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Faculty of Mathematics, Physics and Informatics

Solutions with moving singularities for nonlinear diffusion equations

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prof. RNDr. Ján Filo, CSc.
garant študijného programu

....
študent školiteľ

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Abstract

This thesis focuses on the study of singular solutions of nonlinear diffusion equations, specifically the fast diffusion and porous medium equations. It consists of three papers and two additional chapters that provide context, summarize results, and clarify the current state of the topic. Chapter 1 formulates and describes the problem and provides an overview of the literature on singular solutions of nonlinear diffusion equations. This chapter also summarizes the research on asymptotically radially symmetric solutions with a moving singularity. Chapter 2 provides a summary of the papers included in this thesis. Building on work on asymptotically radially symmetric solutions in space dimensions higher than two, we study the existence of such solutions in two space dimensions. We also explore a different type of singularity that we call an anisotropic singularity and discuss the existence of solutions with such a singularity, as well as open problems and possibilities for further analysis. Finally, we extend the knowledge of the properties of asymptotically radially symmetric solutions in space dimensions higher than two, focusing on their uniqueness and the equation they satisfy in the sense of distributions. This equation involves a moving Dirac source term, which is also found in parabolic systems used in various biological applications.

Keywords: nonlinear diffusion, porous medium equation, fast diffusion, singular solution, moving singularity

Abstrakt

Táto dizertačná práca sa zameriava na štúdium singulárnych riešení nelineárnych difúznych rovníc, konkrétne rovnice rýchlej difúzie a difúzie v pórovitom médiu. Práca sa skladá z troch článkov a dvoch ďalších kapitol, ktoré poskytujú kontext, zhŕňajú výsledky a objasňujú súčasný stav témy. Kapitola 1 formuluje a popisuje problém a poskytuje prehľad literatúry o singulárnych riešeniach nelineárnych difúznych rovníc. Táto kapitola tiež sumarizuje výskum asymptoticky radiálne symetrických riešení s pohybujúcou sa singularitou. V Kapitole 2 sa nachádza súhrn článkov obsiahnutých v tejto dizertačnej práci. Na základe práce na asymptoticky radiálne symetrických riešeniach v priestorových dimenziách vyšších ako dva skúmame existenciu takýchto riešení v priestorovej dimenzii rovnej dvom. Ďalej skúmame iný typ singularity známy ako anizotropná singularita a diskutujeme o existencii riešení s takouto singularitou, ako aj o otvorených problémoch a možnostiach pre ďalší výskum. Nakoniec rozširujeme znalosti o vlastnostiach asymptoticky radiálne symetrických riešení v priestorových dimenziách vyšších ako dva, zameriavajúc sa na ich jednoznačnosť a rovnicu, ktorú spĺňajú v zmysle distribúcií. Táto rovnica zahŕňa pohybujúci sa Diracov zdrojový člen, ktorý sa taktiež dá nájsť v parabolických systémoch používaných v rôznych biologických aplikáciách.

Kľúčové slová: nelineárna difúzia, rovnica v pórovitom médiu, rýchla difúzia, singulárne riešenie, pohybujúca sa singularita

Contents

Introduction

This thesis is concerned with the study of singular solutions of a nonlinear diffusion equation. Depending on the value of a parameter m, this equation describes either the diffusion in a porous medium, fast diffusion, or in the simplest case, it is reduced to the linear heat equation. We specifically focus on solutions that maintain their singularity at all times and examine various aspects of their behavior.

This work consists of two published papers [8, 9] and a paper to be submitted for publication [7], which delve into the topic of the dissertation thesis. All three papers are included in the Appendix. Additionally, the thesis is supplemented by two chapters that provide context for the research problem, clarify the current state of the topic, and summarize the results of this thesis.

The problem we study is formulated and described in Chapter 1. We give a brief overview of the literature on singular solutions of nonlinear diffusion equations and related problems in Sections 1.1–1.3. In Section 1.4, we discuss the existence of a class of asymptotically radially symmetric solutions for the porous medium and fast diffusion equations with a singularity that moves in time along a prescribed curve, as proven in [10].

In Chapter 2, we provide a summary of the results from papers [7–9]. Our findings build upon the work on asymptotically radially symmetric solutions from [10]. Since the authors of [10] focused on spatial dimensions greater than two, in [8] we study the case where the spatial dimension is equal to two. Using similar methods to [10], we prove the existence of solutions with a moving singularity and compare our results to [10], see Section 2.1. In Section 2.2, we explore a different type of singularity known as the anisotropic singularity. We discuss the existence of solutions with such a singularity, open problems in this area of research, and possibilities for further analysis, see also [9]. Finally, in Section 2.3, we further extend the knowledge of properties of solutions from [10]. We put the emphasis on uniqueness and what equation they satisfy in the sense of distributions. As it turns out, they satisfy the corresponding problem with a moving Dirac source term. Interestingly, source terms of this form also appear in parabolic systems used in various biological applications, such as axon growth or angiogenesis.

Singular solutions of nonlinear diffusion equations: overview

We study singular solutions of nonlinear diffusion equations, specifically those that arise in the fast diffusion and porous medium equations. Let $m > 0$, $n \geq 2$, and $\xi : \mathbb{R} \to \mathbb{R}^n$ be a given curve in $\mathbb{R}^n$. We consider the initial value problem

$$u_t = \Delta u^m, \qquad x \in \mathbb{R}^n \setminus \{\xi(t)\}, \quad t \in (0, \infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \setminus \{\xi(0)\}. \tag{1.2}$$

When $m = 1$, equation (1.1) reduces to the linear heat equation. For $m > 1$, it is known as the porous medium equation, and for $0 < m < 1$, it is called the fast diffusion equation. This is a consequence of the fact that (1.1) can be written in the form

$$u_t = \nabla \cdot (D(u)\nabla u), \qquad D(u) := mu^{m-1},$$

where the diffusion coefficient $D(u)$ approaches infinity as u approaches zero if $m < 1$, and $D(u) \to 0$ as $u \to 0$ if $m > 1$.

Our main interest lies in positive solutions of (1.1)-(1.2) that maintain a time-dependent singularity, i.e. they are singular along a curve ξ:

$$u(x,t) \to \infty \quad \text{as} \quad x \to \xi(t) \quad \text{for each } \ t \geq 0.$$

If $\xi \in C^1([0,\infty); \mathbb{R}^n)$, we can use the transformation $y = x - \xi(t)$, which simplifies (1.1)-(1.2) to a problem with a singularity fixed at 0. For $v(y,t) = u(x,t)$ we obtain

$$v_t = \Delta v^m + \xi'(t) \cdot \nabla v, \qquad y \in \mathbb{R}^n \setminus \{0\}, \quad t \in (0, \infty), \tag{1.3}$$

$$v(y,0) = v_0(y), \qquad y \in \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

There are known results in the literature related to the existence, uniqueness, and regularity of solutions to problems of the form (1.1)-(1.2) and (1.3)-(1.4). In the rest of this chapter, we provide a brief overview of these results and literature focusing on singular solutions.

1.1 Explicit solutions

Explicit solutions to the nonlinear problem (1.1)-(1.2) play an important role in the literature. To introduce some known examples, we begin with some notation. For $x_0 \in \mathbb{R}^n$ let $B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$ denote the open ball in $\mathbb{R}^n$ with radius $R > 0$, centered in x_0. Let $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ denote the unit $(n-1)$-sphere and $|S^{n-1}|$ the surface area of S^{n-1}. Moreover, defining a critical exponent

$$m_c := \begin{cases} \frac{n-2}{n} & \text{if } n \geq 2, \\ 0 & \text{if } n = 1, \end{cases}$$

we distinguish three different cases for the fast diffusion equation, namely subcritical ($0 < m < m_c$), critical ($m = m_c$), and supercritical ($m_c < m < 1$). The exponent m_c is of great significance, see, for example Vázquez [34].

Assume that the singularity ξ is standing, i.e. let $\xi(t) \equiv \xi_0$ for some ξ_0 in $\mathbb{R}^n$. Then for dimensions $n \geq 3$, problem (1.1)-(1.2) has a family of radially symmetric positive steady states

$$\tilde{u}(x) := \kappa |x - \xi_0|^{-\frac{n-2}{m}}, \quad \kappa > 0. \tag{1.5}$$

It is well known that these solutions satisfy the corresponding problem with a weighted Dirac measure on $\mathbb{R}^n$ giving unit mass to the point $\xi_0 \in \mathbb{R}^n$, i.e.

$$0 = \Delta \tilde{u} + \kappa^m (n-2)|S^{n-1}|\delta_{\xi_0}(x) \quad \text{in} \quad D'(\mathbb{R}^n). \tag{1.6}$$

We remark that if $m = 1$ and $\kappa = ((n-2)|S^{n-1}|)^{-1}$, $\tilde{u}$ is the fundamental solution of the Laplace equation. For $n = 2$ and $m = 1$, the steady states have a logarithmic singularity at the point ξ_0, and they are given by the formula

$$\tilde{u}(x) := \kappa \log |x - \xi_0|^{-1}, \quad \kappa > 0. \tag{1.7}$$

The fundamental solution of the Laplace equation is obtained by choosing $\kappa = (2\pi)^{-1}$. In the case $n = 2$ and $m \neq 1$, raising $\tilde{u}$ from (1.7) to the power of $1/m$ creates a family of positive singular steady states that exist in the domain $B_1(\xi_0) \setminus \{\xi_0\}$.

Provided that $m_c < m < 1$, another explicit solution of (1.1)-(1.2) with a standing singularity at ξ_0 is given by

$$\hat{u}(x,t) = \left(\frac{\hat{\kappa}\, t}{|x - \xi_0|^2} \right)^{\frac{1}{1-m}}, \quad \hat{\kappa} = \frac{2mn(m - m_c)}{1-m}, \tag{1.8}$$

see [3] and references therein. Function $\hat{u}$ is a classical solution for $x \neq \xi_0$, but it is not a distributional solution, since it is never locally integrable near $x = \xi_0$, i.e.

$$\int_{B_R(0)} \hat{u}(x,t)\, dx = +\infty, \qquad t > 0, \quad R > 0.$$

Let $m^* = 0$ if $n = 2$ and $m^* = (n-3)/(n-1)$ if $n \geq 3$. In reference [6], Fila, King, Takahashi, and Yanagida study a positive entire-in-time solution for $n \geq 2$ and $m^* < m < 1$, which is referred to as a traveling wave solution. This solution is singular on the set $\Gamma(t) = \{s\omega; -\infty < s < ct\}$, where $\omega \in S^{n-1}$ is a unit vector and $c > 0$ is a constant that determines the speed of propagation. The traveling wave solution is expressed as

$$\bar{u}(x,t) = \bar{\kappa}\left(|a||x-ta| + a\cdot(x-ta)\right)^{-\frac{1}{1-m}}, \quad \bar{\kappa} = \left(\frac{m(n-1)(m-m^*)}{1-m}\right)^{\frac{1}{1-m}}, \tag{1.9}$$

where $a = c\omega$ represents the direction of the wave.

1.2 Related results: existence and behavior of solutions

In [15], Kan and Takahashi constructed singular solutions with time-dependent singularities for the heat equation

$$u_t = \Delta u, \qquad x \in \mathbb{R}^n \setminus \{\xi(t)\}, \quad t \in (0,T), \tag{1.10}$$

assuming that $n \geq 2$, $T \in (0,\infty]$ and $\xi \in C([0,T]; R^n)$. More specifically, they considered the equation

$$u_t = \Delta u + w(t)\delta_{\xi(t)}(x) \quad \text{in} \quad D'(\mathbb{R}^n \times (0,T)),$$

where $w \in L^1((0,t))$ for each $t \in (0,T)$. The solutions presented in [15] do not always behave like the fundamental solution of the Laplace equation near the singularity, and the profile loses asymptotic radial symmetry. The removability of time-dependent singularities for solutions of (1.10) with $\Omega \setminus \{\xi(t)\}$ instead of $\mathbb{R}^n \setminus \{\xi(t)\}$, where $\Omega \subset \mathbb{R}^n$, was studied in [30]. We say that the singularity of u at the point $x = \xi(t)$ is removable if there exists a solution v which satisfies the heat equation in $\Omega \times (0,T)$ in the classical sense and $u \equiv v$ in $\Omega \setminus \{\xi(t)\} \times (0,T)$. Imposing some additional regularity conditions on ξ, it was shown that a singularity is removable if it is weaker than the order of the fundamental solution of the Laplace equation. Further results concerning the heat equation can be found in [16]. Here, Kan and Takahashi showed that for every non-negative solution u of (1.10) there is a non-negative Radon measure M on $(0,T)$ such that

$$u_t = \Delta u + \delta_{\xi(t)}(x) \otimes M(t) \quad \text{in} \quad D'(\mathbb{R}^n \times (0,T)),$$

where $\delta_{\xi(t)}(x) \otimes M(t)$ is a product measure of $M(t)$ and $\delta_{\xi(t)}(x)$. Moreover, u behaves like the fundamental solution of the Laplace equation as $x \to \xi(t)$.

Article [3] presents a classification of non-negative solutions of (1.1)-(1.2) for $0 < m < 1$, $n \geq 2$, and with a fixed singularity at $\xi(t) \equiv 0$. These solutions are continuous in $\mathbb{R}^n \times [0,\infty)$, unbounded at $x = 0$, and satisfy the initial condition $u(x,0) = 0$ for $x \in \mathbb{R}^n \setminus \{0\}$. In the supercritical exponent range, that is if $m_c < m < 1$, these solutions either have a singularity of the same type as $\hat{u}$ defined in (1.8), or there is $\tau \in (0, \infty]$ such that they satisfy

$$u_t = \Delta u^m + \delta_0(x) \otimes M(t) \quad \text{in} \quad D'(\mathbb{R}^n \times (0, \tau)),$$

for some positive Radon measure M, while they have a singularity of the same type as $\hat{u}$ for $t > \tau$. For $M(t) = t^\sigma$ with some $\sigma \in [0, m/(1-m)]$, the solution is of the self-similar form

$$u(x,t) := t^\alpha f(xt^{-\beta}), \quad \alpha := \frac{2\sigma + 2 - n}{n(m-1)+2}, \quad \beta := \frac{m - \sigma(1-m)}{n(m-1)+2}.$$

Here, f is a solution to an elliptic equation, see [3] for further details. The author of [3] also considered a more general filtration equation.

If $m_c < m < 1$, the authors of [11] proved that all solutions of

$$u_t = \Delta u^m \quad \text{in} \quad D'(\mathbb{R}^n \times (0, \infty)),$$

with $u_0 \in L^1_{loc}(\mathbb{R}^n)$ become bounded and continuous, i.e. $u \in C([0, \infty); L^1_{loc}(\mathbb{R}^n))$.

On the other hand, in the same range $m_c < m < 1$, the authors of [4] concluded the monotonicity of strongly singular sets of extended continuous solutions, i.e. that it cannot shrink in time. Hence, the singularity of such solutions persists at all times. Here, an extended continuous solution satisfies the equation pointwise (it is a classical solution) in the regular set $\{(x,t) \in \mathbb{R}^n \times (0, \infty);\ u(x,t) < \infty\}$, and is continuous with values in $(0, \infty]$. If u is an extended solution, then the strongly singular set $S(t)$ at a time t is defined as the set of points $x \in \mathbb{R}^n$ at which $u(\cdot, t)$ is not locally integrable. The existence of extended continuous solutions with expanding strongly singular sets is also established in [4].

For $n \geq 3$ and $0 < m \leq m_c$, the evolution of initial standing singularities of proper solutions of $u_t = \Delta u^m$ on a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ was studied in [35]. Proper solutions are constructed as limits of increasing bounded approximations, and they play the role of the minimal solutions that appear in other elliptic and parabolic theories. It follows from [35] that if $\xi_0 \in \Omega$ and the singularity of $u_0(x)$ satisfies

$$a_1|x - \xi_0|^{-\gamma_1} \leq u_0(x) \leq a_2|x - \xi_0|^{-\gamma_2}, \qquad x \in \Omega \setminus \{\xi_0\},$$

for some $\xi_0 \in \mathbb{R}^n$, $a_1, a_2 > 0$ and $\gamma_2 \geq \gamma_1 > 0$, then the following situations may occur:

(a) if $\gamma_1 \leq \gamma_2 < 2/(1-m)$, then u becomes smooth instantaneously,

(b) if $\gamma_1 = \gamma_2 = 2/(1-m)$, then finite-time blow-down occurs. That is, there is $T > 0$ such that $u(\cdot, t) \notin L^\infty(\Omega)$ for $t < T$ but $u(\cdot, t) \in L^\infty(\Omega)$ for $t > T$,

(c) if $2/(1-m) < \gamma_1 \leq \gamma_2 < (n-2)/m$, then u maintains its singularity at the point ξ_0 for all times but each member of the ω-limit set of the solution is bounded (infinite-time blow-down),

(d) if $\gamma_1 = \gamma_2 = (n-2)/m$, then $u(\xi_0, t) = \infty$ for all $t > 0$ and the singularity persists even up to $t = \infty$, i.e. each element in the ω-limit set of u has an isolated singularity at the point ξ_0 (singular stabilization),

(e) if $(n-2)/m < \gamma_1 \leq \gamma_2$, then the singularity persists at all times and $u(x,t) \to \infty$ as $t \to \infty$ for all $x \in \Omega$ (infinite-time blow-up).

Here, we define the ω-limit set as the set $\omega(u) := \{w : \Omega \to \mathbb{R};\ \exists t_k \to \infty : u^m(\cdot, t_k) \to w \text{ a.e. in } \Omega\}$. Note that finite-time blow-down, as in (b), was previously shown in [34] when $\Omega = \mathbb{R}^n$, $n \geq 3$, and $0 < m < m_c$.

For $m > 1$, $n \geq 2$ and $C > 0$, a singular self-similar radial function which is a solution to the problem

$$u_t = \Delta u^m + C\delta_0(y) \quad \text{in} \quad D'(\mathbb{R}^n \times (0, \infty))$$

with zero initial data was constructed in [25]. Near $x = 0$, this solution behaves like the singular stationary solution (1.5) or (1.7) and it is compactly supported in x for all $t > 0$.

Later, Lukkari studied solutions of the porous medium and supercritical fast diffusion equation ($m_c < m < 1$), respectively. Assuming that the forcing term f is a non-negative Radon measure on $\mathbb{R}^{n+1}$ such that $f(\Omega \times (0,T)) < \infty$, he proved the existence of a specific class of weak solutions of

$$u_t = \Delta u^m + f(x,t),$$

on finite or infinite cylinders $\Omega \times (0,T)$, $0 < T \leq \infty$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. These solutions satisfy $u \in L^q(0, T : W_0^{1,q}(\Omega))$, where q is any number such that $1 < q < 1 + 1/(1+mn)$ in the case of the porous medium equation, and $1 \leq q < 1 + 1/(1+mn)$ in the case of the fast diffusion equation.

In [6], the authors investigated the fast diffusion equation when $m^* < m < 1$ and $n \geq 2$, and constructed positive entire-in-time solutions with snaking singularities. We recall that $m^* = (n-3)/(n-1)$ when $n \geq 3$ and $m^* = 0$ when $n = 2$. For each $t \in \mathbb{R}$, these solutions have a singularity on the set $\Gamma(t) := \{\xi(s); -\infty < s < ct\}$ where $c > 0$ and $\xi : \mathbb{R} \to \mathbb{R}^n$ satisfies Condition 1.1 in [6]. The authors based their construction on the existence of the explicit singular traveling wave solution (1.9) which has a cylindrical symmetry.

For more literature regarding similar investigations for semilinear heat equations, see for instance [12, 13, 16, 17, 26–28, 31]. For the Navier-Stokes equations, see [18, 19].

1.3 Related results: uniqueness

We begin this section by recalling standard uniqueness results for the porous medium equation. The classical result by Pierre from 1982 [24] guarantees the uniqueness of solutions u satisfying

(a) $u_t = \Delta(u^m)$ in $\mathcal{D}'(\mathbb{R}^n \times (0,T))$ for $m > 1$,

(b) $u \in L^1((\mathbb{R}^n \times [0,T))$,

(c) $u \in L^\infty((\mathbb{R}^n \times (\tau, T))$ for each $\tau \in (0,T)$.

A similar result can also be found in [2] by Daskalopoulos and Kenig, where they assume that $\sup_{t>0} \int (u_1(x,t) + u_2(x,t))\, dx < \infty$ instead of condition (b). The proof relies on choosing a smooth, compactly supported test function that solves a backward problem.

Vázquez's book [33] contains additional techniques to prove uniqueness of solutions of the nonlinear equation

$$u_t = \Delta u^m + f(x,t), \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^n \times (0,T)),$$

where $0 < T \leq \infty$ and $f \in L^1_{loc}(\mathbb{R}^n \times (0,\infty))$. Assuming that $u_{1,2} \in L^2_{loc}(\mathbb{R}^n \times (0,T))$ and $u^m_{1,2} \in L^2_{loc}(0,T; H^1_0(\mathbb{R}^n))$, one can use a test function φ of the form

$$\varphi(x,t) = \begin{cases} \int_t^T \big(u_1^m(x,s) - u_2^m(x,s)\big) ds & \text{if } 0 < t < T, \\ 0 & \text{if } t \geq T, \end{cases}$$

which was introduced by Oleinik [22]. If we relax the assumption on the gradient of u^m and assume that both $u, u^m \in L^2_{loc}(\mathbb{R}^n \times (0,T))$, a similar method to that in [2] can be used to prove uniqueness. This involves choosing a compactly supported test function that solves a backward problem. For a more detailed explanation, we refer to [33]. Since Lukkari's weak solutions with a non-negative Radon measure mentioned at the end of Section 1.2 lack the L^2-integrability assumed by Vázquez in [33] or L^∞ by Pierre in [24], their uniqueness was left as an open problem.

Regarding uniqueness results for the fast diffusion equation, the study [11] by Herrero and Pierre dates back to 1985. They proved that if u satisfies the following conditions:

(a) $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^n))$,

(b) $u_t = \Delta(u|u|^{m-1})$ in $\mathcal{D}'(\mathbb{R}^n \times (0,\infty))$,

(c) $u_t \in L^1_{loc}(\mathbb{R}^n \times (0,\infty))$,

then $u_1(\cdot,0) \equiv u_2(\cdot,0)$ implies that $u_1 \equiv u_2$.

Recently, there have been new findings on the uniqueness of subcritical fast diffusion. In their study presented in [32], Takahashi and Yamamoto focused on the scenario when $n \geq 3$ and $0 < m < m_c$. They established the uniqueness of signed solutions of the initial value problem

$$u_t = \Delta(u|u|^{m-1}), \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}, \quad t \in (0,T), \tag{1.11}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}, \tag{1.12}$$

with $0 < T \leq \infty$ and $\xi_0 \in \mathbb{R}^n$. More specifically, the authors of [32] showed that if two functions u_1, u_2 satisfy (1.11)-(1.12) pointwise and $u_1(\cdot,0) = u_2(\cdot,0)$ on $\mathbb{R}^n \setminus \xi_0$, where u_1, u_2 belong to $C^{2,1}(\{(\mathbb{R}^n \setminus \{\xi_0\}) \times (0,T)\}) \cap C(\{(\mathbb{R}^n \setminus \{\xi_0\}) \times [0,T)\})$, then $u_1 \equiv u_2$ on $(\mathbb{R}^n \setminus \xi_0) \times (0,T)$.

Moreover, Hui's work [14] established that for $n \geq 3$, $0 < m < m_c$, and under suitable conditions on initial data, solutions that have a finite number of standing singularities are also uniquely determined. By solutions with finitely many standing singularities, we mean that these solutions satisfy equations (1.11)-(1.12) with $\mathbb{R}^n \setminus \{\xi_0, \xi_1, \dots, \xi_i\}$ instead of $\mathbb{R}^n \setminus \{\xi_0\}$. Here, $i \in \mathbb{N}$ and $\xi_0, \xi_1, \dots, \xi_i \in \mathbb{R}^n$.

1.4 Asymptotically radially symmetric solutions

In their study [10] of supercritical fast diffusion, Fila, Takahashi, and Yanagida investigated the existence of positive, asymptotically radially symmetric solutions with a moving singularity for the initial value problem (1.1)-(1.2). Along the given curve ξ with suitable properties, these solutions keep a singularity at all times, i.e. $u(x,t) \to \infty$ as $x \to \xi(t)$ for each $t \geq 0$. We will describe these solutions in detail since many of the results in this thesis build on this research. Let $m_* := (n-2)/(n-1)$. Assume that $n \geq 3$, $m > m_*$, and

(A1) $\xi \in C^1([0,\infty);\mathbb{R}^n)$, ξ' is locally Hölder continuous, and there exist positive constants Ξ, β such that
$$|\xi'(t)| \leq \Xi e^{-\beta t} \quad \text{for} \quad t \geq 0, \tag{1.13}$$

(A2) $k \in C^1([0,\infty))$ satisfies $\kappa^{-1} \leq k(t) \leq \kappa$ and $|k'(t)| \leq \kappa'$ for $t \geq 0$ and some positive constants κ and κ',

(A3) $u_0(x) \in C(\mathbb{R}^n \setminus \{\xi(0)\})$ is positive and there exist λ, μ and ν satisfying
$$\max\{(n-2)/m - 1, 0\} < \lambda < \mu < n-2 < \nu$$
such that $u_0(x)^m = k(t)^m |x - \xi(0)|^{-n+2} + O(|x-\xi(0)|^{-\lambda})$ as $x \to \xi(0)$, and $C^{-1}|x - \xi(0)|^{-\nu} \leq u_0(x)^m \leq C|x-\xi(0)|^{-\mu}$ for $|x - \xi(0)| \geq 1$ with some constant $C > 1$.

Under these assumptions, [10] implies the existence of a function $u > 0$ satisfying:

(i) $u \in C^{2,1}(\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\ t \in (0,\infty)\}) \cap C(\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\ t \in [0,\infty)\})$ and u satisfies (1.1)-(1.2) pointwise,

(ii) $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^n))$,

(iii) for each $t \geq 0$, u has the asymptotic behavior
$$u(x,t)^m = k(t)^m |x - \xi(t)|^{-n+2} + O(|x - \xi(t)|^{-\lambda}) \quad \text{as} \quad x \to \xi(t).$$

Moreover, the authors of [10] showed that m_* is a critical exponent for the existence of such solutions and there are no such solutions if $m < m_*$. See also Theorem 1 in [8] in Appendix. The main idea behind constructing these solutions was to investigate whether there exist solutions with a moving singularity that have the same type of singularity as stationary solutions (1.5) in the sense of (iii).

Summary of the main results, discussion and open problems

The chapter is divided into three sections, each one summarizing the results of our papers [7–9], which focus on the topic of this dissertation. These papers are included in the Appendix for reference.

2.1 Moving singularities for nonlinear diffusion equations in two space dimensions

At the end of the previous chapter, we discussed research conducted in [10], which constructed solutions for nonlinear diffusion equations in dimensions $n \geq 3$. These solutions behave as the steady state (1.5) near the moving singularity in the sense of (iii). In our article [8], we aimed to extend this work to two dimensions and answer the question of whether we could construct solutions as in Section 1.4 but now with a logarithmic singularity as the steady state (1.7) for $n = 2$.

2.1.1 Existence of solutions and their properties

We addressed the question above in article [8] and successfully constructed solutions with the desired properties, with some differences depending on whether $m_* = 0 < m < 1$ and $m > 1$. Here, we give a brief overview of this result. Assuming that $n = 2$ and $0 < m < 1$, we imposed the following assumptions:

(B1) ξ is as in (A1) but instead of (1.13) there exists a constant $\varepsilon > 0$ such that $|\xi'(t)| \leq \varepsilon(1+t)^{-1}$ for $t \geq 0$,

(B2) k is a given positive constant,

(B3) $u_0 \in C(\mathbb{R}^n \setminus \{\xi(0)\})$ is positive and is close to the singular steady state $\tilde{u}(x)$ in the following sense. There exists $0 < \lambda < 1$ such that

$$u_0^m(x) = k^m \log(|x - \xi(0)|^{-1}) + O(\log^\lambda(|x - \xi(0)|^{-1}))$$

as $x \to \xi(0)$, and

$$C^{-1}|x-\xi(0)|^{-\frac{2}{1-m}} \le u_0^m(x) \le C|x-\xi(0)|^{-\frac{2}{1-m}} \tag{2.1}$$

for $|x-\xi(0)| \ge 1$ with a constant $C > 1$,

and proved the existence of a positive function u satisfying properties (i) and

(iii') for each $t \ge 0$, u has the asymptotic behavior

$$u(x,t)^m = k^m \log(|x-\xi(t)|^{-1}) + O(\log^\lambda(|x-\xi(t)|^{-1})) \quad \text{as} \quad x \to \xi(t).$$

See also Theorem 2 in [8] in Appendix. When $n = 2$ and $m > 1$, we assumed that:

(B1') ξ is as in (A1) but instead of (1.13) there exists a constant $\Xi > 0$ such that $|\xi'(t)| \le \Xi$ for $t \ge 0$,

(B2') k is as in (B2),

(B3') u_0 is as in (B3) but instead of $0 < \lambda < 1$ assume that $0 < \lambda < 1/m$ and instead of (2.1) let

$$C^{-1} \le u_0(x) \le C \tag{2.2}$$

for $|x-\xi(0)| \ge 1$ and some constant $C > 1$.

Theorem 3 in [8] implies the existence of a function $u > 0$ satisfying properties (i), (iii') when $n = 2$ and $m > 1$. For more details, see [8] in Appendix.

2.1.2 Discussion and open problems

When comparing the results from [10] for $n \ge 3$, $m > m_*$ described in Section 1.4 to the case where $n = 2$ in [8], there are natural differences due to the different behavior near the singularity. The method of sub- and supersolutions used for the existence proofs relies on comparison functions, which differ substantially in the three cases: $n \ge 3$, $m > m_*$, $n = 2, 0 < m < 1$, and $n = 2, m > 1$. Because of this, the initial datum u_0 behaves differently far from the singularity $x = \xi(0)$.

Moreover, in (A2), we assume that k is a positive and bounded function of time, but for $n = 2$, we impose conditions (B2) and (B2') that require k to be a positive constant. This difference leads to weaker conditions on the growth of ξ' when $n = 2$ compared to $n \ge 3$, where exponential decay (1.13) of ξ is required. Additionally, the condition on ξ' is much weaker in (B1') than in (B1). As noted in article [8], this suggests that it may be easier for the singularity to follow a given curve ξ if the diffusion coefficient is large near the curve. On the other hand, local Hölder continuity of ξ' is assumed in all three cases: $n \ge 3$, $m > m_*$, $n = 2, 0 < m < 1$, and $n = 2, m > 1$, as it is a technical requirement in the existence proofs.

In future work, it is possible to study the optimality of conditions (A1)-(A3), (B1)-(B3), and (B1')-(B3'), under which the existence of solutions was proved. The differences between them suggest that they could be further improved.

2.2 Solutions with anisotropic standing singularities

Sections 1.4 and 2.1 discuss the asymptotically radially symmetric behavior of solutions near a moving singularity. This raises the question of whether there exist non-negative singular solutions of the fast diffusion and porous medium equation that exhibit a different behavior as they approach the singularity and are not radially symmetric. Focusing on standing singularities, we addressed this question in article [9]. The following subsections provide a summary and discussion of this topic.

2.2.1 Existence of solutions and their properties

Let $\xi_0 \in \mathbb{R}^n$, $r := |x-\xi_0|$, $\omega := (x-\xi_0)/|x-\xi_0|$, $\lambda > 0$, and $\alpha \in C^{2,1}(S^{n-1} \times [0,\infty))$ be positive. In [9], we were interested in the construction of positive solutions with the singular behavior of the form

$$u(x,t) = \alpha(\omega,t)r^{-\lambda} + o(r^{-\lambda}) \quad \text{as} \quad r \to 0, \tag{2.3}$$

for $\omega \in S^{n-1}$ and $t \geq 0$. We say that if $\alpha(\omega,t)$ depends non-trivially on the space variable ω, the corresponding solution u of problem (1.1)-(1.2) has an anisotropic singularity at ξ_0, otherwise it is asymptotically radially symmetric. The main result in [9] regarding the existence of such solutions is presented in Theorem 1.1 (see Appendix). In order to prove the existence of such solutions, we assume that $n \geq 2$, $0 < m < 1$, and the following conditions are satisfied:

(C1) $\xi(t) \equiv \xi_0$ for some $\xi_0 \in \mathbb{R}^n$,

(C2) $\alpha \in C^2(S^{n-1})$ is positive,

(C3) $u_0 \in C(\mathbb{R}^n \setminus \{\xi_0\})$ is positive and there exist λ, ν satisfying $\lambda > 2/(1-m)$, $(1-m)\lambda - 2 - m(\lambda-\nu) > 0$, and $\lambda > \nu > 0$ such that

$$u_0^m(x) = \alpha^m(\omega)|x-\xi_0|^{-m\lambda} + O\big(|x-\xi_0|^{-m\nu}\big) \quad \text{as} \quad x \to \xi_0,$$

for each $\omega \in S^{n-1}$, and

$$C^{-1} \leq u_0(x) \leq C$$

for $|x-\xi_0| \geq 1$ and some constant $C > 1$.

Under these assumptions, Theorem 1.1 in [9] guarantees the existence of functions that satisfy (i) and have the asymptotic behavior given by:

(iii”) for each $t \geq 0$ and $\omega \in S^{n-1}$, u has the asymptotic behavior

$$u(x,t)^m = \alpha^m(\omega)|x-\xi_0|^{-m\lambda} + O(|x-\xi_0|^{-m\nu}) \quad \text{as} \quad x \to \xi_0.$$

The uniqueness of these solutions follows from [32] (see Section 1.3 for more details). When $\lambda < n$, $0 < m < m_c$, and $n > 2$, Theorem 1.2 in [9] shows that a solution u that meets conditions (i), (iii") and also (ii) is a distributional solution of the original problem

$$u_t = \Delta u^m \quad \text{in} \quad D'(\mathbb{R}^n \times (0, \infty)). \tag{2.4}$$

However, if $\lambda \geq n$, the function u that satisfies (i) and (iii") is not locally integrable in space-time. It is worth mentioning that in [7], we showed a more general result for solutions of the subcritical fast diffusion equation. Corollary 1.1 in [7] demonstrates that if $n \geq 3$, $0 < m < m_c$ and $\xi \in C([0, \infty); \mathbb{R}^n)$, then for every function u that satisfies (i) with $u_0 \in L^1_{loc}(\mathbb{R}^n)$, it holds that $u \in L^1_{loc}(\mathbb{R}^n \times (0, \infty))$. Moreover, such a function is a distributional solution of equation (2.4).

2.2.2 Formal computations, discussion and open problems

The assumption $\lambda > 2/(1-m)$ in (C3) was made after conducting formal computations, which we introduce here, as they provide additional results and raise further questions. If we assume that u is given by (2.3), for $u(x,t) = w(r, \omega, t)$ we can formally compute

$$\begin{aligned} w_t &= \alpha_t r^{-\lambda} + o(r^{-\lambda}), \\ \Delta w^m &= \big(\Delta_\omega \alpha^m - m\lambda(n-2-m\lambda)\alpha^m\big) r^{-m\lambda-2} + o(r^{-m\lambda-2}). \end{aligned} \tag{2.5}$$

Here, Δ_ω denotes the Laplace-Beltrami operator on S^{n-1}. The leading term is different in each of the three cases: $\lambda > 2/(1-m)$, $\lambda = 2/(1-m)$, and $\lambda < 2/(1-m)$. In this subsection, we will focus mainly on the first two cases that lead to solutions with anisotropic singularities. The last case leads to asymptotically radially symmetric solutions and will be dealt with in more detail in the following section.

First, taking $\lambda > 2/(1-m)$ implies that the leading term is w_t. This has led us to the assumption $\alpha = \alpha(\omega)$ in the previous subsection.

In the critical case $\lambda = 2/(1-m)$, the terms w_t and Δw^m are balanced, leading us to an initial value problem

$$\alpha_t(\omega, t) = \Delta_\omega \alpha^m(\omega, t) + A\alpha^m(\omega, t), \qquad \omega \in S^{n-1}, \quad 0 < t < T, \tag{2.6}$$

$$\alpha(\omega, 0) = \alpha_0(\omega) > 0, \qquad \omega \in S^{n-1}, \tag{2.7}$$

where $T \in (0, \infty]$ and $A = 2mn(m - m_c)/(1-m)^2$. By proving the existence of a positive classical solution α of (2.6)-(2.7) for some $T > 0$, we obtain a positive classical solution of (1.1)-(1.2) of the form

$$u(x,t) = \alpha(\omega, t)|x - \xi_0|^{-\frac{2}{1-m}}, \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}, \quad 0 < t < T. \tag{2.8}$$

An example of a function that solves (2.6)-(2.7) classically is

$$\tilde{\alpha}(t) = \big((1-m)A\,t + t_0\big)^{\frac{1}{1-m}},$$

where t_0 is an arbitrary positive constant. Note that taking $\tilde{\alpha}$ in (2.8) produces the radially symmetric function $\hat{u}(x,t)$ mentioned in (1.8) in Chapter 1. In order to obtain solutions of (1.1)-(1.2) with an anisotropic singularity, we are interested in solutions of (2.6)-(2.7) that, unlike $\tilde{\alpha}$, depend non-trivially on the space variable ω. In [8] we provide an example of such a function in a separated form if $0 < m < m_c$. The existence of a more general class of solutions of the form (2.8) with an anisotropic singularity is left as an open problem. For more details, see Section 2 in [9]. Another open problem is the extension of these results for moving singularities.

Finally, in the case $\lambda < 2/(1-m)$, the leading term in (2.5) is Δw^m, which implies that α must be a solution of

$$-\Delta_\omega \alpha^m = -m\lambda(n-2-m\lambda)\alpha^m.$$

Eigenvalues of $-\Delta_\omega$ are non-negative and start with zero (the constant 1 is the corresponding eigenfunction), other eigenfunctions change sign, see [29]. Since we are looking for positive solutions, the only choice is

$$\lambda = \frac{n-2}{m}, \quad m > m_c, \quad n \geq 3, \quad \text{and} \quad \alpha = \alpha(t). \tag{2.9}$$

We will derive a conclusion based on this observation in what follows.

2.3 Asymptotically radially symmetric solutions in space dimensions $n \geq 3$

In this section, we expand on the analysis of asymptotically radially symmetric solutions with a moving singularity, previously discussed in Section 1.4.

2.3.1 Existence up to $m > m_c$ for a standing singularity

The existence of asymptotically radially symmetric solutions of problem (1.1)-(1.2) with a moving singularity, which correspond to (2.9), was discussed in Section 1.4. Here, it was assumed that $n \geq 3$ and $m > m_*$. The authors of [10] showed that m_* is a critical exponent for the existence of such solutions and there are no such solutions if $m < m_*$. However, it appears from (2.9) that the case of a standing singularity (i.e., using condition (C1) instead of (A1)) allows for the existence of such solutions for $m > m_c$. Indeed, this can be verified by an inspection of the proof of Theorem 1.1 in [10], as we pointed out in paper [9], see the Appendix.

2.3.2 Distributional solutions with a weighted Dirac measure

In their paper [10], the authors investigated the existence of solutions with a moving singularity that have the same type of singularity as the stationary solutions (1.5) in the sense of (iii). It is known that stationary solutions satisfy problem (1.6) with a weighted Dirac measure. This raises an interesting question. Do asymptotically radially symmetric solutions from [10] also satisfy a corresponding problem with a moving Dirac measure?

In our paper [9], we attempted to answer this question and found that solutions from [10] indeed satisfy the equation

$$u_t = \Delta u^m + (n-2)k^m(t)\delta_{\xi(t)}(x) \quad \text{in} \quad D'(\mathbb{R}^n \times (0,\infty)), \tag{2.10}$$

where k is a weight function satisfying (A2), and $\delta_{\xi(t)}$ is a moving Dirac measure. This is analogous to equation (1.6) for stationary solutions (1.5).

A moving Dirac measure on the right-hand side of parabolic systems also appears in several biological applications, such as the growth of axons or angiogenesis. References [5] and [1] provide more detailed insights on these applications, respectively. In [23], the Cattaneo telegraph equation with a moving time-harmonic source is studied in the context of the Doppler effect, where a moving Dirac measure is also used.

2.3.3 Uniqueness for the fast diffusion

In this study, we also established the uniqueness of solutions from [10] in the case of the fast diffusion equation, i.e. when $m < 1$. The corresponding Theorem 1.1 can be found in article [7], we refer to the Appendix for details. Moreover, we extended a uniqueness result from [32] (see Section 1.3) from standing to moving singularities, although the existence of such solutions remains an open problem.

2.3.4 Discussion and open problems

Having established the uniqueness of distributional solutions of the fast diffusion equation with a moving Dirac measure, it raises the question of whether the same can be achieved for solutions with more general source terms. A possible direction for future research would be to investigate the open problem of the uniqueness of Lukkari's weak solutions [21], which involve a Radon measure as a source term, as discussed in Section 1.2.

Another natural question concerns the uniqueness of distributional solutions of the porous medium equation with a measure source term. While the existence of solutions in [10] was established also in the case $m > 1$, the uniqueness was so far only shown for $m < 1$. In [20], Lukkari also considered the case of the porous medium equation. He left the question of uniqueness open, as his solutions lack the L^2-integrability, and the uniqueness proofs for $m > 1$ from [2, 33] rely on this condition, see Section 1.3. This is also a problem for asymptotically radially symmetric solutions from [10] due to their behavior near the singularity. It holds that

$$\int_{B_1(\xi(t))} \left(|x-\xi(t)|^{-\frac{n-2}{m}}\right)^2 dx = \int_0^1 r^{n-1-2\frac{n-2}{m}}\, dr < \infty \quad \text{if} \quad m > \frac{2(n-2)}{n} = 2m_c,$$

but $2m_c < 1$ only if $n < 4$. In order to establish the uniqueness of such solutions for all $m > 1$, a new strategy is needed.

Conclusion

This thesis is concerned with the study of singular solutions of nonlinear diffusion equations with a parameter m, which can describe diffusion in a porous medium, fast diffusion, or reduce to the linear heat equation. We investigate solutions that maintain their singularity at all times and examine various aspects of their behavior. The work includes two published papers [8, 9] and a paper submitted for publication [7], all included in the Appendix. Moreover, two chapters providing context, summarizing the results, and discussing open problems are included.

Chapter 1 introduces the problem, provides an overview of the literature on singular solutions of nonlinear diffusion equations and related problems in Sections 1.1–1.3, and discusses the existence of asymptotically radially symmetric solutions with a moving singularity from [10] in Section 1.4.

Chapter 2 summarizes the results of the papers included in the Appendix and extends the results from [10]. In our article [8], we extended this work to two space dimensions and answered the question of whether we could construct solutions as in Section 1.4 but now with a logarithmic moving singularity as the steady state (1.7) for $n = 2$. For more details concerning the existence and properties of such solutions, we refer to Section 2.1. In Section 2.2, we explored a different type of singularity known as the anisotropic singularity. In [9], we examined the existence of solutions with such singularity with a fixed position in space. We also performed a formal analysis, leading to new questions and open problems. Lastly, in Section 2.3, we extended the knowledge of the properties of solutions from [10]. More specifically, in [9], we established that solutions with a moving singularity from [10] satisfy the corresponding problem with a moving Dirac source term. Interestingly, source terms of this form also appear in parabolic systems used in various biological applications. Moreover, in [7], we investigated the uniqueness of these solutions and established it in the case of the fast diffusion equation, leaving the case of the porous medium equation as an open problem.

Overall, our work provides new insights into the behavior of singular solutions of nonlinear diffusion equations with moving singularities and opens up possibilities for further research.

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Moving singularities for nonlinear diffusion equations in two space dimensions

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Abstract

We construct solutions with prescribed moving singularities for equations of porous medium type in two space dimensions. This complements a previous study of the problem where only dimensions higher than two were considered.

Keywords Porous medium equation · Fast diffusion · Singular solution · Moving singularity

Mathematics Subject Classification 35K65 · 35K67 · 35A02 · 35B40

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1 Introduction

Consider the equation

$$v_t = \Delta_y v^m, \quad y \in \mathbb{R}^n \setminus \{\xi(t)\}, \quad t > 0, \tag{1}$$

where $m > 0$ and $\xi \in C^1([0,\infty);\mathbb{R}^n)$ with the initial condition

$$v(y,0) = v_0(y), \quad y \in \mathbb{R}^n \setminus \{\xi(0)\}. \tag{2}$$

We are interested in positive solutions that are singular at $\xi(t)$, that is,

$$v(y,t) \to \infty \quad \text{as } y \to \xi(t), \quad t > 0.$$

For example, when $\xi \equiv 0$ and $n \geq 3$, then (1) has a singular steady state

$$\tilde{v}(y) = K|y|^{-\frac{n-2}{m}}, \quad y \in \mathbb{R}^n \setminus \{0\},$$

where K is an arbitrary positive constant. Using the variable $x = y - \xi(t)$ and setting $u(x,t) = v(y,t)$, Eq. (1) is transformed into

$$u_t = \Delta u^m + \xi'(t) \cdot \nabla u, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad t > 0. \tag{3}$$

The main aim of [1] was to construct solutions of (1) which for a given function k behave near $y = \xi(t)$ as $k(t)|y - \xi(t)|^{-(n-2)/m}$. To recall the result from [1] we first introduce the assumptions from there:

(A1) $k \in C^1([0,\infty))$ is a given positive function of t.
(A2) $\xi \in C^1([0,\infty);\mathbb{R}^n)$ and the derivative ξ' is locally Hölder continuous.
(A3) The initial value $v_0(y)$ is positive and is close to the singular steady state $\tilde{v}(y)$ in the following sense: There exist constants λ, μ and ν with

$$\max\left\{\frac{n-2}{m} - 1, 0\right\} < \lambda < \mu < n-2 < \nu$$

such that

$$v_0(y)^m = k(0)^m|y - \xi(0)|^{-n+2} + O(|y - \xi(0)|^{-\lambda})$$

as $y \to \xi(0)$, and

$$C^{-1}|y - \xi(0)|^{-\nu} \leq v_0(y)^m \leq C|y - \xi(0)|^{-\mu}$$

for $|y - \xi(0)| \geq 1$ with a constant $C > 1$.

To construct global-in-time solutions, the following conditions on the velocity of the singular point ξ', the weight of the leading term k and its derivative k' were imposed:

(B1) There exist positive constants Ξ and β such that

$$|\xi'(t)| \leq \Xi e^{-\beta t}, \quad t \geq 0.$$

(B2) There exist positive constants κ and κ' such that

$$\kappa^{-1} \le k(t) \le \kappa, \quad |k'(t)| \le \kappa', \quad t \ge 0.$$

Now we are ready to state the following:

Theorem 1 [1] *Let $n \ge 3$ and $m > m_* := (n-2)/(n-1)$. Assume (A1), (A2), (A3), (B1) and (B2). Then there exists a positive constant β_0 such that the following holds: If $\beta \ge \beta_0$ in (B1), then there exists a function*

$$v \in C^{2,1}(\{(y,t) \in \mathbb{R}^{n+1} \;:\; y \ne \xi(t),\, t \in (0,\infty)\})$$
$$\cap\, C(\{(y,t) \in \mathbb{R}^{n+1} \;:\; y \ne \xi(t),\, t \in [0,\infty)\}) \cap C([0,\infty); L^1_{loc}(\mathbb{R}^n))$$

which satisfies (1), (2) pointwise and

$$v(y,t)^m = k(t)^m |y-\xi(t)|^{-n+2} + O(|y-\xi(t)|^{-\lambda}) \tag{4}$$

for each $t \ge 0$ as $y \to \xi(t)$.

It was also shown in [1] that there is no solution with a behavior as in (4) if $m < m_*$.

The main aim of this paper is to construct solutions which for $n = 2$ and a given constant $k > 0$ behave near $y = \xi(t)$ as $k\psi^{1/m}(|y-\xi(t)|)$ where ψ is the fundamental solution of the Laplace equation, i.e. $\psi(r) = -\log r$, $r = |y| > 0$. We distinguish two cases: $0 < m < 1$ and $m \ge 1$. Our results read as follows.

Theorem 2 *Let $n = 2$ and $0 < m < 1$. Assume that (A2) holds. Let $v_0 \in C(\mathbb{R}^n \setminus \{\xi(0)\})$ be positive and such that*

$$v_0^m(y) = k^m \psi(|y-\xi(0)|) + O(\psi^\lambda(|y-\xi(0)|)) \quad \text{as } y \to \xi(0),$$

for some $k > 0$ and $0 < \lambda < 1$, and

$$C^{-1}|y-\xi(0)|^{-\frac{2}{1-m}} \le v_0(y)^m \le C|y-\xi(0)|^{-\frac{2}{1-m}} \tag{5}$$

for $|y-\xi(0)| \ge 1$ and some constant $C > 1$. Then there is a constant $\varepsilon > 0$ such that if $|\xi'(t)| \le \varepsilon(1+t)^{-1}$ for $t \ge 0$ then there is a function

$$v \in C^{2,1}(\{(y,t) \in \mathbb{R}^{n+1} \;:\; y \ne \xi(t),\, t \in (0,\infty)\})$$
$$\cap\, C(\{(y,t) \in \mathbb{R}^{n+1} \;:\; y \ne \xi(t),\, t \in [0,\infty)\}) \cap C([0,\infty); L^1_{loc}(\mathbb{R}^n))$$

which satisfies (1), (2) pointwise and

$$v(y,t)^m = k^m \psi(|y-\xi(t)|) + O(\psi^\lambda(|y-\xi(t)|))$$

for each $t \ge 0$ as $y \to \xi(t)$.

Theorem 3 *Let $n = 2$, $m \geq 1$ and $0 < \lambda < 1/m$. Assume that (A2) holds and ξ' is bounded. Let v_0 be as in Theorem 2 but instead of (5) assume that $C^{-1} \leq v_0(y) \leq C$ for $|y - \xi(0)| \geq 1$ and some constant $C > 1$. Then there is a function v as in Theorem 2.*

The condition on ξ' is much weaker in Theorem 3 than in Theorem 2. It seems to indicate that it might be easier for the singularity to follow a given curve ξ if the diffusion coefficient u^{m-1} is large near the curve. The weight k is a function in Theorem 1 while it is a constant in Theorems 2 and 3. On the other hand, exponential decay of $|\xi'|$ is required in Theorem 1.

The method of proof of Theorems 1–3 is the same. It uses suitable sub- and supersolutions, but if $n = 2$ these comparison functions differ substantially from those from [1].

For various results on solutions with moving singularities for the heat equation ($m = 1$) we refer to [4, 5, 14], for semilinear heat equations, see [2, 3, 5, 6, 9–13, 15] and also [7, 8] for the Navier–Stokes system.

This paper is organized as follows. Sub- and supersolutions which will be used in the proofs of Theorems 2 and 3 are constructed in Sects. 3 and 2, respectively. The proofs are then completed in Sect. 4.

2 Comparison functions in case $m \geq 1$

This section is devoted to construction of comparison functions for the proof of Theorem 3. For positive constants $B, b_0 > 1$ we define

$$b(t) := b_0 e^{Bt}, \quad t \geq 0,$$

and

$$\phi(r) := \log(e + r^{-1}), \quad r > 0.$$

Let $k > 0$, $\lambda \in (0, 1)$ and $\mu \in (\lambda, 1)$. We construct supersolutions and subsolutions of the following form:

$$\begin{cases} u^+_{in}(x,t) := \left[b^m(t) + k^m\left[\phi(r) + b(t)\phi^\lambda(r)\right]\right]^{\frac{1}{m}}, \\ u^+_{out}(x,t) := \left[b^m(t) + k^m c(t)\phi^\mu(r)\right]^{\frac{1}{m}}, \quad r = |x| > 0,\ t \geq 0, \end{cases}$$

and

$$\begin{cases} u^-_{in}(x,t) := k\left[\phi(r) - b_0\phi^\lambda(r)\right]_+^{\frac{1}{m}}, \\ u^-_{out}(x,t) := kd^{\frac{1}{m}}, \quad r = |x| > 0,\ t \geq 0. \end{cases} \tag{6}$$

For $\mu > \lambda$ we choose

$$\begin{cases} c(t) := \phi^{-\mu+1}(r_0) + b(t)\phi^{-\mu+\lambda}(r_0), & t \geq 0, \\ d := \phi(r_1) - b_0\phi^{\lambda}(r_1), \end{cases} \tag{7}$$

so that the functions u^+_{in} and u^+_{out} intersect at the point $r = r_0$, whereas u^-_{in} and u^-_{out} at the point $r = r_1$.

We claim that the following two lemmata hold true.

Lemma 1 *Let $n = 2$. Assume that (A2) holds and let $\Xi > 0$ be such that $|\xi'| \leq \Xi$. Then there exist constants $B, b_0 > 1$ and $r_0 > 0$ such that*

$$u^+(x,t) := \begin{cases} u^+_{in}(x,t) & \text{for } |x| \in (0, r_0], \\ u^+_{out}(x,t) & \text{for } |x| \in [r_0, \infty), \end{cases}$$

is a supersolution for $|x| > 0$ and $t > 0$.

Lemma 2 *Let $n = 2$ and $0 < \lambda < 1/m$. Assume that (A2) holds and let $\Xi > 0$ be such that $|\xi'| \leq \Xi$. Then there exist constants $B, b_0 > 1$, $r_1 > 0$, and $0 < d \ll 1$ of the form (7) such that*

$$u^-(x,t) := \begin{cases} u^-_{in}(x,t) & \text{for } |x| \in (0, r_1], \\ u^-_{out}(x,t) & \text{for } |x| \in (r_1, \infty), \end{cases}$$

is a subsolution for $|x| > 0$ and $t > 0$.

2.1 Inner part of a supersolution

Computing the time derivative of u^+_{in} we obtain

$$\begin{aligned} (u^+_{in})_t &= \frac{1}{m}\left[b^m + k\big(\phi + b\phi^{\lambda}\big)\right]^{\frac{1}{m}-1}\left[mb^{m-1}b' + kb'\phi^{\lambda}\right] \\ &= \frac{B}{m}\left[b^m + k\big(\phi + b\phi^{\lambda}\big)\right]^{\frac{1}{m}-1}\left[mb^m + kb\phi^{\lambda}\right] \\ &\geq 0. \end{aligned}$$

By $b, \phi \geq 1$ and $\lambda < 1$ we have

$$\begin{aligned} -\Delta(u^+_{in})^m &= -k^m(\Delta\phi + b\Delta\phi^{\lambda}) \\ &= \frac{k^m r^{-2}\phi^{\lambda-2}}{(er+1)^2}\left[\lambda b((1-\lambda) - er\phi) - er\phi^{2-\lambda}\right] \\ &\geq \frac{k^m r^{-2} b}{(er+1)^2}\left[\lambda(1-\lambda)\phi^{\lambda-2} - (1+\lambda)er\right] \\ &= \frac{r^{-2}\phi^{\lambda-2} b}{(er+1)^2}\left[c_1 - c_2 r\phi^{2-\lambda}\right], \end{aligned}$$

where $c_1 = k^m\lambda(1-\lambda)$ and $c_2 = k^m(1+\lambda)e$ are positive constants. Denoting $\omega := x/|x|$ we have

$$\begin{aligned}
-\xi'\cdot\nabla u_{in}^+ &= -(\xi'\cdot\omega)\frac{k^m}{m}\left[b^m + k^m\left(\phi+b\phi^\lambda\right)\right]^{\frac{1}{m}-1}\left(1+\lambda b\phi^{\lambda-1}\right)\phi' \\
&\geq -\frac{k^m r^{-1}}{m(er+1)}|\xi'|\left[b^m + k^m\left(\phi+b\phi^\lambda\right)\right]^{\frac{1}{m}-1}\left(1+\lambda b\phi^{\lambda-1}\right) \\
&\geq -\frac{(1+\lambda)(1+2k^m)^{\frac{1}{m}-1}k^m r^{-1}}{m(er+1)}|\xi'|b \\
&= -\frac{c_3 r^{-1}}{er+1}|\xi'|b,
\end{aligned}$$

where $c_3 = (1+\lambda)(1+2k^m)^{\frac{1}{m}-1}k^m m^{-1} > 0$. Thus, for $|\xi'| \leq \Xi$ and some $r_0 = r_0(m,\lambda,k,\Xi)$ we obtain

$$\begin{aligned}
&(u_{in}^+)_t - \Delta(u_{in}^+)^m - \xi'\cdot\nabla u_{in}^+ \\
&\quad\geq \frac{r^{-2}\phi^{\lambda-2}b}{(er+1)^2}\left[c_1 - c_2 r\phi^{2-\lambda} - c_3|\xi'|r(er+1)\phi^{2-\lambda}\right] \\
&\quad\geq 0
\end{aligned}$$

for all $0 < r \leq r_0$.

2.2 Matching condition for a supersolution

Denoting $\phi_0 = \phi(r_0)$ and $\phi_0' = \phi'(r_0)$ we have

$$\begin{aligned}
\frac{\partial(u_{in}^+)^m}{\partial r} &= k^m\left(1+\lambda b\phi^{\lambda-1}\right)\phi', \\
\frac{\partial(u_{out}^+)^m}{\partial r} &= \mu k^m c\phi^{\mu-1}\phi' = \mu k^m\left(\phi_0^{-\mu+1} + b\phi_0^{-\mu+\lambda}\right)\phi^{\mu-1}\phi'.
\end{aligned}$$

Hence, at $r = r_0$ for $\mu > \lambda$ and b_0 large enough it holds that

$$\frac{\partial(u_{in}^+)^m}{\partial r} - \frac{\partial(u_{out}^+)^m}{\partial r} = |\phi_0'|k^m\left((\mu-\lambda)\phi_0^{\lambda-1}b + \mu - 1\right) \geq 0.$$

2.3 Outer part of a supersolution

Note that since $r \in [r_0,\infty)$ it follows that $1 \leq \phi(r) \leq \phi_0$. Using the notation $\phi_0 = \phi(r_0)$ and assuming that $r \geq r_0$, we compute

$$
\begin{aligned}
-\xi' \cdot \nabla u_{out}^{+} &= -(\xi' \cdot \omega)\frac{\mu}{m}\left[b^m + k^m c\phi^\mu\right]^{\frac{1}{m}-1} k^m c\phi^{\mu-1}\phi' \\
&\geq -|\xi'|\frac{\mu k^m r^{-1}\phi^{\mu-1}}{m(er+1)}\left[b^m + k^m\left(\phi_0^{-\mu+1} + b\phi_0^{-\mu+\lambda}\right)\phi^\mu\right]^{\frac{1}{m}-1} \\
&\quad \times \left(\phi_0^{-\mu+1} + b\phi_0^{-\mu+\lambda}\right) \\
&\geq -\frac{c_1 r^{-1}}{er+1}|\xi'|(b^m + b)^{\frac{1}{m}-1} b \\
&\geq -\frac{c_1 r^{-1}}{er+1}|\xi'| b^{\frac{1}{m}},
\end{aligned}
$$

where c_1 is a positive constant depending on m, μ, λ, k, r_0 and $\varXi$. Moreover, we have

$$
\begin{aligned}
-\varDelta(u_{out}^{+})^m &= -k^m c\varDelta\phi^\mu \\
&= -\frac{\mu k^m r^{-2}\phi^{\mu-2}}{(er+1)^2}\left(\phi_0^{-\mu+1} + b\phi_0^{-\mu+\lambda}\right)(er\phi - (1-\mu)) \\
&\geq -\frac{c_2 r^{-1}}{(er+1)^2} b,
\end{aligned}
$$

and

$$
\begin{aligned}
(u_{out}^{+})_t &= \frac{1}{m}\left[b^m + k^m c\phi^\mu\right]^{\frac{1}{m}-1}\left(mb^{m-1}b' + k^m c'\phi^\mu\right) \\
&= \frac{B}{m}\left[b^m + k^m\left(\phi_0^{-\mu+1} + b\phi_0^{-\mu+\lambda}\right)\phi^\mu\right]^{\frac{1}{m}-1}\left(mb^m + k^m\phi_0^{-\mu+\lambda} b\phi^\mu\right) \\
&\geq c_3 B(b^m + b)^{\frac{1}{m}} \\
&\geq c_3 Bb,
\end{aligned}
$$

where c_2 and c_3 are positive constants depending on m, μ, λ, k, and r_0. Provided that $|\xi'| \leq \varXi$ and B is large enough, for $r \in [r_0, \infty)$ we have

$$
\begin{aligned}
&(u_{out}^{+})_t - \varDelta(u_{out}^{+})^m - \xi' \cdot \nabla u_{out}^{+} \\
&\geq \frac{r^{-1}}{(er+1)^2}\left[c_3 Br(er+1)^2 b - c_2 b - c_1(er+1)|\xi'| b^{\frac{1}{m}}\right] \\
&\geq 0.
\end{aligned}
$$

2.4 Inner part of a subsolution

We define

$$
\theta := \frac{1}{1-\lambda}. \tag{8}
$$

We see that $u_{in}^{-} > 0$ when $\phi(r) > b_0^{\theta}$. Thus we choose

$$r_1 \leq \left(e^{b_0^\theta} - e\right)^{-1}.$$

Note that the right-hand side of this inequality can be made arbitrarily small choosing b_0 large. Now we estimate the following expressions. We have $(u_{in}^-)_t = 0$. Since $\phi - b_0\phi^\lambda \geq 0$, $b_0 > 1$ and $\lambda \in (0,1)$, we obtain

$$\begin{aligned}
-\Delta(u_{in}^-)^m &= -k^m\left(\Delta\phi - b_0\Delta\phi^\lambda\right) \\
&= -\frac{k^m r^{-2}}{(er+1)^2}\left[\lambda(1-\lambda)b_0\phi^{\lambda-2} + er(1-\lambda b_0\phi^{\lambda-1})\right] \\
&\leq -\frac{\lambda(1-\lambda)k^m b_0 r^{-2}\phi^{\lambda-2}}{(er+1)^2} \\
&= -\frac{c_1 b_0 r^{-2}\phi^{\lambda-2}}{(er+1)^2},
\end{aligned}$$

where $c_1 = \lambda(1-\lambda)k^m$. For $m \geq 1$ we have $\frac{1}{m} - 1 \leq 0$. Let $\delta \ll 1$. For $\phi \geq (1-\delta)^{-\theta} b_0^\theta$ it holds that

$$\phi - b_0\phi^\lambda \geq \delta\phi.$$

Hence, we choose

$$r_1 := \left(e^{(1-\delta)^{-\theta}b_0^\theta} - e\right)^{-1}.$$

We take $\delta = \delta_0 b_0^{-\theta}$, so that the outer part of the subsolution u_{out}^- defined in (6) is equal to $u_{out}^- = k[\delta_0(1-\delta_0 b_0^{-\theta})^{-\theta}]^{1/m}$, which can be made arbitrarily small choosing δ_0 small and independent of b_0. We have

$$\begin{aligned}
-\xi' \cdot \nabla u_{in}^- &= (\xi' \cdot \omega)\frac{kr^{-1}}{m(er+1)}\left(\phi - b_0\phi^\lambda\right)^{\frac{1}{m}-1}(1-\lambda b_0\phi^{\lambda-1}) \\
&\leq \frac{k\delta^{\frac{1}{m}-1} r^{-1}\phi^{\frac{1}{m}-1}}{m(er+1)}|\xi'| \\
&= \frac{c_2\delta^{\frac{1}{m}-1} r^{-1}\phi^{\frac{1}{m}-1}}{er+1}|\xi'|,
\end{aligned}$$

where $c_2 = k/m$. For $\lambda < 1/m$, $|\xi'| \leq \Xi$, $r \in (0, r_1]$ and b_0 large enough we obtain

$$\begin{aligned}
&(u_{in}^-)_t - \Delta(u_{in}^-)^m - \xi' \cdot \nabla u_{in}^- \\
&\leq -\frac{r^{-2}\phi^{\lambda-2}}{(er+1)^2}\left[c_1 b_0 - c_2\delta_0^{\frac{1}{m}-1} b_0^{\theta\left(1-\frac{1}{m}\right)}|\xi'|r(er+1)\phi^{1+\frac{1}{m}-\lambda}\right] \\
&\leq 0,
\end{aligned}$$

since $\theta\left(\frac{1}{m} - 1\right) < 1$.

3 Comparison functions in case $m \in (0, 1)$

In this section we construct comparison functions for the proof of Theorem 3. For a positive constant $p_0 > 1$ we define

$$p(t) := p_0(1+t)^{\frac{1}{1-m}}, \quad t \geq 0,$$

and for $K > 0$ we set

$$\varphi(r) := \log(e + Kr^{-1}), \quad r > 0.$$

With $k > 0$ and $\lambda \in (0, 1)$ we construct supersolutions and subsolutions of the following form:

$$\begin{cases} u_{in}^+(x,t) := k\left[\varphi(r) + p^m(t)\varphi^\lambda(r)\right]^{\frac{1}{m}}, \\ u_{out}^+(x,t) := Kp(t)r^{-\frac{2}{1-m}}, \quad r = |x| > 0,\ t \geq 0, \end{cases}$$

and

$$\begin{cases} u_{in}^-(x,t) := k\left[\varphi(r) - p_0^m\varphi^\lambda(r)\right]_+^{\frac{1}{m}}, \quad r = |x| > 0, \\ u_{out}^-(x,t) := (t_0+t)^{-\alpha}\left[1 + A(t_0+t)^{-\alpha(1-m)-1}r^2\right]^{-\frac{1}{1-m}}, \quad t \geq 0. \end{cases} \tag{9}$$

Here $t_0 > 1$ and

$$A > \frac{1-m}{4m^2}, \quad \alpha > \frac{4mA}{1-m}. \tag{10}$$

Notice that u_{out}^- is a modified Barenblatt solution for the fast diffusion equation. Choosing t_0 large, we can make u_{out}^- and its derivative with respect to r arbitrarily small for any $r \geq 0$ and $t \geq 0$. Thus, there is a unique intersection point $r = \rho_1(t)$ of the functions u_{in}^- and u_{out}^- and the matching condition is satisfied, i.e.

$$(u_{out}^-)_r(\rho_1(t),t) > (u_{in}^-)_r(\rho_1(t),t).$$

The following lemma concerns the first intersection point of the functions u_{in}^+ and u_{out}^+.

Lemma 3 *There exists a constant $K_1 = K_1(m,\lambda,k) > 0$ such that for any $K \geq K_1$ the first intersection point $r = \rho_0(t)$ of the functions u_{in}^+ and u_{out}^+ satisfies $1 < \rho_0(t) < R := (K/k)^{(1-m)/2}$.*

Proof By the definitions of u_{in}^+ and u_{out}^+ we have

$$u_{in}^+(1,t) \leq 2^{\frac{1}{m}}kp(t)\log^{\frac{1}{m}}(e+K),$$
$$u_{out}^+(1,t) = Kp(t).$$

Then there exists a constant $K_{1,1} = K_{1,1}(m, k) > 0$ such that $u^+_{in}(1, t) < u^+_{out}(1, t)$ for any $K \geq K_{1,1}$. On the other hand, simple computations show that

$$\begin{aligned} u^+_{in}(R, t) &= k\left[\log\left(e + k^{\frac{1-m}{2}} K^{\frac{1+m}{2}}\right) + p^m(t) \log^\lambda\left(e + k^{\frac{1-m}{2}} K^{\frac{1+m}{2}}\right)\right]^{\frac{1}{m}} \\ &> kp(t) = u^+_{out}(R, t) \end{aligned}$$

when $K \geq K_{1,2}$, where $K_{1,2} = K_{1,2}(m, \lambda, k)$ is a sufficiently large positive constant. Moreover, we have

$$\begin{aligned} \frac{\partial u^+_{in}}{\partial r} &= \frac{k}{m}\left[\varphi(r) + p^m(t)\varphi^\lambda(r)\right]^{\frac{1}{m}-1}\left[1 + \lambda p^m(t)\varphi^{\lambda-1}(r)\right]\varphi'(r), \\ \frac{\partial u^+_{out}}{\partial r} &= -\frac{2K}{1-m} p(t) r^{-\frac{2}{1-m}-1}, \end{aligned}$$

and for $r \in [0, 1]$ it holds that

$$\begin{aligned} \frac{\partial u^+_{out}}{\partial r} - \frac{\partial u^+_{in}}{\partial r} &= -\frac{Kp(t)r^{-1}}{er + K}\left[\frac{2}{1-m} r^{-\frac{2}{1-m}}(er + K)\right. \\ &\qquad \left. - \frac{k}{m}\left[p^{-m}(t)\varphi(r) + \varphi^\lambda(r)\right]^{\frac{1}{m}-1}\left[p^{-m}(t) + \lambda\varphi^{\lambda-1}(r)\right]\right] \\ &\leq -\frac{Kp(t)r^{-1}}{er + K}\left[\frac{2K}{1-m} r^{-\frac{2}{1-m}} - \frac{2^{\frac{1}{m}-1}(1+\lambda)k}{m}\varphi^{\frac{1}{m}-1}(r)\right] \\ &\leq 0, \end{aligned}$$

for $K \geq K_{1,3}$, where $K_{1,3} = K_{1,3}(m, \lambda, k) > 0$ is sufficiently large. We set $K_1 = \max\{K_{1,1}, K_{1,2}, K_{1,3}\}$ so that the first intersection point of the functions u^+_{in} and u^+_{out} satisfies the desired inequality. □

In the subsequent subsections we prove the following two lemmata.

Lemma 4 *Let $n = 2$ and assume that (A2) holds. Then there exist constants $K > 0$ and $0 < \varepsilon \ll 1$ such that if $|\xi'| \leq \varepsilon(1 + t)^{-1}$, then*

$$u^+(x, t) := \begin{cases} u^+_{in}(x, t) & \text{for } |x| \in (0, \rho_0(t)], \\ u^+_{out}(x, t) & \text{for } |x| \in (\rho_0(t), \infty), \end{cases}$$

is a supersolution for $|x| > 0$ and $t > 0$.

Lemma 5 *Let $n = 2$. Assume that (A2) holds and let $\Xi > 0$ be such that $|\xi'| \leq \Xi$. Then there exist constants $p_0, t_0 > 1$ such that*

$$u^-(x, t) := \begin{cases} u^-_{in}(x, t) & \text{for } |x| \in (0, \rho_1(t)], \\ u^-_{out}(x, t) & \text{for } |x| \in (\rho_1(t), \infty), \end{cases}$$

is a subsolution for $|x| > 0$ and $t > 0$.

3.1 Inner part of a supersolution

Computing the time derivative of u_{in}^+ we obtain $(u_{in}^+)_t \geq 0$. For $r \in [0, R]$ and $\lambda < 1$ we have

$$\begin{aligned}
-\Delta(u_{in}^+)^m &= -k^m(\Delta\varphi + p^m \Delta\varphi^\lambda) \\
&= \frac{k^m K r^{-2} \varphi^{\lambda-2}}{(er+K)^2}\left[\lambda p^m((1-\lambda)K - er\varphi) - er\varphi^{2-\lambda}\right] \\
&\geq \frac{k^m K r^{-2} \varphi^{\lambda-2} p^m}{(er+K)^2}\left[\lambda(1-\lambda)K - (1+\lambda)er\varphi^{2-\lambda}\right] \\
&\geq \frac{k^m K r^{-2} \varphi^{\lambda-2} p^m}{(er+K)^2}\left[\lambda(1-\lambda)K - 2er\varphi^2\right].
\end{aligned}$$

For $r \in [0, R]$, the function $r \mapsto r\log^2(e + Kr^{-1})$ is monotone increasing, provided that we choose K sufficiently large depending on m and k. Thus,

$$\lambda(1-\lambda)K - 2er\varphi^2 \geq \lambda(1-\lambda)K - 2eK^{\frac{1-m}{2}}k^{-\frac{1-m}{2}}\log^2\left(e + K^{\frac{1+m}{2}}k^{\frac{1-m}{2}}\right)$$

for $r \in [0, R]$. Since $(1-m)/2 < 1$, for $K \geq K_2 = K_2(m, k, \lambda)$ we have

$$-\Delta(u_{in}^+)^m \geq \frac{\lambda(1-\lambda)k^m K^2 r^{-2}\varphi^{\lambda-2}p^m}{2(er+K)^2} = \frac{c_1 K^2 r^{-2}\varphi^{\lambda-2}p^m}{(er+K)^2},$$

where $c_1 = 2^{-1}\lambda(1-\lambda)k^m$. Furthermore, it holds that

$$\begin{aligned}
-\xi' \cdot \nabla u_{in}^+ &= -(\xi' \cdot \omega)\frac{k}{m}\left(\varphi + p^m\varphi^\lambda\right)^{\frac{1}{m}-1}\left(1 + \lambda p^m \varphi^{\lambda-1}\right)\varphi' \\
&\geq -\frac{kKr^{-1}}{m(er+K)}|\xi'|\left(\varphi + p^m\varphi^\lambda\right)^{\frac{1}{m}-1}\left(1 + \lambda p^m\varphi^{\lambda-1}\right) \\
&\geq -\frac{2^{\frac{1}{m}-1}(1+\lambda)kKr^{-1}\varphi^{\frac{1}{m}-1}}{m(er+K)}|\xi'|p \\
&= -\frac{c_2 K r^{-1}\varphi^{\frac{1}{m}-1}}{er+K}|\xi'|p,
\end{aligned}$$

where $c_2 = 2^{\frac{1}{m}-1}(1+\lambda)km^{-1}$. Thus, we obtain

$$\begin{aligned}
&(u_{in}^+)_t - \Delta(u_{in}^+)^m - \xi' \cdot \nabla u_{in}^+ \\
&\geq \frac{Kr^{-2}\varphi^{\lambda-2}p}{(er+K)^2}\left[c_1 K p^{m-1} - c_2|\xi'|r(er+K)\varphi^{\frac{1}{m}+1-\lambda}\right] \\
&\geq 0
\end{aligned}$$

for $0 < r \leq R$ and $|\xi'| \leq \varepsilon(1+t)^{-1}$, where $\varepsilon = \varepsilon(m, \lambda, k, K, p_0) > 0$ is sufficiently small.

3.2 Outer part of a supersolution

Assuming that $r \in [1, \infty)$ we compute

$$-\xi' \cdot \nabla u^+_{out} = (\xi' \cdot \omega) \frac{2K}{1-m} p(t) r^{-\frac{2}{1-m}-1} \geq -|\xi'| \frac{2K}{1-m} p(t) r^{-\frac{2}{1-m}-1},$$

and the time derivative

$$(u^+_{out})_t = Kp'(t) r^{-\frac{2}{1-m}} = \frac{K}{1-m} p(t)(1+t)^{-1} r^{-\frac{2}{1-m}}.$$

Since $p_0 > 1$ we have

$$\begin{aligned} -\Delta(u^+_{out})^m &= -K^m p^m(t) \Delta r^{-\frac{2m}{1-m}} = -\frac{4m^2 K^m p_0^{m-1}}{(1-m)^2} p(t)(1+t)^{-1} r^{-\frac{2}{1-m}} \\ &\geq -\frac{4m^2 K^m}{(1-m)^2} p(t)(1+t)^{-1} r^{-\frac{2}{1-m}}. \end{aligned}$$

Assuming that $|\xi'| < \varepsilon(1+t)^{-1}$ and $K \geq K_3 = K_3(m)$ we obtain

$$\begin{aligned} &(u^+_{out})_t - \Delta(u^+_{out})^m - \xi' \cdot \nabla u^+_{out} \\ &\quad \geq \frac{K}{1-m} p(t) r^{-\frac{2}{1-m}} \left[(1+t)^{-1} - \frac{4m^2 K^{m-1}}{1-m}(1+t)^{-1} - 2|\xi'| r^{-1} \right] \\ &\quad \geq 0. \end{aligned}$$

3.3 Inner part of a subsolution

As in Sect. 2.4 we have $u^-_{in} > 0$ when $\varphi(r) > p_0^{\theta m}$ which is equivalent to

$$r < \tilde{r} := K\left(e^{p_0^{\theta m}} - e\right)^{-1}, \tag{11}$$

where θ is defined in (8). Throughout the rest of this subsection we assume that the above inequality (11) holds. We have $(u^-_{in})_t = 0$, and for $\lambda \in (0,1)$ we obtain

$$\begin{aligned} -\Delta(u^-_{in})^m &= -k^m \left(\Delta\varphi - p_0^m \Delta\varphi^\lambda \right) \\ &= -\frac{k^m K r^{-2}}{(er+K)^2} \left[K\lambda(1-\lambda) p_0^m \varphi^{\lambda-2} + er(1 - \lambda p_0^m \varphi^{\lambda-1}) \right] \\ &\leq -\frac{\lambda(1-\lambda) k^m K^2 p_0^m r^{-2} \varphi^{\lambda-2}}{(er+K)^2} \\ &= -\frac{c_1 K^2 p_0^m r^{-2} \varphi^{\lambda-2}}{(er+K)^2}, \end{aligned}$$

where $c_1 = \lambda(1-\lambda)k^m$. For $m \in (0,1]$ the inequality $\frac{1}{m} - 1 \geq 0$ yields

$$
\begin{aligned}
-\xi' \cdot \nabla u_{in}^{-} &= (\xi' \cdot \omega)\frac{kKr^{-1}}{m(er+K)}\left(\varphi - p_0^m \varphi^\lambda\right)^{\frac{1}{m}-1}(1 - \lambda p_0^m \varphi^{\lambda-1}) \\
&\le \frac{kKr^{-1}}{m(er+K)}\varphi^{\frac{1}{m}-1}|\xi'|.
\end{aligned}
$$

We denote $c_2 = k/m$ and assume that $|\xi'| \le \Xi$ for some $\Xi > 0$. Then for any $K > 0$ there exists $p_0 > 1$ sufficiently large such that

$$
\begin{aligned}
&(u_{in}^{-})_t - \Delta(u_{in}^{-})^m - \xi' \cdot \nabla u_{in}^{-} \\
&\le -\frac{Kr^{-2}\varphi^{\lambda-2}}{(er+K)^2}\left[c_1 K p_0^m - c_2|\xi'|r(er+K)\varphi^{1+\frac{1}{m}-\lambda}\right] \\
&\le 0
\end{aligned}
$$

for all $r \le \tilde{r}$.

3.4 Outer part of a subsolution

By choosing t_0 large, the intersection point $r = \rho_1(t)$ satisfies $\tilde{r}/2 \le \rho_1(t) \le \tilde{r}$ for any $t \ge 0$, where $\tilde{r}$ is given by (11). We fix such a t_0 and set

$$
\Phi(r,t) := 1 + A(t_0+t)^{-\beta} r^2, \quad \beta := \alpha(1-m) + 1
$$

Direct computations show that

$$
\begin{aligned}
(u_{out}^{-})_t &= \frac{A\beta}{1-m}(t_0+t)^{-\alpha-\beta-1} r^2 \Phi^{-\frac{1}{1-m}-1} - \alpha(t_0+t)^{-\alpha-1}\Phi^{-\frac{1}{1-m}}, \\
\Delta(u_{out}^{-})^m &= \frac{4mA^2}{(1-m)^2}(t_0+t)^{-\alpha m-2\beta} r^2 \Phi^{-\frac{1}{1-m}-1} \\
&\quad - \frac{4mA}{1-m}(t_0+t)^{-\alpha m-\beta}\Phi^{-\frac{1}{1-m}}, \\
\xi' \cdot \nabla u_{out}^{-} &= -(\xi' \cdot \omega)\frac{2A}{1-m}(t_0+t)^{-\alpha-\beta} r \Phi^{-\frac{1}{1-m}-1}.
\end{aligned}
$$

By $\beta = \alpha(1-m) + 1$, we have

$$
\begin{aligned}
&(u_{out}^{-})_t - \Delta(u_{out}^{-})^m - \xi' \cdot \nabla u_{out}^{-} \\
&= (t_0+t)^{-\alpha m-2\beta}\Phi^{-\frac{1}{1-m}-1}\left(F_0 + F_1 r + F_2 r^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
F_0 &:= \left(-\alpha + \frac{4mA}{1-m}\right)(t_0+t)^{\beta}, \\
F_1 &:= (\xi' \cdot \omega)\frac{2A}{1-m}(t_0+t), \\
F_2 &:= \frac{A}{1-m} - \frac{4m^2A^2}{(1-m)^2}.
\end{aligned}
$$

The assumption $|\xi'(t)| \le \varepsilon(1+t)^{-1}$ implies that

$$|F_1| \le \frac{2A\varepsilon(t_0+t)}{(1-m)(1+t)}.$$

Hence $|F_1|$ is bounded for each t_0. Then there exist large constants A and α satisfying (10) such that $F_0 + F_1 r + F_2 r^2 \le 0$ for $r \ge \tilde{r}/2$. This shows that

$$(u^-_{out})_t - \Delta (u^-_{out})^m - \xi' \cdot \nabla u^-_{out} \le 0$$

for $r \ge \tilde{r}/2$.

4 Completion of the proofs of Theorems 2 and 3

Proposition 1 *Let $n = 2$ and $k > 0$. Assume that (A2) holds and $0 < \lambda < 1$.*

(i) *When $m \in (0,1)$ then there is a constant $\varepsilon > 0$ such that if $|\xi'(t)| \le \varepsilon(1+t)^{-1}$ for $t \ge 0$ then there exist a supersolution u^+ and a subsolution u^- of (3) satisfying*

$$u^+(x,t)^m = k^m \psi(|x|) + O(\psi^\lambda(|x|)), \quad u^-(x,t)^m = k^m \psi(|x|) + O(\psi^\lambda(|x|))$$

for each $t \ge 0$ as $x \to 0$, and

$$u^-(x,t) \le u^+(x,t) \quad \text{for } |x| > 0 \text{ and } t \ge 0.$$

Moreover, if v_0 is as in Theorem 2 then

$$u^-(x,0) \le v_0(x+\xi(0)) \le u^+(x,0) \quad \text{for } |x| > 0.$$

(ii) *When $m \ge 1$ assume that $0 < \lambda < 1/m$ and $|\xi'| \le \Xi$ for some $\Xi > 0$. If v_0 is as in Theorem 3 then there are functions u^-, u^+ as in (i).*

Proof For $m \ge 1$ we choose u^+ and u^- as in Lemmata 1 and 2, and for $m \in (0,1)$ as in Lemmata 4 and 5, respectively. It is not difficult to see that one can change the constants b_0, d if $m \ge 1$ or p_0, t_0 if $0 < m < 1$ so that $v_0(x + \xi(0))$ is squeezed in between $u^-(x,0)$ and $u^+(x,0)$. □

The rest of the proofs of Theorems 2 and 3 is the same as in Section 5 in [1]. It was proved there that the existence of global-in-time comparison functions, i.e. super- and subsolution of (3) which are positive and bounded on each compact subset of $(\mathbb{R}^n \setminus \{0\}) \times (0,\infty)$, implies the existence of a global-in-time solution. Here, the local Hölder continuity of ξ' is needed.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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ANISOTROPIC AND ISOTROPIC PERSISTENT SINGULARITIES OF SOLUTIONS OF THE FAST DIFFUSION EQUATION

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Abstract. The aim of this paper is to study a class of positive solutions of the fast diffusion equation with specific persistent singular behavior. First, we construct new types of solutions with anisotropic singularities. Depending on parameters, either these solutions solve the original equation in the distributional sense, or they are not locally integrable in space-time. We show that the latter also holds for solutions with snaking singularities, whose existence has been proved recently by M. Fila, J.R. King, J. Takahashi, and E. Yanagida. Moreover, we establish that in the distributional sense, isotropic solutions whose existence was proved by M. Fila, J. Takahashi, and E. Yanagida in 2019, actually solve the corresponding problem with a moving Dirac source term. Last, we discuss the existence of solutions with anisotropic singularities in a critical case.

1. Introduction

Let $n \geq 2$ and $m \in (0,1)$. We study positive singular solutions of the fast diffusion equation

$$u_t = \Delta u^m, \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}, \quad t > 0, \tag{1.1}$$

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with an initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}. \tag{1.2}$$

Here, $\xi_0 \in \mathbb{R}^n$ is a given point at which solutions are singular, i.e.,

$$u(x,t) \to \infty \quad \text{as} \quad x \to \xi_0, \quad t > 0.$$

Let $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ denote the unit $(n-1)$-sphere and set

$$r := |x - \xi_0| \quad \text{and} \quad \omega := (x - \xi_0)/|x - \xi_0|. \tag{1.3}$$

Let $\lambda > 0$ and $\alpha \in C^{2,1}(S^{n-1} \times [0,\infty))$ be positive. The aim of this paper is to study positive solutions with the persistent singular behavior of the form

$$u(x,t) = \alpha(\omega,t) r^{-\lambda} + o(r^{-\lambda}) \quad \text{as} \quad r \to 0, \tag{1.4}$$

for $\omega \in S^{n-1}$ and $t \geq 0$. We say that if $\alpha(\omega,t)$ depends non-trivially on the space variable ω, the corresponding solution u has an anisotropic singularity, otherwise it is asymptotically radially symmetric.

Our main result formulated in Theorem 1.1 concerns the existence of solutions of (1.1)-(1.2) with anisotropic singularities. In order to prove the existence of such solutions, we introduce the following assumptions.

(A1) Let $\alpha \in C^2(S^{n-1})$ be positive.
(A2) Let $0 < m < 1$ and let λ, ν satisfy

$$\lambda > \frac{2}{1-m}, \qquad (1-m)\lambda - 2 - m(\lambda - \nu) > 0, \qquad \lambda > \nu > 0.$$

(A3) Let $u_0 \in C(\mathbb{R}^n \setminus \{\xi_0\})$ be positive and such that it has the asymptotic behavior

$$u_0^m(x) = \alpha^m(\omega)|x - \xi_0|^{-m\lambda} + O\big(|x - \xi_0|^{-m\nu}\big) \quad \text{as} \quad x \to \xi_0,$$

for each $\omega \in S^{n-1}$, and

$$C^{-1} \leq u_0(x) \leq C$$

for $|x - \xi_0| \geq 1$ and some constant $C > 1$.

Note that condition (A2) implies that ν is sufficiently close to λ.

Theorem 1.1. *Let $n \geq 2$ and assume* (A1), (A2), *and* (A3). *Then there is a function*

$$u \in C^{2,1}(\{(x,t) \in (\mathbb{R}^n \setminus \{\xi_0\}) \times (0,\infty)\}) \cap C(\{(x,t) \in (\mathbb{R}^n \setminus \{\xi_0\}) \times [0,\infty)\}),$$

which satisfies (1.1)-(1.2) *pointwise and*

$$u(x,t)^m = \alpha^m(\omega)|x - \xi_0|^{-m\lambda} + O(|x - \xi_0|^{-m\nu}) \quad \textit{as} \quad x \to \xi_0,$$

for each $\omega \in S^{n-1}$ and $t \geq 0$.

A subclass of solutions from Theorem 1.1 has been also studied in [14]. The authors of [14] focused on radially symmetric solutions of (1.1)-(1.2) with $n \geq 3$, $0 < m < m_c := (n-2)/n$ and with the initial condition $u_0(x) = (c_1^m|x-\xi_0|^{-m\lambda} + c_2^m)^{1/m}$, where $2/(1-m) < \lambda < (n-2)/m$, $c_1 > 0$, and $c_2 \geq 0$. In addition to the existence, several interesting properties of these solutions have been proved, among them their uniqueness.

In our next result, we show that, depending on parameters, solutions constructed in Theorem 1.1 either solve the original fast diffusion equation in the distributional sense, i.e.,

$$u_t = \Delta u^m \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times (0,\infty)), \tag{1.5}$$

or they are not locally integrable in space-time.

Theorem 1.2. *Let the conditions from Theorem* 1.1 *be satisfied.*

(i) *If $\lambda < n$, $0 < m < m_c$, and $n > 2$, then for the solution u from Theorem* 1.1 *it holds that $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^n))$, and it satisfies* (1.5) *in the distributional sense, i.e.,*

$$\int_0^\infty \int_{\mathbb{R}^n} \big(u\varphi_t + u^m \Delta\varphi\big)\, dy\, dt = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$.

(ii) *If $\lambda \geq n$, then the solution u from Theorem* 1.1 *satisfies $u \notin L^p_{loc}(\mathbb{R}^n \times [0,\infty))$ for any $p \geq 1$.*

We note that in the supercritical exponent range $m_c < m < 1$, the authors of [7] proved that all solutions of (1.5) with $u_0 \in L^1_{loc}(\mathbb{R}^n)$ become locally bounded and continuous for all $t > 0$.

A further related result concerning anisotropic singularities can be found in [4]. Here, the authors constructed positive entire-in-time solutions with snaking singularities for the fast diffusion equation (in the range $m^* < m < 1$ and $n \geq 2$, where $m^* := (n-3)/(n-1)$ when $n \geq 3$ and $m^* := 0$ when $n = 2$). In particular, these solutions have a singularity on a set $\Gamma(t) := \{\xi(s); -\infty < s < ct\}$ for $c > 0$ and each $t \in \mathbb{R}$. Here, $\xi : \mathbb{R} \to \mathbb{R}^n$ satisfies Condition 1.1 in [4]. Their construction was based on the existence of the following explicit singular traveling wave solution with cylindrical symmetry

$$U(x,t) = C\big(|a||x-ta| + a\cdot(x-ta)\big)^{-\frac{1}{1-m}}, \tag{1.6}$$

where a is a velocity vector, and C is an explicitly computable constant. As in Theorem 1.2 1.2, the solution U is also an example of a function with no local integrability in space-time. Namely, in Section 6, we show the following.

Remark 1.3. Let $n \geq 2$ and $m^* < m < 1$. Then for the function U from (1.6) it holds that $U \notin L^p_{loc}(\mathbb{R}^n \times \mathbb{R})$ for any $p \geq 1$.

To extend the idea of various possibilities of distributional solutions of the fast diffusion and porous medium equation, we present our last result in Theorem 1.5. Here, a class of asymptotically radially symmetric singular solutions satisfies the corresponding equation with a moving Dirac source term in the distributional sense. The existence of such solutions of the initial value problem

$$u_t = \Delta u^m, \qquad x \in \mathbb{R}^n \setminus \{\xi(t)\}, \quad t \in (0, \infty), \tag{1.7}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \setminus \{\xi(0)\}, \tag{1.8}$$

was established in Theorem 1.1 in [6]. Assuming $n \geq 3$ and $m > m_* := (n-2)/(n-1)$, the authors of [6] constructed singular solutions of (1.7)-(1.8), which for some given C^1 function $k(t)$ behave as

$$u^m(x,t) = k^m(t)|x-\xi(t)|^{-(n-2)} + o(|x-\xi(t)|^{-(n-2)}) \quad \text{as} \quad x \to \xi(t). \tag{1.9}$$

It was shown in [6] that m_* is a critical exponent for the existence of such solutions, and there are no such solutions if $m < m_*$. To construct global-in-time solutions of this form, suitable conditions on ξ', k, and k' were imposed.

Remark 1.4. The existence of solutions from [6] can be extended to the parameter range $n \geq 3$ and $m_c < m \leq m_*$ if the singularity is not moving, i.e., if $\xi(t) \equiv \xi_0$. This can be verified by an inspection of the proof of Theorem 1.1 in [6], which is in this case simpler since all terms containing ξ' vanish.

We also remark that the results from [6] have been extended previously in a different way in [5]. Here, the authors treated the case $n = 2$, $m > m_* = 0$. They established the existence of solutions that, near the singularity, behave like the fundamental solution of the Laplace equation to the power $1/m$.

In Theorem 1.5, we show that solutions from [6] satisfy

$$u_t = \Delta u^m + (n-2)|S^{n-1}|k^m(t)\delta_{\xi(t)}(x) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times (0,\infty)). \tag{1.10}$$

Here, $|S^{n-1}|$ is the surface area of the $(n-1)$-dimensional unit sphere and $\delta_{\xi(t)}$ denotes the Dirac measure on $\mathbb{R}^n$, giving unit mass to the point $\xi(t) \in \mathbb{R}^n$.

Theorem 1.5. *Let $n \geq 3$ and assume that conditions on $k(t)$ and $u_0(x)$ from Theorem 1.1 in [6] hold. Let either $m > m_*$ and $\xi(t)$ be as in Theorem 1.1* *in* [6]*, or* $m_c < m \le m_*$ *and* $\xi(t) \equiv \xi_0$*. Then the solution* u *satisfies equation* (1.10) *in the distributional sense, i.e.,*

$$-\int_0^\infty \int_{\mathbb{R}^n} \big(u\varphi_t + u^m \Delta\varphi\big)\, dx\, dt = \int_0^\infty (n-2)|S^{n-1}| k^m(t)\varphi(\xi(t), t)\, dt$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$*.*

Equations (1.1) and (1.7) with $n \ge 3$ have radially symmetric stationary solutions of the form

$$\tilde{u}(x) = K|x-\xi_0|^{-(n-2)/m}, \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}, \tag{1.11}$$

where K is an arbitrary positive constant, and these solutions satisfy

$$-\Delta\tilde{u}(x) = (n-2)|S^{n-1}|K\delta_{\xi_0}(x) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n). \tag{1.12}$$

Hence, the result of Theorem 1.5 can be expected. In [8], the authors constructed singular solutions with time-dependent singularities for the heat equation

$$u_t = \Delta u + w(t)\delta_{\xi(t)}(x) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times (0,T)),$$

where $n \ge 2$, $T \in (0,\infty]$, and $w \in L^1((0,t))$ for each $t \in (0,T)$. The behavior of solutions from [8] near the singularity does not always have to be like that of the fundamental solution of the Laplace equation, and the profile loses the asymptotic radial symmetry. Further results concerning the heat equation

$$u_t = \Delta u + \delta_{\xi(t)}(x) \otimes M(t) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times (0,T)),$$

where $\delta_{\xi(t)}(x) \otimes M(t)$ is a product measure of $M(t)$ and $\delta_{\xi(t)}(x)$, can be found in [9]. Solutions of the porous medium ($m > 1$) and fast diffusion equation in the supercritical range ($m_c < m < 1$) with singularities which are not necessarily standing were analyzed in [10] and [11], respectively. If M is a nonnegative Radon measure on $\mathbb{R}^{n+1}$, which satisfies $M(\Omega \times (0,T)) < \infty$ for $T > 0$ and a bounded domain $\Omega \subset \mathbb{R}^n$, then there exists a function u such that

$$u_t = \Delta u^m + M(x,t) \quad \text{in} \quad \mathcal{D}'(\Omega \times (0,T)),$$

and $u^m \in L^q((0,T); W_0^{1,q}(\Omega))$ with $1 < q < 1 + 1/(1+mn)$.

A moving Dirac measure on the right-hand side of parabolic systems also appears in several biological applications concerning, for example, the growth of axons or angiogenesis. See [3] and [1], respectively. A moving Dirac measure also appears in [12], where the authors studied the Cattaneo telegraph equation with a moving time-harmonic source in the context of the Doppler effect.

We also mention the following two results, which can be applied to solutions with anisotropic singularities. When $n \geq 3$, $0 < m < m_c$, and the singularity of the initial function satisfies $a_1|x - \xi_0|^{-2/(1-m)} \leq u_0(x) \leq a_2|x - \xi_0|^{-2/(1-m)}$ for some $a_1, a_2 > 0$ and for all $x \in \Omega$, then from [15] (for $\Omega = \mathbb{R}^n$) and [16] (for smoothly bounded domain $\Omega \subset \mathbb{R}^n$) it results that finite-time blow-down occurs. More specifically, there is a $T > 0$ such that $u(\cdot, t) \notin L^\infty(\Omega)$ for $t < T$ but $u(\cdot, t) \in L^\infty(\Omega)$ for $t > T$, i.e., that the singularity disappears after a time T. On the other hand, if m is in the range $m_c < m < 1$, the authors of [2] concluded the monotonicity of strongly singular sets of extended solutions, i.e., that it cannot shrink in time. Hence, the singularity of such solutions persists for all times.

This paper is organized as follows. A formal analysis of solutions with the asymptotic behavior (1.4) is given in Section 2. The last part of this section is devoted to a critical case that is left as an open problem. The existence result in Theorem 1.1 is then proved in Section 3. Formal computations in Section 2 suggest the choice of comparison functions in Subsections 3.1 and 3.2. We leave the question of extending the results from Theorem 1.1 from standing to moving singularities open. We see no problem in using the methods employed in this text, however, different critical exponents and technical difficulties may arise. We continue with the proof of Theorem 1.5 in Section 4, and the proof of Theorem 1.2 in Section 5. Finally, computations concerning Remark 1.3 are given in Section 6.

2. Formal computations

Let $u(x, t)$ be given by (1.4), i.e.,

$$u(x, t) = \alpha(\omega, t) r^{-\lambda} + o(r^{-\lambda}),$$

where $r > 0$, $\omega \in S^{n-1}$, $t \geq 0$, and recall notation (1.3) for r and ω. For $u(x, t) = w(r, \omega, t)$, the fast diffusion equation (1.1) is transformed into

$$w_t = r^{1-n} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial w^m}{\partial r} \right) + \frac{1}{r^2} \Delta_\omega w^m. \tag{2.1}$$

Here, Δ_ω denotes the Laplace-Beltrami operator on S^{n-1}. Simple computations show that

$$\begin{aligned} w_t &= \alpha_t r^{-\lambda} + o(r^{-\lambda}), \\ \Delta w^m &= \big(\Delta_\omega \alpha^m - m\lambda(n - 2 - m\lambda)\alpha^m\big) r^{-m\lambda-2} + o(r^{-m\lambda-2}). \end{aligned} \tag{2.2}$$

The leading term is different in each of the three cases: $\lambda > m\lambda + 2$, $\lambda < m\lambda + 2$, and $\lambda = m\lambda + 2$.

2.1. $\boldsymbol{\lambda > 2/(1-m)}$. The most singular case is $\lambda > m\lambda+2$, which is equivalent to $\lambda > 2/(1-m)$. This implies that the leading term in (2.2) is w_t, hence, we set $\alpha = \alpha(\omega)$. The existence result in Theorem 1.1 is based on this observation.

2.2. $\boldsymbol{\lambda < 2/(1-m)}$. The case $\lambda < m\lambda + 2$ is equivalent to $\lambda < 2/(1-m)$. The leading term in (2.2) is Δw^m, which implies that α must be a solution of

$$-\Delta_\omega \alpha^m = -m\lambda(n-2-m\lambda)\alpha^m.$$

Eigenvalues of $-\Delta_\omega$ are non-negative and start with zero (the constant 1 is the corresponding eigenfunction), other eigenfunctions change sign, see [13]. Since we are looking for positive solutions, we obtain conditions

$$\lambda = \frac{n-2}{m}, \quad m > m_c, \quad \text{and} \quad n \geq 3.$$

As we pointed out in the introduction, the existence of the corresponding asymptotically radially symmetric solutions for $n \geq 3$ and $m > m_*$ in the case of a moving singularity was established in Theorem 1.1 in [6]. Moreover, in Remark 1.4, we explain that in the case of a standing singularity, the proof of Theorem 1.1 in [6] is valid also in the parameter range $n \geq 3$, $m_c < m \leq m_*$. Our result extending the qualitative analysis of these solutions can be found in Theorem 1.5.

2.3. **Critical case $\boldsymbol{\lambda = 2/(1-m)}$, open problem.** In the critical case $\lambda = 2/(1-m)$, the terms w_t and Δw^m are balanced. Let

$$A := m\lambda(m\lambda - n + 2) = \frac{2mn(m-m_c)}{(1-m)^2}.$$

Balancing the leading terms in (2.2) leads us to an initial value problem

$$\alpha_t(\omega,t) = \Delta_\omega \alpha^m(\omega,t) + A\alpha^m(\omega,t), \qquad \omega \in S^{n-1}, \quad 0 < t < T, \tag{2.3}$$

$$\alpha(\omega,0) = \alpha_0(\omega) > 0, \qquad \omega \in S^{n-1}, \tag{2.4}$$

where $T \in (0,\infty]$. If we prove the existence of a positive classical solution α of (2.3)-(2.4) for some $T > 0$, we obtain a positive classical solution of (1.1)-(1.2) of the form

$$u(x,t) = \alpha(\omega,t)|x-\xi_0|^{-\lambda}, \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}, \quad 0 < t < T.$$

At the end of this section, we present some examples of solutions of (2.3)-(2.4). A well-known explicit solution is

$$\tilde\alpha(t) = \big((1-m)A\,t + t_0\big)^{\frac{1}{1-m}},$$

where t_0 is an arbitrary positive constant. In order to obtain solutions of (1.1)-(1.2) with an anisotropic singularity, we are interested in solutions of (2.3)-(2.4) that, unlike $\tilde{\alpha}$, depend non-trivially on the space variable ω. Such solutions can be obtained by looking for solutions of the form $\alpha(\omega,t) = \tau(t)\beta^{1/m}(\omega)$, where β is non-constant. Using the method of separation of variables, we have $\tau(t) = ((1-m)Ct + t_0)^{\frac{1}{1-m}}$, where $t_0 > 0$ and C is a constant from the separation of variables. For $\beta(\omega)$, we obtain a semilinear elliptic equation on a sphere

$$\Delta_\omega \beta(\omega) + A\beta(\omega) = C\beta^{1/m}(\omega), \qquad \omega \in S^{n-1}. \tag{2.5}$$

We briefly examine the existence of a class of solutions of (2.5) depending only on an angle $\theta \in [0, 2\pi)$. In this case, equation (2.5) becomes

$$\ddot{\beta}(\theta) + A\beta(\theta) = C\beta^{1/m}(\theta).$$

It represents a Hamiltonian system

$$\begin{cases} \dot{\beta} = v, \\ \dot{v} = -\beta(A - C\beta^{1/m-1}), \end{cases}$$

with a relevant critical point $P = ((A/C)^{\frac{m}{1-m}}, 0)$ if $C \neq 0$ has the same sign as $A \neq 0$. Notice that this condition guarantees the same asymptotic behavior of τ as that of $\tilde{\alpha}$, which is consistent with the results from [15, 2] described in the introduction. Finally, the existence of periodic trajectories results from a standard ODE theory: the critical point P is a center, i.e., all trajectories close to it are closed orbits if $A < 0$.

The existence of a more general class of classical positive solutions of (2.3)-(2.4), which depend non-trivially on ω, is left as an open problem.

3. PROOF OF THEOREM 1.1

3.1. **Construction of supersolutions.** We set $a(t) := Ae^{At}$, where $A \geq 1$ is a sufficiently large constant chosen later, and define a function

$$w^+(r,\omega,t) := \left(\alpha^m(\omega) r^{-m\lambda} + a(t) r^{-m\nu} + A\right)^{\frac{1}{m}}. \tag{3.1}$$

In what follows, we prove that w^+ is a supersolution of (2.1).

Lemma 3.1. *Let $n \geq 2$ and assume* (A1) *and* (A2)*. Then there exists constant $A \geq 1$, such that the function $w^+(r,\omega,t)$ defined in* (3.1) *is a supersolution of equation* (2.1) *for $r > 0$, $\omega \in S^{n-1}$, and $t > 0$.*

Proof. We define a bounded function

$$\sigma(\omega) := \Delta_\omega \alpha^m(\omega) + m\lambda(m\lambda - n + 2)\alpha^m(\omega)$$

and compute

$$\begin{aligned} w_t^+ &= \frac{1}{m} A a r^{-m\nu} \big(\alpha^m r^{-m\lambda} + a r^{-m\nu} + A\big)^{\frac{1}{m}-1}, \\ -\Delta(w^+)^m &= -\sigma r^{-m\lambda-2} - m\nu(m\nu - n + 2) a r^{-m\nu-2}. \end{aligned}$$

Since

$$\begin{aligned} &(1-m)\lambda - 2 - m(\lambda - \nu) > 0, \\ &\alpha \geq \alpha_{\min} := \min_{\omega \in S^{n-1}} \alpha(\omega) > 0, \\ &\sigma \leq \sigma_{\max} := \max_{\omega \in S^{n-1}} \sigma(\omega) < \infty, \end{aligned}$$

and $A \geq 1$, for $r \leq 1$, we obtain

$$\begin{aligned} w_t^+ - \Delta(w^+)^m &= \frac{1}{m} A a \big(\alpha^m + a r^{m(\lambda-\nu)} + A r^{m\lambda}\big)^{\frac{1}{m}-1} r^{-\lambda + m(\lambda-\nu)} \\ &\quad - \sigma r^{-m\lambda-2} - m\nu(m\nu - n + 2) a r^{-m\nu-2} \\ &\geq \frac{1}{m} A a \alpha^{1-m} r^{-m\lambda-2} - \sigma r^{-m\lambda-2} - m\nu(m\nu+2) a r^{-m\nu-2} \\ &\geq \left(\Big(\frac{1}{m} A \alpha_{\min}^{1-m} - m\nu(m\nu+2) \Big) a - \sigma_{\max} \right) r^{-m\lambda-2}. \end{aligned}$$

Thus, for

$$A \geq m\alpha_{\min}^{-(1-m)} (\sigma_{\max} + m\nu(m\nu+2)),$$

it holds that

$$w_t^+ - \Delta(w^+)^m \geq 0 \quad \text{for all } r \leq 1,\ \omega \in S^{n-1}, \text{ and } t > 0.$$

Similarly, for $r > 1$, $\omega \in S^{n-1}$, and $t > 0$ we have

$$\begin{aligned} w_t^+ - \Delta(w^+)^m &\geq \frac{1}{m} A^{\frac{1}{m}} a r^{-m\nu} - \sigma_{\max} r^{-m\lambda-2} - m\nu(m\nu - n + 2) a r^{-m\nu-2} \\ &\geq \left(\Big(\frac{1}{m} A - m\nu(m\nu - n + 2) \Big) a - \sigma_{\max} \right) r^{-m\nu-2}. \end{aligned}$$

This completes the proof that for any $A \geq 1$ sufficiently large, the function w^+ defined in (3.1) is a supersolution of (2.1) for $t > 0$ in the whole space. $\square$

3.2. **Construction of subsolutions.** Let $\mu > \lambda$ satisfy

$$m\mu(m\mu + 2 - n) \min_{\omega \in S^{n-1}} \alpha^m - \max_{\omega \in S^{n-1}} |\Delta_\omega(\alpha^m)| > 0. \tag{3.2}$$

Note that (3.2) implies $\mu > (n-2)/m$. Let $\delta > 0$ satisfy

$$0 < \delta < \frac{\lambda - \nu}{\mu - \nu},$$

and define

$$b(t) := b_0 e^{Bt}, \qquad \rho(t) := (1-\delta)^{\frac{1}{m(\lambda-\nu)}} b^{-\frac{1}{m(\lambda-\nu)}}(t),$$

where $b_0, B > 1$ are sufficiently large constants chosen later. We set

$$\begin{aligned} w_{in}^-(r,\omega,t) &:= \alpha(\omega) r^{-\lambda}(1 - b(t) r^{m(\lambda-\nu)})^{\frac{1}{m}}, \\ w_{out}^-(r,\omega,t) &:= \alpha(\omega)\delta^{\frac{1}{m}} \rho^{\mu-\lambda}(t) r^{-\mu}. \end{aligned}$$

Note that the zero point of w_{in}^- is $b^{-\frac{1}{m(\lambda-\nu)}}(t)$ and that w_{in}^- intersects w_{out}^- at $r = \rho(t) < 1$. Now, we can construct a subsolution of the form

$$w^-(r,\omega,t) = \begin{cases} w_{in}^-(r,\omega,t) & \text{for } r \le \rho(t),\ t \ge 0, \\ w_{out}^-(r,\omega,t) & \text{for } r > \rho(t),\ t \ge 0. \end{cases} \tag{3.3}$$

Lemma 3.2. *Let $n \ge 2$ and assume* (A1) *and* (A2)*. Then there exist constants $b_0, B > 1$, such that the function w^- defined in* (3.3) *is a subsolution of equation* (2.1) *for $r > 0$, $\omega \in S^{n-1}$, and $t > 0$.*

Proof. Inner part: Let $t > 0$. We consider the inner part $r \le \rho(t)$. Straightforward computations show that

$$\begin{aligned} (w_{in}^-)_t &= \frac{1}{m}\alpha r^{-\lambda}(1 - br^{m(\lambda-\nu)})^{\frac{1}{m}-1}(-b' r^{m(\lambda-\nu)}) \\ &= -\frac{1}{m} Bb\alpha r^{-\lambda+m(\lambda-\nu)}(1 - br^{m(\lambda-\nu)})^{\frac{1}{m}-1} \end{aligned}$$

and

$$\begin{aligned} \Delta(w_{in}^-)^m = {}& m\alpha^m((m\lambda + 2 - n)\lambda r^{-m\lambda-2} - (m\nu + 2 - n)\nu b r^{-m\nu-2}) \\ &+ (r^{-m\lambda-2} - br^{-m\nu-2})\Delta_\omega(\alpha^m). \end{aligned}$$

By the definition of ρ, we have

$$\begin{aligned} &(w_{in}^-)_t - \Delta(w_{in}^-)^m \\ &= -\frac{1}{m} Bb\alpha r^{-\lambda+m(\lambda-\nu)}(1 - br^{m(\lambda-\nu)})^{\frac{1}{m}-1} - (r^{-m\lambda-2} - br^{-m\nu-2})\Delta_\omega(\alpha^m) \end{aligned}$$

$$
\begin{aligned}
&- m\alpha^m((m\lambda+2-n)\lambda r^{-m\lambda-2} - (m\nu+2-n)\nu b r^{-m\nu-2})\\
&\le -\frac{1}{m}Bb\alpha r^{-\lambda+m(\lambda-\nu)}(1-b\rho^{m(\lambda-\nu)})^{\frac{1}{m}-1} + Cr^{-m\lambda-2} + Cbr^{-m\nu-2}\\
&= -\frac{1}{m}Bb\alpha r^{-\lambda+m(\lambda-\nu)}\delta^{\frac{1}{m}-1} + Cr^{-m\lambda-2} + Cbr^{-m\nu-2}
\end{aligned}
$$

for $r \le \rho$, where $C > 0$ is a constant independent of b. By $\alpha_{\min} > 0$, $\lambda > \nu$, $b > 1$, and $r \le \rho < 1$, we have

$$
\begin{aligned}
(w_{in}^-)_t - \Delta(w_{in}^-)^m &\le -\frac{1}{m}Bb\delta^{\frac{1}{m}-1}\alpha_{\min} r^{-\lambda+m(\lambda-\nu)} + Cbr^{-m\lambda-2}\\
&= -br^{-\lambda+m(\lambda-\nu)}\Big(\frac{1}{m}\delta^{\frac{1}{m}-1}\alpha_{\min}B - Cr^{(1-m)\lambda-2-m(\lambda-\nu)}\Big).
\end{aligned}
$$

Recall that $(1-m)\lambda - 2 - m(\lambda-\nu) > 0$. Thus,

$$
(w_{in}^-)_t - \Delta(w_{in}^-)^m \le -br^{-\lambda+m(\lambda-\nu)}\Big(\frac{1}{m}\delta^{\frac{1}{m}-1}\alpha_{\min}B - C\Big)
$$

for $r \le \rho$. Hence, by choosing $B > 1$ large, we conclude that w_{in}^- is a subsolution for $r \le \rho(t)$.

Matching condition: Since both w_{in}^- and w_{out}^- are of the separated form $\alpha(\omega)f(r,t)$, it is sufficient to check that

$$
\frac{\partial}{\partial r}(w_{in}^-)^m\Big|_{r=\rho(t)} < \frac{\partial}{\partial r}(w_{out}^-)^m\Big|_{r=\rho(t)}.
$$

By the definition of ρ and the choice of δ, we have

$$
\begin{aligned}
&\frac{\partial}{\partial r}(w_{out}^-)^m\Big|_{r=\rho(t)} - \frac{\partial}{\partial r}(w_{in}^-)^m\Big|_{r=\rho(t)}\\
&= -\alpha^m m\mu\delta\rho^{-m\lambda-1} - \alpha^m(-m\lambda\rho^{-m\lambda-1} + m\nu b\rho^{-m\nu-1})\\
&= \alpha^m m\rho^{-m\lambda-1}\Big(-\mu\delta + \lambda - \nu b\rho^{m(\lambda-\nu)}\Big)\\
&= \alpha^m m\rho^{-m\lambda-1}\big(-\mu\delta + \lambda - (1-\delta)\nu\big)\\
&= \alpha^m m\rho^{-m\lambda-1}\big(\lambda - \nu - (\mu-\nu)\delta\big) > 0.
\end{aligned}
$$

Outer part: Note that

$$
\rho'(t) = -\frac{1}{m(\lambda-\nu)}B\rho(t) \le 0.
$$

From this, it follows that

$$
(w_{out}^-)_t = \alpha\delta^{\frac{1}{m}}(\mu-\lambda)\rho^{\mu-\lambda-1}\rho' r^{-\mu} \le 0.
$$

By direct computations, we have

$$\Delta(w_{out}^-)^m = \alpha^m \delta \rho^{m(\mu-\lambda)} m\mu(m\mu+2-n) r^{-m\mu-2} + \delta \rho^{m(\mu-\lambda)} r^{-m\mu-2} \Delta_\omega(\alpha^m).$$

Then (3.2) implies

$$\begin{aligned} &(w_{out}^-)_t - \Delta(w_{out}^-)^m \\ &\le -\alpha^m \delta \rho^{m(\mu-\lambda)} m\mu(m\mu+2-n) r^{-m\mu-2} - \delta \rho^{m(\mu-\lambda)} r^{-m\mu-2} \Delta_\omega(\alpha^m) \\ &= -\delta \rho^{m(\mu-\lambda)} r^{-m\mu-2} \left(\alpha^m m\mu(m\mu+2-n) + \Delta_\omega(\alpha^m)\right) \\ &\le -\delta \rho^{m(\mu-\lambda)} r^{-m\mu-2} \Big(\alpha_{\min}^m m\mu(m\mu+2-n) - \max_{\omega\in S^{n-1}} |\Delta_\omega(\alpha^m)|\Big) \le 0. \end{aligned}$$

Hence, w_{out}^- is a subsolution for $r \ge \rho(t)$. □

3.3. **Completion of the proof of Theorem 1.1.**

Proposition 3.3. *Let* $n \ge 2$ *and assume* (A1), (A2), *and* (A3). *Then there exist a supersolution* w^+ *and a subsolution* w^- *of* (2.1), *which have for each* $\omega \in S^{n-1}$ *and* $t \ge 0$ *the asymptotic behavior*

$$\begin{aligned} w^+(r,\omega,t)^m &= \alpha^m(\omega) r^{-m\lambda} + O(r^{-m\nu}), \\ w^-(r,\omega,t)^m &= \alpha^m(\omega) r^{-m\lambda} + O(r^{-m\nu}) \end{aligned}$$

as $r \to 0$. *Moreover,*

$$w^-(r,\omega,t) \le w^+(r,\omega,t)$$

and

$$w^-(r,\omega,0) \le u_0(x) \le w^+(r,\omega,0)$$

for all $r > 0$, $\omega \in S^{n-1}$, *which are defined in* (1.3), *and* $t \ge 0$.

Proof. We choose w^+ and w^- as in Lemmata 3.1 and 3.2, respectively. Note that

$$\begin{aligned} w^+(r,\omega,0) &= \left(\alpha^m(\omega) r^{-m\lambda} + A r^{-m\nu} + A\right)^{\frac{1}{m}}, \\ w^-(r,\omega,0) &= \begin{cases} \alpha(\omega) r^{-\lambda} (1 - b_0 r^{m(\lambda-\nu)})^{\frac{1}{m}} & \text{for } r \le \rho(0), \\ \alpha(\omega) \delta^{\frac{1}{m}} \rho^{\mu-\lambda}(0) r^{-\mu} & \text{for } r > \rho(0). \end{cases} \end{aligned}$$

Moreover, $\rho(0) < 1$. Then by choosing A and b_0 sufficiently large and δ sufficiently small, we see that the function u_0 satisfying (A3) can be always squeezed in between comparison functions $w^-(r,\omega,0)$ and $w^+(r,\omega,0)$. □

Proof of Theorem 1.1. In Proposition 3.3, we proved the existence of a global-in-time sub- and supersolution of problem (2.1), which implies the existence of sub- and supersolution of (1.1) with the desired asymptotic behavior. The rest of the proof of Theorem 1.1 is the same as in Section 5 in [6]. Here, it was proved that the existence of global-in-time comparison functions, i.e., sub- and supersolution of (1.1), which are positive and bounded on each compact subset of $(\mathbb{R}^n \setminus \{\xi_0\}) \times (0, \infty)$, implies the existence of a global-in-time solution of (1.1)-(1.2). □

4. Proof of Theorem 1.5

Proof of Theorem 1.5. For simplicity, let $B_R := B_R(\xi(t))$ denote an open ball in $\mathbb{R}^n$ of radius R centered at $\xi(t)$. For $\varepsilon > 0$ let $\eta_\varepsilon \in C^2(\mathbb{R})$ be a nonnegative cut-off function such that $\eta_\varepsilon(r) \equiv 0$ for $r \leq \varepsilon$, $\eta_\varepsilon(r) \equiv 1$ for $r \geq 3\varepsilon$, $\eta''_\varepsilon \geq 0$ for $r \in [\varepsilon, 2\varepsilon]$, $\eta''_\varepsilon \leq 0$ for $r \in [2\varepsilon, 3\varepsilon]$, and $0 \leq \eta'_\varepsilon(r) \leq \eta'_\varepsilon(2\varepsilon) = \tilde{c}_1 \varepsilon^{-1}$ for some $\tilde{c}_1 > 0$ and $|\eta''_\varepsilon| \leq \tilde{c}_2 \varepsilon^{-2}$ for some $\tilde{c}_2 > 0$.

Let u be from Theorem 1.1 in [6], that means that u is a classical solution of (1.7)-(1.8) such that $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^n))$. Let $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and set $\varphi_\varepsilon(x,t) := \eta_\varepsilon(|x - \xi(t)|)\varphi(x,t)$. Without loss of generality, we may assume that there is a nonempty open time interval $I \subset (0,\infty)$ such that $\xi(t) \in \operatorname{supp}\varphi(\cdot,t)$ for all $t \in I$. We can fix ε sufficiently small so that $B_{3\varepsilon} \subset \operatorname{supp}\varphi(\cdot,t)$ for all $t \in I$. Multiplying now equation (1.7) by φ_ε and integrating it over $\mathbb{R}^n \times (0,\infty)$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} u_t \varphi_\varepsilon \, dx\, dt = \int_0^\infty \int_{\mathbb{R}^n} \Delta u^m \varphi_\varepsilon \, dx\, dt. \tag{4.1}$$

Let us denote

$$I_\varepsilon := \int_{B_{3\varepsilon}\setminus B_\varepsilon} u^m \eta_\varepsilon \Delta\varphi \, dx, \quad J_\varepsilon := 2\int_{B_{3\varepsilon}\setminus B_\varepsilon} u^m \nabla\eta_\varepsilon \cdot \nabla\varphi \, dx,$$
$$K_\varepsilon := \int_{B_{3\varepsilon}\setminus B_\varepsilon} u^m \varphi \Delta\eta_\varepsilon \, dx, \quad H_\varepsilon := \int_{B_{3\varepsilon}\setminus B_\varepsilon} u \eta_\varepsilon \varphi_t \, dx.$$

Since φ is smooth and compactly supported in $\mathbb{R}^n \times (0,\infty)$, integrating the right-hand side of (4.1) by parts, we have

$$\int_0^\infty \int_{\mathbb{R}^n} \Delta u^m \varphi_\varepsilon \, dx\, dt = \int_0^\infty \int_{\mathbb{R}^n \setminus B_{3\varepsilon}} u^m \Delta\varphi \, dx\, dt + \int_0^\infty (I_\varepsilon + J_\varepsilon + K_\varepsilon)\, dt. \tag{4.2}$$

Similarly, we analyze the left-hand side of (4.1) and obtain

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} u_t \varphi_\varepsilon \, dx \, dt &= -\int_0^\infty \int_{\mathbb{R}^n} u(\varphi_\varepsilon)_t \, dx \, dt \\ &= -\int_0^\infty \int_{\mathbb{R}^n \setminus B_{3\varepsilon}} u\varphi_t \, dx \, dt - \int_0^\infty H_\varepsilon \, dt.\end{aligned}$$

Hence, equation (4.1) can be written as

$$-\int_0^\infty \int_{\mathbb{R}^n \setminus B_{3\varepsilon}} \left(u\varphi_t + u^m \Delta\varphi\right) dx \, dt = \int_0^\infty \left(H_\varepsilon + I_\varepsilon + J_\varepsilon + K_\varepsilon\right) dt.$$

In the following, we show that

$$H_\varepsilon + I_\varepsilon + J_\varepsilon + K_\varepsilon \to (n-2)|S^{n-1}| k^m(t) \varphi(\xi(t), t) \tag{4.3}$$

locally uniformly for t as $\varepsilon \to 0$. In order to do that, we choose ε sufficiently small so that the method of sub- and supersolutions in [6] provides estimates of the form

$$\begin{aligned} u^m(x,t) &\le k^m(t) \left(|x - \xi(t)|^{2-n} + b(t)|x - \xi(t)|^{-\lambda} \right), \\ u^m(x,t) &\ge k^m(t) \left(|x - \xi(t)|^{2-n} - b(t)|x - \xi(t)|^{-\lambda} \right)_+, \end{aligned} \tag{4.4}$$

for all $(x,t) \in B_{3\varepsilon} \times I$. Here, $b(t) = b_0 e^{Bt}$ for some constants B, $b_0 > 1$ and $\lambda < n-2$. First, we deal with the integrals $H_\varepsilon, I_\varepsilon, J_\varepsilon$ and show that they converge to zero locally uniformly for t. In what follows, by c, we will denote a large enough but otherwise arbitrary constant independent of t and ε. Given that $|\eta_\varepsilon| \le 1$, for $m > m_c$ and $t < T$ for some $T > 0$ it holds that

$$\begin{aligned} |H_\varepsilon| &= \Big| \int_{B_{3\varepsilon} \setminus B_\varepsilon} u \eta_\varepsilon \varphi_t \, dx \Big| \le \sup_{B_{3\varepsilon} \setminus B_\varepsilon} |\varphi_t| \int_{B_{3\varepsilon} \setminus B_\varepsilon} u \, dx \\ &\le c \int_\varepsilon^{3\varepsilon} \left(r^{2-n} + b(t) r^{-\lambda}\right)^{\frac{1}{m}} r^{n-1} \, dr \\ &\le c \int_\varepsilon^{3\varepsilon} r^{\frac{2-n}{m} + n - 1} \, dr \to 0 \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

Similarly,

$$\begin{aligned} |I_\varepsilon| &= \Big| \int_{B_{3\varepsilon} \setminus B_\varepsilon} u^m \eta_\varepsilon \Delta\varphi \, dx \Big| \le \sup_{B_{3\varepsilon} \setminus B_\varepsilon} |\Delta\varphi| \int_{B_{3\varepsilon} \setminus B_\varepsilon} u^m \, dx \\ &\le c \int_\varepsilon^{3\varepsilon} \left(r^{2-n} + b(t) r^{-\lambda}\right) r^{n-1} \, dr \end{aligned}$$

$$\leq c \int_{\varepsilon}^{3\varepsilon} r\,dr \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Moreover, using $0 \leq \eta_\varepsilon' \leq \tilde{c}_1 \varepsilon^{-1}$, we obtain

$$\begin{aligned}
|J_\varepsilon| &= \Big| 2 \int_{B_{3\varepsilon} \setminus B_\varepsilon} u^m \nabla \eta_\varepsilon \cdot \nabla \varphi \, dx \Big| \\
&\leq 2\tilde{c}_1 \varepsilon^{-1} \sup_{B_{3\varepsilon} \setminus B_\varepsilon} |\omega \cdot \nabla \varphi| \int_{B_{3\varepsilon} \setminus B_\varepsilon} u^m \, dx \\
&\leq c\,\varepsilon^{-1} \int_{\varepsilon}^{3\varepsilon} \big(r^{2-n} + b(t) r^{-\lambda} \big) r^{n-1} \, dr \\
&\leq c\varepsilon^{-1} \int_{\varepsilon}^{3\varepsilon} r\,dr \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Now, we deal with the integral K_ε. Denoting

$$\begin{aligned}
K_\varepsilon^1 &:= \int_{B_{3\varepsilon} \setminus B_\varepsilon} u^m \varphi |x - \xi(t)|^{-1} \eta_\varepsilon' \, dx, \\
K_\varepsilon^2 &:= \int_{B_{2\varepsilon} \setminus B_\varepsilon} u^m \varphi \eta_\varepsilon'' \, dx, \\
K_\varepsilon^3 &:= \int_{B_{3\varepsilon} \setminus B_{2\varepsilon}} u^m \varphi (-\eta_\varepsilon)'' \, dx,
\end{aligned}$$

we can split K_ε into

$$K_\varepsilon = (n-1) K_\varepsilon^1 + K_\varepsilon^2 - K_\varepsilon^3.$$

By means of the non-negativity of u^m and the properties $\eta_\varepsilon' \geq 0$, $\eta_\varepsilon'' \geq 0$ on $B_{2\varepsilon} \setminus B_\varepsilon$, and $\eta_\varepsilon'' \leq 0$ on $B_{3\varepsilon} \setminus B_{2\varepsilon}$, we obtain

$$\begin{aligned}
&\inf_{B_{3\varepsilon} \setminus B_\varepsilon} \varphi \int_{B_{3\varepsilon} \setminus B_\varepsilon} u^m |x - \xi(t)|^{-1} \eta_\varepsilon' \, dx \leq K_\varepsilon^1 \\
&\qquad \leq \sup_{B_{3\varepsilon} \setminus B_\varepsilon} \varphi \int_{B_{3\varepsilon} \setminus B_\varepsilon} u^m |x - \xi(t)|^{-1} \eta_\varepsilon' \, dx, \\
&\inf_{B_{2\varepsilon} \setminus B_\varepsilon} \varphi \int_{B_{2\varepsilon} \setminus B_\varepsilon} u^m \eta_\varepsilon'' \, dx \leq K_\varepsilon^2 \leq \sup_{B_{2\varepsilon} \setminus B_\varepsilon} \varphi \int_{B_{2\varepsilon} \setminus B_\varepsilon} u^m \eta_\varepsilon'' \, dx, \\
&\inf_{B_{3\varepsilon} \setminus B_{2\varepsilon}} \varphi \int_{B_{3\varepsilon} \setminus B_{2\varepsilon}} u^m (-\eta_\varepsilon)'' \, dx \leq K_\varepsilon^3 \leq \sup_{B_{3\varepsilon} \setminus B_{2\varepsilon}} \varphi \int_{B_{3\varepsilon} \setminus B_{2\varepsilon}} u^m (-\eta_\varepsilon)'' \, dx.
\end{aligned}$$

We only consider the case where $\sup\limits_{B_{3\varepsilon}\setminus B_\varepsilon} \varphi$ and $\inf_{B_{3\varepsilon}\setminus B_\varepsilon} \varphi$ are non-negative. The other cases can be handled in the same way by using (4.5) below. Indeed, a change of the sign of $\sup_{B_{3\varepsilon}\setminus B_\varepsilon} \varphi$ and $\inf_{B_{3\varepsilon}\setminus B_\varepsilon} \varphi$ will not change the limit value.

For $a := n-2-\lambda > 0$, we set

$$\begin{aligned} L_\varepsilon^1 &:= b(t)\int_\varepsilon^{2\varepsilon} r^{a+1}\eta_\varepsilon''\,dr, \\ L_\varepsilon^2 &:= b(t)\int_{2\varepsilon}^{3\varepsilon} r^{a+1}(-\eta_\varepsilon)''\,dr, \\ L_\varepsilon^3 &:= b(t)\int_\varepsilon^{3\varepsilon} r^{a}\eta_\varepsilon'\,dr. \end{aligned}$$

Since $0 \le \eta_\varepsilon' \le \tilde{c}_1\varepsilon^{-1}$, for $t < T$ with some $T > 0$ it holds that

$$|L_\varepsilon^3| = \left| b(t)\int_\varepsilon^{3\varepsilon} r^{a}\eta_\varepsilon'\,dr \right| \le c\varepsilon^a,$$

and by $|\eta_\varepsilon''| \le \tilde{c}_2\varepsilon^{-2}$, we have

$$|L_\varepsilon^1| + |L_\varepsilon^2| = \left| b(t)\int_\varepsilon^{2\varepsilon} r^{a+1}\eta_\varepsilon''\,dr \right| + \left| b(t)\int_{2\varepsilon}^{3\varepsilon} r^{a+1}(-\eta_\varepsilon)''\,dr \right| \le c\varepsilon^a.$$

Hence,

$$L_\varepsilon^1 \to 0, \quad L_\varepsilon^2 \to 0, \quad L_\varepsilon^3 \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{4.5}$$

locally uniformly for t. Using inequalities (4.4), we have

$$\begin{aligned} K_\varepsilon^1 &\le |S^{n-1}| k^m(t) \sup_{B_{3\varepsilon}\setminus B_\varepsilon} \varphi \int_\varepsilon^{3\varepsilon} \big(1 + b(t) r^a\big)\eta_\varepsilon'\,dr \\ &= |S^{n-1}| k^m(t) \sup_{B_{3\varepsilon}\setminus B_\varepsilon} \varphi \left(1 + L_\varepsilon^3\right), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} K_\varepsilon^1 &\ge |S^{n-1}| k^m(t) \inf_{B_{3\varepsilon}\setminus B_\varepsilon} \varphi \int_\varepsilon^{3\varepsilon} \big(1 - b(t) r^a\big)\eta_\varepsilon'\,dr \\ &= |S^{n-1}| k^m(t) \inf_{B_{3\varepsilon}\setminus B_\varepsilon} \varphi \left(1 - L_\varepsilon^3\right). \end{aligned} \tag{4.7}$$

Integrating by parts and estimating integrals K_ε^2 and K_ε^3, we have

$$\begin{aligned}K_\varepsilon^2 &\le |S^{n-1}|k^m(t)\sup_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\int_\varepsilon^{2\varepsilon}\big(r+b(t)r^{a+1}\big)\eta_\varepsilon''\,dr\\ &=|S^{n-1}|k^m(t)\sup_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\left(2\varepsilon\eta_\varepsilon'(2\varepsilon)-\eta_\varepsilon(2\varepsilon)+L_\varepsilon^1\right),\end{aligned}$$

and

$$\begin{aligned}K_\varepsilon^3 &\ge |S^{n-1}|k^m(t)\inf_{B_{3\varepsilon}\setminus B_{2\varepsilon}}\varphi\int_{2\varepsilon}^{3\varepsilon}\big(r-b(t)r^{a+1}\big)(-\eta_\varepsilon)''\,dr\\ &=|S^{n-1}|k^m(t)\inf_{B_{3\varepsilon}\setminus B_{2\varepsilon}}\varphi\left(2\varepsilon\eta_\varepsilon'(2\varepsilon)+1-\eta_\varepsilon(2\varepsilon)-L_\varepsilon^2\right).\end{aligned}$$

Thus,

$$\begin{aligned}K_\varepsilon^2-K_\varepsilon^3 \le -|S^{n-1}|k^m(t)\Big[&(1-L_\varepsilon^2)\inf_{B_{3\varepsilon}\setminus B_{2\varepsilon}}\varphi-L_\varepsilon^1\sup_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\\ &-\Big(\sup_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi-\inf_{B_{3\varepsilon}\setminus B_{2\varepsilon}}\varphi\Big)\big(2\varepsilon\eta_\varepsilon'(2\varepsilon)-\eta_\varepsilon(2\varepsilon)\big)\Big].\end{aligned}\tag{4.8}$$

Analogously,

$$\begin{aligned}K_\varepsilon^2 &\ge |S^{n-1}|k^m(t)\inf_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\int_\varepsilon^{2\varepsilon}\big(r-b(t)r^{a+1}\big)\eta_\varepsilon''\,dr\\ &=|S^{n-1}|k^m(t)\inf_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\left(2\varepsilon\eta_\varepsilon'(2\varepsilon)-\eta_\varepsilon(2\varepsilon)-L_\varepsilon^1\right),\end{aligned}$$

and

$$\begin{aligned}K_\varepsilon^3 &\le |S^{n-1}|k^m(t)\sup_{B_{3\varepsilon}\setminus B_{2\varepsilon}}\varphi\int_{2\varepsilon}^{3\varepsilon}\big(r+b(t)r^{a+1}\big)(-\eta_\varepsilon)''\,dr\\ &=|S^{n-1}|k^m(t)\sup_{B_{3\varepsilon}\setminus B_{2\varepsilon}}\varphi\left(2\varepsilon\eta_\varepsilon'(2\varepsilon)+1-\eta_\varepsilon(2\varepsilon)+L_\varepsilon^2\right).\end{aligned}$$

Hence,

$$\begin{aligned}K_\varepsilon^2-K_\varepsilon^3 \ge -|S^{n-1}|k^m(t)\Big[&(1+L_\varepsilon^2)\sup_{B_{3\varepsilon}\setminus B_{2\varepsilon}}\varphi+L_\varepsilon^1\inf_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\\ &-\Big(\inf_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi-\sup_{B_{3\varepsilon}\setminus B_{2\varepsilon}}\varphi\Big)\big(2\varepsilon\eta_\varepsilon'(2\varepsilon)-\eta_\varepsilon(2\varepsilon)\big)\Big].\end{aligned}\tag{4.9}$$

Finally, using (4.5), inequalities (4.6), (4.7), (4.8), and (4.9) yield

$$K_\varepsilon\to(n-2)|S^{n-1}|k^m(t)\varphi(\xi(t),t)\quad\text{as}\quad\varepsilon\to0$$

locally uniformly for t. Thus, the assertion (4.3) is proved, which completes the proof. $\square$

5. Proof of Theorem 1.2

Proof of Theorem 1.2 (i). The functions from Theorem 1.1 satisfy $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^n))$. The proof of this statement is analogous to the proof of Lemma 5.2 in [6], and so we omit it here.

Since in this case $\lambda < (n-2)/m$ by $\lambda < n$, it necessarily holds that the singularity of solutions from Theorem 1.1 is weaker than the singularity of solutions of the type (1.11) and (1.9). It suggests that in the distributional sense, unlike solutions of the type (1.11) and (1.9) satisfy equations (1.12) and (1.10) with a singular source term, solutions from Theorem 1.2 (i) satisfy equation (1.5) with no source term on the right-hand side. Rigorously, the proof of this theorem can be carried out analogously to the proof of Theorem 1.5 in Section 4 (and it is less technical due to standing versus moving singularity). $\square$

Proof of Theorem 1.2 (ii). To prove that $u \notin L^p_{loc}(\mathbb{R}^n \times [0,\infty))$ for any $p \geq 1$ if $\lambda \geq n$, we integrate over $B_1(\xi_0) \times [0,1]$, and use the comparison function w^- to estimate the integral

$$I := \int_0^1 \int_{B_1(\xi_0)} u^p(x,t)\,dx\,dt \geq \int_0^1 \int_{B_1(\xi_0)} (w^-(r,\omega,t))^p\,dx\,dt.$$

By the definition of $w^- \geq 0$ in (3.3), $\rho(t) < 1$, and $\alpha \geq \alpha_{\min} > 0$, we have

$$\begin{aligned} I &\geq \int_0^1 \int_{B_{\rho(t)}(\xi_0)} (w^-(r,\omega,t))^p\,dx\,dt \\ &\geq \alpha_{\min}^p \int_0^1 \int_0^{\rho(t)} r^{n-1-p\lambda}\big(1-b(t)r^{m(\lambda-\nu)}\big)^{\frac{p}{m}}\,dr\,dt. \end{aligned}$$

Substituting $z = b(t)r^{m(\lambda-\nu)}$, by $b \geq 1$, we obtain

$$\begin{aligned} I &\geq \frac{\alpha_{\min}^p}{m(\lambda-\nu)} \int_0^1 b^{\frac{p\lambda-n}{m(\lambda-\nu)}}(t) \int_0^{1-\delta} z^{-1-\frac{p\lambda-n}{m(\lambda-\nu)}}(1-z)^{\frac{p}{m}}\,dz\,dt \\ &\geq \frac{\alpha_{\min}^p \delta^{\frac{p}{m}}}{m(\lambda-\nu)} \int_0^1 \int_0^{1-\delta} z^{-1-\frac{p\lambda-n}{m(\lambda-\nu)}}\,dz\,dt. \end{aligned}$$

This integral is infinite exactly when $\lambda \geq n$ for any $p \geq 1$, which completes the proof. $\square$

6. Non-integrability of the singular traveling wave

In this section, we show that for U from (1.6) it holds that $U \notin L^p_{loc}(\mathbb{R}^n \times \mathbb{R})$ for any $p \geq 1$. Without loss of generality, we may take $a = e_n = (0, \dots, 0, 1)$. Indeed, given any velocity vector $a \in \mathbb{R}^n$, we could transform the coordinate system and proceed as below. Let $B_1' := \{x' \in \mathbb{R}^{n-1}; |x'| < 1\}$. For $p \geq 1$, we examine the integrability over $[0,1] \times B_1' \times [-1, 0]$, i.e.,

$$\begin{aligned} I :&= \int_0^1 \int_{B_1' \times [-1,0]} U^p(x,t)\, dx\, dt \\ &= \int_0^1 \int_{-1}^0 \int_{B_1'} C^p \big(\sqrt{|x'|^2 + (x_n - t)^2} + (x_n - t) \big)^{-\frac{p}{1-m}} dx'\, dx_n\, dt. \end{aligned}$$

By the change of variables $y_n = -(x_n - t)$ and $|x'| = y_n r$, we obtain

$$\begin{aligned} I &= C^p |S^{n-2}| \int_0^1 \int_{1+t}^t \int_0^{1/y_n} (y_n r)^{n-2} \big(\sqrt{y_n^2 r^2 + y_n^2} - y_n \big)^{-\frac{p}{1-m}} y_n dr (-dy_n) dt, \\ &= C^p |S^{n-2}| \int_0^1 \int_t^{1+t} y_n^{n-1-\frac{p}{1-m}} \int_0^{1/y_n} r^{n-2} \big(\sqrt{r^2+1} - 1 \big)^{-\frac{p}{1-m}} dr\, dy_n dt, \\ &= C^p |S^{n-2}| \int_0^1 \int_t^{1+t} y_n^{n-1-\frac{p}{1-m}} \int_0^{1/y_n} r^{n-2-\frac{2p}{1-m}} \big(\sqrt{r^2+1} + 1 \big)^{\frac{p}{1-m}} dr\, dy_n dt. \end{aligned}$$

We have $1/y_n \geq 1/2$, hence

$$I \geq C^p \int_0^1 \int_1^{1+t} y_n^{n-1-\frac{p}{1-m}} \int_0^{1/2} r^{n-2-\frac{2p}{1-m}} dr\, dy_n\, dt,$$

which is finite if $p < (1-m)(n-1)/2$. Since we assumed that $p \geq 1$, $m > (n-3)/(n-1) = m^*$ for $n \geq 3$ and $m > 0 = m^*$ for $n = 2$, this condition for p cannot be satisfied. This implies the conclusion.

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FAST DIFFUSION EQUATION: UNIQUENESS OF SOLUTIONS WITH A MOVING SINGULARITY

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Abstract. We focus on open questions regarding the uniqueness of distributional solutions of the fast diffusion equation (FDE) with a given source term. When the source is sufficiently smooth, the uniqueness follows from standard results. Assuming that the source term is a measure, the existence of different classes of solutions is known, but in many cases, their uniqueness is an open problem. In our work, we focus on the supercritical FDE and prove the uniqueness of distributional solutions with a Dirac source term that moves along a prescribed curve. Moreover, we extend a uniqueness results for the subcritical FDE from standing to moving singularities.

1. Introduction

Let $0 < m < 1$, $n \geq 3$, and $0 < T \leq \infty$. We study the uniqueness of distributional solutions of the fast diffusion equation

$$u_t = \Delta u^m + f(x,t), \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^n \times (0,T)), \tag{1}$$

where f is a given source term. More specifically, we are interested in solutions of (1) that satisfy $u \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ and the integral equality

$$\int_0^T \int_{\mathbb{R}^n} \big(u\varphi_t + u^m \Delta\varphi + f\varphi\big)\, dx\, dt = 0 \tag{2}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,T))$. If, moreover, $\nabla u^m \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ and u satisfies

$$\int_0^T \int_{\mathbb{R}^n} \big(u\varphi_t - \nabla u^m \cdot \nabla\varphi + f\varphi\big)\, dx\, dt = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,T))$ then we call it a weak solution of (1).

Some techniques to prove uniqueness of solutions of (1) can be found in the book [16] by Vázquez. Focusing on weak solutions and assuming that $u \in L^2_{loc}(\mathbb{R}^n \times (0,T))$, $u^m \in$

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$L^2_{loc}(0,T;H^1_0(\mathbb{R}^n))$, and $f \in L^1_{loc}(\mathbb{R}^n \times (0,T))$, one can use a test function φ of the form

$$\varphi(x,t) = \begin{cases} \int_t^T \big(u_1^m(x,s) - u_2^m(x,s)\big)ds & \text{if } 0 < t < T, \\ 0 & \text{if } t \geq T, \end{cases}$$

which was introduced by Oleinik [12].

The critical exponent $m_c := (n-2)/n$ plays an important role in the theory of the fast diffusion equation. See, for example Vázquez [17]. In [11], Lukkari studies solutions of the fast diffusion equation in the range $m_c < m < 1$ with Ω instead of $\mathbb{R}^n$, where Ω is a bounded domain with a smooth boundary. Assuming that the forcing term f is a non-negative Radon measure on $\mathbb{R}^{n+1}$ such that $f(\Omega \times (0,T)) < \infty$, he proves the existence of a specific class of weak solutions of (1) in cylinders of the form $\Omega \times (0,T)$. These solutions satisfy $u \in L^q((0,T);W_0^{1,q}(\Omega))$, where q is any number such that $1 \leq q < 1 + 1/(1+mn)$. Since the upper bound on q is always less than 2, Lukkari's weak solutions lack the L^2-integrability conditions assumed by Vázquez in [16], hence, their uniqueness was left as an open problem.

A standard uniqueness result for $0 < m < 1$ by Herrero and Pierre can be found in [8]. Here, the authors prove the uniqueness of distributional solutions of the signed fast diffusion equation, i.e.

$$u_t = \Delta(u|u|^{m-1}) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times (0,\infty)), \tag{3}$$

assuming that $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^n))$ and the time derivative satisfies $u_t \in L^1_{loc}(\mathbb{R}^n \times (0,\infty))$.

More recently, new results concerning uniqueness of subcritical fast diffusion have been found. In [15], Takahashi and Yamamoto focused on the case when $n \geq 3$ and $0 < m < m_c$. They showed the uniqueness of signed solutions of the initial value problem

$$u_t = \Delta(u|u|^{m-1}), \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}, \quad t \in (0,T), \tag{4}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \setminus \{\xi_0\}, \tag{5}$$

with $0 < T \leq \infty$ and $\xi_0 \in \mathbb{R}^n$. More precisely, the authors of [15] proved that for two functions u_1, u_2 such that $u_1, u_2 \in C^{2,1}((\mathbb{R}^n \setminus \{\xi_0\}) \times (0,T)) \cap C((\mathbb{R}^n \setminus \{\xi_0\}) \times [0,T))$ that satisfy (4)-(5) pointwise and $u_1(\cdot,0) = u_2(\cdot,0)$ on $\mathbb{R}^n \setminus \{\xi_0\}$, it holds that $u_1 \equiv u_2$ on $(\mathbb{R}^n \setminus \{\xi_0\}) \times (0,T)$. Hui demonstrated in [9] that if $n \geq 3$ and $0 < m < m_c$, under suitable conditions on initial data, solutions that have a finite number of standing singularities are also uniquely determined. By solutions with finitely many standing singularities, we mean that these solutions satisfy equations (4)-(5) with $\mathbb{R}^n \setminus \{\xi_0, \xi_1, \dots, \xi_i\}$ instead of $\mathbb{R}^n \setminus \{\xi_0\}$. Here, $i \in \mathbb{N}$ and $\xi_0, \xi_1, \dots, \xi_i \in \mathbb{R}^n$.

More generally, we can assume that $\xi : [0,T) \to \mathbb{R}^n$ is a given curve and study the problem

$$u_t = \Delta(u|u|^{m-1}), \qquad x \in \mathbb{R}^n \setminus \{\xi(t)\}, \quad t \in (0,T), \tag{6}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \setminus \{\xi(0)\}, \tag{7}$$

with a moving singularity $\xi(t) \not\equiv \xi(0)$ for some $t \in (0,T)$ and $0 < T \le \infty$. In the case $m > m_c$ and $T = \infty$, positive asymptotically radially symmetric solutions of the initial value problem (6)-(7) were studied in [4, 5, 6]. Along the given curve ξ with suitable properties, these solutions keep a singularity at all times, i.e. $u(x,t) \to \infty$ as $x \to \xi(t)$ for each $t \in [0,T)$. Our main result concerns the uniqueness of these solutions in the supercritical fast diffusion case. In order to formulate this result, we give a precise description of solutions from [4, 6]. Let $n \ge 3$ and $T = \infty$. Assume that either

(A0) $m > m_c$ and $\xi(t) \equiv \xi_0$ for some $\xi_0 \in \mathbb{R}^n$,

or

(A1) $m > m_* := (n-2)/(n-1)$ and $\xi \in C^1([0,\infty);\mathbb{R}^n)$, ξ' is locally Hölder continuous, and there exist positive constants Ξ, β such that $|\xi'(t)| \le \Xi e^{-\beta t}$ for $t \ge 0$.

Assume, moreover, that

(A2) $k \in C^1([0,\infty))$ satisfies $\kappa^{-1} \le k(t) \le \kappa$ and $|k'(t)| \le \kappa'$ for $t \ge 0$ and some positive constants κ and κ',

(A3) $u_0(x) \in C(\mathbb{R}^n \setminus \{\xi(0)\})$ is positive and there exist λ, μ and ν satisfying

$$\max\{(n-2)/m - 1, 0\} < \lambda < \mu < n-2 < \nu \tag{8}$$

such that $u_0(x)^m = k(t)^m|x-\xi(0)|^{-n+2} + O(|x-\xi(0)|^{-\lambda})$ as $x \to \xi(0)$, and $C^{-1}|x-\xi(0)|^{-\nu} \le u_0(x)^m \le C|x-\xi(0)|^{-\mu}$ for $|x-\xi(0)| \ge 1$ with some constant $C > 1$.

Under these assumptions, [4] implies the existence of a function $u > 0$ satisfying the following:

(i) $u \in C^{2,1}(\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\, t \in (0,\infty)\}) \cap C(\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\, t \in [0,\infty)\})$ and u satisfies (6)-(7) pointwise,

(ii) $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^n))$,

(iii) for each $t \ge 0$, u has the asymptotic behavior

$$u(x,t)^m = k(t)^m|x-\xi(t)|^{-n+2} + O(|x-\xi(t)|^{-\lambda}) \quad \text{as} \quad x \to \xi(t),$$

(iv) for $t \ge 0$ and $|x-\xi(t)| \ge 1$, it holds that

$$C^{-1}e^{-Ct}|x-\xi(t)|^{-\nu} \le u(x,t)^m \le Ce^{Ct}|x-\xi(t)|^{-\mu}$$

with some constant $C > 1$.

We note that [4] dealt with moving singularities, i.e. the existence was proved under assumptions (A1), (A2), (A3). Later, in [6] it was remarked that the existence from [4] is valid in the whole supercritical parameter range $m > m_c$ if the singularity is standing (i.e. assuming (A0), (A2), (A3)).

Moreover, it was established in [6] that a function u from [4] satisfying (i)-(iii) is a distributional solution of problem (1) with a weighted moving Dirac source term

$$u_t = \Delta u^m + (n-2)|S^{n-1}|k^m(t)\delta_{\xi(t)}(x) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times (0,\infty)).$$

More precisely, u satisfies (2) for all $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ with

$$f(x,t) = (n-2)|S^{n-1}|k^m(t)\delta_{\xi(t)}(x).$$

Here, $\delta_{\xi(t)}$ gives unit mass to the point $\xi(t) \in \mathbb{R}^n$ for each $t \geq 0$, and $|S^{n-1}|$ denotes the surface area of the $(n-1)$-dimensional unit sphere. A Dirac measure that moves with time can be also found as a source in parabolic systems, and this phenomenon has been used to model various biological scenarios, such as axon growth or angiogenesis, as discussed in [2] and [1], respectively. We summarize our main result in the theorem below.

Theorem 1.1. *Let $n \geq 3$ and $T = \infty$. Assume that either (A0) or (A1) holds. Assume, moreover, that conditions (A2), and (A3) are satisfied, and that functions u_1, u_2 satisfy* (i)–(iii). *Then the equality $u_1(\cdot,0) = u_2(\cdot,0)$ on $\mathbb{R}^n \setminus \{\xi(0)\}$ implies that $u_1 \equiv u_2$ on $\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\, t \in (0,\infty)\}$.*

In this paper, we show that with a modification of the proof of Theorem 2.2 in [15], the uniqueness result of Takahashi and Yamamoto can be extended from solutions with standing singularities to solutions with moving singularities. More precisely, we will prove the following.

Theorem 1.2. *Let $n \geq 3$, $0 < m < m_c$, $0 < T \leq \infty$, and $\xi \in C([0,T);\mathbb{R}^n)$. If functions u_1, u_2 belong to the function space $C^{2,1}(\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\, t \in (0,T)\}) \cap C(\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\, t \in [0,T)\})$, they satisfy* (6)-(7) *pointwise, and $u_1(\cdot,0) = u_2(\cdot,0)$ on $\mathbb{R}^n \setminus \{\xi(0)\}$, then $u_1 \equiv u_2$ on $\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\, t \in (0,T)\}$.*

Moreover, given a particular condition on the initial function u_0, an approach from [15] leads to the following result.

Corollary 1.3. *Let $n \geq 3$, $0 < m < m_c$, $0 < T \leq \infty$, and $\xi \in C([0,T);\mathbb{R}^n)$. Assume that function u satisfies* (6)-(7) *pointwise, $u \in C^{2,1}(\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\, t \in (0,T)\}) \cap C(\{(x,t) \in \mathbb{R}^{n+1} : x \neq \xi(t),\, t \in [0,T)\})$, and $u_0 \in L^1_{loc}(\mathbb{R}^n)$. Then $u \in L^1_{loc}(\mathbb{R}^n \times (0,T))$.*

Remark 1.4. Solutions with standing singularities that satisfy the assumptions of both Theorem 1.2 and Corollary 1.3 have been shown to exist, such as a class of solutions with so-called anisotropic singularities from [6]. However, the existence of solutions with moving singularities when $n \geq 3$ and $0 < m < m_c$ remains an open problem.

Remark 1.5. It is worth noting that by applying arguments from [6], Theorem 1.2, and Corollary 1.3, it can be shown that the function u mentioned in Corollary 1.3 is a distributional solution of equation (3), i.e.

$$\int_0^T \int_{\mathbb{R}^n} \left(u\varphi_t + u|u|^{m-1}\Delta\varphi\right) dx\, dt = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,T))$.

Uniqueness results for the porous medium equation can be found in [3, 13, 16]. For the uniqueness of solutions of a semi-linear parabolic equation with singularity moving along a prescribed curve, see [14], where similar conditions to (i)-(iv) were considered. For non-uniqueness examples for a semilinear heat equation, see e.g. [7] and references therein.

This paper is organized as follows. Section 2 contains the proof of Theorem 1.1. This proof is based on ideas from [15], [6], and [8]. In Section 3, we present the proof of Theorem 1.2. Finally, Section 4 consists of the proof of Corollary 1.3.

2. Proof of Theorem 1.1

Proof. This proof is based on ideas of Takahashi, Yamamoto, F., M., Yanagida, Herrero, and Pierre, see [15], [6], and [8].

Step 1. Set $\mathrm{sign}(f) = f/|f|$ for $f \neq 0$ and $\mathrm{sign}(f) = 0$ for $f = 0$. We recall that for a locally integrable function f such that $\Delta f \in L^1_{loc}(D)$ in $D \subseteq \mathbb{R}^n$, Kato proved the distributional inequality

$$\mathrm{sign}(f)\Delta f \leq \Delta|f|. \tag{9}$$

Let u_1, u_2 be two functions satisfying assumptions (i)-(iii) and $u_1(\cdot,0) = u_2(\cdot,0)$ on $\mathbb{R}^n \setminus \{\xi(0)\}$. Then it holds that

$$\partial_t|u_1 - u_2| = \mathrm{sign}(u_1-u_2)\partial_t(u_1-u_2) = \mathrm{sign}(u_1-u_2)\Delta(u_1^m - u_2^m) \leq \Delta|u_1^m - u_2^m|$$

for $x \in \mathbb{R}^n \setminus \{\xi(\tau)\}$ and $\tau \in [0,\infty)$. For $\varepsilon > 0$ we let $\eta_\varepsilon \in C^\infty(\mathbb{R})$ be a non-negative cut-off function such that

$$\eta_\varepsilon = \begin{cases} 0 & \text{if } r \leq \varepsilon, \\ 1 & \text{if } r \geq 2\varepsilon, \end{cases} \tag{10}$$

and

$$0 \leq \eta_\varepsilon \leq 1, \quad |\eta_\varepsilon'| \leq c_0\varepsilon^{-1}, \quad |\eta_\varepsilon''| \leq c_0\varepsilon^{-2} \tag{11}$$

for some $c_0 > 0$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function and set $\varphi_\varepsilon(x,\tau) := \eta_\varepsilon(|x - \xi(\tau)|)\varphi(x)$. For $R > 0$ and $z \in \mathbb{R}^n$, we let $B_R(z) := \{x \in \mathbb{R}^n; |x-z| < R\}$. For simplicity, by $B_R := B_R(\xi(\tau))$ we will denote an open ball with radius R centered at $\xi(\tau)$. We note that $\varphi_\varepsilon = \varphi$ for $x \in \mathbb{R}^n \setminus B_{2\varepsilon}$.

Testing with φ_ε and integrating the right-hand side by parts, we have

$$\int_{\mathbb{R}^n} \eta_\varepsilon \partial_\tau \big(\varphi |u_1 - u_2|\big)\, dx \le \int_{\mathbb{R}^n} |u_1^m - u_2^m| \Delta\big(\varphi \eta_\varepsilon\big)\, dx.$$

We now focus on the left-hand side of the equation above. Changing the variable to $y = x - \xi(\tau)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B_\varepsilon} \eta_\varepsilon \partial_\tau \big(\varphi |u_1 - u_2|\big)\, dx \\
&\qquad = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \eta_\varepsilon(|y|) \partial_\tau \big(\varphi(y + \xi(\tau)) |u_1(y + \xi(\tau), \tau) - u_2(y + \xi(\tau), \tau)|\big)\, dy \\
&\qquad = \partial_\tau \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \eta_\varepsilon(|y|) \varphi(y + \xi(\tau)) |u_1(y + \xi(\tau), \tau) - u_2(y + \xi(\tau), \tau)|\, dy \\
&\qquad = \partial_\tau \int_{\mathbb{R}^n \setminus B_\varepsilon} \eta_\varepsilon(|x - \xi(\tau)|) \varphi(x) |u_1(x, \tau) - u_2(x, \tau)|\, dx.
\end{aligned}$$

This gives us

$$\partial_\tau \int_{\mathbb{R}^n} \varphi_\varepsilon |u_1 - u_2|\, dx \le \int_{\mathbb{R}^n} |u_1^m - u_2^m| \Delta \varphi_\varepsilon\, dx. \tag{12}$$

Furthermore, from (12) we can derive a useful estimate that will be needed later. In order to do so, we recall the reverse triangle inequality $|a|a|^{m-1} - b|b|^{m-1}| \le 2|a - b|^m$ with exponent $m < 1$ and $a, b \in \mathbb{R}$. Together with the Hölder inequality, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |u_1^m - u_2^m| \Delta \varphi_\varepsilon\, dx &\le 2 \int_{\mathbb{R}^n} (|u_1 - u_2| \varphi_\varepsilon)^m |\Delta \varphi_\varepsilon| \varphi_\varepsilon^{-m}\, dx \\
&\le 2C[\varphi_\varepsilon]^{1-m} \left(\int_{\mathbb{R}^n} \varphi_\varepsilon |u_1 - u_2|\, dx \right)^m,
\end{aligned}$$

where

$$C[\varphi_\varepsilon] := \int_{\mathbb{R}^n} |\Delta \varphi_\varepsilon|^{\frac{1}{1-m}} \varphi_\varepsilon^{-\frac{m}{1-m}}\, dx. \tag{13}$$

Equation (12) can be now written as $f'(t) \le 2C[\varphi_\varepsilon]^{1-m} f^m(t)$ with $f(0) = 0$, and so

$$\int_{\mathbb{R}^n} \varphi_\varepsilon |u_1 - u_2|\, dx \le C[\varphi_\varepsilon] \left(2(1 - m)t\right)^{\frac{1}{1-m}}. \tag{14}$$

We fix $t > 0$. Since $u_1(\cdot, 0) = u_2(\cdot, 0)$ on $\mathbb{R}^n \setminus \{\xi(0)\}$, integrating equation (12) with respect to τ from 0 to t gives

$$\int_{\mathbb{R}^n} \varphi_\varepsilon(x, t) |u_1(x, t) - u_2(x, t)|\, dx \le \int_0^t \int_{\mathbb{R}^n} |u_1(x, \tau)^m - u_2(x, \tau)^m| \Delta \varphi_\varepsilon(x, \tau)\, dx\, d\tau.$$

This can be written as

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_{2\varepsilon}} \varphi(x) |u_1(x, t) - u_2(x, t)|\, dx &\le \int_0^t \int_{\mathbb{R}^n \setminus B_{2\varepsilon}} |u_1^m - u_2^m| \Delta \varphi\, dx\, d\tau \\
&\quad + \int_0^t \left(I_\varepsilon + J_\varepsilon + K_\varepsilon \right)\, d\tau + H_\varepsilon,
\end{aligned}$$

where we use similar notation as in [6], i.e. we denote

$$
\begin{aligned}
H_\varepsilon &:= -\int_{B_{2\varepsilon}\setminus B_\varepsilon} \eta_\varepsilon(|x-\xi(t)|)\varphi(x)|u_1(x,t)-u_2(x,t)|\,dx,\\
I_\varepsilon &:= \int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1^m - u_2^m|\eta_\varepsilon \Delta\varphi\,dx,\\
J_\varepsilon &:= 2\int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1^m - u_2^m|\nabla\eta_\varepsilon\cdot\nabla\varphi\,dx,\\
K_\varepsilon &:= \int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1^m - u_2^m|\varphi\Delta\eta_\varepsilon\,dx.
\end{aligned}
\tag{15}
$$

Step 2. We want to pass to the limit as $\varepsilon \to 0$ and prove that

$$
H_\varepsilon,\ I_\varepsilon,\ J_\varepsilon,\ K_\varepsilon\ \to 0. \tag{16}
$$

As in [6], we choose ε sufficiently small so that the method of sub- and supersolutions in [4] provides estimates of the form

$$
\begin{aligned}
u^m(x,\tau) &\le k^m(\tau)\left(|x-\xi(\tau)|^{2-n} + b(\tau)|x-\xi(\tau)|^{-\lambda}\right),\\
u^m(x,\tau) &\ge k^m(\tau)\left(|x-\xi(\tau)|^{2-n} - b(\tau)|x-\xi(\tau)|^{-\lambda}\right)_+,
\end{aligned}
\tag{17}
$$

for all $(x,\tau)\in B_{2\varepsilon}\times[0,t]$. Here, $b(\tau)=b_0e^{B\tau}$ for some constants B, $b_0>1$, $\lambda<n-2$ by (8), and we recall that k is a given function satisfying (A2). In what follows, by c we will denote a large enough but otherwise arbitrary constant independent of t,τ and ε. Inspecting the proof of Theorem 1.5 in [6], we see that for $\tau\in[0,t]$ we have

$$
\begin{aligned}
|I_\varepsilon| &\le \int_{B_{2\varepsilon}\setminus B_\varepsilon} (u_1^m+u_2^m)\eta_\varepsilon|\Delta\varphi|\,dx \le c\int_\varepsilon^{2\varepsilon} r\,dr \to 0 \quad \text{as} \quad \varepsilon\to 0,\\
|J_\varepsilon| &\le 2\int_{B_{2\varepsilon}\setminus B_\varepsilon} (u_1^m+u_2^m)|\nabla\eta_\varepsilon\cdot\nabla\varphi|\,dx \le c\varepsilon^{-1}\int_\varepsilon^{2\varepsilon} r\,dr \to 0 \quad \text{as} \quad \varepsilon\to 0.
\end{aligned}
$$

By (17) and $|\eta_\varepsilon''|\le c_0\varepsilon^{-2}$ for some $c_0>0$, we obtain

$$
|K_\varepsilon| \le \int_{B_{2\varepsilon}\setminus B_\varepsilon} \varphi|u_1^m-u_2^m||\Delta\eta_\varepsilon|\,dx \le c\sup_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\varepsilon^{-2}\int_\varepsilon^{2\varepsilon} r^{n-1-\lambda}\,dr \to 0 \quad \text{as} \quad \varepsilon\to 0.
$$

Finally, by (17), $|\eta_\varepsilon|\le 1$, and $m>m_c$, we have

$$
\begin{aligned}
|H_\varepsilon| &\le \sup_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\int_{B_{2\varepsilon}\setminus B_\varepsilon}|u_1-u_2|\,dx \le \sup_{B_{2\varepsilon}\setminus B_\varepsilon}\varphi\int_{B_{2\varepsilon}\setminus B_\varepsilon}(u_1+u_2)\,dx\\
&\le c\int_\varepsilon^{2\varepsilon} r^{\frac{2-n}{m}+n-1} \le c\,\varepsilon^{\frac{n}{m}(m-m_c)} \to 0 \quad \text{as} \quad \varepsilon\to 0.
\end{aligned}
$$

Hence, for any nonnegative function $\varphi\in C_0^\infty(\mathbb{R}^n)$ it holds that

$$
\int_{\mathbb{R}^n}\varphi|u_1-u_2|\,dx \le \int_0^t\int_{\mathbb{R}^n}|u_1^m-u_2^m|\Delta\varphi\,dx\,d\tau. \tag{18}
$$

Step 3. The rest of the proof is the same as the latter part of the proof of Theorem 2.2 by Takahashi and Yamamoto in [15] and Theorem 2.3 by Herrero and Pierre in [8]. We present it for completeness. Set

$$w(x,t) := \int_0^t |u_1^m - u_2^m|\, d\tau.$$

Since $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^n))$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, Fubini's theorem gives

$$\int_{\mathbb{R}^n} \varphi |u_1 - u_2|\, dx \le \int_{\mathbb{R}^n} w(x,t)\Delta\varphi(x)\, dx.$$

Then, $\int_{\mathbb{R}^n} w(x,t)\Delta\varphi(x)\, dx \ge 0$ and so $-\Delta w(x,t) \le 0$ in $\mathcal{D}'(\mathbb{R}^n)$. Hence, the following mean value inequality for subharmonic functions holds

$$w(z,t) \le \frac{1}{|B_1|R^n}\int_{B_R(z)} w(x,t)\, dx =: L_R,$$

where $z \in \mathbb{R}^n$, $|B_1|$ is the volume of a unit ball, and $R > 0$. Thus, $u_1 \equiv u_2$ will be proved once we show $L_R \to 0$ as $R \to \infty$. For $R \ge 1$ we define $\phi_R \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \phi_R \le 1$, $\phi_R = 0$ if $|x-z| \ge 2R$, and $\phi_R = 1$ if $|x-z| \le R$. Let $\tilde{\phi}_R := \phi_R^k$ for $k > 2/(1-m)$. We proceed by using the reverse triangle inequality, Hölder inequality, and (14) with $C[\tilde{\phi}_R]$, which was defined in (13). We obtain

$$\begin{aligned}
L_R &\le \frac{2}{|B_1|R^n}\int_0^t \int_{B_R(z)} |u_1 - u_2|^m\, dx\, d\tau \\
&\le \frac{2R^{n(1-m)}}{|B_1|R^n}\int_0^t \left(\int_{B_R(z)} |u_1 - u_2|\, dx\right)^m d\tau \\
&\le \frac{2R^{-nm}}{|B_1|}\int_0^t \left(\int_{\mathbb{R}^n} \tilde{\phi}_R |u_1 - u_2|\, dx\right)^m d\tau \\
&\le \frac{(2(1-m))^{\frac{1}{1-m}}}{|B_1|} R^{-nm} C[\tilde{\phi}_R]^m t^{\frac{1}{1-m}}.
\end{aligned}$$

Substituting $x - z = R(y-z)$, it holds that

$$C[\tilde{\phi}_R] = \int_{B_{2R}(z)} |\Delta\tilde{\phi}_R|^{\frac{1}{1-m}} \tilde{\phi}_R^{-\frac{m}{1-m}}\, dx = R^{n-\frac{2}{1-m}} \int_{B_2(z)} |\Delta\tilde{\phi}_1|^{\frac{1}{1-m}} \tilde{\phi}_1^{-\frac{m}{1-m}}\, dy = R^{n-\frac{2}{1-m}} C[\tilde{\phi}_1].$$

Since $k > 2/(1-m)$, we have

$$C[\tilde{\phi}_1] = \int_{\mathbb{R}^n} |k(k-1)\phi_1^{k(1-m)-2}|\nabla\phi_1|^2 + k\phi_1^{k(1-m)-1}\Delta\phi_1|^{\frac{1}{1-m}}\, dx < \infty. \tag{19}$$

Thus,

$$L_R \le \frac{(2(1-m))^{\frac{1}{1-m}}}{|B_1|} R^{-\frac{2m}{1-m}} C[\tilde{\phi}_1]^m t^{\frac{1}{1-m}} \to 0 \quad \text{as } R \to \infty.$$

This shows that $u_1 \equiv u_2$, which completes the proof. □

3. Proof of Theorem 1.2

Proof. *Step 1.* With a few modifications, the first step is almost the same as in the proof of Theorem 1.1 in Section 2. First, we substitute $|u_1^m - u_2^m|$ with $|u_1|u_1|^{m-1} - u_2|u_2|^{m-1}|$. We will be also choosing a slightly different test function. Let $\eta_\varepsilon, \varphi$ be as in Section 2, i.e. let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function, and for $\varepsilon > 0$ let $\eta_\varepsilon \in C^2(\mathbb{R})$ be a non-negative cut-off function such that (10) and (11) hold for some $c_0 > 0$. We define $\tilde{\eta}_\varepsilon := \eta_\varepsilon^k$ and $\tilde{\varphi} := \varphi^k$ where $k > 1/(1-m)$. Then we choose $\tilde{\varphi}_\varepsilon(x,t) := \varphi_\varepsilon(x,t)^k = \eta_\varepsilon(|x - \xi(t)|)^k \varphi(x,t)^k$ with $k > 1/(1-m)$. With these two changes, the first step is the same as in Section 2 and we proceed directly to the second step.

Step 2. In order to pass to the limit as $\varepsilon \to 0$, we need to show that $|u_1 - u_2| \in L^1_{loc}(\mathbb{R}^n \times (0,t))$ for $t > 0$ fixed. This will be done in the same spirit as in the proof of Theorem 2.2 in [15] by Takahashi and Yamamoto, who considered this problem with a standing singularity. We use estimate (13) and show that $C[\tilde{\varphi}_\varepsilon] < \infty$. In what follows, by c we will denote a positive, large enough constant independent of ε but otherwise arbitrary. By the definition of $C[\tilde{\varphi}_\varepsilon]$, $\eta_\varepsilon \le 1$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, $k > 1/(1-m)$, (19), and change of variables $y = x/\varepsilon$, we have

$$
\begin{aligned}
C[\tilde{\varphi}_\varepsilon] &= \int_{\mathbb{R}^n} |\Delta \tilde{\varphi}_\varepsilon|^{\frac{1}{1-m}} \tilde{\varphi}_\varepsilon^{-\frac{m}{1-m}}\, dx \\
&= \int_{\mathbb{R}^n} |\tilde{\eta}_\varepsilon \Delta \tilde{\varphi} + 2\nabla \tilde{\eta}_\varepsilon \cdot \nabla \tilde{\varphi} + \tilde{\varphi} \Delta \tilde{\eta}_\varepsilon|^{\frac{1}{1-m}} \tilde{\varphi}^{-\frac{m}{1-m}} \tilde{\eta}_\varepsilon^{-\frac{m}{1-m}}\, dx \\
&\le C[\tilde{\varphi}] + 2k^2 \int_{\mathbb{R}^n} |\nabla \eta_\varepsilon \cdot \nabla \varphi|^{\frac{1}{1-m}} \varphi^{k - \frac{1}{1-m}} \eta_\varepsilon^{k-\frac{1}{1-m}}\, dx + cC[\tilde{\eta}_\varepsilon] \\
&\le C[\tilde{\varphi}] + c\int_{\mathbb{R}^n} |\nabla \eta_\varepsilon|^{\frac{1}{1-m}} \eta_\varepsilon^{k-\frac{1}{1-m}}\, dx + cC[\tilde{\eta}_\varepsilon] \\
&= C[\tilde{\varphi}] + c\varepsilon^{n-1/(1-m)} \int_{\mathbb{R}^n} |\nabla \eta_1|^{\frac{1}{1-m}} \eta_1^{k-\frac{1}{1-m}}\, dy + c\varepsilon^{n-2/(1-m)} C[\tilde{\eta}_1] \\
&= C[\tilde{\varphi}] + \varepsilon^{n-1/(1-m)} c + \varepsilon^{n-2/(1-m)} c.
\end{aligned} \tag{20}
$$

The condition $m < m_c$ is equivalent to $n > 2/(1-m)$. Letting $\varepsilon \to 0$, we thus obtain that (14) holds for all $\tilde{\varphi}$ such that $\tilde{\varphi} = \varphi^k$ with $k > 2/(1-m)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. This means that $|u_1 - u_2| \in L^1_{loc}(\mathbb{R}^n \times (0,t))$ for $t > 0$ fixed. We further estimate

$$
\begin{aligned}
\tilde{I}_\varepsilon &:= \int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1|u_1|^{m-1} - u_2|u_2|^{m-1}| \tilde{\eta}_\varepsilon \Delta \tilde{\varphi}\, dx \le 2\int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1 - u_2|^m |\Delta \tilde{\varphi}|\, dx \\
&\le \left(2\int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1 - u_2||\Delta \tilde{\varphi}|^{\frac{1}{m}}\, dx\right)^m \left(\int_{B_{2\varepsilon}\setminus B_\varepsilon} dx\right)^{1-m} \le c\varepsilon^{n(1-m)},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{J}_\varepsilon &:= 2\int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1|u_1|^{m-1} - u_2|u_2|^{m-1}|\nabla\tilde{\eta}_\varepsilon \cdot \nabla\tilde{\varphi}\,dx \\
&\le \left(4\int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1-u_2||\nabla\tilde{\varphi}|^{\frac{1}{m}}\,dx\right)^m \left(\int_{B_{2\varepsilon}\setminus B_\varepsilon} |\nabla\tilde{\eta}_\varepsilon|^{\frac{1}{1-m}}\,dx\right)^{1-m} \le c\varepsilon^{\left(n-\frac{1}{1-m}\right)(1-m)}, \\
\tilde{K}_\varepsilon &:= \int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1|u_1|^{m-1} - u_2|u_2|^{m-1}|\tilde{\varphi}\Delta\tilde{\eta}_\varepsilon\,dx \\
&\le \left(2\int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_1-u_2|\tilde{\varphi}^{\frac{1}{m}}\,dx\right)^m \left(\int_{B_{2\varepsilon}\setminus B_\varepsilon} |\Delta\tilde{\eta}_\varepsilon|^{\frac{1}{1-m}}\,dx\right)^{1-m} \le c\varepsilon^{\left(n-\frac{2}{1-m}\right)(1-m)}, \\
\tilde{H}_\varepsilon &:= -\int_{B_{2\varepsilon}\setminus B_\varepsilon} \tilde{\eta}_\varepsilon(|x-\xi(t)|)\tilde{\varphi}(x)|u_1(x,t)-u_2(x,t)|\,dx \le c\varepsilon^n.
\end{aligned}
$$

Letting $\varepsilon \to 0$, we see that by $m < m_c$, the asserion (16) holds. This completes the second step.

Step 3. Finally, in the last step, we proceed as in the proof of Theorem 1.1.

□

4. Proof of Corollary 1.3

Proof. Corollary 1.3 follows from estimates at the end of the first step of proof of Theorem 1.1 in Section 2. We can repeat these estimates with u^m instead of $|u_1^m - u_2^m|$, u instead of $|u_1 - u_2|$, and $\tilde{\varphi}_\varepsilon$ instead of φ_ε, where $\tilde{\varphi}_\varepsilon$ is as in Section 3. Then we can write equation (12) as $f'(t) \le 2C[\tilde{\varphi}_\varepsilon]^{1-m} f^m(t)$ with $f(0) = \int_{\mathbb{R}^n} \tilde{\varphi}_\varepsilon(x,0)|u(x,0)|\,dx$. Hence, instead of (14), we now have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \tilde{\varphi}_\varepsilon |u|\,dx &\le C[\tilde{\varphi}_\varepsilon]\left(2(1-m)t + \int_{\mathbb{R}^n} \tilde{\varphi}_\varepsilon(x,0)|u(x,0)|\,dx\right)^{\frac{1}{1-m}} \\
&\le C[\tilde{\varphi}_\varepsilon]\left(2(1-m)t + \int_{\mathbb{R}^n} \tilde{\varphi}|u_0|\,dx\right)^{\frac{1}{1-m}}.
\end{aligned}
$$

As was shown in (20), we can let $\varepsilon \to 0$ and obtain

$$
\int_{\mathbb{R}^n} \tilde{\varphi}|u|\,dx \le C[\tilde{\varphi}]\left(2(1-m)t + \int_{\mathbb{R}^n} \tilde{\varphi}|u_0|\,dx\right)^{\frac{1}{1-m}} < \infty
$$

for all $\tilde{\varphi}$ such that $\tilde{\varphi} = \varphi^k$ with $k > 2/(1-m)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. The claim that $u \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ follows.

□

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