COMENIUS UNIVERSITY BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Dynamic portfolio optimization with risk management and strategy constraints

Dissertation Thesis

Bratislava 2013 Mgr. Csilla Krommerová

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Declaration

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Abstract

We investigate the field of portfolio optimization where the agent maximizes the expected utility from the terminal wealth. The aim of this work is to obtain dynamic portfolio strategies where risk management and strategy constraints apply. When requiring a guaranteed floor with probability one, we provide two admissible solutions, the option based portfolio insurance in the constrained model and the alternative method. When the floor is guaranteed partially, we provide conditions under which the Limited-Expected-Losses based risk management is optimal. We show that the Value-at-Risk based risk management is not admissible and provide an admissible alternative to it, the portfolio insurance with spreads.

Keywords: power utility maximization, optimal strategy, risk management, convex constraints

Abstrakt

Práca sa zaoberá oblasťou optimalizácie portfólia, pričom sa maximalizuje očakávaná užitočnosť z majetku na konci investície. Našim cieľom je nájsť dynamické stratégie, kde uvažujeme riadenie rizika a ohraničenia na váhy. V prípade, že uvažujeme garantované dno so 100% pravdepodobnosťou, uvedieme dve prípustné riešenia: OBPI s obmedzenými stratégiami a alternatívnu metódu. V prípade, že uvažujeme čiastočne garantované dno, uvedieme podmienky, pri ktorých je metóda Limited-Expected-Losses based risk management optimálna. Ukážeme, že metóda Value-at-Risk based risk management je neprípustná a predstavíme k nej alternatívnu metódu, zaistenie portfólia pomocou spreadov.

Kľúčové slová: maximalizácia očakávaného užitku, optimálna stratégia, risk management, konvexné ohraničenia

Contents

Introduction

The problem of maximizing the expected utility over a given time horizon is one of the most frequently examined problems in financial mathematics. One can achieve the maximum expected utility by choosing the proper portfolio strategy, i.e. by optimal allocation of the available funds among risky and risk-free assets.

The problem was first examined by P. A. Samuelson. In his work [33], Samuelson investigates the lifetime consumption-planning and the investment decisions, considering one risk-free and one risky asset. The return on the risky asset is stochastic, having a general probability distribution. Samuelson presents the model in a discrete form and interprets it as a problem of dynamic stochastic programming, solving the Bellman equation. He states that for power utility functions, the optimal portfolio strategy is independent from the consumption decision. Moreover, the optimal strategy is constant over time and the consumption is given recursively.

Merton [24] confirms Samuelsons results for a continuous-time case. He develops a multi-asset model considering one risky-free and more risky assets. In addition, he specifies that the returns on the risky assets are generated by a Brownian motion.

J. Lehoczky, S. Sethi and S. Shreve [17] expand the model of Samuelson [33] and Merton [24] by allowing constraints on the consumption. Moreover, they admit the possibility of bankruptcy, i.e. the wealth level of the agent may reach zero in finite time.

While the previously mentioned works of Samuelson [33], Merton [24] and Lehoczky, Sethi and Shreve [17] consider constant risk-free interest rate and constant returns on the risky assets, Karatzas, Lehoczky and Shreve [15] present the utility maximization model with time-dependent interest rate and asset-returns.

In his work, Nutz [26] expands the power utility maximization problem using a special case, when the prices follow the exponential Lévy process. His approach to the problem is also based on the construction of the corresponding Bellman equation. Nutz proves that the results of Samuelson and

Merton hold in this case too. Additionally, Nutz examines the case when the portfolio strategy is constrained by a fixed convex set and shows that in such case the optimal portfolio strategy is constant and the intermediate consumption is deterministic. In [27], Nutz considers stochastic portfolio constraints and shows that the portfolio strategy can be obtained as the argmax of a predictable function.

No portfolio with risky assets guarantees any return. The aim of the portfolio insurance is to limit the losses and simultaneously to allow the participation on the rising market. The idea of insuring the portfolio against losses was first introduced by H. Leland and M. Rubinstein in 1976. They developed the option based portfolio insurance, also referred to as OBPI. The OBPI consists of a risky asset and a put option written on it. The strike price of the put option represents the floor such that the value of the investment at the maturity is higher than the floor with 100% probability.

There is a possibility that the required put option is not available on the market. By Leland and Rubinstein [18], in such case one can synthesize the put option with a replication portfolio that consists of the underlying asset and a risk-free bond. Using the replication portfolio the OBPI becomes dynamic, so that one can guarantee the discounted level of the floor at any time from the beginning until the maturity.

In 1986, Perold [30] introduced another type of dynamic portfolio insurance, the constant proportion portfolio insurance, also referred to as CPPI (see also [31]). The CPPI agent first determines the floor under which the portfolio is not allowed to fall at the terminal date. At each time he calculates the difference between the discounted level of the floor and the actual value of the portfolio, the so called cushion. The exposure to the risky assets is calculated as the cushion multiplied by a predefined constant multiplier. Both the floor and the multiplier are the characteristics of the agents risk-tolerance. In case of no constraints on the exposure, the pay-off expressed in terms of the terminal price of the risky asset, is path-independent. One can explicitly calculate the expected terminal value and the variance of the strategy.

There are numerous modifications of the CPPI model. Considering continuous time, the CPPI strategy protects the portfolio from falling below the predefined floor. In reality, the dynamic relocation is realized at discrete times. If there is a sharp drop of the market before the agent has a chance to trade, the portfolio falls below the floor. The risk of violating the floor protection is called gap risk.

Balder, Brandl and Mahayni [1] examine the effectiveness of CPPI strategies in terms of the gap risk under discrete-time trading. They define the shortfall probability, that is the probability of falling under the predefined floor at the terminal date. The shortfall probability is dependent on the expected return and the volatility of the risky asset and the multiplier. Using these parameters, the agent is able to determine a minimal number of re-hedges such that the CPPI strategy is considered to be effective in discrete time. In our case, effectiveness means that once the number of re-hedges exceeds the minimal number of re-hedges, the shortfall probability decreases with the increase of the number of re-hedges. Moreover, for every given confidence level of the shortfall probability and every number of re-hedges, the agent is able to determine the multiplier.

Lundvik [19] states that if the agent determines the multiplier to be less than the multiplicative inverse of the maximum historical asset fall during one trading period, then the portfolio cannot fall under the floor.

The basic model of CPPI does not consider any constraints on the exposure, i.e. the method supposes unlimited credit. Boulier and Kanniganti [5] discuss the CPPI with constraints on the exposure. In such case the value of the portfolio is path-dependent at any time and one cannot determine the expected terminal value analytically.

In the standard CPPI method, the higher the multiplier is, the more actively the portfolio participates in the rising market. Simultaneously, the portfolio approaches the floor faster when the market is decreasing.

In case of a significant market climb, the floor can become irrelevant in relation to the portfolio-value. In such case, it is logical to modify the floor. Boulier and Kanniganti [5] suggest to incorporate a "ratchet" effect. If the calculated exposure exceeds the constraint on the exposure, the difference between the calculated exposure and the constraint in addition to the original floor represents the new floor.

The exposure to risky assets approaches zero when the cushion approaches zero. In such case the agent invests only in the risk-free asset till the maturity and the portfolio has no chance to participate in any further market-rising. Boulier and Kanniganti [5] suggest to increase the initial floor by a margin. Once the cushion approaches zero, the agent can extend the cushion by a proportion of the margin, allowing the portfolio to participate in the rising of the market.

Mlynarovič [25] suggests different modifications of the CPPI strategy, considering asset management fees. His main result is the modification of CPPI for pension funds with moving investment horizon. By the legislation of Slovak Republic, the current average value of the pension fund defines the guaranteed value of the pension fund six months later, thus the guaranteed level of the pension fund creates a moving horizon.

Khuman and Constantinou [16] apply the CPPI strategy under the assumption that the price returns are modelled by a standard GARCH process that captures time varying volatility and excess kurtosis.

The comparison of the OBPI and the CPPI models became a topic of several papers. Bookstaber and Langsam [4] states that the option-replicating portfolios are path-independent. By Bertrand and Prigent [3], the continuous version of the CPPI model is also path-independent, however exposure constraints or the variable floor cause path-dependency.

Brennan and Schwartz [7] define the portfolio to be time-invariant if the fraction of the wealth invested in the risky asset is at most the function of the previous level of the portfolio. The advantage of time-invariant strategies is that the strategy does not have to be bound to a given terminal date. According to [7], the continuous version of the CPPI model is time-invariant. The opposite is true for the OBPI method, since when hedging the put option, the agent needs to know the terminal date to calculate the next proportion invested in the risky asset at any time.

Bertrand and Prigent [3] compares the OBPI and the CPPI methods in terms of dominance. They state that neither of the strategies dominates the other for all terminal values of the risky asset, i.e. the payoff functions intersect each other. Moreover, none of the strategies stochastically dominates the other at first order. Bertrand and Prigent [3] also show that if the multiplicator is allowed to vary, the OBPI model is a generalized version of the CPPI model. Zagst and Kraus [35] extend the question of stochastic dominance up to the third order. They provide certain conditions for the market parameters under which the CPPI method dominates the OBPI method.

The aim to improve the portfolio performance led to the question of risk measuring. The Value-at-Risk is a risk measure that calculates the maximum expected loss of the portfolio over a given time-horizon with a given level of confidence. Even though the concept was known from the 1920s, it became a main topic of interest after the stock market crash in 1987 and has been analyzed by many authors, such as the American multinational banking corporation JP Morgan [13] or P. Jorion [14].

While the OBPI and CPPI methods require a guaranteed floor with probability one, the Value-at-Risk based risk management guarantees the floor with a given probability less than one. Basak and Shapiro [2] introduce the power utility optimization model using the Value-at-Risk based risk management (also called VaR-RM). The VaR-RM divides the behavior of the agent: he insures the portfolio for most cases, but leaves the worst cases uninsured. Basak and Shapiro derive the optimal portfolio strategy and find that it is a proportion of the optimal portfolio strategy of the benchmark agent, who does not consider any risk management. An interesting result is that when losses occur, the terminal wealth of the insured portfolio is less than the terminal wealth of the benchmark agent. Hence, Basak and Shapiro [2] suggest another type of portfolio insurance, the Limited-Expected-Losses based risk management (LEL-RM), which controls the magnitude of the loss instead of the probability of the loss. The optimal portfolio strategy using the LEL-RM is also a proportion of the optimal portfolio strategy of the benchmark agent. The advantage of the LEL-RM is that when losses occur, the terminal wealth is higher than the terminal wealth of the benchmark agent.

Even though both the optimal portfolio selection and the portfolio insurance were examined by many scientists, it still offers many research opportunities. The aim of this work is to bring together these two areas, specifically, we investigate how to insure the portfolio when convex constraints are imposed on the portfolio strategy. We intend to provide either optimal or admissible solutions for the problem of dynamic portfolio optimization with risk management and strategy constraints.

Our work is organized as follows. In Chapter 1 we introduce the basic settings and define the sets of convex constraints, which we will use in the further chapters. Since we evaluate each strategy by its certainty equivalent, we briefly describe its principles.

In the next three chapters, we review the relevant literature on the basic models that we use as a starting point of our investigations. In Chapter 2 we recall the power utility maximization problem. First, we consider that the portfolio strategy is constant over the investment period. We derive the optimal portfolio strategy in both cases, with no constraints and with constraints imposed on the portfolio strategy. Naturally, the certainty equivalent of the unconstrained model appears to be higher than of the constrained model. Second, we examine the dynamic portfolio strategy. We review the results of Samuelson and Nutz, who state that the optimal dynamic strategies of both the unconstrained and constrained models are constant over time.

In Chapter 3 we briefly describe two types of portfolio insurances with guaranteed floor, the option based portfolio insurance and the constant proportion portfolio insurance. If the required put option is available on the market, the OBPI is a static strategy. We introduce the dynamic strategy of the OBPI, which is used when the required put option is not available on the market and the agent synthesizes it. The constant proportion portfolio insurance is a dynamic strategy.

Chapter 4 gives an insight into the results of Basak and Shapiro who examined the portfolio insurances with partially guaranteed floor. We highlight their conclusion, that insuring the portfolio using the Value-at-Risk based risk management or the Limited-Expected-Losses based risk management yield an optimal portfolio strategy, which is a proportion of the portfolio strategy of the benchmark agent.

Our goal is to investigate how to apply the previously mentioned insurance methods when the portfolio strategies are constrained. Chapter 5 examines the portfolio insurance with guaranteed floor in the constrained model, where short-selling of both risky and risk-free assets is prohibited. We provide an admissible solution for the OPBI in the constrained model and compare it with an alternative method.

Chapter 6 presents the VaR-RM and the LEL-RM in the constrained model. We show that the LEL-RM is optimal under certain conditions, but the VaR-RM is not admissible in the constrained model. Hence, in Chapter 7 we provide an alternative to the VaR-RM in the constrained model, called the portfolio insurance with spreads. The portfolio is insured based on the VaR-constraint: if the constraint is not satisfied, in addition to the portfolio the agent buys a put option with the strike price equal to the required floor and sells another put option with a strike, so that the VaR-constraint attains equality. If the constraint is naturally satisfied, the agent does not insure the portfolio at all. Such a strategy is admissible in the model with strategy constraints.

Chapter 1

Preliminaries

First, we establish the basic economic settings and define some sets and relations that we will use throughout this work.

1.1 Economic settings

Let $T > 0$ represent the time horizon and let the triplet $(\Omega, \mathcal{F}, P)$ represent the probability space. We use d risky assets and one risk-free bond to construct our portfolio.

For a given quantity, we use the upper index $i = 1, 2, ..., d$ to represent a particular asset and the lower index $t \in \langle 0, T \rangle$ to express the time dependence.

We denote the expected return on the asset i by μ^i, the positive definite volatility matrix by $\sigma = \{\sigma^{ij}, i = 1, ..., d, j = 1, ..., d\}$, the covariance matrix by $c^R = \sigma\sigma^\top$ and the risk-free interest rate by r. We consider these parameters to be constant over the time.

Let $w_t = (w_t^1, w_t^2, ..., w_t^d)^\top$ be an $\mathbb{R}^d$-valued Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Then the prices of the risky assets and the non-risky bond follows

$$dS_t^i = S_t^i[\mu_i dt + \sigma_i dw_t^i], \qquad \text{for } i = 1, 2, ..., d, \tag{1.1}$$

$$dB_t = B_t r dt. \tag{1.2}$$

We define the portfolio strategy as $\beta_t = (\beta_t^1, \beta_t^2, ..., \beta_t^d)^\top$, where β_t^i represents the proportion of the total wealth invested in the i-th asset at time t. For simplicity we fix the initial capital X_0. The wealth process then follows

$$dX_t = X_t[r + \beta_t^\top(\mu - r\mathbf{1})]dt + X_t\beta_t^\top \sigma dw_t, \tag{1.3}$$

where $\mathbf{1} = (1, 1, ..., 1)^\top$.

The existence of the state price density process ξ_t ensures the market completeness (under no-arbitrage). The stochastic differential equation for ξ_t is given as

$$d\xi_t = -\xi_t[rdt + \kappa^\top dw_t], \tag{1.4}$$

where $\kappa = \sigma^{-1}(\mu - r\mathbf{1})$ is the market price of the risk process and is also considered to be constant over time. In all cases we consider the portfolio to be self-financing

$$E[\xi_T X_T] \leq \xi_0 X_0,$$

i.e. after the initial investment, no further investments are needed (the assumption of zero net investments), and buying or selling one type of asset is balanced by selling or buying other assets (the principle of self-financing).

The agent strives to utilize the expected terminal wealth $U(X_T)$. The utility function U is assumed to be increasing, concave and twice continuously differentiable. In our work, we focus on the power utility functions of the form

$$U(X) = \frac{X^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0. \tag{1.5}$$

We exclude the case when $\gamma = 1$, as in this case the utility function is logarithmic.

By Prigent [32], the power utility functions have a constant Arrow-Pratt measure of relative risk-aversion in the form

$$R(W_T) = -W_T \frac{U(W_T)''}{U(W_T)'} = \gamma.$$

Mehra and Prescott [21] state that a reasonable relative risk-aversion takes values between $\gamma \in \langle 2, 10 \rangle$. The higher the parameter of the risk aversion is, the more conservative the agent is.

Note that in the literature, the power utility function can also be referred to as isoelastic function or CRRA (Constant Relative Risk Aversion) function.

1.2 Convex constraints

Often, there are constraints imposed on the portfolio strategies. Let π represent any portfolio strategy in general. When short-selling of the risky assets is prohibited, i.e.

$$\pi^i \geq 0 \qquad \text{for } i = 1, 2, ..., d,$$

the set of admissible strategies can be defined as

$$\mathcal{C}_0 = \{\pi \in \mathbb{R}^d : \pi^i \geq 0, i = 1, 2, ..., d\}. \tag{1.6}$$

Another common restriction is that the agent is not allowed to borrow risk-free bonds or cash to finance the purchase of further risky assets, i.e.

$$\sum_{i=1}^{d} \pi^i \leq 1.$$

In such case, the set of admissible strategies is given as

$$\mathcal{C}_1 = \{\pi \in \mathbb{R}^d : \sum_{i=1}^{d} \pi^i \leq 1\}. \tag{1.7}$$

Finally, if both restrictions apply, i.e. the short-selling of both risky and risk-free assets is prohibited, the set of admissible strategies takes the form

$$\mathcal{C} = \{\pi \in \mathbb{R}^d : \pi^i \geq 0, i = 1, 2, ..., d; \sum_{i=1}^{d} \pi^i \leq 1\}. \tag{1.8}$$

Note that all sets of admissible strategies $\mathcal{C}_1$, $\mathcal{C}_0$ and $\mathcal{C}$ contain the origin.

1.3 The certainty equivalent

The certainty equivalent is defined as the amount that yields to the same utility as the risky portfolio X, i.e.

$$U(C_X) = E[U(X_T)].$$

Using the power utility function, the certainty equivalent of the risky portfolio X can be calculated as

$$\begin{aligned} \frac{C_X^{1-\gamma}}{1-\gamma} &= E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right] \\ C_X &= \left((1-\gamma)E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}. \end{aligned} \tag{1.9}$$

Chapter 2

Portfolio optimization

The aim of this chapter is to provide an insight into the power utility maximization problem under various conditions.

After establishing the basic settings, we first examine the simplified model. By solving a quadratic programming problem, we obtain the optimal portfolio strategy. We consider two types of strategies, one with no constraints and one with convex constraints.

We provide an example, where we examine the portfolio performance under different conditions.

In the next part, we consider a multi-period model. We describe the results of Samuelson [33], Merton [24] and Nutz [26] and [27], who approached the problem through dynamic stochastic programming.

2.1 Power utility maximization with constant-proportion strategies

Consider a portfolio with constant proportions β invested in the risky assets. Then the portfolio process follows

$$dX_t = X_t[r + \beta^\top(\mu - r\mathbf{1})]dt + X_t\beta^\top\sigma dw_t.$$

For a given X_0, one can express the terminal wealth by Ito's lemma as

$$X_T = X_0 e^{(r+\beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\beta^\top c^R \beta)T + \beta^\top \sigma w_T},$$

where $w_T^i \sim N(0,T)$ for $i = 1, 2, ..., d$.

Using the power utility function, we can describe our maximization problem as

$$\max_\beta E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right],$$

where the expected utility from the terminal wealth can be expressed as

$$E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right] = \frac{X_0^{1-\gamma}}{1-\gamma} e^{[r+\beta^\top(\mu-r\mathbf{1})-\frac{1}{2}\gamma\beta^\top c^R\beta](1-\gamma)T}. \tag{2.1}$$

First, we differentiate (2.1) with respect to β and set the derivative to zero

$$\frac{X_0^{1-\gamma}}{1-\gamma} e^{[r+\beta^\top(\mu-r\mathbf{1})-\frac{1}{2}\gamma\beta^\top c^R\beta](1-\gamma)T}(1-\gamma)T[(\mu-r\mathbf{1})-\gamma c^R\beta] = \mathbf{0},$$

where $\mathbf{0} = (0, 0, ..., 0)^\top$.

Then we can express the optimal portfolio strategy as

$$\hat{\beta} = \frac{1}{\gamma}(c^R)^{-1}(\mu - r\mathbf{1}). \tag{2.2}$$

Let $\mathcal{A}$ be any set of all admissible constant portfolio strategies, containing the origin. Assume that $\mathcal{A}$ is convex. Then the power utility maximization problem with strategy constraints can be described as

$$\max_{\beta\in\mathcal{A}} E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right].$$

One can see from (2.1) that in order obtain the optimal strategy on the set of admissible strategies, it is sufficient to minimize the exponent. Hence the optimal strategy $\hat{\beta}$ is obtained as

$$\hat{\beta} \in \arg\max_{\beta\in\mathcal{A}} \beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\gamma\beta^\top c^R\beta. \tag{2.3}$$

Let us now examine the portfolio performance using different strategies.

Table 2.1 includes 10 risky assets representing each sector of the Global Industry Classification Standard (GICS - developed by Standard & Poor's). We use these 10 assets to construct our portfolio.

Table 2.2 provides the yearly returns and the covariance matrix, estimated from the daily data from 4th October 2011 till 2nd October 2012. The assets with the highest returns are Apple, Bank of America Corp. and AT&T, the assets with the lowest returns are Ford Motor, United States Steel Corp. and the Southern Co. In our case, there are no assets with negative returns.

Let the initial value invested in the portfolio be $X_0 = 1$ and the risk-free interest rate $r = 2\%$.

We calculate the optimal portfolio strategy with no constraints by (2.2). Table 2.3 summarizes the optimal strategies for $\gamma = 2, 3, ..., 10$.

Ticker	Company	Sector
F	Ford Motor	Consumer Discretionary
KO	The Coca-Cola Company	Consumer Staples
XOM	Exxon Mobil Corp.	Energy
BAC	Bank of America Corp.	Financials
JNJ	Johnson & Johnson	Health Care
BA	Boeing Company	Industrials
AAPL	Apple Inc.	Information Technology
X	United States Steel Corp.	Materials
T	AT&T, Inc.	Telecommunications Services
SO	Southern Co.	Utilities

Table 2.1: Assets included in the portfolio.

Name	F	KO	XOM	BAC	JNJ	BA	AAPL	X	T	SO
Returns	0.0128	0.1704	0.2457	0.5470	0.1113	0.1879	0.6091	0.0383	0.2988	0.0924
Covariance matrix										
F	0.0841	0.0155	0.0286	0.0840	0.0162	0.0347	0.0356	0.0985	0.0167	0.0081
KO	0.0155	0.0187	0.0150	0.0257	0.0101	0.0184	0.0127	0.0310	0.0112	0.0069
XOM	0.0286	0.0150	0.0319	0.0490	0.0146	0.0281	0.0152	0.0566	0.0146	0.0077
BAC	0.0840	0.0257	0.0490	0.2226	0.0288	0.0572	0.0475	0.1676	0.0318	0.0108
JNJ	0.0162	0.0101	0.0146	0.0288	0.0151	0.0173	0.0109	0.0372	0.0106	0.0058
BA	0.0347	0.0184	0.0281	0.0572	0.0173	0.0538	0.0279	0.0713	0.0163	0.0085
AAPL	0.0356	0.0127	0.0152	0.0475	0.0109	0.0279	0.0745	0.0629	0.0079	0.0015
X	0.0985	0.0310	0.0566	0.1676	0.0372	0.0713	0.0629	0.3194	0.0360	0.0108
T	0.0167	0.0112	0.0146	0.0318	0.0106	0.0163	0.0079	0.0360	0.0211	0.0078
SO	0.0081	0.0069	0.0077	0.0108	0.0058	0.0085	0.0015	0.0108	0.0078	0.0132

Table 2.2: Yearly returns and the covariance matrix.

For all values of γ, there are five assets in a long position and five assets in a short position. Interestingly, despite the fact that the Southern Co. has one of the lowest returns, it is in long position.

Note that all portfolio strategies can be written as $\beta = \frac{1}{\gamma}\beta_{fix}$, where

$$\beta_{fix} = (c^R)^{-1}(\mu - r\mathbf{1}),$$

i.e. the ratio of the particular assets in the portfolio is equal to β_{fix} for all γ.

Let the set of admissible strategies $\mathcal{C}_0$ be defined as in (1.6). In such case, we either invest in a long position or we do not invest at all in the particular assets. Table 2.4 summarizes the optimal strategies calculated by (2.3), using $\mathcal{A} = \mathcal{C}_0$.

We can see that according to the optimal strategies we only invest in the assets of Apple and At & T for all levels of γ. Interestingly we do not invest in the asset Bank of America Corp, even tough it has the second highest

γ	F	KO	XOM	BAC	JNJ	BA	AAPL	X	T	SO
2	-3.5767	-1.2997	4.7377	1.2851	-3.5904	-1.9841	5.7100	-1.4962	8.6153	0.1696
3	-2.3845	-0.8665	3.1585	0.8567	-2.3936	-1.3227	3.8067	-0.9974	5.7435	0.1131
4	-1.7884	-0.6499	2.3689	0.6425	-1.7952	-0.9920	2.8550	-0.7481	4.3076	0.0848
5	-1.4307	-0.5199	1.8951	0.5140	-1.4362	-0.7936	2.2840	-0.5985	3.4461	0.0678
6	-1.1922	-0.4332	1.5792	0.4284	-1.1968	-0.6614	1.9033	-0.4987	2.8718	0.0565
7	-1.0219	-0.3713	1.3536	0.3672	-1.0258	-0.5669	1.6314	-0.4275	2.4615	0.0485
8	-0.8942	-0.3249	1.1844	0.3213	-0.8976	-0.4960	1.4275	-0.3740	2.1538	0.0424
9	-0.7948	-0.2888	1.0528	0.2856	-0.7979	-0.4409	1.2689	-0.3325	1.9145	0.0377
10	-0.7153	-0.2599	0.9475	0.2570	-0.7181	-0.3968	1.1420	-0.2992	1.7231	0.0339

Table 2.3: Optimal portfolio strategies with no constraints.

γ	F	KO	XOM	BAC	JNJ	BA	AAPL	X	T	SO
2	0	0	0	0	0	0	3.3868	0	5.3479	0
3	0	0	0	0	0	0	2.2579	0	3.5653	0
4	0	0	0	0	0	0	1.6934	0	2.6739	0
5	0	0	0	0	0	0	1.3547	0	2.1392	0
6	0	0	0	0	0	0	1.1289	0	1.7826	0
7	0	0	0	0	0	0	0.9677	0	1.5280	0
8	0	0	0	0	0	0	0.8467	0	1.3370	0
9	0	0	0	0	0	0	0.7526	0	1.1884	0
10	0	0	0	0	0	0	0.6774	0	1.0696	0

Table 2.4: Optimal portfolio strategies for $\mathcal{C}_0$.

return.

Let us now consider the case when the agent is not allowed to borrow risk-free assets. Then the set of admissible strategies $\mathcal{C}_1$ is given by (1.7) and the optimal portfolio strategies are calculated by (2.3), using $\mathcal{A} = \mathcal{C}_1$. Table 2.5 summarizes the strategies.

γ	F	KO	XOM	BAC	JNJ	BA	AAPL	X	T	SO
2	-3.7233	-2.2376	4.5477	1.5833	-6.6516	-1.3391	5.0073	-1.1286	7.7835	-2.8416
3	-2.4757	-1.4505	3.0402	1.0424	-4.2996	-0.9211	3.3692	-0.7686	5.2256	-1.7618
4	-1.8520	-1.0569	2.2864	0.7720	-3.1236	-0.7122	2.5501	-0.5886	3.9467	-1.2219
5	-1.4777	-0.8207	1.8341	0.6097	-2.4180	-0.5868	2.0586	-0.4806	3.1793	-0.8980
6	-1.2282	-0.6633	1.5326	0.5015	-1.9476	-0.5032	1.7310	-0.4086	2.6677	-0.6820
7	-1.0500	-0.5508	1.3173	0.4242	-1.6116	-0.4435	1.4970	-0.3571	2.3023	-0.5278
8	-0.9163	-0.4665	1.1558	0.3663	-1.3596	-0.3987	1.3214	-0.3186	2.0283	-0.4121
9	-0.8123	-0.4009	1.0301	0.3212	-1.1636	-0.3638	1.1849	-0.2886	1.8151	-0.3221
10	-0.7292	-0.3484	0.9296	0.2852	-1.0068	-0.3360	1.0757	-0.2646	1.6446	-0.2501

Table 2.5: Optimal portfolio strategies for $\mathcal{C}_1$.

We can see that for all levels of γ, the portfolio is constructed from six assets in short position and the four assets with the highest returns are in long position.

Finally, let us examine the portfolio strategy when the set of admissible strategies is represented by $\mathcal{C}$ given in (1.8), i.e. short-selling of both risky and risk-free assets is forbidden. We obtain the optimal portfolio strategy from (2.3), using $\mathcal{A} = \mathcal{C}$. The results are summarized in Table 2.6 for different levels of γ.

γ	F	KO	XOM	BAC	JNJ	BA	AAPL	X	T	SO
2	0	0	0	0	0	0	1.0000	0	0	0
3	0	0	0	0.0311	0	0	0.9689	0	0	0
4	0	0	0	0.0567	0	0	0.9433	0	0	0
5	0	0	0	0.0689	0	0	0.9186	0	0.0125	0
6	0	0	0	0.0422	0	0	0.7985	0	0.1593	0
7	0	0	0	0.0231	0	0	0.7127	0	0.2642	0
8	0	0	0	0.0088	0	0	0.6484	0	0.3429	0
9	0	0	0	0	0	0	0.5975	0	0.4025	0
10	0	0	0	0	0	0	0.5542	0	0.4458	0

Table 2.6: Optimal portfolio strategies for $\mathcal{C}$.

The risk aversion $\gamma = 2$ represents the agent that is less averse, i.e. considers returns more than risk. There is no surprise that the agent invests all his wealth in the asset with the highest return, Apple. For $\gamma = 3$ and 4, the agent is still considered to be risk-favouring, therefore the portfolios include the two most risky assets, Apple and Bank of America Corp. For $\gamma = 5, 6, 7$ and 8 the portfolio includes an additional asset with the third highest expected return, AT&T. The most risk averse agent, with $\gamma = 9$ and 10 constructs the portfolio only from two assets, Apple and AT&T.

For comparison, in Table 2.7 we provide the certainty equivalents for the unconstrained portfolio and for the portfolio constrained by $\mathcal{C}$.

γ	$C_{unconstr}$	C_{constr}
2	28.596864	1.706843
3	9.414001	1.644938
4	5.401348	1.586408
5	3.870249	1.530487
6	3.099061	1.484235
7	2.644205	1.448436
8	2.347437	1.419052
9	2.139843	1.393897
10	1.987066	1.371557

Table 2.7: Certainty equivalents calculated by different methods.

We conclude that for all levels of γ, the certainty equivalent of the unconstrained portfolio is significantly higher than the certainty equivalent of the portfolio constrained by $\mathcal{C}$.

2.2 Dynamic power utility maximization

The usage of the optimization model with constant-proportion strategies is reasonable if the agent invests for short term. However, in case of long term investments, the agent needs to adjust his portfolio strategy to the developing conditions of the market. Moreover, the agent does not only invest, he might also consider consuming.

The multi-period model aims to solve the problem of maximizing the expected utility from consumption and terminal wealth.

2.2.1 Dynamic power utility maximization with no constraints

P. A. Samuelson [33] was one of the first economists, who introduced the lifetime planning of consumption and investment decisions and provided for a discrete model.

For Samuelson, the point of departure is the Ramsey model. Let X_0 represent the initial investment in the portfolio that consists of one risky and one risk-free asset. The income of the agent is generated by the returns on his portfolio, given in a discrete-time form as

$$X_{t+1} = [(1-\beta_t)(1+r) + \beta_t Z_t] X_t,$$

where β_t represents the proportion invested in the risky asset, r represents the risk-free interest rate and Z_t represents the return on the risky asset. Z_t is a random variable with the probability distribution $P(Z_t \leq z) = P(z)$ for $z \geq 0$. Moreover, the values of Z_t for different times t are mutually independent, i.e. $P(z_0, z_1, ..., z_T) = P(z_0)P(z_1)...P(z_T)$.

At every time $t \in \langle 0, T \rangle$, the agent has choice to consume k_t or reinvest in his portfolio, choosing the proper portfolio strategy β_t. The aim of the agent is to maximize the expected utility from the consumption. In our case the terminal wealth is considered to be the last consumption, i.e. $X_T = k_T$. The aim of the agent is to reach the maximum discounted utility from the consumption

$$\max_{X_t} \sum_{t=0}^{T} (1+\rho)^{-t} U(k_t)$$
$$s.t. \quad k_t = X_t - \frac{X_{t+1}}{1+r},$$

where ρ is the discount rate, $U(.)$ is a strictly increasing concave utility function and the second equation represents the consumption constraint.

Using dynamic stochastic programming, Samuelson constructs the Bellman equation and solves the problem backwards. We provide his main conclusion,considering the power utility function.

Theorem 1 ([33]). *For power utility functions given by (2.1), the optimal portfolio decision is independent of wealth at each stage and independent of all consumption-savings decisions, leading to a constant β^*, that is the solution to*

$$0 = \int_0^\infty [(1-\beta)(1+r) + \beta Z]^{-\gamma}(Z - 1 - r)dP(Z).$$

Merton [24] confirmes the results of Samuelson [33] for a continuous model, specifying that the returns on the risky assets follow a Brownian-motion process. Moreover, while Samuelson provides a two-asset model, Merton expands it to a multi-asset model.

2.2.2 Dynamic power utility maximization with constraints

Nutz [26] develops the power utility maximization problem assuming that the asset returns follow Lévy processes. In our work, we consider no intermediate consumption, therefore the utility is obtained from the terminal wealth. When using the power utility function, our aim is to find

$$\max_\beta E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right], \tag{2.4}$$

where we maximize through all dynamic strategies β.

Additionally, Nutz restricts the optimal portfolio strategies by a deterministic set of convex constraints, then (2.4) is solved through the set of admissible strategies.

First we introduce the Lévy process, then we define the set of deterministic constraints, construct the opportunity process, provide the assumptions and finally we summarize the results in a theorem.

The Lévy Process

Let R be an $\mathbb{R}^d$-valued Lévy process. Let (b^R, c^R, F^R) represent the Lévy triplet of R, where $b^R \in \mathbb{R}^d$ is the drift term, $c^R \in \mathbb{R}^{d\times d}$ is the nonnegative definite covariance matrix and F^R is the Lévy measure such that

$$F^R\{0\} = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^d} \min(1, |x|^2)F^R(dx) < \infty.$$

The Lévy process at time t is represented as

$$R_t = b^R t + R_t^c + h(x) * (\mu_t^R - \nu_t^R) + (x - h(x)) * \mu_t^R, \tag{2.5}$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is called the truncation function such that h is bounded and $h(x) = x$ in the neighborhood of $x = 0$. The value μ_t^R is associated with the number of jumps in R, ν_t^R is the compensator of the bigger jumps and R_t^c is the continuous martingale part such that $R_t^c = \sigma w_t$, where $\sigma \in \mathbb{R}^{d\times d}$ is the volatility matrix satisfying $\sigma\sigma^T = c^R$ and $w_t \in \mathbb{R}^d$ is a Brownian motion. For further information about the Lévy processes, we refer to [12] and [29].

The portfolio strategy at time t is represented by $\beta_t \in \mathbb{R}^d$. Since the returns follow the Lévy process, the portfolio X_t follows (1.3), using dR_t instead of $(\mu dt + \sigma dw_t)$.

Deterministic constraints

In this section we restrict the set of optimal strategies.

Let $\mathcal{S} \subseteq \mathbb{R}^d$ be the *set of constraints* imposed on the agent. Then the *set of admissible strategies* according to the initial wealth X_0 is

$$\mathcal{A}(X_0) := \{\beta : X_t > 0 \quad \text{and} \quad \beta_t \in \mathcal{S} \quad \text{for all} \quad t \in \langle 0, T\rangle\}.$$

In case of fixed X_0, we simply write $\mathcal{A}$ instead of $\mathcal{A}(X_0)$ and we optimize

$$\max_{\beta\in\mathcal{A}} E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right]. \tag{2.6}$$

The set of "compatible" controls is defined for every fixed $t \in \langle 0, T\rangle$ as $\mathcal{A}(\beta, t) := \{\tilde{\beta} \in \mathcal{A} : \tilde{\beta} = \beta_t \text{ on } \langle 0, t\rangle\}$.

By Proposition 3.1. in [28], there exists a unique càdlàg semimartingale L, called the *opportunity process* such that

$$L_t \frac{1}{1-\gamma} X_t^{1-\gamma} = \operatorname*{ess\ sup}_{\mathcal{A}(\beta,t)} U(X_T). \tag{2.7}$$

The opportunity process describes both the value function and the optimal strategy. If there is an optimal strategy, [27] states that the drift rate of the opportunity process L satisfies the Bellman equation. Note that in our settings the opportunity process is deterministic.

To provide the main results, we first need to define a deterministic function

$$\begin{aligned}\eta(\beta) = {} & r + \beta^T(b^R - r\mathbf{1}) - \frac{\gamma}{2}\beta^T c^R \beta \\ & + \int_{\mathbb{R}^d} \{(1-\gamma)^{-1}(1+\beta^T x)^{(1-\gamma)} - (1-\gamma)^{-1} - \beta^T h(x)\} F^R(dx).\end{aligned}$$

Theorem 2 ([26], Theorem 3.2.). *Assume that $\mathcal{S}$ is convex and there is no arbitrage on the market. Then, there exists an optimal strategy $\hat{\beta}$ such that $\hat{\beta}$ is a constant vector and is characterized by*

$$\hat{\beta} \in \arg\max_{\beta \in \mathcal{S}} \eta(\beta) \tag{2.8}$$

and the opportunity process is given by

$$L_t = e^{a\gamma(T-t)},$$

where

$$a = \frac{1-\gamma}{\gamma} \max_{\beta \in \mathcal{S}} \eta(\beta).$$

In case of the Brownian motion, function $o(.)$ is given in the form

$$\eta(\beta) = r + \beta^T(\mu - r\mathbf{1}) - \frac{\gamma}{2}\beta^T c^R \beta.$$

In case of no constraints, the optimal strategy can be obtained as

$$\hat{\beta} = \frac{1}{\gamma}(c^R)^{-1}(\mu - r\mathbf{1}). \tag{2.9}$$

We conclude that the power utility maximization with deterministic convex constraints leads to a constant vector. Moreover, the optimal portfolio strategy obtained from the power utility maximization with constant-proportion strategies corresponds to the optimal portfolio strategy obtained from the much wider dynamic power utility maximization problem.

Chapter 3

Portfolio insurance with guaranteed floor

The main idea of insuring the portfolio against losses is to guarantee a minimum return and simultaneously allow the portfolio to participate on the uprising market. In this chapter we introduce two types of portfolio insurances that guarantee a predefined amount at the terminal date: the option based portfolio insurance, also known as OBPI and the constant proportion portfolio insurance, also called CPPI.

To distinguish between the portfolio strategies, let the portfolio strategy of the uninsured portfolio X be represented by $\beta = (\beta^1, \beta^2, ..., \beta^d)^\top$. We suppose β to be constant. Let the strategy of the insured portfolio W be represented by $\theta_t = (\theta_t^1, \theta_t^2, ..., \theta_t^d)^\top$. The floor, the predefined level under which the investor is not allowed to fall, is denoted by $\underline{W}$.

3.1 The option based portfolio insurance

The OBPI strategy consist of a portfolio covered by a put option written on it. The put option has the same maturity T as the portfolio and its strike price $\underline{W}$ is the predefined floor. The basic overview of OBPI can be found in [3].

Let the risky portfolio X, invested in d risky assets and a non-risky bond follow the process

$$dX_t = X_t \mu_X dt + X_t \sigma_X dw_t,$$

where $\mu_X = r + \beta^\top(\mu - r)$ is the drift of the portfolio, $\sigma_X = \sqrt{\beta^\top c^R \beta}$ is the volatility of the portfolio and w_t is a one-dimensional Brownian motion.

Let V_t^{put} be the price of the put option and V_t^{call} be the price of the call option with maturity T and strike price $\underline{W}$ at time $t \in \langle 0, T \rangle$. The value of the insured portfolio W_t at time t is given as

$$\begin{aligned} W_t &= X_t + V_t^{put} \\ &= \underline{W}e^{-r(T-t)} + V_t^{call} \end{aligned}$$

due to the put-call parity. One can see that the value of the insured portfolio W_t is always above the deterministic level $\underline{W}e^{-r(T-t)}$ at any time t.

Using the Black-Scholes pricing, the prices of V_t^{put} and V_t^{call} at time t can be calculated as

$$\begin{aligned} V_t^{put} &= \underline{W}e^{-r(T-t)}\Phi\left(-d_2(\underline{W})\right) - X_t\Phi\left(-d_1(\underline{W})\right. \\ V_t^{call} &= X_t\Phi\left(d_1(\underline{W})\right) - \underline{W}e^{-r(T-t)}\Phi\left(d_2(\underline{W})\right), \end{aligned} \tag{3.1}$$

with

$$\begin{aligned} d_1(\underline{W}) &= \frac{\ln\frac{X_t}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X\sqrt{T-t}} \\ d_2(\underline{W}) &= d_1 - \sigma_X\sqrt{T-t}, \end{aligned}$$

where $\Phi(.)$ is the standard normal distribution function (see [34]).

Possible difficulties might occur when the desired put option cannot be found on the market. In such case the put option can be synthesized by a replication portfolio invested in the risk-free asset and the underlying portfolio. The replication portfolio should have the same characteristics as the put option (e.g. the value, payoff and risk).

The replication portfolio at time t can be expressed as

$$V_t = \varphi_t X_t + \psi_t B_t, \tag{3.2}$$

where $\varphi_t = \dfrac{\partial V_t}{\partial X_t}$ is the so called delta of the option, in other words the sensitivity of the option-value on the value of the underlying portfolio. The delta of the put option φ_t can be computed as

$$\varphi_t = \Phi(d_1(\underline{W})) - 1 \tag{3.3}$$

and one can easily see that $-1 < \varphi_t < 0$, for every t (see [23]).

Then the value of the insured portfolio can be expressed as

$$\begin{aligned} W_t &= X_t + V_t^{put} \\ &= X_t + \varphi_t X_t + \psi_t B_t \\ &= (1+\varphi_t)X_t + \psi_t B_t. \end{aligned}$$

Because the portfolio weights are calculated as

$$weight^i = \frac{money\ invested\ in\ the\ asset\ i}{total\ money\ invested},$$

the new portfolio strategy can be expressed as

$$\theta_t^i = \frac{(1+\varphi_t)\beta^i X_t}{W_t}, \qquad i = 1, ..., d. \tag{3.4}$$

Subsequently, the portfolio process follows

$$dW_t = W_t \mu_W dt + W_t \sigma_W dw_t,$$

where the drift is $\mu_W = r + \theta_t^\top(\mu - r\mathbf{1})$, the volatility is $\sigma_W = \sqrt{\theta_t^\top c^R \theta^\top}$ and w_t is a one-dimensional Brownian motion.

The OBPI ensures that the terminal wealth is always over the floor

$$\begin{aligned} W_T &= X_T + V_T^{Put} \\ &= X_T + \max(0, \underline{W} - X_T) \\ &= \max(X_T, \underline{W}). \end{aligned}$$

3.2 Constant proportion portfolio insurance

The CPPI allocates the risky asset dynamically. In our settings the risky asset is represented by the risky portfolio X^{CPPI}, that follows the process (1.3).

First, the agent determines the required minimum value at the terminal date, the floor $F_T = \underline{W}$. The value of the floor at time t is given as

$$F_t = \underline{W} e^{-r(T-t)}.$$

At each time t, the agent calculates the cushion K_t that is the excess of the portfolio value over the floor, i.e.

$$K_t = W_t^{CPPI} - F_t$$

and determines the exposure E_t to the risky portfolio X^{CPPI} as a multiple of the cushion by a predetermined constant multiplier m

$$E_t = mK_t.$$

Note that when the cushion reaches zero, the exposure also becomes zero and the entire portfolio is invested in the risk-free asset.

By Bouyé [6], the value of the CPPI portfolio can be calculated at each time as

$$W_t^{CPPI} = F_t + K_0 \left(\frac{X_t}{X_0} \right)^m e^{\left(r - m(r - \frac{\sigma_X^2}{2}) - m^2 \frac{\sigma_X^2}{2} \right) t}.$$

Boulier and Kanniganti [5] provide the expected terminal wealth of the insured portfolio as

$$E[W_T^{CPPI}] = \underline{W} + K_0 e^{(r + m(\mu_X - r))T}.$$

One can see that the expected terminal wealth is independent of the volatility of the risky portfolio and it is an increasing function of the multiplier m. The variance of the strategy can be expressed as

$$Var[W_T^{CPPI}] = K_0^2 e^{2(r + m(\mu_X - r))T} \left(e^{m^2 \sigma_X^2 T} - 1 \right).$$

The portfolio variance is increasing in both the multiplier and the volatility of the risky asset.

By Bertrand and Prigent [3], the OBPI is a generalized version of the CPPI. If the multiplier is allowed to vary, precisely its value is

$$m^{OBPI} = \frac{X_t \Phi \left(d_1(\underline{W}) \right)}{V_t^{Call}},$$

then the OBPI method is equivalent to the CPPI method.

Chapter 4

Portfolio insurance with partially guaranteed floor

Both portfolio insurances mentioned in the previous chapter guarantee the floor with probability one. In this chapter, we allow the portfolio to fall under the guaranteed floor with a given probability. We provide two insurance methods, the Value-at-Risk-based risk management, called as VaR-RM and the Limited-Expected-Losses-based risk management, called as LEL-RM. In both models, the portfolio process follows the process

$$dW_t = W_t[r + \theta_t^\top(\mu - r\mathbf{1})]dt + W_t\theta_t^\top\sigma dw_t.$$

4.1 Value-at-Risk based risk management

In case of insuring the portfolio with a put option, the terminal value W_T of the portfolio does not fall under the predefined floor, i.e. $W_T \geq \underline{W}$ with the probability of 100%. Now, let us investigate the case of relaxing the condition

$$P(W_T \geq \underline{W}) = 1$$

and consider instead the probability of falling under the predefined floor to be less than α, i.e.

$$P(W_T \geq \underline{W}) \geq 1 - \alpha. \tag{4.1}$$

Inequality (4.1) represents the so-called Value-at-Risk constraint.

The Value-at-Risk based risk management (shortly VaR-RM) is defined to measure the loss, which is exceeded with some given probability α.

Let $VaR(\alpha)$ represent the loss which is exceeded with some probability α over a given time horizon T. If we denote the magnitude of the loss as

$W_0 - W_T$, then

$$P(W_0 - W_T \geq VaR(\alpha)) \equiv \alpha, \qquad \alpha \in [0,1],$$

or equivalently,

$$P(W_0 - W_T \leq VaR(\alpha)) \equiv 1 - \alpha, \qquad \alpha \in [0,1]. \tag{4.2}$$

Since our aim is to keep the loss under $VaR(\alpha)$, we can define the minimum terminal wealth $\underline{W}$, the so-called "floor"such that the condition

$$VaR(\alpha) = W_0 - \underline{W} \tag{4.3}$$

holds. Combining the equations (4.2) and (4.3) we obtain the VaR-constraint as

$$P(W_T \geq \underline{W}) \geq 1 - \alpha.$$

Note that for $\alpha = 1$ the investor behaves as a benchmark agent, who does not consider any risk management. If $\alpha = 0$, the investor behaves as a portfolio insurer (securing with put options). In such case the terminal wealth will exceed the "floor" at all states.

4.1.1 Portfolio maximization under the VaR-RM

Our goal is to maximize the expected utility from the terminal wealth under VaR-RM. Let $U(.)$ be any increasing and concave utility function. We consider the following problem

$$\begin{aligned} &\max_{\theta} E[U(W_T)] \\ s.t. \qquad &P(W_T \geq \underline{W}) \geq 1 - \alpha, \end{aligned} \tag{4.4}$$

where we maximize through all dynamic strategies θ and the initial is given as W_0.

Remark 1. *Since we optimize on the complete market, where all possible terminal wealths W_T can be constructed using the appropriate portfolio strategy β, the problem (4.4) is equivalent to*

$$\begin{aligned} &\max_{W_T} E[U(W_T)] \\ s.t. \qquad &E[\xi_T W_T] \leq \xi_0 W_0, \\ &P(W_T \geq \underline{W}) \geq 1 - \alpha. \end{aligned}$$

Despite the fact that the VaR-constraint (4.1) violates the concavity of the problem (4.4), Basak and Shapiro [2] introduce the optimal terminal wealth at time T as following:

Proposition 1. *[[2], Proposition 1.] The optimal wealth of the VaR agent at time T is*

$$W_T^{VaR} = \begin{cases} I(y\xi_T) & \text{if} \quad \xi_T < \underline{\xi}, \\ \underline{W} & \text{if} \quad \underline{\underline{\xi}} \leq \xi_T \leq \overline{\xi}, \\ I(y\xi_T) & \text{if} \quad \overline{\xi} < \xi_T, \end{cases} \tag{4.5}$$

where $I(.)$ is the inverse function of $U'(.)$, $\underline{\xi} \equiv \frac{U'(\underline{W})}{y}$, the value of $\overline{\xi}$ is such that $P(\xi_T > \overline{\xi}) \equiv \alpha$ and $y \geq 0$ solves $E[\xi_T W_T^{VaR}(y)] = \xi_0 X_0$. The VaR-constraint (4.1) is binding if and only if $\underline{\xi} < \overline{\xi}$. Moreover, the Lagrange multiplier y is decreasing in α, so that $y \in [y^B, y^{PI}]$ (proof: see [2]).

Remark 2. *In case the VaR-constraint (4.1) is not binding, i.e. if*

$$P(W_T < \underline{W}) < \alpha,$$

we have $\underline{\xi} \geq \overline{\xi}$ and the terminal wealth is determined as of the benchmark agent, i.e.

$$W_T^{VaR} = I(y\xi_T).$$

Figure 4.1 illustrates the optimal terminal wealth W_T^{VaR} depending on the terminal state price density ξ_T.
The dotted line represents the portfolio insurer, the dashed line represents the benchmark agent and the full line represents the VaR-RM agent. The value $\underline{\underline{W}}$ is defined as

$$\underline{\underline{W}} \equiv \begin{cases} I(y\overline{\xi}) & \text{if} \quad \underline{\xi} < \overline{\xi}, \\ \underline{W} & \text{otherwise.} \end{cases} \tag{4.6}$$

As one can see, the terminal state price density ξ_T is divided into three intervals:

- for $\xi_T < \underline{\xi}$, the VaR-RM agent acts as the benchmark agent,
- for $\underline{\xi} \leq \xi_T \leq \overline{\xi}$, the VaR-RM agent insures the portfolio against losses as the portfolio insurer,
- for $\overline{\xi} < \xi_T$, the VaR-RM agent leaves the portfolio is uninsured.

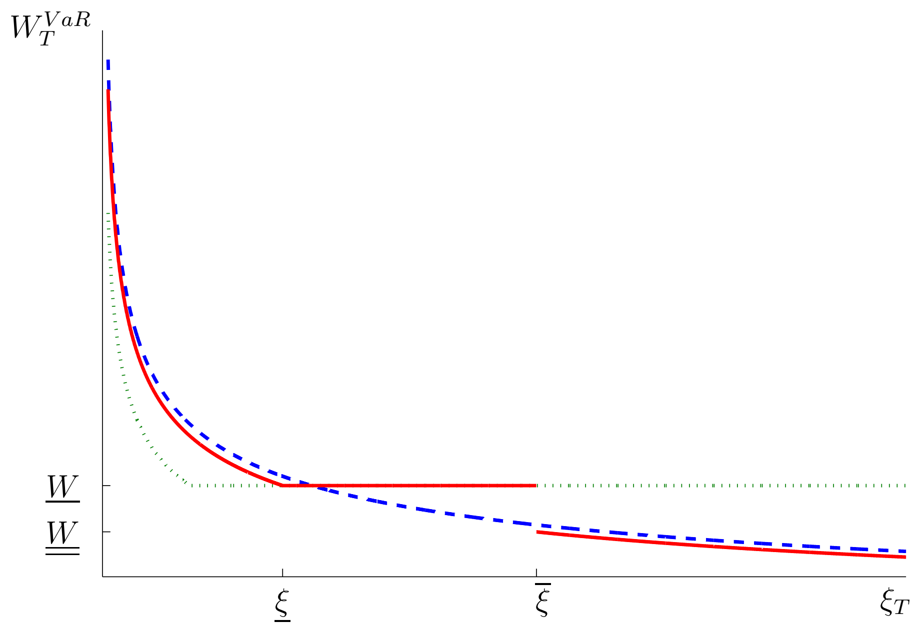

Figure 4.1: Optimal terminal wealth of the VaR-RM agent.

The VaR-RM agent controls the probability of the loss. At the state price densities $\overline{\xi} < \xi_T$, the probability of loss is high, the insurance is more expensive, therefore the agent chooses to leave the portfolio completely uninsured at these states.

Note that when $\overline{\xi} < \xi_T$, the loss of the VaR-RM agent is greater than the loss of the benchmark agent, who did not consider any risk management.

4.1.2 Properties of maximizing under VaR-RM

Let the state price density be log-normal with constant interest rate r, constant market price of risk κ and let the utility function be isoelastic, i.e.

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0.$$

The next proposition introduces the optimal wealth and portfolio strategy before the horizon T.

Proposition 2. *[[2], Proposition 3.] Assume that* $U(W) = \dfrac{W^{1-\gamma}}{1-\gamma}$ *for* $\gamma > 0$ *and that* r *and* κ *are constants. Then* *i) The optimal wealth at time t is given by*

$$
\begin{aligned}
W_t^{VaR} \quad &= \frac{e^{\Gamma_t^{VaR}}}{(y\xi_t)^{\frac{1}{\gamma}}} - \\
&- \left[\frac{e^{\Gamma_t^{VaR}}}{(y\xi_t)^{\frac{1}{\gamma}}} \Phi\left(-d_1^{VaR}(\underline{\xi})\right) - \underline{W} e^{-r(T-t)} \Phi\left(-d_2^{VaR}(\underline{\xi})\right) \right] \\
&+ \left[\frac{e^{\Gamma_t^{VaR}}}{(y\xi_t)^{\frac{1}{\gamma}}} \Phi\left(-d_1^{VaR}(\overline{\xi})\right) - \underline{W} e^{-r(T-t)} \Phi\left(-d_2^{VaR}(\overline{\xi})\right) \right],
\end{aligned}
$$

where $\Phi(.)$ is the standard-normal cumulative distribution function, y is as in Proposition 1 and

$$
\begin{aligned}
\underline{\xi} &= \frac{1}{y\,\underline{W}^{\gamma}}, \\
\Gamma_t^{VaR} &= \frac{1-\gamma}{\gamma}\left(r + \frac{\|\kappa\|^2}{2}\right)(T-t) + \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\|\kappa\|^2}{2}(T-t), \\
d_2^{VaR}(x) &= \frac{\ln \dfrac{x}{\xi_t} + \left(r - \dfrac{\|\kappa\|^2}{2}\right)(T-t)}{\|\kappa\|\sqrt{T-t}}, \\
d_1^{VaR}(x) &= d_2^{VaR}(x) + \frac{1}{1-p}\|\kappa\|\sqrt{T-t}.
\end{aligned}
$$

ii) The fraction of wealth invested in stocks is

$$\theta_t^{VaR} = q_t^{VaR}\hat{\beta},$$

where $\hat{\beta}$ is the portfolio strategy of the benchmark agent, calculated by (2.9) and

$$
\begin{aligned}
q_t^{VaR} \quad &= \quad 1 - \frac{\underline{W} e^{-r(T-t)} \left[\Phi\left(-d_2^{VaR}(\underline{\xi})\right) - \Phi\left(-d_2^{VaR}(\overline{\xi})\right)\right]}{W_t^{VaR}} \\
&\qquad + \frac{\gamma\left(\underline{W} - \underline{\underline{W}}\right) e^{-r(T-t)} \phi\left(d_2^{VaR}(\overline{\xi})\right)}{W_t^{VaR}\|\kappa\|\sqrt{T-t}},
\end{aligned}
$$

where $\phi(.)$ is the standard-normal probability function.

iii) The exposure to risky assets relative to the benchmark is bounded below, namely $q_t^{VaR} \geq 0$ and

$$\lim_{\xi_t \to 0} q_t^{VaR} = \lim_{\xi_t \to \infty} q_t^{VaR} = 1$$

(proof: see [2]).

Note that W_t^{VaR} is a decreasing function of ξ_t for all $t \in \langle 0, T \rangle$.

The advantage of focusing on power utility functions is that knowing the optimal strategy of the benchmark agent, $\hat{\beta}$ and the ratio q^{VaR}, which can be calculated from the model settings, one can easily determine the optimal strategy θ_t^{VaR} of the maximizing problem under VaR-RM at each time t.

4.2 Limited-Expected-Losses based risk management

As it has been highlighted before, the VaR-RM agent controls only the probability of the loss that can occur. As a result, at the worst states, i.e. when $\xi_T > \overline{\xi}$, the terminal wealth of the VaR-RM agent is lower than the terminal wealth of the benchmark agent. Basak and Shapiro [2] suggest another model of risk management, called the Limited-Expected-Losses based risk management (shortly LEL-RM), that controls the magnitude of the loss.

LEL-RM controls the present value of the agent's losses, which is equivalent to the price of the put option against losses. The LEL-RM agent strives to keep this value under a predefined level $\epsilon \geq 0$. We define the LEL-constraint as

$$E[\xi_T(\underline{W} - W_T)1_{\{W_T \leq \underline{W}\}})] \leq \epsilon. \tag{4.7}$$

For $\epsilon = \infty$, the value of the insurance is not controlled and the LEL-RM agent acts as the benchmark agent. For $\epsilon = 0$, the LEL-RM agent does not accept any loss, therefore acts as the portfolio insurer.

4.2.1 Portfolio maximization under the LEL-RM

We replace the VaR-constraint (4.1) with the LEL-constraint (4.7). Let $U(.)$ be any increasing and concave utility function. Then our maximization problem takes the following form

$$\begin{aligned} & \max_{\theta} E[U(W_T)] \\ s.t. \quad & E[\xi_T(\underline{W} - W_T)1_{\{W_T \leq \underline{W}\}})] \leq \epsilon, \end{aligned} \tag{4.8}$$

where we maximize through all dynamic strategies θ and the initial is given as W_0.

Remark 3. *Just as in the case of the VaR-RM, we optimize on the complete market, therefore the problem (4.8) is equivalent to the problem*

$$\begin{aligned} &\max_{W_T} E[U(W_T)] \\ s.t. \quad &E[\xi_T W_T] \leq \xi_0 W_0, \\ &E[\xi_T(\underline{W} - W_T)1_{\{W_T \leq \underline{W}\}})] \leq \epsilon, \end{aligned}$$

The terminal wealth W_T^{LEL} is similar to the terminal wealth W_T^{VaR} obtained from the maximization under the VaR-RM in Proposition 1, except when $\overline{\xi}_\epsilon < \xi_T$. Proposition 3 summarizes the results:

Proposition 3. *[[2], Proposition 4] The optimal wealth of the LEL-RM agent at time T is*

$$W_T^{LEL} = \begin{cases} I(z_1\xi_T) & \text{if} \quad \xi_T < \underline{\xi}_\epsilon, \\ \underline{W} & \text{if} \quad \underline{\xi}_\epsilon \leq \xi_T \leq \overline{\xi}_\epsilon, \\ I((z_1 - z_2)\xi_T) & \text{if} \quad \overline{\xi}_\epsilon < \xi_T, \end{cases} \tag{4.9}$$

where $\underline{\xi}_\epsilon \equiv \dfrac{U'(\underline{W})}{z_1}$, $\overline{\xi}_\epsilon \equiv \dfrac{U'(\underline{W})}{z_1 - z_2}$, *and* $(z_1 \geq 0, z_2 \geq 0)$ *solve the following system:*

$$\begin{cases} E[\xi_T W_T^{LEL}(z_1, z_2)] = \xi_0 X_0, \\ E[\xi_T \left(\underline{W} - W_T^{LEL}(z_1, z_2)\right) 1_{\{W_T^{LEL}(z_1,z_2) \leq \underline{W}\}}] = \epsilon \text{ or } z_2 = 0. \end{cases} \tag{4.10}$$

The LEL-constraint (4.7) is binding if and only if $\underline{\xi}_\epsilon < \overline{\xi}_\epsilon$. Moreover, the Lagrange multiplier z_1 is decreasing in ϵ, so that $z_1 \in [z_1^B, z_1^{PI}]$. It also holds that $z_1 - z_2 \leq z_1^B$ (proof: see [2]).

Remark 4. *If the LEL-contraint (4.7) is not binding, then $z_2 = 0$, and the critical values are equal, i.e. $\underline{\xi}_\epsilon = \overline{\xi}_\epsilon$. Then the terminal wealth of the LEL-RM agent is determined as of the benchmark agent, i.e.*

$$W_T^{LEL} = I(z_1\xi_T).$$

Figure 4.2 represents the terminal wealth W_T^{LEL} as a function of the terminal state price density ξ_T.
Just as in the case of VaR-RM, the dotted line represents the portfolio insurer, the dashed line represents the benchmark agent and the full line represents the LEL-RM agent. It is easy to see that the graph can be divided into three parts:

- for $\xi_T < \underline{\xi}_\epsilon$, the LEL-RM agent acts as the benchmark agent,

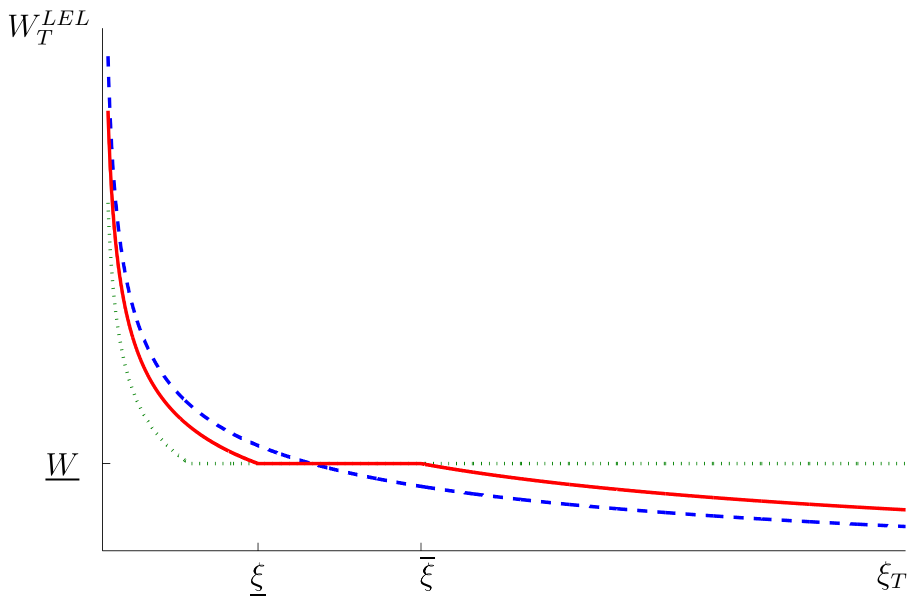

Figure 4.2: Optimal terminal wealth of the LEL-RM agent.

- for $\underline{\xi}_\epsilon \leq \xi_T \leq \overline{\xi}_\epsilon$, the LEL-RM agent insures the portfolio against losses as the portfolio insurer,
- for $\overline{\xi}_\epsilon < \xi_T$, the LEL-RM agent partially insures the portfolio against losses.

Note that in case of the LEL-RM, the loss at $\overline{\xi}_\epsilon < \xi_T$ is smaller than the loss of the benchmark agent.

4.2.2 Properties of maximizing under LEL-RM

Just as for the VaR-RM model, we consider the state price density to be log-normal with constant interest rate r and market price of risk κ. Let the utility function be isoelastic. The next proposition introduces the optimal wealth and portfolio strategy before the horizon T.

Proposition 4. *[[2], Proposition 5.] Assume that* $U(W) = \dfrac{W^{1-\gamma}}{1-\gamma}$ *for* $\gamma > 0$ *and that* r *and* κ *are constants. Then* *i) The optimal wealth at time t is given by*

$$
\begin{aligned}
W_t^{LEL} = & \frac{e^{\Gamma_t^{LEL}}}{(z_1 \xi_t)^{\frac{1}{\gamma}}} \\
& - \left[\frac{e^{\Gamma_t^{LEL}}}{(z_1 \xi_t)^{\frac{1}{\gamma}}} \Phi\left(-d_1^{LEL}(\underline{\xi}_\epsilon)\right) - \underline{W} e^{-r(T-t)} \Phi\left(-d_2^{LEL}(\underline{\xi}_\epsilon)\right) \right] \\
& + \left[\frac{e^{\Gamma_t^{LEL}}}{((z_1 - z_2)\xi_t)^{\frac{1}{\gamma}}} \Phi\left(-d_1^{LEL}(\overline{\xi}_\epsilon)\right) - \underline{W} e^{-r(T-t)} \Phi\left(-d_2^{LEL}(\overline{\xi}_\epsilon)\right) \right],
\end{aligned}
$$

where $\Gamma_t^{LEL} = \Gamma_t^{VaR}, d_1^{LEL}(x) = d_1^{VaR}(x), d_2^{LEL}(x) = d_2^{VaR}(x)$ are as in Proposition 2, (z_1, z_2) are as in Proposition 3 and

$$
\begin{aligned}
\underline{\xi}_\epsilon &= \frac{1}{z_1 \, \underline{W}^\gamma}, \\
\overline{\xi}_\epsilon &= \frac{1}{(z_1 - z_2) \, \underline{W}^\gamma}.
\end{aligned}
$$

ii) The fraction of wealth invested in stocks is

$$\theta_t^{LEL} = q_t^{LEL} \hat{\beta},$$

where $\hat{\beta}$ is calculated by (2.9) and q^{LEL} is given as

$$q_t^{LEL} = 1 - \frac{\underline{W} e^{-r(T-t)} \left[\Phi\left(-d_2^{LEL}(\underline{\xi}_\epsilon)\right) - \Phi\left(-d_2^{LEL}(\overline{\xi}_\epsilon)\right) \right]}{W_t^{LEL}}.$$

iii) The exposure to risky assets relative to the benchmark is bounded below and above, namely $0 \leq q_t^{LEL} \leq 1$ and

$$\lim_{\xi_t \to 0} q_t^{LEL} = \lim_{\xi_t \to \infty} q_t^{LEL} = 1$$

(proof: see [2]).

From i), one can easily see that W_t^{LEL} is a decreasing function of ξ_t for all $t \in \langle 0, T \rangle$.

Just as in the model of the VaR-RM, the advantage of using the power utility functions is that one can easily determine the optimal portfolio strategy θ_t^{LEL} at each time t by calculating the strategy $\hat{\beta}$ of the benchmark agent and evaluating q_t^{LEL} using the LEL-RM settings.

Chapter 5

Portfolio insurance with a guaranteed floor in the constrained model

In this chapter we examine the possibilities to insure the portfolio with options when short-selling is prohibited for both risky and risk-free assets. First, we review the dynamic optimal portfolio strategy of the portfolio insured with OBPI when no constraints are imposed. Then we provide a dynamic portfolio strategy when constraints apply and show that such strategy is admissible for the OBPI in the constrained model. In addition, we offer an alternative method to the OBPI in the constrained model and we compare the two methods through sensitivity analysis.

5.1 Option based portfolio insurance in the constrained model

Assume that the portfolio X consists of one risk-free bond and d risky assets with a constant strategy β, hence itself represents a risky asset. From now on, we denote the risky portfolio X to be one risky asset following the process

$$dX_t = X_t[r + \beta^\top(\mu - r\mathbf{1})]dt + X_t\,\sigma_X dw_t, \tag{5.1}$$

where $\sigma_X = \sqrt{\beta^\top c^R \beta}$ is the volatility of the risky asset X and w_t is a one-dimensional Brownian motion.

The stochastic process for the insured portfolio can be described as

$$dW_t = W_t[r + \theta_t^\top(\mu - r\mathbf{1})]dt + W_t\sqrt{\theta_t^\top c^R \theta_t}dw_t. \tag{5.2}$$

5.1.1 OBPI in the unconstrained model

The portfolio manager aims to maximize the utility from the expected terminal wealth of the insured portfolio

$$\begin{aligned} &\max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ s.t. \quad & W_T \geq \underline{W}, \end{aligned} \tag{5.3}$$

where the maximum is taken through all dynamic strategies θ. Note that in order to avoid immediate arbitrage situations, the floor must satisfy the condition $\underline{W} < W_0 e^{rT}$, where $W_0 > 0$ is the initial amount invested in the portfolio insured with OBPI.

The option based portfolio insurance is a special case of the VaR-RM model, therefore according to [2], the following theorem holds:

Theorem 3. *The optimal portfolio strategy for the problem (5.3) is*

$$\hat{\theta}_t = \frac{1}{\gamma}[c^R]^{-1}(\mu - r\boldsymbol{1})\frac{(1+\varphi_t)X_t}{W_t},$$

where X_t is given by (5.1), and W_t follows (5.2). The fraction of wealth invested in stocks can be expressed as

$$\hat{\theta}_t = q_t \hat{\beta}, \tag{5.4}$$

where $\hat{\beta}$ is the optimal portfolio strategy of the uninsured model without constraints (2.4), calculated by (2.9) and

$$q_t = \frac{(1+\varphi_t)X_t}{W_t}. \tag{5.5}$$

5.1.2 OBPI in the constrained model

Now, let us consider a portfolio with convex constraints on the portfolio strategy and simultaneously is insured by a put option. Mathematically, our model can be written as

$$\begin{aligned} &\max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ s.t. \quad & W_T \geq \underline{W}, \\ & \mathcal{C} = \{\theta^i \geq 0,\ i = 1,2,...,d; \sum \theta^i \leq 1\}. \end{aligned} \tag{5.6}$$

Theorem 4. *Let $\hat{\beta}$ be the optimal portfolio strategy for the portfolio with convex constraints, computed as*

$$\hat{\beta} = \arg\max_{\beta \in \mathcal{C}} \; \beta^\top(\mu - r\boldsymbol{1}) - \frac{1}{2}\gamma\beta^\top c^R \beta,$$

where $\mathcal{C}$ is given by (1.8). Let X_t follow (5.1) and W_t follow (5.2). Let φ_t be the delta of the put option calculated by (3.3). Then the portfolio strategy

$$\theta_t = (1 + \varphi_t)\hat{\beta}\frac{X_t}{W_t} \tag{5.7}$$

is admissible for problem (5.6).

Proof. The optimal portfolio strategy $\hat{\beta}$ for the model with convex constraints and no insurance satisfies

$$\begin{aligned} &\hat{\beta}^i \geq 0, \qquad \text{for } i = 1, 2, ...d, \\ &\sum_{i=1}^{d} \hat{\beta}^i \leq 1. \end{aligned}$$

Since $-1 < \varphi_t < 0$, $\hat{\beta}^i \geq 0$ for $i = 1, 2, ...d$ and $0 \leq \frac{X_t}{W_t} \leq 1$, then $\theta_t^i \geq 0$ for $i = 1, 2, ...d$ and

$$\sum_{i=1}^{d} \theta_t^i = \sum_{i=1}^{d} (1 + \varphi_t)\hat{\beta}^i \frac{X_t}{W_t} \leq (1 + \varphi_t)\sum_{i=1}^{d} \hat{\beta}^i \leq \sum_{i=1}^{d} \hat{\beta}^i \leq 1.$$

Therefore the strategy θ_t is admissible for the Problem 5.6. □

Corollary 1. *Let the solution $\hat{\beta}$ computed by (2.9) be optimal for the problem (2.4). In case that the optimal solution $\hat{\beta}$ with no constraints on the portfolio strategy satisfies $\hat{\beta}^i \geq 0$ for $i = 0, ..., d$ and $\sum_{i=1}^{d} \hat{\beta}^i \leq 1$, the portfolio strategy $\theta_t = \frac{(1 + \varphi_t)X_t}{W_t}\hat{\beta}$ is optimal for the Problem (5.6).*

5.1.3 Optimal distribution of the wealth W_t at time t

The portfolio strategy of the OBPI at time $t \in \langle 0, T \rangle$ is given as

$$\theta_t = (1 + \varphi_t)\hat{\beta}\frac{X_t}{W_t}.$$

At each time t, the agent determines the amount invested in the risky asset X_t according to

$$W_t = X_t + Put_t(X_T \geq \underline{W}). \tag{5.8}$$

Let the function $f_t(X_t)$ be the right-handside of (5.8) expressed as

$$\begin{aligned} f_t(X_t) &= X_t \Phi\left(\frac{\ln\frac{X_t}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X\sqrt{T-t}}\right) \\ &+ \underline{W}e^{-r(T-t)}\Phi\left(-\frac{\ln\frac{X_t}{\underline{W}} + \left(r - \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X\sqrt{T-t}}\right). \end{aligned} \tag{5.9}$$

We show that for every wealth W_t there exists a unique amount X_t invested in the risky asset.

Proposition 5. *At every time $t \in \langle 0, T\rangle$, the function $f_t(X_t)$ is continuous and increasing in X_t. Hence for every given wealth $W_t \geq \underline{W}e^{-r(T-t)}$ there exists a unique X_t such that $W_t = f_t(X_t)$.*

Proof. Let us first examine the limits, when X_t approaches 0 or ∞. For the first limit we have

$$\lim_{X_t \to 0} f_t(X_t) = \underline{W}e^{-r(T-t)}.$$

Note that $\underline{W}e^{-r(T-t)} \leq W_t$ because of the no-arbitrage condition. The second limit is

$$\lim_{X_t \to \infty} f_t(X_t) = \infty.$$

The function $f_t(X_t)$ is continuous, because it is a composition of continuous functions. The first derivative of $f_t(X_t)$ is positive for all X_t:

$$\frac{df_t(X_t)}{dX_t} = \Phi\left(\frac{\ln\frac{X_t}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X\sqrt{T-t}}\right) > 0, \qquad \forall X_t > 0.$$

Considering the facts that $f_t(X_t) \in \left(\underline{W}e^{-r(T-t)}, \infty\right)$ and $f_t(X_t)$ is increasing in X_t, there exists a unique X_t such that $W_t = f_t(X_t)$. $\square$

5.1.4 Expected terminal values

Now, we take a closer look at the expected terminal values, such as the expected terminal value of the risky asset, the expected terminal utility from the risky asset, the expected terminal wealth of the insured portfolio, the expected utility of the terminal wealth and the certainty equivalent.

The expected terminal value of the risky asset X

The risky asset X follows (5.1). The terminal value of the risky asset can be expressed by Ito's lemma as

$$X_T = X_0 e^{\left(r+\beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\sigma_X^2\right)T + \sigma_X\sqrt{T}Z}, \tag{5.10}$$

where Z is a normally distributed random variable, $Z \sim N(0,1)$.

The expected terminal value of the risky asset X_T is

$$\begin{aligned} E[X_T] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X_0 e^{\left(r+\beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\sigma_X^2\right)T + \sigma_X\sqrt{T}z} e^{-\frac{z^2}{2}} dz \\ &= X_0 e^{\left(r+\beta^\top(\mu - r\mathbf{1})\right)T}. \end{aligned} \tag{5.11}$$

Expected utility of the terminal value of the uninsured portfolio

We calculate the expected utility $E[U(X_T)]$ from the terminal value of the risky asset as

$$\begin{aligned} E[U(X_T)] &= E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{X_0^{1-\gamma}}{1-\gamma} e^{\left(r+\beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\sigma_X^2\right)(1-\gamma)T + (1-\gamma)\sigma_X\sqrt{T}z} e^{-\frac{z^2}{2}} dz \\ &= \frac{X_0^{1-\gamma}}{1-\gamma} e^{\left(r+\beta^\top(\mu - r\mathbf{1})\right)(1-\gamma)T + \frac{1}{2}\gamma(\gamma-1)\sigma_X^2 T}. \end{aligned} \tag{5.12}$$

The expected terminal value of the insured portfolio

The terminal value of the insured portfolio W_T with the floor $\underline{W}$ is given as

$$W_T = \max(\underline{W}, X_T),$$

where X_T is given in (5.10). The condition $X_T \geq \underline{W}$ is equivalent to

$$Z \geq \frac{\ln\frac{\underline{W}}{X_0} - \left[r + \beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\sigma_X^2\right]T}{\sigma_X\sqrt{T}} = M. \tag{5.13}$$

Then the expected terminal value of the portfolio is

$$\begin{aligned} E[W_T] &= E[\max(\underline{W}, X_T)] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{M} \underline{W} e^{-\frac{z^2}{2}} dz \\ &+ \frac{1}{\sqrt{2\pi}} \int_{M}^{\infty} X_0 e^{\left(r+\beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\sigma_X^2\right)T + \sigma_X\sqrt{T}z} e^{-\frac{z^2}{2}} dz \\ &= \underline{W}\Phi(M) + X_0 e^{\left(r+\beta^\top(\mu - r\mathbf{1})\right)T}\Phi(-M + \sigma_X\sqrt{T}). \end{aligned} \tag{5.14}$$

Expected utility of the terminal wealth

Now, we calculate the expected utility of the terminal value of the insured portfolio

$$E[U(W_T)] = E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right],$$

where $W_T = \max(\underline{W}, X_T)$. The condition for $X_T \geq \underline{W}$ is defined in (5.13). Then the expected utility from terminal value is calculated as

$$\begin{aligned}
E[U(W_T)] &= E\left[\max\left(\frac{\underline{W}^{1-\gamma}}{1-\gamma}, \frac{X_T^{1-\gamma}}{1\gamma}\right)\right] \\
&= \frac{\underline{W}^{1-\gamma}}{1-\gamma}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M} e^{-\frac{z^2}{2}} dz \\
&+ \frac{1}{\sqrt{2\pi}}\int_{M}^{\infty} \frac{X_0^{1-\gamma}}{1-\gamma} e^{\left(r+\beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\sigma_X^2\right)(1-\gamma)T + (1-\gamma)\,\sigma_X\sqrt{T}z} e^{-\frac{z^2}{2}} dz \\
&= \frac{\underline{W}^{1-\gamma}}{1-\gamma}\Phi(M) + \frac{X_0^{1-\gamma}}{1-\gamma} e^{\left(r+\beta^\top(\mu - r\mathbf{1})\right)(1-\gamma)T + \frac{1}{2}\gamma(\gamma-1)\,\sigma_X^2 T} \\
&\quad \times\Phi(-M + (1-\gamma)\,\sigma_X\sqrt{T}). \qquad (5.15)
\end{aligned}$$

Certainty equivalents

The certainty equivalent C is the amount that yields the same utility as the uncertain portfolio. The certainty equivalent of the risky asset X can be calculated as

$$C_X = \left((1-\gamma)E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}},$$

where $E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right]$ is given as in (5.12). The certainty equivalent of the insured portfolio W_T can be expressed as

$$C_W = \left((1-\gamma)E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}},$$

where $E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]$ is given by (5.15).

5.1.5 Examples

Example - Expected terminal values

Let us now calculate the expected terminal values for both unconstrained and constrained models, i.e. for problems (5.3) and (5.6), respectively.

We use the following settings: risk-free interest rate $r = 2\%$, parameter of the utility function $\gamma = 5$ and initial wealth $W_0 = 1$. We determine the expected returns and the covariance matrix based on data analysis, considering three risky assets, namely McDonald's Corp. (MCD), Johnson & Johnson (JNJ) and Toyota Motor Corporation (TM). We estimate the yearly returns, volatilities and correlations using the daily data from 4th October 2011 to 2nd October 2012. The expected yearly returns are $\mu = (0.06626, 0.1113, 0.1625)$ and the covariance matrix is $c^R = \begin{pmatrix} 0.02155 & 0.00825 & 0.00749 \\ 0.00825 & 0.01517 & 0.01190 \\ 0.00749 & 0.01190 & 0.05011 \end{pmatrix}$.

For simplicity, we use different notations for the unconstrained portfolio. Let $\hat{\beta u}$ represent the portfolio strategy of the unconstrained problem, calculated by (2.9), and $\hat{\beta}$ represent the portfolio strategy of the constrained problem, calculated by (2.8). Table 5.1 shows the optimal strategies for both models.

	β_1	β_2	β_3
$\hat{\beta u}$	-0.060424	0.962767	0.349074
$\hat{\beta}$	0	0.674372	0.325628

Table 5.1: Optimal portfolio strategies of the unconstrained and constrained portfolios.

The volatilities of the risky assets are $\sigma_{Xu} = \sqrt{\hat{\beta u}^\top c^R \hat{\beta u}} = 0.164225$ for the unconstrained model and $\sigma_X = \sqrt{\hat{\beta} c^R \hat{\beta}} = 0.132058$ for the constrained model. Based on volatility, the unconstrained portfolio is more risky in this case.

First, we calculate the initial amount invested in the risky asset X_0 for a given initial wealth W_0. In Matlab, we use the function 'fsolve' to find X_0 such that $f_0(X_0) = W_0$. The function $f_0(X_0)$ is defined by (5.9), setting $t = 0$. We denote Xu_0 and X_0 the initial amounts invested in the risky assets in the unconstrained and the constrained models, respectively.

Table 5.2 summarizes the initial amounts invested in the risky asset, their expected terminal values and the expected terminal values of the insured portfolios for different levels of the floor $\underline{W}$.

T=1

$\underline{W}$	Xu_0	$E[\,Xu_T]$	$E[\,Wu_T]$	X_0	$E[X_T]$	$E[W_T]$
0.980	0.910522	1.063017	1.096366	0.937820	1.065853	1.094841
0.990	0.892117	1.041530	1.085379	0.922808	1.048791	1.078094
1	0.866556	1.011687	1.071833	0.901545	1.024625	1.066506
1.010	0.825268	0.963485	1.053954	0.866476	0.984769	1.050851
1.015	0.788062	0.920047	1.041803	0.834331	0.948235	1.039994

T=3

0.980	0.869972	1.384379	1.402298	0.909869	1.335708	1.346274
0.990	0.856470	1.362893	1.384375	0.898798	1.319456	1.332459
1	0.840987	1.338256	1.364255	0.885945	1.300587	1.316748
1.010	0.822940	1.309536	1.341410	0.870771	1.278312	1.298672
1.015	0.812685	1.293218	1.328734	0.862068	1.265536	1.288541

T=5

0.980	0.858070	1.861110	1.869097	0.902820	1.711942	1.716255
0.990	0.847399	1.837965	1.847246	0.894113	1.695433	1.700557
1	0.835757	1.812713	1.823542	0.884517	1.677236	1.683350
1.010	0.822985	1.785013	1.797713	0.873883	1.657072	1.664405
1.015	0.816115	1.770111	1.783897	0.868119	1.646142	1.654193

T=10

0.980	0.855471	4.024423	4.025418	0.904120	3.250888	3.251325
0.990	0.848319	3.990780	3.991900	0.898397	3.230311	3.230812
1	0.840829	3.955545	3.956805	0.892356	3.208591	3.209166
1.010	0.832977	3.918607	3.920028	0.885976	3.185648	3.186308
1.015	0.828908	3.899462	3.900971	0.882649	3.173689	3.174396

Table 5.2: Portfolio development for different floors $\underline{W}$.

For all $T \in \{1, 3, 5, 10\}$, one can see that the higher the floor $\underline{W}$ is, the lower the initial amounts invested in the risky assets Xu_0 and X_0 are. Equivalently, the higher the required floor is, the more the insurance costs.

Since the unconstrained portfolio has a higher volatility, i.e. $\sigma_{Xu} > \sigma_X$, the initial value invested in the risky asset is lower for the unconstrained model than for the constrained one, i.e. $Xu_0 < X_0$. This holds for all levels of the maturity T and all levels of the floor $\underline{W}$. Moreover, for all T, the difference between the initial values $X_0 - Xu_0$ grows with the increasing floor $\underline{W}$.

For all levels of the maturity T, the expected terminal values $E[\,Xu_T]$ and $E[X_T]$ of the risky assets grow with $\underline{W}$.

When $T = 1$, for all levels of the floor $\underline{W}$, the expected terminal value of the risky asset is higher for the constrained model, i.e. $E[Xu_T] < E[X_T]$. The difference $E[X_T] - E[Xu_T]$ grows with $\underline{W}$. Interestingly, for the other levels of maturity, $T \in \{3, 5, 10\}$ and for all levels of the floor $\underline{W}$, the expected terminal value of the risky asset is higher for the unconstrained model, i.e. $E[X_T] < E[Xu_T]$. Moreover, the difference $E[Xu_T] - E[X_T]$ is decreasing with $\underline{W}$.

For all levels of the maturity T, the expected terminal wealths $E[Wu_T]$ and $E[W_T]$ are decreasing with $\underline{W}$.

For all T, the expected terminal wealth is higher for the unconstrained model, i.e. $E[Wu_T] > E[W_T]$. Moreover, the difference $E[Wu_T] - E[W_T]$ grows with the floor $\underline{W}$.

Naturally, for all levels of the floor $\underline{W}$, the expected terminal values of the risky assets, $E[Xu_T]$ and $E[X_T]$ and also the expected terminal wealths $E[Wu_T]$ and $E[W_T]$ are increasing with T.

Now, let us take a look at the certainty equivalents for different levels of the floor. Table 5.3 summarizes the results.

	T=1		T=3	
$\underline{W}$	C_{Xu}	C_X	C_{Xu}	C_X
0.980	1.063275	1.062431	1.225402	1.221527
0.990	1.057925	1.057235	1.217636	1.214068
1	1.051013	1.050492	1.208832	1.205582
1.010	1.041309	1.040983	1.198725	1.195808
1.015	1.034266	1.034059	1.193069	1.190325

	T=5		T=10	
$\underline{W}$	C_{Xu}	C_X	C_{Xu}	C_X
0.980	1.433902	1.424897	2.180488	2.145512
0.990	1.425031	1.416477	2.169222	2.135158
1	1.415372	1.407280	2.157403	2.124260
1.010	1.404808	1.397188	2.144992	2.112780
1.015	1.399142	1.391764	2.138553	2.106810

Table 5.3: Certainty equivalents for different floors $\underline{W}$.

One can notice that for all levels of T, the certainty equivalents C_{Xu} and C_X are decreasing with the floor $\underline{W}$. Also, the certainty equivalents are higher for the unconstrained model than for the constrained one, i.e. $C_{Xu} > C_X$. Moreover, for all T, the difference $C_{Xu} - C_X$ is decreasing with

$\underline{W}$. Naturally, the for all values of $\underline{W}$, the certainty equivalents are increasing with T.

In this example, we consider the certainty equivalent as the determining criteria for comparison.

Example - Discrete time

In theory, we consider the portfolio development to be continuous. In reality, we can control the portfolio performance only at discrete times.

Let us take a look at the terminal wealth of the insured portfolio, comparing two different ways of discrete computation.

Let $\hat{\beta}$ represent the optimal strategy of the risky asset X. In this case, we require convex constraints on the portfolio strategy, therefore let $\hat{\beta}$ be calculated by (2.8).

Just as in the previous example, we first calculate the initial amount invested in the risky asset X_0 for a given initial wealth W_0.

The first method represents the case when the put option is not available on the market and we have to synthesize it.

At time t, we calculate the portfolio strategy θ_t by (5.7). The wealth at time $t + \Delta t$ is given by the Ito's lemma as

$$W_{t+\Delta t} = W_t e^{\left(r+\theta_t^\top(\mu-r\mathbf{1})-\frac{\theta_t^\top c^R \theta_t}{2}\right)\Delta t+\sqrt{\theta_t^\top c^R \theta_t}\sqrt{\Delta t}Z},$$

where $Z \sim N(0,1)$.

If $W_{t+\Delta t} \geq \underline{W}e^{-r(T-(t+\Delta t))}$, we calculate $X_{t+\Delta t}$, representing the new amount invested in the risky asset. To find $X_{t+\Delta t}$ such that $W_{t+\Delta t} = f_{t+\Delta t}(X_{t+\Delta t})$, we use the function 'fsolve' in Matlab (for $f_{t+\Delta t}(X_{t+\Delta t})$, see (5.9)). Because $W_{t+\Delta t}$ is generated randomly at a discrete time, it may happen that $W_{t+\Delta t} < \underline{W}e^{-r(T-(t+\Delta t))}$, in which case we set $X_{t+\Delta t} = 0$, investing only in the risk-free asset till the maturity.

The value W_T represents the terminal wealth of the portfolio insured with a put option with strike $\underline{W}$.

The second method represents the case when the put option is available on the market. To determine the terminal wealth W_T, we do not need to evaluate the portfolio strategy θ_t at each time t. For the calculated initial amount X_0 we follow the development of the risky asset, where the value $X_{t+\Delta t}$ is calculated by Ito's lemma as

$$X_{t+\Delta t} = X_t e^{\left(r+\hat{\beta}^\top(\mu-r\mathbf{1})-\frac{\hat{\beta}^\top c^R \hat{\beta}}{2}\right)\Delta t+\sqrt{\hat{\beta}^\top c^R \hat{\beta}}\sqrt{\Delta t}Z}.$$

We use the same values of the random variable Z as we used for the discrete-time simulation of W_t.

X_T represents the terminal value of the risky asset. We apply the put option with strike price $\underline{W}$. Therefore the terminal value of the insured portfolio is

$$\tilde{W}_T = \max(X_T, \underline{W}).$$

Note that X_T can also be calculated explicitly

$$X_T = X_0 e^{\left(r+\hat{\beta}^\top(\mu - r\mathbf{1}) - \frac{\hat{\beta}^\top c^R \hat{\beta}}{2}\right)T + \sqrt{\hat{\beta}^\top c^R \hat{\beta}}\sqrt{T}Z}.$$

However, in order to compare the two methods correctly, we choose to generate X_T step by step.

Our aim is to show that the terminal wealths of the insured portfolios calculated in these two different ways are close enough for a well chosen step-length Δt, moreover they approach the theoretical value, calculated in the previous example.

We use the same settings as in the previous example and set the maturity $T = 1$ and the level of the floor $\underline{W} = 1$.

Table 5.4 summarizes the average values of the terminal wealth calculated by both methods according to the chosen step for 20000 simulations. We choose the number of steps to be 12 (re-balancing once a month), 52 (re-balancing once a week), 250 (re-balancing every day, considering that the number of trading days in one year is 250) and 500 (re-balancing twice a day).

Number of steps	$E[W_T]$	$E[\tilde{W}_T]$
12	1.065477	1.066522
52	1.064463	1.065052
250	1.064994	1.065150
500	1.065591	1.065669

Table 5.4: Expected terminal wealths of the insured portfolio with a put option.

As we can see, $E[\tilde{W}_T]$ is higher for all numbers of steps. The distance between the computed values $E[W_T]$ and $E[\tilde{W}]$ decreases with the number of steps. For 500 steps, the expected terminal values are approaching the theoretical value 1.066506.

Using the favorable feature that the expected values calculated by two different methods are almost the same, we can continue to use the second method in our further calculations, as it is less time-consuming.

5.2 Alternative method in the constrained model

Now, we provide an alternative strategy for the problem (5.6). Denote the risky asset by Xa with a given constant portfolio strategy βa and let the insured portfolio be represented by $W\!a$ with a given dynamic portfolio strategy θa_t.

Let the set of constraints restrict only the short positions of the risky assets. Then the set of admissible strategies for βa can be described as $Ca = \{\, \beta a^i \geq 0,\ i = 1, 2, ..., d\}$. We determine the optimal portfolio strategy $\hat{\beta a}$ of the risky asset Xa from (2.8), using $\mathcal{S} = Ca$. The volatility of the risky asset Xa is $\sigma_{Xa} = \sqrt{\hat{\beta a}^\top c^R \hat{\beta a}}$. Then Xa follows the process

$$dXa_t = Xa_t[r + \hat{\beta a}^\top(\mu - r\mathbf{1})]dt + Xa_t\,\sigma_{Xa}dw_t, \tag{5.16}$$

where w_t is a one-dimensional Brownian motion.

The insured portfolio $W\!a$ consists of the risky asset Xa and a put option written on it. Its value at time t can be expressed as

$$W\!a_t = Xa_t + Va_t^{put},$$

where Va_t^{put} is the value of the put option at time t. Let $\underline{W}$ be the strike price of the put option and T be the maturity. Because the particular put option might not be available on the market, we synthesize it.

At time t, the delta of the put option can be calculated as

$$\varphi a_t = \Phi\left(\frac{\ln\dfrac{Xa_t}{\underline{W}} + \left(r + \dfrac{\sigma_{Xa}^2}{2}\right)(T-t)}{\sigma_{Xa}\sqrt{T-t}}\right) - 1$$

and the candidate for the portfolio strategy is

$$h_t = (1 + \varphi a_t)\,\hat{\beta a}\frac{Xa_t}{W\!a_t}.$$

The problem (5.6) requires that the sum of the portfolio weights does not access the upper bound 1, therefore we define the new portfolio strategy as

$$\theta a_t = \begin{cases} (1 + \varphi a_t)\,\hat{\beta a}\dfrac{Xa_t}{W\!a_t} & \text{if} \quad \sum_{i=1}^d h_t^i \leq 1, \\[2ex] \dfrac{(1 + \varphi a_t)\,\hat{\beta a}\dfrac{Xa_t}{W\!a_t}}{\sum_{i=1}^d h_t^i} & \text{if} \quad \sum_{i=1}^d h_t^i \geq 1. \end{cases}$$

Portfolio Wa then follows

$$dWa_t = Wa_t[r + \theta a_t^\top(\mu - r\mathbf{1})]dt + Wa_t\sqrt{\theta a_t^\top c^R \theta a_t}dw_t. \tag{5.17}$$

Note that since $\hat{\beta} a \in \mathcal{C}a$, the portfolio strategy $\theta a_t \geq \mathbf{0}$.

At each time $t \in \langle 0, T\rangle$ we determine the amount Xa_t invested in the risky asset from the equation

$$\begin{aligned} Wa_t = f_t(Xa_t) = {} & Xa_t \Phi\left(\frac{\ln\frac{Xa_t}{\underline{W}} + \left(r + \frac{\sigma_{Xa}^2}{2}\right)(T-t)}{\sigma_{Xa}\sqrt{T-t}}\right) \\ & + \underline{W}e^{-r(T-t)}\Phi\left(-\frac{\ln\frac{Xa_t}{\underline{W}} + \left(r - \frac{\sigma_{Xa}^2}{2}\right)(T-t)}{\sigma_{Xa}\sqrt{T-t}}\right). \end{aligned} \tag{5.18}$$

Proposition 6. *For all $t \in \langle 0, T\rangle$, the function $f_t(Xa_t)$ is continuous and increasing in Xa_t and hence, for each $Wa_t \geq \underline{W}e^{-r(T-t)}$, there exists a unique Xa_t such that $Wa_t = f_t(Xa_t)$.*

The proof is analogous to the Proof of Proposition 5.

Theorem 5. *Let $\mathcal{C}a = \{\beta a^i \geq 0,\ i = 1, 2, ..., d\}$ and $\hat{\beta} a$ is calculated by*

$$\hat{\beta} a = \arg\max_{\beta a \in \mathcal{C}a} \ \beta a^\top(\mu - r\mathbf{1}) - \frac{1}{2}\gamma\, \beta a^\top c^R \beta a. \tag{5.19}$$

Let Wa_t be the value of the portfolio at time t. For $Wa_t \geq \underline{W}e^{-r(T-t)}$, we define the portfolio strategy θa_t as

$$\theta a_t = \begin{cases} (1 + \varphi a_t)\, \hat{\beta} a \frac{Xa_t}{Wa_t} & \text{if} \quad \sum_{i=1}^d h_t^i \leq 1, \\ \dfrac{(1 + \varphi a_t)\, \hat{\beta} a \frac{Xa_t}{Wa_t}}{\sum_{i=1}^d h_t^i} & \text{if} \quad \sum_{i=1}^d h_t^i > 1, \end{cases} \tag{5.20}$$

where $h_t^i = (1 + \varphi a_t)\, \hat{\beta} a^i \frac{Xa_t}{Wa_t}$ and Xa_t is a solution of (5.18). If $Wa_0 \geq \underline{W}e^{-rT}$, then θa_t is admissible for the problem (5.6) and Wa_t satisfies (5.17). Moreover, $Wa_t \geq \underline{W}e^{-r(T-t)}$ for all $t \geq 0$ with probability 1.

Proof. First, we show that the portfolio strategy θa_t is admissible for the problem (5.6), i.e. $\theta a_t \in \mathcal{C}$ for all $t \in \langle 0, T\rangle$, where $\mathcal{C}$ is defined in (1.8). It is clear that $\theta a_t \geq \mathbf{0}$, since $\hat{\beta} a \in \mathcal{C}a$ indicates $\hat{\beta} a \geq \mathbf{0}$. Moreover, $Xa_t \geq 0$, $Wa_t \geq 0$ and $(1 + \varphi a_t) \geq 0$.

If

$$\theta a_t = (1 + \varphi a_t)\, \hat{\beta} a \frac{Xa_t}{Wa_t},$$

then we have $\sum_{i=1}^{d} \theta a_t^i \leq 1$. Otherwise, by the definition of θa_t, it holds that $\sum_{i=1}^{d} \theta a_t^i = 1$.

Second, we show that the dynamic portfolio strategy θa_t guarantees the floor $\underline{W}$. Let $\{ Xa_s\}_{t \leq s \leq T}$ be the solution of (5.16). Then, we have by definition

$$Wa_s = Xa_s + Put_s(Xa_T \geq \underline{W}) = \underline{W} e^{-r(T-s)} + Call_s(Xa_T \geq \underline{W}), \tag{5.21}$$

where $Put_s(Xa_T \geq \underline{W})$ and $Call_s(Xa_T \geq \underline{W})$ denote the values of the put and call options on the asset Xa_s with maturity T and strike price $\underline{W}$.

If $\sum_{i=1}^{d} h_s^i \leq 1$, then the strategy $\theta a_s = (1 + \varphi a_s)\, \hat{\beta} a \dfrac{Xa_s}{Wa_s}$ replicates the portfolio (5.21). In this case, at least $\underline{W} e^{-r(T-s)}$ is invested in the risk-free asset. The remaining part of the portfolio is the value the call option, which cannot fall below 0.

If $\sum_{i=1}^{d} h_s^i > 1$, then the strategy is expressed as

$$\theta a_s = \frac{(1 + \varphi a_s)\, \hat{\beta} a \dfrac{Xa_s}{Wa_s}}{\sum_{i=1}^{d} h_s^i} = \frac{\Phi\left(\dfrac{\ln \dfrac{Xa_s}{\underline{W}} + \left(r + \dfrac{1}{2}\, \sigma_{Xa}^2 \right) (T-s)}{\sigma_{Xa} \sqrt{T-s}} \right) \hat{\beta} a \dfrac{Xa_s}{Wa_s}}{\sum_{i=1}^{d}, h_s^i}$$

and replicates the portfolio

$$Call_s(Xa_T \geq \widetilde{\underline{W}}) + [\underline{W} e^{-r(T-s)} + (Call_s(Xa_T \geq \underline{W}) - Call_s(Xa_T \geq \widetilde{\underline{W}}))],$$

where $\widetilde{\underline{W}}$ represents a level of the floor for which the delta of the call option with the strike $\widetilde{\underline{W}}$ is

$$\Phi\left(\frac{\ln \dfrac{Xa_s}{\widetilde{\underline{W}}} + \left(r + \dfrac{1}{2}\, \sigma_{Xa}^2 \right) (T-s)}{\sigma_{Xa} \sqrt{T-s}} \right) = \frac{\Phi\left(\dfrac{\ln \dfrac{Xa_s}{\underline{W}} + \left(r + \dfrac{1}{2}\, \sigma_{Xa}^2 \right) (T-s)}{\sigma_{Xa} \sqrt{T-s}} \right)}{\sum_{i=1}^{d} h_s^i}.$$

It is clear that $\underline{\widetilde{W}} > \underline{W}$ and hence $Call_s(\, Xa \geq \underline{W}) > Call_s(\, Xa \geq \underline{\widetilde{W}})$. Therefore, in this case we invest a higher amount in the risk-free asset, namely $\underline{W}e^{-r(T-s)} + (Call_s(\, Xa_T \geq \underline{W}) - Call_s(\, Xa_T \geq \underline{\tilde{W}}))$. Moreover, the value of the $Call_s(\, Xa_T \geq \underline{\widetilde{W}})$ cannot fall below 0.

One can conclude that in any case, the strategy θa_t invests at least $\underline{W}e^{-r(T-t)}$ in the risk-free asset and the remaining part of the portfolio does not fall below 0. Therefore it holds that $Wa_t \geq \underline{W}e^{-r(T-t)}$ at every time $t \geq 0$ with probability 1. $\square$

5.3 Sensitivity analysis

Let us now examine the portfolio performance of the OPBI in the constrained model and the portfolio performance of the alternative method for different settings.

We compare the two methods by changing the values of the risk-free interest rate r, the parameter of the power utility function γ, the floor $\underline{W}$ and the expected returns on the risky assets μ. We use three risky assets and one risk-free bond to examine whether one method dominates the other one.

We change only one parameter at the time, the remaining variables are kept fixed. We set the initial wealth $W_0 = 1$ and the covariance matrix as $c^R = \begin{pmatrix} 0.02155 & 0.00825 & 0.00749 \\ 0.00825 & 0.01517 & 0.01190 \\ 0.00749 & 0.01190 & 0.05011 \end{pmatrix}$, based on data analysis.

By default, we set the risk-free interest rate $r = 0.02$, the power utility parameter $\gamma = 5$, the floor $\underline{W} = 1$ and the vector of the expected returns $\mu = (0.06626, 0.1113, 0.1625)$, based on data analysis. We investigate all cases for $T = 1$ and $T = 3$.

In all our calculations the market is represented by the same random variable $Z \sim N(0, 1)$.

We approximate the continuous model with a discrete one. Considering the time complexity of the computation, we divide the time to maturity to 500 subintervals and use 20000 simulations.

First, we compute the optimal portfolio strategies $\hat{\beta}$ by (2.8) and $\hat{\beta}a$ by (5.19). The initial wealth for both methods is equal, given as $Wa_0 = W_0 = 1$. The initial amounts invested in the risky assets, X_0 for the OBPI in the constrained model and Xa_0 for the alternative method, are obtained from (5.9) and from (5.18) respectively, using $t = 0$.

The portfolio strategies θ and θa of the insured models are calculated by (5.7) and (5.20), respectively. Using Ito's lemma, the new values of the insured portfolios are

$$
\begin{aligned}
W_{t+\frac{T}{500}} &= W_t e^{\left(r+\theta_t^\top(\mu-r\mathbf{1})-\frac{\theta_t^\top c^R \theta_t}{2}\right)\left(\frac{T}{500}\right)+\sqrt{\theta_t^\top c^R \theta_t}\sqrt{\frac{T}{500}}Z}, \\
Wa_{t+\frac{T}{500}} &= Wa_t e^{\left(r+\theta a_t^\top(\mu-r\mathbf{1})-\frac{\theta a_t^\top c^R \theta a_t}{2}\right)\left(\frac{T}{500}\right)+\sqrt{\theta a_t^\top c^R \theta a_t}\sqrt{\frac{T}{500}}Z}.
\end{aligned}
$$

We determine the new optimal exposures in the risky asset, $X_{t+\Delta t}$ from (5.9) and $Xa_{t+\Delta t}$ from (5.18), using the time $t+\Delta t$. Finally, at time T we obtain the terminal wealths W_T and Wa_T.

Once we obtained the values of the terminal wealth for each simulation, we compare the methods by their certainty equivalents

$$
\begin{aligned}
C &= \left((1-\gamma)E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} = \left(E\left[W_T^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}, \\
Ca &= \left((1-\gamma)E\left[\frac{Wa_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} = \left(E\left[Wa_T^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}.
\end{aligned}
$$

Sensitivity analysis by changing the risk-free interest rate r

In this section we use three different values of the risk-free interest rate r, namely $r = 1\%$, 2% and 4%.

Table 5.5 summarizes the certainty equivalents for the different levels of the interest rate r.

	T=1		T=3	
r	C	Ca	C	Ca
1%	1.033717	1.033929	1.154550	1.156464
2%	1.050159	1.050284	1.204953	1.206191
4%	1.070968	1.070909	1.258078	1.258081

Table 5.5: Certainty equivalents for different levels of r.

One can see that the certainty equivalents of the two methods are very close to each other. When $T = 1$, for the lower risk-free interest rates ($r = 1\%$ and 2%), the alternative method performs better. However for the value of $r = 4\%$, the OBPI in the constrained model results in a slightly higher certainty equivalent. When $T = 3$, the alternative method dominates the OBPI in the constrained model.

One can see that for both $T = 1$ and $T = 3$, the certainty equivalents C and Ca are increasing with r, moreover the difference between the certainty equivalent decreases with r.

Sensitivity analysis by changing γ

In our study, we use three different values γ, numerically 3, 5 and 8, representing three different types of absolute risk aversions.Table 5.6 summarizes the certainty equivalents.

	T=1		T=3	
γ	C	Ca	C	Ca
3	1.054366	1.056546	1.236916	1.249344
5	1.050159	1.050284	1.204953	1.206191
8	1.043913	1.043865	1.165668	1.165578

Table 5.6: Certainty equivalents for different values of γ.

One can see that for both $T = 1$ and $T = 3$, the certainty equivalents C and Ca are decreasing with γ.

When $T = 1$, for the lower levels of the absolute risk-aversion, $\gamma = 3$ and 5, the alternative method results in higher certainty equivalents, than the OBPI in constrained models. The opposite result occurs for the most risk-averse parameter $\gamma = 8$, i.e. the OBPI in constrained models shows a slightly higher result.

When $T = 3$, for all $\gamma \in \{3, 5, 8\}$ the certainty equivalent Ca of the alternative method is higher than the certainty equivalent C of the OBPI in the constrained model. Moreover, the difference between the certainty equivalents decreases with γ.

Sensitivity analysis by changing the floor $\underline{W}$

In this section we change the level of the insurance. When $T = 1$, the floor can not be higher than $W_0 e^{rT} = 1.0202$. We choose three different levels of the floor. The level $\underline{W} = 0.98$ represents the floor under the initial investment W_0, the level $\underline{W} = 1$ represents the desire to keep the value of the terminal wealth at least on the level of the initial investment W_0 and the level $\underline{W} = 1.01$ represents the floor that is higher than the initial investment W_0. Table 5.7 shows the results.

For both $T = 1$ and $T = 3$, the certainty equivalents C and Ca are decreasing with the floor $\underline{W}$. For all levels of the floor $\underline{W}$, the certainty equivalent of the alternative method is higher than the certainty equivalent of the OBPI in the constrained model, i.e. $Ca > C$. Moreover, the difference between the certainty equivalents increases with T.

	T=1		T=3	
$\underline{W}$	C	Ca	C	Ca
0.98	1.062126	1.062395	1.220557	1.222028
1	1.050159	1.050284	1.204953	1.206191
1.01	1.040706	1.040776	1.195318	1.196375

Table 5.7: Certainty equivalents for different levels of $\underline{W}$.

Sensitivity analysis by changing μ

We choose different combinations of μ in such way that the different strategies $\hat{\beta}$ obtained by (2.8) using μ, have different ratios between the assets. Table 5.8 shows $\hat{\beta}$ for different settings of μ.

$\mu(1)$	$\hat{\beta}(1)$	$\mu(2)$	$\hat{\beta}(2)$	$\mu(3)$	$\hat{\beta}(3)$
0.06626	0	0.06626	0	0.06626	0
0.11130	0.509783	0.09000	0.338605	0.09000	0.197966
0.16250	0.490217	0.16250	0.661395	0.18000	0.802034

Table 5.8: Portfolio strategies using different expected returns μ.

Table 5.9 summarizes the certainty equivalents for different combinations of μ

	T=1		T=3	
μ	C	Ca	C	Ca
$\mu(1)$	1.050159	1.050284	1.204953	1.206191
$\mu(2)$	1.044167	1.044124	1.174015	1.173824
$\mu(3)$	1.047472	1.047426	1.191322	1.191154

Table 5.9: Certainty equivalents for different combinations of μ.

For both $T = 1$ and $T = 3$, the dominance varies. For $\mu(1)$, the alternative method dominates the OBPI in the constrained model, i.e. $Ca > C$. For $\mu(2)$ and $\mu(3)$, the OBPI in the constrained model dominates the alternative method, i.e. $C > Ca$.

Conclusion of the sensitivity analysis

We see that there is no exact answer whether one should choose the OBPI in the constrained model or the alternative method. In other words, the OBPI is not optimal in the constrained model.

When changing the interest rate r, the parameter of the absolute risk-aversion γ or the expected returns on the risky assets μ, none of the methods dominate the other one. When changing the floor $\underline{W}$, in our specific settings, the alternative method dominates the OBPI in constrained model.

Chapter 6

Portfolio insurance with a partially guaranteed floor in the constrained model

In this chapter we describe the Value-at-Risk based risk management and the Limited-Expected-Losses based risk management more in detail. For each risk management strategy, we first derive the critical values and the expected terminal values. Then we examine whether such a management strategy can be used in case of short-selling restrictions. Finally, we provide an example where we calculate the certainty equivalent of the portfolio insured by the given risk management strategy.

6.1 Value-at-Risk based risk management

According to Proposition 1, the terminal value of the VaR-agent is given as

$$W_T^{VaR} = \begin{cases} I(y\xi_T) & \text{if} \quad \xi_T < \underline{\xi}, \\ \underline{W} & \text{if} \quad \underline{\xi} \leq \xi_T \leq \overline{\xi}, \\ I(y\xi_T) & \text{if} \quad \overline{\xi} < \xi_T, \end{cases}$$

where $y = \dfrac{U'(\underline{W})}{\underline{\xi}}$. Using the power utility function $U(W_T) = \dfrac{W_T^{1-\gamma}}{1-\gamma}$, we obtain

$$y = \frac{1}{\underline{\xi}}\underline{W}^{-\gamma} \qquad \text{and} \qquad I(y\xi_T) = \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}}\xi_T^{-\frac{1}{\gamma}}.$$

In order to calculate the expected terminal wealth, we need to determine the critical values $\underline{\xi}$ and $\overline{\xi}$ first.

6.1.1 The critical values $\overline{\xi}$ and $\underline{\xi}$

To find the critical values $\overline{\xi}$ and $\underline{\xi}$, let us first take a look at the state price density process given by (1.4). By Ito's lemma, we can express the terminal value of the state price density as

$$\xi_T = \xi_0 e^{-(r+\frac{1}{2}\|\kappa\|^2)T - \|\kappa\|\sqrt{T}Z},$$

where $Z \sim N(0,1)$.
The upper critical value $\overline{\xi}$ is given as

$$P(\xi_T > \overline{\xi}) = \alpha.$$

Then we can express $\overline{\xi}$ as

$$\overline{\xi} = \xi_0 e^{-(r+\frac{1}{2}\|\kappa\|^2)T - \|\kappa\|\sqrt{T}\Phi^{-1}(\alpha)}.$$

Using the power utility function, the lower critical value $\underline{\xi}$ is defined as

$$\underline{\xi} = \frac{1}{y}\underline{W}^{-\gamma},$$

where $y \geq 0$ solves $E[\xi_T W_T^{VaR}] = \xi_0 W_0$.

The inequality defining the lower boundary can be expressed in the following form

$$\begin{aligned} \xi_T &< \underline{\xi}, \\ \xi_0 e^{-(r+\frac{1}{2}\|\kappa\|^2)T - \|\kappa\|\sqrt{T}Z} &< \underline{\xi}, \\ M = \frac{-\ln\frac{\underline{\xi}}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}} &< Z. \end{aligned}$$

Similarly, the inequality defining the upper boundary is

$$\begin{aligned} \xi_T &< \overline{\xi}, \\ \xi_0 e^{-(r+\frac{1}{2}\|\kappa\|^2)T - \|\kappa\|\sqrt{T}Z} &< \xi_0 e^{-(r+\frac{1}{2}\|\kappa\|^2)T - \|\kappa\|\sqrt{T}\Phi^{-1}(\alpha)}, \\ \Phi^{-1}(\alpha) &< Z. \end{aligned}$$

According to the new boundaries, the terminal wealth is given as

$$W_T^{VaR} = \begin{cases} \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}}\xi_T^{-\frac{1}{\gamma}} & \text{if} \quad M < Z, \\ \underline{W} & \text{if} \quad \Phi^{-1}(\alpha) \leq Z \leq M, \\ \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}}\xi_T^{-\frac{1}{\gamma}} & \text{if} \quad Z \leq \Phi^{-1}(\alpha). \end{cases} \tag{6.1}$$

The expected terminal value $E[\xi_T W_T]$ consists of three integrals

$$\begin{aligned}
E[\xi_T W_T^{VaR}] &= \frac{1}{\sqrt{2\pi}} \int_M^\infty \underline{W}\, \underline{\xi}^{\frac{1}{\gamma}} \xi_T^{\frac{\gamma-1}{\gamma}} e^{-\frac{z^2}{2}} dz \\
&+ \frac{1}{\sqrt{2\pi}} \int_{\Phi^{-1}(\alpha)}^M \underline{W} \xi_T e^{-\frac{z^2}{2}} dz \\
&+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(\alpha)} \underline{W}\, \underline{\xi}^{\frac{1}{\gamma}} \xi_T^{\frac{\gamma-1}{\gamma}} e^{-\frac{z^2}{2}} dz.
\end{aligned}$$

First we calculate the integral $J(a, b, p)$ in general, using boundaries a and b and the exponent p

$$\begin{aligned}
J(a,b,p) &= \frac{1}{\sqrt{2\pi}} \int_a^b \left(\xi_0 e^{-\left(r + \frac{1}{2}\|\kappa\|^2\right)T - \|\kappa\|\sqrt{T} z} \right)^p e^{-\frac{z^2}{2}} dz \\
&= \xi_0^p e^{-\left(r+\frac{1}{2}\|\kappa\|^2\right)pT} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\|\kappa\|\sqrt{T} p z} e^{-\frac{z^2}{2}} dz \\
&= \xi_0^p e^{-\left(r+\frac{1}{2}\|\kappa\|^2\right)pT} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{z^2 + 2\|\kappa\|\sqrt{T} p z + \|\kappa\|^2 T p^2 - \|\kappa\|^2 T p^2}{2}} dz \\
&= \xi_0^p e^{-\left(r+\frac{1}{2}\|\kappa\|^2\right)pT} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{(z + \|\kappa\|\sqrt{T} p)^2 - \|\kappa\|^2 T p^2}{2}} dz \\
&= \xi_0^p e^{-\left(r+\frac{1}{2}\|\kappa\|^2\right)pT + \frac{1}{2}\|\kappa\|^2 T p^2} \frac{1}{\sqrt{2\pi}} \int_{a+\|\kappa\|\sqrt{T}p}^{b+\|\kappa\|\sqrt{T}p} e^{-\frac{z^2}{2}} dz \\
&= \xi_0^p e^{-\left(r+\frac{1}{2}\|\kappa\|^2\right)pT + \frac{1}{2}\|\kappa\|^2 T p^2} \\
&\quad \times \left[\Phi\left(b + \|\kappa\|\sqrt{T} p\right) - \Phi\left(a + \|\kappa\|\sqrt{T} p\right) \right].
\end{aligned}$$

Now, the value $E[\xi_T W_T]$ can be calculated as

$$\begin{aligned}
E[\xi_T W_T^{VaR}] &= \underline{W}\, \underline{\xi}^{\frac{1}{\gamma}} J\left(M, \infty, \frac{\gamma-1}{\gamma}\right) \\
&+ \underline{W} J\left(\Phi^{-1}(\alpha), M, 1\right) \\
&+ \underline{W}\, \underline{\xi}^{\frac{1}{\gamma}} J\left(-\infty, \Phi^{-1}(\alpha), \frac{\gamma-1}{\gamma}\right) \\
&= \underline{W}\, \underline{\xi}^{\frac{1}{\gamma}} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2}\|\kappa\|^2 T} \left[1 - \Phi\left(M + \frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T}\right) \right. \\
&\quad \left. + \Phi\left(\Phi^{-1}(\alpha) + \frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T}\right) \right] \\
&+ \underline{W} \xi_0 e^{-rT} \left[\Phi\left(M + \|\kappa\|\sqrt{T}\right) - \Phi\left(\Phi^{-1}(\alpha) + \|\kappa\|\sqrt{T}\right) \right].
\end{aligned}$$

We calculate $\underline{\xi}$ numerically from

$$E[\xi_T W_T^{VaR}] - \xi_0 W_0 = 0.$$

Proposition 7. *There exists a unique* $\underline{\xi}$ *such that*

$$E[\xi_T W_T^{VaR}] - \xi_0 W_0 = 0$$

holds.

Proof. First, we define the function

$$\begin{aligned}
F(\underline{\xi}) &= E[\xi_T W_T] - \xi_0 W_0 \\
&= \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2}\|\kappa\|^2 T} \\
&\quad \times \left[1 - \Phi\left(\frac{-\ln\frac{\underline{\xi}}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} + \frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T} \right) \right. \\
&\qquad \left. + \Phi\left(\Phi^{-1}(\alpha) + \frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T} \right) \right] \\
&+ \underline{W}\xi_0 e^{-rT} \left[\Phi\left(\frac{-\ln\frac{\underline{\xi}}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} + \|\kappa\|\sqrt{T} \right) \right. \\
&\quad \left. - \Phi\left(\Phi^{-1}(\alpha) + \|\kappa\|\sqrt{T} \right) \right] \\
&- \xi_0 W_0.
\end{aligned}$$

The first derivative of F is

$$\begin{aligned}
\frac{dF(\underline{\xi})}{d\underline{\xi}} &= \frac{1}{\gamma}\underline{W}\,\underline{\xi}^{\frac{1-\gamma}{\gamma}} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2}\|\kappa\|^2 T} \\
&\quad \times \left[\Phi\left(\frac{\ln\frac{\underline{\xi}}{\xi_0} + \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} - \frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T} \right) \right. \\
&\qquad \left. + \Phi\left(\Phi^{-1}(\alpha) + \frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T} \right) \right] \\
&+ \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}-1} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2}\|\kappa\|^2 T} \frac{1}{\|\kappa\|\sqrt{T}} \\
&\quad \times \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\ln\frac{\underline{\xi}}{\xi_0} + \left(r + \frac{1}{2}\|\kappa\|^2\right)T + \frac{1-\gamma}{\gamma}\|\kappa\|^2 T\right)^2}{2\|\kappa\|^2 T}} \\
&- \underline{W}\,\underline{\xi}^{-1} \xi_0 e^{-rT} \frac{1}{\|\kappa\|\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\ln\frac{\underline{\xi}}{\xi_0} + rT - \frac{1}{2}\|\kappa\|^2 T\right)^2}{2\|\kappa\|^2 T}}.
\end{aligned}$$

Using the fact that

$$e^{-\frac{\left(\ln\frac{\underline{\xi}}{\xi_0}+\left(r+\frac{1}{2}\,\|\kappa\|^2\right)T+\frac{1-\gamma}{\gamma}\,\|\kappa\|^2T\right)^2}{2\,\|\kappa\|^2T}} = \underline{\xi}^{-\frac{1}{\gamma}}\xi_0^{\frac{1}{\gamma}}e^{-\frac{1}{\gamma}rT+\frac{\gamma-1}{2\gamma^2}\,\|\kappa\|^2T}e^{-\frac{\left(\ln\frac{\underline{\xi}}{\xi_0}+rT-\frac{1}{2}\,\|\kappa\|^2T\right)^2}{2\,\|\kappa\|^2T}}, \tag{6.2}$$

we show that the sum of the second and the third addends is zero:

$$\begin{aligned} II+III &= \underline{W}\,\underline{\xi}^{-1}\xi_0e^{-rT}\frac{1}{\|\kappa\|\sqrt{T}}\,\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(\ln\frac{\underline{\xi}}{\xi_0}+rT-\frac{1}{2}\,\|\kappa\|^2T\right)^2}{2\,\|\kappa\|^2T}} \\ &\quad\times\left[\underline{\xi}^{\frac{1}{\gamma}}\xi_0^{-\frac{1}{\gamma}}e^{\frac{1}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}\,\|\kappa\|^2T}\,\underline{\xi}^{-\frac{1}{\gamma}}\xi_0^{\frac{1}{\gamma}}e^{-\frac{1}{\gamma}rT+\frac{\gamma-1}{2\gamma^2}\,\|\kappa\|^2T}-1\right] \\ &= 0. \end{aligned}$$

Hence the first derivative is positive

$$\begin{aligned} \frac{dF(\underline{\xi})}{d\underline{\xi}} &= \frac{1}{\gamma}\,\underline{W}\,\underline{\xi}^{\frac{1-\gamma}{\gamma}}\xi_0^{\frac{\gamma-1}{\gamma}}\,e^{\frac{1-\gamma}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}\,\|\kappa\|^2T} \\ &\quad\times\left[\Phi\left(\frac{\ln\frac{\underline{\xi}}{\xi_0}+\left(r+\frac{1}{2}\,\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}}-\frac{\gamma-1}{\gamma}\,\|\kappa\|\sqrt{T}\right)\right. \\ &\quad\left.+\,\Phi\left(\Phi^{-1}(\alpha)+\frac{\gamma-1}{\gamma}\,\|\kappa\|\sqrt{T}\right)\right] \\ &> 0. \end{aligned}$$

The function $F(\underline{\xi})$ is increasing, hence there exists a unique $\underline{\xi}$ such that

$$E[\xi_TW_T]-\xi_0W_0=0.$$

□

6.1.2 Expected terminal values

Once we evaluated the critical values $\underline{\xi}$ and $\overline{\xi}$, we can also calculate the expected terminal wealth $E[W_T^{VaR}]$ and the expected utility $E\left[\frac{(W_T^{VaR})^{1-\gamma}}{1-\gamma}\right]$ from the terminal wealth, where W_T^{VaR} is given in (6.1).

Expected terminal wealth

In case the VaR-constraint (4.1) holds, the expected terminal wealth is

$$
\begin{aligned}
E[W_T^{VaR}] = {} & \frac{1}{\sqrt{2\pi}} \int_M^\infty \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} e^{-\frac{z^2}{2}} dz \\
& + \frac{1}{\sqrt{2\pi}} \int_{\Phi^{-1}(\alpha)}^M \underline{W} e^{-\frac{z^2}{2}} dz \\
& + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(\alpha)} \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} e^{-\frac{z^2}{2}} dz \\
& = \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} J\left(M, \infty, -\frac{1}{\gamma}\right) + \underline{W} J\left(\Phi^{-1}(\alpha), M, 0\right) \\
& + \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} J\left(-\infty, \Phi^{-1}(\alpha), -\frac{1}{\gamma}\right) \\
& = \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} \xi_0^{-\frac{1}{\gamma}} e^{\frac{1}{\gamma} rT + \frac{\gamma+1}{2\gamma^2} \|\kappa\|^2 T} \\
& \quad \times \left[1 - \Phi\left(\frac{-\ln \frac{\xi}{\xi_0} - \left(r + \frac{1}{2} \|\kappa\|^2\right) T}{\|\kappa\| \sqrt{T}} - \frac{1}{\gamma} \|\kappa\| \sqrt{T} \right) \right. \\
& \qquad \left. + \Phi\left(\Phi^{-1}(\alpha) - \frac{1}{\gamma} \|\kappa\| \sqrt{T} \right) \right] \\
& + \underline{W} \left[\Phi\left(\frac{-\ln \frac{\xi}{\xi_0} - \left(r + \frac{1}{2} \|\kappa\|^2\right) T}{\|\kappa\| \sqrt{T}} \right) - \alpha \right].
\end{aligned}
$$

In case the VaR-constraint (4.1) does not hold, the expected terminal wealth is

$$
\begin{aligned}
E[W_T^{VaR}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} e^{-\frac{z^2}{2}} dz \\
&= \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} J\left(-\infty, \infty, -\frac{1}{\gamma}\right) \\
&= \underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} \xi_0^{-\frac{1}{\gamma}} e^{\frac{1}{\gamma} rT + \frac{\gamma+1}{2\gamma^2} \|\kappa\|^2 T}.
\end{aligned}
$$

Expected utility from the terminal wealth

In case the VaR-constraint (4.1) holds, the expected utility from the terminal wealth of the VaR-RM agent is

$$
\begin{aligned}
E[U(W_T^{VaR})] = & \frac{1}{\sqrt{2\pi}} \int_M^\infty \frac{(\underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}})^{1-\gamma}}{1-\gamma} e^{-\frac{z^2}{2}} dz \\
& + \frac{1}{\sqrt{2\pi}} \int_{\Phi^{-1}(\alpha)}^{M} \frac{\underline{W}^{1-\gamma}}{1-\gamma} e^{-\frac{z^2}{2}} dz \\
& + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(\alpha)} \frac{(\underline{W}\,\underline{\xi}^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}})^{1-\gamma}}{1-\gamma} e^{-\frac{z^2}{2}} dz \\
& = \frac{\underline{W}^{1-\gamma}}{1-\gamma} \underline{\xi}^{\frac{1-\gamma}{\gamma}} \left[J\left(M, \infty, \frac{\gamma-1}{\gamma}\right) + J\left(-\infty, \Phi^{-1}(\alpha), \frac{\gamma-1}{\gamma}\right) \right] \\
& + \frac{\underline{W}^{1-\gamma}}{1-\gamma} J(\Phi^{-1}(\alpha), M, 0) \\
& = \frac{\underline{W}^{1-\gamma}}{1-\gamma} \underline{\xi}^{\frac{1-\gamma}{\gamma}} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2} \|\kappa\|^2 T} \\
& \quad \times \left[1 - \Phi\left(\frac{-\ln \frac{\underline{\xi}}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} + \frac{\gamma-1}{\gamma} \|\kappa\| \sqrt{T} \right) \right. \\
& \qquad \left. + \Phi\left(\Phi^{-1}(\alpha) + \frac{\gamma-1}{\gamma} \|\kappa\| \sqrt{T} \right) \right] \\
& + \frac{\underline{W}^{1-\gamma}}{1-\gamma} \left[\Phi\left(\frac{-\ln \frac{\underline{\xi}}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} \right) - \alpha \right].
\end{aligned}
$$

In case the VaR-constraint does not hold, the expected utility from the terminal wealth of the VaR-RM agent is

$$
\begin{aligned}
E[U(W_T^{VaR})] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\underline{W}^{1-\gamma}}{1-\gamma} \underline{\xi}^{\frac{1-\gamma}{\gamma}} \xi_T^{\frac{\gamma-1}{\gamma}} e^{-\frac{z^2}{2}} dz \\
&= \frac{\underline{W}^{1-\gamma}}{1-\gamma} \underline{\xi}^{\frac{1-\gamma}{\gamma}} J\left(-\infty, \infty, \frac{\gamma-1}{\gamma}\right) \\
&= \frac{\underline{W}^{1-\gamma}}{1-\gamma} \underline{\xi}^{\frac{1-\gamma}{\gamma}} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2} \|\kappa\|^2 T}.
\end{aligned}
$$

6.1.3 VaR-RM in the constrained model

Basak and Shapiro [2] derived the VaR-RM model for the portfolio with no strategy constraints. We show that when the portfolio strategy is constrained, the VaR-RM is not admissible.

Let $\mathcal{C}$ be the set of all admissible portfolio strategies where short-selling is prohibited and the agent is not allowed to borrow risk-free bonds or cash to finance the purchase of further risky assets:

$$\mathcal{C} = \{\theta^i \geq 0, i = 1, 2, ...d; \sum_1^d \theta^i \leq 1\}.$$

Using the power utility function, we can define the VaR-RM problem in the constrained model as

$$\begin{aligned} & \max_{\theta^{VaR}} E\left[\frac{(W_T^{VaR})^{1-\gamma}}{1-\gamma}\right] \qquad (6.3) \\ s.t. \quad & P(W_T^{VaR} \geq \underline{W}) \geq 1-\alpha, \\ & \mathcal{C} = \{(\theta_t^{VaR})^i \geq 0, i = 1, 2, ...d; \sum_1^d (\theta_t^{VaR})^i \leq 1, \forall t \in (0, T)\}, \end{aligned}$$

with a given initial W_0.

Theorem 6. *Let θ_t^{VaR} represent the portfolio strategy of the VAR-RM agent. For any θ_t^{VaR}, the sum exceeds one, i.e.*

$$\sum_{i=1}^d (\theta_t^{VaR})^i \geq 1,$$

with positive probability. Hence the strategy θ_t^{VaR} is not admissible for the problem (6.3).

Proof. The part *ii*) of Proposition 2 states that the portfolio strategy of the VaR-RM agent can be calculated as

$$\theta_t^{VaR} = q_t^{VaR} \hat{\beta},$$

where $\hat{\beta}$ is the portfolio strategy of the benchmark agent calculated by (2.9) and

$$\begin{aligned} q_t^{VaR} &= 1 - \frac{\underline{W} e^{-r(T-t)} \left(\Phi(-d_2^{VaR}(\underline{\xi})) - \Phi(-d_2^{VaR}(\overline{\xi}))\right)}{W_t^{VaR}} \\ &\quad + \frac{\gamma(\underline{W} - \underline{\underline{W}}) e^{-r(T-t)} \phi(d_2^{VaR}(\overline{\xi}))}{W_t^{VaR} \|\kappa\| \sqrt{T-t}}. \end{aligned}$$

By the part $iii)$ of Proposition 2, $q_t^{VaR} \geq 0$.

In case there exists at least one $i \in \{1, 2, ..., d\}$ such that $\hat{\beta}^i < 0$, the VaR-RM strategy is not admissible because $(\theta^{VaR})_t^i \notin \mathcal{C}$. Let us assume that $\hat{\beta}^i \geq 0$ for all $i = 1, 2, ..., d$.

When the time approaches the maturity and $\xi_t = \overline{\xi}$, we have

$$\lim_{t \to T} q_t^{VaR} = \infty.$$

Hence for every $H \geq 0$ there exist a time t such that $q_t \geq H$. Let H be defined as

$$H = \frac{1}{\sum_{i=1}^{d} \hat{\beta}},$$

then

$$\sum_{i=1}^{d} (\theta^{VaR})_t^i = q_t^{VaR} \sum_{i=1}^{d} \hat{\beta} \geq H \sum_{i=1}^{d} \hat{\beta} > 1.$$

The strategy θ_t^{VaR} is not admissible for the problem (6.3). □

6.1.4 Example - Expected terminal values

We calculate the value of the expected terminal wealth and the certainty equivalent for the portfolio with no constraints using the VaR-RM.

We use the same settings as in Section 5.1.5. We set the initial state price density $\xi_0 = 1$ and the probability of falling under the floor $\alpha = 0.05$.

First, we evaluate the critical values using the function 'fsolve' in Matlab. The upper critical value is identical for all levels of the floor. Table 6.1 shows both the upper and the lower critical values for different levels of the floor.

$\underline{W}$	$\overline{\xi}$	$\underline{\xi}$
0.980	2.700685	1.461108
0.990	2.700685	1.362692
1	2.700685	1.267172
1.010	2.700685	1.174205
1.015	2.700685	1.128622

Table 6.1: Critical values of the state price density for different levels of the floor.

We can see that $\underline{\xi}$ is decreasing with the level of floor.

Table 6.2 summarizes the expected terminal values and the certainty equivalents of the portfolios insured with VaR-RM for different levels of the floor.

$\underline{W}$	$E[W_T^{VaR}]$	C^{VaR}
0.980	1.157530	1.089074
0.990	1.155156	1.088262
1	1.152462	1.087253
1.010	1.149427	1.086015
1.015	1.147785	1.085303

Table 6.2: Expected terminal values and certainty equivalents of the VaR-RM agent for different levels of the floor.

We conclude that both the expected terminal value and the certainty equivalent are decreasing with the floor.

6.2 Limited-Expected-Losses based risk management

According to Proposition 3, the terminal wealth of the LEL-RM agent is given as

$$W_T^{LEL} = \begin{cases} I(z_1\xi_T) & \text{if} \quad \xi_T < \underline{\xi}_\epsilon, \\ \underline{W} & \text{if} \quad \underline{\xi}_\epsilon \leq \xi_T \leq \overline{\xi}_\epsilon, \\ I((z_1 - z_2)\xi_T) & \text{if} \quad \overline{\xi}_\epsilon < \xi_T. \end{cases}$$

For the power utility function, the parameters z_1 and $z_1 - z_2$ are given as

$$z_1 = \frac{1}{\underline{\xi}_\epsilon \, \underline{W}^\gamma} \qquad \text{and} \qquad z_1 - z_2 = \frac{1}{\overline{\xi}_\epsilon \, \underline{W}^\gamma}.$$

Therefore

$$I(z_1\xi_T) = \underline{W}\, \underline{\xi}_\epsilon^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} \qquad \text{and} \qquad I((z_1 - z_2)\xi_T) = \underline{W}\, \overline{\xi}_\epsilon^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}}.$$

To calculate the expected terminal wealth and the utility from the terminal wealth, we first derive the critical values $\underline{\xi}_\epsilon$ and $\overline{\xi}_\epsilon$.

6.2.1 The critical values $\overline{\xi}_\epsilon$ and $\underline{\xi}_\epsilon$

Let the terminal value of the state price density be expressed as

$$\xi_T = \xi_0 e^{-(r + \frac{1}{2}\|\kappa\|^2)T - \|\kappa\|\sqrt{T}Z},$$

where $Z \sim N(0,1)$.

Similarly, as in the model of VaR-RM, the inequality constraint defining the upper boundary can be expressed as

$$\begin{aligned} \xi_T &< \overline{\xi}_\epsilon, \\ M(\overline{\xi}_\epsilon) = \frac{-\ln \frac{\overline{\xi}_\epsilon}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} &< Z \end{aligned}$$

and the lower boundary is

$$\begin{aligned} \xi_T &< \underline{\xi}_\epsilon, \\ M(\underline{\xi}_\epsilon) = \frac{-\ln \frac{\underline{\xi}_\epsilon}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} &< Z. \end{aligned}$$

Then the terminal wealth W_T^{LEL} can be then expressed as

$$W_T^{LEL} = \begin{cases} \underline{W}\,\underline{\xi}_\epsilon^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} & \text{if} \quad M(\underline{\xi}_\epsilon) < Z, \\ \underline{W} & \text{if} \quad M(\overline{\xi}_\epsilon) \le Z \le M(\underline{\xi}_\epsilon), \\ \underline{W}\,\overline{\xi}_\epsilon^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} & \text{if} \quad Z < M(\overline{\xi}_\epsilon). \end{cases}$$

Considering that $z_2 \neq 0$, we calculate $\overline{\xi}_\epsilon$ from the second equation of (4.10), namely

$$E\left[\xi_T \left(\underline{W} - W_T^{LEL}(z_1, z_2)\right) 1_{\{W_T^{LEL}(z_1,z_2) \le \underline{W}\}}\right] = \epsilon.$$

Since W_T^{LEL} is a decreasing function of ξ_T and $\underline{W} - W_T^{LEL} = 0$ on the interval $M(\overline{\xi}_\epsilon) \le Z \le M(\underline{\xi}_\epsilon)$, the condition $W_T^{LEL} \le \underline{W}$ implies that we only use W_T^{LEL} for $Z < M(\overline{\xi}_\epsilon)$. Then

$$\begin{aligned} \epsilon &= E\left[\xi_T \left(\underline{W} - \underline{W}\,\overline{\xi}_\epsilon^{\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}}\right)_{\{Z < M(\overline{\xi}_\epsilon)\}}\right] \\ &= \underline{W}\left[J\left(-\infty, M(\overline{\xi}_\epsilon), 1\right) - \overline{\xi}_\epsilon^{\frac{1}{\gamma}} J\left(-\infty, M(\overline{\xi}_\epsilon), \frac{\gamma-1}{\gamma}\right)\right] \\ &= \underline{W}\xi_0 e^{-rT}\Phi\left(M(\overline{\xi}_\epsilon) + \|\kappa\|\sqrt{T}\right) \\ &- \underline{W}\,\overline{\xi}_\epsilon^{\frac{1}{\gamma}} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2}\|\kappa\|^2 T}\Phi\left(M(\overline{\xi}_\epsilon) + \frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T}\right). \end{aligned} \qquad (6.4)$$

We determine the unique $\overline{\xi}_\epsilon$ numerically from (6.4).

Proposition 8. *There exists a unique* $\overline{\xi}_\epsilon$ *such that*

$$E\left[\xi_T\left(\underline{W}-\underline{W}\,\overline{\xi}_\epsilon^{\frac{1}{\gamma}}\xi_T^{-\frac{1}{\gamma}}\right)_{\{Z<M(\overline{\xi}_\epsilon)\}}\right]-\epsilon=0.$$

Proof. First, we define the function

$$\begin{aligned}
F_\epsilon(\overline{\xi}_\epsilon) &= E\left[\xi_T\left(\underline{W}-\underline{W}\,\overline{\xi}_\epsilon^{\frac{1}{\gamma}}\xi_T^{-\frac{1}{\gamma}}\right)_{\{Z<M(\overline{\xi}_\epsilon)\}}\right]-\epsilon\\
&= \underline{W}\xi_0 e^{-rT}\Phi\left(\frac{-\ln\frac{\overline{\xi}_\epsilon}{\xi_0}-\left(r+\frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}}+\|\kappa\|\sqrt{T}\right)\\
&- \underline{W}\,\overline{\xi}_\epsilon^{\frac{1}{\gamma}}\xi_0^{\frac{\gamma-1}{\gamma}}e^{\frac{1-\gamma}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}\|\kappa\|^2T}\\
&\quad\times\Phi\left(\frac{-\ln\frac{\overline{\xi}_\epsilon}{\xi_0}-\left(r+\frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}}+\frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T}\right)\\
&-\epsilon.
\end{aligned}$$

The first derivative of $F_\epsilon(\overline{\xi}_\epsilon)$ is

$$\begin{aligned}
\frac{dF_\epsilon(\overline{\xi}_\epsilon)}{d\overline{\xi}_\epsilon} &= -\underline{W}\,\overline{\xi}_\epsilon^{-1}\xi_0e^{-rT}\frac{1}{\|\kappa\|\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(\ln\frac{\overline{\xi}_\epsilon}{\xi_0}+rT-\frac{1}{2}\|\kappa\|^2T\right)^2}{2\|\kappa\|^2T}}\\
&\quad-\frac{1}{\gamma}\underline{W}\,\overline{\xi}_\epsilon^{\frac{1-\gamma}{\gamma}}\xi_0^{\frac{\gamma-1}{\gamma}}e^{\frac{1-\gamma}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}\|\kappa\|^2T}\\
&\quad\quad\times\Phi\left(\frac{-\ln\frac{\overline{\xi}_\epsilon}{\xi_0}-\left(r+\frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}}+\frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T}\right)\\
&\quad+\underline{W}\,\overline{\xi}_\epsilon^{\frac{1-\gamma}{\gamma}}\xi_0^{\frac{\gamma-1}{\gamma}}e^{\frac{1-\gamma}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}\|\kappa\|^2T}\\
&\quad\quad\times\frac{1}{\|\kappa\|\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(\ln\frac{\overline{\xi}_\epsilon}{\xi_0}+\left(r+\frac{1}{2}\|\kappa\|^2\right)T-\frac{\gamma-1}{\gamma}\|\kappa\|^2T\right)^2}{2\|\kappa\|^2T}}.
\end{aligned}$$

Using $\overline{\xi}_\epsilon$ instead of $\underline{\xi}$ in (6.2) we show that the sum of the first and the third addends is zero:

$$\begin{aligned}
I+III &= \underline{W}\,\overline{\xi}_\epsilon^{-1}\xi_0e^{-rT}\frac{1}{\|\kappa\|\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(\ln\frac{\overline{\xi}_\epsilon}{\xi_0}rT-\frac{1}{2}\|\kappa\|^2T\right)^2}{2\|\kappa\|^2T}}\\
&\quad\left[-1+\overline{\xi}_\epsilon^{\frac{1}{\gamma}}\xi_0^{-\frac{1}{\gamma}}e^{\frac{1}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}\|\kappa\|^2T}\,\overline{\xi}_\epsilon^{-\frac{1}{\gamma}}\xi_0^{\frac{1}{\gamma}}e^{-\frac{1}{\gamma}rT+\frac{\gamma-1}{2\gamma^2}\|\kappa\|^2T}\right]\\
&= 0.
\end{aligned}$$

Hence, the first derivative is negative

$$\begin{aligned}\frac{dF_\epsilon(\overline{\xi}_\epsilon)}{d\overline{\xi}_\epsilon} &= -\frac{1}{\gamma}\underline{W}\,\overline{\xi}_\epsilon^{\frac{1-\gamma}{\gamma}}\xi_0^{\frac{\gamma-1}{\gamma}}e^{\frac{1-\gamma}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}\|\kappa\|^2T}\\ &\qquad\times\Phi\left(\frac{-\ln\frac{\overline{\xi}_\epsilon}{\xi_0}-\left(r+\frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}}+\frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T}\right)\\ &< 0.\end{aligned}$$

The negative first derivative indicates that $F_\epsilon(\overline{\xi}_\epsilon)$ is a decreasing function of $\overline{\xi}_\epsilon$, hence there exists a unique $\overline{\xi}_\epsilon$ such that $F_\epsilon(\overline{\xi}_\epsilon) = 0$.

□

We calculate the lower critical value $\underline{\xi}_\epsilon$ from the first equation of (4.10), using the fixed $\overline{\xi}_\epsilon$ obtained from the second equation of (4.10):

$$\begin{aligned}\xi_0W_0 &= E[\xi_TW_T^{LEL}]\\ &= \frac{1}{\sqrt{2\pi}}\int_{M(\underline{\xi}_\epsilon)}^{\infty}\underline{W}\,\underline{\xi}_\epsilon^{\frac{1}{\gamma}}\xi_T^{\frac{\gamma-1}{\gamma}}e^{-\frac{z^2}{2}}dz+\frac{1}{\sqrt{2\pi}}\int_{M(\overline{\xi}_\epsilon)}^{M(\underline{\xi}_\epsilon)}\underline{W}\xi_Te^{-\frac{z^2}{2}}dz\\ &+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M(\overline{\xi}_\epsilon)}\underline{W}\,\overline{\xi}_\epsilon^{\frac{1}{\gamma}}\xi_T^{\frac{\gamma-1}{\gamma}}e^{-\frac{z^2}{2}}dz\\ &= \underline{W}\left[\underline{\xi}_\epsilon^{\frac{1}{\gamma}}J\left(M(\underline{\xi}_\epsilon),\infty,\frac{\gamma-1}{\gamma}\right)+\overline{\xi}_\epsilon^{\frac{1}{\gamma}}J\left(-\infty,M(\overline{\xi}_\epsilon),\frac{\gamma-1}{\gamma}\right)\right]\\ &+\underline{W}J\left(M(\overline{\xi}_\epsilon),M(\underline{\xi}_\epsilon),1\right)\\ &= \underline{W}\xi_0^{\frac{\gamma-1}{\gamma}}e^{\frac{1-\gamma}{\gamma}rT+\frac{1-\gamma}{2\gamma^2}\|\kappa\|^2T}\left[\underline{\xi}_\epsilon^{\frac{1}{\gamma}}\left(1-\Phi\left(M(\underline{\xi}_\epsilon)+\frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T}\right)\right)\right.\\ &\quad\left.+\overline{\xi}_\epsilon^{\frac{1}{\gamma}}\Phi\left(M(\overline{\xi}_\epsilon)+\frac{\gamma-1}{\gamma}\|\kappa\|\sqrt{T}\right)\right]\\ &+\underline{W}\xi_0e^{-rT}\left[\Phi\left(M(\underline{\xi}_\epsilon)+\|\kappa\|\sqrt{T}\right)-\Phi\left(M(\overline{\xi}_\epsilon)+\|\kappa\|\sqrt{T}\right)\right].\end{aligned}\tag{6.5}$$

We calculate $\underline{\xi}_\epsilon$ numerically from (6.5).

Proposition 9. *There exists a unique $\underline{\xi}_\epsilon$ such that*

$$E[\xi_TW_T^{LEL}]-\xi_0W_0 = 0.$$

The proof is analogous to the Proof of Proposition 7, using $M(\overline{\xi}_\epsilon)$ instead of $\Phi^{-1}(\alpha)$ and $M(\underline{\xi}_\epsilon)$ instead of M.

6.2.2 Expected terminal values

Once we determined the critical values $\underline{\xi}_\epsilon$ and $\overline{\xi}_\epsilon$, we can calculate the expected terminal values. Using $\underline{\xi}_\epsilon$ instead of $\underline{\xi}$, $M(\overline{\xi}_\epsilon)$ instead of $\Phi^{-1}(\alpha)$ and $M(\underline{\xi}_\epsilon)$ instead of M, we derive them in the analogous way as the expected terminal values for the VaR-RM model. Hence we only provide the final formulas.

Expected terminal wealth

In case the LEL-constraint (4.7) holds, the expected terminal wealth of the LEL-RM agent is

$$
\begin{aligned}
E[W_T^{LEL}] = {} & \underline{W}\xi_0^{-\frac{1}{\gamma}} e^{\frac{1}{\gamma}rT+\frac{\gamma+1}{2\gamma^2}\|\kappa\|^2 T} \\
& \times \left[\underline{\xi}_\epsilon^{\frac{1}{\gamma}} \left(1 - \Phi\left(\frac{-\ln\frac{\underline{\xi}_\epsilon}{\xi_0} - \left(r+\frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}} - \frac{1}{\gamma}\|\kappa\|\sqrt{T} \right) \right) \right. \\
& \left. + \overline{\xi}_\epsilon^{-\frac{1}{\gamma}} \Phi\left(\frac{-\ln\frac{\overline{\xi}_\epsilon}{\xi_0} - \left(r+\frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}} - \frac{1}{\gamma}\|\kappa\|\sqrt{T} \right) \right] \\
& + \underline{W}\left[\Phi\left(\frac{-\ln\frac{\underline{\xi}_\epsilon}{\xi_0} - \left(r+\frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}} \right) \right. \\
& \left. -\Phi\left(\frac{-\ln\frac{\overline{\xi}_\epsilon}{\xi_0} - \left(r+\frac{1}{2}\|\kappa\|^2\right)T}{\|\kappa\|\sqrt{T}} \right) \right].
\end{aligned}
$$

In case the LEL-constraint (4.7) does not hold, the expected terminal wealth of the LEL-RM agent is

$$
E\left[W_T^{LEL}\right] = \underline{W}\,\underline{\xi}_\epsilon^{\frac{1}{\gamma}} \xi_0^{-\frac{1}{\gamma}} e^{\frac{1}{\gamma}rT+\frac{\gamma+1}{2\gamma^2}\|\kappa\|^2 T}.
$$

Expected utility from the terminal wealth

Using the power utility function, the expected utility from the terminal wealth when (4.7) holds is

$$
\begin{aligned}
E[U(W_T^{LEL})] = & \frac{\underline{W}^{1-\gamma}}{1-\gamma} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2} \|\kappa\|^2 T} \\
& \times \left[\underline{\xi}_\epsilon^{\frac{1-\gamma}{\gamma}} \left(1 - \Phi \left(\frac{-\ln \frac{\underline{\xi}_\epsilon}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} + \frac{\gamma-1}{\gamma} \|\kappa\|\sqrt{T} \right) \right) \right. \\
& \left. + \overline{\xi}_\epsilon^{\frac{1-\gamma}{\gamma}} \Phi \left(\frac{-\ln \frac{\overline{\xi}_\epsilon}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} + \frac{\gamma-1}{\gamma} \|\kappa\|\sqrt{T} \right) \right] \\
& + \frac{\underline{W}^{1-\gamma}}{1-\gamma} \left[\Phi \left(\frac{-\ln \frac{\underline{\xi}_\epsilon}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} \right) \right. \\
& \left. - \Phi \left(\frac{-\ln \frac{\overline{\xi}_\epsilon}{\xi_0} - \left(r + \frac{1}{2}\|\kappa\|^2\right) T}{\|\kappa\|\sqrt{T}} \right) \right].
\end{aligned}
$$

The expected utility from the terminal wealth of the LEL-RM agent when (4.7) does not hold is

$$
E\left[U(W_T^{LEL})\right] = \frac{\underline{W}^{1-\gamma}}{1-\gamma} \underline{\xi}_\epsilon^{\frac{1-\gamma}{\gamma}} \xi_0^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma} rT + \frac{1-\gamma}{2\gamma^2} \|\kappa\|^2 T}.
$$

6.2.3 LEL-RM in the constrained model

Basak and Shapiro [2] derived the LEL-RM model when there are no constraints imposed on the portfolio strategy. Let us now examine the case when constraints are required.

Let $\mathcal{C}$ represent the set of admissible strategies of the LEL-RM agent such that the short-selling of both risky and risk-free assets is prohibited

$$
\mathcal{C} = \{\theta^i \geq 0, i = 1, 2, ...d; \sum_1^d \theta^i \leq 1\}.
$$

Using the power utility function, let us define the LEL-RM problem in the constrained model as

$$\max_{\theta^{LEL}} E\left[\frac{(W_T^{LEL})^{1-\gamma}}{1-\gamma}\right] \tag{6.6}$$
$$s.t. \quad P(W_T^{LEL} \geq \underline{W}) \geq 1-\alpha,$$
$$\mathcal{C} = \{(\theta_t^{LEL})^i \geq 0, i = 1, 2, ...d; \sum_1^d (\theta_t^{LEL})^i \leq 1, \forall t \in (0,T)\},$$

with a given initial W_0. The part *ii*) of Proposition 4 defines the optimal portfolio strategy of the LEL-RM agent as

$$\theta_t^{LEL} = q_t^{LEL}\hat{\beta}$$

and the part *iii*) of Proposition 4 states that $0 \leq q_t^{LEL} \leq 1$.

Corollary 2. *Let the optimal portfolio strategy $\hat{\beta}$ of the benchmark agent be calculated by (2.9). In case that $\hat{\beta}^i \geq 0$ for $i = 1, ..., d$ and $\sum_{i=1}^d \hat{\beta}^i \leq 1$, the portfolio strategy of the LEL-RM agent $(\theta_t^{LEL})^i$ is optimal for the problem (6.6).*

6.2.4 Example - Expected terminal values

Let us now calculate the expected terminal values for different levels of the floor. We use the same settings as in the section 5.1.5, additionally we set the initial state price density $\xi_0 = 1$ and $\epsilon = 0.05$.

Using the function 'fsolve' in Matlab, we calculate the upper critical value $\overline{\xi}_\epsilon$. Then, using $\overline{\xi}_\epsilon$, we calculate the lower critical value $\underline{\xi}_\epsilon$. Table 6.3 summarizes the critical values for different levels of the floor.

$\underline{W}$	$\overline{\xi}$	$\underline{\xi}$
0.980	1.496211	1.496211
0.990	1.505954	1.482543
1	1.515625	1.352968
1.010	1.525227	1.225192
1.015	1.530001	1.161579

Table 6.3: Critical values of the state price density for different levels of the floor.

We see that for the floor $\underline{W} = 0.98$, the LEL-constraint (4.7) is not binding. Also, the upper critical value $\overline{\xi}_\epsilon$ is increasing with $\underline{W}$, while the lower critical value $\underline{\xi}_\epsilon$ is decreasing with $\underline{W}$. Therefore the middle interval increases with the floor.

Let us now take a look at the expected terminal wealth and the certainty equivalent for different levels of the floor. Table 6.4 summarizes the results.

$\underline{W}$	$E[W_T^{LEL}]$	C^{LEL}
0.980	1.174179	1.097621
0.990	1.166543	1.091353
1	1.160454	1.091032
1.010	1.153498	1.090171
1.015	1.149639	1.089507

Table 6.4: Expected terminal values and certainty equivalents of the LEL-RM agent for different levels of the floor.

As we can see, both the expected terminal value and certainty equivalent are decreasing with the floor $\underline{W}$.

Chapter 7

Portfolio insurance with spreads

The Value-at-Risk based risk management was developed for portfolios with no constraints on the portfolio strategy. In the previous chapter, we showed that such a strategy is useless when constraints, such as restricting the short selling of all risky or risk-free assets, are required. Insuring the portfolio with a put spread can eliminate this problem.

According to the VaR-constraint

$$P(W_T \geq \underline{W}) \geq 1 - \alpha,$$

we adjust our strategy in a following way:

- in case the risky asset X_T satisfies the condition, we do not insure the portfolio at all,
- in case the risky asset X_T does not satisfy the condition, we modify the portfolio by buying a put option with the strike price $\underline{W}$ and selling a put option with strike price $\underline{\underline{W}}$ such that $P(X_T \geq \underline{\underline{W}}) = 1 - \alpha$.

Formally, we can express the above strategy as

$$W = \begin{cases} X + Put(X_T \geq \underline{W}) - Put(X_T \geq \underline{\underline{W}}) & \text{if } P(X_T \geq \underline{W}) < 1 - \alpha, \\ X & \text{if } P(X_T \geq \underline{W}) \geq 1 - \alpha. \end{cases} \quad (7.1)$$

According to this strategy, we leave the worst $\alpha\%$ cases uninsured. Figure 7.1 depicts the payoff diagram of (7.1).

Let us now take a closer look at the condition

$$P(X_T \geq \underline{W}) \geq 1 - \alpha. \quad (7.2)$$

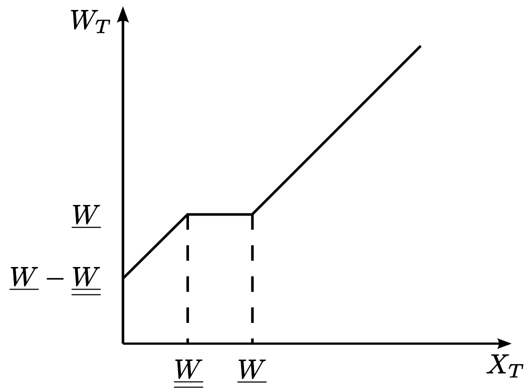

Figure 7.1: Payoff diagram of the strategy (7.1).

Using the fact that X_T is given by (5.10), we can express (7.2) as

$$P\left(X_0 e^{\left(r+\beta^\top(\mu-r\mathbf{1})-\frac{1}{2}\sigma_X^2\right)T+\sigma_X\sqrt{T}Z} \geq \underline{W}\right) \geq 1-\alpha,$$

from which we obtain

$$X_0 e^{\left(r+\beta^\top(\mu-r\mathbf{1})-\frac{1}{2}\sigma_X^2\right)T-\sigma_X\sqrt{T}\Phi^{-1}(1-\alpha)} \geq \underline{W}.$$

For simplicity, we use the following notation

$$\Gamma = \left(r+\beta^\top(\mu-r\mathbf{1})-\frac{1}{2}\sigma_X^2\right)T-\,\sigma_X\sqrt{T}\Phi^{-1}(1-\alpha), \tag{7.3}$$

hence the above condition can be written as

$$\underline{W} \leq X_0 e^{\Gamma}.$$

Since $\underline{\underline{W}}$ is defined by the condition $P(X_T \geq \underline{\underline{W}}) = 1-\alpha$, by analogy we obtain that $\underline{\underline{W}} = X_0 e^{\Gamma}$. Therefore, the condition (7.2) can be expressed as

$$\underline{W} \leq \underline{\underline{W}}.$$

Now, we can reformulate the strategy (7.1) to be $W_t = g_t(X_t)$, where

$$g_t(X) = \begin{cases} X + Put_t(X_T \geq \underline{W}) - Put_t(X_T \geq \underline{\underline{W}}) & \text{if} \quad \underline{\underline{W}} < \underline{W}, \\ X & \text{if} \quad \underline{\underline{W}} \geq \underline{W} \end{cases} \tag{7.4}$$

and the values of the put options are calculated according to $X = X_t$.

In case that $\underline{\underline{W}} < \underline{W}$, the put options can be synthesized as

$$
\begin{aligned}
W_t &= X_t + Put_t(X_T \geq \underline{W}) - Put_t(X_T \geq \underline{\underline{W}}) \\
&= X_t + \varphi_t(\underline{W})X_t + \psi_t(\underline{W})B_t - \varphi_t(\underline{\underline{W}})X_t - \psi_t(\underline{\underline{W}})B_t \\
&= \left[1 + \varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})\right] X_t + [\psi_t(\underline{W}) - \psi_t(\underline{\underline{W}})]B_t,
\end{aligned}
$$

where $\varphi_t(\underline{W}) = \Phi(d_1(\underline{W})) - 1$ is the delta of the option with strike $\underline{W}$ and $\varphi_t(\underline{\underline{W}}) = \Phi(d_1(\underline{\underline{W}})) - 1$ is the delta of the option with strike $\underline{\underline{W}}$. The difference $\varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})$ is called the hedging ratio.

Note that is case when $\underline{\underline{W}} \geq \underline{W}$, it holds that $W_t = X_t$. Let us now define the new portfolio strategy as

$$
\theta_t^i = \begin{cases} \frac{\left[1+\varphi_t(\underline{W})-\varphi_t(\underline{\underline{W}})\right]\beta^i X_t}{W_t} & \text{if} \quad \underline{\underline{W}} < \underline{W}, \\ \beta^i & \text{if} \quad \underline{\underline{W}} \geq \underline{W}. \end{cases} \tag{7.5}
$$

Then the portfolio process follows

$$
dW_t = W_t[r + \theta_t^\top(\mu - r\mathbf{1})]dt + W_t\theta_t^\top \sigma dw_t.
$$

7.1 Portfolio insurance with spreads in the unconstrained models

We investigate the problem

$$
\begin{aligned}
& \max_\theta E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \qquad (7.6) \\
s.t. \quad & P(W_T \geq \underline{W}) \geq 1 - \alpha.
\end{aligned}
$$

In this case, there are no constraints required on the portfolio strategy.

Theorem 7. *Let $\hat{\beta}$ be the optimal portfolio strategy, computed by (2.9). Then the portfolio strategy defined by*

$$
\theta_t = \begin{cases} \frac{\left[1+\varphi_t(\underline{W})-\varphi_t(\underline{\underline{W}})\right]\hat{\beta} X_t}{W_t} & \textit{if} \quad \underline{\underline{W}} < \underline{W}, \\ \hat{\beta} & \textit{if} \quad \underline{\underline{W}} \geq \underline{W} \end{cases} \tag{7.7}
$$

guarantees that $P(W_T \geq \underline{W}) \geq 1 - \alpha$.

The proof is clear from the derivation of the strategy.

7.2 Portfolio insurance with spreads in the constrained models

We investigate the problem

$$
\begin{aligned}
& \max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \qquad\qquad (7.8)\\
s.t. \quad & P(W_T \geq \underline{W}) \geq 1-\alpha,\\
& \mathcal{C} = \{\theta^i \geq 0,\ i = 1, 2, ..., d; \sum \theta^i \leq 1\}.
\end{aligned}
$$

We provide an admissible solution for the problem (7.8) in the following theorem.

Theorem 8. *Let $\hat{\beta}$ be the optimal portfolio strategy with convex constraints, computed by*

$$\hat{\beta} = \arg\max_{\beta \in \mathcal{C}} \ \beta^\top(\mu - r\boldsymbol{1}) - \frac{1}{2}\gamma\beta^\top c^R \beta, \qquad (7.9)$$

where $\mathcal{C}$ is defined as in problem (7.8). Then the portfolio strategy

$$\theta_t = \begin{cases} \frac{\left[1+\varphi_t(\underline{W})-\varphi_t(\underline{\underline{W}})\right]\hat{\beta}X_t}{W_t} & \text{if} \quad \underline{\underline{W}} < \underline{W}, \\ \hat{\beta} & \text{if} \quad \underline{\underline{W}} \geq \underline{W} \end{cases}$$

is admissible for the problem (7.8).

Proof. Since $\hat{\beta}$ is calculated by (7.9), it holds that $\hat{\beta}^i \geq 0$ for $i = 1, ..., d$ and $\sum_{i=1}^d \hat{\beta}^i \leq 1$. In case when $\underline{\underline{W}} < \underline{W}$, the hedging ratio is

$$\varphi(\underline{W}) - \varphi(\underline{\underline{W}}) = \Phi(d_1(\underline{W})) - 1 - \Phi(d_1(\underline{\underline{W}})) + 1 = \Phi(d_1(\underline{W})) - \Phi(d_1(\underline{\underline{W}})).$$

It can easily be shown that

$$-1 < \Phi(d_1(\underline{W})) - \Phi(d_1(\underline{\underline{W}})) < 0 \quad \text{and} \quad 0 < \frac{X_t}{W_t} < 1.$$

Therefore the vector θ_t satisfies

$$\theta_t = \frac{\left[1 + \varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})\right]\hat{\beta}X_t}{W_t} \geq 0.$$

Now, we show that $\sum_{i=1}^{d} \theta_t^i \leq 1$:

$$
\begin{aligned}
\sum_{i=1}^{d} \theta_t^i &= \sum_{i=1}^{d} \left[1 + \varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})\right] \hat{\beta}^i \frac{X_t}{W_t} \\
&\leq \sum_{i=1}^{d} \left[1 + \varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})\right] \hat{\beta}^i \\
&\leq \sum_{i=1}^{d} \hat{\beta}^i \\
&\leq 1.
\end{aligned}
$$

In case when $\underline{\underline{W}} \geq \underline{W}$, $\theta_t = \hat{\beta}$ is admissible for the problem (7.8).

$\square$

7.3 Optimal distribution of the initial wealth W_0

Since we are interested in the behavior at the initial $t = 0$, we use the notation $X = X_0$ in (7.4) and take a closer look at $g_0(X_0)$ when $\underline{\underline{W}} < \underline{W}$:

$$
\begin{aligned}
g_0(X_0) &= X_0 + \underline{W} e^{-rT} \Phi(-d_2(\underline{W})) - X_0 \Phi(-d_1(\underline{W})) \\
&- \underline{\underline{W}} e^{-rT} \Phi(-d_2(\underline{\underline{W}})) + X_0 \Phi(-d_1(\underline{\underline{W}})).
\end{aligned}
$$

We show that both $d_1(\underline{\underline{W}})$ and $d_2(\underline{\underline{W}})$ are independent of X_0. Using (7.3), we obtain

$$
\ln \frac{X_0}{\underline{\underline{W}}} = \ln \frac{X_0}{X_0 e^{\Gamma}} = -\Gamma.
$$

Therefore

$$
d_1(\underline{\underline{W}}) = \frac{\ln \frac{X_0}{\underline{\underline{W}}} + \left(r + \frac{\sigma_X^2}{2}\right) T}{\sigma_X \sqrt{T}} = \frac{-\Gamma + \left(r + \frac{\sigma_X^2}{2}\right) T}{\sigma_X \sqrt{T}}
$$

and

$$
d_2(\underline{\underline{W}}) = \frac{\ln \frac{X_0}{\underline{\underline{W}}} + \left(r - \frac{\sigma_X^2}{2}\right) T}{\sigma_X \sqrt{T}} = \frac{-\Gamma + \left(r - \frac{\sigma_X^2}{2}\right) T}{\sigma_X \sqrt{T}}.
$$

The values $d_1(\underline{W})$ and $d_2(\underline{W})$ both depend on X_0, therefore we express them in full form to explicitly show the dependency of $g_0(X_0)$ on X_0. Then, for

$\underline{\underline{W}} < \underline{W}$ our function takes the following form

$$
\begin{aligned}
g_0(X_0) = {} & X_0 + \underline{W} e^{-rT} \Phi \left(- \frac{\ln \frac{X_0}{\underline{W}} + \left(r - \frac{\sigma_X^2}{2} \right) T}{\sigma_X \sqrt{T}} \right) \\
& - X_0 \Phi \left(- \frac{\ln \frac{X_0}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2} \right) T}{\sigma_X \sqrt{T}} \right) - X_0 e^{\Gamma} e^{-rT} \Phi \left(-d_2(\underline{\underline{W}}) \right) \\
& + X_0 \Phi \left(-d_1(\underline{\underline{W}}) \right) \\
= {} & X_0 \left[\Phi \left(\frac{\ln \frac{X_0}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2} \right) T}{\sigma_X \sqrt{T}} \right) - e^{\Gamma - rT} \Phi(-d_2(\underline{\underline{W}})) + \Phi(-d_1(\underline{\underline{W}})) \right] \\
& + \underline{W} e^{-rT} \Phi \left(- \frac{\ln \frac{X_0}{\underline{W}} + \left(r - \frac{\sigma_X^2}{2} \right) T}{\sigma_X \sqrt{T}} \right). \qquad (7.10)
\end{aligned}
$$

Proposition 10. *The function $g_0(X_0)$ in (7.4) is a continuous function of X_0 at time $t = 0$. Moreover, the following limit holds*

$$\lim_{X_0 \to 0} g_0(X_0) = \underline{W} e^{-rT}.$$

Proof. In case $\underline{\underline{W}} \geq \underline{W}$, the function $g_0(X)$ is linear and hence continuous. Otherwise $g_0(X)$ is constructed from elementary functions, therefore is continuous. We show the continuity in case when $\underline{W} = \underline{\underline{W}}$: the put options have equal values, i.e. $Put_0(X_T \geq \underline{W}) = Put_0(X_T \geq \underline{\underline{W}})$ and therefore $g_0(X) = X$.

In case $X_0 \to 0$, we have $\underline{\underline{W}} = X_0 e^{\Gamma} \to 0$, which implies $\underline{\underline{W}} < \underline{W}$, so we can determine the limit from (7.10) as

$$\lim_{X_0 \to 0} g_0(X_0) = \underline{W} e^{-rT}.$$

□

Note that $\underline{W} e^{-rT} \leq W_0$, otherwise the portfolio W could not be insured with a put option with strike $\underline{W}$.

Proposition 11. *For every given initial wealth $W_0 \geq \underline{W} e^{-rT}$, there exists a unique initial amount X_0 invested in the risky asset such that $W_0 = g_0(X_0)$. Namely,*

$$
X_0 = \begin{cases} W_0 & \text{if} \quad W_0 > \underline{W} e^{-\Gamma}, \\ \text{determined from (7.10)} & \text{if} \quad W_0 \leq \underline{W} e^{-\Gamma}. \end{cases} \qquad (7.11)
$$

Proof. To investigate the behavior of $g_0(.)$, we calculate its derivative in case when $\underline{\underline{W}} < \underline{W}$:

$$\begin{aligned}
\frac{dg_0(X_0)}{dX_0} &= 1 + \Phi(d_1(\underline{W})) - 1 + \Phi(-d_1(\underline{\underline{W}})) - e^{\Gamma - rT}\Phi(-d_2(\underline{\underline{W}})) \\
&= \Phi(d_1(\underline{W})) + \Phi(-d_1(\underline{\underline{W}})) - e^{\Gamma - rT}\Phi(-d_2(\underline{\underline{W}})).
\end{aligned}$$

The function $g_0(X_0)$ has at most one local extreme on the interval $\left(0,\, \underline{W}e^{-\Gamma}\right)$, precisely when the first derivative equals zero:

$$\Phi(d_1(\underline{W})) + \Phi(-d_1(\underline{\underline{W}})) - e^{\Gamma - rT}\Phi(-d_2(\underline{\underline{W}})) = 0.$$

Let us denote

$$k = e^{\Gamma - rT}\Phi(-d_2(\underline{\underline{W}})) - \Phi(-d_1(\underline{\underline{W}})).$$

If $k \notin (0,1)$, the function $g_0(X_0)$ does not attain its extreme on $\left(0,\, \underline{W}e^{-\Gamma}\right)$, otherwise the extreme can be expressed as

$$X_0 = \underline{W}e^{\sigma_X\sqrt{T}\Phi^{-1}(k) - \left(r + \frac{\sigma_X^2}{2}\right)T}.$$

Let us now take a look at the derivatives at $X_0 \to 0$ and $X_0 = \underline{W}e^{-\Gamma}$. The derivative at $X_0 \to 0$ is

$$\begin{aligned}
\left.\frac{dg_0(X_0)}{dX_0}\right|_{X_0 \to 0} &= \Phi(-d_1(\underline{\underline{W}})) - e^{\Gamma - rT}\Phi(-d_2(\underline{\underline{W}})) \\
&= \Phi\left(\frac{\Gamma - rT - \frac{\sigma_X^2}{2}T}{\sigma_X\sqrt{T}}\right) - e^{\Gamma - rT}\Phi\left(\frac{\Gamma - rT + \frac{\sigma_X^2}{2}T}{\sigma_X\sqrt{T}}\right).
\end{aligned}$$

The derivative at $X_0 = \underline{W}e^{-\Gamma}$ is

$$\begin{aligned}
\left.\frac{dg_0(X_0)}{dX_0}\right|_{X_0 = \underline{W}e^{-\Gamma}} &= \Phi\left(\frac{\ln\frac{\underline{W}e^{-\Gamma}}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2}\right)T}{\sigma_X\sqrt{T}}\right) \\
&+ \Phi\left(\frac{\Gamma - rT - \frac{\sigma_X^2}{2}T}{\sigma_X\sqrt{T}}\right) - e^{\Gamma - rT}\Phi\left(\frac{\Gamma - rT + \frac{\sigma_X^2}{2}T}{\sigma_X\sqrt{T}}\right) \\
&= 1 - \Phi\left(-\frac{-\Gamma + \left(r + \frac{\sigma_X^2}{2}\right)T}{\sigma_X\sqrt{T}}\right) \\
&+ \Phi\left(\frac{\Gamma - rT - \frac{\sigma_X^2}{2}T}{\sigma_X\sqrt{T}}\right) - e^{\Gamma - rT}\Phi\left(\frac{\Gamma - rT + \frac{\sigma_X^2}{2}T}{\sigma_X\sqrt{T}}\right) \\
&= 1 - e^{\Gamma - rT}\Phi\left(\frac{\Gamma - rT + \frac{\sigma_X^2}{2}T}{\sigma_X\sqrt{T}}\right).
\end{aligned}$$

We investigate 3 possible cases:

1. $W_0 \geq \underline{W}e^{-rT} \geq \underline{W}e^{-\Gamma}$:
   In this case $\Gamma - rT \geq 0$. Near $X_0 = 0$, the derivative of $g_0(.)$ is negative, that means $g_0(.)$ is decreasing near zero. Approaching $\underline{W}e^{-\Gamma}$, we are not able to tell whether $g_0(.)$ is increasing or decreasing.

   In case when (7.10) attains the extreme value (the local minimum) on $\left(0, \underline{W}e^{-\Gamma}\right)$, $g_0(.)$ is increasing around $\underline{W}e^{-\Gamma}$. If (7.10) does not attain its extreme on $\left(0, \underline{W}e^{-\Gamma}\right)$, $g_0(.)$ is decreasing around $\underline{W}e^{-\Gamma}$. Figure 7.2 plots the possible cases.

   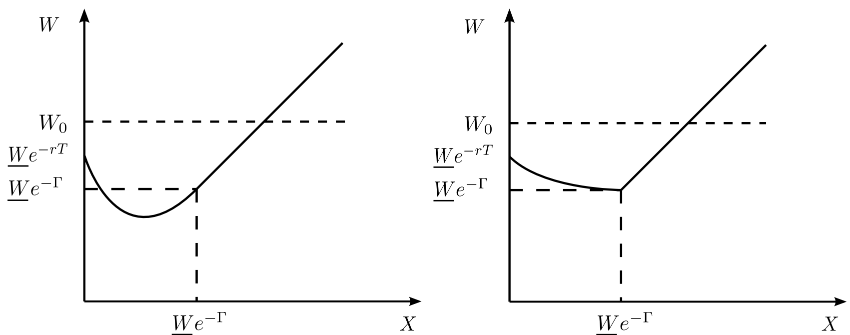

   Figure 7.2: The case of $W_0 \geq \underline{W}e^{-rT} \geq \underline{W}e^{-\Gamma}$.

   In both cases, (7.10) remains lower than W_0 on the entire interval of $\left(0, \underline{W}e^{-\Gamma}\right)$. Therefore, the equality $W_0 = g_0(X_0)$ holds only if $X_0 = W_0$.

2. $W_0 \geq \underline{W}e^{-\Gamma} \geq \underline{W}e^{-rT}$:
   In this case $\Gamma - rT \leq 0$. We are not able to tell whether $g_0(.)$ is increasing or decreasing near $X_0 = 0$. The derivative at $\underline{W}e^{-\Gamma}$ is positive, therefore $g_0(.)$ increases there. Figure 7.3 shows the possible behaviors of the strategy (7.4).

   In case when $g_0(.)$ attains the extreme value (the local minimum), it descends around 0, otherwise increases around 0. In both cases, (7.10) remains under W_0 on $\left(0, \underline{W}e^{-\Gamma}\right)$ and therefore the equality $W_0 = g_0(X_0)$ can hold only if $X_0 = W_0$.

3. $\underline{W}e^{-\Gamma} \geq W_0 \geq \underline{W}e^{-rT}$:
   Similar to the previous case, we have $\Gamma - rT \leq 0$. Therefore, the behavior of (7.10) is the same as in the previous case. Figure 7.4 shows the possible behaviors of the strategy (7.4).

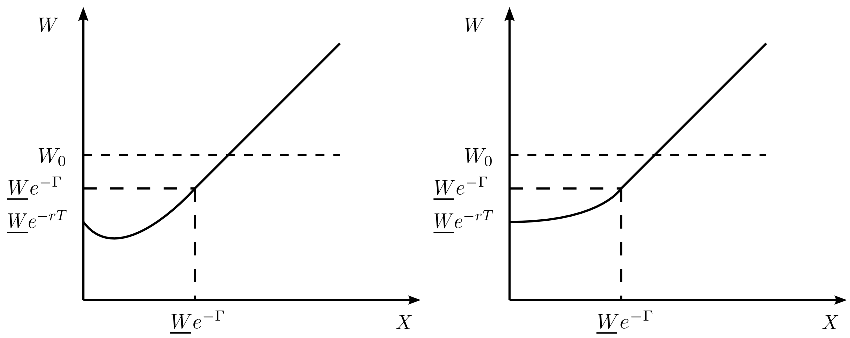

Figure 7.3: The case of $W_0 \geq \underline{W}e^{-\Gamma} \geq \underline{W}e^{-rT}$.

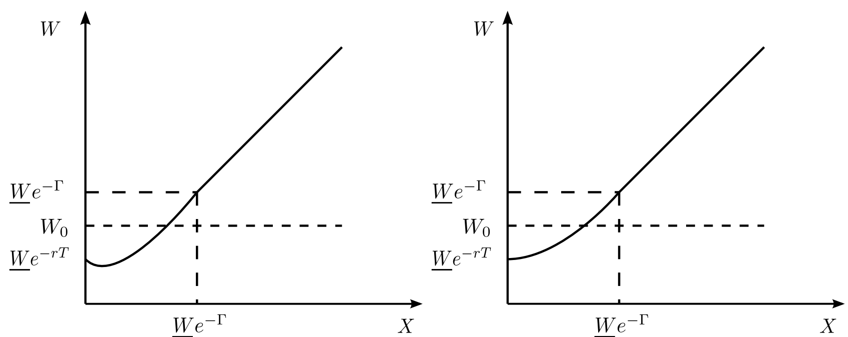

Figure 7.4: The case of $\underline{W}e^{-\Gamma} \geq W_0 \geq \underline{W}e^{-rT}$.

Since $W_0 \leq \underline{W}e^{-\Gamma}$, we have that X_0 must be the solution of $W_0 = g_0(X_0)$.

$\square$

As we mentioned before, the initial wealth must be greater than the discounted value of the floor, i.e. $\underline{W}e^{-rT} \leq W_0$. In order to apply the insurance with the spreads, the initial wealth is bounded above, i.e. $W_0 \leq \underline{W}e^{-\Gamma}$. Combining the two conditions,

$$\begin{aligned}
\underline{W}e^{-rT} &\leq \underline{W}e^{-\Gamma}, \\
-rT &\leq -rT - \left(\beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\,\sigma_X^2\right)T + \sigma_X\sqrt{T}\Phi^{-1}(1-\alpha), \\
T &\leq \left(\frac{\sigma_X\Phi^{-1}(1-\alpha)}{\beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\,\sigma_X^2}\right)^2.
\end{aligned}$$

Remark 5. *The insurance with spreads is not applicable in case when the* *maturity is*

$$T > \left(\frac{\sigma_X \Phi^{-1}(1-\alpha)}{\beta^\top(\mu - r\boldsymbol{1}) - \frac{1}{2}\sigma_X^2} \right)^2 . \tag{7.12}$$

7.4 Optimal distribution of the wealth W_t at time t

To take a look at the optimal distribution of the wealth W_t at time t, we consider the case when $\underline{\underline{W}} < \underline{W}$.

Taking in account the fact that the auxiliary floor $\underline{\underline{W}}$ depends on X_0, we determine the initial amount invested in the risky asset X_0 from (7.10). At time t, the value of $\underline{\underline{W}}$ is already given, hence we rewrite (7.10) as

$$\begin{aligned} g_t(X_t) &= X_t + \underline{W} e^{-r(T-t)} \Phi \left(- \frac{\ln \frac{X_t}{\underline{W}} + \left(r - \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X \sqrt{T-t}} \right) \\ &- X_t \Phi \left(- \frac{\ln \frac{X_t}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X \sqrt{T-t}} \right) \\ &- \underline{\underline{W}} e^{-r(T-t)} \Phi \left(- \frac{\ln \frac{X_t}{\underline{\underline{W}}} + \left(r - \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X \sqrt{T-t}} \right) \\ &+ X_t \Phi \left(- \frac{\ln \frac{X_t}{\underline{\underline{W}}} + \left(r + \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X \sqrt{T-t}} \right) . \end{aligned} \tag{7.13}$$

Proposition 12. *At every time $t \in \langle 0, T \rangle$, the function $g_t(X_t)$ given in (7.13) is increasing and hence there exists a unique X_t for every given W_t such that $W_t = g_t(X_t)$. Moreover, $W_t \geq (\underline{W} - \underline{\underline{W}}) e^{-r(T-t)}$ for all $t \in \langle 0, T \rangle$.*

Proof. Let us first take a look at $g_t(X_t)$ in cases when X_t approaches 0 and ∞:

$$\begin{aligned} \lim_{X_t \to 0} g_t(X_t) &= (\underline{W} - \underline{\underline{W}}) e^{-r(T-t)}, \\ \lim_{X_t \to \infty} g_t(X_t) &= \infty. \end{aligned}$$

The first derivative of $g_t(X_t)$ is positive:

$$\begin{aligned} \frac{dg_t(X_t)}{dX_t} &= 1 + \Phi(d_1(\underline{W})) - 1 - \Phi(d_1(\underline{\underline{W}})) + 1 \\ &= 1 + \Phi(d_1(\underline{W})) - \Phi(d_1(\underline{\underline{W}})) > 0, \end{aligned}$$

based on the fact that $1-\Phi(d_1(\underline{\underline{W}}))\geq 0$.

The function $g_t(X_t)\,:\,\langle 0,\infty)\to\langle(\underline{W}-\underline{\underline{W}})e^{-r(T-t)},\infty)$ is increasing, hence there exists a unique X_t such that $W_t=g_t(X_t)$. $\square$

7.5 Expected terminal values

As in the case of the option based portfolio insurance, we evaluate the expected terminal value of the insured portfolio and the expected utility from the terminal wealth.

The expected terminal value of the insured portfolio

Let us now derive the expected terminal value of the insured portfolio $E[W_T]$ when $\underline{\underline{W}}<\underline{W}$. The terminal value is given as

$$W_T=X_T+\max(\underline{W}-X_T,0)-\max(\underline{\underline{W}}-X_T,0).$$

One can express W_T as

$$W_T=\begin{cases} X_T+\underline{W}-\underline{\underline{W}} & \text{if} \quad X_T<\underline{\underline{W}}<\underline{W},\\ \underline{W} & \text{if} \quad \underline{\underline{W}}\leq X_T<\underline{W},\\ X_T & \text{if} \quad \underline{\underline{W}}<\underline{W}\leq X_T.\end{cases}$$

Let us take a look at the conditions $X_T\geq\underline{W}$ and $X_T\geq\underline{\underline{W}}$. The former is already given in (5.13), i.e. it is equivalent to

$$Z\geq\frac{\ln\frac{\underline{W}}{X_0}-\left(r+\beta^\top(\mu-r\mathbf{1})-\frac{1}{2}\sigma_X^2\right)T}{\sigma_X\sqrt{T}}=M(\underline{W}).$$

The latter condition can be expressed as

$$\begin{aligned} X_0e^{\left(r+\beta^\top(\mu-r\mathbf{1})-\frac{1}{2}\sigma_X^2\right)T+\sigma_X\sqrt{T}Z} &\geq X_0e^{\left(r+\beta^\top(\mu-r\mathbf{1})-\frac{1}{2}\sigma_X^2\right)T-\sigma_X\sqrt{T}\Phi^{-1}(1-\alpha)},\\ Z &\geq -\Phi^{-1}(1-\alpha)=M(\underline{\underline{W}}).\end{aligned}$$

Then the expected terminal value of the insured portfolio is

$$\begin{aligned} E[W_T] &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{M(\underline{\underline{W}})}\left(X_T+\underline{W}-\underline{\underline{W}}\right)e^{-\frac{z^2}{2}}dz\right.\\ &\qquad\left.+\int_{M(\underline{\underline{W}})}^{M(\underline{W})}\underline{W}e^{-\frac{z^2}{2}}dz+\int_{M(\underline{W})}^{\infty}X_Te^{-\frac{z^2}{2}}dz\right]\\ &= \underline{W}\Phi\left(M(\underline{W})\right)-\underline{\underline{W}}\Phi\left(M(\underline{\underline{W}})\right)+X_0e^{\left(r+\beta^\top(\mu-r\mathbf{1})\right)T}\\ &\quad\times\left[\Phi\left(M(\underline{\underline{W}})-\sigma_X\sqrt{T}\right)+1-\Phi\left(M(\underline{W})-\sigma_X\sqrt{T}\right)\right].\end{aligned}$$

Note that in case $W_0 = X_0$, the portfolio is not insured and the expected terminal wealth of the portfolio is calculated by (5.11).

Expected utility of the terminal wealth

The expected utility from the terminal wealth can be calculated as

$$
\begin{aligned}
E[U(W_T)] \quad = \quad & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{M(\underline{\underline{W}})} \frac{(X_T + \underline{W} - \underline{\underline{W}})^{1-\gamma}}{1-\gamma} e^{-\frac{z^2}{2}} dz \\
& + \frac{1}{\sqrt{2\pi}} \int_{M(\underline{\underline{W}})}^{M(\underline{W})} \frac{\underline{W}^{1-\gamma}}{1-\gamma} e^{-\frac{z^2}{2}} dz + \frac{1}{\sqrt{2\pi}} \int_{M(\underline{W})}^{\infty} \frac{X_T^{1-\gamma}}{1-\gamma} e^{-\frac{z^2}{2}} dz.
\end{aligned}
$$

The first integral can not be expressed analytically, hence we provide it in the integral form:

$$
\begin{aligned}
E[U(W_T)] = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{M(\underline{\underline{W}})} \frac{(X_T + \underline{W} - \underline{\underline{W}})^{1-\gamma}}{1-\gamma} e^{-\frac{z^2}{2}} dz \\
& + \frac{\underline{W}^{1-\gamma}}{1-\gamma} \left[\Phi(M(\underline{W})) - \Phi(M(\underline{\underline{W}})) \right] \\
& + \frac{X_0^{1-\gamma} e^{\left(r + \beta^\top (\mu - r\mathbf{1})\right)(1-\gamma)T + \frac{1}{2}\gamma(\gamma-1)\,\sigma_X^2 T}}{1-\gamma} \\
& \quad \times \Phi\left(-M(\underline{W}) + \sigma_X \sqrt{T}(1-\gamma) \right).
\end{aligned}
\tag{7.14}
$$

Note that in case $W_0 = X_0$, the portfolio is not insured and the expected utility from the terminal wealth is calculated by (5.12).

7.6 Examples

Example - Expected terminal values

Let us now examine the expected terminal value $E[W_T]$ and the expected utility from the terminal value $E[U(W_T)]$ of the portfolio insured with spreads, for both unconstrained and constrained models.

Let $\hat{\beta}u$ represent the portfolio strategy of the unconstrained problem, calculated by (2.9) and let $\hat{\beta}$ represent the portfolio strategy of the constrained problem, calculated by (2.8). Table 5.1 summarizes the optimal strategies for both models.

We calculate the volatilities as $\sigma_{Xu} = \sqrt{\hat{\beta u}^{\top} c^R \hat{\beta u}}$ for the unconstrained model and $\sigma_X = \sqrt{\hat{\beta}^{\top} c^R \hat{\beta}}$ for the constrained model. Then

$$\begin{aligned}
\Gamma_{Xu} &= \left(r + \hat{\beta u}^{\top} (\mu - r\mathbf{1}) - \frac{1}{2}\sigma_{Xu}^2 \right) T - \sigma_{Xu}\sqrt{T}\Phi^{-1}(1-\alpha), \\
\Gamma_X &= \left(r + \hat{\beta}^{\top} (\mu - r\mathbf{1}) - \frac{1}{2}\sigma_X^2 \right) T - \sigma_X\sqrt{T}\Phi^{-1}(1-\alpha).
\end{aligned}$$

The initial amounts invested in the risky asset are calculated by (7.11), using Γ_{Xu} for the unconstrained model and Γ_X for the constrained model.

In this example we use the same settings as in the Example 5.1.5.

First, we take a look at the condition (7.12) for the maturity T for different levels of α. Table 7.1 summarizes the lower bounds of T, over which the insurance with spreads is not applicable.

α	T_{Xu}	T_X
0.01	9.909419	9.580635
0.02	7.723134	7.466889
0.03	6.477108	6.262205
0.04	5.611971	5.425772
0.05	4.953970	4.789603
0.07	3.987940	3.855624
0.10	3.007260	2.907483
0.15	1.966897	1.901638

Table 7.1: The boundaries of the maturity T.

We can see that the higher the level of the probability α is, the lower the boundaries of the maturity T are. In other words, we can use the insurance with spreads for a longer maturity if we set the probability lower.

Let us choose the probability level $\alpha = 0.05$. Table 7.2 summarizes the floors of the put options, $\underline{W}$, $\underline{\underline{W}}_{Xu}$ and $\underline{\underline{W}}_X$; the initial amounts invested in the risky assets, Xu_0 and X_0; and the expected terminal wealths of the insured portfolios, $E[Wu_T]$ and $E[W_T]$, for both unconstrained and constrained models.

For $\alpha = 0.05$, the insurance with spreads is applicable for the maturity levels $T = 1$ and $T = 3$. For the other levels of the maturity, $T = 5$ and $T = 10$, we provide only the expected terminal wealths, $E[Wu_T]$ and $E[W_T]$.

For $T = 1$ and $T = 3$, the auxiliary floors $\underline{\underline{W}}_{Xu}$ and $\underline{\underline{W}}_X$, the initial amounts Xu_0 and X_0 invested in the risky assets and the expected terminal wealths $E[Wu_T]$ and $E[W_T]$ are decreasing with $\underline{W}$.

$\underline{W}$	$\underline{\underline{W}}_{Xu}$	Xu_0	$E[Wu_T]$	$\underline{\underline{W}}_X$	X_0	$E[W_T]$
T=1						
0.980	0.828653	0.942526	1.121798	0.872670	0.962487	1.107088
0.990	0.816809	0.929054	1.112772	0.862571	0.951349	1.099555
1	0.801595	0.911750	1.102001	0.849259	0.936667	1.090355
1.010	0.780779	0.888073	1.088703	0.830476	0.915950	1.078709
1.015	0.766738	0.872102	1.080643	0.817420	0.901550	1.071492
T=3						
0.980	0.942197	0.984363	1.568590	n/a	1	1.468023
0.990	0.934573	0.976397	1.557156	0.976400	0.994603	1.460824
1	0.926161	0.967609	1.544645	0.969320	0.987391	1.451280
1.010	0.916830	0.957860	1.530893	0.961353	0.979276	1.440646
1.015	0.911772	0.952576	1.523496	0.956986	0.974827	1.434863
T=5						
	n/a	1	2.168948	n/a	1	1.896217
T=10						
	n/a	1	4.704338	n/a	1	3.595639

Table 7.2: Portfolio development for different levels of the floor $\underline{W}$.

Moreover, the auxiliary floors are higher for the constrained model, i.e. $\underline{\underline{W}}_X > \underline{\underline{W}}_{Xu}$ and the difference $\underline{\underline{W}}_X - \underline{\underline{W}}_{Xu}$ is growing with the increase of the floor $\underline{W}$. Also, the initial values invested in the risky asset are higher for the constrained model, i.e. $X_0 > Xu_0$ and the difference $X_0 - Xu_0$ is growing with the increase of the floor $\underline{W}$. On the other hand, the expected terminal wealths are higher for the unconstrained model, i.e. $E[Wu_T] > E[W_T]$ and the difference $E[Wu_T] - E[W_T]$ is decreasing with the increase of the floor $\underline{W}$.

In case when $T = 3$ and $\underline{W} = 0.98$, the insurance with spreads is not applicable for the constrained model because $\underline{W}e^{-\Gamma_X} = 0.998270 < 1 = W_0$.

Naturally, for all levels of the floor $\underline{W}$ the expected terminal wealths $E[Wu_T]$ and $E[W_T]$ are increasing with the maturity T. Moreover, the expected terminal wealth is higher for the unconstrained model, i.e. $E[Wu_T] > E[W_T]$ and the difference $E[Wu_T] - E[W_T]$ increases with T.

To calculate the certainty equivalent, we need to enumerate the expected utility from the terminal wealth. To evaluate the integral in (7.14), we use the function 'quad' in Matlab.

Table 7.3 summarizes the values of the certainty equivalents for different levels of the maturity T and different levels of the floor $\underline{W}$.

T=1

$\underline{W}$	C_{Xu}	C_X
0.980	1.078479	1.075588
0.990	1.074683	1.071753
1	1.069859	1.066867
1.010	1.063500	1.060392
1.015	1.059436	1.056223

T=3

0.980	1.298313	1.288040
0.990	1.296581	1.285885
1	1.294152	1.282696
1.010	1.290954	1.278792
1.015	1.289044	1.276544

T=5

	1.548244	1.524810

T=10

	2.397059	2.325046

Table 7.3: Certainty equivalents for different floors $\underline{W}$.

For both $T = 1$ and $T = 3$, the certainty equivalents C_{Xu} and C_X are decreasing with T. Moreover, the certainty equivalents are higher for the unconstrained model, i.e. $C_{Xu} > C_X$, and the difference $C_{Xu} - C_X$ is increasing with $\underline{W}$.

For all levels of the floor $\underline{W}$, the certainty equivalents are increasing with the maturity T.

The question of comparing the Value-at-Risk based risk management, the Limited-Expected-Losses based risk management and the portfolio insurance with spreads arises naturally. The insurance strategies VaR-RM and LEL-RM are optimal strategies, however they are applicable only when there are no constraints imposed on the portfolio strategy. The portfolio insurance with spreads is admissible for the strategy with constraints, though it is not an optimal strategy.

We compare the certainty equivalents of the VaR-RM given in Table 6.2, of the LEL-RM given in Table 6.4 and of the portfolio insurance with spreads in the constrained model given in Table 7.3. One can see that for all levels of the floor, the certainty equivalents of the portfolio insured with LEL-RM are higher than the certainty equivalents of the portfolio insured with VaR-RM. The average difference between the two values is about 0.5%. The certainty equivalents of the portfolio insurance with spreads are significantly lower than those of the VaR-RM and the LEL-RM. The average difference between the certainty equivalents of the VaR-RM and of the portfolio insurance with spreads in the constrained model is approximately 1.8%.

Example - Discrete time

We derived our model with the assumption that the portfolio development is continuous. However, in reality, one can control the portfolio development only at discrete times.

Let us consider the case when $W_0 \leq \underline{W} e^{-\Gamma}$. In addition to our portfolio, we buy a put option with the strike price $\underline{W}$ and sell a put option with the strike price $\underline{\underline{W}}$. Just as in the section of the option based portfolio insurance 5.1.5, we examine two discrete ways of calculating the terminal wealth W_T.

Since we require convex constraints on the portfolio strategy of the risky asset X, we determine $\hat{\beta}$ by (2.8). We calculate the initial amount X_0 invested in the risky asset by (7.11).

The first method represents the case when the required put options are not available on the market. We evaluate the portfolio strategy θ_t by (7.7). Then we calculate $W_{t+\Delta t}$, the value of the wealth at time $t + \Delta t$ by Ito's lemma as

$$W_{t+\Delta t} = W_t e^{\left(r+\theta_t^\top(\mu - r\mathbf{1}) - \frac{\theta_t^\top c^R \theta_t}{2}\right)\Delta t + \sqrt{\theta_t^\top c^R \theta_t}\sqrt{\Delta t} Z},$$

where $Z \sim N(0, 1)$.

We divide the wealth $W_{t+\Delta t}$ to the amount invested in the risky asset $X_{t+\Delta t}$ and the insurance with spreads, according to Proposition 12. Finally, the amount W_T represents the terminal value of the insured portfolio.

In the second method, we consider that the required put options are available in the market. We observe the development of the risky asset and at the end we evaluate the put options. The value of the risky asset $X_{t+\Delta t}$ at time $t + \Delta t$ is given by Ito's lemma as

$$X_{t+\Delta t} = X_t e^{\left(r+\hat{\beta}^\top(\mu - r\mathbf{1}) - \frac{\hat{\beta}^\top c^R \hat{\beta}}{2}\right)\Delta t + \sqrt{\hat{\beta}^\top c^R \hat{\beta}}\sqrt{\Delta t} Z}.$$

The amount X_T represents the terminal value of the risky asset. The terminal value of the insured portfolio is

$$\tilde{W}_T = X_T + \max(\underline{W} - X_T, 0) - \max(\underline{\underline{W}} - X_T, 0).$$

We use the same settings as in the section of the OBPI and compare W_T and $\tilde{W}_T$ for different numbers of steps, numerically 12, 52, 250 and 500.

First, let us examine the percentage of falling under the floor $\underline{W}$. In the case when we consider that the required put options are available on the market, we generate the terminal value of the risky asset X_T and at time T we apply the put options. Table 7.4 summarizes in how many cases the terminal wealth ended below the floor $\underline{W}$ and the percentage that this value represents out of 20000 simulations.

number of steps	$\tilde{W}_T < \underline{W}$	α^*
12	990	4.950%
52	1037	5.185%
250	1031	5.155%
500	1029	5.145%

Table 7.4: Percentage of $\tilde{W}_T$ falling under the floor $\underline{W}$.

One can see that based on simulations, the percentages α^* are very close to the target probability $\alpha = 0.05$, under which the terminal wealth is not allowed to fall.

In case when the required put options are not available on the market, we calculate the portfolio strategy θ_t for all $t \in \langle 0, T)$. The discrete approach that we use in our calculations naturally generates numerical errors. The histograms in Figure 7.5 show that the majority of the simulated terminal wealths are around or greater than the floor $\underline{W}$.

The step of the histograms is 0.01. One way to examine the success of the strategy is to count how many times the terminal value W_T is in the interval $\langle 0, 0.99)$, allowing the maximum error of 0.01. The other way is to organize all terminal values in an increasing order and take a look at the particular one that represents the probability $\alpha = 0.05$, i.e. the worst 5% of the terminal values being under the 1000^{th} value in the order. Table 7.5 represents the results for different numbers of the step.

number of steps	# of $W_T < 0.99$	$W_T(1000)$
12	1729	0.977773
52	1218	0.985684
250	925	0.992539
500	892	0.994202

Table 7.5: Number of cases when the terminal wealths $W_T < 0.99$.

One can see that the higher the number of the terminal values in the interval $\langle 0, 0.99)$ is decreasing with the number of the steps. In addition, the

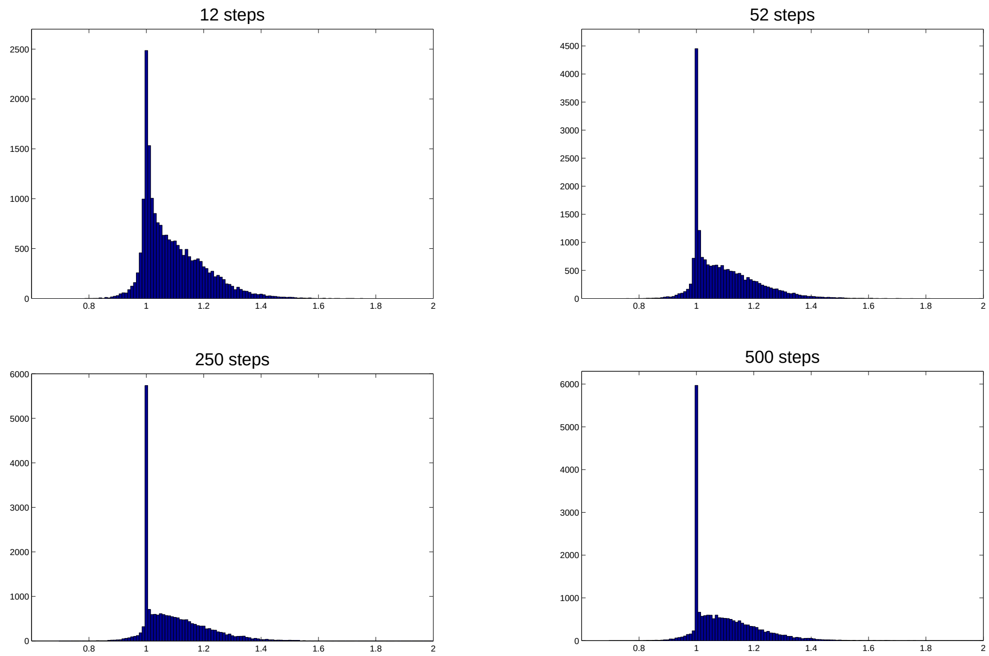

Figure 7.5: Histograms of W_T for different number of steps.

upper bound representing the worst $\alpha\%$ cases is increasing with the number of steps. Moreover, this number approaches the floor $\underline{W}$, therefore we state that our calculations are consistent with the theory.

Table 7.6 provides the average values of the terminal wealths of the insured portfolios for 20000 simulations.

number of steps	$E[W_T]$	$E[\tilde{W}_T]$
12	1.090867	1.090519
52	1.089360	1.089110
250	1.089556	1.089484
500	1.089891	1.089873

Table 7.6: Average terminal wealths of the insured portfolio with spreads.

One can easily see that as the number of steps increases, the average terminal wealths are closer to each other. Due to time complexity, we recommend to conduct the calculations assuming that the put options are available on the market.

Conclusions

The main objective of our work was to examine the portfolio insurance when short-selling of both risky and risk-free assets is prohibited. Our goal was to provide a dynamic portfolio strategy that satisfies such constraints and maximizes the expected utility from the partially guaranteed terminal wealth.

Nutz [26] examined the power utility maximization problem with convex strategy constraints, not considering insurance and concluded that the optimal portfolio strategy is constant over time.

Basak and Shapiro [2] investigated the power utility maximization applying the Value-at-Risk based risk management with no strategy constraints and stated that the optimal portfolio strategy is a proportion of the optimal benchmark portfolio strategy.

To our knowledge, no one studied the combination of these two problems so far. That directed our interest to the power utility maximization problem with risk management and strategy constraints.

Assuming that the terminal wealth of the portfolio is not allowed to fall under the predefined level with probability one and that short-selling is prohibited, we provided two admissible strategies, the OBPI in the constrained model and the alternative method. Based on the results of sensitivity analysis, we concluded that none of the methods dominates the other.

Under the assumption that the terminal wealth is partially allowed to fall under the predefined floor, we found conditions under which the Limited-Expected-Losses based risk management is optimal. The Value-at-Risk based risk management turned out not to be admissible, hence we provided an alternative to it, the portfolio insurance with spreads, which is an admissible solution. The portfolio insurance with spreads is not an optimal strategy, hence its certainty equivalents were significantly lower that the certainty equivalents of the VaR-RM and the LEL-RM in the unconstrained model.

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