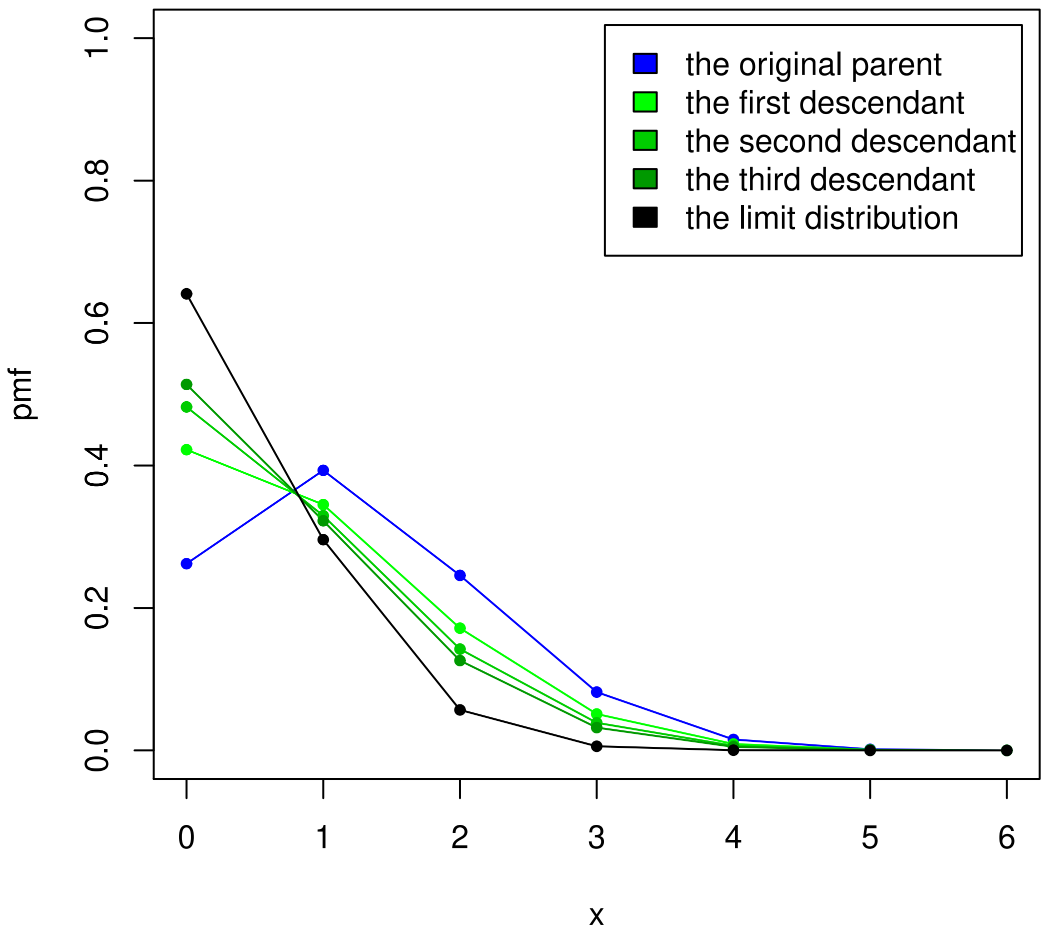

Figure 3.2: *Descendants of the binomial distribution under the iterated Poisson partial summations*

3.2 Power method

In this section we present analysis of the power method's use in the field of iterated partial summations, which is also the scope of our recently published paper Koščová et al. (2020).

Let $\{P_j^*\}_{j=0}^{\infty}$ and $\{P_j\}_{j=0}^{\infty}$ be discrete probability distributions defined on the set of non-negative integers. A general form of partial-sums distributions was introduced in Mačutek (2003). The distribution $\{P_j\}_{j=0}^{\infty}$ is the result of a partial summation applied to $\{P_j^*\}_{j=0}^{\infty}$ if

$$P_x = c \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0, 1, 2, \dots, \tag{3.2}$$

where $g(j)$ is a real function and c a normalization constant (which ensures that the sequence $\{P_j\}_{j=0}^{\infty}$ is a proper probability distribution, i.e., it sums to 1). The distributions $\{P_j^*\}_{j=0}^{\infty}$ and $\{P_j\}_{j=0}^{\infty}$ are called parent and descendant, respectively. Some special cases of (3.2) are mentioned also in the comprehensive monograph on discrete distributions Johnson et al. (2005), pp. 508-512.

Hereafter, we restrict our considerations to parent distributions

$$\{P_0^*, P_1^*, P_2^*, \ldots, P_{S-1}^*\},$$

i.e. to discrete distributions defined on a finite support of the size S. As the probabilities P_j^* in (3.2) are zero for $j \geq S$ in this case, the partial summation (3.2) can be written as

$$P_x = c\sum_{j=x}^{S-1} g(j)P_j^*, \qquad x = 0, 1, \ldots, S-1,$$

or equivalently as

$$\mathbb{P} = cA\mathbb{P}^*, \tag{3.3}$$

where $\mathbb{P}$ and $\mathbb{P}^*$ are the vectors of probabilities

$$(P_0, P_1, \ldots, P_{S-1})^\top$$

and

$$(P_0^*, P_1^*, \ldots, P_{S-1}^*)^\top,$$

respectively. Matrix A is of dimension $S \times S$, with the following structure:

$$A = \begin{pmatrix} g(0) & g(1) & g(2) & g(3) & \ldots & g(S-1) \\ 0 & g(1) & g(2) & g(3) & \ldots & g(S-1) \\ 0 & 0 & g(2) & g(3) & \ldots & g(S-1) \\ 0 & 0 & 0 & g(3) & \ldots & g(S-1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & g(S-1) \end{pmatrix}. \tag{3.4}$$

The partial summation (3.2) can be applied iteratively as we describe in the beginning of this chapter, however here we are restricted to finite-support parents only. The descendant distribution becomes a parent of another distribution, i.e.

$$P_x^{(1)} = c_1 \sum_{j=x}^{S-1} g(j) P_j^*, \qquad x = 0, 1, \ldots, S-1 \ ,$$

$$P_x^{(2)} = c_2 \sum_{j=x}^{S-1} g(j) P_j^{(1)}, \qquad x = 0, 1, \ldots, S-1 \ ,$$

$$\vdots$$

$$P_x^{(n)} = c_n \sum_{j=x}^{S-1} g(j) P_j^{(n-1)}, \qquad x = 0, 1, \ldots, S-1 \ ,$$

$$\vdots$$

with c_i, $i = 1, 2, 3, \ldots$ being normalization constants. The distribution

$\{P_x^*\}_{x=0}^{S-1}$

will be called the original parent. We will now investigate properties of the sequence of the descendant distributions, especially the question under which conditions the limit of this sequence exists. We remind that he existence of the limit for iterated partial summations applied to discrete distributions with infinite supports for a constant function $g(j)$ was proved in Mačutek (2006).

In the following, we will not consider the normalization constants. Then the matrix notation (see (3.3)) of the iterated partial summations is

$$\mathbb{Q}^{(1)} = A\mathbb{P}^*,$$

$$\mathbb{Q}^{(2)} = A\mathbb{Q}^{(1)} = AA\mathbb{P}^* = A^2\mathbb{P}^*,$$

$$\mathbb{Q}^{(3)} = A\mathbb{Q}^{(2)} = AA^2\mathbb{P}^* = A^3\mathbb{P}^*,$$

$$\vdots$$

$$\mathbb{Q}^{(n)} = A\mathbb{Q}^{(n-1)} = A^n\mathbb{P}^*,$$

$$\vdots$$

The i-th descendant probability distribution can be obtained by the normalization of the vector $\mathbb{Q}^{(i)} = (Q_0^{(i)}, Q_1^{(i)}, \ldots, Q_{S-1}^{(i)})^\top$.

Denote $\|U\|_1$, $\|U\|_2$ the L1-norm and the L2-norm of vector U, respectively. If the limit of the sequence of the descendant distributions exists, it can be written as

$$\mathbb{P}^{(\infty)} = \lim_{n\to\infty} \frac{\mathbb{Q}^{(n)}}{\|\mathbb{Q}^{(n)}\|_1} = \lim_{n\to\infty} \frac{A^n\mathbb{P}^*}{\|A^n\mathbb{P}^*\|_1}. \tag{3.5}$$

The power method is one of computational approaches to the problem of finding matrix eigenvalues (see e.g. Golub and Van Loan (1996), pp. 330-332). We apply it to matrix A (denote its eigenvalues by $\lambda_0, \ldots, \lambda_{S-1}$; we remind that in general they need not be distinct) and vector $\mathbb{P}^*$ from (3.3). To satisfy the conditions of the method, suppose that A is diagonalizable and that it has a unique dominant eigenvalue λ_k (i.e., there exists k such that $|\lambda_k| > |\lambda_i|$, $i \neq k$).

If $\mathbb{P}^*$ is not a non-dominant eigenvector of A and, in addition, if $\mathbb{P}^*$ is such a linear combination of the eigenvectors of A that the coefficient corresponding to the dominant eigenvector is non-zero, then

$$\lim_{n\to\infty} \frac{A^n\mathbb{P}^*}{\|A^n\mathbb{P}^*\|_2} = V,$$

where V is the dominant eigenvector of A (i.e., the one which corresponds to the dominant eigenvalue). Under these conditions, the power method implies the existence of $\lim_{n\to\infty} \mathbb{P}^{(n)}$, see (3.5), with

$$\mathbb{P}^{(\infty)} = \lim_{n\to\infty} \mathbb{P}^{(n)} = \frac{V}{\|V\|_1}.$$

The matrix A from (3.4) is an upper triangular matrix, which means that its eigenvalues are its diagonal entries, i.e.

$$\lambda_j = g(j), \qquad j = 0, 1, \ldots, S-1.$$

Consequently, to determine the dominant eigenvalue λ_D of A it is necessary to find

$$D = \arg \max_{j\in\{0,1,\ldots,S-1\}} |g(j)|.$$

Let $D = k$, i.e., let the dominant eigenvalue be $\lambda_k = g(k)$. The eigenvector corresponding to the dominant eigenvalue λ_k is the solution of the of linear equation

$$AV = \lambda_k V,$$

or, equivalently,

$$(A - \lambda_k I)V = 0.$$

For matrix A from (3.3) we obtain

$$\begin{pmatrix} g(0)-g(k) & g(1) & g(2) & \dots & g(k-1) & g(k) & g(k+1) & \dots & g(S-2) & g(S-1) \\ 0 & g(1)-g(k) & g(2) & \dots & g(k-1) & g(k) & g(k+1) & \dots & g(S-2) & g(S-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & g(k-1)-g(k) & g(k) & g(k+1) & \dots & g(S-2) & g(S-1) \\ 0 & 0 & 0 & \dots & 0 & 0 & g(k+1) & \dots & g(S-2) & g(S-1) \\ 0 & 0 & 0 & \dots & 0 & 0 & g(k+1)-g(k) & \dots & g(S-2) & g(S-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & g(S-2)-g(k) & g(S-1) \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & g(S-1)-g(k) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{k-1} \\ v_k \\ v_{k+1} \\ \vdots \\ v_{S-2} \\ v_{S-1} \end{pmatrix} = 0.$$

This system of linear equations yields the solution

$$V = \begin{pmatrix} t \\ t\left(1-\frac{g(0)}{g(k)}\right) \\ t\left(1-\frac{g(0)}{g(k)}\right)\left(1-\frac{g(1)}{g(k)}\right) \\ \vdots \\ t\prod_{j=1}^{k}\left(1-\frac{g(j-1)}{g(k)}\right) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}. \tag{3.6}$$

3.3 Demonstration of the power method's use

We present a simple example of finding the limit distribution $\mathbb{P}^{(\infty)}$ for iterated Poisson partial summation when the original parent $\mathbb{P}^*$ is of the support size $S = 2$. The iterated Poisson partial summation can be written as

$$P_x^{(1)} = c_1 \sum_{j=x}^{1} \left(1 - \frac{a}{j+1}\right) P_j^*, \qquad x = 0, 1\ ,$$

$$P_x^{(2)} = c_2 \sum_{j=x}^{1} \left(1 - \frac{a}{j+1}\right) P_j^{(1)}, \qquad x = 0, 1\ ,$$

$$\vdots$$

$$P_x^{(n)} = c_n \sum_{j=x}^{1} \left(1 - \frac{a}{j+1}\right) P_j^{(n-1)}, \qquad x = 0, 1\ ,$$

$$\vdots$$

where a is the parameter of the Poisson distribution. In this example the matrix A is

$$A = \begin{pmatrix} 1-a & 1-\frac{a}{2} \\ 0 & 1-\frac{a}{2} \end{pmatrix}$$

and its eigenvalues are $\lambda_1 = 1-a$, $\lambda_2 = 1-\frac{a}{2}$. For $a > \frac{4}{3}$, λ_1 is the dominant eigenvalue, while for $a < \frac{4}{3}$, λ_2 is dominant.

Consider $a > \frac{4}{3}$. The aim is to find the eigenvector V correspondent to the dominant eigenvalue, which is now $\lambda_D = \lambda_1 = 1 - a$,

$$(A - \lambda_1 I)V = 0,$$

$$\left(\begin{pmatrix} 1-a & 1-\frac{a}{2} \\ 0 & 1-\frac{a}{2} \end{pmatrix} - \begin{pmatrix} 1-a & 0 \\ 0 & 1-a \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

$$\begin{pmatrix} 0 & 1-\frac{a}{2} \\ 0 & \frac{a}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

$$\begin{pmatrix} \left(1-\frac{a}{2}\right) v_2 \\ \frac{a}{2} v_2 \end{pmatrix} = 0,$$

therefore

$$V = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Under the conditions that both elements of the original parent are non-zero and the original parent is not equal to the non-dominant eigenvector of A, the distribution $\mathbb{P}^{(\infty)}$ exists and it holds

$$\mathbb{P}^{(\infty)} = \lim_{n\to\infty} \mathbb{P}^{(n)} = \frac{V}{\|V\|_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The consequences of using power method while the mentioned conditions are not fulfilled are discussed later in this chapter.

Now let $a < \frac{4}{3}$. The dominant eigenvalue in this case is $\lambda_D = \lambda_2 = 1 - \frac{a}{2}$ and for its correspondent eigenvector it holds

$$(A - \lambda_2 I)W = 0,$$

$$\left(\begin{pmatrix} 1-a & 1-\frac{a}{2} \\ 0 & 1-\frac{a}{2} \end{pmatrix} - \begin{pmatrix} 1-\frac{a}{2} & 0 \\ 0 & 1-\frac{a}{2} \end{pmatrix}\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0,$$

$$\begin{pmatrix} -\frac{a}{2} & 1-\frac{a}{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0,$$

$$-\frac{a}{2} w_1 + \left(1 - \frac{a}{2}\right) w_2 = 0,$$

$$w_2 = \frac{a}{2-a} w_1,$$

which means

$$W = \begin{pmatrix} t \\ \frac{a}{2-a} t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Again, under the conditions that both elements of the original parent are non-zero and the original parent is not equal to the non-dominant eigenvector of A, the $\mathbb{P}^{(\infty)}$ exists, with

$$\mathbb{P}^{(\infty)} = \lim_{n \to \infty} \mathbb{P}^{(n)} = \frac{W}{\|W\|_1} = \frac{W}{w_1 + w_2} = \begin{pmatrix} \frac{2-a}{2} \\ \frac{a}{2} \end{pmatrix}.$$

If $a = \frac{4}{3}$, then $\lambda_1 = \lambda_2$. It means that the dominant eigenvalue of A is not unique. Hence the power method does not work and cannot be used to find the limit distribution.

3.4 Katz family

Discrete distribution $\{R_j^*\}_{j=0}^{n/\infty}$ belongs to the Katz family (see e.g. Wimmer and Altmann (1999), pp. 324-325) with the parameters $\alpha \geq 0$, $\beta < 1$ if

$$\frac{R_{x+1}}{R_x} = \frac{\alpha + \beta x}{x+1}, \qquad x = 0, 1, 2, \dots .$$

When $\alpha + \beta n < 0$, $n \in \{1, 2, \dots\}$ then $R_{n+j} = 0$, for $j = 1, 2, \dots$. The Katz partial summation, i.e. the summation

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0, 1, 2, \dots$$

with

$$g(j) = 1 - \frac{R_{j+1}}{R_j} = 1 - \frac{\alpha + \beta j}{j+1} = \frac{(1-\alpha) + (1-\beta) j}{j+1}, \qquad j = 0, 1, 2, \dots \tag{3.7}$$

was analyzed in Mačutek (2001).

Consider a finite-support discrete distribution

$$\{P_0^*, P_1^*, P_2^*, \dots, P_{S-1}^*\}.$$

As for function $g(j)$ from (3.7) it holds

$$\frac{\partial g(j)}{\partial j} = \frac{\alpha - \beta}{(j+1)^2},$$

the function $g(j)$ is increasing if $\alpha > \beta$, constant if $\alpha = \beta$, and decreasing if $\alpha < \beta$. Thus, if $\alpha \neq \beta$, all eigenvalues of matrix A are distinct, which is a sufficient condition for its diagonalizability (see Horn and Johnson (2013)). If, in addition, $|g(0)| \neq |g(S-1)|$, there exists the unique dominant eigenvalue and the power method can be applied. The dominant eigenvector V from (3.6) can be expressed as

$$V = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_{S-1} \end{pmatrix} = \begin{pmatrix} t \\ t\frac{\alpha-\beta}{(1-\alpha)+(1-\beta)k}k \\ t\left(\frac{\alpha-\beta}{(1-\alpha)+(1-\beta)k}\right)^2 \frac{k(k-1)}{2} \\ \vdots \\ t\left(\frac{\alpha-\beta}{(1-\alpha)+(1-\beta)k}\right)^k \frac{k!}{k!} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

We remind that k is the position of the dominant eigenvalue λ_D. We use the parameter t to scale the vector V so that the sum of the vector elements is equal to 1, i.e.

$$t = \left(\frac{(1-\alpha)+(1-\beta)k}{(1-\beta)(k+1)}\right)^k.$$

Therefore,

$$\mathbb{P}^{(\infty)} = \begin{pmatrix} P_0^{(\infty)} \\ P_1^{(\infty)} \\ P_2^{(\infty)} \\ \vdots \\ P_k^{(\infty)} \\ P_{k+1}^{(\infty)} \\ \vdots \\ P_{S-1}^{(\infty)} \end{pmatrix} = \begin{pmatrix} \left(1-\frac{\alpha-\beta}{(1-\beta)(k+1)}\right)^k \\ k\frac{\alpha-\beta}{(1-\beta)(k+1)}\left(1-\frac{\alpha-\beta}{(1-\beta)(k+1)}\right)^{k-1} \\ \binom{k}{2}\left(\frac{\alpha-\beta}{(1-\beta)(k+1)}\right)^2\left(1-\frac{\alpha-\beta}{(1-\beta)(k+1)}\right)^{k-2} \\ \vdots \\ \left(\frac{\alpha-\beta}{(1-\beta)(k+1)}\right)^k \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which means that

$$\mathbb{P}^{(\infty)} \sim Bin\left(k; \frac{\alpha-\beta}{(1-\beta)(k+1)}\right).$$

However, as the function $g(j)$ from (3.7) is strictly monotonic in j if $\alpha \neq \beta$, there are only two possible values of k, either 0 or $S-1$. If $k = S-1$, the iterated partial summations have the limit which is the binomial distribution with parameters $S-1$

and $\frac{\alpha-\beta}{(1-\beta)S}$. On the other hand, if $k = 0$, the distribution $\mathbb{P}^{(\infty)}$ degenerates to the deterministic distribution.

To inspect whether k is equal to 0 or to $S-1$ for a particular choice of parameters of the Katz family - i.e. for the parameters which appear in function $g(j)$ from (3.7) - it is sufficient to compare the values of $|g(0)|$ and $|g(S-1)|$, which means to analyze the inequalities

$$|1-\alpha| \lesseqqgtr \left|\frac{(1-\alpha)+(1-\beta)(S-1)}{S}\right|. \tag{3.8}$$

(A) If $\alpha = \beta$, the conditions under which the power method can be applied are not satisfied; however, this case is solved in Mačutek (2006)[1].

(B) If $\alpha < \beta$, the limit distribution is deterministic.

(C) If $\alpha > \beta$

  (1) and if $\alpha \leq 1$, the limit distribution is binomial,

  (2) while if $\alpha > 1$, additional conditions come into play:

    (a) If $\alpha + \beta - 2 \geq 0$, the limit distribution is deterministic.

    (b) If $\alpha + \beta - 2 < 0$, the limit behaviour depends on the support size S.

      (i) If $\beta > \frac{2S-(S+1)\alpha}{S-1}$, the limit distribution is deterministic, while

      (ii) for $\beta < \frac{2S-(S+1)\alpha}{S-1}$ the iterated summations converge to the binomial distribution.

      (iii) In case $\beta = \frac{2S-(S+1)\alpha}{S-1}$, the power method cannot be used (the dominant eigenvalue of the matrix A is not unique).

Figure 3.3 depicts the parametric space of the Katz partial summations. The iterated Katz partial summations with parameters from the dark grey area (triangle with vertices in (0,0), (1,1), (0,1) and the area above the half-line $\beta = 2-\alpha$ and below the half-line $\beta = 1$) result in the deterministic distribution. Those with parameters from the light grey area result in the binomial distribution

$$Bin\left(S-1; \frac{\alpha-\beta}{S(1-\beta)}\right).$$

[1]The power method cannot be applied if $\alpha = \beta$; however, it was shown in Mačutek (2006) that in this particular summation the sequence of the descendant distributions converges to the geometric distribution for a wide family of original parents. Specifically, if the original parent is a distribution with a finite support, the limit distribution is deterministic (which can be considered a special case of the geometric distribution).

We remind that these results are valid regardless of the original parent distribution

$$\{P_0^*, P_1^*, P_2^*, \ldots, P_{S-1}^*\}$$

except that it is a non-dominant eigenvector of A. The limit behaviour of the iterated Katz partial summations with parameters from the dotted area depends also on the size of support S. If

$$\beta = \frac{2S - (S+1)\alpha}{S-1}, \tag{3.9}$$

the existence of the limit distribution remains an open question (as the power method cannot be applied in this case). Figures 3.4 and 3.5 show the situation for fixed support sizes $S = 4$ and $S = 50$, respectively. The white line represents the cases where the power method cannot be applied (equation (3.9) is true).

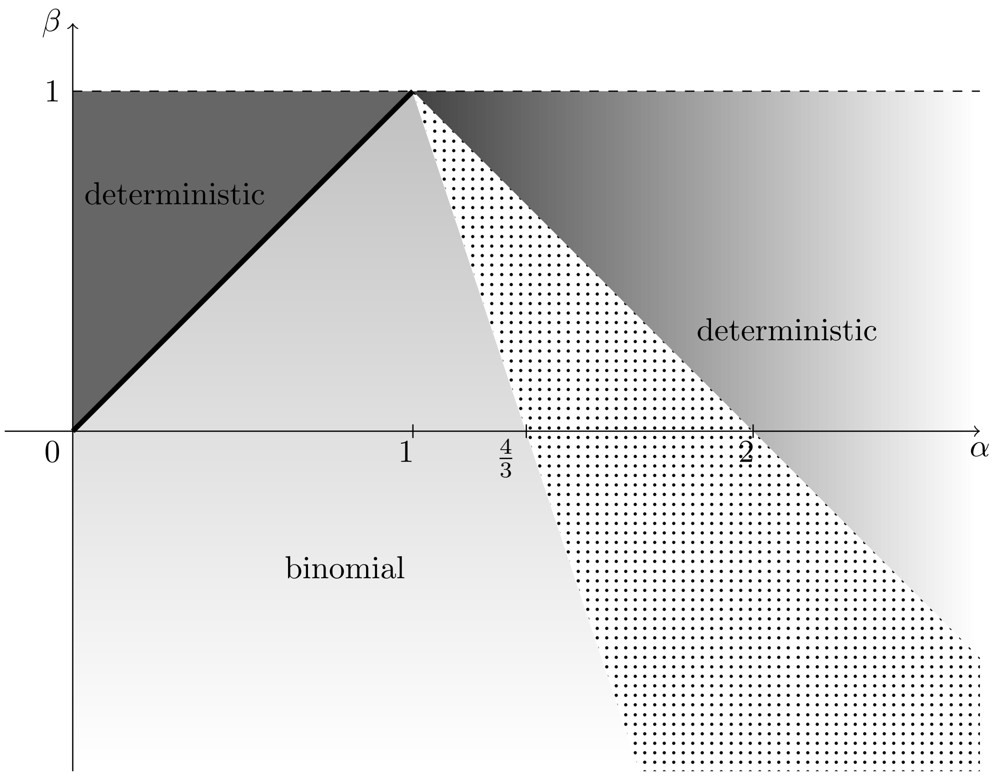

Figure 3.3: *The parametric space of the Katz family*

In the next four subsections, we present detailed results of finding the limit distribution for particular types of iterated partial summation from the Katz family of discrete distributions. We focus on summations related to the most common distributions, namely iterated Poisson partial summation, iterated binomial partial summation, iterated negative binomial partial summation and iterated geometric partial summation.

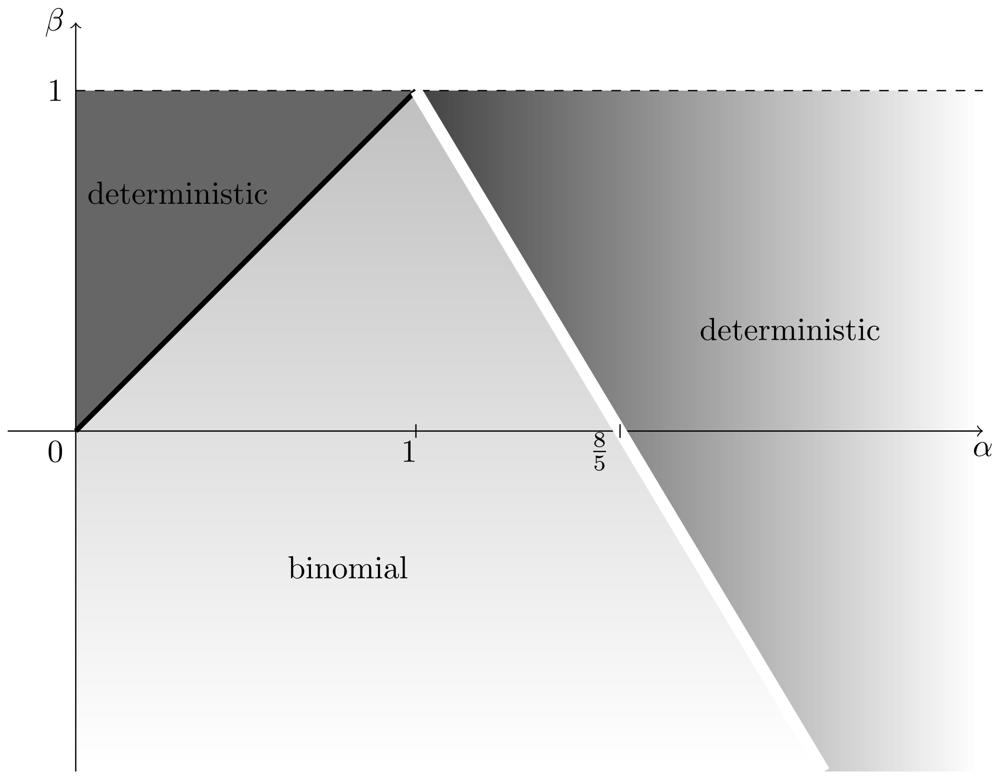

Figure 3.4: *The parametric space of the Katz family,* $S = 4$

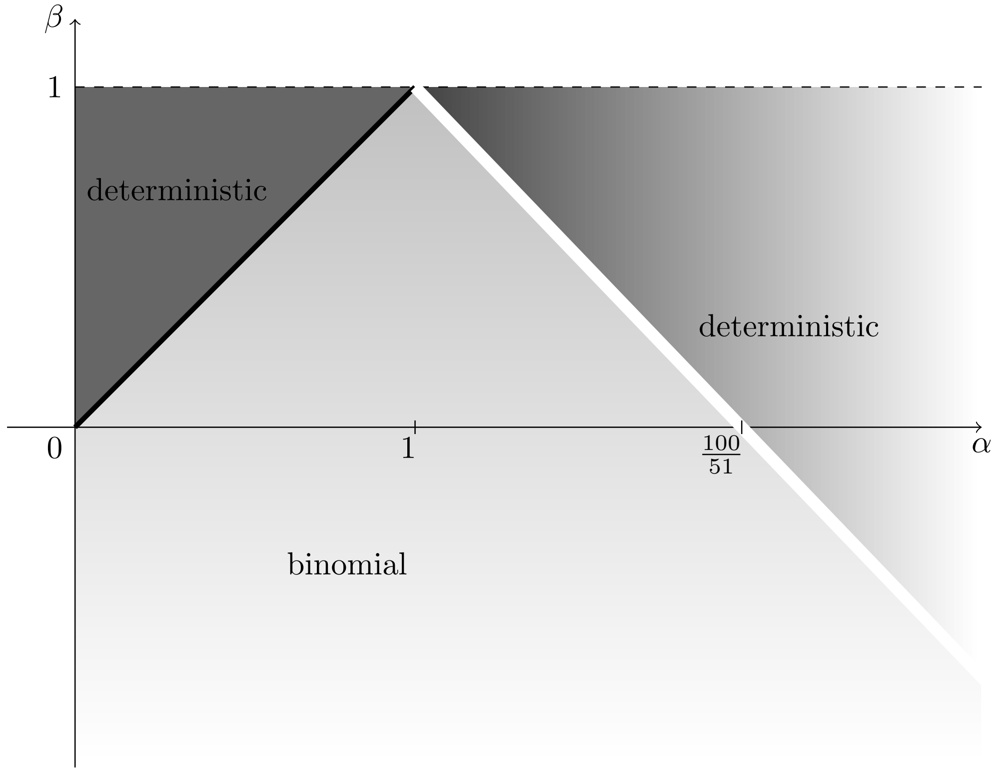

Figure 3.5: *The parametric space of the Katz family,* $S = 50$

3.4.1 Iterated Poisson partial summation

The Poisson distribution with

$$Q_x = \frac{e^{-a}a^x}{x!}, \qquad a \geq 0, \quad x = 0, 1, 2, \dots$$

belongs to the Katz family of discrete distributions for $\alpha = a$ and $\beta = 0$. The function $g(\cdot)$ corresponding to Poisson partial summation is

$$g(j) = 1 - \frac{Q_{j+1}}{Q_j} = 1 - \frac{a}{j+1}, \qquad j = 0, 1, 2, \dots \ .$$

The iterated Poisson partial summation is then

$$P_x^{(1)} = c_1 \sum_{j=x}^{S-1} \left(1 - \frac{a}{j+1}\right) P_j^*, \qquad x = 0, 1, \dots, S-1 \ ,$$

$$P_x^{(n)} = c_n \sum_{j=x}^{S-1} \left(1 - \frac{a}{j+1}\right) P_j^{(n-1)}, \qquad x = 0, 1, \dots, S-1 \ , \qquad n = 2, 3, \dots \ .$$

According to the results from the previous section,

(1) if $a \leq 1$, the limit distribution is binomial with parameters $S-1$ and $\dfrac{a}{S}$,

(2) if $a \geq 2$, the limit distribution is deterministic,

(3) if $1 < a < 2$, the limit behaviour depends on the relation between a and the support size S:

   (i) if $a < \dfrac{2S}{S+1}$, the limit distribution is binomial with parameters $S-1$ and $\dfrac{a}{S}$,

   (ii) if $a > \dfrac{2S}{S+1}$, the limit distribution is deterministic,

   (iii) if $a = \dfrac{2S}{S+1}$, the power method cannot be used as the dominant eigenvalue of the matrix A is not unique.

We can inspect the results obtained by computational study performed in Chapter 3.1. The parameter of the Poisson partial summation was set to $a = 0.5$ and the original parent is binomial distribution with $n = 6$ and $p = 0.2$. The support size of the original parent is thus $S = 7$. As $a = 0.5 \leq 1$, the limit distribution is $Bin(S-1, \frac{a}{S})$. The probability mass function of $Bin(6, \frac{0.5}{7})$ is

$$\{P_x\}_{x=0}^{6} = \binom{6}{x}\left(\frac{0.5}{7}\right)^x\left(1-\frac{0.5}{7}\right)^{6-x} \doteq$$

$$\doteq (0.6410500, 0.2958692, 0.0568979, 0.0058357, 0.0003367, 0.0000104, 0.0000001),$$

which corresponds to (3.1).

3.4.2 Iterated binomial partial summation

The binomial distribution with parameters p, N and the probability mass function

$$Q_x = \binom{N}{x} p^x (1-p)^{N-x}, \quad 0 \le p \le 1, \quad N = 0, 1, 2, \dots, \quad x = 0, 1, \dots, N$$

is another member of the Katz family of discrete distributions. It is the case when $\alpha = \frac{Np}{1-p}$ and $\beta = -\frac{p}{1-p}$. The function $g(\cdot)$ defining the type of partial summation is now

$$g(j) = 1 - \frac{\frac{p}{1-p}(N-j)}{j+1}, \quad j = 0, 1, \dots, N$$

and we define

$$g(j) = 0, \quad j = N+1, N+2, N+3, \dots .$$

Let us denote $M = \min\{S-1, N\}$. The iterated binomial partial summation is

$$P_x^{(1)} = c_1 \sum_{j=x}^{M} \left(1 - \frac{\frac{p}{1-p}(N-j)}{j+1}\right) P_j^*, \qquad x = 0, 1, \dots, M,$$

$$P_x^{(n)} = c_n \sum_{j=x}^{M} \left(1 - \frac{\frac{p}{1-p}(N-j)}{j+1}\right) P_j^{(n-1)}, \qquad x = 0, 1, \dots, M, \qquad n = 2, 3, \dots .$$

It leads to the following results.

(1) If $p \le \dfrac{1}{N+1}$, the limit distribution is binomial with the parameters M and $\dfrac{p(N+1)}{M+1}$.

(2) If $p > \dfrac{1}{N+1}$, additional conditions come into consideration:

(a) If $p \geq \dfrac{2}{N+1}$, the limit distribution is deterministic.

(b) If $p < \dfrac{2}{N+1}$, the limit behaviour depends on the support size S:

   (i) If $S < \dfrac{p(N+1)}{2-p(N+1)}$, the limit distribution is deterministic, while

   (ii) for $S > \dfrac{p(N+1)}{2-p(N+1)}$ the iterated summations converge to the binomial distribution with the parameters M and $\dfrac{p(N+1)}{M+1}$.

   (iii) In case $S = \dfrac{p(N+1)}{2-p(N+1)}$, the power method cannot be used (the dominant eigenvalue of the matrix A is not unique).

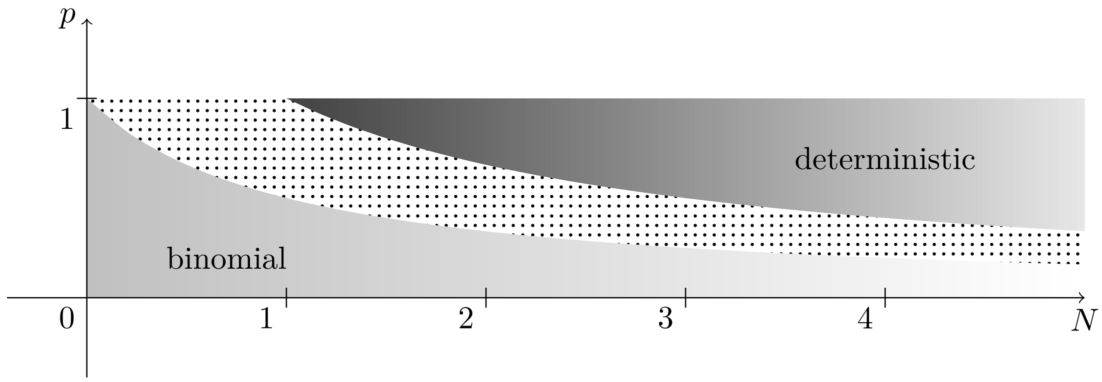

Figure 3.6: *The parametric space of the binomial distribution*

Figure 3.6 depicts the parametric space of the binomial partial summations. The iterated binomial partial summations with parameters from the dark grey area (the area above curve $p = \frac{2}{n+1}$) result in the deterministic distribution. Those with parameters from the light grey area (the area below curve $p = \frac{1}{n+1}$) result in the binomial distribution

$$Bin\left(M; \frac{p(N+1)}{M+1}\right).$$

We remind that these results are valid regardless of the original parent distribution

$$\{P_0^*, P_1^*, P_2^*, \ldots, P_{S-1}^*\}$$

except that it is a non-dominant eigenvector of A. The limit behaviour of the iterated binomial partial summations with parameters from the dotted area (between curves $p = \frac{1}{n+1}$ and $p = \frac{2}{n+1}$) depends also on the size of support S. If

$$S = \frac{p(N+1)}{2-p(N+1)}, \tag{3.10}$$

the existence of the limit distribution remains an open question (as the power method cannot be applied in this case). However, only vertical cross-sections of this parametric space are relevant because the parameter N of binomial distribution is a natural number.

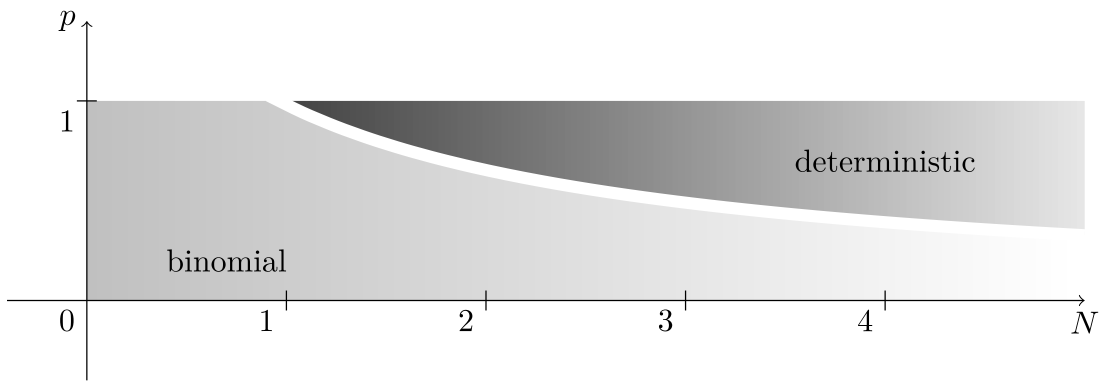

Figure 3.7: *The parametric space of the binomial distribution, S=50*

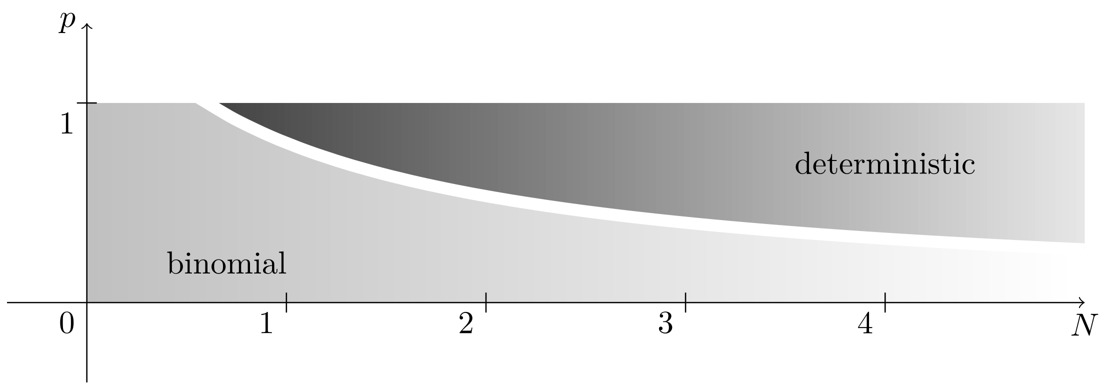

Figure 3.8: *The parametric space of the binomial distribution, S=4*

Figures 3.7 and 3.8 show the situation for fixed support sizes $S = 50$ and $S = 4$, respectively. The white line represents the cases where the power method cannot be applied (equation (3.10) is true).

3.4.3 Iterated negative binomial partial summation

The next member of the Katz family of discrete distributions is the negative binomial distribution with the probability mass function

$$Q_x = \binom{k+x-1}{x} p^k (1-p)^x, \qquad 0 < p \leq 1, \quad k \geq 0, \quad x = 0, 1, 2, \dots \; .$$

Its parameters $k \geq 0$ and $0 < p \leq 1$ are related to α and β as $\alpha = (1-p)k$ and $\beta = 1-p$. The partial summation under examination defined by

$$g(j) = 1 - \frac{Q_{j+1}}{Q_j} = \frac{1-(1-p)k+pj}{j+1}$$

is then of the form

$$P_x^{(1)} = c_1 \sum_{j=x}^{S-1} \frac{1-(1-p)k+pj}{j+1} P_j^*, \quad x = 0,1,\ldots,S-1\ ,$$

$$P_x^{(n)} = c_n \sum_{j=x}^{S-1} \frac{1-(1-p)k+pj}{j+1} P_j^{(n-1)}, \qquad x = 0,1,\ldots,S-1\ , \qquad n = 2,3,\ldots\ .$$

The resulting limit distribution can be determined by the following conditions.

(A) If $k = 1$, the power method is not applicable; however this case is solved in Mačutek (2006).

(B) If $k < 1$, the limit distribution is deterministic.

(C) If $k > 1$ and

  (1) if $p \geq 1 - \frac{1}{k}$, the limit distribution is binomial with the parameters $S-1$ and $\frac{(1-p)(k-1)}{Sp}$ while

  (2) if $p < 1 - \frac{1}{k}$, additional conditions come into consideration:

    (a) If $p \leq 1 - \frac{2}{k+1}$, the limit distribution is deterministic.

    (b) If $p > 1 - \frac{2}{k+1}$, the limit behaviour depends on the support size S:

      (i) If $S < \frac{(k-1)(1-p)}{1-k+p+pk}$, the limit distribution is deterministic, while for

      (ii) $S > \frac{(k-1)(1-p)}{1-k+p+pk}$ the iterated summations converge to the binomial distribution with the parameters $S-1$ and $\frac{(1-p)(k-1)}{Sp}$.

      (iii) In case $S = \frac{(k-1)(1-p)}{1-k+p+pk}$, the power method cannot be used (the dominant eigenvalue of the matrix A is not unique).

Figure 3.9 depicts the parametric space of the negative binomial distribution. The iterated negative binomial partial summations with parameters from the dark grey area (square with vertices in (0,0), (1,0), (1,1), (0,1) and the area below curve $p = 1 - \frac{2}{k+1}$)

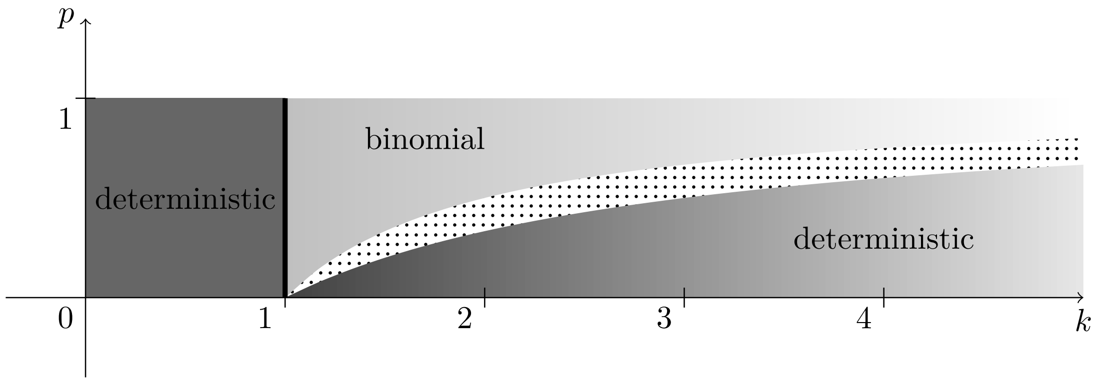

Figure 3.9: *The parametric space of the negative binomial distribution*

result in the deterministic distribution. Those with parameters from the light grey area (above curve $p = 1 - \frac{1}{k}$) result in the binomial distribution

$Bin\left(S-1; \frac{(1-p)(k-1)}{Sp}\right)$.

We remind that these results are valid regardless of the original parent distribution

$\{P_0^*, P_1^*, P_2^*, \ldots, P_{S-1}^*\}$

except that it is a non-dominant eigenvector of A. The limit behaviour of the iterated negative binomial partial summations with parameters from the dotted area (between curves $p = 1 - \frac{2}{k+1}$ and $p = 1 - \frac{1}{k}$) depends also on the size of support S. If

$$S = \frac{(k-1)(1-p)}{1-k+p+pk} \tag{3.11}$$

the existence of the limit distribution remains an open question (as the power method cannot be applied in this case).

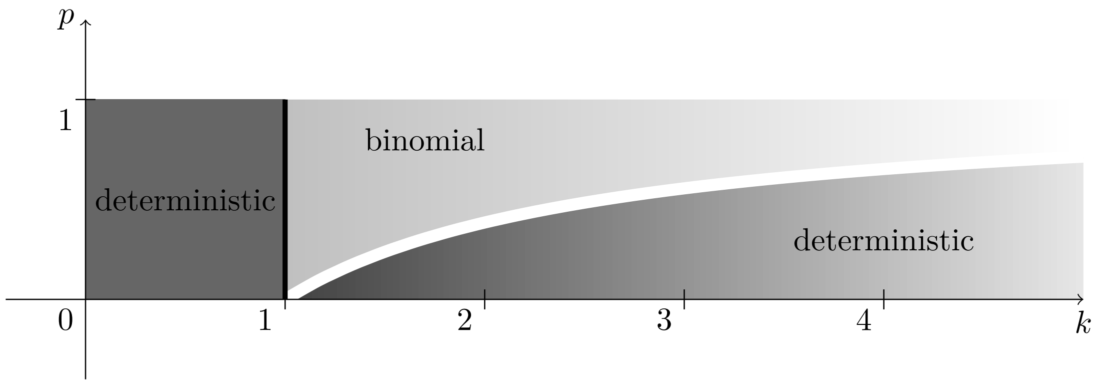

Figure 3.10: *The parametric space of the negative binomial distribution, S=4*

Figures 3.10 and 3.11 show the situation for fixed support sizes $S = 4$ and $S = 500$, respectively. The white line represents the cases where the power method cannot be applied (equation (3.11) is true).

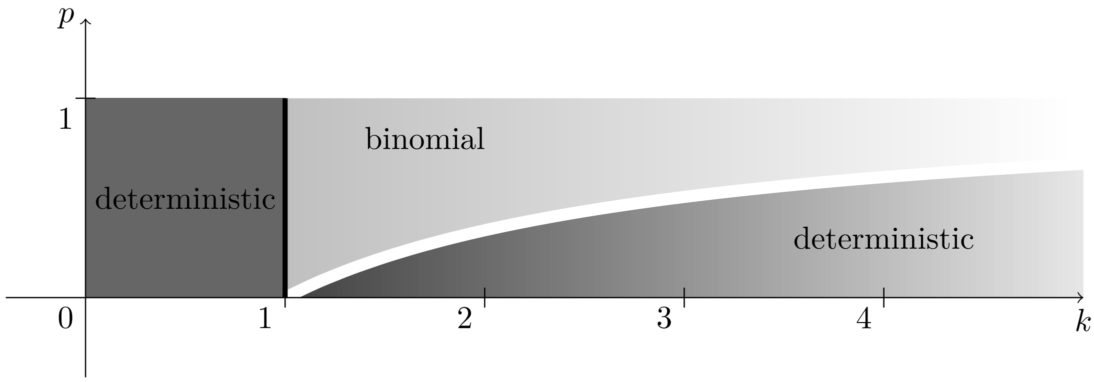

Figure 3.11: *The parametric space of the negative binomial distribution, S=500*

3.4.4 Iterated geometric partial summation

Another frequently used distribution from the Katz family is the geometric distribution. It is the case when $\alpha = 1 - p$ and $\beta = 1 - p$. The probability mass function of the geometric distribution is

$$Q_x = p(1-p)^x, \qquad 0 < p \leq 1, \quad x = 0, 1, 2, \dots \ ,$$

which leads to the function $g(\cdot)$ of a very simple form

$$g(j) = p.$$

The iterated geometric partial summation is then

$$P_x^{(1)} = c_1 \sum_{j=x}^{S-1} p P_j^*, \quad x = 0, 1, \dots, S-1 \ ,$$

$$P_x^{(n)} = c_n \sum_{j=x}^{S-1} p P_j^{(n-1)}, \qquad x = 0, 1, \dots, S-1 \ , \qquad n = 2, 3, \dots \ .$$

As $\alpha = \beta$ here, the power method is not applicable, because the dominant eigenvalue is not unique. It can be seen also in (3.8). As it was already mentioned before, the iterated geometric partial summation is inspected in Mačutek (2006). With the use of probability generating functions Mačutek (2006) proves the limit distribution is deterministic for the original parent with a finite support.

3.5 The trinity of parent, descendant and type of partial summation

In the first Chapter we emphasize that the partial summations consist of three elements, the parent distribution, the descendant distribution and the function $g(\cdot)$. Wimmer and Mačutek (2012) showed that if we fix two of the three elements, the remaining one is uniquely given. According to our results from Chapter 3.4.1, the limit distribution $\mathbb{P}^{(\infty)}$ of the Poisson partial summation is binomial for $a \leq 1$. Hence, we have the elements

1. the parent distribution $\mathbb{P}^{(\infty)}$ is binomial with the parameters $S-1$ and $\dfrac{a}{S}$,
2. the descendant distribution $\mathbb{P}^{(\infty+1)}$ is binomial with the same parameters,
3. the function $g(\cdot)$ corresponds to the Poisson distribution with the parameter a.

On the other hand, the only partial summation that leaves the binomial distribution unaltered should be the binomial partial summation with the exact same parameters, thus the three elements might be also

1. the binomial parent with the parameters $S-1$ and $\dfrac{a}{S}$,
2. the binomial descendant with the same parameters,
3. the function $g(\cdot)$ corresponds to the binomial distribution, again with the same parameters.

This is why the finding of Wimmer and Mačutek (2012) can seem to be false. However, it is not. Consider the function $g(\cdot)$ corresponding to the binomial distribution with the parameters $S-1$ and $\frac{a}{S}$,

$$g_b(x) = 1 - \frac{\binom{S-1}{x+1}\left(\frac{a}{S}\right)^{x+1}\left(1-\frac{a}{S}\right)^{S-2-x}}{\binom{S-1}{x}\left(\frac{a}{S}\right)^{x}\left(1-\frac{a}{S}\right)^{S-1-x}} = \frac{S}{S-a}\left(1-\frac{a}{x+1}\right),$$

and on the other hand the function $g(\cdot)$ corresponding to the Poisson distribution with the parameter a,

$$g_P(x) = 1 - \frac{\frac{e^{-a}a^{x+1}}{(x+1)!}}{\frac{e^{-a}a^{x}}{x!}} = 1 - \frac{a}{x+1}.$$

These functions $g(\cdot)$ differ only by the multiplication constant $\frac{S}{S-a}$. When the Poisson partial summation is applied to the parent with finite support, we have theoretically the infinite number of values of function $g(\cdot)$ available, however, not needed. The resulting descendant then must be normalized in order to obtain a proper probability distribution. When the normalization constant is included in the function $g(\cdot)$, we obtain $g(\cdot)$ corresponding to the binomial distribution in this case.

3.6 Miscellanea

The power method solves problems related to iterated partial summations for a wide spectrum of the partial summation types and of the original parent distributions. However, the power method has its conditions under which it can be applied. They are

1. the dominant eigenvalue must be unique,
2. the original parent must differ from all the non-dominant eigenvectors and
3. the original parent is such a linear combination of the eigenvectors of the matrix A that the coefficient corresponding to the dominant eigenvector is non-zero,

as was also stated in Chapter 3.2. One can notice that the condition number 2 is redundant when the condition 3 is fulfilled. However, it is an interesting special case. We discuss all three conditions in the following subsections, together with some examples. Even though the power method cannot be applied when these conditions are not satisfied, the problem of iterated partial summation might have a solution (i.e., the limit descendant distribution can exist and can be easily identified in some cases).

3.6.1 Non-unique dominant eigenvalue

This condition is the only one which actually causes problems if it is not satisfied. We demonstrate two types of problems on a simple example. Consider the iterated Poisson partial summation (see Subsection 3.1.2) with the parameter $a = \frac{4}{3}$ applied to the original parent with the support size $S = 2$. The corresponding matrix A is

$$A = \begin{pmatrix} 1-a & 1-\frac{a}{2} \\ 0 & 1-\frac{a}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

with eigenvalues 1/3 and -1/3. Its eigenvectors are

$$V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$W = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix},$$

as was shown in Chapter 3.3. Let the original parent be the alternative distribution with the parameter $\frac{1}{2}$,

$$\mathbb{P}^* = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The first descendant $P^{(1)}$ is then

$$\mathbb{P}^{(1)} = c_1 A \mathbb{P}^* = c_1 \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the second descendant yields

$$\mathbb{P}^{(2)} = c_2 A \mathbb{P}^{(1)} = c_2 \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_2 \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

which leads to the oscillation between two distributions, $\mathbb{P}^*$ and $\mathbb{P}^{(1)}$. The limit distribution does not exist in this case. Leššová (2019) brings an insight into the problematic of oscillating iterated partial summations.

Another type of problematic behaviour can be seen if we choose

$$\mathbb{P}^* = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

as the original parent. Trying to identify the first descendant we compute

$$\mathbb{P}^{(1)} = c_1 A \mathbb{P}^* = c_1 \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = c_1 \begin{pmatrix} -\frac{1}{9} \\ \frac{1}{9} \end{pmatrix}$$

and we obtain a vector which is not even a probability distribution (and cannot become one by means of a normalization). Not every original parent is compatible with such a type of partial summation. In the following we consider the condition of the dominant eigenvalue uniqueness to be satisfied.

3.6.2 Original parent identical with a non-dominant eigenvector

Let us apply the iterated Poisson partial summation (see Subsection 3.1.2) to the original parent of the support size $S = 2$. The respective matrix A is then

$$A = \begin{pmatrix} 1-a & 1-\frac{a}{2} \\ 0 & 1-\frac{a}{2} \end{pmatrix}$$

with the eigenvectors

$$V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$W = \begin{pmatrix} \frac{2-a}{2} \\ \frac{a}{2} \end{pmatrix},$$

as was shown in Chapter 3.3. Let the parameter of the Poisson distribution defining the type of partial summation be $a < \frac{4}{3}$, which means that W is the dominant eigenvector. Let the original parent be

$$\mathbb{P}^* = \{P_0^*, P_1^*\} = \{1, 0\} = V.$$

Then

$$\mathbb{P}^{(1)} = c_1 A \mathbb{P}^* = c_1 \begin{pmatrix} 1-a & 1-\frac{a}{2} \\ 0 & 1-\frac{a}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1-a \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbb{P}^*,$$

$$\mathbb{P}^{(2)} = c_2 A \mathbb{P}^{(1)} = c_2 A \mathbb{P}^* = \mathbb{P}^*,$$

etc., c_i being the normalization constant ensuring that $\mathbb{P}^{(i)}$ is a proper probability distribution. This leads to

$$\mathbb{P}^{(\infty)} = \mathbb{P}^* = V,$$

even though the dominant eigenvector of A is W.

In general, if $\mathbb{P}^*$ is an eigenvector of A, the definition of the eigenvector says

$$A\mathbb{P}^* = \lambda \mathbb{P}^*,$$

where λ is the eigenvalue corresponding to eigenvector $\mathbb{P}^*$. The first descendant is then

$$\mathbb{P}^{(1)} = c_1 A \mathbb{P}^* = c_1 \lambda \mathbb{P}^* = \mathbb{P}^*,$$

assuming $c_1 \lambda = 1$, as $\mathbb{P}^*$ is already a proper distribution. All successive descendants are obviously identical with the original parent too.

The power method is not able to find the dominant eigenvector of A if the starting point is equal to some non-dominant eigenvector because in such a case the iterated multiplication by A is not able to move from the given eigenvector. Applying this to iterated partial summations, one cannot automatically claim the dominant eigenvector of A to be the limit distribution, but all eigenvectors of A must be found first. Then it must be checked whether the original parent is among them. If yes, then the limit distribution of iterated partial summation is identical with the original parent distribution. On the other hand, if the original parent is different from all of the eigenvectors of A, then the limit distributions is equal to the dominant eigenvector of A.

3.6.3 Inconvenient linear combination of eigenvectors as the original parent

The original parent $\mathbb{P}^*$ is supposed to be such a linear combination of the eigenvectors of A that the coefficient corresponding to the dominant eigenvector is non-zero. For the purpose of this section let us denote $V_0, V_1, \ldots, V_{S-1}$ the eigenvectors of the matrix A ordered by their dominance, V_0 being the dominant eigenvector. It means that the corresponding eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{S-1}$ are in the relation

$$|\lambda_0| > |\lambda_1| \geq \cdots \geq |\lambda_{S-1}|.$$

Now consider a violation of this condition, which means that the coefficient k_0, by which the dominant eigenvector is multiplied in the linear combination, is equal to zero, i.e.,

$$\mathbb{P}^* = 0 \cdot V_0 + k_1 V_1 + \cdots + k_{S-1} V_{S-1}.$$

For the sake of simplicity, suppose $k_1 \neq 0$ and $|\lambda_1| > |\lambda_2|$. The vector $A^n \mathbb{P}^*$ can be expressed as

$$A^n \mathbb{P}^* = 0 + k_1 A^n V_1 + \cdots + k_{S-1} A^n V_{S-1} = 0 + k_1 \lambda_1^n V_1 + \sum_{j=2}^{S-1} k_j \lambda_j^n V_j.$$

In accordance with Kuttler (2012), for large n, the term $k_1 \lambda_1^n V_1$ determines quite well the direction of the vector $A^n \mathbb{P}^*$. This is because $|\lambda_1| > |\lambda_j|$ for $j \geq 2$ and so for a large n, the sum

$$\sum_{j=2}^{S-1} k_j \lambda_j^n V_j$$

is fairly insignificant. It indicates that the sequence of the descendant distributions converges to the eigenvector V_1 after the normalization. In general, the limit descendant is

$$\mathbb{P}^{(\infty)} = V_L,$$

for

$$L = \arg \max_{i \in \{0,1,\ldots,S-1\}; k_i \neq 0} |\lambda_i|,$$

but only if L is unique. Otherwise we might encounter the same problems as demonstrated in the Chapter 3.6.1.

The situation when

$$\mathbb{P}^* = V_L = 0 \cdot V_0 + \cdots + 0 \cdot V_{L-1} + 1 \cdot V_L + 0 \cdot V_{L+1} + \cdots + 0 \cdot V_{S-1}.$$

is exactly the same as the one analysed in the previous section. The limit descendant here is obviously

$$\mathbb{P}^{(\infty)} = V_L = \mathbb{P}^*.$$

Conclusion

The field of partial summations within the theory of discrete probability distributions offers a wide spectrum of unsolved problems. Our research managed to solve some of them.

First, we bring an answer to the problem of invariance when the partial summation is parametrized. The parametrization of partial summations defines the family of resistant discrete distributions and the family of sensitive discrete distributions. We formulate the sufficient and necessary condition for a distribution to belong to the family of resistant distributions. So far only the geometric and the Salvia-Bolinger distributions are known to belong to this family. We also provide a useful necessary condition for a distribution to belong to the resistant family. It is easy to evaluate it and therefore to identify or to reject other candidates. We attempted to find other resistant distributions from the rich Kemp-Dacey-hypergeometric family. However, all distributions from that family turned out to be sensitive. Finding more distributions from the resistant family, or proving that the resistant family consists of the two above mentioned distributions only, is one of the possible directions of future research in this field.

Second, we suggest an approach to solve the problem of the existence of the limit for iterated partial summation if the original parent has a finite support. Results of our computational study revealed that the iterated application of the Poisson partial summation to a parent with a finite support yields a limit distribution which is not deterministic. This fact encouraged a deeper research on the existence of the limit descendant distribution for finite-supported parent distributions. The idea to reformulate the whole partial summation process as a matrix multiplication allowed us to use the power method. It proves the existence of the limit descendant distribution for a very wide class of parent distributions only with minor exceptions, which we also define. A detailed analysis of the iterated Katz partial summation is provided. The Katz partial summation includes e.g. geometric partial summation, Poisson partial summation, binomial partial summation or negative binomial partial summation, all of which we address. We also identify some specific cases, when the power method cannot be applied. The existence of the limit in these cases remains questionable. In future, these results will be generalized to parent distributions with the infinite support.

References

Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. New York: Dover.

Baker, R. (2019). Creating new distributions using integration and summation by parts. arXiv:1904.01859 [math.ST].

Bissinger, B. H. (1965). A type resisting distribution generated from consideration of an inventory decision model. In Patil, G. P., editor, *Classical and Contagious Discrete Distributions*, pages 15–17. Calcutta: Statistical Publishing House and Oxford: Pergamon Press.

Brookes, B. C. (1977). Theory of the Bradford law. *Journal of Documentation*, 33/3:180–209.

Golub, G. H. and Van Loan, C. F. (1996). *Matrix Computations*. Baltimore, London: The John Hopkins University Press.

Horn, R. A. and Johnson, C. R. (2013). *Matrix Analysis*. Cambridge: Cambridge University Press.

Johnson, N. L., Kemp, A. W., and Kotz, S. (2005). *Univariate Discrete Distributions, 3rd edition*. Hoboken: Wiley.

Johnson, N. L. and Kotz, S. (1970). *Continuous Univariate Distributions I*. New York: Wiley.

Kotz, S. and Johnson, N. L. (1991). A note on renewal (partial sums) distributions for discrete variables. *Statistics & Probability Letters*, 12:229–231.

Koščová, M., Mačutek, J., and Harman, R. (2020). Iterated partial summations applied to finite-support discrete distributions. *Mathematica Slovaca*, 70:489–496.

Kuttler, K. (2012). Elementary linear algebra. https://resources.saylor.org/wwwresources/archived/site/wp-content/uploads/2012/02/Elementary-Linear-Algebra-1-30-11-Kuttler-OTC.pdf.

Leššová, L. (2019). Oscillating sequences of partial-sums discrete probability distributions. In Milošević, B. and Obradović, M., editors, *Proceedings of the 21st European Young Statisticians Meeting*, pages 41–45. Belgrade: Faculty of Mathematics.

Leššová, L. and Mačutek, J. (2020). On the limit behaviour of finite-support bivariate discrete probability distributions under iterated partial summations. *Acta Mathematica Universitatis Comenianae (to appear).* arXiv:1903.03316v1 [math.PR].

Li, S. (2005). On a class of discrete time renewal risk models. *Scandinavian Actuarial Journal*, 4:241–260.

Li, S. and Garrido, J. (2002). On the time value of ruin in the discrete time risk model. *Universidad Carlos III, Departamento de Economía de la Empresa, Business Economics Working Papers.*

Li, S., Lu, Y., and Garrido, J. (2009). A review of discrete-time risk models. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 103/2:321–337.

Lin, X. S. and Willmot, G. E. (1999). Analysis of a defective renewal equation arising in ruin theory. *Insurance: Mathematics and Economics*, 25:63–84.

Mačutek, J. (2001). Katz partial summations family. *Tatra Mountains Mathematical Publications*, 22:143–148.

Mačutek, J. (2002). Discrete probability distributions generated by the generalized STER summation. *Brazilian Journal of Probability and Statistics*, 16:1–6.

Mačutek, J. (2003). On two types of partial summations. *Tatra Mountains Mathematical Publications*, 26:403–410.

Mačutek, J. (2005). Discrete distributions connected by partial summations. *Glottometrics*, 11:45–52.

Mačutek, J. (2006). A limit property of the geometric distribution. *Theory of Probability and Its Applications*, 50/2:316–319.

Mačutek, J. and Altmann, G. (2007). Discrete and continuous modelling in quantitative linguistics. *Journal of Quantitative Linguistics*, 14/1:81–94.

Naranan, S. and Balasubrahmanyan, V. K. (2005). Power laws in statistical linguistics and related systems. In Köhler, R., Altmann, G., and Piotrowski, R. G., editors, *Quantitative Linguistics. An International Handbook*, pages 716–738. Berlin: Mouton de Gruyter.

Salvia, A. A. and Bolinger, R. C. (1982). On discrete hazard functions. *IEEE Transactions on Reliability*, R-31:458–459.

Unnikrishnan Nair, N. and Hitha, N. (1989). Characterization of discrete models by distribution based on their partial sums. *Statistics & Probability Letters*, 8:335–337.

Unnikrishnan Nair, N., Sankaran, P. G., and Preeth, M. (2012). Reliability aspects of discrete equilibrium distributions. *Communications in Statistics - Theory and Methods*, 41/3:500–515.

Willmot, G. (1986). Mixed compound Poisson distribution. *ASTIN Bulletin*, 16:59–79.

Wimmer, G. and Altmann, G. (1999). *Thesaurus of univariate discrete probability distributions*. Essen: STAMM.

Wimmer, G. and Altmann, G. (2000). On the generalization of the STER distribution applied to generalized hypergeometric parents. *Acta Universitatis Palackianae Olomoucensis. Facultas Rerum Naturalium. Mathematica*, 39/1:215–247.

Wimmer, G. and Altmann, G. (2001a). Models of rank-frequency distributions in language and music. In Uhlířová, L., Wimmer, G., Altmann, G., and Köhler, R., editors, *Text as a Linguistic Paradigm: Levels, Constituents, Constructs*, pages 283–294. Trier: WVT.

Wimmer, G. and Altmann, G. (2001b). A new type of partial-sums distributions. *Statistics & Probability Letters*, 52:359–364.

Wimmer, G. and Altmann, G. (2005). Unified derivation of some linguistic laws. In Köhler, R., Altmann, G., and Piotrowski, R. G., editors, *Quantitative Linguistics. An International Handbook*, pages 791–807. Berlin: Mouton de Gruyter.

Wimmer, G. and Kalas, J. (1999). A characterization of the geometric distribution. *Tatra Mountains Mathematical Publications*, 17:325–329.

Wimmer, G. and Mačutek, J. (2012). New integrated view at partial-sums distributions. *Tatra Mountains Mathematical Publications*, 51:183–190.

Xekalaki, E. (1983). A property of the Yule distribution and its applications. *Communications in Statistics - Theory and Methods*, 12/10:1181–1189.

Appendix A - Computational study

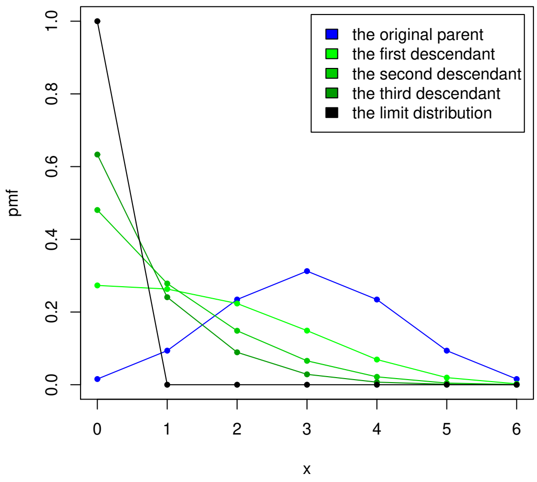

Figure 12: *Descendants of the binomial distribution* $(n = 6,\ p = 0.5)$ *under the iterated Salvia-Bolinger partial summation* $(a = 0.1)$

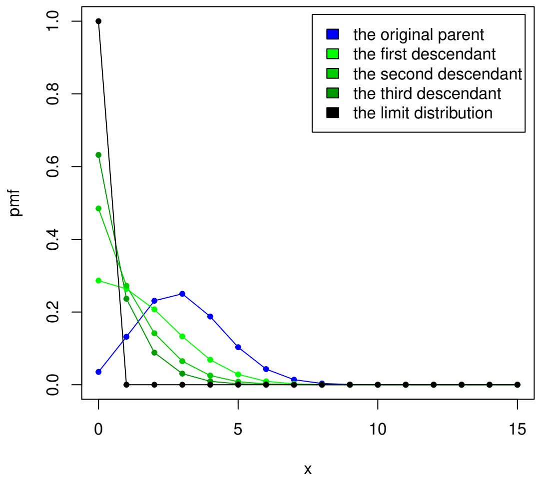

Figure 13: *Descendants of the binomial distribution* $(n = 15,\ p = 0.2)$ *under the iterated Salvia-Bolinger partial summation* $(a = 0.1)$

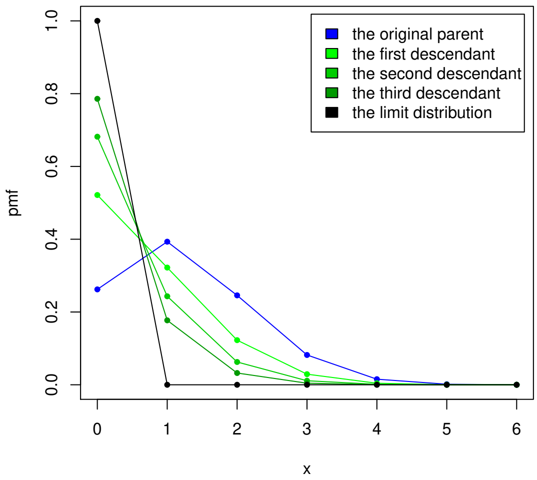

Figure 14: *Descendants of the binomial distribution* $(n = 6,\ p = 0.2)$ *under the iterated Salvia-Bolinger partial summation* $(a = 0.8)$

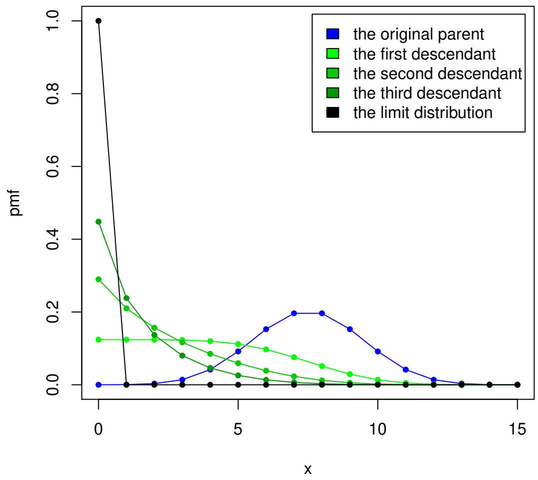

Figure 15: *Descendants of the binomial distribution* $(n = 15,\ p = 0.5)$ *under the iterated Salvia-Bolinger partial summation* $(a = 0.8)$

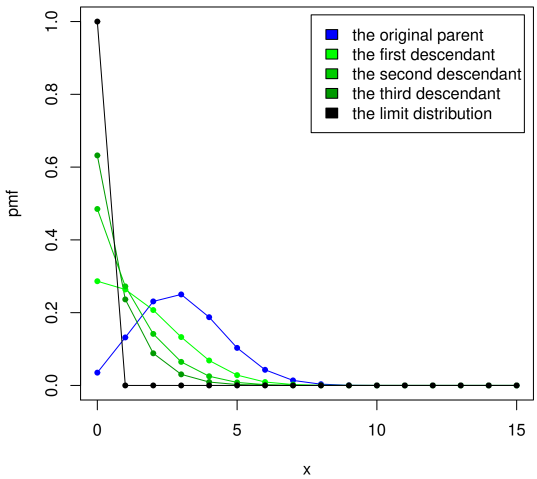

Figure 16: *Descendants of the binomial distribution* $(n = 15,\ p = 0.2)$ *under the iterated Salvia-Bolinger partial summation* $(a = 0.9)$

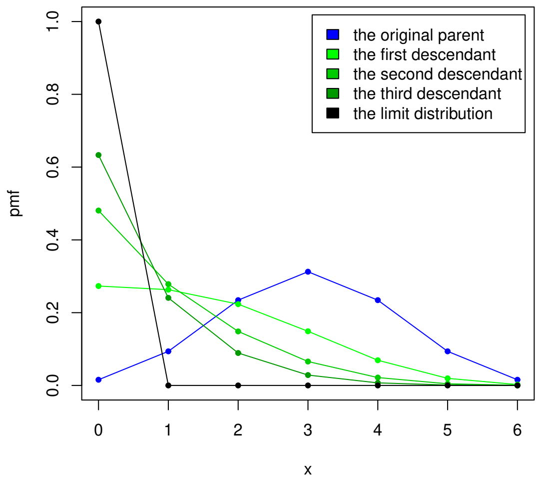

Figure 17: *Descendants of the binomial distribution* $(n = 6,\ p = 0.5)$ *under the iterated Salvia-Bolinger partial summation* $(a = 0.9)$

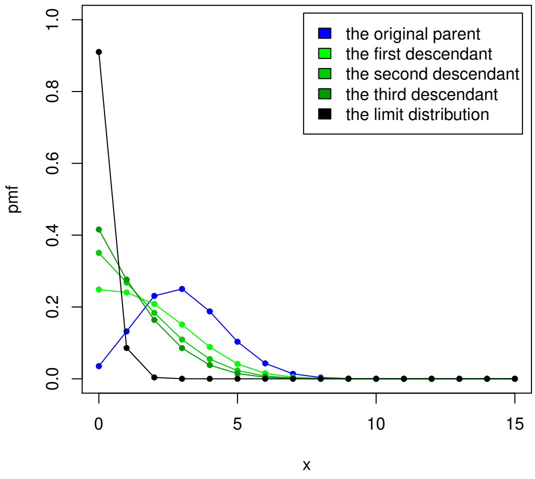

Figure 18: *Descendants of the binomial distribution ($n = 15$, $p = 0.2$) under the iterated Poisson partial summation ($a = 0.1$)*

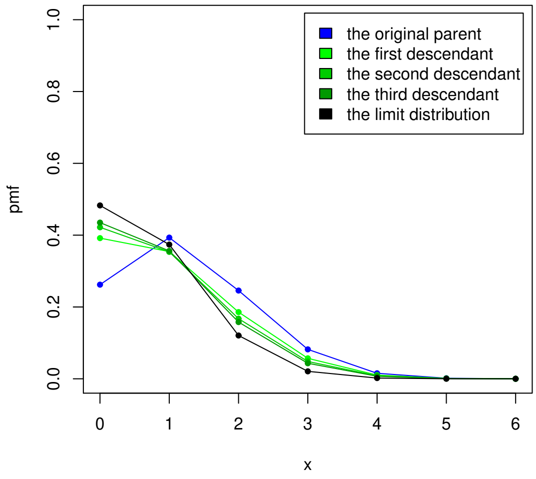

Figure 19: *Descendants of the binomial distribution ($n = 6$, $p = 0.2$) under the iterated Poisson partial summation ($a = 0.8$)*

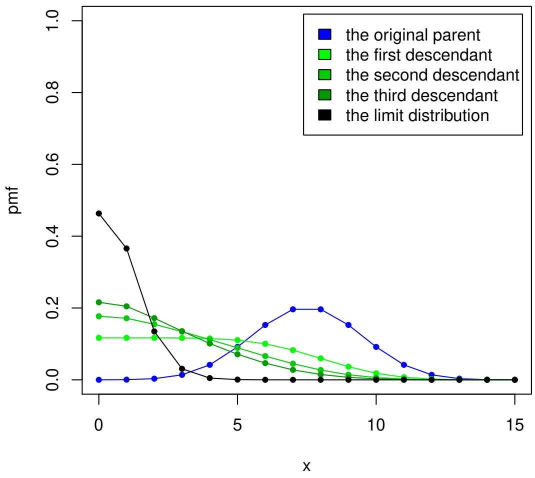

Figure 20: *Descendants of the binomial distribution ($n = 15$, $p = 0.5$) under the iterated Poisson partial summation ($a = 0.8$)*

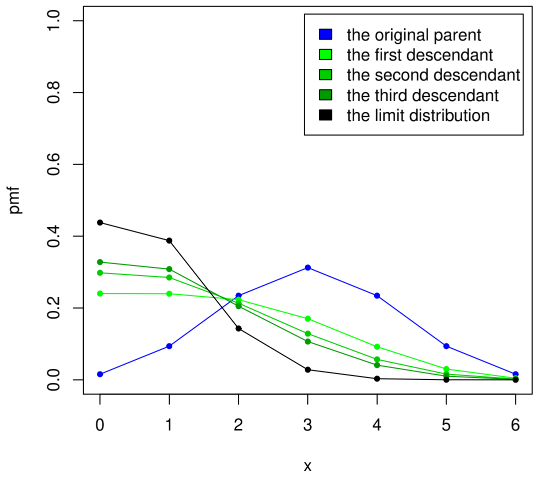

Figure 21: *Descendants of the binomial distribution ($n = 6$, $p = 0.5$) under the iterated Poisson partial summation ($a = 0.9$)*