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The risk sensitive dynamic accumulation model and optimal pension saving management

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1 Introduction

The key objective of this study is to determine and investigate the optimal strategy that the future pensioner – the participant of the Second pillar of the Slovak pension system – should follow in order to attain to maximize their expected future pension income from the Second pillar with respect to their specific risk aversion. Based on their personal characteristics, legislative regulations and financial market data we derived the analytic model that formulates the optimal decision for the investor about the specific pension fund selection. Furthermore, besides the model advisory role in the investor's optimal fund selection strategy, this model also helps future pensioners to perceive the aspects impacting the level of their retirement pensions.

The decision about the optimal allocation is made in perspective of the investor interested in the portfolio terminal value, via their utility criterion with both the portfolio expected terminal utility and risk combined. The problem is postulated in terms of the solution to the following Hamilton–Jacobi–Bellman equation derived from continuous version of the dynamic stochastic optimization model for the portfolio value function $V = V(t,y)$

$$\begin{cases} 0 = \dfrac{\partial V}{\partial t} + \max\limits_{\theta \in \Delta_t^T} \left\{ A_\varepsilon(\theta,t,y)\dfrac{\partial V}{\partial y} + \dfrac{1}{2}B^2(\theta,t,y)\left[\dfrac{\partial^2 V}{\partial y^2} - \lambda\left[\dfrac{\partial V}{\partial y}\right]^2\right]\right\}, & (t,y) \in [0,T) \times \mathbb{R}^+ \\ V(T,y) = U(y) = -y^{d-1} + \dfrac{\lambda}{2}y^{2(d-1)}, & y \in \mathbb{R}^+, \end{cases}$$

where all $U = U(y)$, $A_\varepsilon \equiv A_\varepsilon(\theta,t,y)$ and $B \equiv B(\theta,t,y)$ are smooth, investor's regular contribution rate ε and risk sensitivity parameter λ are small, $0 < \varepsilon, \lambda \ll 1$ and y denotes wealth already allocated on saver's private pension account measured relatively to his/her salary. Presence of the function $V(t,y)$ space derivative squared term, $[\partial_y V]^2$ above is not obvious in the standard formulation of such problem (Bellman [2], Bertsekas [5], Evans [12], Fletcher [14], Kilianová et al. [22], [23], Macová and Ševčovič [26], Melicherčík and Ševčovič [28], Múčka [31], Oksendal [33]). It caused by a special choice of the risk–sensitive investor's utility function $U(y)$ arising from his/her criterion reflecting both the expected portfolio return and volatility.

We show how the foregoing Hamilton–Jacobi–Bellman equation can be transformed into a quasi–linear parabolic differential equation. The weak solution to the problem is approached by its double asymptotic expansion with respect to small model parameters and utilized to estimate the unconstrained optimal investment strategy. We present key attributes of the optimal allocation policy determined by our model and illustrate it on the problem of optimal fund selection in the Second Pillar of the Slovak Pension System.

What makes the optimal pension fund portfolio selection issue so attractive? Obviously, it is a nice non–deterministic highly complex mathematical puzzle, a challenge with many extremely interesting extensions.
But from a practical point of view, this brain–teaser has a very useful real–world application, with many consequences and policy implications. The inevitable economy and

social care reforms (e.g. tax reform, the pension system reform, healthcare & long–term care reforms) are being ultimately underwent in many Western culture countries and remain particularly relevant in Slovakia due to two ticking time bombs – poor demography trends and long–term public finance sustainability. The projected dramatic changes in the population structure, demographic prospects (characterized by drop in fertility rate, longevity increase and extreme raise of dependency ratio) and economic effects of ageing populations causing a significant pressure on public finance (due to high share of ageing and demographic structure related share on expenditure), slowing potential economic growth and labour market permanent structural changes have strong implications for pensions and overall budgetary effects of ageing populations.

Hence all the reasons mentioned above prod policy–makers to rebuilt paradigms about the participation rôle and responsibility of current generation of active and pre–active individuals on their future income. Therefore facing definitely not rosy future, they also aim their attention to the optimal long–term saving schemes, investment decisions and possible financial instruments that can bring additional cash–flow for future pensioners and thus at least partially reduce the future load of claim on public finance. Thus nowadays the momentous question of optimal and safe saving on pension emerges and it is posed not only by policy makers, financier and economists but also by a non–myopic part of currently active population.

As we are also concerned about this issue the purpose of this dissertation thesis is to ask and look for a proper solution to this problem, derive a optimal pension strategy model that will fit the Slovak pension system, namely, its private, defined–contribution pillar. The individual's private pension at retirement is substantially susceptible to investment allocation policy preferred during the active life of the pensioner as Slovak private pension scheme is built on defined contribution idea - pension benefits depend on returns of the pension fund's portfolio financed via fixed regular contributions of future pensioner during the accumulation period who borne financial risk associated with investment. Therefore, the optimal wealth allocation strategy is the fundamental issue of this thesis.

Thesis Objectives

This dissertation thesis stakes out the following fundamental targets:

- Formulate the continuous–time pension investment portfolio selection problem that encounters any participant of the Second pillar of the Slovak pension system properly, and find the relationship between optimal portfolio allocation policy and its intermediate value function;

- Provide (at least approximative) a simple explicit analytic decision mechanism estimating a future pensioner's optimal portfolio selection strategy that based on a saver's time to retirement and already allocated wealth advice him/her how to allocate his/her wealth optimally between unlimited number of more or less risky securities;

  – The decision formula should reflect individual characteristics of a risk–sensitive investor (risk aversion attitude, gross wage growth rate), existing government restrictions (retirement age, contribution rate) and financial market data;

- Analyse properly the optimal investment strategy decision tool from a qualitative and quantitative perspective; and highlight the resulting policy implications;
- Calibrate the model on Slovak data and illustrate its behaviour;
  – Show how both the optimal allocation policy and the expected terminal portfolio wealth are affected by varying model parameters;
  – Accentuate the effects of changes in fiscal policy parameters – prescribed retirement age and contribution rate;

Beside them, our aim is also top provide a deep explanation of the three–pillar Slovak pension system and its undergoing reforms, legislative framework and key concepts:

- Clarify and support with data reasons for the pension system reform and describe the main aspects of the reform in the First pillar;
- elucidate the scope of the private Second pillar, the scope of available wealth allocation policies and existing government regulations and support with data the actual investment decisions of its participants.

2 Methodology and Literature Review

2.1 Literature Review

Technically we are focused on approximative analytic solution to a specific Hamilton–Jacobi–Bellman equation arising from stochastic dynamic programming for trading the optimal investment decision technique for an individual investor during accumulation of pension savings.
Such an optimization problem often emerges in optimal dynamic portfolio selection and asset allocation policy for an investor who is concerned about the performance of a portfolio relative to the performance of a given benchmark. We take as our baseline the standard continuous–time settings pioneered by Bodie et al. [8], Bodie et al. [7], Browne [9], Samuelson [37], Merton [29] who were interested in optimal consumption–portfolio strategies, life–cycle model or Songzhe [41].

Obviously, there are numerous recent very practically oriented models preferring discrete–time defined contributions pension scheme framework e.g. Gao [15], Kim and Noh [24], or Noh [32]. Within our work we use the principles of investor's risk–sensitivity deeply studied in Bielecki et al. [6].

In this work we refer to novel papers of Múčka [31], Kilianová et al. [22], Macová and Ševčovič [26] and Kilianová and Ševčovič [23]. In Kilianová et al. [22] the baseline dynamic accumulation model for the private second defined–contribution pillar of the Slovak pension system was firstly introduced.
This model was extended and studied later in Melicherčík and Ševčovič [28]. Furthermore, in Macová and Ševčovič [26] a simplified analytic tool to determine the optimal investment strategy for a participant of the second pillar of the Slovak pension system was developed and its very first quantitative and qualitative analysis was provided.

This instrument along with a similar one obtained via transformation of the original Hamilton–Jacobi–Bellman problem into a quasi–linear equation presented by Kilianová and Ševčovič [23] and very new paper of Múčka [31] studying so–called *one–stock–one–bond* (portfolio composition problem is limited to only one pair of quite risky and relatively safe securities) problem employing the portfolio value function method, inspired us to build a new extended model. Its solution was estimated applying the techniques of Riccati transformation used by e.g. Abe and Ishimura [1], Ishimura and Ševčovič [20], Ishimura and Mita [18] and asymptotic expansion method (see Holmes [17], Bender and Orszag [3], O'Malley [34]) allowed us to determine the explicit approximative analytic optimal allocation policy formula.
In opposed to previously assumed models, the investor's utility criterion ponder also the the aspect of the portfolio returns volatility – to endow this attribute into our model we utilized the approach of e.g. Sharpe [38], Bielecki et al. [6], or Songzhe [41]. Finally this dissertation thesis is built on fundamentals of the author's dissertation project text.

2.2 Used Methodology

In order to derive the model determining the explicit approximative analytic optimal allocation policy formula for a future pensioner we start with a simple discrete–time optimal portfolio composition problem on finite time–horizon, which was studied in Kilianová et al. [22], Macová and Ševčovič [26] or Kilianová [21].

Each period a typical sever transfers a fraction ε of his/her salary with a deterministic growth rate β to a his/her portfolio consisting of only one risky stock and one quite safe bond instrument and has to make a decision about proper proportion of risky stock proportion in this portfolio. For the sake of simplicity, we presume that the investment strategy of the pension fund at time t is given by the proportion $\theta \in [0,1]$ of stocks and $1-\theta$ of bonds and the portfolio return $r_t = r_t(\theta) \sim \mathscr{N}(\mu_t(\theta), \sigma_t^2(\theta))$ is normally distributed for any choice of the stock to bond proportion θ.
Thus, in terms of the quantity y_t representing the number of yearly salaries already saved at time $t = 0,1,\ldots,T-1$, the budget–constraint equation can be reformulated recurrently as follows:

$$
\begin{aligned}
&y_1 = \varepsilon, \qquad \text{and} \qquad y_{t+1} = G_t^1(y_t, r_t(\theta_t)), \\
&\text{for} \qquad G_t^1(y, r_t) = \varepsilon + y\frac{1+r_t}{1+\beta_t}, \qquad t = 1,2,\ldots,T-1.
\end{aligned} \tag{2.1}
$$

Assuming the knowledge of the saver's utility function $U = U(y)$, our aim is to determine the optimal value of the weight θ at each time t that maximizes the contributor's utility from the terminal wealth allocated on their pension account. Thus, the problem of discrete stochastic dynamic programming can be formulated as

$$
\max_{\theta \in \Delta} \mathbb{E}(U(y_T) \,|\, y_t = y), \tag{2.2}
$$

subject to the constraint (2.1) where the maximum in the stochastic dynamic problem is taken over all non-anticipative strategies, stocks proportions $\{\theta\}_t^T$ taken from

$$
\Delta_t^T = \Delta \equiv \{\theta : [t,T] \times \mathbb{R}^+ \mapsto \mathbb{R},\ \theta \in [0,1]\}, \qquad \forall t \in [0,T].
$$

Therefore under the Bellman's optimality principle (see Bellman [2], Fletcher [14] or Bertsekas [5]) the optimal strategy of the problem (2.1)–(2.2) is the solution to the Bellman equation of the dynamic programming

$$
W(t,y) = \begin{cases} U(y), & t = T, \\ \max\limits_{\theta \in \Delta} \mathbb{E}_Z\left(W(t+1, F_t^1(\theta, y, Z))\right), & t = T-1,\ldots,2,1, \end{cases} \tag{2.3}
$$

where $Z \in \mathscr{N}(0,1)$ and

$$
F_t^1(\theta, y, z) \equiv G_t^1(y, \mu_t(\theta) + \sigma_t(\theta)z) = y\frac{1+\mu_t(\theta)+\sigma_t(\theta)z}{1+\beta_t} + \varepsilon. \tag{2.4}
$$

In this baseline model setting, investor's utility function expresses his/her time t expectations about the terminal value of the pension fund portfolio (e.g. Bergman [4], Pflug and Romisch [35], Fishburn [13], Markowitz [27] or Sharpe [38]). This discrete–time model is discussed deeply in e.g. Kilianová et al. [22], Macová and Ševčovič [26], Kilianová [21] or Múčka [31].

As we are interested in continuous–time strategies, we assume that given a small time increment $0 < \tau \leq 1$ the proportion of size $\varepsilon\tau$ of saving deposits is transferred to the saver's pension account on short time intervals $[0,\tau], [\tau, 2\tau], \ldots, [T-\tau, T]$. Next, taking into consideration the investor's natural risk–aversion we extend our perception of the saver's utility and by the aspect of the portfolio returns volatility, so that at time t a typical participant of the second pillar of the Slovak pension system strives to maximize the terminal wealth–to–salary ratio y_T:

$$
\max_{\theta \in \Delta_t^T|_{[0,T)}} \{\mathscr{K}[y_T^\theta \,|\, y_t^\theta = \bar{y}]\}, \qquad \text{where} \qquad \mathscr{K}(Y) = \mathbb{E}[U(Y)] - \frac{\lambda}{2}\mathbb{D}[Y]. \tag{2.5}
$$

where $\{y_t^\theta\}_{t=0}^\infty$ in the finite time horizon Ito's process $\bar{y}$ a given initial state of $\{y_t^\theta\}$ evaluated at time t and $\mathscr{K}$ denotes a utility criterion functional assumed for a given utility function $U = U(y)$.

Then, applying the Bellman's optimality principle the optimal strategy for the problem of stochastic dynamic programming for $0 < \tau \ll 1$ can be formulated in using the concept of the saver's portfolio intermediate value function $V = V(t,y)$ similarly to the case of $\tau = 1$ (see (2.3)–(2.4)) as follows:

$$
V(t,y) = \begin{cases} U(y), & t = T; \\ \max\limits_{\theta \in \Delta_t^{t+\tau}} \{\mathscr{K}[V(t+\tau, y_{t+\tau}(\theta)) \,|\, y_t = y]\}, & 0 \leq t < t+\tau \leq T, \end{cases}
$$

and similarly to (2.1), for any $y > 0$,

$$
\begin{aligned}
y_{t+\tau}(\theta) &= F_t^\tau(\theta, y_t, Z), \qquad Z \sim N(0,1),\ 0 < \tau \ll 1, \\
F_t^\tau(\theta, y, z) &= y\exp\{[\mu(\theta) - \beta - \tfrac{1}{2}\sigma^2(\theta)]\tau + \sigma(\theta)z\sqrt{\tau}\} + \varepsilon\tau.
\end{aligned}
$$

Then, letting $\tau \equiv dt \to 0^+$, using basic properties of random variable mean and variance, applying stochastic calculus and Itô lemma (see Kwok [25], Oksendal [33], Chiang [10], Múčka [31], Epps [11], or Macová and Ševčovič [26]) we find out that the intermediate value function $V(t,y)$ satisfies the subsequent fully non–linear Hamilton–Jacobi–Bellman equation

$$
\begin{cases} 0 = \frac{\partial V}{\partial t} + \max\limits_{\theta \in \Delta_t^T} \left\{ A_\varepsilon(\theta,t,y)\frac{\partial V}{\partial y} + \frac{1}{2}B^2(\theta,t,y)\left[\frac{\partial^2 V}{\partial y^2} - \lambda\left[\frac{\partial V}{\partial y}\right]^2\right]\right\}, & (t,y) \in [0,T) \times R^+, \\ V(T,y) = U(y), & y \in \mathbb{R}^+, \end{cases} \tag{2.6}
$$

and

$$
A_\varepsilon(\theta,t,y) = \varepsilon + \left[\mu(\theta) - \beta\right]y, \qquad \text{and} \qquad B(\theta,t,y) = \sigma(\theta)y.
$$

Next, recalling to Abe and Ishimura [1], Ishimura and Nakamura [19], Ishimura and Ševčovič [20], Macová and Ševčovič [26] and Múčka [31] we introduce the Riccati transformation

$$\varphi(s,x) = -\frac{\partial_{xx}\mathscr{V}(s,x)}{\partial_x \mathscr{V}(s,x)}, \qquad \text{for} \quad s = T-t\,,\; x=\ln y\,,\; \mathscr{V}(s,x) = V(t,y)\,, \tag{2.7}$$

for all $x \in \mathbb{R}$ and $s \in [0,T]$ where φ refers to the coefficient of absolute risk aversion of the (s,x) domain transformed intermediate value function $\mathscr{V}$.

Therefore assuming that both φ and $\mathscr{V}$ are positive on $[0,T]\times\mathbb{R}$ the originally stated Hamilton–Jacobi–Bellman equation (2.6) is transformed as follows

$$\frac{\partial \mathscr{V}}{\partial s} = \mathscr{G}(s,x)\frac{\partial \mathscr{V}}{\partial x}, \qquad \text{for} \qquad \mathscr{G}(s,x) \equiv \varepsilon e^{-x} - \beta - \phi(\zeta(\varphi(s,x)))\,, \tag{2.8a}$$

with $\phi = \phi(\zeta(\varphi))$ the value function of the parametric optimization problem

$$\phi(\zeta) = \min_{\theta\in\Delta}\left\{-\mu(\theta) + \frac{1}{2}\sigma^2(\theta)\zeta\right\}. \tag{2.8b}$$

and the auxiliary function ζ satisfying the subsequent relationship

$$\zeta(\varphi(s,x)) = 1 + \varphi(s,x) + \lambda\,\omega(\varphi(s,x))\,, \quad \omega(\varphi(s,x)) = \partial_x\mathscr{V}(s,x) = \kappa e^{-\int_{x_0}^{x}\varphi(s,z)dz} \tag{2.8c}$$

for some $x_0 \in \mathbb{R}$ and $\kappa \equiv \mathscr{V}'(s,x_0)$ finite.

Then φ is a solution to the Cauchy–type quasi–linear parabolic equation (see Kilianová and Ševčovič [23])

$$\begin{cases} \dfrac{\partial \varphi}{\partial s} = \dfrac{\partial^2 \phi(\zeta(\varphi))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\varphi)\left(\varepsilon e^{-x} - \beta\right) - \varphi\,\phi(\zeta(\varphi))\right], & x\in\mathbb{R}, s \in (0,T]\,, \\ \varphi(0,x) = -\dfrac{U''(e^x)}{U'(e^x)}e^x, & x \in \mathbb{R}\,. \end{cases} \tag{2.9}$$

and problems (2.8a)–(2.8b) and (2.9) are equivalent. Furthermore, referring to Kilianová and Ševčovič [23] in case of $\mu \in \mathbb{R}^n$ and Σ positive definite matrix, the optimal value function $\phi(\zeta)$ given by (2.8b) is $C^{1,1}$ continuous, $\zeta \mapsto \phi(\zeta)$ is strictly increasing and for the unique minimizer $\hat{\theta} = \hat{\theta}(\zeta) \in \Delta$ of (2.8b) it holds that

$$\phi'(\zeta) = \frac{1}{2}\hat{\theta}^T(\zeta)\Sigma\hat{\theta}(\zeta)\,. \tag{2.10}$$

Furthermore, recalling (2.8c), we see that $\zeta'(\varphi) = 1 + \lambda\dfrac{\varphi}{\partial_x\varphi}\omega(\varphi(s,x))\,.$

Recalling he unique minimizer $\hat{\theta}(\zeta) \in \Delta$ of (2.8b) for any subset S of $\{1,\ldots,N\}$ the set $\mathscr{I}_S$ of all functions $\zeta > 0$ for which the index set of $\hat{\theta}(\zeta) \in \Delta$ zero components coincide with S we define:

$$\mathscr{I}_\emptyset = \left\{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) > 0\,, \forall i = 1,\ldots,N\right\}, \quad \mathscr{I}_S = \left\{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) = 0 \iff i \in S\right\}.$$

Then, concerning the future pensioner's optimal investment strategy problem we need to distinguish between two cases. In case of $\zeta \in \mathscr{I}_\emptyset$ we directly employ the technique of Lagrange multiplier (see e.g. Smith [40], Fletcher [14], Chiang [10], Smith [39], or Walde [42]) whereas providing that that $\zeta \in \mathscr{I}_S$ for some non–empty subset S then we may reduce the problem dimension to a lower $N - |S|$ dimensional simplex Δ_S.

Thus, $\phi(\zeta)$ is C^∞ on the open set $\bigcup_{0\le|S|\le N-1} int\,(\mathscr{I}_S)$ for any $S \subset \{0,\ldots,N\}$ and

$$\phi(\zeta) = \begin{cases} \dfrac{\zeta}{2a} - \dfrac{b}{a} - \dfrac{ac-b^2}{2a}\zeta^{-1}, & \zeta \in \mathscr{I}_\emptyset\,, \\ \dfrac{\zeta}{2a_S} - \dfrac{b_S}{a_S} - \dfrac{a_S c_S - b_S^2}{2a_S}\zeta^{-1}, & \zeta \in int(\mathscr{I}_S)\,, \end{cases} \tag{2.11}$$

where $a = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$, $b = \mu^T\Sigma^{-1}\mathbf{1}$, $c = \mu^T\Sigma^{-1}\mu$ and a_S, b_S and c_S are obtained as projections of a, b, c when the the corresponding rows and columns elements from the matrix Σ and vector μ are nullified.

Assume that $\zeta \in \mathscr{I}_\emptyset$. Therefore employing (2.11) with $\zeta = \zeta(\varphi)$ given by (2.8c), the quasi–linear initial value problem (2.9) takes the subsequent form for unknown $\varphi = \varphi(s,x)$ and $\gamma = (ac-b^2)^{-1/2}$:

$$\begin{cases} \dfrac{\partial\varphi}{\partial s} = \dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\dfrac{\partial\varphi}{\partial x}\left[1 + \dfrac{1}{\gamma^2\zeta^2(\varphi)}\right]\zeta'(\varphi)\right. & (s,x) \in (0,T]\times\mathbb{R}\,, \\ \qquad\qquad \left. +2a(1+\varphi)\left(\varepsilon e^{-x} - \beta\right) - \varphi\left[\zeta(\varphi) - 2b - \dfrac{1}{\gamma^2\zeta(\varphi)}\right]\right\}, & \\ \varphi(0,x) = -e^x\dfrac{U''(e^x)}{U'(e^x)}, & x\in\mathbb{R}\,. \end{cases} \tag{2.12}$$

Now it is time to specify the utility function as a linear combination of two CRRA–type (Bergman [4], Pflug and Romisch [35], Pratt [36] or Sharpe [38])utility functions:

$$U(y) = -y^{1-d} + \frac{\lambda}{2}y^{2(1-d)}, \qquad y > 0\,,\; 0 < \lambda \ll 1\,,\; d \gg 1\,.$$

Next, we write φ and U in terms of their asymptotic expansions (see e.g. Holmes [17], Bender and Orszag [3], Hinch [16] or O'Malley [34]) with respect to parameter λ as follows for any $x \in \mathbb{R}$ and $s \in [0,T]$.

$$\varphi(s,x) = \sum_{n=0}^{\infty}\lambda^n\varphi_n(s,x)\,, \qquad \text{and} \qquad U(e^x) = \sum_{n=0}^{\infty}\lambda^n U_n(e^x)\,. \tag{2.13}$$

Thus, the absolute and linear terms φ_0 and φ_1 of (2.13) can be achieved gradually by solving the following pair of sub–problems for the function $\psi = \psi(s,x) = \gamma(1+\varphi(s,x))$

defined $\psi(s,x)=\psi_0(s,x)+\lambda\,\psi_1(s,x)$ for all $s\in[0,T]$ and $x\in\mathbb{R}$:

$$[\mathbf{P_0}]\quad \begin{cases} \dfrac{\partial\psi_0}{\partial s}=\dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\left[1+\dfrac{\partial}{\partial x}\right]\left[\psi_0-\dfrac{1}{\psi_0}\right]\dfrac{\partial\psi_0}{\partial x}\right. \\ \qquad\qquad \left. +2a\,(\varepsilon e^{-x}+p_0)\,\psi_0-\dfrac{\psi_0^2}{\gamma}\right\}, & (s,x)\in(0,T]\times\mathbb{R}, \\ \psi_0(0,x)=\gamma d\,, & x\in\mathbb{R}, \end{cases} \tag{2.14}$$

$$[\mathbf{P_1}]\quad \begin{cases} \dfrac{\partial\psi_1}{\partial s}=\dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\left[1+\dfrac{1}{\psi_0^2}\right]\left[\dfrac{\partial\psi_1}{\partial x}-q_1\psi_1\right]\right. \\ \qquad\qquad \left. +2a[\varepsilon e^{-x}+p_1]\psi_1+2\left[1+\dfrac{1}{\psi_0^2}\right]\gamma q_1 e^{-q_1 x}\right\}, & (s,x)\in(0,T]\times\mathbb{R}, \\ \psi_1(0,x)=\gamma(1-d)e^{(1-d)x}, & x\in\mathbb{R}, \end{cases} \tag{2.15}$$

where $p_0=\frac{b}{a}-\beta$, $p_1(s,x)=-\beta+\frac{b}{a}-\frac{1}{2a}\frac{\psi_0^2-1}{\gamma\psi_0}$ and $q_1=\frac{\psi_0}{\gamma}-1\equiv\varphi_0$.

Firstly, in order to solve approximately the problem $[\mathbf{P_0}]$ (see (2.14)) we apply again the technique of $\psi_0(s,x)$ asymptotic expansion with respect to $0<\varepsilon\ll 1$, hence estimate

$$\psi_0(s,x)\approx\psi_{0,0}(s,x)+\varepsilon\,\psi_{0,1}(s,x)\,.$$

Then evidently, $\psi_{0,0}=\gamma d$ and so what remains is to find the solution to the subsequent Cauchy problem for $\psi_{0,1}(s,x)$

$$\begin{cases} \dfrac{\partial\psi_{0,1}}{\partial s}=\dfrac{1}{2a}\left[1+\dfrac{1}{\psi_{0,0}^2}\right]\dfrac{\partial^2\psi_{0,1}}{\partial x^2} \\ \qquad\qquad +\dfrac{1}{2a}\left[1+\dfrac{1}{\psi_{0,0}^2}+2a\delta\right]\dfrac{\partial\psi_{0,1}}{\partial x}-\psi_{0,0}e^{-x}, & (s,x)\in(0,T]\times\mathbb{R}; \\ \psi_{0,1}(0,x)=0\,, & x\in\mathbb{R}. \end{cases}$$

The linear approximation to the solution of the problem $[\mathbf{P_0}]$ defined by (2.14) is given as

$$\psi_0(s,x)=\gamma d\left(1+\varepsilon\frac{e^{-\delta s}-1}{\delta}e^{-x}\right)+o(\varepsilon^2)\,,\qquad \delta=\frac{b-d}{a}-\beta\,. \tag{2.16}$$

Next, plugging (2.16) into problem $[\mathbf{P_1}]$ (see (2.15)) and setting $\varepsilon=0$ in the resulting problem leads to the following initial value problem for the unknown $\psi_{1,0}=\psi_{1,0}(s,x)$

$$\begin{cases} \dfrac{\partial\psi_{1,0}}{\partial s}=\dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\left(1+\dfrac{1}{\psi_{0,0}^2}\right)\dfrac{\partial\psi_{1,0}}{\partial x}+\left(1+\dfrac{1}{\psi_{0,0}^2}+2a\delta\right)\psi_{1,0}\right. \\ \qquad\qquad \left. +2\gamma(d-1)\left(1+\dfrac{1}{\psi_{0,0}^2}\right)e^{(1-d)x}\right\}, & (0,T]\times\mathbb{R}; \\ \psi_1(0,x)=\gamma(1-d)e^{(1-d)x}, & x\in\mathbb{R}, \end{cases} \tag{2.17}$$

where $\psi_{0,0}$ stands for γd and the parameter δ is prescribed by (2.16). The solution to problem above can be found in the time–space separable form.

It is inevitable to remark that our approximative solution to the unconstrained problem (2.12) is in fact the super–solution to the original problem (2.9) and it is given regardless the number of securities in the investment portfolio as

$$\begin{aligned} \theta^*(s,x)&=\frac{\Sigma^{-1}}{a}\left[\mathbf{1}+(a\mu-b\mathbf{1})[\zeta(s,x)]^{-1}\right], \qquad (s,x)\in\Omega, \\ \text{where}\qquad \zeta^{-1}(s,x)&\approx\frac{1}{d}+\varepsilon\frac{\Phi_\varepsilon(s)}{d^2}e^{-x}+\lambda\frac{\Phi_\lambda(s)+1}{d^2}e^{-(d-1)x}, \end{aligned} \tag{2.18}$$

on the region Ω defined as follows:

$$\begin{aligned} \Omega&\equiv\{(s,x)\in[0,T]\times(\Lambda,\infty)\,,\ d-\varepsilon\Phi_\varepsilon(s)e^{-x}-\lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x}>0\}, \\ \text{for}\qquad \Lambda&\equiv\frac{1}{d-1}\ln\left[\lambda\frac{2d-1}{d}\right], \end{aligned} \tag{2.19}$$

with the auxiliary functions $\Phi_\varepsilon=d\delta^{-1}(1-e^{-\delta s})$ and $\Phi_\lambda=(d-1)[(1+\widetilde{\phi})e^{\widetilde{\delta}s}-\widetilde{\phi}]$ for $\widetilde{\delta}$ and $\widetilde{\phi}$ arising from the unique solution to (2.17).

3 Results

We have derived a simple analytic mechanism that in general for unrestrained number of securities considered in the saver's portfolio, enables us to estimate the optimal investment policy with active natural ban on short positions. Hence, a typical future pensioner is advised to consider the investment strategy determined by the following formula while deciding how to allocate the the wealth already accumulated on his/her pension account among various pension funds.

$$\hat{\theta}(t,y) = \begin{cases} \frac{1}{a}\Sigma^{-1}\left\{\mathbf{1} + (a\mu - b\mathbf{1})\frac{1}{\zeta(T-t,\ln y)}\right\}, & \zeta \in \mathscr{I}_{\emptyset}, \\ \frac{1}{a_S}\Sigma_S^{-1}\left\{\mathbf{1} + (a_S\mu_S - b_S\mathbf{1})\frac{1}{\zeta(T-t,\ln y)}\right\}, & \zeta \in int(\mathscr{I}_S), \end{cases} \tag{3.1a}$$

where

$$\zeta^{-1}(s,x) \approx \frac{1}{d} + \varepsilon\frac{\Phi_\varepsilon(s)}{d^2}e^{-x} + \lambda\frac{\Phi_\lambda(s)+1}{d^2}e^{-(d-1)x} \tag{3.1b}$$
$$\text{for} \quad \Phi_\varepsilon = d\delta^{-1}(1-e^{-\delta s}), \quad \text{and} \quad \Phi_\lambda = (d-1)[(1+\widetilde{\phi})e^{\widetilde{\delta}s} - \widetilde{\phi}].$$

Furthermore, providing that we are interested in unconstrained optimal policy, in (3.1a) we apply the branch associated with $\mathscr{I}_{\emptyset}$. Otherwise, for any (even empty) subset $S \subset \{0,\ldots,N\}$ the sets $\bigcup_{0\le|S|\le N-1} int\,(\mathscr{I}_S)$ are defined such that

- $\mathscr{I}_{\emptyset}$ the set of all $\zeta > 0$ for which the unique minimizer $\hat{\theta}(\zeta) \in \Delta$ has positive components only,
$$\mathscr{I}_{\emptyset} = \left\{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) > 0,\ \forall i = 1,\ldots,N\right\},$$
- For any subset S of $\{1,\ldots,N\}$ the set $\mathscr{I}_S$ of all functions $\zeta > 0$ for which the index set of $\hat{\theta}(\zeta) \in \Delta$ zero components coincide with S;
$$\mathscr{I}_S = \left\{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) = 0 \iff i \in S\right\}.$$

The coefficients a_s, b_S, c_s are determined for the problem dimension reduced to lower $N-|S|$ dimensional simplex Δ_S with nullified rows and columns elements from the matrix Σ and vector μ corresponding to components with index belonging to S and satisfy

$$a = \mathbf{1}^T\Sigma^{-1}\mathbf{1}, \quad b = \mu^T\Sigma^{-1}\mathbf{1}, \quad c = \mu^T\Sigma^{-1}\mu,$$
$$a_S = \mathbf{1}^T\Sigma_S^{-1}\mathbf{1}, \quad b_S = \mu_S^T\Sigma_S^{-1}\mathbf{1}, \quad c_S = \mu_S^T\Sigma_S^{-1}\mu_S.$$

Moreover, in case of the *one–stock–one–bond* problem, one can simplify the constrained optimal policy into the subsequent explicit decision tool:

$$\widehat{\theta}^{(s)}(t,y) = \begin{cases} \frac{1}{\alpha_\sigma}\left[\beta_\sigma + \frac{\Delta\mu}{\zeta(T-t,\ln y)}\right], & \zeta(T-t,\ln y) \in \Omega_2^*, \\ 1 & 0 < \zeta(T-t,\ln y), \quad \zeta(T-t,\ln y) \notin \Omega_2^*, \end{cases} \tag{3.2}$$
$$\alpha_\sigma \equiv \left[\sigma^{(s)}\right]^2 - 2\rho\sigma^{(s)}\sigma^{(b)} + \left[\sigma^{(b)}\right]^2 > \beta_\sigma \equiv \sigma^{(b)}\left[\sigma^{(b)} - \rho\sigma^{(s)}\right].$$

and the region on which ζ follows the prescription (3.1b),

$$\Omega_2^* \equiv \left\{(s,x) \in [0,T]\times(\Lambda,\infty),\ \varepsilon\Phi_\varepsilon(s)e^{-x} + \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x} < d - \frac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}\right\}. \tag{3.3}$$

3.1 Sensitivity Analysis

For the case of *one–stock–one–bond* problem we have shown that the unconstrained optimal policy planar problem approximate solution exhibits the subsequent very intuitive qualities which are in consistence with the reality observed:

- falls monotonically in both *wealth–to–salary* y and time t and it is strictly convex in y,
- descends in risk aversion coefficient d,
- raises in both small model parameters: contribution rate ε and return volatility sensitivity parameter λ,
- augments in both gross wage growth rate β and retirement age T,
- enlarges in stock returns $\mu^{(s)}$ and drops in bond returns $\mu^{(b)}$,
- decreases in both stock returns volatility $\sigma^{(s)}$ and the coefficient of correlation between the returns of stocks and bonds, ρ, while grows with bond returns volatility $\sigma^{(b)}$.

Therefore the solution attributes mentioned above have led in several policy implications and recommendations summarized below.

3.2 Applications

The goal of first application presented in this paper is to establish the saver's optimal strategy in pension fund selection, conditioned primarily by their time to retirement, intermediate *wealth-to-salary ratio* and various model parameters.

Referring to Slovak pension system and its private Second pillar from the saver's point of view the investment decision essence lies in detecting the best fitting ratio between resources allocated to the *Equity–Linked Index Fund* (symbolizes stocks) and the *Bond Fund* (depicts bonds). In this sense, even the *Equity–Linked Index Fund* allocation strategy applied when replicating the performance of the benchmark prescribed by the pension fund management, is unlimited in the choice of stocks, financial derivatives or exchange traded funds, for the sake of simplicity we assume the fund investment decisions restricted in stocks only. Moreover we remark that in this problem there exists an obvious restriction on the stocks and bonds proportions - naturally, both ratios must be non-negative, so that no short–selling is allowed.

Asset	2009-2012			2003-2012		
	mean	st.deviation	correlation	mean	st.deviation	correlation
MSCI World	0.1053	0.1423	-0.8344	0.0763	0.2050	-0.1688
10-Y Slovak Bonds	0.0439	0.0036		0.0447	0.0047	
DAX	0.1328	0.1738	-0.1727	0.1286	0.2270	-0.0951
10-Y German Bunds	0.02552	0.0063		0.0335	0.0085	
S&P500	0.1242	0.0964	-0.1518	0.0667	0.1799	-0.0535
10-Y US treasuries	0.0276	0.0068		0.0367	0.0092	

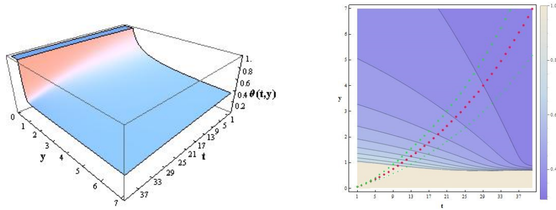
Source: Bloomberg, MSCI, ECB, EuroStat, US Treasury

Table 1: Descriptive statistics of selected market data observed in periods 2009–2012 and 2003–2012

Next, we have brought the model to Slovak data. According to recently changed Slovak legislature, in September 2012 the regular contribution level of a private scheme participant dropped from their original value of 9% to 4% of his/her gross wage. This rate prescription is valid until 2017 and then gradually raises by 0.25 p.p. such that in 2024 it attains the value of 6%. Hence in the baseline scenario we set $\varepsilon = 0.06$. As ε plays the key role not only in this model, but in its actual application to Slovak pension system, we have tested several levels of ε to scrutinize the model outcomes for various ε values and study how its value affect both the portfolio component weights and the expected terminal *wealth–to–salary* payoffs. Moreover, since each private asset management company charges fund management fees defined as 1% of an investor's contribution, within our model we use the effective contribution rate in all scenarios. We have assumed the overall time period $T = 40$ of saving of an individual pensioner, the value of their risk aversion attitude coefficient was estimated on 0.04, i.e. $\lambda = 0.04$ and the Arrow–Pratt risk aversion related coefficient $d = 10$. Regarding the average gross wage growth rate we adopted the expert judgement taken from the from the Slovak Institute of Financial Policy macroeconomic forecast (see Ministry of Finance of the Slovak Republic [30]) and estimated it as for 3.5% p.a., i.e. $\beta = 0.035$. Concerning financial market data, we pay attention to the recent time periods: 2009–2012 and 2003–2012. Next, the investment portfolio consists of two securities: 10−Year zero coupon Slovak Government Bonds and *MSCI All Country World Index*. Our choice of these financial assets comes from real composition of pension funds in Slovakia. For the comparison purpose, we provide another two pair of investment options, namely 10-Year US Treasury Bonds versus S&P500 index, and 10-Year German Bunds versus DAX index, with the descriptive statistics summarized in Table 1.

On Figure 2 we present the 3D plots as well as the contour plots of the constrained optimal share of assets (represented by the MSCI World index) in the pension fund portfolio consisting of 10-Year zero coupon Slovak Government Bonds and MSCI World index calculated based on financial market data in time periods 2009–2012 and 2003–2012, respectively. This constrained optimal share $\widehat{\theta}^{(s)}$ is modelled as a function of time $t \in [0, T]$ and *wealth–to–salary ratio* y. The optimal investment strategy is constrained as the share of assets cannot exceeds 100% since borrowings are forbidden.

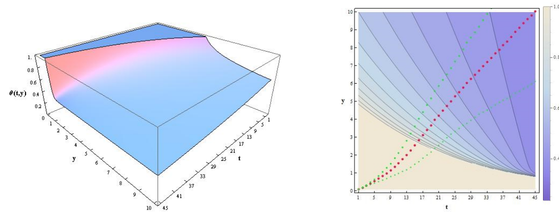
On all contour plots (Figures 2b, 3b, 4b) the mean portfolio wealth $\mathbb{E}[y_t]$ (red dot line)



(a) 3D plot of the constrained optimal share of MSCI All Country World Index (2009–2012 data)

(b) Contour plot of the constrained optimal share of MSCI All Country World Index (2009–2012 data)

Figure 2: 3D plot and Contour plot of the constrained optimal share of MSCI World index in the portfolio of 10-Year Slovak Government Bonds and MSCI All Country World Index, based on data between 2009-2012



(a) 3D plot of the constrained optimal share of DAX Index (2009–2012)

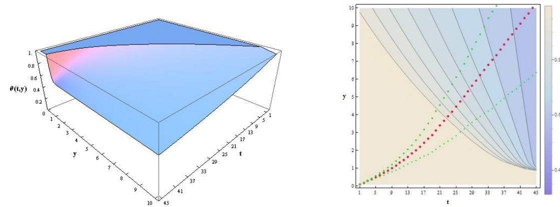
(b) Contour plot of the constrained optimal share of DAX Index (2009–2012)

Figure 3: 3D plot and Contour plot of the constrained optimal share of DAX Index in the portfolio of 10-Year German Bunds and DAX World Index, based on data between 2009–2012

is obtained by performing 10000 Monte-Carlo simulations calculated according to the recurrent equation (2.1). The green dot lines depict the mean wealth plus/minus one standard deviation of the random variable. The simulations were attained employing the optimal share of stocks in the pension fund portfolio $\widehat{\theta}^{(s)} = \widehat{\theta}^{(s)}(t, y)$ depending on the value of simulated yearly accumulated wealth y_t at time t and at the terminal time $t = T$.

For the comparison purpose we present similar plots for alternative investment strategies: DAX index versus 10–Year zero coupon German Bunds (Figure 3), and the S&P500 Index with 10–Year US Treasuries (Figure 4). We consider the baseline setting for all model parameters but the financial market data which are taken from Table 1 and evaluate the constrained optimal policies with financial market data observed between 2009–2012.

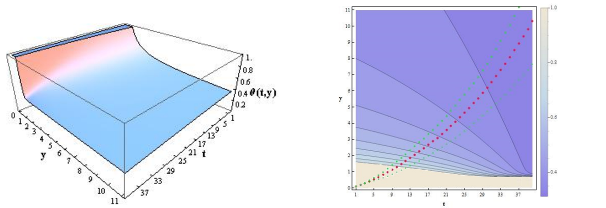
Providing that financial market data from 2009–2012 were applied and the portfolio consisting of Slovak bonds and MSCI All Country Worlds Index was assumed, we observe that at the end of simulation period, $t = T$, the average accumulated *wealth–to–salary ratio*



(a) 3D plot of the constrained optimal share of S&P500 Index (2009–2012)

(b) Contour plot of the constrained optimal share of S&P500 Index (2009–2012)

Figure 4: 3D plot and Contour plot of the constrained optimal share of S&P500 Index in the portfolio of 10-Year US Treasuries and S&P500 Index, based on data between 2009–2012



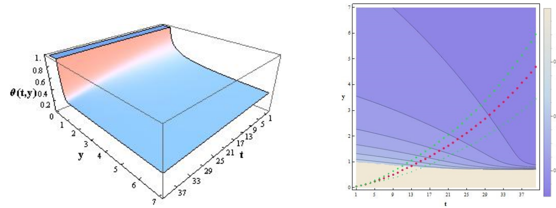
(a) 3D plot of the constrained optimal share of MSCI Index for $\varepsilon = 0.09$

(b) Contour plot of the constrained optimal share of MSCI Index for $\varepsilon = 0.09$

Figure 5: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the saver's regular contribution rate ε raises to 9%, based on financial market data from 2009-2012.

$\mathbb{E}[y_T] \approx 7.05$ meaning that the future pensioner following the optimal investment strategy given by $\widehat{\theta}^{(s)}$ has accumulated approximately 7.05 multiples of saver's last yearly salary. During 2009–2012 both DAX and S&P500 indices outperformed MSCI index and Slovak bond yield surpassed the yields of foreign bonds with comparable volatilities. So it should be better for a saver to prefer more dynamic (i.e. risky) investment strategy in both alternative portfolios and thus on average accumulate much higher *wealth–to–salary ratio*, more than 10 multiples of saver's last yearly salary.

Within our dissertation theses we have provided the description of the effects of changes in all key model parameters on the constrained optimal share of the MSCI All Country World index in the investment portfolio $\widehat{\theta}^{(s)}$, and on the terminal average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T]$. Inasmuch as we want to emphasize the consequences of variations in model factors that are directly predetermined by policy–makers and legislative rules, we aim our attention particularly on the sequels of fluctuations in prescribed con-



(a) 3D plot of the constrained optimal share of MSCI Index for $\varepsilon = 0.04$

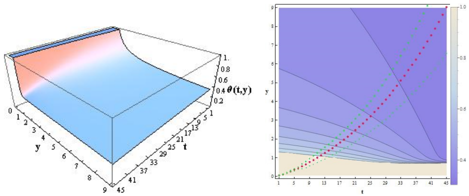
(b) Contour plot of the constrained optimal share of MSCI Index for $\varepsilon = 0.04$

Figure 6: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the saver's regular contribution rate drops to 4%, based on financial market data from 2009-2012.

tribution rate ε and retirement age T. Within the model we considered financial market data from the period 2009–2012.

Saver's Contribution Rate ε. Firstly, on Figure 5 we propose the illustration of the optimal policy behaviour under the crucial model structural parameter variation – we ponder the *2012 - no policy change scenario* increase the saver's regular contribution rate ε from 6% to 9% per year and observe higher share of risky investment during the whole accumulation period in compare to the case of $\varepsilon = 0.6$ (Figure 5a) and an essential rise in the terminal average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T] \approx 10.65$ (Figure 5b). Hence, assuming the saver's equal contribution to both mandatory pillars of the Slovak pension scheme generating the same expected future pay–offs, the future pensioner may expect to be able to cover the expenses during approximately 21 years of his/her retirement with the lower level of government implicit liabilities. Furthermore, his/her investment strategy is aimed more on risky assets in compare to the baseline scenario, as in the first half of the the accumulation period more than 3/4 of his wealth is stored in the MSCI Index – and even in the 10 years this share does not decline below 40% of the portfolio.

On the other side, considering the temporal drop of the contribution rate ε to 4% of yearly salary for a permanent leads to a core conservative strategy in terms of a substantial fall in MSCI Index weight in the pension fund investment portfolio accompanied with decline in the terminal average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T] \approx 4.65$ (see Figures 6a–6b). In our concrete application, 4% saver's regular contribution to the private pension scheme represent only 22% of his/her overall pension system payments. Then, assuming the proportional expected future pay–offs from both public and private mandatory schemes the future pensioner may expect to be able to cover the expenses during approximately 21 years of his/her retirement with the substantially higher level of government implicit liabilities.



(a) 3D plot of the constrained optimal share of MSCI Index for $T = 45$

(b) Contour plot of the constrained optimal share of MSCI Index for $T = 45$

Figure 7: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI All Country World Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the accumulation period length T increases to 45 years, based on financial market data from 2009-2012.

Accumulation Period Length T: Next, on Figure 7 we provide the depiction of the optimal policy behaviour under the changes the accumulation period length T.

Elongated working life (equivalent for accumulation period T prolongation) to 45 years (in contract to 40 years in the baseline scenario) has an effect on raising share of risky asset in the portfolio (there is a slower shift towards less risky bond and even in the last decade of the accumulation period more than 40% of wealth is placed in the risky stock) and thus causes even larger expected terminal wealth–to–salary ratio $\mathbb{E}[y_T] \approx 9.05$.

This is an intuitive scenario as the return volatility accompanying stocks is spread over time while the portfolio value is expected to raise above the one with lower share of stocks. Hence a more aggressive investment strategy is allowed as there is more time to wipe off possible losses associated with risky investment – therefore retirement age delay brings in higher expected terminal payoffs for the investor.

4 Conclusion

The main objectives of this dissertation thesis were to formulate properly the continuous–time pension investment portfolio selection problem that encounters any participant of the Second pillar of the Slovak pension system properly, and find the relationship between optimal portfolio allocation policy and its intermediate value function. We were aimed to build a simple explicit analytic decision mechanism estimating a future pensioner's optimal portfolio selection strategy that based on a saver's time to retirement and already allocated wealth advice him/her how to allocate his/her wealth optimally between unlimited number of more or less risky securities. The decision formula derived in this thesis reflects individual characteristics of a risk–sensitive investor (risk aversion attitude, gross wage growth rate), existing government restrictions (retirement age, contribution rate) and financial market data.
Furthermore we concentrated our attention to provide a deep analysis of the optimal investment strategy decision tool from a qualitative and quantitative perspective on which basis we emphasized fundamental policy implications and recommendations. We calibrated the model on Slovak data and illustrate its behaviour on various examples.

We certify that the fundamental objectives of this dissertation thesis stated in Section 1 were accomplished, the obtained results are in consistence with intuition and reality observations.

Furthermore, based on the qualitative and quantitative properties of the unconstrained optimal policy, we formulate the subsequent policy recommendations: In order to increase the Second pillar retirement benefit of a future pensioner we recommend the policy–makers to increase regular contribution rate ε, elevate the retirement age and reduce fees charged by the private asset management companies. A future pensioner is advised to be more aggressive in his/her investment decision in the beginning of the active life and as time approaches the planned retirement age and the amount of allocated wealth on his/her pension account raises, decline gradually the share of investment in risky assets while moving towards more safe financial market instruments. Hence, a typical saver should start with risky stocks (or stock indices) and then in very last years before retirement switch to highly rated bonds. Furthermore we suggest a saver to ponder carefully his/her risk aversion attitude, so that very risk–aware investor should choose more conservative investment strategy with higher share of bonds in the investment portfolio. On the other side, providing that a saver's gross wage growth rate increased he/she should follow more dynamic investment strategy. Finally, due to volatile financial markets active portfolio management is essential – hence, the pension fund portfolio weight of the financial instrument which appreciates or its returns are getting more stable (i.e. returns are higher or less volatile) raises.

Moreover, as the derived theoretical model has been calibrated on Slovak data, its results can be applied on a typical participant of the Second pillar of Slovak pension system deciding how to split optimally the wealth already allocated on his/her pension account between the Index Fund (represented by MSCI All Country World Index) and Bond Fund (deputized by 10–year zero coupon Slovak government bonds). Providing that the regular contribution rate ε is 6%, the accumulation period length $T = 40$ and 2009–2012 financial market data are taken, saver is advised to place more that 50% of his wealth in the Index Fund in the first 30 years of the accumulation period (more than 65% during the first half of the active life) and even in the last decade this proportion should not fall under 30%. Following such strategy would bring him/her approximately 7 yearly salaries. Furthermore, if the contribution rate increases to 9%, he/she might expect to earn around 10.65 yearly salaries emulating more aggressive strategy with more than 3/4 of investment allocated in the Index fund during the first half of active life and more than 40% in the last decade. A similar effect can be observed when the retirement age is elevated – it is optimal for a future pensioner to choose more dynamic strategy with high share of wealth invested in the Index Fund and slower shift towards Bond Fund yielding in 9 yearly salaries saved.

Abstract

This dissertation thesis analyses solutions to a specific fully nonlinear Hamilton–Jacobi–Bellman equation arising from the problem of optimal investment portfolio construction that encounters a risk sensitive future pensioner, a typical participant of the private defined–contribution based Second pillar of the Slovak pension system.
We show how the Hamilton–Jacobi–Bellman equation can be converted using the Riccati transform into a Cauchy–type quasi–linear parabolic differential equation and solve the associated parametric convex optimization problem. The weak solution to the studied problem is approached by its double asymptotic expansion with respect to small model parameters and utilized to build the analytical model which serves us to estimate the investor's optimal pension fund selection strategy. We provide the analysis of the optimal policy from qualitative as well as quantitative point of view and formulate main policy implications and recommendations that are applicable for all: policy–makers, pension fund managers, and the Second pillar participants.
Finally, we bring to model to Slovak data and illustrate how the optimal investment strategies and saver's expected terminal wealth accumulated on his/her pension account change depending on model calibration and its key parameters.

Abstrakt

Táto dizertačná práca analyzuje riešenie špeciálnej plne nelineárnej Hamilton–Jacobi–Bellmanovej rovnice vyplývajúcej z problému tvorby optimálneho portfólia ktorému čelí typický budúci dôchodca, rizikoaverzný účastník druhého piliera slovenského penzijného systému.
Použitím Riccatiho trasformácie ukazujeme premenu pôvodnej Hamilton–Jacobi–Bellmanovej rovnice na začiatočnú kvázilineárnu parabolickú úlohu a riešime príslušný parametrický konvexný optimalizačný problém. Využitím techniky dvojitého asymptotického rozvoja aproximujeme slabé riešenie študovaného problému a vzniknuý analytický model použijeme na určenie sporiteľovej optimálnej investičnej stratégie. Model optimálnej investičnej stratégie analyzujeme z kvalitatívneho aj kvantitatívneho hľadiska a vyvodzujeme hlavné politické závery a odporúčania určné tvorcom legislatívy, správcom penzijných fondov aj sporiteľom v druhom pilieri slovenského penzijného systému.
Nakoniec model nakalibrujeme na slovenské dáta. Pomocou neho ilustrujeme zmeny v sporiteľovej optimálnej investičnej stratégii a očakávanom majetku naakumulovanom na jeho osobnom penzijnom účte, ako dôsledok rôznych nastavení kľúčových parametrov modelu.

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