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Boundedness, a priori estimates and existence of solutions of nonlinear elliptic problems

Dissertation Thesis

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Mgr. Ivana Kosírová

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Abstract.

Consider the semilinear elliptic system $-\Delta u = f(x,u,v)$, $-\Delta v = g(x,u,v)$, $x \in \Omega$, complemented by the homogeneous Dirichlet boundary conditions or by the nonlinear boundary conditions: $\partial_\nu u = \tilde{f}(y,u,v)$, $\partial_\nu v = \tilde{g}(y,u,v)$, $y \in \partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ and ∂_ν denotes the derivative with respect to the outer unit normal ν. In this thesis, we are mainly interested in regularity, boundedness and a priori estimates of very weak solutions of such elliptic systems. In the first part, we improve recent results of Y. Li [32] on L^∞-regularity and a priori estimates for non-negative very weak solutions of elliptic systems complemented by Dirichlet boundary conditions. The proof is based on an alternate-bootstrap procedure in the scale of weighted Lebesgue spaces. In the next part, we show that any positive very weak solution of elliptic problem complemented by the nonlinear boundary conditions belongs to L^∞ provided the functions $f, g, \tilde{f}, \tilde{g}$ satisfy suitable polynomial growth conditions. In addition, all positive solutions are uniformly bounded provided the right-hand sides are bounded in L^1. We also prove that our growth conditions are optimal. Finally, we show that our results remain true for problems involving nonlocal nonlinearities and we use our a priori estimates to prove existence of positive solutions.

Key words and phrases: very weak solutions, elliptic system, a priori estimates, regularity

Abstrakt.

Uvažujme semilineárny eliptický systém $-\Delta u = f(x,u,v)$, $-\Delta v = g(x,u,v)$, $x \in \Omega$, doplnený homogénnymi Dirichletovými okrajovými podmienkami alebo nelineárnymi okrajovými podmienkami : $\partial_\nu u = \tilde{f}(y,u,v)$, $\partial_\nu v = \tilde{g}(y,u,v)$, $y \in \partial\Omega$, kde Ω je hladká ohraničená oblasť v $\mathbb{R}^N$ a ∂_ν označuje deriváciu vzhľadom k vonkajšej jednotkovej normále ν. V tejto práci sa zaoberáme regularitou a apriórnymi odhadmi kladných veľmi slabých riešení takýchto eliptických systémov. V prvej časti práce vylepšíme nedávne výsledky Y. Liho [32] o L^∞-regularite a apriórnych odhadoch pre nezáporné veľmi slabé riešenia eliptických systémov s Dirichletovými okrajovými podmienakmi. Dôkaz je založený na metóde striedavého „bootstrapu" vo vážených Lebesgueových priestoroch. V ďaľšej časti ukážeme, že všetky kladné veľmi slabé riešenia eliptického systému s nelineárnymi okrajovými podmienkami patria do L^∞ pokiaľ funkcie $f, g, \tilde{f}, \tilde{g}$ spĺňajú vhodné polynomiálne rastové podmienky. Naviac, všetky kladné riešenia sú rovnomerne ohraničené pokiaľ sú pravé strany ohraničené v L^1. Taktiež ukážeme, že naše rastové podmienky sú optimálne. Napokon ukážeme, že naše výsledky ostávajú v platnosti aj pre problémy zahŕňajúce nelokálne nelinearity a použijeme naše apriórne odhady na dôkaz existencie kladných riešení.

Kľúčové slová a vety: veľmi slabé riešenia, eliptické systémy, apriórne odhady, regularita

Contents

Introduction

We are interested in existence and a priori estimates of nonnegative very weak solutions of superlinear elliptic systems. Such problems arise in a variety of situations in biology, chemistry or physics.

A priori estimate (also called a priori bound) is an estimate for the size of a solution. "A priori" is a Latin expression which means "from before" and refers to the fact that the estimate for the solution is derived before the solution is known to exist. In this thesis by a priori estimate we mean an estimate guaranteeing that all possible nonnegative solutions (u, v) (in a given set of functions) of an elliptic problem are bounded by some positive constant C independent of (u, v).

A priori estimates play important role in the proof of existence of the solution. Indeed, when problem does not possess variational structure so that variational methods cannot be used, the existence of a solution can be proved using other (for instance topological) methods as far as we can derive a priori estimates for all possible solutions. Moreover, a priori estimates can provide information about multiplicity and bifurcation branches of solutions.

The main aim of this thesis is to prove boundedness and a priori estimates of positive very weak solutions of elliptic systems of the form:

$$\left.\begin{aligned} -\Delta u &= f(.\,, u, v) \\ -\Delta v &= g(.\,, u, v) \end{aligned}\right\} \quad \text{in } \Omega, \tag{1}$$

complemented by Dirichlet boundary conditions:

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \end{aligned}\right\} \quad \text{on } \partial\Omega, \tag{2}$$

or nonlinear boundary conditions:

$$\left.\begin{aligned} \partial_\nu u &= \tilde{f}(.\,, u, v) \\ \partial_\nu v &= \tilde{g}(.\,, u, v) \end{aligned}\right\} \quad \text{on } \partial\Omega, \tag{3}$$

where $f, g, \tilde{f}, \tilde{g}$ are Caratheódory functions satisfying suitable polynomial growth conditions.

There exist various methods which provide a priori estimates of solutions of elliptic problems: The technique called "blow-up" was first introduced by B. Gidas and J. Spruck in [25]. In order to obtain a priori estimates, one can proceed by contradiction by assuming there exists a sequence of solutions which is not bounded. Rescaling argument and choosing suitable subsequence then leads to a subsequence which can be proved to converge to a positive solution of elliptic problem in the whole space (or the half space). The existence of such a solution contradicts to some known Liouville-type theorem. This method often yields optimal results provided optimal Liouville-type theorems exist. In case of system (1), (2), this is often an open problem.

Another method is the method of Rellich-Pohozaev identities and moving planes. This technique was first introduced by D. G. de Figueiredo, P.-L. Lions and R. D. Nussbaum in [21] for the scalar case. In case of system (1), (2), this method proceeds as follows. First, (u, v) is estimated near the boundary of Ω by moving-plane method which requires f, g to be nondecreasing and independent of x. Next, identities of Rellich-Pohozaev type are applied which restricts this technique to the case $f = f(v)$ and $g = g(u)$. In addition, Ω has to be convex or certain further technical conditions on f and g have to be satisfied. This method yields optimal results in the model case $f(v) = v^p$ and $g(u) = u^q$ but is often not applicable in more general cases because of requirements on f and g.

Method of Hardy-Sobolev inequalities were first used in [14] in the scalar case where authors studied a priori estimates of variational solutions of scalar problem. This method is based on using the first eigenfunction of the Laplacian as a test function which derives an estimate on the nonlinearity. This estimate together with suitable growth assumptions on nonlinearity yields H^1 bound using Hardy-Sobolev inequalities. In case of system (1), (2), this method requires only upper bounds on the growth of nonlinearities f, g, but it doesn't provide optimal results in terms of growth rates.

Finally, the bootstrap procedure for deriving a priori bounds can be used. This procedure consists of the fulfillment of certain initial condition which guarantees initiation of self-sustaining process leading to desired result. The key idea is as follows. Suppose that we know that regularity of solutions (and certain growth conditions) imply better regularity of right-hand sides and vice versa. Then once we prove better regularity of nonlinearities, we can iterate over and over using the bootstrap procedure until desired regularity of solutions is obtained in finite number of iteration steps.

The bootstrap procedure was used to prove a priori estimates of scalar problems and systems in [29, 32, 33, 34, 41, 42]. In [42], P. Quittner and Ph. Souplet presented new alternate-bootstrap method yielding a priori estimates of very weak solutions of system (1). This method can be applied under weak regularity assumptions unlike the scaling methods or the method of moving planes which require variational or classical solutions (see Chapter 1 for definitions of different types of solutions of (1) complemented by (2)). Recently, Y. Li [32] obtained a priori estimates of very weak solutions of (1) and (2) under more general assumptions on f, g as in [42] (see also Chapter 2 for related known results).

In Chapter 3, we improve results on regularity and a priori estimates obtained in [32]. We consider elliptic system (1) complemented by Dirichlet boundary conditions (2) where f, g satisfy growth assumptions

$$\begin{aligned} 0 \leq f(x,u,v) &\leq C_1(1+|u|^{r_1}|v|^{p_1}+|u|^{r_2}|v|^{p_2}+|u|^{\gamma}),\\ 0 \leq g(x,u,v) &\leq C_1(1+|u|^{q_1}|v|^{s_1}+|u|^{q_2}|v|^{s_2}+|v|^{\sigma}). \end{aligned}$$

We derive conditions on growth exponents $p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, \gamma, \sigma$ guaranteeing essential boundedness of all possible positive very weak solutions of (1) and (2) and their a priori estimates. Similarly to [42], [32], our proof is based on alternate-bootstrap arguments. Our results hold true if we treat variational solutions or L^1-solutions of (1) and (2). We just have to replace critical growth exponent for very weak solutions by corresponding critical growth exponent for variational or L^1-solutions. In order to show that we improved results in [32], we present an example of system (1) complemented by (2) whose positive very weak solutions are a priori bounded thanks to our results but f, g do not satisfy assumptions required by [32].

In Chapter 4, we consider elliptic systems (1) complemented by nonlinear boundary conditions (3). Corresponding scalar problem was recently studied in [41]. We assume following polynomial growth of $f, g, \tilde{f}, \tilde{g}$:

$$\begin{aligned} |f(x,u,v)| &\leq C_f(1+|u|^r+|v|^p),\\ |g(x,u,v)| &\leq C_g(1+|u|^q+|v|^s),\\ |\tilde{f}(y,u,v)| &\leq C_{\tilde{f}}(1+|u|^{\tilde{r}}+|v|^{\tilde{p}}),\\ |\tilde{g}(y,u,v)| &\leq C_{\tilde{g}}(1+|u|^{\tilde{q}}+|v|^{\tilde{s}}), \end{aligned}$$

for all $x \in \Omega$, $y \in \partial\Omega$ and $u, v \in \mathbb{R}$. We derive optimal conditions on growth exponents $p, q, r, s, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$ guaranteeing a priori estimates of positive very weak solutions of (1) with (3). Similarly to [41], [42], our proofs are based on regularity results for linear problems and alternate-bootstrap arguments. In fact, in order to obtain optimal results in the case of systems with homogeneous boundary conditions, the bootstrap arguments in [42, 32, 33, 29] were quite complicated and technical: one has first to make a bootstrap in one of the equations, then in the second equation, then in the first equation again etc. The presence of nonlinear boundary conditions represents another difficulty, since one has to prove simultaneous estimates for the solutions and their traces on the boundary $\partial\Omega$. Due to the complexity of problem (1) and (3), our proofs are far from a trivial modification of the proofs in [41] and [42].

We also prove that our results are optimal. We show that there exist Ω and $f, g, \tilde{f}, \tilde{g}$ which do not satisfy required conditions on growth, such that problem (1) with (3) possesses a positive unbounded very weak solution.

Similarly as in [42], our results on a priori estimates can be used to prove existence of nontrivial solutions, provided one can estimate the right-hand sides in L^1. We provide a few typical problems where the L^1-bounds and existence of positive solutions can be proved.

One of the advantages of our approach is its robustness. Unlike other methods yielding a priori estimates for positive solutions of elliptic systems (for example, those based on scaling arguments and Liouville theorems), our method does require neither scaling properties nor variational or local structure. In particular, it can also be applied for problems with nonlocal nonlinearities. In Section 4.6 below we formulate a generalization of our results for nonlocal problems and we also show its applications in the study of some particular nonlocal problems.

This thesis is organized as follows: In Chapter 1, we state some notational conventions and we will provide definitions of different types of solutions of (1). Then in Chapter 2 we state some related known results on regularity and a priori estimates of positive solutions of elliptic scalar problems and systems. Chapter 3 contains our improvement of results in [32]. In Chapter 4, we state our results on regularity and a priori estimates of very weak solutions of elliptic systems (1) complemented by nonlinear boundary conditions (3). We proof optimality of these results and we treat some nonlocal problems as well. We outline our results in Summary.

Chapter 1

Preliminaries

Throughout this thesis, we will assume that

$$\boxed{\Omega \text{ is a bounded domain in } \mathbb{R}^N, \;\; N \geq 3, \text{ with a smooth boundary } \partial\Omega,}$$

unless explicitly stated otherwise. By λ_1 and φ_1 we will denote the first eigenvalue of the negative Dirichlet Laplacian in Ω and the corresponding positive eigenfunction (normalized in $L^\infty(\Omega)$), respectively. We denote by ν the exterior unit normal on $\partial\Omega$ and by δ we denote the distance to the boundary $\partial\Omega$.

For integer $k \geq 1$, $L^k(\Omega)$ denotes the Lebesgue space $L^k(\Omega, dx)$ endowed with the norm

$$||\varphi||_{L^k(\Omega)} = \left(\int_\Omega |\varphi(x)|^k dx\right)^{\frac{1}{k}},$$

$L^k_\delta(\Omega)$ denotes the weighted Lebesgue space $L^k(\Omega, \delta(x)dx)$ endowed with the norm

$$||\varphi||_{L^k_\delta(\Omega)} = \left(\int_\Omega |\varphi(x)|^k \delta(x) dx\right)^{\frac{1}{k}}.$$

For $k = \infty$, the space $L^\infty(\Omega) = L^\infty_\delta(\Omega)$ is the set of all measurable functions φ from Ω to $\mathbb{R}$ which are essentially bounded, i.e. bounded up to a set of measure zero. For φ in $L^\infty(\Omega)$, its essential supremum serves as an appropriate norm:

$$\|\varphi\|_\infty := \inf\{C \geq 0 : |\varphi(x)| \leq C \text{ for almost every } x\}.$$

$W^{1,k}(\Omega)$ denotes the space of functions such that

$$\{\varphi \in L^k(\Omega), \nabla\varphi \in (L^k(\Omega))^N\},$$

where all the derivatives are interpreted in the weak sense. $W^{1,k}(\Omega)$ is endowed with the norm

$$||\varphi||_{W^{1,k}(\Omega)} = \left(\int_\Omega |\varphi(x)|^k dx + \int_\Omega |\nabla \varphi(x)|^k dx \right)^{\frac{1}{k}}.$$

We will denote by $H_0^1(\Omega)$ the closure in $W^{1,2}(\Omega)$ of infinitely differentiable functions compactly supported in Ω. By $H^{-1}(\Omega)$ we will denote corresponding dual space with respect to the duality pairing

$$\langle \phi, \varphi \rangle := \int_\Omega \phi \varphi \, dx \qquad \phi \in L^2(\Omega), \ \varphi \in H_0^1(\Omega).$$

By T we will denote the trace operator

$$T : W^{1,k}(\Omega) \to L^k(\partial\Omega).$$

For references and further properties of such spaces and trace operator, see for instance [1].

As we have mentioned already, we state here definitions of different types of solutions. First, let us consider the following scalar problem

$$\left.\begin{aligned} -\Delta u &= f(\cdot, u) \qquad \text{in } \Omega, \\ u &= 0 \qquad\quad\ \ \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where f is a superlinear, non-dissipative function satisfying $|f(\cdot, u)| \leq C(1 + |u|^p)$, $p > 1$. Let u be a solution of (1.1) and $\tilde{f}(x) := f(x, u(x))$. Then u solves the linear problem

$$\left.\begin{aligned} -\Delta u &= \tilde{f}(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

Definition 1.0.1. *u is a classical solution of* (1.1) *if $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $\tilde{f} \in C(\Omega)$ and u satisfies the equation and the boundary condition in* (1.2) *pointwise.*

Definition 1.0.2. *u is a variational $H_0^1(\Omega)$-solution of* (1.2) *if $u \in H_0^1(\Omega)$, $\tilde{f} \in H^{-1}(\Omega)$ and*

$$\int_\Omega \nabla u . \nabla \varphi \, dx = \langle \tilde{f}, \varphi \rangle \text{ for all } \varphi \in H_0^1(\Omega).$$

Definition 1.0.3. *u is called a very weak solution of* (1.2) *if $u, \tilde{f}\delta \in L^1(\Omega)$ and*

$$\int_\Omega (u\Delta\varphi + \tilde{f}\varphi) dx = 0 \text{ for all } \varphi \in C^2(\overline{\Omega}) \text{ such that } \varphi = 0 \text{ on } \partial\Omega.$$

Classical, variational and very weak solutions of (1) are defined analogously as in the scalar case. In particular, let (u, v) be a solution of (1) and $\tilde{f}(x) := f(x, u(x), v(x))$ and $\tilde{g}(x) := g(x, u(x), v(x))$. Then (u, v) solves the linear problem

$$\left.\begin{array}{rcl} \left.\begin{array}{rcl} -\Delta u &=& \tilde{f}(x) \\ -\Delta v &=& \tilde{g}(x) \end{array}\right\} & \text{in } \Omega, \\ \left.\begin{array}{rcl} u &=& 0 \\ v &=& 0 \end{array}\right\} & \text{on } \partial\Omega, \end{array}\right\} \tag{1.3}$$

Definition 1.0.1. *(u, v) is a classical solution of (1) if $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$, $\tilde{f}, \tilde{g} \in C(\Omega)$ and (u, v) satisfies equations and the boundary conditions in (1.3) pointwise.*

Definition 1.0.4. *(u, v) is a variational $H_0^1(\Omega)$-solution of (1) if $u, v \in H_0^1(\Omega)$, $\tilde{f}, \tilde{g} \in H^{-1}(\Omega)$ and*

$$\int_\Omega \nabla u . \nabla \varphi \, dx = \langle \tilde{f}, \varphi \rangle,$$

$$\int_\Omega \nabla v . \nabla \varphi \, dx = \langle \tilde{g}, \varphi \rangle,$$

for all $\varphi \in H_0^1(\Omega)$.

Definition 1.0.5. *(u, v) is called a very weak solution of (1) if $u, v, \tilde{f}, \tilde{g}\delta \in L^1(\Omega)$ and*

$$\int_\Omega (u\Delta\varphi + \tilde{f}\varphi) dx = 0,$$

$$\int_\Omega (v\Delta\varphi + \tilde{g}\varphi) dx = 0$$

for all $\varphi \in C^2(\overline{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$.

Chapter 2

Known results

2.1 Existence and nonexistence

Let us first mention some known results for problem (1.1) in the case $f(x,u) = |u|^{p-1}u$. Due to the classical result of S. I. Pohozaev [39], p_S defined by

$$p_S := \begin{cases} \infty, & \text{if } N < 3, \\ \frac{N+2}{N-2}, & \text{if } N \geq 3 \end{cases} \tag{2.1}$$

is a critical exponent for non-existence of classical positive solutions of (1.1), assuming that Ω is starshaped. (We say that Ω is starshaped with respect to some point $x_0 \in \Omega$ if the segment $[x_0, x]$ is a subset of Ω for any $x \in \Omega$.) Under assumptions

$$f(x,u) = |u|^{p-1}u, \quad p \geq p_S, \tag{2.2}$$

he proved that (1.1) does not possess classical nontrivial solutions. The proof is based on contradiction using Pohozaev's identity

$$\frac{N-2}{2} \int_\Omega |\nabla u|^2 dx - N \int_\Omega \frac{|u|^{p+1}}{p+1} dx + \frac{1}{2} \oint_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right| x \cdot \nu \, d\sigma = 0$$

which is true for any classical solution of (1.1) with $f(x,u) = |u|^{p-1}u$. S. I. Pohozaev obtained this result also for more general f independent of x.

In 1973, Ambrosetti and Rabinowitz [3] proved the existence of a positive solution of (1.1) with f satisfying some technical conditions and the growth assumption

$$|f(x,u)| \leq C(1 + |u|^p), \quad p < p_S,$$

using variational methods, but they did not obtain any a priori bound.

2.2 A priori estimates

In 1974, R.E.L. Turner [47] studied problem (1.1) for $N = 2$. He proved that if a continuous function f defined on $\Omega \times [0, \infty)$ satisfies

$$C_1 u^p \leq f(x, u) \leq C_2(1 + u^p), \quad p < 3,$$

for some constants $C_1, C_2 > 0$, then any nonnegative classical solution of (1.1) satisfies the a priori bound

$$||u||_{L^\infty} \leq C.$$

His proof was based on the existence of a conformal mapping of Ω to the unit disc D and estimates of the Green function for the Dirichlet Laplacian in D.

One year later, R. Nussbaum [37] proved that any positive classical solution of (1.1) with f satisfying

$$|f(x, u)| \leq C(1 + |u|^p), \quad p < \frac{N}{N-1},$$

satisfies the a priori bound

$$||u||_{L^\infty} \leq C.$$

His proof was based on the Sobolev inequality and an L^1 bound for $f(u)\varphi_1$. Moreover, if Ω is a ball in R^N, he obtained a priori estimates for all positive radially symmetric solutions of (1.1) for $p < p_S$.

In 1977, H. Brezis and R.E.L. Turner used Hardy-Sobolev inequalities and a bootstrap argument in order to obtain a uniform a priori bound for all nonnegative H_0^1-solutions of the scalar elliptic problem (2.4) bellow, under the following growth restriction on f:

$$0 \leq f(x, u) \leq C(1 + |u|^p), \quad p < p_{BT},$$

$$p_{BT} := \begin{cases} \infty, & \text{if } N < 2, \\ \frac{N+1}{N-1}, & \text{if } N \geq 2 \end{cases} \tag{2.3}$$

They studied elliptic problems of the form

$$\left.\begin{aligned} -\Delta u &= f(\cdot, u) + t\varphi_1 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.4}$$

and

$$\left.\begin{aligned} -\Delta u &= f(\cdot, u, \nabla u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.5}$$

The following two theorems are also valid with $-\Delta$ replaced by a more general elliptic operator.

Theorem 2.2.1. *Let $f = f(x,u)$ be a continuous, nonnegative function defined on $\overline{\Omega} \times [0,\infty)$ and suppose:*

$$\liminf_{u\to\infty} \frac{f(x,u)}{u} > \lambda_1, \tag{2.6}$$

$$\lim_{u\to\infty} \frac{f(x,u)}{u^{p_{BT}}} = 0, \tag{2.7}$$

both conditions holding uniformly for $x \in \overline{\Omega}$. Then there is a constant C such that any nonnegative solution $u \in H_0^1$ of (2.4) *with $t \geq 0$ satisfies the a priori bound*

$$||u||_\infty \leq C \text{ where } C \text{ is independent of } t.$$

Theorem 2.2.2. *Let $f = f(x,u,s)$ be a continuous, nonnegative function defined on $\overline{\Omega} \times [0,\infty) \times \mathbb{R}^N$ and suppose:*

$$\liminf_{u\to\infty} \frac{f(x,u,s)}{u} > \lambda_1, \tag{2.8}$$

$$\lim_{u\to\infty} \frac{f(x,u,s)}{u^{p_{BT}}} = 0, \tag{2.9}$$

$$\limsup_{u\to 0} \frac{f(x,u,s)}{u} < \lambda_1, \tag{2.10}$$

the three conditions holding uniformly for $x \in \overline{\Omega}$ and $s \in \mathbb{R}^N$. Then there exists a positive variational solution of (2.5) *such that $u \in W^{2,q}(\Omega)$ for any $q < \infty$.*

In 1981, B. Gidas and J. Spruck [25] derived an a priori estimate for positive solutions of (1.1) in the optimal range of exponents. Their proof was done by contradiction using a "blow-up" argument and a Liouville-type theorem. The result is as follows:

Theorem 2.2.3. *Let u be a classical positive solution of the boundary value problem* (1.1)*. Suppose $f(x,u)$ is continuous in $x \in \overline{\Omega}$ and for some $1 < p < p_S$*

$$\lim_{u\to\infty} \frac{f(x,u)}{u^p} = h(x) \tag{2.11}$$

uniformly in $x \in \overline{\Omega}$. Here h is continuous and strictly positive in Ω. Then

$$\|u\|_{L^\infty} \leq C$$

for some uniform constant $C = C(\Omega, p)$ independent of u.

Independently of them, D.G. Figueiredo, P.-L. Lions and R.D. Nussbaum [21] obtained the following a priori estimates for classical positive solution of (1.1) with $f = f(u)$:

Theorem 2.2.4. *Assume $f \in C(\mathbb{R}_+)$ and f is locally Lipschitz,*

$$\liminf_{u\to\infty} \frac{f(u)}{u} > \lambda_1, \tag{2.12}$$

$$\lim_{u\to\infty} \frac{f(u)}{u^{p_S}} = 0, \tag{2.13}$$

$$\limsup_{u\to\infty} \frac{uf(u) - \theta F(u)}{u^2 f(u)^{2/N}} \leq 0 \text{ for some } 0 \leq \theta < \frac{2N}{N-2}, \tag{2.14}$$

where $F(u) = \int_0^u f(s)ds$. Assume further that either $f(u)/u^{p_S}$ is nonincreasing for $t \geq 0$, or Ω is convex. Then we have

$$||u||_{L^\infty(\Omega)} \leq C,$$

where C depends only on Ω and on the behavior of f in the limits arising in (2.12), (2.13) *and* (2.14).

Their proof used estimates of $f(u)\varphi_1$ in $L^1(\Omega)$, local estimates of u close to the boundary $\partial\Omega$ based on the moving plane arguments of B. Gidas, W.-M. Ni and L. Nirenberg [24] and the Pohozaev identity.

Hence critical growth exponent in the scalar case problem (1.1) is the Sobolev exponent p_S (see (2.1) for its definition).

2.3 Very weak solutions

A priori estimates and critical exponent for very weak solutions of (1.1) were obtained later. The exponent p_{BT}, which appeared first in the work of Brezis and Turner [14] and seemed to be technical exponent in the proof of regularity, a priori bounds and existence for classical solutions, turned out to be critical exponent in the $L^1_\delta-$solutions case.

In 2004, P. Quittner and Ph. Souplet [42] showed that any very weak solution of (1.1), where

$$|f(x,u)| \leq C(1 + |u|^p), \quad p < p_{BT}, \tag{2.15}$$

is in $L^\infty(\Omega)$, and thus is a classical solution if f is smooth enough.

Surprisingly, in 2005, Ph. Souplet [45] showed that exponent p_{BT} in the growth assumption (2.15) is optimal:

Theorem 2.3.1. *Assume $p > p_{BT}$. There exist a function $a \in L^\infty(\Omega)$, $a \geq 0$, such that the problem*

$$\left.\begin{aligned} -\Delta u = a(\cdot)u^p \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.16}$$

admits a positive very weak solution $u \notin L^\infty(\Omega)$.

If $p_{BT} < p < p_S$, then problem (2.16) possesses at least two different nonnegative very weak solutions, one variational ($u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$) and a second one u_2 which is unbounded and not variational ($u_2 \notin H_0^1(\Omega)$).

In 2007, M. del Pino, M. Musso and F. Pacard [18] constructed positive very weak solutions of (1.1) with $f(x,u) = u^p$, $p \geq p_{BT}$, which vanish in the sense of traces on $\partial\Omega$, but which are singular at prescribed points of $\partial\Omega$. The result can be formulated using the following definition.

Definition 2.3.2. *Let u be a function defined in Ω and $y \in \partial\Omega$. We say that*

$$u(x) \to \mu \text{ as } x \to y \text{ nontangentially}$$

if

$$\lim_{\Gamma_\alpha(y) \ni x \to y} u(x) = \mu \text{ for all } \alpha \in \left[0, \frac{\pi}{2}\right),$$

where $\Gamma_\alpha(y)$ denotes the cone with vertex in y, and angle α with respect to his axis, the inner normal to $\partial\Omega$ at y.

Theorem 2.3.3. *There exists a number $\hat{p} = \hat{p}(N) > p_{BT}$, such that given $p \in [p_{BT}, \hat{p})$ and given points $y_1, y_2, \ldots, y_k \in \partial\Omega$, there exists a positive very weak solution u to the problem*

$$\left.\begin{aligned} -\Delta u = u^p \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.17}$$

such that $u \in C^2(\overline{\Omega} \backslash \{y_1, \ldots, y_k\})$ and

$$u(x) \to \infty \text{ as } x \to y_i \text{ nontangentially} \quad \text{for } i = 1, \ldots, k.$$

The proof of Theorem 2.3.3 was based on the construction of a singular solution in the halfspace $\{x \in \mathbb{R}^N : x_N > 0\}$ (if $p > p_{BT}$) or the half-ball $\{x \in \mathbb{R}^N : x_N > 0, |x| < 1\}$ (if $p = p_{BT}$), with singularity at $x = 0$.

The behavior of positive solutions of (2.17) near an isolated singularity at the boundary has been recently studied by M.-F. Bidaut-Véron, A. Ponce and L. Véron [7]:

Theorem 2.3.4. *If $1 < p < p_S$ then any positive solution of*

$$-\Delta u = u^p \text{ in } \Omega, \tag{2.18}$$

which is continuous in $\overline{\Omega}\backslash\{x_0\}$ and coincides on $\partial\Omega\backslash\{x_0\}$ with some function $\zeta \in C(\partial\Omega)$, satisfies

$$u(x) \leq C|x - x_0|^{-\frac{2}{p-1}} \text{ for all } x \in B_{R_0}(x_0) \cap \Omega \tag{2.19}$$

for some $R_0 > 0$ and $C = C(N, \Omega, p, ||\zeta||_{L^\infty}) > 0$.

The proof was based upon a doubling lemma and the method introduced by P. Poláčik, P. Quittner and Ph. Souplet [40]. Moreover, in [7], the authors also proved the following result on the asymptotic behavior of singular solutions of (2.18): Let $p_{BT} \leq p \leq \frac{N+1}{N-3}$, $p \neq p_S$. Assuming that $x_0 = 0$ and the outward normal unit vector to $\partial\Omega$ at 0 is $-e_N$ than

- $u(x) = |x|^{\frac{-2}{p-1}}(w_0(x/|x|) + o(1))$ if $p > p_{BT}$,
- $u(x) = C_N|x|^{1-N}(\ln(1/|x|))^{\frac{1-N}{2}}(x_N/|x| + o(1))$ if $p = p_{BT}$,

where w_0 satisfies a particular nonlinear elliptic equation on an hemisphere S of the unit sphere S^{N-1}. The existence of a singular solution u in supercritical case with $\zeta = 0$ is guaranteed by Theorem 2.3.3.

Finally, let us mention the result of P.J. McKenna and W. Reichel [34], who derived a priori bounds for positive very weak solutions of semilinear elliptic boundary value problems (1.1) on a Lipschitz domain Ω. They introduced two exponents $p_* \leq p^*$, which depend on the boundary behavior of the Green function and on the smallest interior opening angle of $\partial\Omega$. They proved that all positive very weak solutions are a priori bounded in $L^\infty(\Omega)$ for $1 < p < p_*$. Moreover they constructed problems with $f(x, u) = a(x)u^p$, where

$$p > p^* \text{ and } 0 \leq a \in L^\infty(\Omega),$$

which possess positive very weak solutions $u \notin L^\infty(\Omega)$. Finally, they found several classes of domains for which $p_* = p^*$. For example, $p_* = p^* = p_{BT}$ in the case of smooth domains and $p_* = p^* = N/(N-1)$ in the case of N-dimensional hypercubes.

Recently, J. Horák, P.J. McKenna and W. Reichel [27] proved the existence of at least two positive unbounded very weak solutions of problem (2.17), where Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ with a cone-like corner at $0 \in \partial\Omega$, for any exponent p slightly larger than the exponent p^* mentioned above:

Theorem 2.3.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with a conical boundary piece of cross section $\omega \subset S^{N-1}$ at $0 \in \partial\Omega$. Then there is $\varepsilon > 0$ such that for $p \in (p^*, p^* + \varepsilon)$ there exist at least two positive, unbounded, very weak solutions of* (2.17) *blowing up at 0.*

2.4 Nonlinear boundary conditions

The following problem with nonlinear Neumann boundary conditions

$$\left.\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ \partial_\nu u &= \tilde{f}(\cdot, u) - u && \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.20}$$

was studied recently by P. Quittner and W. Reichel [41]. They found sufficient conditions on $\tilde{f}$ guaranteeing boundedness and a priori bounds for any very weak solution of (2.20) as stated in Theorem 4.1.1. Moreover, they have found a domain Ω and $\varepsilon > 0$ such that problem (2.20) with $\tilde{f}(x, u) = u^p$ possess at least two positive, unbounded, very weak solutions for $p \in \left(\frac{N-1}{N-2}, \frac{N-1}{N-2} + \varepsilon\right)$. Hence they showed that $\frac{N-1}{N-2}$ is a sharp critical growth exponent in nonlinear boundary condition case. We will state their results more precisely in Chapter 4.

2.5 Very weak solutions to elliptic systems

In the case of systems

$$\left.\begin{aligned} &\left.\begin{aligned} -\Delta u &= f(.\,, u, v) \\ -\Delta v &= g(.\,, u, v) \end{aligned}\right\} \text{ in } \Omega, \\ &\left.\begin{aligned} u &= 0 \\ v &= 0 \end{aligned}\right\} \text{ on } \partial\Omega, \end{aligned}\right\} \tag{2.21}$$

very weak solutions of (1) are defined analogously as in the scalar case, see Definition 1.0.5 in Preliminaries for details. The boundedness of very weak solutions of systems and their a priori estimates have been studied in [9], [29], [32], [33], [42] and [45]. Let us mention some related results from [32], [42] and [45].

In 2004, P. Quittner and Ph. Souplet [42] showed that any nonnegative L^1_δ-solution (u, v) of system (2.21) belongs to $L^\infty(\Omega)$ and has the a priori bound

$$||u||_\infty + ||v||_\infty \leq C(\Omega, p, q, \gamma, \sigma, N, C_1, M) \tag{2.22}$$

provided

$$||u||_{L^1_\delta} + ||v||_{L^1_\delta} \leq M, \tag{2.23}$$

$$\begin{array}{rcl} 0 \leq & f(x,u,v) & \leq C_1(1 + |v|^p + |u|^\gamma), \\ 0 \leq & g(x,u,v) & \leq C_1(1 + |u|^q + |v|^\sigma), \end{array}$$

where

$$\max\{p+1, q+1\} > \frac{pq-1}{p_{BT}-1}, \tag{2.24}$$

$$1 \leq \gamma, \sigma < p_{BT} \tag{2.25}$$

and $p, q > 0$. Their proof was based on a bootstrap argument using L^p_δ-regularity of the Dirichlet Laplacian, see [22] and Lemma 3.2.1 below. They also found sufficient conditions on f, g guaranteeing estimate (2.23). They proved similar results for very weak solutions of elliptic systems complemented with the Neumann boundary conditions as well. We will mention some of these results in Theorem 4.1.2 in Chapter 4.

In 2005, Ph. Souplet [45] showed that exponent p_{BT} appearing in (2.24), (2.25) is optimal. Assuming

$$\max\{p+1, q+1\} < \frac{pq-1}{p_{BT}-1}, \tag{2.26}$$

he constructed functions $a, b \in L^\infty(\Omega)$, $a, b \geq 0$ such that the problem

$$\left.\begin{array}{l} \left.\begin{array}{rcl} -\Delta u &=& av^p \\ -\Delta v &=& bu^q \end{array}\right\} \text{ in } \Omega, \\ \left.\begin{array}{rcl} u &=& 0 \\ v &=& 0 \end{array}\right\} \text{ on } \partial\Omega, \end{array}\right\} \tag{2.27}$$

admits a positive very weak solution such that $u \notin L^\infty(\Omega)$ and $v \notin L^\infty(\Omega)$.

Recently, Y.-X. Li [32] presented another bootstrap procedure for elliptic systems (2.21) which yields optimal L^∞-regularity conditions for three types of weak solutions:

H_0^1-solutions, L^1-solutions and L_δ^1-solutions. He proved that (2.23) implies (2.22) for positive L_δ^1-solutions under more general assumptions on f, g than in [42]:

$$\left.\begin{array}{rcccl} 0 & \leq & f(x,u,v) & \leq & C_1(1+|u|^r|v|^p+|u|^\gamma), \\ 0 & \leq & g(x,u,v) & \leq & C_1(1+|u|^q|v|^s+|v|^\sigma), \end{array}\right\} \tag{2.28}$$

where

$$r, s, \min\{p+r, q+s\} \in [0, p_c)\,, \tag{2.29}$$

$$\max\{p+1-s, q+1-r\} > \frac{pq-(1-r)(1-s)}{p_c-1}, \tag{2.30}$$

$$1 \leq \gamma, \sigma < p_c \tag{2.31}$$

and $p, q > 0$ is true. In conditions (2.29), (2.30) and (2.31), p_c denotes p_S, p_{sg}, p_{BT}, respectively, for the case of H_0^1-solutions, L^1-solutions and L_δ^1-solutions. (See Remark (3.1.4) for details.) Notice that if $r = s = 0$, then the assumptions (2.29), (2.30) are equivalent to (2.24) in case of L_δ^1-solutions (since (2.24) guarantees that $\min\{p, q\} < p_{BT}$). Similarly to [45], Li also constructed an example showing that his results are optimal in some sense.

Later, Y.-X. Li [33] extended this result to the case of systems with n components, where $n \geq 3$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be Carathéodory functions and denote $\mathbf{u} = (u_1, u_2, \ldots, u_n) : \Omega \to \mathbb{R}^n$. He studied systems of the form

$$\left.\begin{array}{rcll} -\Delta \mathbf{u} & = & \mathbf{f}(x,u) & \text{in } \Omega, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \partial\Omega, \end{array}\right\} \tag{2.32}$$

and obtained optimal conditions for a priori estimates of nonnegative solutions of the form

$$\sum_{i=1}^{n} ||u_i||_{L^\infty} \leq C. \tag{2.33}$$

Chapter 3

Elliptic systems with Dirichlet boundary conditions

3.1 Main results

The aim of this chapter is to extend some recent results of Li [32] on L^∞-regularity and a priori estimates for very weak solutions of elliptic systems:

$$\left.\begin{array}{rcl}\left.\begin{array}{rcl}-\Delta u &=& f(.\,,u,v)\\ -\Delta v &=& g(.\,,u,v)\end{array}\right\} & \text{in } \Omega, \\ \left.\begin{array}{rcl}u &=& 0\\ v &=& 0\end{array}\right\} & \text{on } \partial\Omega.\end{array}\right\} \tag{3.1}$$

where f and g are nonnegative Carathéodory functions satisfying growth assumptions

$$\left.\begin{array}{rcl} f(x,u,v) &\le& C_1(1+|u|^{r_1}|v|^{p_1}+|u|^{r_2}|v|^{p_2}+|u|^{\gamma}),\\ g(x,u,v) &\le& C_1(1+|u|^{q_1}|v|^{s_1}+|u|^{q_2}|v|^{s_2}+|v|^{\sigma}).\end{array}\right\} \tag{3.2}$$

Recall from Chapter 2 that f, g in [32] satisfy less general growth assumptions (2.28):

$$\left.\begin{array}{rcccl} 0 &\le& f(x,u,v) &\le& C_1(1+|u|^{r}|v|^{p}+|u|^{\gamma}),\\ 0 &\le& g(x,u,v) &\le& C_1(1+|u|^{q}|v|^{s}+|v|^{\sigma}),\end{array}\right\}$$

It is well known, (see [8] and [42]), that all very weak solutions of corresponding scalar problem (1.1) belong to $L^\infty(\Omega)$ provided

$$f(x,u) \le C_1(1+|u|^p), \quad p < p_{BT},$$

where p_{BT} is defined by

$$p_{BT} := \begin{cases} \infty, & \text{if } N < 2, \\ \frac{N+1}{N-1}, & \text{if } N \geq 2. \end{cases}$$

On the other hand, unbounded very weak solutions of (1.1) were constructed for $p \geq p_{BT}$ in [18], [45], see also [6], [7].

In this chapter, we obtain the following improvement of results in [32].

Theorem 3.1.1. *Let $f, g : \Omega \times \mathbb{R}^2 \to [0, \infty)$ be Carathéodory functions satisfying* (3.2) *where $p_i, q_i, r_i, s_i \geq 0$ for $i = 1,\ 2$, $\max\{p_1, p_2\}, \max\{q_1, q_2\} > 0$ and* (2.25) *is true. Assume also that*

$$\left.\begin{aligned} \min\{\max\{p_1 + r_1,\ p_2 + r_2\},\ \max\{q_1 + s_1,\ q_2 + s_2\}\} < p_{BT}, \\ r_i, s_i < p_{BT}, \end{aligned}\right\} \quad i = 1, 2, \tag{3.3}$$

$$\max\{p_i + 1 - s_j,\ q_j + 1 - r_i\} > \frac{p_i q_j - (1 - r_i)(1 - s_j)}{p_{BT} - 1}, \qquad i, j = 1, 2, \tag{3.4}$$

and (u, v) is a nonnegative very weak solution of (1) *satisfying*

$$||u||_{L^1_\delta} + ||v||_{L^1_\delta} \leq M. \tag{3.5}$$

Then (u, v) belongs to $L^\infty(\Omega) \times L^\infty(\Omega)$ and

$$||u||_{L^\infty} + ||v||_{L^\infty} \leq C(\Omega, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, \gamma, \sigma, N, C_1, M). \tag{3.6}$$

Remark 3.1.2. *Actually, if we replace growth assumption* (3.2) *by*

$$\left.\begin{aligned} f(x, u, v) &\leq C_1(1 + (1 + |u|)^{r_1}(1 + |v|)^{p_1} + (1 + |u|)^{r_2}(1 + |v|)^{p_2} + |u|^\gamma), \\ g(x, u, v) &\leq C_1(1 + (1 + |u|)^{q_1}(1 + |v|)^{s_1} + (1 + |u|)^{q_2}(1 + |v|)^{s_2} + |v|^\sigma), \end{aligned}\right\} \tag{3.7}$$

the results in Theorem 3.1.1 *remain valid.*

Remark 3.1.3. *If we set $p_2 = q_2 = r_2 = s_2 = 0$, Theorem* 3.1.1 *recovers Li's result* [32] *since* (3.3), (3.4) *are equivalent to* (2.29), (2.30) *in this case. We provide also an example of problem such that all assumptions of Theorem* 3.1.1 *are satisfied for $N = 3$ but f, g do not satisfy Li's assumptions* (2.25), (2.28), (2.29) *and* (2.30).

Remark 3.1.4. *Similarly as in Li's paper* [32], *the same argument as in the proof of Theorem* 3.1.1 *can be used in order to get* L^∞ *regularity of* H_0^1*- or* L^1*-solutions of* (1) *(see Definitions* 1.0.1, 1.0.4 *in Preliminaries for precise definitions of such a solutions). In the case of* H_0^1*-solutions,* p_{BT} *has to be replaced by the Sobolev exponent* p_S *and in the case of* L^1*-solutions* p_{BT} *has to be replaced by the singular exponent* p_{sg} *defined by*

$$p_{sg} := \begin{cases} \infty, & \text{if } N < 3, \\ \frac{N}{N-2}, & \text{if } N \geq 3. \end{cases} \tag{3.8}$$

Notice that in the case of H_0^1*-solutions, the* L^∞ *a priori bound* (3.5) *requires the estimate*

$$||u||_{H_0^1} + ||v||_{H_0^1} \leq M$$

instead of (3.5) *and obtaining this estimate (unlike estimate* (3.5) *in the case of* L_δ^1*-solutions) is far from easy, see* [42], [43] *and the references therein, for example.* L^1*-solutions are in particular important in the case of Neumann or Newton boundary conditions where the bootstrap argument works as well and, in addition, one can easily find conditions on* f, g *guaranteeing the necessary initial bound*

$$||u||_{L^1} + ||v||_{L^1} \leq M,$$

see [42].

A significant difference between H_0^1*-solutions and* L^1*- (or* L_δ^1*-) solutions can be observed in the critical case: While* H_0^1*-solutions of the scalar problem* (1.1) *are regular in the critical case* $p = p_S$*, see* [13] *or* [18, *Corollary* 3.4], *singular* L^1*- or* L_δ^1*-solutions of* (1.1) *exist if* $p = p_{sg}$ *or* $p = p_{BT}$ *respectively, see* [5], [36], [38] *and* [18].

3.2 Proof of Theorem 3.1.1

In order to give a complete proof of Theorem 3.1.1, we will need the following regularity results for very weak solutions of the scalar problem

$$\left.\begin{aligned} -\Delta u &= \phi \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.9}$$

see [42] and [22].

Lemma 3.2.1. *Let $1 \le m \le k \le \infty$ satisfy*

$$\frac{1}{m} - \frac{1}{k} < \frac{1}{p'_{BT}},$$

where p'_{BT} satisfies $\frac{1}{p_{BT}} + \frac{1}{p'_{BT}} = 1$. Let $u \in L^1_\delta(\Omega)$ be the unique L^1_δ-solution of (3.9). If $\phi \in L^m_\delta(\Omega)$, then $u \in L^k_\delta(\Omega)$ and u satisfies the estimate $||u||_{L^k_\delta} \le C(\Omega, m, k)||\phi||_{L^m_\delta}$.

Now, we can give the proof of Theorem 3.1.1:

Proof. Without loss of generality, we can assume

$$p_2 + r_2 \le p_1 + r_1, \quad q_2 + s_2 \le q_1 + s_1 \tag{3.10}$$

and

$$p_1 + r_1 \le q_1 + s_1, \tag{3.11}$$

which together with (3.3) implies

$$p_1 + r_1 < p_{BT}. \tag{3.12}$$

Moreover, we can assume $p_1 \neq p_{BT} - 1$, $p_2 \neq p_{BT} - 1$, otherwise we can increase the values of exponents p_1 and/or p_2 (and q_1 if necessary) in such a way that (3.3), (3.4), (3.10) and (3.11) remain true.

We will denote by C a constant, which may vary from line to line, but is independent of (u, v). For simplicity, we denote by $|\cdot|_k$ the norm $||\cdot||_{L^k_\delta}$. Let $\varphi_1 > 0$ be the first eigenfunction of the negative Dirichlet Laplacian.. Notice that there exist $c_1, c_2 > 0$ such that

$$c_1\delta \le \varphi_1 \le c_2\delta. \tag{3.13}$$

Testing both equations of (1) with φ_1 and using Green's Theorem implies

$$\int_\Omega f\varphi_1 dx = \lambda_1 \int_\Omega u\varphi_1 dx \qquad \int_\Omega g\varphi_1 dx = \lambda_1 \int_\Omega v\phi_1 dx.$$

Thus (3.13) and the non-negativity of f, g, u, v yield

$$|f|_1 \le C|u|_1 \quad \text{and} \quad |g|_1 \le C|v|_1.$$

Then, application of Lemma 3.2.1 and (3.5) imply

$$|u|_k + |v|_k \le C, \quad \forall k \in [1, p_{BT}).$$

We distinguish several cases:

Case 1: $r_2 \le r_1$ and $p_2 \ge p_1$

1a. If $p_2 < p_{BT} - 1$, using bootstrap on the first equation of (1), we will obtain $|u|_\infty \le C$.

(i) First assume $r_1 < 1$. (2.25), (3.10) and (3.12) imply that there exists k such that

$$\max\{\gamma, p_1 + r_1\} < k < p_{BT}, \quad \frac{p_2}{k} < \frac{1}{p'_{BT}}. \tag{3.14}$$

For such a fixed k, we can find ε small enough to satisfy

$$\left.\begin{aligned} \frac{\gamma}{k+m\varepsilon} - \frac{1}{k+(m+1)\varepsilon} &< \frac{1}{p'_{BT}}, \quad \text{for any } m \in \mathbb{N}_0 = \{0,1,2,\ldots\}, \\ \frac{r_i}{k+m\varepsilon} + \frac{p_i}{k} - \frac{1}{k+(m+1)\varepsilon} &< \frac{1}{p'_{BT}}, \quad \text{for } i = 1,2 \text{ and any } m \in \mathbb{N}_0. \end{aligned}\right\} \tag{3.15}$$

For $m \in \mathbb{N}_0$, set

$$\begin{aligned} \frac{1}{\rho_m} &= \frac{r_1}{k+m\varepsilon} + \frac{p_1}{k}, \\ \frac{1}{\nu_m} &= \frac{r_2}{k+m\varepsilon} + \frac{p_2}{k}, \\ \frac{1}{\varrho_m} &= \frac{\gamma}{k+m\varepsilon}. \end{aligned}$$

Using (3.10) and (3.14), we obtain that $\rho_m, \nu_m, \varrho_m > 1$. Denote

$$m_0 = \min\{m : \min\{\rho_m, \nu_m, \varrho_m\} > p'_{BT}\}.$$

We claim that after m_0-th bootstrap on the first equation, we arrive at the desired result.

Assume the estimate $|u|_{k+m\varepsilon} \le C$ holds for some $m \in [0, m_0] \cap \mathbb{N}_0$ (which is true for $m = 0$). Then (3.15) implies

$$\frac{1}{\min\{\rho_m, \nu_m, \varrho_m\}} - \frac{1}{k+(m+1)\varepsilon} < \frac{1}{p'_{BT}},$$

hence Lemma 3.2.1 together with (3.2) and the Hölder inequality imply

$$\begin{aligned} |u|_{k+(m+1)\varepsilon} &\le C|f|_{\min\{\rho_m, \varrho_m, \nu_m\}} \\ &\le C(||u|^{r_1}|v|^{p_1}|_{\rho_m} + ||u|^{r_2}|v|^{p_2}|_{\nu_m} + ||u|^{\gamma}|_{\varrho_m} + 1) \\ &\le C(|u|^{r_1}_{k+m\varepsilon}|v|^{p_1}_k + |u|^{r_2}_{k+m\varepsilon}|v|^{p_2}_k + |u|^{\gamma}_{k+m\varepsilon} + 1) \\ &\le C \end{aligned}$$

So $|u|_{k+(m_0+1)\varepsilon} \le C$ and another application of Lemma 3.2.1 yields

$$|u|_\infty \le C.$$

(ii) If $r_1 \ge 1$, (2.25), (3.10) and (3.12) imply that there exist k and η,

$$\max\{\gamma, p_1 + r_1\} < k < p_{BT}, \quad \frac{p_2}{k} < \frac{1}{p'_{BT}}, \quad k \text{ close enough to } p_{BT},$$

$$1 < \eta, \quad \eta \text{ close enough to } 1,$$

such that

$$\left.\begin{aligned} \frac{\gamma}{\eta^m k} - \frac{1}{\eta^{m+1}k} &< \frac{1}{p'_{BT}}, \\ \frac{r_i}{\eta^m k} + \frac{p_i}{k} - \frac{1}{\eta^{m+1}k} &< \frac{1}{p'_{BT}}, \qquad i = 1, 2, \end{aligned}\right\} \tag{3.16}$$

for any $m \in \mathbb{N}_0$. Similarly to the case **1a(i)**, we obtain $|u|_\infty \le C$.

Now, we can carry on the bootstrap on the second equation of (1). From (2.25), (3.3), there exist l close enough to p_{BT} and $\eta > 1$ such that

$$\alpha := \max\{\sigma, s_1, s_2\} < l < p_{BT} \quad \text{and} \quad \frac{\alpha}{l} - \frac{1}{\eta l} < \frac{1}{p'_{BT}}.$$

Applying Lemma 3.2.1 we conclude after finitely many steps

$$|v|_\infty \le C.$$

1b. In case $p_{BT} - 1 < p_1 \le p_2$, let us denote by k_1^* and k_2^* the solutions of

$$\frac{r_i}{k_i^*} + \frac{p_i}{p_{BT}} - \frac{1}{k_i^*} = \frac{1}{p'_{BT}}, \qquad i = 1, 2. \tag{3.17}$$

We claim that $|u|_{k'} \le C$, $k' \in [1, k^*)$ where $k^* = \min\{k_1^*, k_2^*\}$. Inequality

$$p_{BT} - 1 < p_1 \le p_2$$

and (3.10), (3.12) imply $r_2 \le r_1 < 1$. Remark that

$$k^* > p_{BT} \tag{3.18}$$

since $p_i + r_i < p_{BT}$ for $i = 1, 2$ due to (3.10) and (3.12). As in [32], let us denote $k_\varepsilon := k^* - \varepsilon$ for any $0 < \varepsilon \ll 1$ and $k_\varepsilon^{\tau^m} := k_\varepsilon - \tau^m(k_\varepsilon - k)$ for $m \in \mathbb{N}_0$. Thanks to (2.25), (3.10), (3.17) and (3.12), we can find $k = k(\varepsilon) < k_\varepsilon$ and $\tau = \tau(\varepsilon)$ such that

$$\max\{\gamma, p_1 + r_1\} < k < p_{BT}, \quad k \text{ close enough to } p_{BT},$$
$$r_2 \le r_1 < \tau < 1, \quad \tau \text{ close enough to } 1,$$
$$r_2 k_\varepsilon^\tau \le r_1 k_\varepsilon^\tau < \tau k,$$

and

$$\left.\begin{aligned} \frac{\gamma}{k} - \frac{1}{k_\varepsilon^\tau} &< \frac{1}{p'_{BT}}, \\ \frac{r_i}{k_\varepsilon} + \frac{p_i}{k} - \frac{1}{k_\varepsilon} &< \frac{1}{p'_{BT}}, \quad i = 1, 2. \end{aligned}\right\} \tag{3.19}$$

Using $r_2 k_\varepsilon^\tau \le r_1 k_\varepsilon^\tau < \tau k$ and $\gamma \ge 1$ we get

$$\left.\begin{aligned} \frac{\gamma}{k_\varepsilon^{\tau^m}} - \frac{1}{k_\varepsilon^{\tau^{(m+1)}}} &\le \frac{\gamma}{k} - \frac{1}{k_\varepsilon^\tau}, \\ \frac{r_i}{k_\varepsilon^{\tau^m}} - \frac{1}{k_\varepsilon^{\tau^{(m+1)}}} &< \frac{r_i}{k_\varepsilon} - \frac{1}{k_\varepsilon}, \quad i = 1, 2, \end{aligned}\right\} \tag{3.20}$$

for all $m \in \mathbb{N}_0$. Now setting

$$\begin{aligned} \frac{1}{\rho_m} &= \frac{r_1}{k_\varepsilon^{\tau^m}} + \frac{p_1}{k}, \\ \frac{1}{\nu_m} &= \frac{r_2}{k_\varepsilon^{\tau^m}} + \frac{p_2}{k}, \\ \frac{1}{\varrho_m} &= \frac{\gamma}{k_\varepsilon^{\tau^m}}, \end{aligned}$$

and using similar bootstrap argument as in case **1a** leads to

$$|u|_{k_\varepsilon^{\tau^{(m+1)}}} \le C, \quad m \in \mathbb{N}_0.$$

As $k_\varepsilon^{\tau^m}$ tends to k_ε with m going to infinity, we obtain

$$|u|_{k'} \le C, \quad k' \in [1, k^*).$$

To continue the bootstrap on the second equation of (1), we first show that

$$\frac{q_i}{k^*} + \frac{s_i}{p_{BT}} < 1, \quad i = 1, 2. \tag{3.21}$$

Inequality (3.21) is true for $i = 1$ thanks to (3.4) and (3.11). Let $j \in \{1, 2\}$ be such that $k^* = k_j^*$. If $i = 2$, then (3.21) follows from (3.4) if $p_j + r_j \le q_2 + s_2$ and from inequality

$$(q_2 + 1 - r_j)(p_{BT} - p_j - r_j) > 0 > (r_j - 1)(p_j + r_j - q_2 - s_2)$$

otherwise.

From the definition of k^*, it is easy to see that

$$\frac{r_i}{k^*} + \frac{p_i}{p_{BT}} - \frac{1}{k^*} \le \frac{1}{p'_{BT}} \qquad \text{for } i = 1, 2. \tag{3.22}$$

Thanks to (2.25), (3.3), (3.10), (3.12), (3.18), (3.21) and (3.22) we can choose l, k_1 and η satisfying

$$\left.\begin{aligned} \max\{p_1 + r_1, \sigma, s_1, s_2\} < l < p_{BT}, &\qquad l \text{ close enough to } p_{BT}, \\ p_{BT} < k_1 < k^*, &\qquad k_1 \text{ close enough to } k^*, \\ 1 < \eta, &\qquad \eta \text{ close enough to } 1, \end{aligned}\right\} \tag{3.23}$$

such that

$$\begin{aligned} \frac{q_i}{k_1} + \frac{s_i}{l} &< 1, \quad i = 1, 2, \\ \frac{\sigma}{l} - \frac{1}{\eta l} &< \frac{1}{p'_{BT}}, \\ \frac{\gamma}{k_1} - \frac{1}{\eta k_1} &< \frac{1}{p'_{BT}}, \\ \frac{q_i}{k_1} + \frac{s_i}{l} - \frac{1}{\eta l} &< \frac{1}{p'_{BT}}, \quad i = 1, 2, \\ \frac{r_i}{k_1} + \frac{p_i}{\eta l} - \frac{1}{\eta k_1} &< \frac{1}{p'_{BT}}, \quad i = 1, 2. \end{aligned}$$

Multiplying the LHS of the inequalities above by $1/\eta^m$, we get

$$\begin{aligned} \frac{q_i}{\eta^m k_1} + \frac{s_i}{\eta^m l} &< 1, \quad i = 1, 2, \\ \frac{\sigma}{\eta^m l} - \frac{1}{\eta^{m+1} l} &< \frac{1}{p'_{BT}}, \\ \frac{\gamma}{\eta^m k_1} - \frac{1}{\eta^{m+1} k_1} &< \frac{1}{p'_{BT}}, \\ \frac{q_i}{\eta^m k_1} + \frac{s_i}{\eta^m l} - \frac{1}{\eta^{m+1} l} &< \frac{1}{p'_{BT}}, \quad i = 1, 2, \\ \frac{r_i}{\eta^m k_1} + \frac{p_i}{\eta^{m+1} l} - \frac{1}{\eta^{m+1} k_1} &< \frac{1}{p'_{BT}}, \quad i = 1, 2, \end{aligned} \tag{3.24}$$

for all $m \in \mathbb{N}_0$. Set

$$\frac{1}{\mu_m} = \frac{q_1}{\eta^m k_1} + \frac{s_1}{\eta^m l}, \quad \frac{1}{\varsigma_m} = \frac{q_2}{\eta^m k_1} + \frac{s_2}{\eta^m l}, \quad \frac{1}{\sigma_m} = \frac{\sigma}{\eta^m l},$$

$$\frac{1}{\rho_m}=\frac{r_1}{\eta^m k_1}+\frac{p_1}{\eta^{m+1}l},\qquad \frac{1}{\nu_m}=\frac{r_2}{\eta^m k_1}+\frac{p_2}{\eta^{m+1}l},\qquad \frac{1}{\varrho_m}=\frac{\gamma}{\eta^m k_1}.$$

It is easy to see that $\mu_m, \varsigma_m, \sigma_m, \rho_m, \nu_m, \varrho_m > 1$ thanks to (2.25), (3.10), (3.12), (3.23) and (3.24). Assume the estimate $|u|_{\eta^m k_1} + |v|_{\eta^m l} \le C$ holds for some $m \in \mathbb{N}_0$ (which is true for $m = 0$). Then the inequalities above imply

$$\frac{1}{\min\{\mu_m, \varsigma_m, \sigma_m\}} - \frac{1}{\eta^{m+1}l} < \frac{1}{p'_{BT}},$$

$$\frac{1}{\min\{\rho_m, \nu_m, \varrho_m\}} - \frac{1}{\eta^{m+1}k_1} < \frac{1}{p'_{BT}}.$$

Hence Lemma 3.2.1 together with (3.2) and the Hölder inequality imply

$$\begin{aligned}
|v|_{\eta^{m+1}l} &\le C|g|_{\min\{\mu_m,\varsigma_m,\sigma_m\}}\\
&\le C(||u|^{q_1}|v|^{s_1}|_{\mu_m} + ||u|^{q_2}|v|^{s_2}|_{\varsigma_m} + ||v|^{\sigma}|_{\sigma_m} + 1)\\
&\le C(|u|^{q_1}_{\eta^m k_1}|v|^{s_1}_{\eta^m l} + |u|^{q_2}_{\eta^m k_1}|v|^{s_2}_{\eta^m l} + |v|^{\sigma}_{\eta^m l} + 1)\\
&\le C
\end{aligned}$$

$$\begin{aligned}
|u|_{\eta^{m+1}k_1} &\le C|f|_{\min\{\rho_m,\varrho_m,\nu_m\}}\\
&\le C(||u|^{r_1}|v|^{p_1}|_{\rho_m} + ||u|^{r_2}|v|^{p_2}|_{\nu_m} + ||u|^{\gamma}|_{\varrho_m} + 1)\\
&\le C(|u|^{r_1}_{\eta^m k_1}|v|^{p_1}_{\eta^{m+1}l} + |u|^{r_2}_{\eta^m k_1}|v|^{p_2}_{\eta^{m+1}l} + |u|^{\gamma}_{\eta^m k_1} + 1)\\
&\le C.
\end{aligned}$$

Denote $m_0 := \min\{m \in \mathbb{N}_0 : \max\{\min\{\rho_m, \varrho_m, \nu_m\},\ \min\{\mu_m, \varsigma_m, \sigma_m\}\} > p'_{BT}\}$. As in [11, Case III in the proof of Theorem 2.4] after m_0-th alternate bootstrap on system (1), we arrive at the desired result $|v|_\infty \le C$ (or $|u|_\infty \le C$). So we also have $|u|_\infty \le C$ (or $|v|_\infty \le C$) thanks to (2.25), (3.3) and Lemma 3.2.1.

1c. In case $p_1 < p_{BT} - 1 < p_2$, we have $r_2 < 1$ from (3.10) and (3.12). Let us denote

$$k^* := k_2^* = \frac{p_{BT}(1-r_2)}{p_2-(p_{BT}-1)},$$

we claim that

$$|u|_{k'} \le C \quad k' \in [1, k^*).$$

(i) If $r_1 < 1$, similarly to case **1b**, due to (2.25), (3.10) and (3.12), there exist k and τ such that

$$\max\{\gamma, p_1 + r_1\} < k < p_{BT}, \quad \frac{p_1}{k} < \frac{1}{p'_{BT}}, \quad k \text{ close enough to } p_{BT},$$

$$r_2 \le r_1 < \tau < 1, \quad \tau \text{ close enough to } 1,$$

$$r_2 k_\varepsilon^\tau \le r_1 k_\varepsilon^\tau < \tau k,$$

where

$$k_\varepsilon = k^* - \varepsilon$$

and (3.19), (3.20) are satisfied. By the same bootstrap on the first equation as in case **1b**, we obtain

$$|u|_{k'} \le C \quad k' \in [1, k^*).$$

(ii) If $r_1 \ge 1$, due to (2.25), (3.10) and (3.12), there exist k and η such that

$$\max\{\gamma, p_1 + r_1\} < k < p_{BT}, \quad \frac{p_1}{k} < \frac{1}{p'_{BT}}, \quad k \text{ close enough to } p_{BT},$$

$$1 < \eta, \quad \eta r_2 < 1, \quad \eta \text{ close enough to } 1,$$

and inequalities

$$\left.\begin{aligned} \frac{\gamma}{\eta^m k} - \frac{1}{\eta^{m+1} k} &< \frac{1}{p'_{BT}}, \\ \frac{r_i}{\eta^m k} + \frac{p_i}{k} - \frac{1}{\eta^{m+1} k} &< \frac{1}{p'_{BT}}, \quad i = 1, 2, \end{aligned}\right\} \tag{3.25}$$

are satisfied for all $m \in \mathbb{N}_0$ such that

$$k' := \eta^{m+1} k < \frac{p_{BT} k (1 - \eta r_2)}{p_2 p_{BT} - k(p_{BT} - 1)}.$$

As the expression on the right-hand side of the last inequality goes to

$$\frac{(1 - \eta r_2) k^*}{1 - r_2} \text{ when } k \text{ approaches } p_{BT},$$

by the bootstrap on the first equation of (1) we obtain

$$|u|_{k'} \le C \quad k' \in [1, k^*),$$

because we can make

$$\frac{(1 - \eta r_2) k^*}{1 - r_2}$$

arbitrarily close to k^* by the choice of η.

Now, we can carry on the alternate bootstrap procedure just like in case **1b** to obtain

$$|u|_\infty + |v|_\infty \le C.$$

Case 2: $r_2 \ge r_1$ and $p_2 < p_1$

Application of the Young inequality implies

$$|u|^{r_2}|v|^{p_2} \le C(|u|^{r_1}|v|^{p_1} + |u|^{\frac{r_2p_1 - r_1p_2}{p_1 - p_2}}).$$

Then (3.3) and (3.10) imply

$$0 < \frac{r_2p_1 - r_1p_2}{p_1 - p_2} < p_{BT},$$

so we can simply set new γ by

$$\gamma := \max\left\{\gamma, \frac{r_2p_1 - r_1p_2}{p_1 - p_2}\right\}.$$

From Lemmas 2.5, 2.6 in [32], we get

$$\left.\begin{aligned} &|u|_\infty \le C, && && \text{if } p_1 < p_{BT} - 1, \\ &|u|_{k_1} \le C, && \forall k_1 \in [1, k^*), && \text{if } p_1 > p_{BT} - 1, \end{aligned}\right\} \tag{3.26}$$

where k^* is the solution of (3.17) with $i = 1$. Using the bootstrap on the second equation similarly to [32] leads to $|v|_\infty \le C$ thanks to (3.3) and (3.4). In particular:

2a. If $p_1 < p_{BT} - 1$ using (2.25), (3.3), similarly to the case **1a**, we obtain $|v|_\infty \le C$.

2b. If $p_1 > p_{BT} - 1$, we first show that

$$\frac{q_i}{k^*} + \frac{s_i}{p_{BT}} < 1, \quad i = 1, 2. \tag{3.27}$$

This inequality holds if $i = 1$ thanks to (3.4) and (3.11). If $i = 2$, then (3.27) is true if $p_1 + r_1 \le q_2 + s_2$ due to (3.4), otherwise it can be derived from the inequality

$$(q_2 + 1 - r_1)(p_{BT} - p_1 - r_1) > 0 > (r_1 - 1)(p_1 + r_1 - q_2 - s_2).$$

We can choose l, k_1 and η satisfying

$$\begin{aligned} \max\{p_1+r_1,\sigma,s_1,s_2\} < l < p_{BT}, &\quad l \text{ close enough to } p_{BT}, \\ p_{BT} < k_1 < k^*, &\quad k_1 \text{ close enough to } k^*, \\ 1 < \eta\,, &\quad \eta \text{ close enough to } 1, \end{aligned}$$

such that

$$\begin{aligned} \frac{q_i}{k_1} + \frac{s_i}{l} &< 1, \quad i = 1, 2, \\ \frac{\sigma}{l} - \frac{1}{\eta l} &< \frac{1}{p'_{BT}}, \\ \frac{\gamma}{k_1} - \frac{1}{\eta k_1} &< \frac{1}{p'_{BT}}, \\ \frac{q_i}{k_1} + \frac{s_i}{l} - \frac{1}{\eta l} &< \frac{1}{p'_{BT}}, \quad i = 1, 2, \\ \frac{r_1}{k_1} + \frac{p_1}{\eta l} - \frac{1}{\eta k_1} &< \frac{1}{p'_{BT}}. \end{aligned}$$

We can carry on the alternate bootstrap procedure to obtain $|v|_\infty \le C$, then we can use the bootstrap on the first equation again to obtain $|u|_\infty \le C$ thanks to (2.25) and (3.3).

Case 3: $r_2 < r_1$ and $p_2 < p_1$

We recall Remark 3.1.2. As $(1+|u|)^{r_2}(1+|v|)^{p_2} \le (1+|u|)^{r_1}(1+|v|)^{p_1}$, we can replace r_2 and p_2 by r_1 and p_1, respectively.

□

3.3 Example

As we have already mentioned in Remark 3.1.3, we will consider system (1) with $N = 3$ and

$$\left.\begin{aligned} f(x,u,v) &= u^{1-\varepsilon}v + v^{\frac{5}{4}-\varepsilon}, \\ g(x,u,v) &= u^4 v, \end{aligned}\right\} \tag{3.28}$$

where

$$\varepsilon \in \left(0, \frac{1}{7}\right).$$

Notice that $p_{BT} = 2$. It is easy to see that any nonnegative very weak solution (u, v) of (3.28) belongs to $L^\infty(\Omega) \times L^\infty(\Omega)$ thanks to Theorem 3.1.1 with $p_1 = 1 - \varepsilon, r_1 = 1, p_2 = \frac{5}{4} - \varepsilon, r_2 = 0, \gamma = 1, q_1 = 4, s_1 = 1, q_2 = s_2 = 0, \sigma = 1$. Next, we will show that f, g do not satisfy Li's assumptions (2.25), (2.28), (2.29) and (2.30). Assume for contradiction

$$u^{1-\varepsilon}v + v^{\frac{5}{4}-\varepsilon} \leq C(u^r v^p + u^2 + 1) \tag{3.29}$$

$$u^4 v \leq C(u^q v^s + v^2 + 1) \tag{3.30}$$

where p, r, s and q satisfy (2.29) and (2.30). If we take $v = 1$ in (3.30) and send u to infinity, we obtain $q \geq 4$. Hence (2.29) guarantees $p + r < 2$. Setting $v = u^{4-\delta}$ with $0 < \delta \ll 1$ in (3.30) yields

$$8 - \delta \leq q + (4 - \delta)s,$$

which (taking $\delta \to 0$) leads to

$$2 - \frac{q}{4} \leq s. \tag{3.31}$$

Since $p + r < 2 < q + s$, (2.30) implies $q + 1 - r > pq - (1 - r)(1 - s)$. This is equivalent to

$$p < 1 + \frac{(1 - r)(2 - s)}{q}. \tag{3.32}$$

Now, setting $u = 1$ in (3.29) and sending v to infinity leads to

$$\frac{5}{4} - \varepsilon \leq p. \tag{3.33}$$

Thus $r < 1$ due to $p + r < 2$. This with (3.31), (3.32) implies

$$p < \frac{5}{4} - \frac{r}{4}. \tag{3.34}$$

Inequalities (3.33), (3.34) lead to $r < 4\varepsilon$. Now we choose $\alpha \in (1 + \varepsilon, 4 - 20\varepsilon)$. This choice of α implies

$$2 < 1 - \varepsilon + \alpha,$$

$$r + \alpha p < 1 - \varepsilon + \alpha.$$

Now, taking $v = u^\alpha$ in inequality (3.29) and sending u to infinity yields a contradiction.

Chapter 4

Elliptic systems with nonlinear boundary conditions

4.1 Introduction

In this chapter we are mainly interested in regularity, a priori estimates and existence of very weak solutions (see Definition 4.2.1) of problems of the form

$$\begin{aligned} -\Delta u &= f(\cdot,u,v), \qquad & -\Delta v &= g(\cdot,u,v) \quad \text{in } \Omega, \\ \partial_\nu u &= \tilde f(\cdot,u,v), \qquad & \partial_\nu v &= \tilde g(\cdot,u,v) \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $f, g, \tilde f, \tilde g$ are Carathéodory functions satisfying suitable polynomial growth conditions. We also consider more general problems involving nonlocal nonlinearities. This chapter is based on [31], to appear.

Regularity and a priori estimates of very weak solutions of the scalar problem

$$\begin{aligned} -\Delta u &= h(\cdot,u) \qquad && \text{in } \Omega, \\ \partial_\nu u &= \tilde h(\cdot,u) \qquad && \text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

have been recently studied in [41]. Denoting

$$N^* := \begin{cases} \frac{N}{N-2} & \text{if } N>2, \\ +\infty & \text{if } N\le 2, \end{cases} \tag{4.3}$$

one of the main results in [41] can be formulated as follows.

Theorem 4.1.1. *Let $r, \tilde{r} \geq 1$ and let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying the growth conditions*

$$|h(x,u)| \leq C_h(1+|u|^r), \qquad |\tilde{h}(y,u)| \leq C_{\tilde{h}}(1+|u|^{\tilde{r}}), \tag{4.4}$$

for all $x \in \Omega$, $y \in \partial\Omega$ and $u \in \mathbb{R}$. If $N > 2$ assume also

$$\max\Big\{r, \frac{N}{N-1}\tilde{r}\Big\} < N^*. \tag{4.5}$$

Let u be a very weak solution of (4.2) satisfying

$$||h(\cdot,u)||_{L^1(\Omega)} + ||\tilde{h}(\cdot,u)||_{L^1(\partial\Omega)} \leq C_1.$$

Then $u \in L^\infty(\Omega)$ and there exists a constant

$$C = C(C_1, C_h, C_{\tilde{h}}, r, \tilde{r}, N, \Omega) > 0$$

such that

$$||u||_{L^\infty(\Omega)} \leq C.$$

It is well known that the condition $r < N^*$ in (4.5) is also necessary for the boundedness of very weak solutions of (4.2) (see [38]). It was shown in [41] that also the second part of condition (4.5) is essentially optimal: if $N > 2$ and $\tilde{r} > (N-1)/(N-2)$ then there exist Ω and function $\tilde{h}$ satisfying the growth condition in (4.4) such that problem (4.2) with $h \equiv 0$ possesses an unbounded solution.

In the case of elliptic systems with homogeneous Dirichlet or Neumann or Dirichlet-Neumann boundary conditions, similar results have been obtained in [9, 42, 45, 32, 33, 30, 29], cf. also the related scalar results in [34, 27, 18, 7]. In particular, in the case of the system

$$\begin{aligned} -\Delta u &= f(\cdot,u,v), & -\Delta v &= g(\cdot,u,v) & &\text{in } \Omega, \\ \partial_\nu u &= 0, & \partial_\nu v &= 0 & &\text{on } \partial\Omega, \end{aligned} \tag{4.6}$$

the following theorem is a consequence of results proved in [42].

Theorem 4.1.2. *Let $p, q, r, s \geq 1$ an let $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be Carathéodory functions satisfying the growth conditions*

$$|f(x,u,v)| \leq C_f(1+|u|^r+|v|^p),$$
$$|g(x,u,v)| \leq C_g(1+|u|^q+|v|^s),$$

for all $x \in \Omega$ *and* $u, v \in \mathbb{R}$. *If* $N > 2$ *assume also*

$$r, s < N^* \tag{4.7}$$

and

$$\min(p,q) + 1 < N^*(1 + 1/\max(p,q)). \tag{4.8}$$

Let (u, v) *be a very weak solution of* (4.6) *satisfying*

$$||f(\cdot, u, v)||_{L^1(\Omega)} + ||g(\cdot, u, v)||_{L^1(\Omega)} \leq C_1.$$

Then $u, v \in L^\infty(\Omega)$ *and there exists a constant*

$$C = C(C_1, C_f, C_g, p, q, r, s, N, \Omega) > 0$$

such that

$$||u||_{L^\infty(\Omega)} + ||v||_{L^\infty(\Omega)} \leq C.$$

It is again known that the conditions (4.7) and (4.8) are essentially optimal, see [45].

In this chapter we will prove generalizations of Theorems 4.1.1 and 4.1.2 for system (4.1) and more general systems. We also prove that our results are optimal and we apply them to the proof of existence of positive solutions of some particular problems.

In order to present a simple presentation of our results, we introduce the following notation. Assuming the growth conditions

$$\left.\begin{aligned} |f(x,u,v)| &\leq C_f(1 + |u|^r + |v|^p), \\ |g(x,u,v)| &\leq C_g(1 + |u|^q + |v|^s), \\ |\tilde{f}(y,u,v)| &\leq C_{\tilde{f}}(1 + |u|^{\tilde{r}} + |v|^{\tilde{p}}), \\ |\tilde{g}(y,u,v)| &\leq C_{\tilde{g}}(1 + |u|^{\tilde{q}} + |v|^{\tilde{s}}), \end{aligned}\right. \tag{4.9}$$

for all $x \in \Omega$, $y \in \partial\Omega$ and $u, v \in \mathbb{R}$, we set

$$\left.\begin{aligned} &\mathsf{P} := \max\Big\{p, \tilde{p} + \frac{1}{N-2}\Big\}, && \mathsf{Q} := \max\Big\{q, \tilde{q} + \frac{1}{N-2}\Big\}, \\ &\mathcal{P} := \max\Big\{p, \frac{N}{N-1}\tilde{p}\Big\}, && \mathcal{Q} := \max\Big\{q, \frac{N}{N-1}\tilde{q}\Big\}, \\ &\mathcal{R} := \max\Big\{r, \frac{N}{N-1}\tilde{r}\Big\}, && \mathcal{S} := \max\Big\{s, \frac{N}{N-1}\tilde{s}\Big\}, \end{aligned}\right\} \tag{4.10}$$

provided $N > 2$. Using this notation, our first main result reads as follows.

Theorem 4.1.3. *Let* $p, q, r, s \geq 1$, $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s} \geq 0$, *and let* $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *and* $\tilde{f}, \tilde{g} : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$ *be Carathéodory functions satisfying* (4.9). *If* $N > 2$ *assume also*

$$\mathcal{R}, \mathcal{S} < N^* \tag{4.11}$$

and

$$\min\{\mathsf{P}, \mathsf{Q}\} + 1 < N^*(1 + 1/\max\{\mathcal{P}, \mathcal{Q}\}). \tag{4.12}$$

Let (u, v) *be a very weak solution of* (4.1) *satisfying*

$$\begin{aligned} &||f(\cdot, u, v)||_{L^1(\Omega)} + ||g(\cdot, u, v)||_{L^1(\Omega)} \\ &\quad + ||\tilde{f}(\cdot, u, v)||_{L^1(\partial\Omega)} + ||\tilde{g}(\cdot, u, v)||_{L^1(\partial\Omega)} \leq C_1. \end{aligned} \tag{4.13}$$

Then $u, v \in L^\infty(\Omega)$ *and there exists a constant*

$$C = C(C_1, C_f, C_g, C_{\tilde{f}}, C_{\tilde{g}}, p, q, r, s, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}, N, \Omega) > 0 \tag{4.14}$$

such that

$$||u||_{L^\infty(\Omega)} + ||v||_{L^\infty(\Omega)} \leq C.$$

Notice that Theorem 4.1.3 implies both Theorem 4.1.1 (by choosing $f = f(x, u)$, $\tilde{f} = \tilde{f}(y, u)$, $g = g(x, v)$, $\tilde{g} = \tilde{g}(y, v)$, $p = q = 1$ and $\tilde{p} = \tilde{q} = 0$) and Theorem 4.1.2 (by choosing $\tilde{f} = \tilde{g} = 0$, $\tilde{p} = \tilde{q} = \tilde{r} = \tilde{s} = 0$).

Results for scalar problems (see [38, 41]) guarantee that condition (4.11) is optimal in the following sense: If $\max\{\mathcal{R}, \mathcal{S}\} > N^*$ then one can find a domain Ω and functions $f, g, \tilde{f}, \tilde{g}$ satisfying the growth conditions (4.9) such that (4.1) possesses an unbounded very weak solution. Similarly, the following theorem shows that the condition (4.12) is optimal (except for the equality case).

Theorem 4.1.4. *Let* $N > 2$, $p, q \geq 1$, $\tilde{p}, \tilde{q} \geq 0$ *and*

$$\min\{\mathsf{P}, \mathsf{Q}\} + 1 > N^*(1 + 1/\max\{\mathcal{P}, \mathcal{Q}\}). \tag{4.15}$$

Then there exist Ω *and* $f, g, \tilde{f}, \tilde{g}$ *satisfying the growth conditions* (4.9) *with* $r = s = 1$ *and* $\tilde{r} = \tilde{s} = 0$ *such that problem* (4.1) *possesses a positive unbounded very weak solution.*

Similarly as in [42], our results on a priori estimates can be used for the proof of existence of nontrivial solutions, provided one can estimate the right-hand sides in L^1.

This is, in general, a non-trivial task (see [42, Section 3] in the case of homogeneous Dirichlet boundary conditions). In what follows we provide a few typical problems where the L^1-bounds and existence of positive solutions can be proved.

Given $K > 0$, we denote by $\lambda_K^{\mathcal{N}} > 0$ the first eigenvalue of the problem

$$-\Delta\varphi + K\varphi = 0 \quad \text{in } \Omega, \qquad \partial_\nu\varphi = \lambda\varphi \quad \text{on } \partial\Omega. \tag{4.16}$$

The proof of the following proposition is based on rather standard arguments which use multiplication of the differential equations in (4.1) with the first eigenfunction in (4.16) and integration by parts.

Proposition 4.1.5. *Let $f, g, \tilde{f}, \tilde{g}$ satisfy the assumptions of Theorem 4.1.3. Assume, in addition, that there exist $t \in [N/(N-1), N^*)$, $\alpha, \beta, \kappa, c_1 > 0$, $K \geq 0$ and $\mu > \lambda_K^{\mathcal{N}}$ ($\mu = 0$ if $K = 0$) such that*

$$\begin{aligned} \alpha f + \beta g + c_1 &\geq \max\big\{-K(\alpha u + \beta v), \kappa \max\{f, g\} - c_1(u^t + v^t)\big\}, \\ \alpha\tilde{f} + \beta\tilde{g} + c_1 &\geq \max\big\{\mu(\alpha u + \beta v), \kappa \max\{\tilde{f}, \tilde{g}\} - c_1(u^{\tilde{t}} + v^{\tilde{t}})\big\}, \end{aligned} \tag{4.17}$$

for all $x \in \Omega$, $y \in \partial\Omega$ and $u, v \geq 0$, where $f = f(x, u, v)$, $g = g(x, u, v)$, $\tilde{f} = \tilde{f}(y, u, v)$, $\tilde{g} = \tilde{g}(y, u, v)$ and $\tilde{t} = t(N-1)/N$. Then there exists a positive constant C depending on $\alpha, \beta, \kappa, c_1, t, K, \mu$ and all the parameters in (4.14) except for C_1 such that any positive very weak solution (u, v) of (4.1) satisfies

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \leq C. \tag{4.18}$$

It is easy to see that, for example, functions $f(x, u, v) = v^p - u^r$ and $g(x, u, v) = u^q - v^s$ satisfy the assumption in (4.17) with $\alpha = \beta = 1$, $K = 0$, $\kappa = 1$ and $c_1 = 2$ provided $\min\{p, q\} > \max\{r, s\}$ (one can choose $t > \max\{r, s\}$). The same is true for $f(u, v) = uv$ and $g(u, v) = -uv$ if $N \leq 3$ (one can choose $t = 2$ and $\alpha = \beta = 1$, $K = 0$, $\kappa = 2$ and $c_1 = 1$). In the following proposition we consider the case $f(x, u, v) = -u$ and $g(x, u, v) = -v$ since this choice corresponds to problems which have been extensively studied by other methods.

Proposition 4.1.6. *Let $N > 2$. Consider problem (4.1) with $f(x, u, v) = -u$, $g(x, u, v) = -v$ and Carathéodory functions $\tilde{f}, \tilde{g} \geq 0$ satisfying the growth conditions in (4.9) with*

$$\tilde{r}, \tilde{s} < \frac{N-1}{N-2}, \qquad \tilde{p} \leq \tilde{q} < \frac{N-1}{N-4}, \qquad \tilde{p}(N-2) < 1 + \frac{N-1}{\tilde{q}}.$$

Assume that there exist $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}, c_1 > 0$ *and* $\varepsilon < \lambda_1^{\mathcal{N}} < \mu$ *such that*

$$\alpha \tilde{f}(y, u, v) + \beta \tilde{g}(y, u, v) \geq \mu(\alpha u + \beta v) - c_1 \tag{4.19}$$

for all $y \in \partial\Omega$ *and* $u, v \geq 0$*, and*

$$\tilde{\alpha} \tilde{f}(y, u, v) + \tilde{\beta} \tilde{g}(y, u, v) \leq \varepsilon(\tilde{\alpha} u + \tilde{\beta} v) \tag{4.20}$$

for all $y \in \partial\Omega$ *and* $u, v \geq 0$ *small. Then* (4.1) *possesses a positive bounded solution* (u, v)*.*

Existence of nontrivial solutions of problem (4.1) with $f(x, u, v) = -u$, $g(x, u, v) = -v$ and superlinear $\tilde{f}, \tilde{g}$ has been studied by several authors, see [10, 11, 12, 26, 44], for example. In [10], the authors proved the existence via a priori estimates of classical positive solutions based on scaling arguments and Liouville-type theorems (cf. also the use of such arguments for related scalar problems in [23], for example). In comparison with Proposition 4.1.6, the scaling arguments require a very specific asymptotic behavior of the nonlinearities for large u, v. On the other hand, in general, scaling arguments and optimal Liouville theorems usually provide a priori estimates in a larger range of exponents (see [43, Chapter I], for example). Unfortunately, optimal Liouville theorems for systems are very difficult to prove (see [46] and the references therein) and the authors of [10] also had to require the technical assumption $\tilde{p}, \tilde{q} \leq N^*$. Notice that we do not need such restriction: if $p = q = 1$ and $\tilde{p}$ is sufficiently small then we only need $\tilde{q} < (N-1)/(N-4)$.

The papers [11, 12, 26, 44] study the existence for the problem in Proposition 4.1.6 in the variational (Hamiltonian or gradient) case by using variational methods which do not yield a priori estimates of solutions. But even working in the restricted class of variational problems, the authors of all those papers except for [12] also assume $\tilde{p}, \tilde{q} \leq N^*$. In [12], the authors just need $\tilde{p}, \tilde{q} < (N+1)/(N-3)$ and the intrinsic assumption

$$\frac{1}{\tilde{p}+1} + \frac{1}{\tilde{q}+1} > \frac{N-2}{N-1}. \tag{4.21}$$

Condition (4.21) seems to be optimal for the existence and a priori estimates of classical positive solutions in the case of the model problem $\tilde{f}(y, u, v) = v^{\tilde{p}}$ and $\tilde{g}(y, u, v) = u^{\tilde{q}}$ (cf. also the analogous condition in [46] in the case of the Lane-Emden system). Notice that our stronger condition (4.24) is essentially optimal just for the boundedness of very weak solutions of the general class of problems, but not for a priori estimates of classical positive solutions of the model problem.

Proposition 4.1.5 requires a linear lower bound on a suitable linear combination of f and g. If this is not true then one can often try other ad-hoc arguments to verify the necessary L^1 bound. One of such arguments can be found in the proof of the following proposition.

Proposition 4.1.7. *Let $f, g, \tilde{f}, \tilde{g}$ be C^1 functions satisfying the assumptions of Theorem 4.1.3. Assume also*

$$0 \geq f(x,u,v), g(x,u,v) \geq -c_1(1+u^t+v^z) \quad \text{for all } x \in \Omega,\ u,v \geq 0,$$

$$\left.\begin{aligned} \tilde{f}(y,u,v), \tilde{g}(y,u,v) &\geq -c_2(1+u^{\tilde{t}}+v^{\tilde{z}}) \\ \tilde{f}(y,u,v)+\tilde{g}(y,u,v) &\geq c_3(u^{\tilde{t}}+v^{\tilde{z}})-c_4 \end{aligned}\right\} \quad \text{for all } y \in \partial\Omega,\ u,v \geq 0,$$

where $\tilde{t} > t \geq 1$, $\tilde{z} > z \geq 1$ and $c_1, c_2, c_3, c_4 > 0$. Then there exists a positive constant C depending on $t, z, \tilde{t}, \tilde{z}, c_1, c_2, c_3, c_4$ and all the parameters in (4.14) except for C_1 such that any positive very weak solution (u,v) of (4.1) satisfies

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \leq C.$$

4.2 Very weak solutions

As in [41], by $L^1(\Omega \times \partial\Omega)$ we denote the space of functions $u : \overline{\Omega} \to \overline{\mathbb{R}}$ such that $u|_\Omega \in L^1(\Omega)$ and $u|_{\partial\Omega} \in L^1(\partial\Omega)$.

Definition 4.2.1. *A couple $(u,v) \in L^1(\Omega \times \partial\Omega) \times L^1(\Omega \times \partial\Omega)$ is called a very weak solution of (4.1) if $f, g \in L^1(\Omega)$, $\tilde{f}, \tilde{g} \in L^1(\partial\Omega)$ and*

$$\int_\Omega (u\Delta\phi + f\phi)\,dx = \int_{\partial\Omega} (u\partial_\nu\phi - \tilde{f}\phi)\,dS$$
$$\int_\Omega (v\Delta\psi + g\psi)\,dx = \int_{\partial\Omega} (v\partial_\nu\psi - \tilde{g}\psi)\,dS$$

for all $\phi, \psi \in C^2(\overline{\Omega})$, where $f = f(\cdot, u|_\Omega, v|_\Omega)$, $g = g(\cdot, u|_\Omega, v|_\Omega)$, $\tilde{f} = \tilde{f}(\cdot, u|_{\partial\Omega}, v|_{\partial\Omega})$ and $\tilde{g} = \tilde{g}(\cdot, u|_{\partial\Omega}, v|_{\partial\Omega})$.

It follows from [41] that if (u,v) is a very weak solution of (4.1) then $u, v \in W^{1,q}(\Omega)$ for $q < N/(N-1)$ and the traces of u and v equal $u|_{\partial\Omega}$ and $v|_{\partial\Omega}$, respectively. In what follows, without fearing confusion, we will use the simple notation u for functions u, $u|_\Omega$, $u|_{\partial\Omega}$ and Tu. It will be always clear from the context, which of the above functions we mean. We will also denote by $\|\cdot\|_{p,\Omega}$ and $\|\cdot\|_{p,\partial\Omega}$ the norms in $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. Finally, throughout this chapter we will also use the notation introduced in (4.10), (4.3).

4.3 A priori estimates

In order to proof Theorem 4.1.3, we will need following lemma which is due to [41], [42].

Lemma 4.3.1. *Let u be a very weak solution of*

$$-\Delta u = h \ \text{in}\ \Omega, \qquad \partial_\nu u = \tilde{h} - u \ \text{on}\ \partial\Omega,$$

where $h \in L^p(\Omega)$ and $\tilde{h} \in L^{\tilde{p}}(\partial\Omega)$ for some $p, \tilde{p} \geq 1$. Let $\tilde{q} \in [1, \infty]$, $q = \tilde{q}N/(N-1)$ and

$$\frac{1}{q} > \max\Big\{\frac{N-1}{N\tilde{p}} - \frac{1}{N}, \frac{1}{p} - \frac{2}{N}\Big\}.$$

Then

$$||u||_{q,\Omega} + ||u||_{\tilde{q},\partial\Omega} \leq C(||h||_{p,\Omega} + ||\tilde{h}||_{\tilde{p},\partial\Omega}).$$

Proof of Theorem 4.1.3. We will assume $N > 2$ (the proof for $N \leq 2$ requires just trivial modifications). In addition, without loss of generality we may assume

$$\mathsf{P} \leq \mathsf{Q}. \tag{4.22}$$

Notice that

$$\mathsf{P} < N^* \text{ if and only if } \mathcal{P} < N^*, \tag{4.23}$$

and similarly for Q and $\mathcal{Q}$. This fact, (4.22) and (4.12) guarantee $\mathsf{P} < N^*$ (otherwise, denoting by (L) and (R) the left- and the right-hand side of (4.12), respectively, we would have $N^* + 1 \leq (\text{L}) < (\text{R}) \leq N^* + 1$). If $\mathcal{Q} < N^*$ then increasing the value of q or $\tilde{q}$ we may achieve $\mathcal{Q} = \mathsf{Q} = N^*$ while (4.12) remains true due to $\min(\mathsf{P}, \mathsf{Q}) < N^*$ and $\max\{\mathcal{P}, \mathcal{Q}\} = N^*$. Consequently, in any case we may assume

$$\max\{\mathsf{P}, \mathcal{P}\} < N^* \leq \min\{\mathsf{Q}, \mathcal{Q}\}.$$

In particular, we have

$$\mathsf{P} < N^* \leq \mathcal{Q} \quad \text{and} \quad \mathsf{P} + 1 < N^*(1 + 1/\mathcal{Q}). \tag{4.24}$$

Without changing the values of $\mathsf{P}, \mathcal{Q}$, we can further increase the value of p (or $\tilde{p}$) and q (or $\tilde{q}$) in such a way that

$$p = \tilde{p} + \frac{1}{N-2} \quad \text{and} \quad q = \frac{N}{N-1}\tilde{q}. \tag{4.25}$$

Due to (4.11) and $\mathsf{P} < N^*$, we can also increase the values of $r, s, \tilde{r}, \tilde{s}$ to have

$$\max\Big\{p, \frac{N}{N-1}\Big\} < r = s = \frac{N}{N-1}\tilde{r} = \frac{N}{N-1}\tilde{s} < N^*. \tag{4.26}$$

In particular, (4.11) remains true. In the rest of the proof we will assume (4.24), (4.25), (4.26).

By C we denote various positive constants which may vary from step to step but which depend only on the parameters in (4.14). Given $k \in [1, N^*)$ and $\tilde{k} := k(N-1)/N$, Lemma 4.3.1 and (4.13) guarantee

$$\|u\|_{k,\Omega} + \|v\|_{k,\Omega} + \|u\|_{\tilde{k},\partial\Omega} + \|v\|_{\tilde{k},\partial\Omega} \le C, \tag{4.27}$$

where C also depends on k. We will show by a bootstrap argument that we can increase the value of k up to $k = \infty$.

The second inequality in (4.24) and (4.25) guarantee

$$\frac{q}{N^*} < \frac{N}{(pN - 2N^*)_+}, \tag{4.28}$$

the inequality $p < N^*$ and (4.25) imply

$$\Big(\frac{N-1}{N}\Big)^2 p - \frac{N^*(N-1)}{N^2} < \frac{\tilde{p}}{N^*}. \tag{4.29}$$

Hence we can find

$$k_0 \in (r, N^*) \tag{4.30}$$

close to N^* such that (4.28) and (4.29) remain true with N^* replaced by k_0. In particular, we can fix

$$\alpha \in \Big(\frac{q}{k_0}, \frac{N}{(pN - 2k_0)_+}\Big), \tag{4.31}$$

and, setting $\tilde{k}_0 := k_0(N-1)/N > 1$ (cf. (4.26) and (4.30)), we obtain

$$\Big(\frac{N-1}{N}\Big)^2 p - \frac{\tilde{k}_0}{N} < \frac{\tilde{p}}{k_0} < \frac{\tilde{p}}{r}. \tag{4.32}$$

Notice that $q \geq N^* > k_0$ due to (4.25) and the first inequality in (4.24), hence

$$\alpha > q/k_0 > 1.$$

The inequality $\alpha > q/k_0$ also implies $\alpha N > q(N-1) - \alpha k_0$, so that we may fix

$$\beta \in \Big(1, \frac{\alpha N}{(q(N-1)-\alpha k_0)_+}\Big). \tag{4.33}$$

Assume that (4.27) is true for some $k \geq k_0$ and $\tilde{k} = k(N-1)/N$. Using a bootstrap argument in the first equation of system (4.1), we will show

$$\|u\|_{\alpha k,\Omega} + \|u\|_{\alpha\tilde{k},\partial\Omega} \leq C. \tag{4.34}$$

Next, using a bootstrap argument in the second equation of system (4.1), we will show that (4.27) and (4.34) guarantee

$$\|v\|_{\beta k,\Omega} + \|v\|_{\beta\tilde{k},\partial\Omega} \leq C. \tag{4.35}$$

In particular, (4.27) is true with k replaced by $\min(\alpha,\beta)k$. Since the factor

$$\min(\alpha,\beta) > 1$$

is independent of k, after finitely many steps we arrive at (4.27) with $k > Nq$. This estimate, (4.9) and Lemma 4.3.1 guarantee (4.27) with $k = \infty$.

Step 1. Assuming (4.27) with $k \geq k_0$, we will prove (4.34). Since $r < k_0 < N^*$, we can fix

$$\kappa \in \Big(1, \frac{N}{(r(N-1)-k_0)_+}\Big). \tag{4.36}$$

This choice of κ implies

$$\frac{1}{\kappa\eta} > \max\Big\{\frac{(N-1)r}{N\eta} - \frac{1}{N}, \frac{r}{\eta} - \frac{2}{N}\Big\} \quad \text{for all } \eta \geq k. \tag{4.37}$$

Notice that (4.26) guarantees $p < r < k$, $\tilde{p} < \tilde{r}$ and $k\tilde{p} \leq \tilde{k}p$. If $k \leq \eta \leq kr/p$ then (4.9), (4.27) imply

$$\begin{aligned} \|f(\cdot,u,v)\|_{\eta/r,\Omega} &\leq C(1 + \|u\|^r_{\eta,\Omega} + \|v\|^p_{\eta p/r,\Omega}) \leq C(1 + \|u\|^r_{\eta,\Omega}), \\ \|\tilde{f}(\cdot,u,v)\|_{\eta/r,\partial\Omega} &\leq C(1 + \|u\|^{\tilde{r}}_{\tilde{\eta},\partial\Omega} + \|v\|^{\tilde{p}}_{\xi,\partial\Omega}) \leq C(1 + \|u\|^{\tilde{r}}_{\tilde{\eta},\partial\Omega}), \end{aligned} \tag{4.38}$$

where $\tilde{\eta} := \eta(N-1)/N \geq 1$ and $\xi := \max\{1, \eta\tilde{p}/r\} \leq \tilde{k}$. If

$$\|u\|_{\eta,\Omega} + \|u\|_{\tilde{\eta},\partial\Omega} \leq C \tag{4.39}$$

(which is true for $\eta = k$ due to (4.27)) then (4.38) implies

$$\|f(\cdot,u,v)\|_{\eta/r,\Omega} + \|\tilde f(\cdot,u,v)\|_{\eta/r,\partial\Omega} \le C \tag{4.40}$$

and (4.37) together with Lemma 4.3.1 guarantee

$$\|u\|_{\kappa\eta,\Omega} + \|u\|_{\kappa\tilde\eta,\partial\Omega} \le C.$$

Consequently, an obvious bootstrap argument shows that estimates (4.39), (4.40) are true for all $\eta \in [k, kr/p]$. Next (4.32) guarantees

$$\frac{\tilde p}{\tilde k r} > \max\Big\{\frac{(N-1)p}{Nk} - \frac{1}{N}, \frac{p}{k} - \frac{2}{N}\Big\},$$

hence (4.40) with $\eta = kr/p$ and Lemma 4.3.1 imply $\|u\|_{\tilde k\tilde r/\tilde p,\partial\Omega} \le C$, where $\tilde k\tilde r/\tilde p := \infty$ if $\tilde p = 0$. This estimate, (4.9), (4.27) and (4.40) with $\eta = kr/p$ yield

$$\|f(\cdot,u,v)\|_{k/p,\Omega} + \|\tilde f(\cdot,u,v)\|_{\tilde k/\tilde p,\partial\Omega} \le C. \tag{4.41}$$

Since (4.31) guarantees

$$\frac{1}{\alpha k} > \max\Big\{\frac{(N-1)\tilde p}{N\tilde k} - \frac{1}{N}, \frac{p}{k} - \frac{2}{N}\Big\},$$

estimate (4.41) and Lemma 4.3.1 imply (4.34).

Step 2. Assuming (4.27) with $k \ge k_0$ and (4.34), we will prove (4.35). If $r \le \eta \le \alpha kr/q$ and $\tilde\eta := \eta(N-1)/N$ then (4.9), (4.26) and (4.34) guarantee

$$\begin{aligned} \|g(\cdot,u,v)\|_{\eta/r,\Omega} &\le C(1 + \|u\|^q_{\eta q/r,\Omega} + \|v\|^r_{\eta,\Omega}) \le C(1 + \|v\|^r_{\eta,\Omega}), \\ \|\tilde g(\cdot,u,v)\|_{\eta/r,\partial\Omega} &\le C(1 + \|u\|^{\tilde q}_{\xi,\partial\Omega} + \|v\|^{\tilde r}_{\tilde\eta,\partial\Omega}) \le C(1 + \|v\|^{\tilde r}_{\tilde\eta,\partial\Omega}), \end{aligned} \tag{4.42}$$

where $\xi := \max\{1, \eta\tilde q/r\} \le \alpha\tilde k$. Notice also that (4.33) and (4.31) imply

$$\frac{1}{\beta k} > \max\Big(\frac{(N-1)q}{N\alpha k} - \frac{1}{N}, \frac{q}{\alpha k} - \frac{2}{N}\Big). \tag{4.43}$$

If $\alpha r/q \le 1$ then taking $\eta = \alpha kr/q$ in (4.42) and using (4.27), (4.43) and Lemma 4.3.1 we obtain (4.35).

If $\alpha r/q > 1$ then we take κ as in (4.36) so that (4.37) is true. Starting with $\eta = k$, the same bootstrap argument as in the case of inequalities (4.39)-(4.40) yields

$$\|v\|_{\eta,\Omega} + \|v\|_{\tilde\eta,\partial\Omega} \le C$$

and

$$\|g(\cdot,u,v)\|_{\eta/r,\Omega} + \|\tilde g(\cdot,u,v)\|_{\eta/r,\partial\Omega} \le C \tag{4.44}$$

for all $\eta \in [k, \alpha kr/q]$. Now (4.44) with $\eta = \alpha kr/q$, (4.43) and Lemma 4.3.1 implies (4.35). □

4.4 Singular solutions

In what follows we will need the following result from [41].

Lemma 4.4.1. *Let* $H = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 > 0\}$*,* $N > 2$ *and*

$$t > (N-1)/(N-2).$$

Then the problem

$$-w = 0 \quad in\ H, \qquad \partial_\nu w = w^t \quad on\ \partial H,$$

possesses a singular solution of the form $w(x) = h(x)|x|^{-1/(t-1)}$*, where* h *is a* C^2 *function satisfying* $0 < c_1 \le h(x) \le c_2$ *for all* $x \in H$*.*

Proof of Theorem 4.1.4. We may assume $\mathsf{P} \le \mathsf{Q}$ without loss of generality. Notice also that (4.15) guarantees $\max\{\mathcal{P}, \mathcal{Q}\} > N^*$, hence $\mathsf{Q} = \max\{\mathsf{P}, \mathsf{Q}\} > N^*$. If $\mathsf{P} > N^*$ then we can decrease the value of p or $\tilde{p}$ to achieve $\mathsf{P} = N^* - \varepsilon$, where $\varepsilon > 0$ is chosen in such a way that (4.15) remains true. Consequently, we may assume

$$\mathsf{P} < N^* < \mathcal{Q} \quad \text{and} \quad \mathsf{P} + 1 > N^*(1 + 1/\mathcal{Q}). \tag{4.45}$$

Let us consider the following cases separately: (i) $\mathsf{P} = p$, $\mathcal{Q} = q$; (ii) $\mathsf{P} = p$, $\mathcal{Q} = \tilde{q}N/(N-1)$; (iii) $\mathsf{P} = \tilde{p}+1/(N-2)$, $\mathcal{Q} = q$; (iv) $\mathsf{P} = \tilde{p}+1/(N-2)$, $\mathcal{Q} = \tilde{q}N/(N-1)$.

(i) If $\mathsf{P} = p$ and $\mathcal{Q} = q$ then (4.45) guarantees

$$p < q, \qquad p + 1 > N^*\Big(1 + \frac{1}{q}\Big).$$

Set $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$, $\alpha = 2(p+1)/(pq-1)$, $\beta = 2(q+1)/(pq-1)$, and

$$(u(x), v(x)) = \Big(r^{-\alpha} + \alpha\frac{r^2}{2}, r^{-\beta} + \beta\frac{r^2}{2}\Big), \quad r = |x|.$$

Then it is easy to check that (u, v) is a very weak solution of (4.1) with $\tilde{f} = \tilde{g} = 0$, $f(x, u, v) = a(x)v^p - \alpha N$, $g(x, u, v) = b(x)u^q - \beta N$, where $a, b \in L^\infty(\Omega)$, cf. also [45, Remark 3.2].

(ii) If $\mathsf{P} = p$ and $\mathcal{Q} = \tilde{q}N/(N-1)$ then (4.45) implies

$$\frac{\tilde{q}N}{N-1} > p > \frac{2}{N-2} + \frac{N-1}{(N-2)\tilde{q}} \tag{4.46}$$

and $\tilde{q} > (N-1)/(N-2)$. Set

$$q_1 = \frac{\tilde{q}(p+2)}{2\tilde{q}+1}$$

and notice that (4.46) implies

$$q_1 > \frac{N-1}{N-2} \quad \text{and} \quad \frac{p}{q_1-1} < N. \tag{4.47}$$

Let H be the halfspace from Lemma 4.4.1 and consider a smooth bounded domain Ω satisfying $\{x \in \Omega : |x| < 1\} = \{x \in H : |x| < 1\}$. Let $w = w_t$ be the singular solution from Lemma 4.4.1 and set $v := w_{q_1}$. There exist $c_1, c_2 > 0$ such that

$$c_1 \le v(x)|x|^{1/(q_1-1)} \le c_2, \qquad x \in \overline{\Omega} \setminus \{0\}, \tag{4.48}$$

hence $v^p \in L^1(\Omega)$ due to (4.47). Let u be the very weak solution of the linear problem

$$-\Delta u + u = v^p \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega.$$

Then $u(x) = \int_\Omega N(x,z)v^p(z)\,dz$, where the Neumann function N satisfies

$$N(x,z) \ge c|x-z|^{2-N},$$

see [29] and the references therein. Due to the lower bound in (4.48), an easy estimate (cf. [29, Section 5] and [45]) shows

$$u(x) \ge c_3|x|^{-\alpha} - c_4, \qquad x \in \overline{\Omega}, \tag{4.49}$$

where $c_3, c_4 > 0$ and $\alpha := p/(q_1-1) - 2$. The choice of q_1 guarantees

$$\alpha\tilde{q} = q_1/(q_1-1). \tag{4.50}$$

Since $\partial_\nu v = v^{q_1}$ on $\{x \in \partial\Omega : |x| < 1\}$ and $\partial_\nu v$ is bounded on $\{x \in \partial\Omega : |x| \ge 1\}$, estimates (4.48), (4.49) and equality (4.50) guarantee the existence of $b_1, b_2 \in L^\infty(\partial\Omega)$ such that $\partial_\nu v = b_1 u^{\tilde{q}} + b_2$. Consequently, it is sufficient to set $f(x,u,v) = -u + v^p$, $\tilde{f} = 0$, $g = 0$ and $\tilde{g}(y,u,v) = b_1(y)u^{\tilde{q}} + b_2(y)$.

(iii) If $\mathsf{P} = \tilde{p} + 1/(N-2)$ and $\mathcal{Q} = q$ then (4.45) implies

$$\tilde{p} + \frac{1}{N-2} < q, \qquad \tilde{p}(N-2) > 1 + \frac{N}{q}, \tag{4.51}$$

$q > N^*$ and $\tilde{p} < (N-1)/(N-2)$. Set

$$p_1 = \frac{\tilde{p}(q+2)}{2\tilde{p}+1}$$

and notice that (4.51) and $\tilde{p} < (N-1)/(N-2)$ imply

$$p_1 > \frac{N-1}{N-2} \quad \text{and} \quad \frac{q}{p_1-1} < N. \tag{4.52}$$

Let Ω and w_t be as in (ii) and set $u := w_{p_1}$. There exist $c_1, c_2 > 0$ such that

$$c_1 \le u(x)|x|^{1/(p_1-1)} \le c_2, \qquad x \in \overline{\Omega} \setminus \{0\}, \tag{4.53}$$

hence $u^q \in L^1(\Omega)$ due to (4.52). Let v be the very weak solution of the linear problem

$$-\Delta v + v = u^q \quad \text{in } \Omega, \qquad \partial_\nu v = 0 \quad \text{on } \partial\Omega.$$

As in (ii) we obtain

$$v(x) \ge c_3|x|^{-\beta} - c_4, \qquad x \in \overline{\Omega}, \tag{4.54}$$

where $c_3, c_4 > 0$ and $\beta := q/(p_1-1) - 2$. The choice of p_1 implies

$$\beta\tilde{p} = p_1/(p_1-1). \tag{4.55}$$

Since $\partial_\nu u = u^{p_1}$ on $\{x \in \partial\Omega : |x| < 1\}$ and $\partial_\nu u$ is bounded on $\{x \in \partial\Omega : |x| \ge 1\}$, estimates (4.53), (4.54) and equality (4.55) guarantee the existence of $b_1, b_2 \in L^\infty(\partial\Omega)$ such that $\partial_\nu u = b_1 v^{\tilde{p}} + b_2$. Consequently, it is sufficient to set $f = 0$, $\tilde{f}(y,u,v) = b_1(y)v^{\tilde{p}} + b_2(y)$, $g(x,u,v) = -v + u^q$, $\tilde{g} = 0$.

(iv) If $\mathsf{P} = \tilde{p} + 1/(N-2)$ and $\mathcal{Q} = \tilde{q}N/(N-1)$ then (4.45) implies

$$\tilde{p} + \frac{1}{N-2} < \frac{\tilde{q}N}{N-1}, \qquad \tilde{p}\tilde{q} - \frac{\tilde{q}}{N-2} > \frac{N-1}{N-2} \tag{4.56}$$

and $\tilde{p} < (N-1)/(N-2) < \tilde{q}$. Set

$$p_1 := (\tilde{q}+1)\frac{\tilde{p}}{\tilde{p}+1} \quad \text{and} \quad q_1 := (\tilde{p}+1)\frac{\tilde{q}}{\tilde{q}+1}.$$

The second inequality in (4.56) and $\tilde{p} < \tilde{q}$ imply $p_1, q_1 > (N-1)/(N-2)$. In addition, the choice of p_1, q_1 also implies

$$\tilde{p} = \frac{p_1}{p_1-1}(q_1-1), \qquad \tilde{q} = \frac{q_1}{q_1-1}(p_1-1). \tag{4.57}$$

Let Ω and w_t be as in (ii) and set $u := w_{p_1}$, $v := w_{q_1}$. Then there exist $c_1, c_2 > 0$ such that

$$c_1 \le u(x)|x|^{1/(p_1-1)},\ v(x)|x|^{1/(q_1-1)} \le c_2, \qquad x \in \partial\Omega \setminus \{0\}. \tag{4.58}$$

Since $\partial_\nu u = u^{p_1}$ and $\partial_\nu v = v^{q_1}$ on $\{x \in \partial\Omega : |x| < 1\}$ and $u, v, \partial_\nu u, \partial_\nu$ are bounded on $\{x \in \partial\Omega : |x| \geq 1\}$, (4.57) and inequalities (4.58) guarantee the existence of $a_1, a_2, b_1, b_2 \in L^\infty(\partial\Omega)$ such that

$$\partial_\nu u = a_1 v^{\tilde{p}} + a_2 \quad \text{and} \quad \partial_\nu v = b_1 u^{\tilde{q}} + b_2 \qquad \text{on } \partial\Omega.$$

Consequently, it is sufficient to set $f = g = 0$ and $\tilde{f}(y, u, v) = a_1(y) v^{\tilde{p}} + a_2(y)$, $\tilde{g}(y, u, v) = b_1(y) u^{\tilde{q}} + b_2(y)$. □

4.5 L^1-bounds and existence

Proof of Proposition 4.1.5. Let $\varphi_K^{\mathcal{N}}$ denote the first (positive) eigenfunction of problem (4.16) normalized by $\sup_\Omega \varphi_K^{\mathcal{N}} = 1$. We will write shortly f instead of $f(x, u, v)$ and similarly for $g, \tilde{f}, \tilde{g}$. We also set $w := \alpha u + \beta v$, $h := \alpha f + \beta g$ and $\tilde{h} := \alpha \tilde{f} + \beta \tilde{g}$ so that w solves the problem

$$-\Delta w = h \quad \text{in } \Omega, \qquad \partial_\nu w = \tilde{h} \quad \text{on } \partial\Omega, \tag{4.59}$$

and (4.17) implies

$$\begin{aligned} h + c_1 &\geq \max\{-Kw, \kappa \max\{f, g\} - c_1(u^t + v^t)\}, \\ \tilde{h} + c_1 &\geq \max\{\mu w, \kappa \max\{\tilde{f}, \tilde{g}\} - c_1(u^{\tilde{t}} + v^{\tilde{t}})\}. \end{aligned}$$

As

$$c_1(u^t + v^t) \leq \frac{c_1}{(\min\{\alpha, \beta\})^t}((\alpha u)^t + (\beta v)^t) \leq \frac{c_1}{(\min\{\alpha, \beta\})^t}(\alpha u + \beta v)^t = \frac{c_1}{(\min\{\alpha, \beta\})^t} w^t,$$

we can increase value of c_1 if necessary to

$$c_2 = \max\left\{c_1, \frac{c_1}{(\min\{\alpha, \beta\})^t}\right\}$$

and we obtain

$$\begin{aligned} h + c_2 &\geq \max\{-Kw, \kappa \max\{f, g\} - c_2 w^t\}, \\ \tilde{h} + c_2 &\geq \max\{\mu w, \kappa \max\{\tilde{f}, \tilde{g}\} - c_2 w^{\tilde{t}}\}. \end{aligned} \tag{4.60}$$

First consider the case $K > 0$. Fix $\varepsilon \in (0, K)$ such that $\mu \geq \lambda_{K+\varepsilon}^{\mathcal{N}} + \varepsilon$ and set $\varphi := \varphi_{K+\varepsilon}^{\mathcal{N}}$. The definition of a very weak solution w of (4.59) with test function φ yields

$$\int_\Omega (w \Delta\varphi + h\varphi)\, dx = \int_{\partial\Omega} w \partial_\nu \varphi - \tilde{h}\varphi \, dS.$$

As $\Delta\varphi = (K+\varepsilon)\varphi$ in Ω and $\partial_\nu\varphi = \lambda^{\mathcal{N}}_{K+\varepsilon}\varphi \le (\mu-\varepsilon)\varphi$ on $\partial\Omega$, we obtain

$$\int_\Omega (\varepsilon w + (Kw+h))\varphi\,dx = \int_\Omega (w\Delta\varphi + h\varphi)\,dx = \int_{\partial\Omega} w\partial_\nu\varphi - \tilde h\varphi\,dS$$
$$= \int_{\partial\Omega} \lambda^{\mathcal{N}}_{K+\varepsilon} w\varphi - \tilde h\varphi\,dS \le -\varepsilon\int_{\partial\Omega} w\varphi\,dS + \int_{\partial\Omega}(\mu w - \tilde h)\varphi\,dS$$

and (4.60) guarantee

$$\begin{aligned}\varepsilon\int_\Omega w\varphi\,dx - c_2|\Omega| &\le \int_\Omega (\varepsilon w + (Kw+h))\varphi\,dx \\ \le -\varepsilon\int_{\partial\Omega} w\varphi\,dS + \int_{\partial\Omega}(\mu w - \tilde h)\varphi\,dS &\le -\varepsilon\int_{\partial\Omega} w\varphi\,dS + c_2|\partial\Omega|.\end{aligned} \tag{4.61}$$

This implies a bound for w (hence for u and v) both in $L^1(\Omega)$ and $L^1(\partial\Omega)$. These bounds, (4.60) and (4.61) also imply

$$\|h\|_{1,\Omega} \le C \quad \text{and} \quad \|\tilde h\|_{1,\partial\Omega} \le C. \tag{4.62}$$

If $K=\mu=0$ then we use the test function $\varphi := 1$ and similarly as above we obtain

$$-c_2|\Omega| \le \int_\Omega h\,dx = -\int_{\partial\Omega}\tilde h\,dS \le c_2|\partial\Omega|. \tag{4.63}$$

Since $h, \tilde h \ge -c_2$,

$$\int_\Omega h^-\,dx \le c_2|\Omega| \text{ and } \int_{\partial\Omega}\tilde h^-\,dS \le c_2|\partial\Omega|.$$

Thus (4.63) implies

$$\int_\Omega h^+\,dx \le \int_\Omega h^-\,dx + c_2|\partial\Omega| \le c_2(|\Omega| + |\partial\Omega|),$$

$$\int_{\partial\Omega}\tilde h^+\,dS \le \int_{\partial\Omega}\tilde h^-\,dS + c_2|\Omega| \le c_2(|\Omega| + |\partial\Omega|)$$

and we obtain (4.62). Hence (4.62) is true for any $K \ge 0$.

Now (4.62) implies that the right-hand sides in the problem (4.59) are bounded in L^1, hence w is bounded in $L^t(\Omega)$ and $L^{\tilde t}(\partial\Omega)$ as $t < N^*$ and the same is true for functions u, v.

Since

$$f, g \le \frac{1}{\kappa}(h + c_2(1+w^t)) := F,$$

$$f \ge \frac{1}{\alpha}(-\beta g - Kw - c_1) \ge \frac{1}{\alpha}(-\beta F - Kw - c_1),$$

$$g \geq \frac{1}{\beta}(-\alpha f - Kw - c_1) \geq \frac{1}{\beta}(-\alpha F - Kw - c_1)$$

and F, w are bounded in $L^1(\Omega)$, we see that f and g are bounded in $L^1(\Omega)$. Analogously we obtain the boundedness of $\tilde{f}, \tilde{g}$ in $L^1(\partial\Omega)$ as

$$\tilde{f}, \tilde{g} \leq \frac{1}{\kappa}(\tilde{h} + c_2(1 + w^{\tilde{t}})) := \tilde{F},$$

$$\tilde{f} \geq \frac{1}{\alpha}(\mu w - \beta \tilde{g} - c_1) \geq \frac{1}{\alpha}(-\beta \tilde{F} - c_1),$$
$$\tilde{g} \geq \frac{1}{\beta}(\mu w - \alpha \tilde{f} - c_1) \geq \frac{1}{\beta}(-\alpha \tilde{F} - c_1)$$

and $\tilde{F}$ is bounded in $L^1(\partial\Omega)$.

Hence we obtain

$$||f||_{1,\Omega} + ||g||_{1,\Omega} + ||\tilde{f}||_{1,\partial\Omega} + ||\tilde{g}||_{1,\partial\Omega} \leq C(\Omega, \alpha, \beta, \kappa, K, t, c_1, \mu).$$

Now Theorem 4.1.3 guarantees estimate (4.18). □

Proof of Proposition 4.1.6 will be modification of the arguments leading to [10, Theorem 3.2]. In this proof, we will apply the following fixed-point theorem (see for instance Theorem 3.1 in [20]):

Theorem 4.5.1. *Let $\mathcal{C}$ be a closed convex cone in a Banach space X and $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ a compact mapping such that $\mathcal{S}(0) = 0$. assume that there are real numbers $0 < r < R$ and t_0 such that*

1. *$x \neq t\mathcal{S}x$ for $t \in [0, 1]$ and $x \in \mathcal{C}$, $||x|| = r$.*

2. *There exists a compact mapping $H : \overline{B}_R \times [0, \infty) \to \mathcal{C}$ (where $B_\rho = \{x \in \mathcal{C} : ||x|| < \rho\}$) such that*

   (a) *$H(x, 0) = S(x)$ for $||x|| = R$.*

   (b) *$H(x, t) = x$ has no solution in $\overline{B}_R$ for $t \geq t_0$.*

   (c) *$H(x, t) \neq x$ for $||x|| = R$ and $t > 0$.*

Then $\mathcal{S}$ has a fixed point in $U = \{x \in \mathcal{C} : r < ||x|| < R\}$.

Now, we will state following lemma:

Lemma 4.5.2. *Let* $\lambda_1^{\mathcal{N}}$ *be first positive eigenvalue and* $\varphi_1^{\mathcal{N}}$ *corresponding positive eigenfunction of* (4.16) *with* $K = 1$*:*

$$\left.\begin{aligned} -\Delta\varphi &= -\varphi \qquad \text{in } \Omega, \\ \partial_\nu\varphi &= \lambda^{\mathcal{N}}\varphi \qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{4.64}$$

If $\mu > \lambda_1^{\mathcal{N}}$ *there is no nonnegative nontrivial solution of* (4.64) *of*

$$\left.\begin{aligned} -\Delta w &= -w \qquad \text{in } \Omega, \\ \partial_\nu w &\geq \mu w \qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{4.65}$$

Proof. Assume that w is a nonnegative solution of (4.65). Testing w with first eigenfunction $\varphi_1^{\mathcal{N}}$ satisfying (4.64) implies

$$0 = \int_\Omega w(\Delta\varphi_1^{\mathcal{N}} - \varphi_1^{\mathcal{N}})\, dx = \int_{\partial\Omega} w\partial_\nu\varphi_1^{\mathcal{N}} - \partial_\nu w\varphi_1^{\mathcal{N}}\, dS.$$

Hence we obtain

$$0 \leq (\lambda_1^{\mathcal{N}} - \mu)\int_{\partial\Omega} w\varphi_1^{\mathcal{N}}\, dS,$$

which is a contradiction unless $w \equiv 0$.

□

Proof of Proposition 4.1.6. To apply Theorem 4.5.1, we proceed as follows. Consider the space

$$X = \{(u, v) : u, v \in C(\overline{\Omega})\}$$

with the norm $||(u, v)|| := ||u||_\infty + ||v||_\infty$. Notice that Theorem 4.1.3 guarantees that all very weak solutions of (4.1) with $f(x, u, v) = -u$, $g(x, u, v) = -v$ are bounded, hence classical by the standard regularity theory. Let $\mathcal{S} : X \to X$ be the solution operator defined by $\mathcal{S}(\phi, \psi) = (u, v)$, where (u, v) is the unique solution of

$$\left.\begin{aligned} -\Delta u &= -u, & -\Delta v &= -v & &\text{in } \Omega, \\ \partial_\nu u &= \tilde{f}(\cdot, \phi, \psi), & \partial_\nu v &= \tilde{g}(\cdot, \phi, \psi) & &\text{on } \partial\Omega. \end{aligned}\right\}$$

We observe that a fixed point of $\mathcal{S}$ is a solution of (4.1) with $f(x, u, v) = -u$ and $g(x, u, v) = -v$.

Now let us prove that $\mathcal{S}$ satisfies the hypothesis of Theorem 4.5.1. Theorem 4.1.3, standard regularity and embedding results ensure that $\mathcal{S}$ is compact, see [2], [41].

Thanks to assumption (4.20), $\tilde{f}(y,0,0) = \tilde{g}(y,0,0) = 0$ for all $y \in \partial\Omega$. We have $\mathcal{S}(0,0) = 0$.

Let $\mathcal{C}$ be the cone $\mathcal{C} := \{(u,v) \in X : u \geq 0 \quad v \geq 0\}$. It follows from the maximum principle that $\mathcal{S}(\mathcal{C}) \subset \mathcal{C}$, see Theorem 8.7 in [2]. To show that (1) in Theorem 4.5.1 is true, we argue by contradiction. Let us assume that for every small $r > 0$ there exists $t \in [0,1]$ and a pair $(U,V) \in \mathcal{C}$ such that $||(U,V)|| = r$ and $(U,V) = tS(U,V)$. Hence (U,V) satisfies

$$\begin{aligned} -\Delta U &= -U, & -\Delta V &= -V & &\text{in } \Omega,\\ \partial_\nu U &= t\tilde{f}(\cdot,U,V), & \partial_\nu V &= t\tilde{g}(\cdot,U,V) & &\text{on } \partial\Omega. \end{aligned}$$

Using $\varphi_1^{\mathcal{N}}$ as a test function in Definition 4.2.1 implies

$$\left.\begin{aligned} 0 = \int_\Omega U(\Delta\varphi_1^{\mathcal{N}} - \varphi_1^{\mathcal{N}})\,dx &= \int_{\partial\Omega} U\partial_\nu\varphi_1^{\mathcal{N}} - t\tilde{f}(y,U,V)\varphi_1^{\mathcal{N}}\,dS\\ &= \int_{\partial\Omega} (\lambda_1^{\mathcal{N}}U - t\tilde{f}(y,U,V))\varphi_1^{\mathcal{N}}\,dS,\\ 0 = \int_\Omega V(\Delta\varphi_1^{\mathcal{N}} - \varphi_1^{\mathcal{N}})\,dx &= \int_{\partial\Omega} V\partial_\nu\varphi_1^{\mathcal{N}} - t\tilde{g}(y,U,V)\varphi_1^{\mathcal{N}}\,dS\\ &= \int_{\partial\Omega} (\lambda_1^{\mathcal{N}}V - t\tilde{g}(y,U,V))\varphi_1^{\mathcal{N}}\,dS. \end{aligned}\right\} \tag{4.66}$$

Now we multiply first equality in (4.66) by $\tilde{\alpha}$ and second equality by $\tilde{\beta}$ and we add them together to obtain

$$0 = \int_{\partial\Omega} (\lambda_1^{\mathcal{N}}(\tilde{\alpha}U + \tilde{\beta}V) - t(\tilde{\alpha}\tilde{f}(y,U,V) + \tilde{\beta}\tilde{g}(y,U,V)))\varphi_1^{\mathcal{N}}\,dS.$$

As r can be arbitrarily small, assumption (4.20) yields to

$$0 \geq (\lambda_1^{\mathcal{N}} - \varepsilon t)\int_{\partial\Omega} (\tilde{\alpha}U + \tilde{\beta}V)\varphi_1^{\mathcal{N}}\,dS,$$

which is a contradiction as $t \in [0,1]$ and $\varepsilon < \lambda_1^{\mathcal{N}}$.

To verify (2) we define H as follows:

$$H((\phi,\psi),t) = \mathcal{S}(\phi + t, \psi + t).$$

As $H((\phi,\psi),0) = \mathcal{S}(\phi,\psi)$ for any $(\phi,\psi) \in \mathcal{C}$, (a) clearly holds true. To see (b), we proceed by contradiction. Assume that for every $t > 0$, we have $(U,V) \in \mathcal{C}$ such that $H((U,V),t) = (U,V)$ which means that (U,V) solves

$$\begin{aligned} -\Delta U &= -U, & -\Delta V &= -V & &\text{in } \Omega,\\ \partial_\nu U &= \tilde{f}(\cdot,U+t,V+t), & \partial_\nu V &= \tilde{g}(\cdot,U+t,V+t) & &\text{on } \partial\Omega. \end{aligned}$$

Let us define $w = \alpha U + \beta V$ and $\tilde{h}(\cdot) = \alpha \tilde{f}(\cdot, U+t, V+t) + \beta \tilde{g}(\cdot, U+t, V+t)$. Hence w solves

$$\begin{aligned} -\Delta w &= -w, \quad \text{in } \Omega, \\ \partial_\nu w &= \tilde{h}, \quad \text{on } \partial\Omega \end{aligned}$$

and $\tilde{h}(\cdot) \geq \mu(\alpha(U+t) + \beta(V+t)) - c_1 = \mu w + \mu t(\alpha+\beta) - c_1$. We observe that for any $t \geq t_0 := c_1/\mu(\alpha+\beta)$, assumption (4.19) guarantees that w satisfies

$$\begin{aligned} -\Delta w &= -w, \quad \text{in } \Omega, \\ \partial_\nu w &\geq \mu w, \quad \text{on } \partial\Omega, \end{aligned}$$

for $\mu > \lambda_1^{\mathcal{N}}$ which contradicts Lemma 4.5.2.

Condition (c) is an immediate consequence of the fact that for given $T > 0$, Proposition 4.1.5 and Theorem 4.1.3 guarantee uniform a priori estimates for positive solutions of (4.1) with $f(x,u,v) = -u$, $g(x,u,v) = -v$ and with $\tilde{f}(\cdot,u,v), \tilde{g}(\cdot,u,v)$ replaced by $\tilde{f}(\cdot, u+\tau, v+\tau), \tilde{g}(\cdot, u+\tau, v+\tau)$, $\tau \in [0,T]$. □

Proof of Proposition 4.1.7. Theorem 4.1.3 guarantees that all very weak solutions are bounded, hence classical by the standard regularity theory. Since $\Delta u^t \geq 0$ due to $f \leq 0$, there exists a constant $C_\Omega > 0$ such that $\int_\Omega u^t\,dx \leq C_\Omega \int_{\partial\Omega} u^t\,dS$, and similarly for v^z (cf. also the same argument in [16, p. 46]). Now the definition of a very weak solution, assumptions of Proposition 4.1.7, the above estimates for u^t, v^z and Hölder's inequality imply

$$\begin{aligned} 0 &= \int_\Omega (f(x,u,v) + g(x,u,v))\,dx + \int_{\partial\Omega} (\tilde{f}(y,u,v) + \tilde{g}(y,u,v))\,dS \\ &\geq -2c_1 \int_\Omega (1 + u^t + v^z)\,dx + \int_{\partial\Omega} (c_3(u^{\tilde{t}} + v^{\tilde{z}}) - c_4)\,dS \\ &\geq -2c_1 c_\Omega \int_{\partial\Omega} (u^t + v^z)\,dS + c_5 \Big[\Big(\int_{\partial\Omega} u^t\,dS \Big)^{\tilde{t}/t} + \Big(\int_{\partial\Omega} v^z\,dS \Big)^{\tilde{z}/z} \Big] - c_6, \end{aligned}$$

where $c_5 > 0$ depends on c_3 and Ω and c_6 depends on c_1, c_4 and Ω. As $\tilde{t}/t$, $\tilde{z}/z > 1$ these estimates imply the boundedness of $\|u\|_{t,\partial\Omega}$ $\|v\|_{z,\partial\Omega}$. As

$$\int_\Omega u^t\,dx + \int_\Omega v^z\,dx \leq C_\Omega \Big(\int_{\partial\Omega} u^t\,dS + \int_{\partial\Omega} v^z\,dS \Big)$$

we obtain the boundedness of $\|u\|_{t,\Omega}$, $\|v\|_{z,\Omega}$. This one together with lower estimates and negativity of f, g imply the boundedness of $\|f\|_{1,\Omega}$ and $\|g\|_{1,\Omega}$ as well. A repeated use of the estimates above and the lower bounds for $\tilde{f}, \tilde{g}$ yield

$$\begin{aligned} C \geq \|f\|_{1,\Omega} + \|g\|_{1,\Omega} &= \int_{\partial\Omega} (\tilde{f}(y,u,v) + \tilde{g}(y,u,v))\, dS \\ &\geq c_3 \int_{\partial\Omega} (u^{\tilde{t}} + v^{\tilde{z}})\, dS - c_4|\partial\Omega|. \end{aligned}$$

Thus we obtain the bound for $\|u\|_{\tilde{t},\partial\Omega}$, $\|v\|_{\tilde{z},\partial\Omega}$. Now the negativity of f, g and the boundedness of $\|f\|_{1,\Omega}$ and $\|g\|_{1,\Omega}$ imply

$$\begin{aligned} \int_{\partial\Omega} (\tilde{f}(y,u,v) + \tilde{g}(y,u,v))\, dS &= -\int_{\Omega} (f(x,u,v) + g(x,u,v))\, dx \leq C, \\ \int_{\partial\Omega} (\tilde{f}^+(y,u,v) + \tilde{g}^+(y,u,v))\, dS &\leq C + \int_{\partial\Omega} (\tilde{f}^-(y,u,v) + \tilde{g}^-(y,u,v))\, dS \\ &\leq C + 2c_2 \int_{\partial\Omega} (1 + u^{\tilde{t}} + v^{\tilde{z}})\, dx. \end{aligned}$$

and finally the boundedness of $\|u\|_{\tilde{t},\partial\Omega}$, $\|v\|_{\tilde{z},\partial\Omega}$ yields the boundedness of $\|\tilde{f}\|_{1,\partial\Omega}$, $\|\tilde{g}\|_{1,\partial\Omega}$. □

4.6 Nonlocal problems

In this section we study problems with nonlocal nonlinearities. Problems involving nonlocal (typically integral) operators appear in many applications in physics, biology or control theory: see the list of references in [17], for example. Nonlocal problems also appear if one reduces a system of local equations by expressing one of the unknowns as a convolution of the corresponding kernel and right-hand side: such reduction has been often used for the Schrödinger-Poisson(-Slater) or FitzHugh-Nagumo systems, for example.

We will consider problems of the form

$$\begin{aligned} -\Delta u &= \mathcal{F}(u,v,Tu,Tv), & -\Delta v &= \mathcal{G}(u,v,Tu,Tv) \quad \text{in } \Omega, \\ \partial_\nu u &= \tilde{\mathcal{F}}(u,v,Tu,Tv), & \partial_\nu v &= \tilde{\mathcal{G}}(u,v,Tu,Tv) \quad \text{on } \Omega, \end{aligned} \tag{4.67}$$

where Tu denotes the trace of u on the boundary $\partial\Omega$ and $\mathcal{F}, \mathcal{G}, \tilde{\mathcal{F}}, \tilde{\mathcal{G}}$ are possibly nonlocal operators. Fix $k \in (1, N^*)$, set $\tilde{k} := k(N-1)/N$, and, given $u, v \in L^k(\Omega)$ and $\tilde{u}, \tilde{v} \in L^{\tilde{k}}(\partial\Omega)$, set

$$\|(u,v,\tilde{u},\tilde{v})\|_k := \|u\|_{L^k(\Omega)} + \|v\|_{L^k(\Omega)} + \|\tilde{u}\|_{L^{\tilde{k}}(\partial\Omega)} + \|\tilde{v}\|_{L^{\tilde{k}}(\partial\Omega)},$$

$$\|(u,v,\tilde u,\tilde v)\|_F := \|\mathcal{F}(u,v,\tilde u,\tilde v)\|_{L^1(\Omega)} + \|\mathcal{G}(u,v,\tilde u,\tilde v)\|_{L^1(\Omega)} \\ + \|\tilde{\mathcal{F}}(u,v,\tilde u,\tilde v)\|_{L^1(\partial\Omega)} + \|\tilde{\mathcal{G}}(u,v,\tilde u,\tilde v)\|_{L^1(\partial\Omega)},$$

and

$$\|(u,v,\tilde u,\tilde v)\|_{k,F} := \|(u,v,\tilde u,\tilde v)\|_k + \|(u,v,\tilde u,\tilde v)\|_F.$$

Repeating word by word the arguments in the proof of Theorem 4.1.3 below one can see that the following theorem is true.

Theorem 4.6.1. *Let $p,q,r,s\ge 1$, $\tilde p,\tilde q,\tilde r,\tilde s\ge 0$, $k\in(1,N^*)$, and let $\mathcal{F},\mathcal{G},\tilde{\mathcal{F}},\tilde{\mathcal{G}}$ satisfy the following growth conditions:*

$$\begin{aligned} |\mathcal{F}(u,v,\tilde u,\tilde v)(x)| &\le C_{\mathcal{F}}(\|(u,v,\tilde u,\tilde v)\|_{k,F})(1+|u(x)|^r+|v(x)|^p),\\ |\mathcal{G}(u,v,\tilde u,\tilde v)(x)| &\le C_{\mathcal{G}}(\|(u,v,\tilde u,\tilde v)\|_{k,F})(1+|u(x)|^q+|v(x)|^s),\\ |\tilde{\mathcal{F}}(u,v,\tilde u,\tilde v)(y)| &\le C_{\tilde{\mathcal{F}}}(\|(u,v,\tilde u,\tilde v)\|_{k,F})(1+|\tilde u(y)|^{\tilde r}+|\tilde v(y)|^{\tilde p}),\\ |\tilde{\mathcal{G}}(u,v,\tilde u,\tilde v)(y)| &\le C_{\tilde{\mathcal{G}}}(\|(u,v,\tilde u,\tilde v)\|_{k,F})(1+|\tilde u(y)|^{\tilde q}+|\tilde v(y)|^{\tilde s}), \end{aligned} \tag{4.68}$$

for almost all $x\in\Omega$, $y\in\partial\Omega$ and all $u,v\in L^k(\Omega)$, $\tilde u,\tilde v\in L^{\tilde k}(\partial\Omega)$, where functions $C_{\mathcal{F}},C_{\mathcal{G}},C_{\tilde{\mathcal{F}}},C_{\tilde{\mathcal{G}}}$ are bounded on bounded sets. If $N>2$ assume also

$$\mathcal{R},\mathcal{S} < N^* \tag{4.69}$$

and

$$\min\{\mathsf{P},\mathsf{Q}\}+1 < N^*(1+1/\max\{\mathcal{P},\mathcal{Q}\}). \tag{4.70}$$

Let (u,v) be a very weak solution of (4.67) *satisfying*

$$\|(u,v,Tu,Tv)\|_F \le C_1.$$

Then $u,v\in L^\infty(\Omega)$ and there exists a constant

$$C = C(C_1,C_{\mathcal{F}},C_{\mathcal{G}},C_{\tilde{\mathcal{F}}},C_{\tilde{\mathcal{G}}},p,q,r,s,\tilde p,\tilde q,\tilde r,\tilde s,N,\Omega) > 0$$

such that

$$||u||_{L^\infty(\Omega)} + ||v||_{L^\infty(\Omega)} \le C.$$

Remark 4.6.2. The growth assumption (4.68) in Theorem 4.6.1 can be weakened in the following way: Instead of the first condition in (4.68) it is sufficient to assume that there exists a nonlocal operator $\mathcal{F}_1$ and a locally bounded function

$$C_{\mathcal{F}} : [1,\infty)\times[0,\infty)\to[0,\infty)$$

such that

$$|\mathcal{F}(u,v,\tilde{u},\tilde{v})(x)| \le \mathcal{F}_1(u,v,\tilde{u},\tilde{v})(x)(1+|u(x)|^r+|v(x)|^p),$$
$$\|\mathcal{F}_1(u,v,\tilde{u},\tilde{v})\|_{L^t(\Omega)} \le C_{\mathcal{F}}(t,\|(u,v,\tilde{u},\tilde{v})\|_{k,F}), \tag{4.71}$$

where $k \in (1,N^*)$ may depend on t so $k=k(t)$ (and similarly for $\mathcal{G},\tilde{\mathcal{F}},\tilde{\mathcal{G}}$). Then the conclusion of Theorem 4.6.1 remain true. In fact, choose $\varepsilon>0$ such that the conditions (4.69) and (4.70) remain true if we replace $p,q,r,s,\tilde{p},\tilde{q},\tilde{r},\tilde{s}$ with $p+\varepsilon, q+\varepsilon, r+\varepsilon, s+\varepsilon, \tilde{p}+\varepsilon, \tilde{q}+\varepsilon, \tilde{r}+\varepsilon, \tilde{s}+\varepsilon$. Then, given $\eta\ge 1$, Minkowski and Hölder inequalities and (4.71) with t large enough guarantee

$$\begin{aligned}
\|\mathcal{F}(u,v,\tilde{u},\tilde{v})\|_{L^\eta(\Omega)} &\le \Big(\int_\Omega (\mathcal{F}_1(u,v,\tilde{u},\tilde{v})(x)(1+|u(x)|^r+|v(x)|^p))^\eta dx\Big)^{\frac{1}{\eta}}\\
&\le \|\mathcal{F}_1(u,v,\tilde{u},\tilde{v})\|_{L^\eta(\Omega)} + \|\mathcal{F}_1(u,v,\tilde{u},\tilde{v})\|_{L^{\frac{(r+\varepsilon)\eta}{\varepsilon}}(\Omega)}\|u\|^r_{L^{\eta(r+\varepsilon)}(\Omega)}+\\
&\quad \|\mathcal{F}_1(u,v,\tilde{u},\tilde{v})\|_{L^{\frac{(p+\varepsilon)\eta}{\varepsilon}}(\Omega)}\|v\|^p_{L^{\eta(p+\varepsilon)}(\Omega)}\\
&\le C_{\mathcal{F}}(\eta,\|(u,v,\tilde{u},\tilde{v})\|_{k(\eta),F})+\\
&\quad C_{\mathcal{F}}\Big((r+\varepsilon)\eta/\varepsilon,\|(u,v,\tilde{u},\tilde{v})\|_{k((r+\varepsilon)\eta/\varepsilon),F}\Big)\|u\|^r_{L^{\eta(r+\varepsilon)}(\Omega)}+\\
&\quad C_{\mathcal{F}}\Big((p+\varepsilon)\eta/\varepsilon,\|(u,v,\tilde{u},\tilde{v})\|_{k((p+\varepsilon)\eta/\varepsilon),F}\Big)\|v\|^p_{L^{\eta(p+\varepsilon)}(\Omega)}\\
&\le C(1+\|u\|^r_{L^{\eta(r+\varepsilon)}(\Omega)}+\|v\|^p_{L^{\eta(p+\varepsilon)}(\Omega)})\\
&\le \tilde{C}(1+\|u\|^{r+\varepsilon}_{L^{\eta(r+\varepsilon)}(\Omega)}+\|v\|^{p+\varepsilon}_{L^{\eta(p+\varepsilon)}(\Omega)})
\end{aligned}$$

with $C,\tilde{C}$ depending on $C_{\mathcal{F}}(t,\|(u,v,\tilde{u},\tilde{v})\|_{k,F})$, and this estimate is sufficient for our bootstrap arguments, cf. the first estimate in (4.38) below. □

Next we show some typical problems where Theorem 4.6.1 or Remark 4.6.2 can be used.

Example 4.6.3. Consider the system

$$\begin{aligned}
-\Delta u &= F(\cdot,u,v,w), \quad -\Delta v = G(\cdot,u,v,w), \quad -\Delta w = H(\cdot,u,v) \quad \text{in } \Omega,\\
\partial_\nu u &= \tilde{f}(\cdot,u,v), \qquad \partial_\nu v = \tilde{g}(\cdot,u,v), \qquad w=0 \qquad \text{on } \partial\Omega,
\end{aligned} \tag{4.72}$$

where $F,G,H,\tilde{f},\tilde{g}$ are Carathéodory functions. If G_D denotes the Green function for the negative Dirichlet Laplacian in Ω then the solution w of the last boundary value problem in (4.72) can be written as $w=K(u,v)$, where

$$K(u,v)(x) := \int_\Omega G_D(x,z)H(z,u(z),v(z))\,dz.$$

Consequently, the first equation in (4.72) can be written as $-\Delta u = \mathcal{F}(u,v)$, where

$$\mathcal{F}(u,v)(x) := F(x, u(x), v(x), K(u,v)(x)),$$

and similarly for the second equation.

Assume for simplicity $N \in \{3,4\}$,

$$\begin{aligned} |F(x,u,v,w)| &\leq |f(x,u,v)|(1+|w|^m), \\ |G(x,u,v,w)| &\leq |g(x,u,v)|(1+|w|^m), \\ |H(x,u,v)| &\leq C(1+|u|^\kappa+|v|^\kappa), \quad \kappa = \begin{cases} 2 & \text{if } N=3, \\ 1 & \text{if } N=4, \end{cases} \end{aligned} \tag{4.73}$$

where $m \geq 1$ and $f, g, \tilde{f}, \tilde{h}$ satisfy the assumptions of Theorem 4.1.3. Then the conclusions of Theorem 4.1.3 remain true for problem (4.72). In fact, due to the well-known estimate $0 \leq G_D(x,z) \leq C|x-z|^{2-N}$ and (4.73) we have for arbitrary $t \in (1,\infty)$

$$\|K(u,v)\|_{L^t(\Omega)} \leq C\Big(\int_\Omega \Big(\int_\Omega |x-z|^{2-N}(1+|u|^\kappa+|v|^\kappa)dz\Big)^t dx\Big)^{\frac{1}{t}}. \tag{4.74}$$

Moreover Lemma (4.3.1) implies that $|u|^\kappa, |v|^\kappa \in L^{\frac{k}{\kappa}}(\Omega)$ for any $k \in [\kappa, N^*)$. According to Theorem 1.33 in [35], the Riesz potential I_2 defined by

$$I_2\varphi(x) := \int_\Omega |x-z|^{2-N}\varphi(z)dz$$

is bounded operator from $L^{\frac{k}{\kappa}}(\Omega)$ to $L^t(\Omega)$ provided $t < \frac{N\frac{k}{\kappa}}{N-2\frac{k}{\kappa}}$, satisfying estimate

$$||I_2\varphi||_{L^t(\Omega)} \leq C||\varphi||L^{\frac{k}{\kappa}}(\Omega).$$

Hence in case $N=3$, which gives $\kappa=2$ and $N^*=3$ we can choose

$$k \in \Big(3 - \frac{4}{2t+3}, 3\Big)$$

and in case $N=4$ with $\kappa=1$ and $N^*=2$ we can choose

$$k \in \Big(\max\Big\{1, 2-\frac{4}{t+2}\Big\}, 2\Big),$$

which is possible for arbitrary $t \geq 1$. Thus (4.74) becomes

$$\begin{aligned} \|K(u,v)\|_{L^t(\Omega)} &\leq C(1+\|I_2|u|^\kappa\|_{L^t(\Omega)} + \|I_2|v|^\kappa\|_{L^t(\Omega)}), \\ &\leq C(1+\||u|^\kappa\|_{L^{\frac{k}{\kappa}}(\Omega)} + \||v|^\kappa\|_{L^{\frac{k}{\kappa}}(\Omega)}), \\ &\leq C(1+\|u\|^\kappa_{L^k(\Omega)} + \|v\|^\kappa_{L^k(\Omega)}). \end{aligned}$$

Hence $\mathcal{F}(u,v)$ satisfies

$$\begin{aligned}|\mathcal{F}(u,v)(x)| &\leq |f(x,u,v)|(1+|K(u,v)(x)|^m),\\ &\leq \mathcal{F}_1(u,v)(x)(1+|u(x)|^r+|u(x)|^p),\end{aligned}$$

where $\mathcal{F}_1(u,v)(x) := C(1+|K(u,v)(x)|^m)$ satisfies

$$\begin{aligned}\|\mathcal{F}_1(u,v)\|_{L^\eta(\Omega)} &\leq C(1+\|K(u,v)\|^m_{L^{m\eta}(\Omega)})\\ &\leq C_{\mathcal{F}}(t,\|(u,v,\tilde{u},\tilde{v})\|_{k,F}) \quad \text{with } t=m\eta\end{aligned}$$

and similar estimates hold true for $\mathcal{G}(u,v)$.

Consequently, the assumptions of Remark 4.6.2 are satisfied. □

Example 4.6.4. Consider the problem

$$\begin{aligned} -\Delta u &= auv+bu, & -\Delta v &= cu & &\text{in } \Omega,\\ \partial_\nu u &= 0, & \partial_\nu v &= -\tilde{g}(v)+\Phi(\tilde{g}(v)) & &\text{on } \partial\Omega,\end{aligned} \tag{4.75}$$

where $N\leq 3$, $\Phi(w)(y) := \int_{\partial\Omega}\varphi(y,z)w(z)\,dS_z$, $\varphi\in L^\infty$, $\tilde{g}$ is a continuous function satisfying the growth condition $|\tilde{g}(v)|\leq C(1+|v|^{\tilde{s}})$ and a,b,c are real constants. The system of equations in (4.75) describes a steady state in a (slight modification of a) nuclear reactor model, where u and v correspond to the neutron flux and reactor temperature, respectively; cf. [28, system (6)–(7)]. The nonlocal nonlinear boundary condition in (4.75) appears in the radiative heat transfer problem: $\tilde{g}(v)$ is the surface radiation flux density ($\tilde{g}(v)=\sigma v^4$ in the case of a black body) and $\Phi(\tilde{g}(v))(y)$ is the radiation flux density absorbed at the point y, see [4, 19] and the references therein. It is easily seen that all the assumptions of Theorem 4.6.1 are satisfied provided $N=2$ and $\tilde{s}$ is arbitrary or $N=3$ and $\tilde{s}<2$. □

Example 4.6.5. In the examples above, just the first part of the norm $\|\cdot\|_{k,F}$ was used to guarantee the required growth condition. To see that the second part can also be useful, let us consider the following model problem

$$\begin{aligned} -\Delta u &= \|v\|^{p_1}_{L^\alpha(\Omega)}f(\cdot,u,v)-au, & -\Delta v &= \|u\|^{q_1}_{L^\beta(\Omega)}g(\cdot,u,v)-bv & &\text{in } \Omega,\\ \partial_\nu u &= \tilde{f}(\cdot,u,v), & \partial_\nu v &= \tilde{g}(\cdot,u,v) & &\text{on } \partial\Omega,\end{aligned} \tag{4.76}$$

where $a,b\in\mathbb{R}$, $\alpha,\beta\geq 1$, $p_1,q_1>0$, $f,g,\tilde{f},\tilde{g}$ satisfy the assumptions of Theorem 4.1.3 and $f,g\geq c_0$ for some $c_0>0$. If $\max(\alpha,\beta)>N^*$ then the norm $\|(u,v,\tilde{u},\tilde{v})\|_k$ does not estimate the nonlocal terms for any $k < N^*$ but the norm $\|(u, v, \tilde{u}, \tilde{v})\|_{k,F}$ does and the growth condition (4.68) is trivially satisfied. Consequently, the conclusion of Theorem 4.1.3 is true. If, for example, $\tilde{f}(x, u, v) = \tilde{g}(x, u, v) = 0$, $a, b > 0$,

$$c_0 \le f = f(x, u) \le C_f(1 + |u|^r), \qquad c_0 \le g = g(x, u) \le C_g(1 + |v|^s), \tag{4.77}$$

where $r, s < N^* < \alpha, \beta$, and either $p_1, q_1 > 1$ or f, g have superlinear growth at infinity, then using constant test functions in the definition of a very weak solution yields the required L^1 bound for the right-hand sides in (4.76). Indeed, using $\phi = 1$ in Definition 4.2.1 implies

$$\begin{aligned} \|v\|^{p_1}_{L^\alpha(\Omega)} \int_\Omega f(x, u, v)\, dx &= a \int_\Omega u\, dx, \\ \|u\|^{q_1}_{L^\beta(\Omega)} \int_\Omega g(x, u, v)\, dx &= b \int_\Omega v\, dx. \end{aligned} \tag{4.78}$$

If $p_1, q_1 > 1$, the Hölder inequality and the boundedness of Ω guarantee the existence of the constant C such that

$$\begin{aligned} \|u\|_{L^1(\Omega)} &\le C(\beta)\|u\|_{L^\beta(\Omega)}, \\ \|v\|_{L^1(\Omega)} &\le C(\alpha)\|v\|_{L^\alpha(\Omega)}. \end{aligned}$$

This together with (4.77) imply

$$\begin{aligned} c_0|\Omega|\, \|v\|^{p_1}_{L^\alpha(\Omega)} &\le a\|u\|_{L^1(\Omega)} \le a\, C(\beta)\|u\|_{L^\beta(\Omega)} = \frac{aC(\beta)}{(c_0\,|\Omega|)^{1/q_1}} \Big(c_0\,|\Omega|\, \|u\|^{q_1}_{L^\beta(\Omega)}\Big)^{1/q_1} \\ &\le \frac{a\, C(\beta)}{(c_0|\Omega|)^{1/q_1}} \Big(\|u\|^{q_1}_{L^\beta(\Omega)} \int_\Omega g(x, v)\, dx\Big)^{1/q_1} \le \frac{a\, C(\beta)}{(c_0|\Omega|)^{1/q_1}} \Big(b\|v\|_{L^1(\Omega)}\Big)^{1/q_1} \\ &\le a\, C(\beta)\Big(\frac{b\, C(\alpha)}{c_0|\Omega|}\Big)^{1/q_1} \|v\|^{1/q_1}_{L^\alpha(\Omega)}. \end{aligned}$$

As $p_1, q_1 > 1$ this implies the boundedness of $\|v\|_{L^\alpha(\Omega)}$, hence the boundedness of $\|u\|_{L^\beta(\Omega)}, \|u\|_{L^1(\Omega)}$ and $\|v\|_{L^1(\Omega)}$ as well.

In case of superlinear growth at infinity of f, g, (4.77) implies the existence of $K > 0$ such that

$$f(x, u) \ge Ku \quad g(x, v) \ge Kv \quad \text{for all } x \in \Omega.$$

This with (4.78) yield

$$\begin{aligned} \Big(K\, \|v\|^{p_1}_{L^\alpha(\Omega)} - a\Big) \int_\Omega u\, dx &\le \int_\Omega \|v\|^{p_1}_{L^\alpha(\Omega)} f(x, u) - au\, dx = 0, \\ \Big(K\, \|u\|^{q_1}_{L^\beta(\Omega)} - b\Big) \int_\Omega v\, dx &\le \int_\Omega \|u\|^{q_1}_{L^\beta(\Omega)} g(x, v) - bv\, dx = 0, \end{aligned}$$

which implies the boundedness of $\|u\|_{L^\beta(\Omega)}$, $\|v\|_{L^\alpha(\Omega)}$, $\|u\|_{L^1(\Omega)}$ and $\|v\|_{L^1(\Omega)}$ again.

Now the boundedness of $\|f\|_{L^1(\Omega)}$ and $\|g\|_{L^1(\Omega)}$ is guaranteed by (4.77) and the boundedness of norms above.

The existence of positive solutions of the system of equations in (4.76) complemented by homogeneous Dirichlet boundary conditions was studied in [15] in the sublinear case. More precisely, if $N \geq 3$ then the authors in [15] assumed (a technical generalization of) (4.77), $a = b = 0$, $1 < \alpha, \beta < 2N^*$, $\max\{p_1 + r, q_1 + s\} < 1$, $r < \beta - 1$, $s < \alpha - 1$. □

Remark 4.6.6. Nonlocal operators frequently appear in the control theory. In particular, feedback operators usually depend on the solution in a nonlocal way. In the case of distributed observation, they often have the form that we consider here. Since we are mainly interested in the boundedness of very weak solutions, it does not have too much sense for us to consider operators depending on the solution in a specific point: this would require to work with solutions which are a priori bounded. On the other hand, one can approximate the value $u(x_0)$ of the solution in a given point $x_0 \in \Omega$ by an integral of the form $\int_\Omega \omega(x_0 - z)u(z)\,dz$ (where ω is a smooth approximation of the Dirac distribution), and such integrals fit well in our theory. □

Summary

In this thesis, we improved results on regularity and a priori estimates of positive very weak solutions of elliptic systems. First, we considered elliptic system complemented by Dirichlet boundary conditions and we derived conditions on growth exponents of right-hand sides guaranteeing essential boundedness of all possible positive very weak solutions and their a priori estimates. Our proof was based on alternate-bootstrap arguments where we were dealing with a significant amount of growth exponents. Similarly to [32], our results hold true if we treat variational solutions or L^1-solutions of (1) and (2) provided we replace critical growth exponent for very weak solutions by corresponding critical growth exponent for variational or L^1-solutions. Our example of system (1), (2) satisfies growth conditions on right-hand sides required by our theorem. Hence all positive very weak solutions of such a system are a priori bounded thanks to our results. However functions f, g of our system do not satisfy assumptions required by [32] that clearly shows that we improved results in [32].

We also considered elliptic systems complemented by nonlinear boundary conditions. We derived optimal conditions on growth of right-hand sides guaranteeing a priori estimates of positive very weak solutions of such systems. Similarly as in the case of [41] and [42], our proofs are based on regularity results for linear problems and alternate-bootstrap arguments. Due to the presence of nonlinear boundary conditions, one has to prove simultaneous estimates for the solutions and their traces on the boundary $\partial\Omega$. This difficulty and also presence of significant amount of growth exponents make our proofs to be far from a trivial modification of the proofs in [41] and [42]. Another justification of our computations comes from the fact that the optimal growth conditions for system (1) and (3) could hardly be guessed just from the the corresponding conditions in [41] and [42].

We also proved that our results are optimal. We showed that there exist a domain

and right-hand sides which do not satisfy required conditions on growth, such that elliptic problem with nonlinear boundary conditions possesses a positive unbounded very weak solution.

We used our results on a priori estimates to prove existence of nontrivial solutions of few typical problems where the L^1-bounds of right-hand sides can be estimated.

One of the advantages of using alternate bootstrap method is its robustness. It does require neither scaling properties nor variational or local structure. Hence, our results could also be applied for problems with nonlocal nonlinearities. We also showed applications of our results in the study of some particular nonlocal problems.

Resumé

V tejto práci sa zaoberáme regularitou a apriórnymi odhadmi kladných veľmi slabých riešení eliptických systémov. Takéto systémy popisujú rôzne situácie v biológii, fyzike alebo chémii.

A priori je latinský výraz, ktorý znamená vopred. Pod pojmom apriórny odhad máme na mysli odhad o veľkosti riešení bez toho, aby sme mali informáciu o existencii riešenia daného sytému. Presnejšie, v tejto práci pod pojmom apriórny odhad myslíme, že všetky možné kladné riešenia (v danej triede funkcií) eliptického systému sú ohraničené kladnou konštantou C nezávislou od riešenia.

Apriórne odhady zohrávajú dôležitú úlohu v dokazovaní existencie riešenia problému. Vskutku, pokiaľ úloha nemá variačnú štruktúru, na existenciu riešenia treba použiť iné, nevariačné metódy ako napríklad topologické, a tie zvyčajne vyžadujú znalosť apriórnych odhadov pre všetky možné riešenia. Naviac, apriórne odhady poskytujú informácie o štruktúre riešení a využívajú sa pri skúmaní bifurkačných vetiev.

Presnejšie, zaujímame sa o systémy tvaru

$$\left.\begin{array}{rcl} -\Delta u &=& f(.\,,u,v) \\ -\Delta v &=& g(.\,,u,v) \end{array}\right\} \quad \text{v } \Omega, \tag{4.79}$$

spolu s Drichletovými okrajovými podmienkami

$$\left.\begin{array}{rcl} u &=& 0 \\ v &=& 0 \end{array}\right\} \quad \text{na } \partial\Omega, \tag{4.80}$$

alebo nelineárnymi okrajovými podmienkami tvaru

$$\left.\begin{array}{rcl} \partial_\nu u &=& \tilde{f}(.\,,u,v) \\ \partial_\nu v &=& \tilde{g}(.\,,u,v) \end{array}\right\} \quad \text{na } \partial\Omega, \tag{4.81}$$

kde $f, g, \tilde{f}, \tilde{g}$ sú Caratheódoryho funkcie s vhodným polynomiálnym rastom a Ω je hladká ohraničená oblasť v $\mathbb{R}^N$.

Existujú rôzne metódy na získanie apriórnych odhadov. Technika zvaná „blow-up" bola prvýkrát použitá v [25] v prípade skalárnej úlohy. Metóda je založená na dôkaze sporom. Predpokladá sa, že existuje postupnosť riešení, ktorá nie je ohraničená. Po vhodnom preškálovaní a vybraní podpostupnosti sa získa postupnosť, ktorá konverguje ku kladnému riešeniu eliptickej úlohy v celom priestore (alebo v polpriestore). Existencia takéhoto riešenia je ale v rozpore s vetou Liouvilleovho typu. Táto metóda vedie k optimálnym výsledkom vzhľadom k rastu pravých strán, pokiaľ sú známe príslušné vety Liouvilleovho typu. V prípade systému (4.79), (4.80) je však znalosť viet Liouvilleovho typu často otvorený problém.

Ďaľšou používanou metódou je metóda Rellichových-Pohozaeovových identít a „moving planes". Prvýkrát bola použitá na dokázanie apriórnych odhadov riešení skalárnej úlohy v [21]. Metóda pozostáva z viacerých krokov. V prípade systému (4.79), (4.80), sa riešenia t najprv odhadnú v blízkosti hranice Ω pomocou metódy „moving planes", načo je potrebné, aby boli nelinearity nezávislé od x a neklesajúce. Následne sa použijú identity Rellichovho-Pohozaevovho typu. Tieto identity obmedzujú použi- teľnosť tejto metódy pre prípad funkcií $f = f(v)$ a $g = g(u)$. Naviac, Ω musí byť konvexná, alebo musia byť splnené ďaľšie technické predpoklady na f a g. Táto metóda vedie k optimálnym výsledkom v modelovom prípade $f(v) = v^p$ a $g(u) = u^q$, ale často sa nedá použiť v prípade všeobecnejších funkcií f a g.

Metóda Hardy-Sobolevových nerovností bola prvýkrát použitá v [14] v prípade skalárnej úlohy, kde H. Brezis a R. E. L. Turner študovali variačné riešenia skalárnej úlohy. Táto metóda je založená na použití prvej vlastnej funkcie Laplaceovej rovnice ako testovacej funkcie. To vedie k odhadu nelinearity, ktorý spolu s vhodnými rastovými predpokladmi a Hardy-Sobolevovými nerovnosťami implikuje H^1 ohraničenosť. V prípade systému (4.79), (4.80), táto metóda vyžaduje iba horné ohraničenie na rast nelinearít f, g, ale nevedie k optimálnym výsledkom vzhľadom k rastu pravých strán.

Na odvodenie apriórnych odhadov sa používa aj takzvaná „bootstrap" metóda. Procedúra spočíva v splnení istých predpokladov, ktoré naštartujú proces, vedúci v konečnom počte krokov k žiadanému výsledku. Presnejšie, pokiaľ informácie o lepšej regularite f a/alebo g zaručia lepšiu regularitu riešenia a následne lepšia regularita riešenia spolu s rastovými predpokladmi na f, g implikuje lepšiu regularitu f, g, stačí dokázať na počiatku lepšiu regularitu f, g a overiť, že sa tým spustí iterovaný proces, ktorý vedie v konečnom počte krokov k žiadanej regularite a apriórnym odhadom riešení. Táto metóda sa použila na odvodenie apriórnych odhadov riešení rôznych úloh či systémov ako napríklad v [29, 32, 33, 34, 41, 42]. V [42], P. Quittner a Ph. Souplet použili nový druh metódy striedavého „bootstrapu”, ktorá viedla k značnému vylepšeniu dovtedy známych výsledkov o apriórnych odhadoch a existencii riešení systému (4.79) spolu s (4.80). Táto metóda sa môže používať za slabších počiatočných predpokladov na riešenie narozdiel od metódy „blow-up” či metódy Rellichových-Pohozaeovových rovností a „moving planes”, ktoré vyžadujú variačné alebo klasické riešenia.

Eliptické systémy s Dirichletovými okrajovými podmienkami

Ako sme už spomínali, P. Quittner a Ph. Souplet v [42] použili novú metódu striedavého „bootstrapu”. Táto bola nedávno vylepšená v [32]. Y. Li [32] získal apriórne odhady veľmi slabých riešení eliptického systému (4.79) s Dirichletovými okrajovými podmienkami (4.80) za všeobecnejších predpokladov na rast f, g ako v [42]:

$$\left.\begin{array}{rcccl} 0 & \leq & f(x,u,v) & \leq & C_1(1+|u|^r|v|^p+|u|^\gamma), \\ 0 & \leq & g(x,u,v) & \leq & C_1(1+|u|^q|v|^s+|v|^\sigma), \end{array}\right\} \tag{4.82}$$

kde

$$r, s, \min\{p+r, q+s\} \in [0, p_c)\,, \tag{4.83}$$

$$\max\{p+1-s, q+1-r\} > \frac{pq-(1-r)(1-s)}{p_c-1}, \tag{4.84}$$

$$1 \leq \gamma, \sigma < p_c \tag{4.85}$$

a platí $p, q > 0$. V podmienkach (4.83), (4.84) a (4.85), p_c predtavuje istý kritický exponent, ktorého hodnota záleží od toho, či skúmame H^1_0-riešenia, L^1-riešenia alebo L^1_δ-riešenia. Y. Li dokázal, že každé kladné L^1_δ-riešenie systému (4.79) spolu s (4.80) spĺňajúce:

$$||u||_{L^1_\delta} + ||v||_{L^1_\delta} \leq M,$$

je apriórne ohraničené:

$$||u||_\infty + ||v||_\infty \leq C, \tag{4.86}$$

kde $C = C(\Omega, p, q, \gamma, \sigma, N, C_1, M)$.

Podarilo sa nám rozšíriť výsledky z článku [32] a dokázali sme ohraničenosť kladných veľmi slabých riešení systému (4.79) s Dirichletovými okrajovými podmienkami (4.80) za všeobecnejších predpokladov na f, g. Nech je p_{BT} definovaný nasledovne:

$$p_{BT} := \begin{cases} \infty, & \text{pre } N < 2, \\ \frac{N+1}{N-1}, & \text{pre } N \geq 2. \end{cases}$$

Výsledky o ohraničenosti a apriórnych odhadoch veľmi slabých riešení systému (4.79) sa dajú zhrnúť do nasledujúcej vety:

Veta 4.6.1. *Nech sú $f, g : \Omega \times \mathbb{R}^2 \to [0, \infty)$ Carathéodoryho funkcie spĺňajúce rastové podmienky* (3.2)

$$\begin{aligned} f(x,u,v) &\leq C_1(1 + |u|^{r_1}|v|^{p_1} + |u|^{r_2}|v|^{p_2} + |u|^{\gamma}), \\ g(x,u,v) &\leq C_1(1 + |u|^{q_1}|v|^{s_1} + |u|^{q_2}|v|^{s_2} + |v|^{\sigma}), \end{aligned}$$

kde $p_i, q_i, r_i, s_i \geq 0$ pre $i = 1,\ 2$, $\max\{p_1, p_2\}, \max\{q_1, q_2\} > 0$ a platí

$$1 \leq \gamma, \sigma < p_{BT}.$$

Predpokladajme tiež, že

$$\left.\begin{aligned} \min\{\max\{p_1 + r_1,\ p_2 + r_2\},\ \max\{q_1 + s_1,\ q_2 + s_2\}\} < p_{BT}, \\ r_i, s_i < p_{BT}, \end{aligned}\right\} \quad i = 1, 2, \tag{4.87}$$

$$\max\{p_i + 1 - s_j,\ q_j + 1 - r_i\} > \frac{p_i q_j - (1 - r_i)(1 - s_j)}{p_{BT} - 1}, \qquad i, j = 1, 2, \tag{4.88}$$

a (u, v) je kladné riešenie systému (4.79) *spĺňajúce*

$$||u||_{L^1_\delta} + ||v||_{L^1_\delta} \leq M. \tag{4.89}$$

Potom patrí (u, v) do $L^\infty(\Omega) \times L^\infty(\Omega)$ a

$$||u||_{L^\infty} + ||v||_{L^\infty} \leq C(\Omega, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, \gamma, \sigma, N, C_1, M). \tag{4.90}$$

Pripomeňme, že p_{BT} je exponent, ktorý sa prvýkrát objavil v práci H. Brezisa a R.E.L. Turnera [14] v prípade apriórnych odhadov variačných riešení skalárnej úlohy. Ukázalo sa (viď [42], [45]), že exponent p_{BT} je kritickým exponentom pre veľmi slabé riešenia eliptických systémov s Dirichletovou okrajovou podmienkou.

Podobne ako v prípade Y. Liho sa kritický exponent pre veľmi slabé riešenia p_{BT} dá nahradiť iným kritickým exponentom, ak skúmame L^1-riešenia alebo variačné riešenia a výsledky Vety 4.6.1 ostanú v platnosti.

Ďalej sme skonštruovali systém (4.79) s pravými stranami:

$$\left.\begin{array}{rcl} f(x,u,v) & = & u^{1-\varepsilon}v + v^{\frac{5}{4}-\varepsilon}, \\ g(x,u,v) & = & u^4 v, \end{array}\right\} \tag{4.91}$$

kde $\varepsilon \in \left(0, \frac{1}{7}\right)$ a $N = 3$. Všimnime si, že $p_{BT} = 2$. Je zrejmé, že každé nezáporné veľmi slabé riešenie (u,v) problému (4.91) patrí do $L^\infty(\Omega) \times L^\infty(\Omega)$ vďaka Vete 4.6.1 pre rastové koeficinety $p_1 = 1 - \varepsilon, r_1 = 1, p_2 = \frac{5}{4} - \varepsilon, r_2 = 0, \gamma = 1, q_1 = 4, s_1 = 1, q_2 = s_2 = 0, \sigma = 1$. Zároveň sme ukázali, že takto definované f, g nespĺňajú Liho podmienky (4.82), (4.83), (4.84) a (4.85).

Eliptické systémy s nelineárnymi okrajovými podmienkami

Ďalej sme sa v práci zaoberali veľmi slabými riešeniami systému (4.79) doplneného nelineárnymi okrajovými podmienkami (4.81).

Regularita a apriórne odhady veľmi slabých riešení príslušnej skalárnej úlohy

$$\begin{aligned} -\Delta u &= h(\cdot, u) \qquad && \text{v } \Omega, \\ \partial_\nu u &= \tilde{h}(\cdot, u) \qquad && \text{na } \partial\Omega, \end{aligned} \tag{4.92}$$

boli nedávno študované v [41]. Označme

$$N^* := \begin{cases} \frac{N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N \le 2, \end{cases} \tag{4.93}$$

jeden z hlavných výsledkov [41] znie nasledovne:

Theorem 4.6.7. *Nech $r, \tilde{r} \ge 1$ a nech sú $h : \Omega \times \mathbb{R} \to \mathbb{R}$ a $\tilde{h} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ Carathéodoryho funkcie spĺňajúce polynomiálny rast*

$$|h(x,u)| \le C_h(1 + |u|^r), \qquad |\tilde{h}(y,u)| \le C_{\tilde{h}}(1 + |u|^{\tilde{r}}), \tag{4.94}$$

pre všetky $x \in \Omega$, $y \in \partial\Omega$ *a* $u \in \mathbb{R}$. *Ak* $N > 2$, *nech naviac platí*

$$\max\Big\{r, \frac{N}{N-1}\tilde{r}\Big\} < N^*. \tag{4.95}$$

Nech je u *veľmi slabé riešenie* (4.92) *také, že*

$$||h(\cdot,u)||_{L^1(\Omega)} + ||\tilde{h}(\cdot,u)||_{L^1(\partial\Omega)} \leq C_1.$$

Potom $u \in L^\infty(\Omega)$ *a existuje konštanta*

$$C = C(C_1, C_h, C_{\tilde{h}}, r, \tilde{r}, N, \Omega) > 0$$

taká, že

$$||u||_{L^\infty(\Omega)} \leq C.$$

Je všeobecne známe, že podmienka $r < N^*$ v (4.95) je zároveň nutnou pre ohraničenosť veľmi slabých riešení (4.92) (viď [38]). P. Quittner a W. Reichel v [41] ukázali, že aj druhá podmienka (4.95) je optimálna: ak $N > 2$ a $\tilde{r} > (N-1)/(N-2)$ potom existuje Ω a funkcia $\tilde{h}$ s rastom (4.94) taká, že problém (4.92) s $h \equiv 0$ má neohraničené riešenie.

V prípade eliptických systémov (4.79) s homogénnou Neumannovou okrajovou podmienkou:

$$\left.\begin{aligned} \partial_\nu u &= 0 \\ \partial_\nu v &= 0 \end{aligned}\right\} \quad \text{na } \partial\Omega, \tag{4.96}$$

vyplýva nasledujúca veta z výsledkov získaných v [42].

Theorem 4.6.8. *Nech* $p, q, r, s \geq 1$ *a nech sú* $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *Carathéodoryho funkcie spĺňajúce polynomiálny rast*

$$|f(x,u,v)| \leq C_f(1 + |u|^r + |v|^p),$$
$$|g(x,u,v)| \leq C_g(1 + |u|^q + |v|^s),$$

pre všetky $x \in \Omega$ *a* $u, v \in \mathbb{R}$. *Ak* $N > 2$, *nech naviac platí*

$$r, s < N^* \tag{4.97}$$

a

$$\min(p,q) + 1 < N^*(1 + 1/\max(p,q)). \tag{4.98}$$

Nech je (u, v) veľmi slabé riešenie systému (4.79), (4.96) *také, že*

$$||f(\cdot, u, v)||_{L^1(\Omega)} + ||g(\cdot, u, v)||_{L^1(\Omega)} \leq C_1.$$

Potom $u, v \in L^\infty(\Omega)$ a existuje konštanta

$$C = C(C_1, C_f, C_g, p, q, r, s, N, \Omega) > 0$$

taká, že

$$||u||_{L^\infty(\Omega)} + ||v||_{L^\infty(\Omega)} \leq C.$$

Je opäť známe, ze podmienky (4.97) a (4.98) sú optimálne, viď [45].

V prípade systémov (4.79) s nelineárnymi okrajovými podmienkami (4.81) sa nám podarilo nájsť optimálne podmienky na rast funkcií $f, g, \tilde{f}, \tilde{g}$ garantujúce apriórnu ohraničenosť veľmi slabých riešení systému:

$$\begin{aligned} -\Delta u &= f(\cdot, u, v), & -\Delta v &= g(\cdot, u, v) \quad \text{in } \Omega, \\ \partial_\nu u &= \tilde{f}(\cdot, u, v), & \partial_\nu v &= \tilde{g}(\cdot, u, v) \quad \text{on } \partial\Omega. \end{aligned} \tag{4.99}$$

Predpokladajme polynomiálny rast funkcií $f, g, \tilde{f}, \tilde{g}$:

$$\begin{aligned} |f(x, u, v)| &\leq C_f(1 + |u|^r + |v|^p), \\ |g(x, u, v)| &\leq C_g(1 + |u|^q + |v|^s), \\ |\tilde{f}(y, u, v)| &\leq C_{\tilde{f}}(1 + |u|^{\tilde{r}} + |v|^{\tilde{p}}), \\ |\tilde{g}(y, u, v)| &\leq C_{\tilde{g}}(1 + |u|^{\tilde{q}} + |v|^{\tilde{s}}), \end{aligned} \tag{4.100}$$

pre všetky $x \in \Omega$, $y \in \partial\Omega$ a $u, v \in \mathbb{R}$. V záujme prehľadnosti označme

$$\left.\begin{aligned} \mathsf{P} &:= \max\Big\{p, \tilde{p} + \frac{1}{N-2}\Big\}, & \mathsf{Q} &:= \max\Big\{q, \tilde{q} + \frac{1}{N-2}\Big\}, \\ \mathcal{P} &:= \max\Big\{p, \frac{N}{N-1}\tilde{p}\Big\}, & \mathcal{Q} &:= \max\Big\{q, \frac{N}{N-1}\tilde{q}\Big\}, \\ \mathcal{R} &:= \max\Big\{r, \frac{N}{N-1}\tilde{r}\Big\}, & \mathcal{S} &:= \max\Big\{s, \frac{N}{N-1}\tilde{s}\Big\}, \end{aligned}\right\} \tag{4.101}$$

pre $N > 2$. Dokázali sme platnosť nasledujúcej vety:

Theorem 4.6.9. *Nech $p, q, r, s \geq 1$, $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s} \geq 0$ a nech sú $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ a $\tilde{f}, \tilde{g} : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$ Carathéodoryho funkcie spĺňajúce* (4.100). *Pokiaľ $N > 2$, nech platí aj*

$$\mathcal{R}, \mathcal{S} < N^* \tag{4.102}$$

a

$$\min\{\mathsf{P},\mathsf{Q}\}+1 < N^*(1+1/\max\{\mathcal{P},\mathcal{Q}\}). \tag{4.103}$$

Nech je (u,v) *veľmi slabé riešnie systému* (4.99) *také, že*

$$\begin{aligned} &||f(\cdot,u,v)||_{L^1(\Omega)}+||g(\cdot,u,v)||_{L^1(\Omega)} \\ &\quad+||\tilde{f}(\cdot,u,v)||_{L^1(\partial\Omega)}+||\tilde{g}(\cdot,u,v)||_{L^1(\partial\Omega)} \le C_1. \end{aligned} \tag{4.104}$$

Potom $u,v\in L^\infty(\Omega)$ *a existuje konštanta*

$$C = C(C_1, C_f, C_g, C_{\tilde{f}}, C_{\tilde{g}}, p, q, r, s, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}, N, \Omega) > 0$$

taká, že

$$||u||_{L^\infty(\Omega)}+||v||_{L^\infty(\Omega)} \le C.$$

Všimnime si, že z Vety 4.6.9 vyplývajú Veta 4.6.7 (výberom $f=f(x,u)$, $\tilde{f}=\tilde{f}(y,u)$, $g=g(x,v)$, $\tilde{g}=\tilde{g}(y,v)$, $p=q=1$ a $\tilde{p}=\tilde{q}=0$) ako aj Veta 4.6.8 (výberom $\tilde{f}=\tilde{g}=0$, $\tilde{p}=\tilde{q}=\tilde{r}=\tilde{s}=0$).

Výsledky pre skalárnu úlohu (viď [38, 41]) garantujú optimálnosť podmienky (4.102) v nasledujúcom zmysle: Ak $\max\{\mathcal{R},\mathcal{S}\} > N^*$, potom existuje Ω a funkcie $f,g,\tilde{f}$ a $\tilde{g}$ s rastom daným podmienkou (4.100) také, že (4.99) má neohraničené veľmi slabé riešenie. Podobne, nasledujúca veta ukazuje, že podmienka (4.103) je optimálna (až na kritický prípad).

Theorem 4.6.10. *Nech* $N>2$, $p,q\ge 1$, $\tilde{p},\tilde{q}\ge 0$ *a*

$$\min\{\mathsf{P},\mathsf{Q}\}+1 > N^*(1+1/\max\{\mathcal{P},\mathcal{Q}\}). \tag{4.105}$$

potom existuje Ω *a* $f,g,\tilde{f},\tilde{g}$ *spĺňajúce rast* (4.100) *s* $r=s=1$ *a* $\tilde{r}=\tilde{s}=0$ *také, že systém* (4.99) *má kladné neohraničené veľmi slabé riešenie.*

Podobne ako v [42], naše výsledky o apriórnych odhadoch sa dajú použiť na dôkaz existencie netriviálnych riešení, pokiaľ môžeme odhadnúť L^1 normy pravých strán. Toto je, vo všeobecnosti, netriviálna úloha (viď [42, Section 3] v prípade homogénnych Dirichletových okrajových podmienok). Ukázali sme niekoľko typických príkladov, kde sa L^1-ohraničenosť nôrm a existencia kladných riešení dá dokázať. Napríklad sme dokázali platnosť nasledujúceho tvrdenia:

Proposition 4.6.11. *Nech je* $N > 2$. *Uvažujme systém* (4.79) *s* $f(x,u,v) = -u$, $g(x,u,v) = -v$ *a nech Carathéodoryho funkcie* $\tilde{f}, \tilde{g} \geq 0$ *spĺňajú rastové predpoklady* (4.100), *kde*

$$\tilde{r}, \tilde{s} < \frac{N-1}{N-2}, \qquad \tilde{p} \leq \tilde{q} < \frac{N-1}{N-4}, \qquad \tilde{p}(N-2) < 1 + \frac{N-1}{\tilde{q}}.$$

predpokladajme, že existujú $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}, c_1 > 0$ *a* $\varepsilon < \lambda_1 < \mu$ *také, že*

$$\alpha\tilde{f}(y,u,v) + \beta\tilde{g}(y,u,v) \geq \mu(\alpha u + \beta v) - c_1 \tag{4.106}$$

pre všetky $y \in \partial\Omega$ *a* $u, v \geq 0$, *naviac nech platí*

$$\tilde{\alpha}\tilde{f}(y,u,v) + \tilde{\beta}\tilde{g}(y,u,v) \leq \varepsilon(\tilde{\alpha}u + \tilde{\beta}v) \tag{4.107}$$

pre všetky $y \in \partial\Omega$ *a* $u, v \geq 0$ *malé. Potom má problém* (4.79), (4.81) *kladné ohraničené riešenie* (u, v).

Existencia netriviálneho riešenia problému (4.1) s $f(x,u,v) = -u$ a $g(x,u,v) = -v$ a superlineárnymi $\tilde{f}, \tilde{g}$ bola skúmaná viacerými autormi, ako napríklad [10, 11, 12, 26, 44]. V [10], autori dokázali existenciu pomocou apriórnych odhadov klasických kladných riešení. Na získanie apriórnych odhadov použili metódy založené na škálovaní a vetách Liouvilleovho typu. V porovnaní s Tvrdením 4.1.6, metóda škálovania vyžaduje špecifické asymptotické správanie sa nelinearít pre veľké u, v. Na druhej strane, vo všeobecnosti, metóda škálovania a použitie optimálnych Liouvilleových viet zvyčajne umožnujú získať apriórne odhady pre väčší rozsah exponentov (viď napríklad [43, Chapter I]). Bohužiaľ, optimálne Liouvilleove vety pre systémy sa ťažko dokazujú (viď [46] a referencie). Naviac, autori [10] museli tiež predpokladať technickú podmienku $\tilde{p}, \tilde{q} \leq N^*$. Všimnime si, ze naše tvrdenie nevyžaduje takéto obmedzenia: ak $p = q = 1$ a $\tilde{p}$ je dostatočne malé, potrebujeme len podmienku $\tilde{q} < (N-1)/(N-4)$.

Články [11, 12, 26, 44] sa zaoberajú existenciou riešenia problému v Tvrdení 4.1.6 vo variačnom prípade a používajú variačné metódy, ktorými však nedosiahnu apriórne odhady. Aj keď sa skúmanie obmedzilo iba na prípad variačných problémov, autori všetkých článkov okrem [12] predpokladali $\tilde{p}, \tilde{q} \leq N^*$.

Jednou z výhod použitia metódy striedavého „bootstrapu" je jej robustnosť. Nevyžaduje ani škálovacie vlastnosti ani variačnú alebo lokálnu štruktúru. Preto sme naše výsledky mohli použiť pre problémy s nelokálnymi nelinearitami. Ukázali sme aj aplikácie našich výsledkov v prípade určitých špecifických nelokálnych problémov. V prípade takýchto systémov

$$\begin{aligned} -\Delta u &= \mathcal{F}(u,v,Tu,Tv), & -\Delta v &= \mathcal{G}(u,v,Tu,Tv) \quad \text{v } \Omega, \\ \partial_\nu u &= \tilde{\mathcal{F}}(u,v,Tu,Tv), & \partial_\nu v &= \tilde{\mathcal{G}}(u,v,Tu,Tv) \quad \text{na } \Omega, \end{aligned} \tag{4.108}$$

kde T je operátor stopy, sa nám podarilo dokázať obmenu Vety (4.6.9). Jej výsledky sa dajú aplikovať napríklad na nasledujúci systém:

$$\begin{aligned} -\Delta u &= auv + bu, & -\Delta v &= cu & &\text{v } \Omega, \\ \partial_\nu u &= 0, & \partial_\nu v &= -\tilde{g}(v) + \Phi(\tilde{g}(v)) & &\text{na } \partial\Omega, \end{aligned} \tag{4.109}$$

kde $N \leq 3$, $\Phi(w)(y) := \int_{\partial\Omega} \varphi(y,z)w(z)\,dS_z$, $\varphi \in L^\infty$, $\tilde{g}$ je spojitá funkcia spĺňajúca rastovú podmienku $|\tilde{g}(v)| \leq C(1+|v|^{\tilde{s}})$ a a,b,c sú reálne konštanty. Systém rovníc v (4.109) opisuje drobnú obmenu modelu nukleárneho reaktora, kde u a v predstavujú tok neutrónov a teplotu reaktora; porovnaj s [28, systém (6)–(7)]. Nelokálna nelineárna okrajová podmienka v (4.109) vystupuje v probléme prenosu radiačného tepla: $\tilde{g}(v)$ je hustota toku povrchovej radiacie ($\tilde{g}(v) = \sigma v^4$ v prípade čierneho telesa) a $\Phi(\tilde{g}(v))(y)$ je hustota toku povrchovej radiácie absorbovaná v bode y, viď [4, 19] a príslušné referencie. Predpoklady našej vety sú splnené pokiaľ $N = 2$ a $\tilde{s}$ je ľubovoľné alebo ak $N = 3$ a $\tilde{s} < 2$.

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