# MULTISTAGE PORTFOLIO OPTIMIZATION AS A STOCHASTIC OPTIMAL CONTROL PROBLEM

Martin Lauko

Each portfolio optimization problem is a trade off between minimizing risk and maximizing expected returns. In this paper we consider a multistage portfolio optimization, where the risk at each stage is expressed by the conditional value-at-risk deviation (CVaRD) risk measure and the terminal returns maximization is replaced by some kind of lower constraint on a function of the terminal return. We analyze possible formulations of this problem as some optimal control problems with different types of terminal conditions. For each proposed formulation we suggest effective way to solve the problem by means of the dynamic programming equation. Numerical results illustrate the solutions for a simple model example of pension savings with two funds within the time horizon of 40 years.

*Keywords:* terminal constraint, stochastic discrete optimal control, risk minimization

*Classification:* 93E20, 49L20

## 1. INTRODUCTION

In this paper, we focus on the following problem which we call *Optimal fund selection problem*: Let us consider an investor, who wants at each stage $i$ to divide his capital between two funds: a risk-free one and a risky one. The risk-free fund is composed mainly from bonds, thus we assume the interest rate $r_i$ is known in advance. The risky fund can be formed out of stocks or indices, so its return $z_i$ is random, but we suppose that the distribution of returns is known and the returns in separate stages (time-periods) are independent.

We assume the long-term time horizon of savings, e.g., $k = 40$ years, but with the possibility for the investor to re-balance (to change the weight in the risky fund) every year based on the current value of capital in each fund. The investor's objective is to minimize risk connected to his investment.

Our motivation for studying this problem is the long-term investments, e.g., the pension savings. This problem was already discussed in [6], [7], [8] and [9], a similar problem of the portfolio risk minimization in [1]. Although, we focus on the simple model case – only a single initial payment without additional contributions – to analyze possibilities for solving such kind of problems by the optimal control technique: the stochastic dynamic programming equation. The extension for the real problems with regular contributions is straightforward.

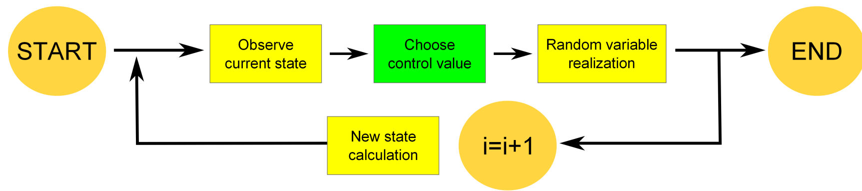



Fig. 1: Scheme of the stochastic discrete optimal control problem solution

There are two distinct areas of mathematics that can be used for solving such problems: the stochastic discrete optimal control and the multistage stochastic programming with recourse. We can find deep comparison of both areas in the paper [4]. As the authors write, despite different technique used, both areas treat very similar real-life problems. Our Optimal fund selection problem can be formulated in terms of the stochastic optimal control as well as the stochastic program. But as Dupačová denotes, if the time horizon is very long, the optimal control allows us to treat such problem efficiently.

Hence we use the stochastic discrete optimal control approach for our problem. Let us recall the basic principles of the optimal control: we suppose there is an object, that is managed by a control during several time periods (stages). The state of the object follows a difference equation that includes a random variable. The main aim is to choose a control value in each period to maximize a selected objective function. The more details can be found in book [5].

The main advantage of the optimal control approach is that we can choose a control value after the current state is known, as we show in Fig. 1. From mathematical point of view, we search for a solution in a form of feedback functions $v_i$, the control value is then a function of the current state $x_i$: $u_i \equiv v_i(x_i)$ for all stages $i$ and states $x_i$. In the optimal control, the sequence of chosen feedback functions is called a strategy.

### 1.1. Problem formulation

Now we focus on the problem formulation. We are using stochastic optimal control, thus we use the following notations:

$i$ – time period (stage), in our case year, $i = 0, \dots, k-1$,

$x_i$ – capital at the beginning of the year $i$, $x_0$ initial capital,

$u_i$ – share in the risky fund in the year $i$, $u_i \in \mathbb{U}_i \equiv [0, 1]$

$r_i$ – known risk-free fund return in the year $i$,

$z_i$ – random risky return (year $i$), it comes from the known distribution $z_i \sim \mathbb{Z}_i$

For the risk measurement, we use the conditional value-at-risk, as it has number of good properties: it is coherent, convex, easy to calculate and widely used in financial risk management. According Pflug in [10], we can define it as follows.

**Definition 1.1. (Conditional Value-at-Risk)** The conditional value-at-risk at probability level $\alpha$, $0 < \alpha < 1$ of a continuous random variable Y with a distribution function $F$ is defined as

$$CVaR_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha F^{-1}(u)\, du, \tag{1}$$

the conditional value-at-risk deviation is

$$CVaRD_\alpha(Y) = E(Y) - CVaR_\alpha(Y). \tag{2}$$

**Definition 1.2. (Multi-period Conditional Value-at-Risk)** Let $Y = (Y_1, \ldots, Y_T)$ be an integrable stochastic process. For a given sequence of constants $c = (c_1, \ldots, c_T)$, probabilities $\alpha = (\alpha_1, \ldots, \alpha_T)$ and a filtration $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_T)$, the multi-period conditional value-at-risk is defined as

$$CVaR_{\alpha,c}(Y; \mathcal{F}) = \sum_{t=1}^{T} c_t\, E\left[\, CVaR_{\alpha_t}(Y_t \,|\, \mathcal{F}_{t-1})\,\right] \tag{3}$$

and the multi-period conditional value-at-risk deviation as

$$CVaRD_{\alpha,c}(Y; \mathcal{F}) = \sum_{t=1}^{T} c_t\, E\left[\, CVaRD_{\alpha_t}(Y_t \,|\, \mathcal{F}_{t-1})\,\right]. \tag{4}$$

As we need to measure a risk during several time periods (stages), we need to use the multi-period version of risk measure. Thus we use the Multi-period Conditional Value-at-Risk Deviation to sum up "pure" risk during several periods. More information about risk measurement in optimal control can be found in [6], [10] or [11].

Now we can formulate our problem in the terms of the optimal control.

**Definition 1.3. (Optimal fund selection problem)** The following stochastic discrete optimal control problem is called Optimal fund selection problem for risk minimization:

$$\begin{aligned}
&\min\ E\left[\sum_{i=0}^{k-1} c_i\, CVaRD_\alpha\big(x_{i+1} \,|\, x_i\big)\right] && \\
&x_{i+1} = x_i\Big[1 + u_i z_i + (1 - u_i)\, r_i\Big], && i = 0, \ldots, k-1, \\
&x_0 = a > 0, && \\
&u_i = v_i(x_i) \in \mathbb{U}_i = \big[0, 1\big], && i = 0, \ldots, k-1, \\
&z_i \sim \mathbb{Z}_i, && i = 0, \ldots, k-1.
\end{aligned} \tag{5}$$

In the problem (5), we did not formulate any conditions or restrictions on the terminal return. Hence the problem (5) represents an unconstrained problem of risk minimization. The unrestricted stochastic optimal control problems can be solved by a standard form of the dynamic programming equation. However, as the certain return is *always better* than any possible risk, there is a trivial solution for our problem: whole capital goes to the risk-free fund, $u_i \equiv 0$ and so the problem (5) itself does not make sense.

To make the problem (5) meaningful we need to add some additional conditions into it, for example a kind of a *terminal constraint* such as – the minimum requested level of terminal capital $\mu$. This, however, will lead to a non-standard stochastic optimal control problem for which the dynamic programming equation must be modified to reflect particular terminal constraints.

## 2. TERMINAL CONSTRAINTS

As discussed above, we want to add an additional terminal constraint to ensure the requested level of the terminal capital $\mu$. There are several ways how to formulate such condition.

1. *Robust Constraint:* $x_k \geq \mu$

   This type of constraint needs to be fulfilled in any case, even for those extreme realizations of the random variable, that have only very low probability. Such certainty might be very expensive, or sometimes even impossible to achieve. This type of constraint is discussed in [2] in connection with the robust optimization. However, it is not very suitable for the stochastic optimal control and we do not discuss it in this paper.

2. *Expected Value Constraint:* $E\,[x_k \,|\, x_{k-1}] \geq \mu$

   This is a common type of constraint in the stochastic optimization, as it respects the fact that $x_k$ is a random variable. This conditions needs to be fulfilled only in the *average case*, we can ignore outliers. This constraint is considered in papers [1], [6].

3. *Probabilistic Constraint:* $P\,[\,x_k \geq \mu\,] \geq \beta$,

   The probabilistic constraint reflects the stochastic character of $x_k$ as well. For this condition, we focus on the probability that the condition $x_k \geq \mu$ would be fulfilled. We want to achieve this probability greater or equal an arbitrary selected $\beta$. This constraint can be also seen as a generalized relaxed version of the robust constraint. It is used in [3] for environment-related problems.

The stochastic discrete optimal control problems are usually solved using the *dynamic programming equation* proposed by Richard Bellman in 1960s. However, if we add a terminal constraint, some strategies $v_i(x_i) \in \mathbb{U}_i$ might become infeasible. We need to somehow define the *set of feasible control values* $W_i(x_i)$ for each stage $i$ and for each state $x_i$, corresponding to the selected terminal constraint, that would contain only such control values, that fulfills the particular constraint.

Then we can search for the problem (5) solution using the dynamic program:

$$\begin{aligned} V_j(x) &= \max_{v_j(x)\in W_j(x)} E_j\left[f_j^0(x, v_j(x), z_j) + V_{j+1}(f_j(x, v_j(x), z_j))\right] \\ &= E_j\left[f_j^0(x, \hat{v}_j(x), z_j) + V_{j+1}(f_j(x, \hat{v}_j(x), z_j))\right], \\ V_k(x) &= \varphi(x), \quad \text{for all } x. \end{aligned}$$

In the equations, $f_j(\cdot)$ stands for the state equation in $j$-th stage, $f_j^0(\cdot)$ stands for objective function in $j$-th stage and $\varphi(\cdot)$ for terminal objective function (in stage $k$).

### 2.1. Expected value constraint

We first focus on an expected value constraint in the form:

$$E\,[\,x_k\,|\,x_{k-1}\,] \geq \mu, \quad \text{for all } x_{k-1} \in X_{k-1} \tag{6}$$

Based on the paper [1] by Brunovský *et al.*, we can rewrite this constraint and derive the following sets of feasible control values for $i = 0, \ldots, k-1$:

$$\begin{aligned}
&W_i(x) = \big\{u_i \in \mathbb{U}_i \,|\, f_i\,(x, u_i, z_i) \in X_{i+1} \;\; \textbf{for all } z_i^s \sim \mathbb{Z}_i\big\},\\
&W_{k-1}(x) = \big\{u_{k-1} \in \mathbb{U}_{k-1} \,|\, E\, f_{k-1}\,(x, u_{k-1}, z_{k-1}) \geq \mu\big\},\\
&X_i = \big\{x \in_i \,|\, W_i(x) \neq \emptyset\big\}
\end{aligned} \tag{7}$$

In this case, the terminal constraint is understood as a priority – if we are not able to fulfill the constraint, the particular state is considered infeasible. Unfortunately, this often leads to infeasibility of the whole problem, especially if the random variable has an approximated normal distribution.

Thus we decided to use a relaxed condition for $W_i(x_i)$. Instead of "for all" possible realizations $z_i^s$ of random variable $\mathbb{Z}_i$, we have used the expected value $E_{z_i}$. The following conditions are called the *alternative formulation of the expected value constraint*:

$$\begin{aligned}
&W_i(x) = \big\{u_i \in U_i \,|\, \boldsymbol{E_{z_i}}\, f_i\,(x, v_i(x), z_i) \in X_{i+1}\big\},\\
&W_{k-1}(x) = \big\{u_{k-1} \in \mathbb{U}_{k-1} \;|\; E_{k-1}\, f_{k-1}\,(x, u_{k-1}, z_{k-1}) \geq \mu\big\},\\
&X_i = \big\{x \in_i \;|\; W_i(x) \neq \emptyset\big\}
\end{aligned} \tag{8}$$

Note that this formulation is not mathematically equivalent with the condition (6), but it might be more suitable for the real applications, as this formulation allows us to use a wider range of the feasible control values. The set of feasible states $X_i$ is bigger as well.

### 2.2. Probabilistic constraint

Another possibility to formulate a terminal constraint is the probabilistic constraint as follows

$$P\,[\,x_k \geq \mu\,] \geq \beta. \tag{9}$$

In this case we define the minimum probability $\beta$, at which the condition $x_k \geq \mu$ must be fulfilled. We understand the probability in (9) corresponding to the random vector $z_0, z_1, \ldots, z_{k-1}$.

First we should solve the special optimal control problem suggested by Doyen in [3] – maximization of the probability $P\,[x_k \geq \mu]$. That can be done using the following dynamic program:

$$\begin{aligned}
\mathcal{P}_k(x) \;&=\; \Phi(x),\\
\mathcal{P}_i(x) \;&=\; \max_{u_i \in \mathbb{U}_i} E_i\left[\mathcal{P}_{i+1}\big(f_i(x, u_i, z_i)\big)\right],\\
\text{where} \qquad &\Phi(x_k) = \begin{cases} 1, & \text{if } x_k \geq \mu,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

We note, the value function $\mathcal{P}_i(x_i)$ is equal to the probability that the condition $x_k \geq \mu$ is fulfilled for the particular stage $i$ and for the particular state $x_i$. Hence we can define the feasible control values and the feasible states for the constraint (9) as follows:

$$\begin{aligned} W_i(x) &= \left\{ u_i \in \mathbb{U}_i \,|\, E_i\left[\mathcal{P}_{i+1}\left[f_i\left(x, u_i, z_i\right)\right]\right] \geq \beta \right\}, \\ X_i &= \left\{ x \,|\, \mathcal{P}_i(x) \geq \beta \right\} \end{aligned} \tag{10}$$

After the calculation of the feasible control value sets (10), we can successfully solve the problem (5) with the probabilistic constraint (9) using the dynamic programming equation.

### 2.3. Problem with Penalization

There is also an alternative way of solving the problem (5) with respect to the terminal value of the savings. Instead of considering the terminal conditions like (6) or (9), we can move the constraint into the objective function as a so-called *penalization factor*:

$$\min\ E\left[\sum_{i=0}^{k-1} CVaRD_\alpha(x_{i+1}\,|\,x_i) \;+\; \delta \cdot \Lambda(x_k)\right] \tag{11}$$

The first term in (11) is the former objective function (the risk minimization) and the second one represents the penalization, with $\Lambda(x_k)$ as the penalization function and $\delta$ its weight. We have used the constant penalization function

$$\Lambda(x) = \begin{cases} 0, & \text{if } x \geq \mu \\ 1, & \text{otherwise,} \end{cases} \tag{12}$$

but in general any function that is equal 0 only if the constraint is fulfilled can be used.

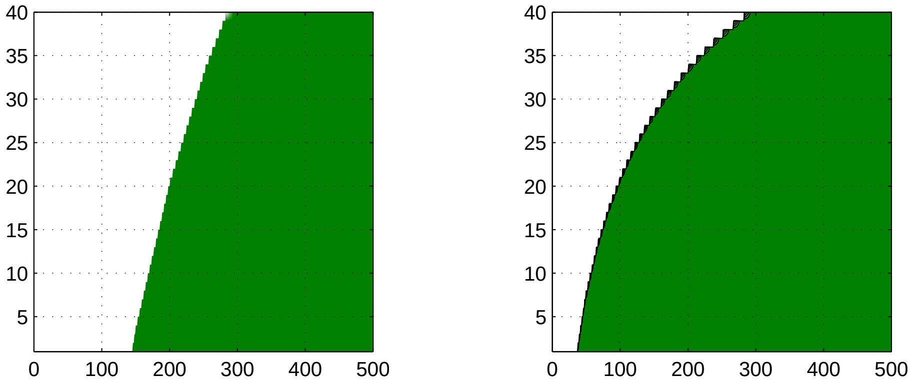
Using the penalization, we can effectively solve the problems with terminal conditions. The main advantage of this approach is the fact that all states and all control values are feasible, so in contrast with the terminal constraint solution, we cannot fall to an infeasible state. Although, the "good" states that lead to the feasible solution are preferred.

In this case, instead of rejecting many infeasible states, we search for a *reasonable compromise solution* to fulfil both our objectives (the risk minimization and the achievement of the predefined terminal value). This solution reflects the fact, that the problem (5) itself is a trade-off between the risk minimization and the terminal value maximization, thus we consider the solution as a trade-off as well.

## 3. NUMERICAL RESULTS

In this section we present numerical results for the Optimal fund selection problem (5) with a different terminal constraints. The optimal control values are calculated using a discrete numerical scheme based on the dynamic programming equation. The set of feasible control values is calculated prior optimization itself.

In the numerical calculations, the following parameters are used:




(a) Former formulation (7) (b) Alternative formulation (8)

Fig. 2: Feasible states comparison – different sets of feasible states for former and alternative formulation. The infeasible states are marked white. Horizontal axis: state $x_i$ (amount of capital), vertical axis: time period $i$ (year)

- Number of periods $k = 40$ years,
- Starting value $x_0 = 100$,
- Requested terminal value $\mu = 300$,
- Risk-free interest rate – $r_i = 2\%$ annually,
- Risk fund random return – discrete approximation of the normal distribution with the expected value $\bar{z}_i = 6\%$, the variance $\sigma = 0.1$

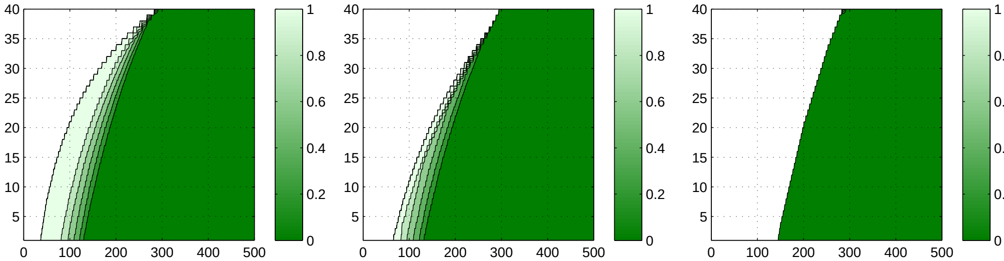
### 3.1. Feasible states comparison

Let us start with a deeper investigation of feasible states for the expected value constraint. We have calculated the sets of feasible states for the former formulation (7) as well as for the alternative one (8). The results are displayed in Fig. (2), thus we can compare them.

In the first case, the starting point $x_0 = 100$ is infeasible. Following the formula (7), the state is considered feasible if and only if we can reach the terminal value $\mu = 300$ with the absolute certainty, i.e. investing only to the risk-free fund with 2% interest rate.

Intuitively, this doesn't make sense – if we invest for example 50% into the risky fund each year, the expected terminal value is 480, which is way more than the requested minimal value $\mu = 300$. Despite the probability of reaching this value is very high, it is not a safety of 100%. Thus from the mathematical point of view, the starting point $x_0 = 100$ is properly considered infeasible. The expected value terminal constraint leads to a very restrictive feasibility area.

Therefore we derived a relaxed alternative formulation (8). In this case, the feasibility area is less restrictive. Any state, that leads to another feasible state in the expected




(a) Area $\mathcal{A}$ (b) Area $\mathcal{B}$ (c) Area $\mathcal{C}$

Fig. 3: Risk minimization results – the problems with terminal conditions. The infeasible states are marked white. Description of areas: $\mathcal{A}$ – expected value alternative formulation (8),
$\mathcal{B}$ – min. probability area with $\beta = 90\%$, $\mathcal{C}$ – expected value former formulation (7).

value, is considered feasible. That means, for example, that the starting point $x_0 = 100$ is considered feasible. The smallest feasible starting point is about 40.

On the other hand, using the alternative formulation, there is a chance of falling into an infeasible state during the saving period if the random return is too low. This is not possible with the former formulation, as the control value is allowed only if it assures we would stay in the feasible area for any random return. If we reach an infeasible state, we can only reconsider the terminal condition, i.e. decrease the value of $\mu$, which is actually a different problem.

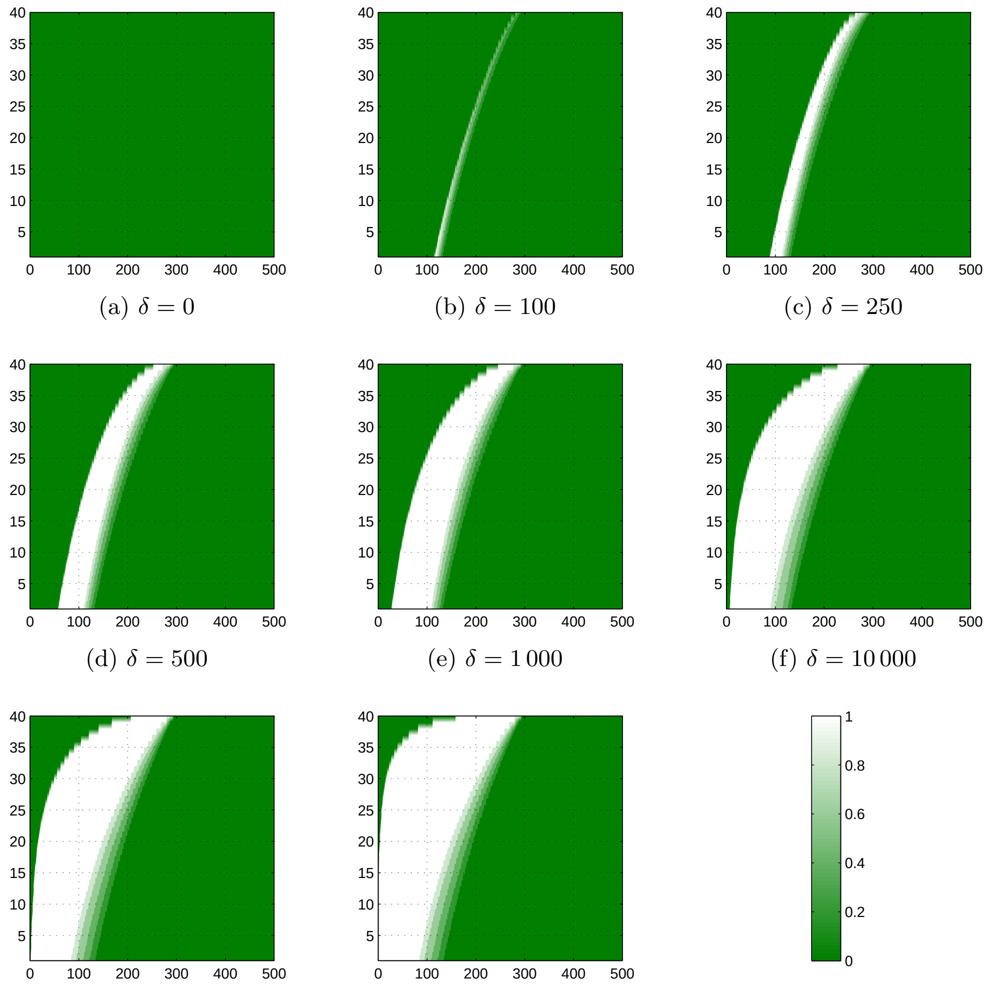
To put it briefly, both formulations for the expected value terminal constraint (7) and (8) have its own disadvantages and it is our decision which to use. In the following subsection, we are comparing the solutions for both formulations.

### 3.2. Risk minimization results

In this part, we provide optimization results for the different problem settings. The results are calculated using the dynamic programming equation with the pre-calculated sets of feasible control values corresponding to the particular constraint.

Firstly, figure 3 presents numerical results for the problems with terminal constraints in a form of expected value (using former and alternative formulation) and the probabilistic constraint. All three solutions are very similar, they are only cropped in a different shape. This shape is determined by the condition for feasible states that is used. The white area in the left hand side of the charts stands for the infeasible state points.

We have calculated results for the penalization problem as well, they are displayed on Fig. 4 with the increasing penalization weight $\delta$. If we use a higher penalization factor, the solution should more likely achieve the requested terminal value $\mu$. In the pictures, we can see the white area (100% risky fund) increasing in the middle. Although, the green area on the right hand side stays the same, because the certainty of achieving the requested terminal value comes with a risk-free investment, so there is no reason for a risky investment.




(a) $\delta = 0$ (b) $\delta = 100$ (c) $\delta = 250$

(d) $\delta = 500$ (e) $\delta = 1\,000$ (f) $\delta = 10\,000$

(g) $\delta = 10^6$ (h) $\delta = 10^{20}$ (i) Colour scale

Fig. 4: Risk minimization results – problem with penalization, different weights $\delta$

To summarize both numerical solutions, we can highlight main differences. First of all – the feasibility. In the case of penalization, all states are considered feasible, hence in any case, we always know the optimal control value that shall be used. Whereas with the terminal conditions, it is possible to fall into an infeasible state, thus the optimal control value in such case is unknown and we can only reconsider the terminal condition. Hence the penalization seems to be an effective way how to handle infeasible points problem occurring in case of the terminal constraint.

On the other hand, in the case of penalization, we do not have any guarantees that the terminal constraint is fulfilled. As we search for a compromise solution, the paths leading to the required terminal value are preferred. However, changing the weight $\delta$ changes the solution, we need to choose a proper weight that represents our objectives regarding the terminal condition.

## 4. CONCLUSION

In this paper, we treated a problem of fund selection in long-term savings and we discussed two main questions: how to formulate the terminal constraints in the case of the stochastic discrete optimal control problem and how to solve such problems using the dynamic programming equation. We showed several types of constraint formulation and we proposed a way how to deal with them using the sets of feasible control values.

We illustrated some disadvantages of the particular constraints: mainly the possibility to reach an infeasible state during the saving period and the problem of an infeasible starting point. We also presented an effective alternative solution for the fund selection problem using the penalization term in the objective function and we showed the advantages of this approach.

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## REFERENCES

[1] Brunovský, P., Halická, M., Kilianová, S.: Stochastic Dynamic Programming for Problems with Expected Value Constraint. Working paper, FMFI, Bratislava 2012.

[2] De Lara, M., Doyen, L.: Sustainable management of natural resources: mathematical models and methods. Springer-Verlag, Berlin 2008.

[3] Doyen, L., De Lara, M.: Stochastic viability and dynamic programming. Systems & Control Letters *59* (2010), 629–634.

[4] Dupačová, J., Sladký, K.: Comparison of Multistage Stochastic Programs with Recourse and Stochastic Dynamic Programs with Discrete Time. ZAMM *82* (2002), 753–765.

[5] Halická, M., Brunovský, P., Jurča, P.: Optimálne riadenie – Viacetapové rozhodovacie procesy v ekonómii a financiách. Epos, Bratislava 2009.

[6] Kilianová, S.: Stochastic dynamic optimization models for pension planning. Dissertation Thesis, FMFI UK, Bratislava 2008.

[7] Kilianová, S., Melicherčík, I., Ševčovič, D.: Dynamic and Static Strategies for the Funded Pillar of the Slovak Pension System. Finance a uver – Czech Journal of Economics and Finance *56* (2006), 506–521.

[8] Kilianová, S., Pflug, G. Ch.: Optimal pension fund management under multi-period risk minimization. Annals of Operations Research *166* (2009), 261–270.

[9] Melicherčík, I., Ševčovič, D.: Dynamic Stochastic Accumulation Model with Application to Pension Savings Management. Yugoslav Journal of Operations Research (2010), 1–24.

[10] Pflug, G. Ch. *et al.*: Modeling, Measuring and Managing Risk. World Scientific, Singapore 2007.

[11] Ruszczyski, A.: Risk-averse dynamic programming for Markov decision processes Math. Program., Ser. B, Vol. 125 (2008), 235–261.

*Martin Lauko, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynska dolina, 842 48 Bratislava, Slovakia*
*e-mail: Martin.Lauko@fmph.uniba.sk*