COMENIUS UNIVERSITY, BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

The risk sensitive dynamic accumulation model and optimal pension saving management

DISSERTATION THESIS

2014 Mgr. Zuzana Múčka

COMENIUS UNIVERSITY, BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

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Abstract

This dissertation thesis analyses solutions to a fully non–linear Hamilton–Jacobi–Bellman equation arising from the problem of optimal investment portfolio construction that encounters a risk sensitive future pensioner, a typical participant of the private defined–contribution based Second pillar of the Slovak pension system. We show how the Hamilton–Jacobi–Bellman equation can be converted using the Riccati transform into a Cauchy–type quasi–linear parabolic differential equation and solve the associated parametric convex optimization problem. The weak solution to the studied problem is approached by its double asymptotic expansion with respect to small model parameters and utilized to build the analytical model which serves us to estimate the investor's optimal pension fund selection strategy. We provide the analysis of the optimal policy from qualitative as well as quantitative point of view and formulate main policy implications and recommendations that are applicable for all – policy–makers, pension fund managers, and the Second pillar participants.
Finally, we bring to model to Slovak data and illustrate how the optimal investment strategies and saver's expected terminal wealth accumulated on his/her pension account change depending on model calibration and its key parameters.

Keywords
Hamilton-Jacobi-Bellman equation, weakly nonlinear analysis, asymptotic expansion, quasi–linear parabolic equation, parametric convex optimization, stochastic dynamic programming, Riccati transformation, pension savings accumulation model.

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Abstrakt

Táto dizertačná práca analyzuje riešenie plne nelineárnej Hamilton–Jacobi–Bellmanovej rovnice vyplývajúcej z problému tvorby optimálneho investičného portfólia ktorému čelí typický budúci dôchodca, rizikoaverzný účastník druhého piliera slovenského penzijného systému.
V práci ukazujeme použitím Riccatiho trasformácie premenu pôvodnej Hamilton–Jacobi–Bellmanovej rovnice na začiatočnú kvázilineárnu parabolickú úlohu a riešime príslušný parametrický konvexný optimalizačný problém. Využitím techniky dvojitého asymptotického rozvoja aproximujeme slabé riešenie študovaného problému a vzniknuý analytický model použijeme na určenie sporiteľovej optimálnej investičnej stratégie. Model optimálnej investičnej stratégie analyzujeme z kvalitatívneho aj kvantitatívneho hľadiska a vyvodzujeme hlavné politické závery a odporúčania určné tvorcom legislatívy, správcom penzijných fondov aj sporiteľom v druhom pilieri slovenského penzijného systému.
Nakoniec model nakalibrujeme na slovenské dáta. Pomocou neho ilustrujeme zmeny v sporiteľovej optimálnej investičnej stratégii a očakávanom majetku naakumulovanom na jeho osobnom penzijnom účte, ako dôsledok rôznych nastavení kľúčových parametrov modelu.

Kľúčové slová

Hamilton-Jacobi-Bellmanova rovnica, slabo nelineárna analýza, asymptotický rozvoj, kvázilineárna parabolická rovnica, parametrická konvexná optimalizácia, stochastické dynamické programovanie, Riccatiho transformácia, akumulačný model penzijného sporenia.

Preface

The key objective of this study is to determine and investigate the optimal strategy that the future pensioner – the participant of the Second pillar of the Slovak pension system – should follow in order to attain to maximize their expected future pension income from the Second pillar with respect to their specific risk aversion. Based on their personal characteristics, legislative regulations and financial market data we derived the analytic model that formulates the optimal decision for the investor about the specific pension fund selection. Furthermore, besides the model advisory role in the investor's optimal fund selection strategy, this model also helps future pensioners to perceive the aspects impacting the level of their retirement pensions.

The decision about the optimal allocation is made in perspective of the investor interested in the portfolio terminal value, via their utility criterion with both the portfolio expected terminal utility and risk combined. The problem is postulated in terms of the solution to the Hamilton–Jacobi–Bellman equation derived from continuous version of the dynamic stochastic optimization model for the portfolio value function. We show how the fully non–linear Hamilton–Jacobi–Bellman equation can be transformed into a quasi–linear parabolic differential equation. The weak solution to the problem is approached by its double asymptotic expansion with respect to small model parameters and utilized to estimate the optimal investment strategy. We present key attributes of the optimal allocation policy determined by our model and illustrate it on the problem of optimal fund selection in the Second Pillar of the Slovak Pension System.

At this place I would like to give thanks to all those who made this work possible. First of all, I would like to express my gratitude towards my supervisor Daniel Ševčovič for his guidance in my research, patience, great support and ideas. Moreover, I very am grateful to my husband for his unreserved and continuing moral support, relentless cheering, understanding and critical remarks. Finally, I would like to appreciate help and support of my colleagues who inspired me a lot by a continuous flow of their valuable comments and fresh ideas. Thank you all.

Bratislava, June 2014 Zuzana Múčka

Contents

Chapter 1

INTRODUCTION

Recently a problem of active portfolio management has transformed from a marginal problem even hardly interested only for some financial plungers considered by the major population more or less for wheeler–dealers, to a new and definitely not expected position. Its conversion to a brain–teaser for various scientists tempted by the observed indeterminism where the number of considered factors can be ostensibly unlimited, high complexity, or many freshly discovered phenomena has stimulated the others to change their attitudes and open their minds. A very important feature that makes the whole problem even more attractive is a surprising observation that any comprehensive, sophisticated prediction on future return sometimes simply cannot beat a human intuition and experience of past history. A globalised world where a proper information may be inestimable, rapid economical, industrial and technological development, lot of significant structural changes in the society imposed by demographical evolution, a possibility of private investment or social welfare sustainability – these are the challenges of today's world that motivates the decision makers to go further in their constructions in order to attain higher returns or lower uncertainty in payoffs.
The inevitable economy and social care reforms (e.g. tax reform, the pension system reform, healthcare & long–term care reforms) are being ultimately underwent in many Western culture countries and remain particularly relevant in Slovakia due to two ticking time bombs – poor demography trends and long–term public finance sustainability. The projected dramatic changes in the population structure, demographic prospects (characterized by drop in fertility rate, longevity increase and extreme raise of dependency ratio) and economic effects of ageing populations causing a significant pressure on public finance (due to high share of ageing and demographic structure related share on expenditure), slowing potential economic growth and labour market permanent structural changes have strong implications for pensions and overall budgetary effects of ageing populations.
Hence all these reasons mentioned above prod policy–makers to rebuilt the paradigms about the participation rôle and responsibility of current generation of active and pre–active individuals on their future income.

Therefore facing definitely not rosy future, they also aim their attention to the optimal long–term saving schemes, investment decisions and possible financial instruments that can bring additional cash–flow for future pensioners and thus at least partially reduce the future load

of claim on public finance. Thus nowadays the momentous question of optimal and safe saving on pension emerges and it is posed not only by policy makers, financier and economists but also by a non–myopic part of currently active population.
As we are also concerned about this issue the purpose of this dissertation thesis is to ask and look for a proper solution to this problem, derive an optimal pension strategy model that will fit the Slovak pension system, namely, its private, defined–contribution pillar. The individual's private pension at retirement is substantially susceptible to investment allocation policy preferred during the active life of the pensioner as Slovak private pension scheme is built on defined contribution idea - pension benefits depend on returns of the pension fund's portfolio financed via fixed regular contributions of future pensioner during the accumulation period who borne financial risk associated with investment. Therefore, the optimal wealth allocation strategy is the fundamental issue of this thesis.

In order to solve the optimal investment strategy puzzle both a portfolio manager and a long–headed future pensioner pose the forthcoming questions. How to design optimally the portfolio allocation policy in the long–term investment plan of a particular saver for the purpose of her/his future pension provided that he/she is eligible to alter this decision continuously any time up to his/her retirement date? How does such strategy changes over time and how the amount of resources already allocated or by volatile financial market data influence it?
Obviously, we need to take into consideration a human–natural risk aversion attitude and his/her personal characteristics. Thus, how varying future pensioner's risk attitude and gross wage growth rate modify the investor's optimal allocation policy? When to prefer risky securities and when to be satisfied by more conservative investment?
Furthermore, we are interested in the function of policy makers. What is their role in defining pension saving legislative norms and place regulations on investment decisions of working people? Are the legislative limitations consistent with optimal investment behaviour or the exiting framework dictates sub–optimal strategies, even containing contra–productive regulations? Mainly, besides various investment regulations how do two key factors prescribed by the government – regular contribution rate and retirement age – affect both the optimal investment strategies and terminal allocated wealth (and hence, saver's pension benefit) under existing legislative framework? And what is the influence of changing managerial fees charged by the private asset management companies operating in Slovakia?

Hence, the aim of this dissertation thesis is to provide suitable answers to the questions posed above. We elaborate this problem in terms of analytical model built in order to describe the optimal pension fund selection problem that encounters a foreseeing participant of the private scheme of the Slovak pension system who struggles to maximize his/her expected future cash flow from the private pension scheme by following the optimal investment strategy determined by our model. This optimal strategy about an employee–specific fund selection is formulated given his/her time to retirement, the amount of resources already allocated measured relatively to his/her income and conditionally of his/her personal attributes (gross wage growth rate, risk attitude, time to retirement age), financial market performance data and legislative framework (investment restrictions, retirement age, contribution rate).

1.1 Thesis Objectives

This dissertation thesis stakes out the following fundamental targets:

- Formulate the continuous–time pension investment portfolio selection problem that encounters any participant of the Second pillar of the Slovak pension system properly, and find the relationship between optimal portfolio allocation policy and its intermediate value function;
- Provide (at least approximative) a simple explicit analytic decision mechanism estimating a future pensioner's optimal portfolio selection strategy that based on a saver's time to retirement and already allocated wealth advice him/her how to allocate his/her wealth optimally between unlimited number of more or less risky securities;
  - The decision formula should reflect individual characteristics of a risk–sensitive investor (risk aversion attitude, gross wage growth rate), existing government restrictions (retirement age, contribution rate) and financial market data;
- Analyse properly the optimal investment strategy decision tool from a qualitative and quantitative perspective; and highlight the resulting policy implications;
- Calibrate the model on Slovak data and illustrate its behaviour;
  - Show how both the optimal allocation policy and the expected terminal portfolio wealth are affected by varying model parameters;
  - Accentuate the effects of changes in fiscal policy parameters – prescribed retirement age and contribution rate;

Beside them, our aim is also top provide a deep explanation of the three–pillar Slovak pension system and its undergoing reforms, legislative framework and key concepts:

- Clarify and support with data reasons for the pension system reform and describe the main aspects of the reform in the First pillar;
- elucidate the scope of the private Second pillar, the scope of available wealth allocation policies and existing government regulations and support with data the actual investment decisions of its participants.

1.2 Literature Review

Technically we are focused on approximative analytic solution to a specific Hamilton–Jacobi–Bellman equation arising from stochastic dynamic programming for trading the optimal investment decision technique for an individual investor during accumulation of pension savings.

Such an optimization problem often emerges in optimal dynamic portfolio selection and asset allocation policy for an investor who is concerned about the performance of a portfolio relative to the performance of a given benchmark. We take as our baseline the standard continuous–time settings pioneered by Bodie et al. [11], Bodie et al. [10], Browne [12], Samuelson [61], Merton [49] who were interested in optimal consumption–portfolio strategies, life–cycle model or Songzhe [66].

Obviously, there are numerous recent very practically oriented models preferring discrete–time defined contributions pension scheme framework e.g. Gao [27], Kim and Noh [39], Haberman and Vigna [28] or Noh [52]. Within our work we use the principles of investor's risk–sensitivity deeply studied in Bielecki et al. [9].

In this work we refer to novel papers of Múčka [51], Kilianová et al. [37], Macová and Ševčovič [44], Macová [43] and Kilianová and Ševčovič [38]. In Kilianová et al. [37] the baseline dynamic accumulation model for the private second defined–contribution pillar of the Slovak pension system was firstly introduced.
This model was extended and studied later in Melicherčík and Ševčovič [47]. Furthermore, in Macová and Ševčovič [44] a simplified analytic tool to determine the optimal investment strategy for a participant of the second pillar of the Slovak pension system was developed and its very first quantitative and qualitative analysis was provided.

This instrument along with a similar one obtained via transformation of the originally stated Hamilton–Jacobi–Bellman problem into a quasi–linear equation presented by Kilianová and Ševčovič [38] and very new paper of Múčka [51] studying so–called *one–stock–one–bond* (portfolio composition problem is limited to only one pair of quite risky and relatively safe securities) problem employing the portfolio value function method, inspired us to build a new extended model. Its solution was estimated applying the techniques of Riccati transformation used by e.g. Abe and Ishimura [1], Ishimura and Ševčovič [35], Ishimura and Mita [33] and Ishimura and Nakamura [34] and asymptotic expansion method (see Holmes [30], Bender and Orszag [6], O'Malley [57] and Hinch [29]) allowed us to determine the explicit approximative analytic optimal allocation policy formula.
In opposed to previously assumed models, the investor's utility criterion ponder also the the aspect of the portfolio returns volatility – to endow this attribute into our model we reutilized the approach of e.g. Sharpe [62], Bielecki et al. [9], Songzhe [66] or Markowitz [45]. Finally this dissertation thesis is built on fundamentals of the author's dissertation project text (see Macová [43]).

1.3 Used Methodology Overview

In order to derive the model determining the explicit approximative analytic optimal allocation policy formula for a future pensioner we start with a simple discrete–time optimal portfolio composition problem on finite time–horizon, which was deeply studied in Kilianová et al. [37], Macová and Ševčovič [44], Kilianová [36] or Macová [43]. Each period a typical sever transfers a fraction ε of his/her salary with a deterministic growth rate β to a his/her portfolio consisting of only one risky stock and one quite safe bond instrument and has to make a decision about proper proportion of risky stock proportion in this portfolio. For the sake of simplicity, we presume that the investment strategy of the pension fund at time t is given by the proportion $\theta \in [0,1]$ of stocks and $1-\theta$ of bonds and the portfolio return $r_t = r_t(\theta) \sim \mathscr{N}(\mu_t(\theta), \sigma_t^2(\theta))$ is normally distributed for any choice of the stock to bond proportion θ. Thus, in terms of the quantity y_t representing the number of yearly salaries already saved at time $t = 0, 1, \ldots, T-1$, the budget–constraint equation can be reformulated recurrently as follows:

$$y_1 = \varepsilon\,, \quad y_{t+1} = G_t^1(y_t, r_t(\theta_t))\,, \quad \text{for} \quad G_t^1(y, r_t) = \varepsilon + y\frac{1+r_t}{1+\beta_t}\,,\ t = 1,2,\ldots,T-1\,. \tag{1.1}$$

Assuming the knowledge of the saver's utility function U, our aim is to determine the optimal value of the weight θ at each time t that maximizes the contributor's utility from the terminal wealth allocated on their pension account. Thus, the problem of discrete stochastic dynamic programming can be formulated as

$$\max_{\theta \in \Delta} \mathbb{E}(U(y_T) \mid y_t = y)\,, \tag{1.2}$$

subject to the constraint (1.1) where the maximum in the stochastic dynamic problem is taken over all non-anticipative strategies, stocks proportions $\{\theta\}_t^T \in \Delta_t^T = \Delta \equiv \{\theta : [t,T] \times \mathbb{R}^+ \mapsto \mathbb{R},\ \theta \in [0,1]\}$. Therefore under the Bellman's optimality principle (see Bellman [5], Fletcher [26] or Bertsekas [8]) the optimal strategy of the problem (1.1)–(1.2) is the solution to the Bellman equation of the dynamic programming

$$W(t,y) = \begin{cases} U(y)\,, & t = T\,, \\ \max\limits_{\theta \in \Delta} \mathbb{E}_Z\left(W(t+1, F_t^1(\theta, y, Z))\right)\,, & t = T-1, \ldots, 2, 1\,, \end{cases} \tag{1.3}$$

where $Z \in \mathscr{N}(0,1)$ and

$$F_t^1(\theta, y, z) \equiv G_t^1(y, \mu_t(\theta) + \sigma_t(\theta)z) = y\frac{1+\mu_t(\theta)+\sigma_t(\theta)z}{1+\beta_t} + \varepsilon. \tag{1.4}$$

In this baseline model setting, investor's utility function expresses his/her time t expectations about the terminal value of the pension fund portfolio (e.g. Bergman [7], Pflug and Romisch [59], Fishburn [25], Markowitz [45] or Sharpe [62]). This discrete–time model is discussed deeply in Section 3.4.

As we are interested in continuous–time strategies, we assume that given a small time increment $0 < \tau \le 1$ the proportion of size $\varepsilon\tau$ of saving deposits is transferred to the saver's pension account on short time intervals $[0,\tau], [\tau, 2\tau], \ldots, [T-\tau, T]$. Next, taking into consideration the investor's natural risk–aversion we extend our perception of the saver's utility and by the aspect of the portfolio returns volatility, so that at time t a typical participant of the second pillar of the Slovak pension system strives to maximize their criterion value of terminal wealth–to–salary ratio y_T:

$$\max_{\theta \in \Delta_t^T|_{[0,T)}} \{\mathscr{K}[y_T^\theta \,|\, y_t^\theta = \bar{y}]\}\,, \qquad \text{where} \qquad \mathscr{K}(Y) = \mathbb{E}[U(Y)] - \frac{\lambda}{2}\mathbb{D}[Y]\,. \tag{1.5}$$

where $\{y_t^\theta\}_{t=0}^\infty$ in the finite time horizon Ito's process (see Section 3), $\bar{y}$ a given initial state of $\{y_t^\theta\}$ evaluated at time t and $\mathscr{K}$ denotes a utility criterion functional assumed for a given utility function $U = U(y)$. The criterion functional takes into consideration both the expected return $\mathbb{E}$ of the portfolio and its volatility $\mathbb{D}$.
Then, applying the Bellman's optimality principle the optimal strategy for the problem of stochastic dynamic programming for $0 < \tau \ll 1$ can be formulated in using the concept of the saver's portfolio intermediate value function $V = V(t,y)$ similarly to the case of $\tau = 1$ (see (1.3)–(1.4)) as follows:

$$V(t,y) = \begin{cases} U(y), & t = T; \\ \max\limits_{\theta\in\Delta_t^{t+\tau}} \{\mathscr{K}[V(t+\tau, y_{t+\tau}(\theta)) \mid y_t = y]\}, & 0 \le t < t+\tau \le T, \end{cases}$$

and similarly to (1.1), for any $Z \sim N(0,1)$, $z \in \mathbb{R}$, $y > 0$ and $0 < \tau \ll 1$,

$$y_{t+\tau}(\theta) = F_t^\tau(\theta, y_t, Z), \quad F_t^\tau(\theta,y,z) = y\exp\{[\mu(\theta) - \beta - \tfrac{1}{2}\sigma^2(\theta)]\tau + \sigma(\theta)z\sqrt{\tau}\} + \varepsilon\tau.$$

Then, letting $\tau \equiv dt \to 0^+$, using basic properties of random variable mean and variance, applying stochastic calculus and Itô lemma (see Kwok [42], Oksendal [56], Chiang [13], Múčka [51], Epps [21], or Macová and Ševčovič [44]) we find out that the intermediate value function $V(t,y)$ satisfies the subsequent fully non–linear Hamilton–Jacobi–Bellman equation

$$\begin{cases} 0 = \frac{\partial V}{\partial t} + \max\limits_{\theta\in\Delta_t^T}\left\{A_\varepsilon(\theta,t,y)\frac{\partial V}{\partial y} + \frac{1}{2}B^2(\theta,t,y)\left[\frac{\partial^2 V}{\partial y^2} - \lambda\left[\frac{\partial V}{\partial y}\right]^2\right]\right\}, & y>0,\ t\in[0,T), \\ V(T,y) = U(y), & y>0,\ t=T \end{cases} \tag{1.6}$$

and

$$A_\varepsilon(\theta,t,y) = \varepsilon + \big[\mu(\theta) - \beta\big]y, \qquad \text{and} \qquad B(\theta,t,y) = \sigma(\theta)y.$$

Next, recalling to Abe and Ishimura [1], Ishimura and Nakamura [34], Ishimura and Ševčovič [35], Macová and Ševčovič [44] and Múčka [51] we introduce the Riccati transformation

$$\varphi(s,x) = -\frac{\partial_{xx}\mathscr{V}(s,x)}{\partial_x\mathscr{V}(s,x)}, \qquad \text{for} \quad s = T-t,\ x = \ln y,\ \mathscr{V}(s,x) = V(t,y), \tag{1.7}$$

for all $x \in \mathbb{R}$ and $s \in [0,T]$ where φ refers to the coefficient of absolute risk aversion of the (s,x) domain transformed intermediate value function $\mathscr{V}$. Therefore assuming that both φ and $\mathscr{V}$ are positive on $[0,T]\times\mathbb{R}$ the originally stated Hamilton–Jacobi–Bellman equation (1.6) is transformed as follows

$$\frac{\partial\mathscr{V}}{\partial s} = \mathscr{G}(s,x)\frac{\partial\mathscr{V}}{\partial x}, \qquad \text{for} \qquad \mathscr{G}(s,x) \equiv \varepsilon e^{-x} - \beta - \phi(\zeta(\varphi(s,x))), \tag{1.8a}$$

with $\phi = \phi(\zeta(\varphi))$ the value function of the parametric optimization problem

$$\phi(\zeta) = \min_{\theta\in\Delta}\left\{-\mu(\theta) + \frac{1}{2}\sigma^2(\theta)\zeta\right\}. \tag{1.8b}$$

and the auxiliary function ζ satisfying the subsequent relationship

$$\zeta(\varphi(s,x)) = 1 + \varphi(s,x) + \lambda\,\omega(\varphi(s,x)), \qquad \omega(\varphi(s,x)) = \partial_x\mathscr{V}(s,x) = \kappa e^{-\int_{x_0}^x \varphi(s,z)dz} \tag{1.8c}$$

for some $x_0 \in \mathbb{R}$ and $\kappa \equiv \mathscr{V}'(s,x_0)$ finite. Then φ is a solution to the Cauchy–type quasi–linear parabolic equation (see Kilianová and Ševčovič [38])

$$\begin{cases} \dfrac{\partial \varphi}{\partial s} = \dfrac{\partial^2 \phi(\zeta(\varphi))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\varphi)(\varepsilon e^{-x} - \beta) - \varphi\,\phi(\zeta(\varphi))\right], & x \in \mathbb{R}, s \in (0,T], \\ \varphi(0,x) = -\dfrac{U''(e^x)}{U'(e^x)} e^x, & x \in \mathbb{R}. \end{cases} \tag{1.9}$$

and problems (1.8a)–(1.8b) and (1.9) are equivalent. Furthermore, referring to Kilianová and Ševčovič [38] in our thesis we will show that for $\mu \in \mathbb{R}^n$ and Σ positive definite matrix, the optimal value function $\phi(\zeta)$ given by (1.8b) is $C^{1,1}$ continuous, $\zeta \mapsto \phi(\zeta)$ is strictly increasing and for the unique minimizer $\hat{\theta} = \hat{\theta}(\zeta) \in \Delta$ of (1.8b) it holds that

$$\phi'(\zeta) = \frac{1}{2}\hat{\theta}^T(\zeta)\Sigma\hat{\theta}(\zeta). \tag{1.10}$$

Furthermore, recalling (1.8c), we see that $\zeta'(\varphi) = 1 + \lambda \dfrac{\varphi}{\partial_x \varphi}\omega(\varphi(s,x))$.

Recalling the unique minimizer $\hat{\theta}(\zeta) \in \Delta$ of (1.8b) for any subset S of $\{1,\ldots,N\}$ the set $\mathscr{I}_S$ of all functions $\zeta > 0$ for which the index set of $\hat{\theta}(\zeta) \in \Delta$ zero components coincide with S we define:

$$\mathscr{I}_\emptyset = \left\{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) > 0,\, \forall i = 1,\ldots,N\right\}, \quad \mathscr{I}_S = \left\{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) = 0 \iff i \in S\right\}.$$

Then, concerning the future pensioner's optimal investment strategy problem we need to distinguish between two cases. In case of $\zeta \in \mathscr{I}_\emptyset$ we directly employ the technique of Lagrange multiplier (see e.g. Smith [65], Fletcher [26], Chiang [13], Smith [64], or Walde [69]) whereas providing that that $\zeta \in \mathscr{I}_S$ for some non–empty subset S then we may reduce the problem dimension to a lower $N - |S|$ dimensional simplex Δ_S. Thus, $\phi(\zeta)$ is C^∞ on the open set $\bigcup\limits_{0\le|S|\le N-1} int\,(\mathscr{I}_S)$ for any $S \subset \{0,\ldots,N\}$ and

$$\phi(\zeta) = \begin{cases} \dfrac{\zeta}{2a} - \dfrac{b}{a} - \dfrac{ac-b^2}{2a}\zeta^{-1}, & \zeta \in \mathscr{I}_\emptyset, \\ \dfrac{\zeta}{2a_S} - \dfrac{b_S}{a_S} - \dfrac{a_S c_S - b_S^2}{2a_S}\zeta^{-1}, & \zeta \in int(\mathscr{I}_S), \end{cases} \tag{1.11}$$

where $a = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$, $b = \mu^T\Sigma^{-1}\mathbf{1}$, $c = \mu^T\Sigma^{-1}\mu$ and a_S, b_S and c_S are obtained as projections of a, b, c when the the corresponding rows and columns elements from the matrix Σ and vector μ are nullified.

Assume that $\zeta \in \mathscr{I}_\emptyset$. Therefore employing (1.11) with $\zeta = \zeta(\varphi)$ given by (1.8c), the quasi–linear initial value problem (1.9) takes the subsequent form for unknown $\varphi = \varphi(s,x)$:

$$\begin{cases} \dfrac{\partial \varphi}{\partial s} = \dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\dfrac{\partial \varphi}{\partial x}\left[1 + \dfrac{1}{\gamma^2\zeta^2(\varphi)}\right]\zeta'(\varphi)\right. \\ \qquad\qquad \left. +2a(1+\varphi)(\varepsilon e^{-x} - \beta) - \varphi\left[\zeta(\varphi) - 2b - \dfrac{1}{\gamma^2\zeta(\varphi)}\right]\right\}, \\ \varphi(0,x) = -e^x\dfrac{U''(e^x)}{U'(e^x)}, \end{cases} \tag{1.12}$$

where $x \in \mathbb{R}$, $s \in (0,T]$ and $\gamma = (ac-b^2)^{-1/2}$.

Firstly, we specify the utility function as a linear combination of two CRRA–type (Bergman [7], Pflug and Romisch [59], Pratt [60] or Sharpe [62])utility functions:

$$U(y) = -y^{1-d} + \frac{\lambda}{2} y^{2(1-d)}, \qquad y > 0\,,\; 0 < \lambda \ll 1\,,\; d \gg 1\,.$$

Next, we write φ and U in terms of their asymptotic expansions (see e.g. Holmes [30], Bender and Orszag [6], Hinch [29] or O'Malley [57]) with respect to parameter λ as follows for any $x \in \mathbb{R}$ and $s \in [0,T]$.

$$\varphi(s,x) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(s,x)\,, \qquad \text{and} \qquad U(e^x) = \sum_{n=0}^{\infty} \lambda^n U_n(e^x)\,. \tag{1.13}$$

Thus, the absolute and linear terms φ_0 and φ_1 of (1.13) can be achieved gradually by solving the following pair of sub–problems for the function $\psi = \psi(s,x) = \gamma(1+\varphi(s,x))$ defined $\psi(s,x) = \psi_0(s,x) + \lambda\, \psi_1(s,x)$ for all $s \in [0,T]$ and $x \in \mathbb{R}$:

$$[\mathbf{P_0}] \qquad \begin{cases} \dfrac{\partial \psi_0}{\partial s} = \dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\left[1+\dfrac{\partial}{\partial x}\right]\left[\psi_0 - \dfrac{1}{\psi_0}\right]\dfrac{\partial \psi_0}{\partial x} + 2a\,(\varepsilon e^{-x} + p_0)\,\psi_0 - \dfrac{\psi_0^2}{\gamma}\right\}, \\ \psi_0(0,x) = \gamma d\,, \end{cases} \tag{1.14}$$

$$[\mathbf{P_1}] \quad \begin{cases} \dfrac{\partial \psi_1}{\partial s} = \dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{[1+\psi_0^{-2}][\dfrac{\partial \psi_1}{\partial x} - q_1\psi_1] + 2a[\varepsilon e^{-x} + p_1]\psi_1 + 2[1+\psi_0^{-2}]\gamma q_1 e^{-q_1 x}\right\}, \\ \psi_1(0,x) = \gamma(1-d)e^{(1-d)x}, \end{cases} \tag{1.15}$$

where $p_0 = \frac{b}{a} - \beta$, $p_1(s,x) = -\beta + \frac{b}{a} - \frac{1}{2a}\frac{\psi_0^2-1}{\gamma\psi_0}$ and $q_1 = \frac{\psi_0}{\gamma} - 1 \equiv \varphi_0$.

Firstly, in order to solve approximately the problem $[\mathbf{P_0}]$ (see (1.14)) we apply again the technique of $\psi_0(s,x)$ asymptotic expansion with respect to $0 < \varepsilon \ll 1$, hence estimate

$$\psi_0(s,x) \approx \psi_{0,0}(s,x) + \varepsilon\, \psi_{0,1}(s,x)\,.$$

Then evidently, $\psi_{0,0} = \gamma d$ and so what remains is to find the solution to the subsequent Cauchy problem for $\psi_{0,1}(s,x)$

$$\begin{cases} \dfrac{\partial \psi_{0,1}}{\partial s} = \dfrac{1}{2a}\left[1+\dfrac{1}{\psi_{0,0}^2}\right]\dfrac{\partial^2 \psi_{0,1}}{\partial x^2} + \dfrac{1}{2a}\left[1+\dfrac{1}{\psi_{0,0}^2} + 2a\delta\right]\dfrac{\partial \psi_{0,1}}{\partial x} - \psi_{0,0}e^{-x}, & (s,x) \in (0,T] \times \mathbb{R}\,; \\ \psi_{0,1}(0,x) = 0\,, & x \in \mathbb{R}\,. \end{cases}$$

The linear approximation to the solution of the problem $[\mathbf{P_0}]$ defined by (1.14) is given as

$$\psi_0(s,x) = \gamma d \left(1 + \varepsilon \frac{e^{-\delta s} - 1}{\delta} e^{-x}\right) + o(\varepsilon^2)\,, \qquad \delta = \frac{b-d}{a} - \beta\,. \tag{1.16}$$

Next, plugging (1.16) into problem $[\mathbf{P_1}]$ (see (1.15)) and setting $\varepsilon = 0$ in the resulting problem leads to the following initial value problem for the unknown $\psi_{1,0} = \psi_{1,0}(s,x)$

$$\begin{cases} \dfrac{\partial \psi_{1,0}}{\partial s} = \dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{(1+\frac{1}{\psi_{0,0}^2})\dfrac{\partial \psi_{1,0}}{\partial x} + (1+\frac{1}{\psi_{0,0}^2} + 2a\delta)\,\psi_{1,0} + 2\gamma(d-1)(1+\frac{1}{\psi_0^2})e^{(1-d)x}\right\}, \\ \psi_1(0,x) = \gamma(1-d)e^{(1-d)x}, \end{cases} \tag{1.17}$$

where ψ_0 stands for γd and the parameter δ is prescribed by (1.16). The solution to problem above can be found in the time–space separable form.

It is inevitable to remark that our approximative solution to the unconstrained problem (1.12) is in fact the super–solution to the original problem (1.9) and it is given as

$$\begin{aligned} \theta^*(s,x) = \frac{\Sigma^{-1}}{a}\left[\mathbf{1} + (a\mu - b\mathbf{1})[\zeta(s,x)]^{-1}\right], \qquad (s,x) \in \Omega, \\ \text{where} \qquad \zeta(s,x) = d - \varepsilon\Phi_\varepsilon(s)e^{-x} - \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x}, \end{aligned} \tag{1.18}$$

on the region Ω defined as follows:

$$\Omega \equiv \{(s,x) \in [0,T] \times (\lambda,\infty),\ d - \varepsilon\Phi_\varepsilon(s)e^{-x} - \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x} > 0\}, \tag{1.19}$$

with the auxiliary functions $\Phi_\varepsilon = d\delta^{-1}(1-e^{-\delta s})$ and $\Phi_\lambda = (d-1)[(1+\widetilde{\phi})e^{\widetilde{\delta}s} - \widetilde{\phi}]$ for $\widetilde{\delta}$ and $\widetilde{\phi}$ arising from the unique solution to (1.17).

1.4 Thesis Structure

This dissertation thesis is organised as follows.

Firstly, Section 2 describing the Slovak pension system is mainly oriented on its private, defined–contribution based Second Pillar. Here we explain its idea, underlying processes, private asset management companies and pension account management, available investment strategies and legislative framework. We provide numerous graphical schemes and figures illustrating the actual investment decisions and characteristics of current participants of the Second pillar. Furthermore, we deeply explain the serious reasons of the Slovak pension reform – bad demographic projections and public finance sustainability issue – and show its consequences in all three pillars of the Slovak pension system – public obligatory PAYG–based First, private Second and mandatory private Third pillar.

Section 3 contains necessary theoretical background. In this passage we prefer explanatory to rigid form of formulations as all the definitions, theorems and ideas presented there – normal distribution and its properties, Itô calculus, utility function concept, Bellman's dynamic programming optimality principle, stochastic optimization, Lagrange multipliers or Hamilton–Jacobi–Bellman equation – accompany us through the whole thesis and should help us to understand the process of optimal investment allocation model derivation. Furthermore, in order to provide better illustration of the model derivation process in Section 3 we demonstrate some practical examples that unfold the motivation and highlight certain interesting attributes of the studied model. The last part of this section is devoted to the simplest variant of our problem – two securities discrete–time stochastic dynamic model.

The core of this dissertation thesis is constituted by Sections 4–6.
In Section 4 we formulate the key problem that we are aimed to solve in this study. We describe the investor's utility criterion employed in our model in order to capture both the expected terminal return of the investor's pension portfolio and the associated volatility that

cannot be separated in the uncertain and turbulent financial worlds. From this perspective we take into consideration a natural investor's risk aversion attitude. Then, we launch the continuous–time version of the model for unlimited number of traded securities and using stochastic calculus show how to determine the corresponding fully non–linear Hamilton–Jacobi–Bellman parabolic equation.

In Section 5 we concentrate ourselves on a specific parametric convex optimization problem. This is obtained from the original Hamilton–Jacobi–Bellman equation employing a Riccati transform and it is equivalent to a particular Cauchy–type quasi–linear parabolic equation. Then under some additional presuppositions we are eligible to prove the existence of unique solution to this convex optimization problem which allows us to determine the general C^∞–smooth formula for the constrained optimal portfolio allocation policy, demonstrate its usage for the case of *one–stock–one–bond* problem and basic properties. Notice that the optimal policy relationship is written as a function of a transformed portfolio value function, hence at this the optimal investment strategy cannot be determined directly. Finally, under some simplifying assumptions we derive the travelling wave type solution to the quasi–linear problem.

The key objective of Section 6 is to determine a simple formula that approximate the optimal investment strategy enough precisely. Therefore, we firstly employ the optimal allocation policy formula determined in Section 5 in the quasi–linear initial value problem. Then we perform a double (λ, ε) asymptotic expansion of the resulting equation up to the second order (formulae for general n–th order terms of the asymptotic expansion are derive in the Appendix) and thus determine a simple approximative prescription of the optimal allocation policy of a future pensioner as a function of his/her time to retirement and already allocated wealth (considered relatively to his/her salary). Furthermore, this prescription takes into account investor's characteristics (gross wage growth rate, risk attitude), legislative framework (retirement age, contribution rate) and financial market performance. The obtained policy is then analysed from a qualitative and quantitative perspective and resulting policy implications are emphasized.

In the application part of this dissertation thesis (Section 7) we bring the Section 6 model on Slovak data. The calibration strategy works with alternative setting of model key parameters and illustrate changes in both the saver's optimal investment allocation policy and the terminal expected wealth allocated on investor's pension account (obtained via Monte–Carlo simulations); resulting from variations in model parameters. We aim our attention particularly on the prescribed contribution rate ε and retirement age T, thus the factors that policy makers can directly affect. The allocation strategy is exemplified through three types of situations studied –the simplest *One–Stock–One–Bond* problem, and in order to clarify the case of higher dimensional problem we present the *Two–Stocks–One–Bond* problem and the *One–Stock–Two–Bonds* problem.

Code snippets used to analyse saver's optimal investment allocation policy and simulate terminal expected wealth allocated on investor's pension account in any of the three examples introduced above are located in the Appendix.

Chapter 2

PRIVATE SCHEME OF THE SLOVAK PENSION MODEL

Pension model of the Slovak Republic consists of three complementary coexisting pension pillars:

1. The traditional first, *pay–as–you–go* philosophy based public, unfunded and mandatory pillar represents a state–guaranteed pension insurance performed by the Social Insurance Company. It is partially earning-related with a markable solidarity element. The participants of the public scheme earn annual pension points indexed to past CPI inflation and gradually downsized lagged average earnings in the economy.
2. The private scheme of the Slovak pension system, so-called the Second Pillar commercially supervised by private asset management companies (hereafter PAMC) establishes a fundamental change in the pension system of Slovakia as it is fully funded from a saver's (i.e. future pensioner) regular contributions and introduces an alternative to save for a pension on an private pension account. Financial resources accumulated on the pension account possesses the ability of value appraising via subsidization allocation into the predefined investment funds and diversify the sources of future income.
3. The fully obligatory third pillar is driven by private companies in the similar way as the second pillar. It represents an interesting opportunity of saving as the saver's contributions are subject to tax reduction with possible extra contribution of the employer and form an employer–based saving scheme.

Our work is aimed on study the optimal investment strategies in the Second Pillar of the Slovak pension system.

2.1 Legislative Framework

The Second Pillar of the Slovak Pension Model was established in January 2005 by law 43/2004 and currently, six pension asset management companies offer the services in Slovakia: AEGON d.s.s.; Allianz d.s.s; AXA d.s.s; DSS Poštovej banky; ING d.s.s; and VÚB

Generali d.s.s. Financial resources accumulated on pension account, a saver's heritable property possess the ability of value appraising by making investment into the pension funds designed and managed by the PAMC and strictly regulated by the legislative norms. As pension funds are managed using defined contribution scheme with fixed regular contributions of individuals and benefits depending on returns of the pension fund's portfolio, the financial risk associated with investment is borne by investors, i.e. future pensioners.

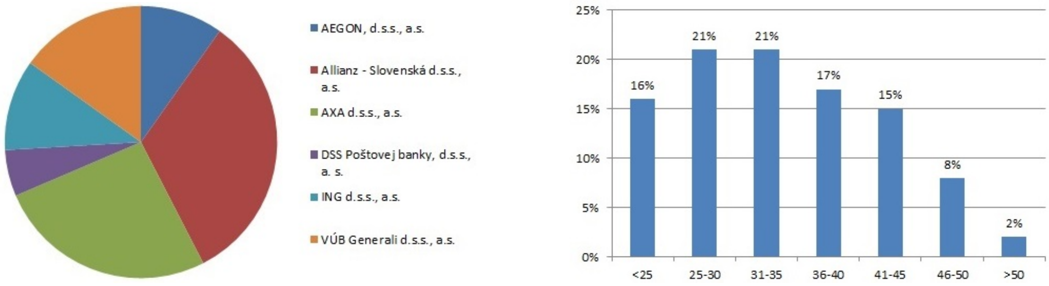
Before the reform, all employees paid the contributions for pensions to the Social Insurance Company obligatorily, so that the Slovak pension system was based on a continuous principle of financing only. Thus, before 2005, the pensions for current pensioners were paid continuously and immediately, from the obligatory contributions paid for the Social Insurance Company – therefore we characterize the pre–reform pension system as PAYG scheme. The pension reform in 2005 has brought an essential change in the pension system philosophy – working individuals have an option to save for their pension through regular contributions on their own private pension accounts in a pension asset management company. Furthermore, participants of this pension system pillar are eligible to choose their own investment allocation strategy by selecting oat most two of defined pension funds managed by private asset management companies and hence invest and increase the wealth accumulated on their pension accounts.

Contribution Rate. Nowadays the regular contribution rate for the Second Pillar scheme is defined as 6% of gross earnings with a temporal drop (valid from September 2012) to 4% with a convergence plan taking place from January 2017. Under this scheme the rate augments by 0.25% on yearly basis until the target of 6% is hit and thereafter remain constant. However until September 2012 the contribution rate used to be 9%. Furthermore, now an employer subsidizes the public scheme by of another 14% of the employee's gross wage so that the overall transfer of the employee and employer to the pension system attains 18% of his/her gross wage.

Entrance Conditions. Originally in 2005 the Second Pillar was designed as mandatory for all labour market entrants who were auto–enrolled with an optional membership for all others. Later on in 2013 entering rules were changed to purely voluntary participation of all (both new and existing before the age of 35) employees.

Retirement benefit of those who join the Second Pillar is formed of two sources:

- Benefit from the First Pillar (public earning–related scheme) calculated as the aliquot part of those employees who participate only in the public pension scheme (hence, 18% of their salary is transferred to the Social Insurance Company only).
- Payoffs from the Second Pillar in the form of annuity or scheduled withdrawal, or combination of both possibilities.



Source: Social Insurance Company

(a) Shares of private asset management companies

Source: Social Insurance Company

(b) Age Structure of Savers in the Second Pillar

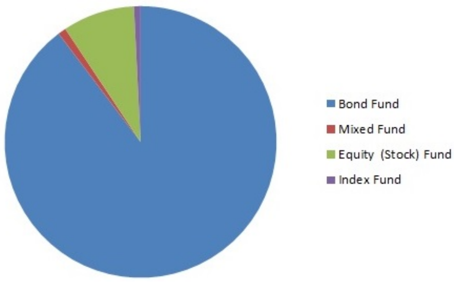
Figure 2.1: Shares of six private asset management companies on the Slovak Pension System Private Scheme market (left) and the age structure of the Second Pillar participants (right)

2.2 Investment Strategies

Any pension fund consists of various more or less risky financial securities in different weights and so it represents the investment portfolio with a certain risk profile. Hence they differ especially by an investment strategy, with which the instruments determined and constrained by law relate, which it is possible to acquire within the restrictions for a property investment for the particular funds:

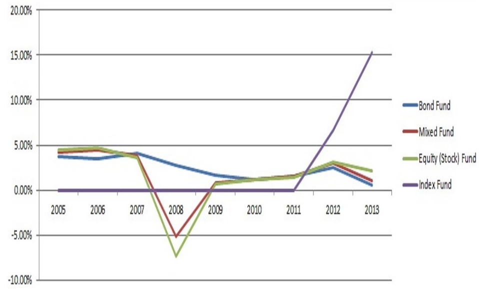
1. *Bond Fund*: investment strategies are strictly restrained to highly rated short–term bonds and money market instruments (mainly money deposits) with full assurance against foreign currency risk. There is a presence of guaranteed return, as in case of no appreciation of the investment (in nominal terms) in ten year time horizon the PAMC has to pay off the balance.
2. *Mixed Fund*: the investment portfolio is restricted to compose of at least 50% of bonds and money market instruments, up to 50% of stocks and up to 20% of precious metal investment instruments. Half of the investment must be secured on foreign currency risk.
3. *Equity (Stock) Fund*: the investment portfolio is formed by stocks (at most 80%), precious metal investments (not more than 20%) and up to 80% of the fund property by bonds and money investment instruments. At least 20% of the investment must be secured on foreign currency risk.
4. *Index Fund*: benchmark of this passively managed fund tracks the performance of one or a pool of selected equity indexes and there are no restrictions imposed on exchange traded funds, assets or derivatives when replicating the benchmark formed initially. If the performance is below the established benchmark, the PAMC looses half of the fund fees.

Furthermore, each PAMC has to establish and manage at least two funds – one of them must have guaranteed yields above the given benchmark (the Bond Fund) and at least one must be without guaranty of returns.



Source: Social Insurance Company

(a) Wealth distribution

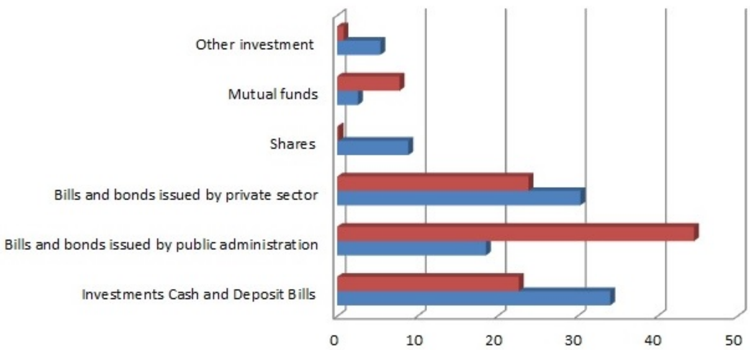


Source: Social Insurance Company

(b) Fund appreciation

Figure 2.2: Distribution of the accumulated wealth among various type of pension funds (left) with their yield path (right)

Investment Dilemma. Any saver already registered in one of PAMC makes his/her investment decision by selecting at most two of the funds mentioned above: in case that two funds are chosen, one of them must be Bond Fund. On the other side, each PAMC as a part of their investment decision specifies a benchmark for each of the fund except the Bond Fund that would satisfy the prescribed restrictions imposed by the government and it is the only one fund which has guaranteed returns. Evidently, each PAMC implements their investment decision by constructing such portfolios that would outperform or at least copy in their return the associated benchmark – otherwise managerial fees charged on savers transfers by PAMC for management services provided by the company are cut.



	Investments Cash and Deposit Bills	Bills and bonds issued by public administration	Bills and bonds issued by private sector	Shares	Mutual funds	Other investment
2012	22.7	44.6	23.9	0.2	7.8	0.8
2007	34.1	18.6	30.4	8.9	2.6	5.4

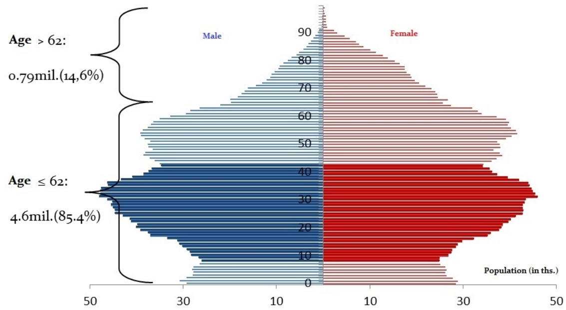
Source: Pension Markets in Focus (2012), OECD [55]

Figure 2.3: Pension fund asset allocation for selected investment categories in Slovakia, observed in 2007 and 2012, expressed as a percentage of total investment

Investment Constraints. The fund selection is not unconstrained with respect to saver's age, as from the age of 50 onwards he/she has to allocate at least 10% of wealth already accumulated on his/her private pension account in the Bond Fund. This prescribed share increases by 10 p.p. each successive year such that in the age of 59 the future pensioner keeps his/her wealth in the Bond fund only (see OECD [53]).

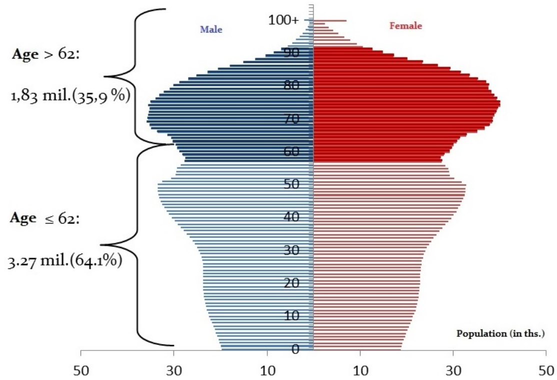
Therefore our aim is to derive the selection of funds thus, for any time to retirement and wealth allocated on an individual's private pension account determine the optimal decision of a representative future pensioner about the weights of the funds introduced above in his/her portfolio. Furthermore, we are interested in the evolution of that fund selection (i.e. how does this decision changes in time and space – wealth already allocated measured relatively to the investor's gross wage) and the effect of other model parameters (contribution rate, risk aversion, financial data, gross wage growth, legislative norms). In our study we will show that for any participant of the Second Pillar the optimal investment strategy is fully replicable by only two of four available pension funds, namely the Index Fund and the Bond Fund; the remaining two are redundant.

But why the Second Pillar exists? Its presence arises as a one part of the Slovak pension system reform motivated by the progressively worsen demography and the necessity of public finance sustainability as described deeply in the following text.



Source: Infostat (VDC), Council for Budget Responsibility [18]

(a) Age pyramid, 2012



Source: Europop 2010, European Commision [23]

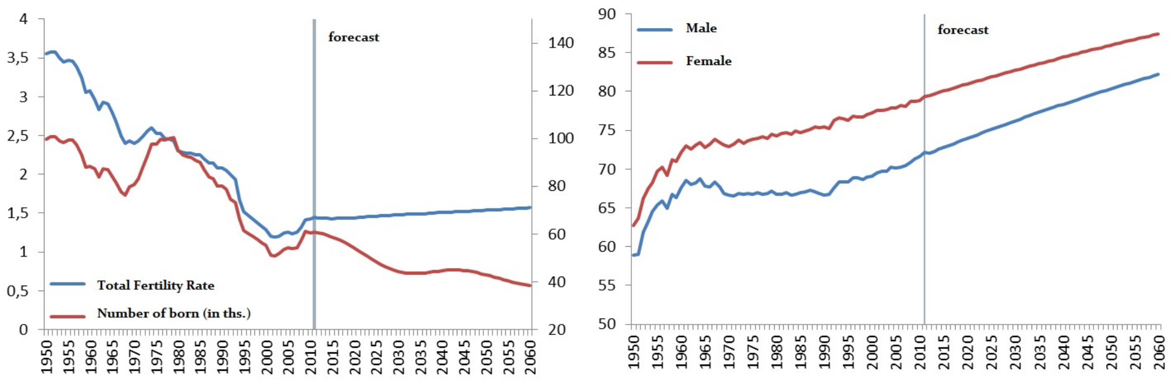
(b) Age pyramid, 2060 (projection)

Figure 2.4: Age pyramids for 2012 (left) and the 2060 projection (right) with highlighted share of elderly people

2.3 Demography and Public Finance Sustainability – Two Slovak Time Bombs

In the coming decades, Slovakia will experience rapidly exacerbating demographic problem with steep increases in the proportion of elderly persons in the total population and a critical decline in the share of young people and those of working age. The gradual change in the demographic structure is caused mainly by the steadily declination of the fertility rate (See Figure 2.5a) since 1950 from 3.6 children per woman to 1.5 children per woman in 2011. Following the Eurostat projections ((see European Commision [23], [22]) we assume that fertility will approach the current EU–average level of 1.6 children per woman. Longevity increased by 15 years from 1950 to 72.2 and 79.4 years for men and woman, respectively (see Figure 2.5b). The Europop 2010 forecasts the augmenting trend with gradual convergence to EU-average which means that in 2060 the expected length of life will attain 82.2 years for men and 87.4 years for women and remain constant thereafter.

Furthermore, based on Europop 2010 prognosis (see European Commision [23], [22], and Infostat [32]) the population group of age 65+ size will double by 2060 and the number of people in productive age (15-64) will shrink by 30% – which means more than 50% increase (as a percentage of GDP) of pension related expenses financed from public sources. In accordance with the Council for Budget Responsibility ([17], [16], and [18]) in 2060 we will face a significant raise in old-age, retirement and widower pension expenses (see Figure 2.4). Obviously the ageing of the population also poses significant challenges for their economies



Source: Infostat (VDC), Europop 2010, Council for Budget Responsibility [16]

(a) Total Fertility rate and number of newborns

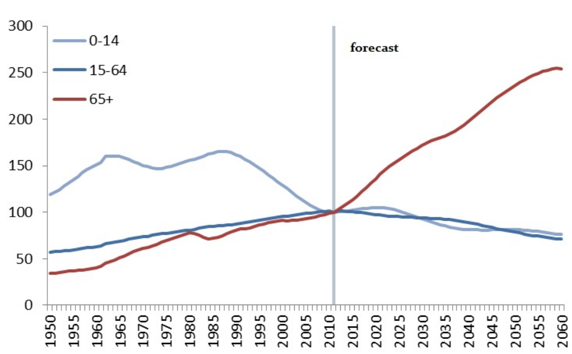
Source: Infostat (VDC), Europop 2010, Council for Budget Responsibility [16]

(b) Life Expectancy

Figure 2.5: Total Fertility rate and number of newborns (left) with expected length of life (right)

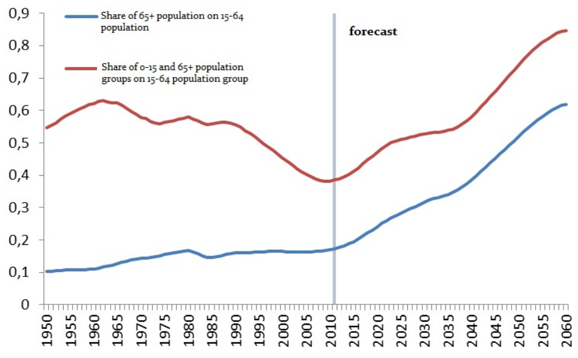
and welfare systems and the demographic projections play the crucial role in the public finance sustainability due to high share of ageing and demographic structure related share on expenditures (healthcare, pension system, long-term care, education and unemployment transfers) and slowing potential economic growth and changes in labour market caused also by different demographic structure. This results in alarming finding - based on the European Commission forecast (see [23], [22]) the Old Age Dependency Ratio will rise from current 0.17 (approximately 5 people in productive age per one pensioner) up to 0.61 (less than two productive people per one pensioner) in 2060. According to Slovak Ministry of Finance and Council for Budget Responsibility [16], in 2012 the share of expenditures depending on demographic situation allocated more than 65% of the primary government budget expenditures and represented 18.4% of Slovak GDP, the Council for Budget Responsibility predicts that in 2060 the share of expenditures sensitive to demographic changes will attain 25.8% of Slovak GDP and the budget revenues will drop by 1.1% of GDP.

Based on the European Commission analysis (see [23], [22]) the public pensions bound 7.4% of GDP and was predicted to rise by 20.3% of GDP in 2050. This increase in current and projected pension spending in Slovakia was deeply analysed and decomposed into four key factors: an extremely strong old-age dependency effect partially offset by the remaining factors - an employment effect, a pension take–up effect, and a benefit effect which is particularly markable owing to the existence of the private pension scheme that based on the contribution rate in 2005 (9%) and relatively high labour productivity. It is inevitable to remark that



Source: Infostat (VDC), Europop 2010, European Commision [22]

(a) Trends in Age Groups

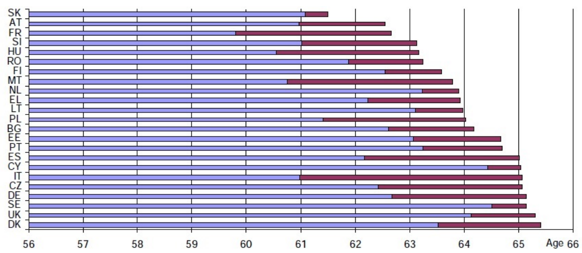


Source: Infostat (VDC), Europop 2010, European Commision [23],

(b) Old Age dependency ratio

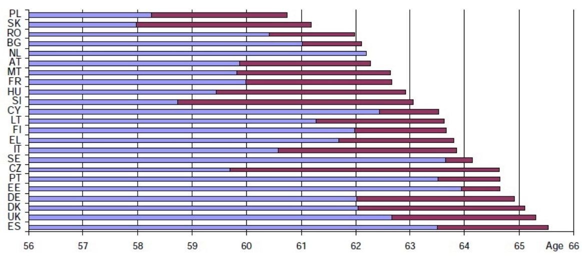
Figure 2.6: Trends in evolution of age groups (left), old age dependency ratio (blue line, right plot) and dependency ratio (red line, right plot)

providing that the offsetting factors are neglected the old–age dependency ratio will double the public pension expenditures.



Source: The 2012 Ageing Report, European Commision [23]

(a) Average exit age from labour force for men in 2060



Source: The 2012 Ageing Report, European Commision [23]

(b) Average exit age from labour force for women in 2060

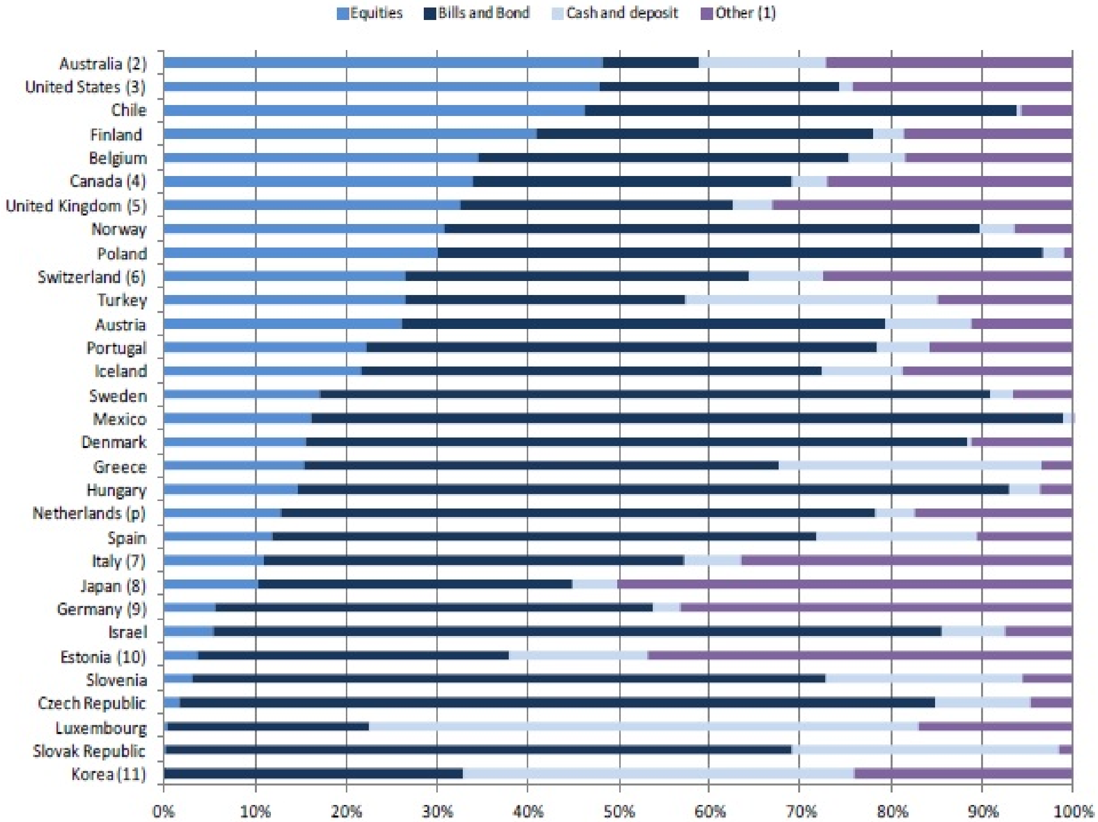
Figure 2.7: Impact of pension reforms on the average effective retirement age from the labour force. Blue colour bars depict the projected average no–reform exit age from labour force in 2060 whereas the brown ones demonstrate the country's pension reform effect on labour market exit age increase in the same time horizon (only pension reforms made between 2001 and 2009 were considered in the calculations). In Slovak Republic, the total average exit age augmented from 57.5 in 2001 to 58.8 in 2009.

Hence we are strongly advice to buttress the offsetting effects by applying the following measures:

- support the private pension scheme;
- lower the pension take-up by gradual elevation of the retirement age and reduction of early pensions possibility (see Figure 2.7);
- shift from earning-based to flat–rate public wages with transition from wage–indexation towards price–indexation implying the constancy of the pensioners' purchasing power;
- undertake measures that adapt pension benefits to expected future demographic or employment changes such adapting the pension benefit to life expectancy of new pensioners and reducing positive discrimination of women in labour market exit age (see Figure 2.7);

- adapt the pension benefit to the relationship between the numbers of the employed and pensioners.

These analyses confirmed that in case that no system change occurs in the pension scheme, the state should not be able to bear the public pensions financing from the long-time point of view without a reform. Therefore besides several parametric changes refining the public finance sustainability adapted in the public pension scheme, in order to increase the potential pension–based income of a future pensioner the private element in the pension system is employed.



Source: Financial Affairs Division Report (2012), OECD [54], OECD [55],

Figure 2.8: Pension fund asset allocation for selected investment categories in selected OECD countries

Above mentioned recommendations have already been partially incorporated into Slovak legislative.

Two private pension schemes were launched so nowadays the pension system of Slovakia consists of three coexisting pension pillars: mandatory public PAYG scheme (so called the First Pillar); mandatory private defined contribution scheme (so called the Second Pillar) and the optional supplementary private defined contribution scheme (so called the Third Pillar).
Then, several changes have been made in the First Pillar the Second Pillar, namely the gradual adjustment of pension age to life expectancy; price-based public pension indexation rule; strengthen solidarity in public pensions (progressive shift towards flat-rate pensions reduc-

ing wedge between the highest and lowest pensions); upper limits imposed on gross wage considered in partially earnings-based pensions; and reduction in private scheme contributions from 9% to 4% of the gross wage.

The reduction in contributions to the private scheme has an ambiguous effect since while it moderates the fall in revenues from 1.8% of GDP to 1.1% of GDP on the other hand creates a significant implicit liability that has to be paid by future generations.

On the other side, the upper limitations on wage considered in partially earning-based pensions and price-based pension indexation rule have a long-term positive effect on PAYG scheme balance while the effect of reduction in contributions to the Second Pillar finally turns out as highly negative as the generated implicit liability takes place in 2060.

Chapter 3

PRELIMINARIES

The forthcoming text is devoted to the general mathematical background needed throughout the whole thesis. We summarizes the basic tools of analysis and probability theory that are needed to develop, solve any analyse the optimal pension strategy model which represents the key issue of this work.

3.1 Itô calculus and Stochastic differential equations

First of all we introduce stochastic processes and normal distribution and underline its connection with the solution to the specific initial value partial differential equation. We also launch the idea of Brownian motion and describe its main attributes. Next, establish the notion of stochastic differential equation, link it with Brownian motion with drift and show its essential properties. Finally we come to Itô lemma and isometry.

3.1.1 Normal distribution and Brownian Motion

The scrutinized phenomenon is said to follow a *stochastic process* if its achieved value changes over time in an uncertain, indeterministic manner and the future values are not pre-visible. The study of stochastic processes is based on the structure of families of random variables X_t investigation, where t is usually interpreted as a time–parameter running over some index set $\mathscr{T}$. If the index set $\mathscr{T}$ is discrete, then the stochastic process $\{X_t\,,\, t \in \mathscr{T}\}$ is referred to as a *discrete stochastic process*, whereas providing a continuous index set $\mathscr{T}$, $\{X_t\,,\, t \in \mathscr{T}\}$ is known a *continuous stochastic process* (for further details see Oksendal [56], Shreve [63], Epps [21], or Chiang [13]).

A *Markovian process* is a stochastic process that, assuming the value of X_s is prescribed, for any choice of $t > s$ the values of X_t, depend only on the given value X_s and are independent of the history of previous random variable X_u taken before time s, i.e. for $u < s$ and henceforth they are characterized by the Markov property, so-called *memoryless* (e.g. Epps [21], or Chiang [13]). If the observed phenomenon follows a Markovian process, then only the present obtained values are relevant for predicting their future values and henceforth for any time

$s \in \mathscr{T}$ we can restart generation of the process $\{X_t\,,\, t > s, t \in \mathscr{T}\}$ by considering s for the process initial value regardless the past history of the process.
For the purpose of concreteness if the asset prices follow a Markovian process, then only the present asset prices are relevant for predicting their future values – this fact is in consistency with the *weak form of market efficiency*, which assumes that the past prices information is already incorporated in the present value of an asset price whereas the particular path taken by the asset price to reach the present value can be ignored. If the past history was indeed relevant, that is, a particular pattern might have a higher chance of price increases, then investors would bid up the asset price once such a pattern occurs and the profitable advantage would be eliminated (Kwok [42]). For more detailed introductory information the reader is recommended to see e.g. Kwok [42], Oksendal [56], Shreve [63], Epps [21], Chiang [13] or Szepessy et al. [68] .

Definition 1. *(Kwok [42], Oksendal [56], Shreve [63], Epps [21]) The random variable X has a normal distribution with parameters μ and $\sigma > 0$ (denotes as $X \sim \mathscr{N}(\mu, \sigma^2)$) if X has the density function f_X such that*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}\,, \qquad \textit{for all } x \in \mathbb{R}\,. \tag{3.1}$$

Furthermore, X is said to be a standard normal random variable providing that X has a normal distribution with parameters $\mu = 0$ and $\sigma = 1$.

Moreover, notice that the linear combination of N arbitrarily chosen independent normally distributed variables $X_i \sim \mathscr{N}(\mu_i, \sigma_i^2)$ for $i = 1, 2, \dots, N$ is also a normally distributed random variable.
Concretely, assuming

$$X = \sum_{i=1}^{N} a_i X_i\,, \qquad \text{then} \qquad X \sim \mathscr{N}(\mu, \sigma^2)\,,$$

where the mean (so-called *expected value*) μ and the variance (or *volatility*) σ^2 satisfy the subsequent prescriptions:

$$\mu = \sum_{i=1}^{N} a_i \mu_i\,, \qquad \text{and} \qquad \sigma^2 = \sum_{i=1}^{N} a_i^2 \sigma_i^2\,.$$

This is particularly needed when evaluating the expected return and volatility of the portfolio consisting of normally distributed financial assets.

Definition 2. *(Kwok [42], Oksendal [56], Shreve [63], Epps [21], Chiang [13]) The random variable Y has a log–normal distribution with parameters μ_Y and $\sigma_Y > 0$ if $X \equiv \ln Y$ is a normally distributed random variable. The probability density function g_Y of the random variable Y satisfies*

$$g_Y(y) = \frac{1}{\sigma_X y\sqrt{2\pi}} \exp\{-\frac{(\ln y - \mu_X)^2}{2\sigma_X^2}\}\,, \qquad \textit{for all } y \in \mathbb{R}^+\,. \tag{3.2}$$

Performing several routine calculations one may straightforwardly derive the expected value μ_Y and the volatility σ_Y associated to the random variable Y:

$$\mu_Y = \exp\{\mu_X + \frac{\sigma_X^2}{2}\}\,, \quad \sigma_Y^2 = \big[-1 + \exp\{\sigma_X^2\}\big]\exp\{2\mu_X + \sigma_X^2\}\,, \quad \text{where } X \sim \mathscr{N}(\mu_X, \sigma_X^2)\,. \tag{3.3}$$

Notice that there is a strong relationship between the random variables and the partial differential equations. From (3.1) the probability density function of a normal random variable X with mean μt and variance $\sigma^2 t$ is given by

$$f(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\{-\frac{(x-\mu_t)^2}{2\sigma^2 t}\}, \qquad \text{for all} \quad x \in \mathbb{R},$$

and it can be checked that $f(x,t)$ is the fundamental solution to the initial value problem formulated for the function $v = v(x,t)$ subsequently

$$\frac{\partial v}{\partial t}(x,t) + \mu \frac{\partial v}{\partial x}(x,t) - \frac{\sigma^2}{2} \frac{\partial^2 v}{\partial x^2}(x,t) = 0, \qquad x \in \mathbb{R},\, t > 0,$$

with the initial condition $v(x,0^+) = \delta(x)$ where $\delta \equiv \delta(x)$ represents the Dirac function.

Definition 3. *(Kwok [42], Oksendal [56], Szepessy et al. [68], Shreve [63], Epps [21], Chiang [13]) The Brownian motion with drift $\{X_t, t \geq 0\}$ is a t- parametric system of random variables, for which*

1. *every increment $X(t+s) - X(s)$ has normal probability distribution with mean μt and variance $\sigma^2 t$, where μ and σ are fixed parameters;*
2. *for any partition $t_0 = 0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$ of the interval $(0,t_n)$, all increments $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are identically distributed independent random variables with parameters according to the point 1,*
3. *$X(0) = 0$ almost surely and the sample paths of $X(t)$ are continuous.*

In particular, providing that the parameters of the Brownian motion proposed in Definition 3 attain the values $\mu = 0$ and $\sigma^2 = 1$, the Brownian motion is called the *standard Brownian motion* (or the *standard Wiener process*).
The related probability distribution function for the standard Wiener process $\{w(t)\,;\, t > 0\}$ is

$$\begin{aligned} P\{w(t) \leq w \,|\, w(t_0) = w\} &= P\{w(t) - w(t_0) \leq w - w_0\} \\ &= \frac{1}{\sqrt{2\pi(t-t_0)}} \int_{-\infty}^{w-w_0} \exp\{-\frac{s^2}{2(t-t_0)}\} ds. \end{aligned} \tag{3.4}$$

For the purpose of this work it is highly desirable to aim the reader's attention to the fact that for the standard Wiener process $\{w(t)\,;\, t > 0\}$ it holds:

$$\mathbb{E}(w(t)) = 0, \qquad \mathrm{Var}(w(t)) = t, \qquad \text{for all } t \geq 0. \tag{3.5}$$

This result can be interpreted in the way that the dispassionate prediction of the phenomenon driven by the standard Wiener process state that can any detached observer make is to expect the instantaneous phenomenon state and the forecast uncertainty grows according to time.

It can be easily seen the point 2 of the Definition 3 succeeds in the independence of the increment $X(t+s)dz - X(s)$ of the path behaviour past history at any time $u \in \mathscr{T}$ for $u < s$, thus the knowledge of $X(u)$ for $u < s$ has no effect on the probability distribution for $X(t+s)dz - X(s)$ (see Shreve [63], Epps [21], Chiang [13], Kwok [42] or Oksendal [56]). Therefore the Markovian property is a characteristic and inextricable Brownian motion feature.

Definition 4. *(Kwok [42], Shreve [63], Epps [21], Chiang [13], Oksendal [56], Szepessy et al. [68]) The stochastic process $Y = Y(t)$ prescribed as $Y(t) = y_0 e^{X(t)}$ for any $t \geq 0$, $y_0 > 0$ and the Brownian motion $X = X(t)$ is called the Geometric Brownian motion.*

Observe that if $X = X(t)$ is a Brownian motion with drift parameter $\mu > 0$ and variance parameter σ^2 then the expected value and variance of the associated geometric Brownian motion $Y = Y(t)$, respectively, are

$$E(Y(t)\,|\,Y(0) = y_0) = y_0 \exp\{\mu t + \frac{\sigma^2 t}{2}\}, \quad \mathrm{Var} = y_0^2\big[-1 + \exp\{t\sigma^2\}\big] \exp\{(2\mu + \sigma^2)t\},$$

and so $Y(t)$ is *log-normally distributed* with the mean and variance parameters presented above.

Furthermore, it the probability density function of $Y(t)$ is given as

$$g(y) = \frac{1}{\sigma y\sqrt{2\pi t}} \exp\{-\frac{(\ln y - \mu t)^2}{2\sigma^2 t}\}, \qquad y > 0. \tag{3.6}$$

Remark that in case of geometric Brownian motion for every time–interval partition $t_1 < \ldots < t_n$, the successive ratios $Y(t_2)/Y(t_1), \ldots, Y(t_n)/Y(t_{n-1})$ are independent random variables, thus the independence of the percentage changes over non-overlapping time intervals is guaranteed (see Kwok [42], Shreve [63], Epps [21], Chiang [13], Szepessy et al. [68], or Oksendal [56]).

3.1.2 Stochastic calculus and Itô's lemma

A Brownian motion $\{X(t), t \geq 0\}$ characterized by parameters μ and σ can be also analysed by means of its increments

$$dX(t) = X(t + dt) - X(t), \qquad t \geq 0; \tag{3.7}$$

where dt is an infinitesimal small quantity, i.e. $dt \to 0^+$. Taking into account the definition of Brownian motion (by virtue of the property 1 stated in the Definition 3), $\mathbb{E}(dX(t)) = \mu t$ and $\mathrm{Var}(dX(t)) = \sigma^2 dt = \sigma^2\,\mathrm{Var}(dw(t))$ (for details we recommend to read Shreve [63], Szepessy et al. [68], Epps [21], Chiang [13], Oksendal [56] or Kwok [42]).

Let $w(t)$ denote the Wiener process and let $\Delta w(t)$ depict the change in $w(t)$ during the time increment Δt. Henceforth using the properties of the Brownian motion, the meaning of $\Delta w(t)$ can be expressed as follows

$$\Delta w(t) = w(t + \Delta t) - w(t) = Z\sqrt{dt}, \tag{3.8a}$$

where Z is a standard normally distributed random variable. Providing that $\Delta t \to 0^+$ the relation above can be reformulated in terms of the differential form

$$dw(t) = Z\sqrt{dt}. \tag{3.8b}$$

Note that under the properties of the Brownian motion, $\mathbb{E}(dw(t)) = 0$ and $\mathrm{Var}(dw(t)) = dt$. Since we are also interested to know the behaviour of the expressions $[\Delta w(t)]^2$ and $\Delta t \Delta w(t)$ and their means and variance, respectively, the ensuing calculations should be performed:

$$\begin{aligned}
\mathbb{E}\left([\Delta w(t)]^2\right) &= \mathrm{Var}(\Delta w(t)) + [\mathbb{E}(\Delta w(t))]^2 = \Delta t\,, \\
\mathrm{Var}([\Delta w(t)]^2) &= \mathbb{E}\left([w(t+\Delta t) - w(t)]^4\right) - \left[\mathbb{E}\left([w(t+\Delta t) - w(t)]^2\right)\right] = o(\Delta t) \\
\mathbb{E}(\Delta t \Delta w(t)) &= \mathbb{E}(\Delta t[w(t+\Delta t) - w(t)]) = 0\,, \\
\mathrm{Var}(\Delta t \Delta w(t)) &= \mathbb{E}\left([\Delta t]2[w(t+\Delta t) - w(t)]^2\right) - [\mathbb{E}(\Delta t[\Delta w(t+\Delta t) - w(t)])]^2 = o((\Delta t)^3)\,.
\end{aligned}$$

Thus

$$\mathbb{E}\left([dw(t)]^2\right) = dt\,, \qquad \mathrm{Var}([dw(t)]^2) = o(dt)\,, \tag{3.9a}$$

$$\mathbb{E}(dt\, dw(t)) = 0\,, \qquad \mathrm{Var}(dt\, dw(t)) = o((dt)^3)\,. \tag{3.9b}$$

Suppose we treat terms of order $o(dt)$ as essentially zero, then we observe that $[dw(t)]^2$ and $dt\, dw(t)$ are both non-stochastic, since their variances are essentially zero. Hence, $[dw(t)]^2 = dt$ and $dt\, dw(t) = 0$ are satisfied not just in expectation but exactly (e.g. Kwok [42], Shreve [63], Szepessy et al. [68], or Epps [21]).

Therefore the Brownian motion can be described by its deterministic and fluctuating components and the increments $dX(t)$ can be expressed in the corresponding form of a total differential as follows

$$dX(t) = \mu dt + \sigma dw(t)\,, \tag{3.10}$$

where $\{w(t)\,;\, t > 0\}$ depicts a Wiener process, μ is the drift rate and σ^2 is the variance rate of the process. Moreover, making use of the results $[dw(t)]^2 = dt$ and $dt\, dw(t) = 0$, one may observe that $[dX(t)]^2 = \sigma^2 dt$, which evidently is *not* a random variable, even thought $dX(t)$ is so. The equation (3.10) is called *stochastic differential equation* (or *Itô's process*)

Proposition 1 (Itô's Lemma)**.** *(Shreve [63], Szepessy et al. [68], Epps [21], Chiang [13], Kwok [42], Oksendal [56]) Let $u(x,t)$ be a smooth, non-random function with continuous partial derivatives and $x(t)$ a stochastic process defined by*

$$dx(t) = \mu(x,t)dt + \sigma(x,t)dw(t), \tag{3.11a}$$

where $w(t)$ is the Wiener process. Then the stochastic process $y(t) = u(x(t),t)$ has the following form of stochastic differential

$$du(x,t) = dy(t) = \Big(\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}\Big)dt + \sigma(x,t)\frac{\partial u}{\partial x}dw(t). \tag{3.11b}$$

We restrict ourselves to the sketch of the proof – for a detailed and rigorous enough version the reader is recommended to see e.g Oksendal [56], Szepessy et al. [68], Shreve [63], Epps [21], Chiang [13], or Kwok [42].

Intuitively, Itô's lemma can be proved utilizing the two dimensional Taylor series expansion up to the second order, since

$$u(x+dx, t+dt) - u(x,t) = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{1}{2}\Big\{\frac{\partial^2 u}{\partial t^2}[dt]^2 + 2\frac{\partial^2 u}{\partial t \partial x}dt\, dx + \frac{\partial^2 u}{\partial x^2}[dx]^2\Big\} + \text{ h.o.t.}.$$

Forasmuch as $dw(t) = Z\sqrt{dt}$ for $Z \sim \mathcal{N}(0,1)$ we achieve that

$$\mathbb{E}\left([dw(t)]^2 - dt\right) = 0\,, \qquad \mathrm{Var}\left([dw(t)]^2 - dt\right) = [\mathbb{E}(Z^4) - (\mathbb{E}(Z^2))^2](dt)^2 = 2(dt)^2\,.$$

The approximation to the term $dw(t)$ by dt can be obtained by omitting the higher order term in expression above and moreover, $dx\,dt = O((dt)^{3/2}) + O((dt)^2)$.Finally the $(dx)^2$ term can be estimated by means of

$$(dx)^2 = \sigma^2(dw)^2 + 2\mu\sigma dw\,dt + \mu^2(dt)^2 \approx \sigma^2 dt + O((dt)^{3/2}) + O((dt)^2).$$

Hence the Taylor expansion of the deterministic function about the deterministic variable t and random variable x up to the second expansion term takes the ensuing form

$$du(x,t) = \frac{\partial u}{\partial x}dx + \left[\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}\right]dt\,.$$

Substituting the expression $dx = \mu(x,t)dt + \sigma(x,t)dw$ valid for the stochastic process $x = x(t)$ for $t \geq 0$ with the variable drift function $\mu(x,t)$ and variance function $\sigma^2(x,t)$, for the differential term one may derive that

$$du(x,t) = dy(t) = \Big(\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}\Big)dt + \sigma(x,t)\frac{\partial u}{\partial x}dw(t)\,. \tag{3.12}$$

Notice that for any small time step $\Delta = Tn^{-1}$ and any equidistant interval partition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ the increments $x(t_1) - x(t_0), \ldots, x(t_n) - x(t_{n-1})$ are independent random variables.

Remark 1. *The Itô's process driven by the stochastic differential equation $dx = \mu(x,t)dt + \sigma(x,t)dw$ can be understood as the limiting case for $n \to \infty$. It is highly desirable to remark that the process of the construction presented above preserves the Wiener process $w \equiv \{w(t)\,,\, t \geq 0\}$ increments $dw(t_i) = w(t_{i+1}) - w(t_i)$ and the random variable $x(t_i)$ independent at any time t_i. Recalling the Brownian motion properties, $\mathbb{E}(w(t_{i+1})) = \mathbb{E}(w(t_i))$ concretely and the linearity of the expected value, we are allowed to conclude the key property of Itô's process :*

$$\mathbb{E}\left(\sigma(x,t)\,dw\right) = 0\,. \tag{3.13}$$

3.1.3 Itô's Integral and Isometry

The definition of the Wiener process $w \equiv \{w(t)\,,\, t \geq 0\}$ (see Definition 3) for a random variable $w(t) \sim \mathcal{N}(0,t)$ can be for any identically constant function $f(s) = a$ formulated as follows

$$\int_0^t f(s)dw(s) = a[w(t) - w(0)] = aw(t) \sim \mathcal{N}(0, a^2 t) = \mathcal{N}(0, \int_0^t f^2(s)ds)\,.$$

The previous observation inspires us to define the *Itô's Integral* for any square–integrable function $f : (0,T) \to \mathbb{R}$ subsequently.

Proposition 2 (Itô's Isometry)**.** *(Shreve [63], Epps [21], Chiang [13], Oksendal [56],Szepessy et al. [68], Kwok [42]) Let be $f : (0,T) \to \mathbb{R}$ an arbitrary square–integrable function. Then there exists Itô's Integral $\int_0^t f(s)dw(s) \sim \mathcal{N}(0, \sigma^2(t))$ where $\sigma(t) = [\int_0^t f^2(s)ds]^{1/2}$. Thus*

$$\mathbb{E}\Big(\int_0^t f(s)dw(s)\Big) = 0\,, \qquad \mathbb{E}\Big(\big[\int_0^t f(s)dw(s)\big]^2\Big) = \int_0^t f^2(s)ds\,. \tag{3.14}$$

Provided the reader is interested it the foregoing lemma proof we advise to read e.g. Oksendal [56], Shreve [63], Epps [21], Chiang [13], or Kwok [42].

3.2 Dynamic Optimization Problem

Many financial instruments allow the holder to make decisions along the way that effect the ultimate value of the instrument. To compute the value of such an instrument, we also seek the optimal decision strategy.

Dynamic programming is a computational method that computes the value and decision strategy at the same time and therefore it affects the ultimate pay–off. The difficulty of such a *multi–period decision problem* is reduced to a hopefully easier *single period problems* sequence that are treated backward in time much as the expectation method does. The principle of the dynamic programming technique consists in the appropriate value function, $f(x,t)$, definition. In the real world, dynamic programming is used to determine *optimal trading strategies* for traders trying to take or unload a big position without moving the market, to find cost efficient hedging strategies when trading costs or other market frictions are significant, and for many other purposes (see Walde [69], Bellman [5], Fletcher [26], Kirk [40], Smith [65]).

3.2.1 Optimal Control Problem

Introducing the Markov chain $X = X(t)$ system where the transition probabilities depend on a *control parameter* θ chosen as a function of a particular time t and system state $X(t)$, and noticing that the knowledge of past history has no effect on the future predictions, we are allowed to formulate a problem of the optimal control policy – uncovering the optimal *feedback control* or *decision strategy*. Instead of trying to choose a whole control trajectory over the time $[0,T]$ we instead try to choose the feedback functions $\theta(X(t),t)$.
The objective of our effort is to maximize the expected payout of any considered strategy, hence find the *optimal decision strategy* θ^* under which the best result is attained conditionally on given initial value x_0 (Bellman [5], Fletcher [26], Walde [69], Kirk [40], Smith [65]):

$$\max_{\theta}\{\mathbb{E}[U(X^{\theta}(T))] \mid X^{\theta}(0) = x_0\} = \mathbb{E}[U(X^{\theta^*}(T)) \mid X^{\theta^*}(0) = x_0] .$$

Furthermore, at any time $t \in [0,T]$ our choice of control variable θ is restricted such that the control trajectory $\{\theta\}_t^T$ over the time horizon $[t,T]$ lies in the set of all admissible strategies

$$\Delta \equiv \Delta_t^T = \{\theta : [t,T] \times \mathbb{R}^+ \mapsto \mathbb{R}^N : \theta^T \mathbf{1} = 1 \, , \, \theta \geq 0\} . \tag{3.15}$$

A practical example what the Dynamic Optimization Problem substantial feature is, is highly needed, therefore we illustrate the above proposed idea on a practical example.
Suppose that an investor has to take a decision about their proportional wealth allocation, i.e. design the investment portfolio – the market offers two possible investments:

1. A risky investment (e.g. a stock), where the price $p_1(t)$ per unit at time t is driven by a stochastic differential equation of the type

$$dp_1(t) = (a + \alpha \cdot \text{noise}) p_1(t)\,, \tag{3.16a}$$

   where $a \in \mathbb{R}^+$ is deterministic and $\alpha \in \mathbb{R}$ is a constant representing the system uncertainty;

2. A safe investment (e.g. a bond, or bank account), where the price $p_2(t)$ per unit at time t fulfils the exponential growth:

$$dp_2(t) = bp_2(t)\,, \tag{3.16b}$$

   where b is a constant symbolizing the growth rate guaranteed on a risky–free financial instrument, such that the obvious restriction takes place – i.e. we presuppose that the condition $0 < b < a$ holds (see Szepessy et al. [68], Oksendal [56], Bellman [5], Fletcher [26], Walde [69], Kirk [40], or Smith [65]).

At each instant t the person has to make a choice of the percentage (or proportion) θ_t of his fortune Y_t that is supposed to be allocated in the risky investment, whereas the rest proportion $1 - \theta_t$ of their wealth is placed automatically in the riskless investment. Forasmuch as we take for granted the particular investor utility function U knowledge and the exact investment period, thus the terminal time T at which the investor requires to attain the pay-off U, the problem consists in the *optimal portfolio risky-to-riskless proportion* $\theta \in [0, 1]$ determination at each time t, i.e. the investment distribution (or the optimal control path) $\{\theta\}_t^T \in \Delta$ derived at time $t \in [0, T]$; which maximizes the expected utility of the corresponding terminal fortune $Y^\theta(T)$ given the initial condition $Y^\theta(0) = y_0$, where y_0 is prescribed (Szepessy et al. [68], Oksendal [56],Bellman [5], Fletcher [26], Walde [69], Kirk [40], or Smith [65]):

$$\max_{0\le\theta\le1} \{\mathbb{E}\left(U(Y^\theta(T)) \mid Y^\theta(0) = y_0\right\}\,. \tag{3.17}$$

We say that the portfolio is *self-financing* providing that it is set up with no initial net investment and no additional funds added or withdrawn afterwards. The additional units acquired for one security in the portfolio is completely financed by the sale of another security in the same portfolio. The portfolio is so-called to be *dynamic* since its composition is allowed to change over time (see Kwok [42], Bellman [5], Fletcher [26], Walde [69], Kirk [40], or Smith [65]).

3.2.2 Examples

Deterministic Problem

We can state our problem in optimal control terms as the maximization of an *objective function* with respect to *control functions* and the set of *feasible controls* that restrict the considered parameters and variables domain of the problem. After introducing the formulation of an optimal control problem the next step is to find its solution. As we shall see, the optimal control is closely related with the solution of a non–linear partial differential equation, known as the Hamilton-Jacobi-Bellman equation. To derive the Hamilton-Jacobi-Bellman equation we shall use the Bellman's dynamic programming principle stating that for an arbitrary initial state (t,y) and initial decision strategy the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision (for further details see Bellman [5], Fletcher [26], Kirk [40], Smith [65] or Bertsekas [8]).

First of all generalize the original definition (3.15) of the set $\Delta = \Delta_t^T$ of control functions on the interval $[t,T]$ for $t \in [0,T]$ as follows

$$\Delta \equiv \Delta_t^T = \{\theta : [t,T] \mapsto A \subset \mathbb{R}^N\}, \qquad t \in [0,T].$$

Above, A represents some compact subset of $\mathbb{R}^N$. Notice that in case of (3.15) A stands for an N–dimensional simplex.
Next, consider the following deterministic ordinary differential equation for the function $v = v(s) \in \mathbb{R}^N$ and flux $f : \mathbb{R}^N \times A \to R^N$ defined on the time interval $[t,T]$, as follows:

$$\begin{aligned} \frac{dv}{ds}(s) &= f(v(s),\theta(s)), \qquad t < s < T; \\ v(t) &= x. \end{aligned} \tag{3.18}$$

Then the solution to the problem (3.18) corresponding optimal controlling mathematical formulation is represented by the ensuing problem - we need to determine

$$\inf_{\theta \in \Delta} \left\{ \int_t^T h(v(s)\theta(s))ds + g(v(T)) \right\}, \tag{3.19}$$

for a prescribed instant cost function $h : \mathbb{R}^N \times [t,T] \to \mathbb{R}$ and given terminal cost function $g : \mathbb{R}^N \to \mathbb{R}$. Optimal control problems can be solved by the Lagrange principle or dynamic programming.

Dynamic Programming Approach The dynamic programming approach uses the value function, defined by

$$u(t,x) = \inf_{\theta \in \Delta} \left\{ \int_t^T h(v(s)\theta(s))ds + g(v(T)) \right\}, \tag{3.20}$$

and leads to solution of a non–linear *Hamilton–Jacobi–Bellman partial differential equation* written as for a shorthand as follows, for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$.

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + H\left(\frac{\partial u}{\partial x}(t,x),x\right) &= 0, \qquad 0 \le t < T, \\ H(p,y) &= \min_{\theta \in \Delta}\{\mathscr{H}(p,y;\theta)\}, \\ \mathscr{H}(p,y;\theta) &= f(y,\theta)\cdot p + h(y,\theta), \\ u(T,x) &= g(x). \end{aligned} \tag{3.21}$$

The idea of the dynamic programming approach to optimal control problems solution dwells in backtracking technique – assuming that at the final time T the value function is known and prescribed as $u(T,x) = g(x)$ and then, recursively for small time step backward, the optimal control θ_t^* is determined as the one under which the transition from the point (t,x) at the time level t to the successive *best possible value* of the value function $u(t+\Delta t, X^\theta(t+\Delta t))$ affected by the control parameter choice, is guaranteed. Thus any path $X^\theta(t+\Delta t)$ on time interval $[t, t+\Delta t]$ that corresponds to a particular choice θ of the control function from $\{\theta\}_t^{t+\Delta t} \in \Delta_t^{t+\Delta t} \equiv \Delta$ satisfies

$$u(x,t) = \inf_{\theta \in \Delta_t^{t+\Delta t}} \left\{ u(t+\Delta t, X^\theta(t+\Delta t)) \right\}.$$

Here we presumed that the function h arising from (3.20) is zero. Furthermore, for a differentiable function u and for an arbitrary control function θ choice made at time t,

$$du(t, X^\theta(t)) = \Big(\partial_t u(t, X^\theta(t)) + \partial_x u(t, X^\theta(t)) \cdot f(X^\theta(t), \theta(t)) \Big)\, dt \geq 0\,; \tag{3.22}$$

and the equality in the relation above holds provided the optimal path $X^*(t) \equiv X^{\theta^*}(t)$ under the corresponding control $\theta^*(t)$ exist, the infimum is attained. Henceforth

$$du(t, X^*(t)) = \left(\partial_t u(t, X^*(t)) + \partial_x u(t, X^*(t)) \cdot f(X^*(t), \theta^*(t)) \right) dt = 0\,. \tag{3.23}$$

Taking (3.22)–(3.23) into account be achieve the Hamilton–Jacobi–Bellman equation in the situation of h, introduced in (3.20) is zero:

$$\begin{aligned} \partial_t u(t,x) + \min_{\theta \in \Delta} \left(\partial_x u(t,x) \cdot f(x,\theta) \right) = 0\,, \qquad & 0 \leq t < T\,; \\ u(T,x) = g(x)\,. & \end{aligned} \tag{3.24}$$

Now coming to h in generally non–zero, notice that

$$0 = \inf_{\theta \in \{\theta\}_t^{t+\Delta t} \in \Delta} \left\{ \int_t^{t+\Delta t} h(X^\theta(t), \theta(t))\, ds + u(t+\Delta t, X^\theta(t+\Delta t)) - u(t,x) \right\}\,; \tag{3.25}$$

and moreover under the presupposition of u differentiability, similarly to (3.24) one can deduce that

$$\begin{aligned} \partial_t u(t,x) + \min_{\theta \in \Delta} \left(\partial_x u(t,x) \cdot f(x,\theta) + h(x,\theta) \right) = 0\,, \qquad & 0 \leq t < T\,; \\ u(T,x) = g(x)\,. & \end{aligned} \tag{3.26}$$

Again, for more detailed information concerning the dynamic programming and the associated Hamilton–Jacobi–Bellman equation for the deterministic case the reader is referred to e.g. Bellman [5], Szepessy et al. [68], Kirk [40], Smith [65], Fletcher [26], Bertsekas [8], Walde [69] or Oksendal [56].

Lagrange Principle The well–known Lagrange principle is aimed on the minimum of the cost function subject to the constraints seeking and thus by uncoupling the characteristics of the foregoing Hamilton–Jacobi–Bellman equation one may obtain the Hamilton system of ordinary differential equations based on Pontryagin Principle (see Bellman [5], Szepessy et al. [68], Fletcher [26], Walde [69], Kirk [40], Smith [65] or Bertsekas [8]). It is particularly useful when the problem dimension $N \gg 1$ but it does not have any efficient implementation providing that the stochastic variables are present.

We introduce the Lagrange function (Szepessy et al. [68],Fletcher [26], Walde [69], Kirk [40], Smith [65])

$$L \equiv \mathscr{L}(\lambda, y, x) = F(x,y) + \lambda \cdot (y - g(x)) ,$$

for the sufficient differentiable objective function $F : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}$ for some $n \in \mathbb{N}$ that is aimed to be minimized subject to the solution feasible region delimiting constraints – $x \in A$ a compact subset of $\mathbb{R}^N$ and $y = g(x)$ for a prescribed function $g : \mathbb{R}^N \to \mathbb{R}^n$; where we employ the Lagrange multiplier $\lambda \in \mathbb{R}^n$.
The problem lead to the usual first order necessary condition for an interior minimum, hence $\nabla \mathscr{L}(\lambda, y, x) = 0$, i.e.

$$\nabla \mathscr{L}(\lambda, y, x) = 0 \Longleftrightarrow \begin{cases} 0 &= \partial_\lambda \mathscr{L}(\lambda, y, x) &= y - g(x) , \\ 0 &= \partial_y \mathscr{L}(\lambda, y, x) &= \partial_y F(x,y) + \lambda , \\ 0 &= \partial_x \mathscr{L}(\lambda, y, x) &= \partial_x F(x,y) - \lambda \partial_x g(y) . \end{cases} \tag{3.27}$$

Note that the first equation represents the constraint whereas the second equation uncover the optimal value the Lagrange multiplier λ inasmuch as it can be easily seen that $\lambda = -\partial_y F(x,y)$.

We recommend the reader interested in Lagrange multiplier issue to see Szepessy et al. [68], Kirk [40], Smith [65], Fletcher [26] or Oksendal [56].

Stochastic Problem

The forthcoming part is devoted to an optimal asset management problem featuring a terminal time expected utility criterion and illustrates the ideas and construction of the continuous stochastic optimal control problem and its formulation in term of dynamic programming technique. The inspiration to the forthcoming problem comes from The Songzhe [66] and Browne [12], and the model was extended and deeply analysed in Bielecki et al. [9]. The market under consideration here consists of $n+1$ underlying processes, in the prescribed time horizon $[0,T]$ continuously traded financial instruments – one riskless asset B called a bond and n possibly correlated risky assets $S^{[1]}, \dots, S^{[n]}$ or stocks. The price process corresponding to the i th asset is depicted as $\{S_t^{[i]}; t \in [0,T]\}_{i=1}^n$ and $\{B_t; t \in [0,T]\}$, respectively with given initial value

$$B_0 = p_0 , \qquad S_0^{[i]} = p_i \qquad i = 1, \dots, n . \tag{3.28a}$$

The abstract portfolio comprising of all assets replicates the market above – at time t each traded financial instrument is presented in $\theta_t^{[i]} \times 100\%$, where $\theta_t^{[i]}$ is not limited. The concrete investor is aimed to invest all possessing initial wealth $Y_0 = y$ into this abstract portfolio. At each time step he is supposed to choose the trading strategy, hence determine the preferred proportion of each existing asset in the portfolio, hence rearrange the portfolio considered in the previous time–step.

Denote $Y_t^{\lambda_t}$ the wealth of the portfolio, also comprehended as the investor's wealth, at time

$t \in [0,T]$, i.e.

$$Y_0 = y\,, \qquad Y_t(\theta_t) = \sum_{i=1}^{n} \theta_t^{[i]} S_t^{[i]} + \Big\{1 - \sum_{i=1}^{n} \theta_t^{[i]}\Big\} B_t \quad 0 < t \leq T\ ,\ \theta_t \in \mathbb{R}^n\,, \tag{3.28b}$$

where θ_t stands for the n-dimensional real-value vector $(\theta_t^{[1]}, \dots, \theta_t^{[n]})$ and $\theta_t^{[i]}, i = 1, \dots, n$ of his wealth at time t signify the ratio be invested in corresponding risky asset at time t with the remainder $\theta_t^{[0]}$ placed in the risk-free bond, thus $\theta_t^{[0]} = 1 - \sum_{i=1}^{n} \theta_t^{[i]}$.
Remark that at each time–step the portfolio wealth is reinvested, i.e. placed back to the market and the existence of no external available financial source is assumed – this reflects the idea of the *self-financing portfolio*. Observe that since no other restrictions on $\theta_t \equiv (\theta_t^{[0]}, \dots, \theta_t^{[n]})$ are set, the short selling and borrowing are permitted. In general several government restrictions may come on stage – for instance, the bond representation ratio in the portfolio cannot tail off below certain appointed level, or short and long positions are not allowed, respectively. Intuitively, in this regard the optimal policy reached under the considered constraints simply cannot be *better* than the one achieved providing that no limitations are presumed.

For the purpose of this paper assume that the risky stock prices are reckoned to be correlated but mutually distinct Brownian motions $\{W_t^{[i]}; t \in [0,T]\}_{i=1}^{n}$, i.e. each stock $S_t^{[i]}, i = 1, \dots, n$ satisfies the following stochastic differential equation associated to its price process

$$\frac{dS_t^{[i]}}{S_t^{[i]}} = \mu_t^{[i]} dt + \sum_{j=1}^{n} \sigma_t^{[ij]} dW_t^{[k]}\,, \qquad\qquad i = 1, \dots, n\,, \tag{3.29a}$$

with the associated appreciation rate $\mu_t^{[i]}$ (known as mean or *expected return* of the relevant stock) and the volatilities $\sigma_t^{[ij]}$, reckoned to be constant and here the values of $\{\mu_t^{[i]}; i = 1, \dots, n\}$ and $\{\sigma_t^{[ij]}; i, j = 1, \dots, n\}$, with constant values (with respect to studied price process of each individual asset) specified at time $t \in [0,T]$.
The price of the risk–free asset is presupposed to evolve fully deterministically according to

$$\frac{dB_t}{B_t} = r_t dt\,, \tag{3.29b}$$

where r_t designates the constant interest rate known at time $t \in [0,T]$.
In practice, bonds also possesses a certain level of the *uncertainty* in attaining the associated expected returns, hence generally the bond volatility should be included – but this is not the case in this illustrative problem. Obviously, it is reasonable to assume that $\mu_t^{[i]} > r_t$ inasmuch as otherwise we would have "money for nothing", or an arbitrage opportunity – therefore we take for granted that the market is *arbitrage–free*. The second postulate is made on the rank of the square matrix σ_t – we presuppose that it is of full rank in any time $t \in [0,T]$, i.e. it is positive defined and thus the existence of the financial instrument with risk profile fully determined by the remaining assets risk profiles is eliminated.

Therefore the wealth of the investor at time $t \in [0,T]$ follows the forthcoming stochastic differential equation, firstly studied in Merton [48]

$$\frac{dY_t^{\theta_t}}{Y_t^{\theta}} = \sum_{i=1}^{n} \theta_t^{[i]} \frac{dS_t^{[i]}}{S_t^{[i]}} + \Big[1 - \sum_{i=1}^{n} \theta_t^{[i]}\Big] \frac{dB_t}{B_t} = \Big\{r_t + \sum_{i=1}^{n} \theta_t^{[i]} \big[\mu_t^{[i]} - r_t\big]\Big\} dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_t^{[i]} \sigma_t^{[ij]} dW_t^{[i]}\,, \tag{3.30a}$$

for all $t \in [0,T]$ and any choice of $\theta_t \in \mathbb{R}^n$, or as for a shorthand,

$$\frac{dY_t^{\theta_t}}{Y_t^{\theta}} = \left\{ r_t + \theta_t^T [\mu_t - r_t \mathbf{1}] \right\} dt + \theta_t^T \sigma_t dW_t \,, \qquad t \in [0,T] \,,\ \theta_t \in \mathbb{R}^n \,. \tag{3.30b}$$

where super–index T in θ_t^T refers to the vector transpose. The utility function $U(x)$ of the concrete investor, understood as the portfolio value at the terminal time, represents the investor's own attitude to risk compared to the profit in final year T, and hence can be construed as strictly increasing and concave function. The aim of the particular investor is to prefer at each time t the investment policy that will lead to the maximal expected terminal value of the continuously traded portfolio, thus we launch the reward function under the investment policy θ chosen at time t as

$$\mathscr{V}^{\theta}(t,y) = \mathbb{E}\left[U(Y_T^{\theta}) \mid Y_t^{\theta} = y \right], \qquad (t,y) \in [0,T) \times \mathbb{R} \,. \tag{3.31a}$$

The objective of this *self–financing optimal investment problem* is to detect at any time $t \in [0,T)$ for the arbitrary investor's wealth y determined by chosen reallocation of his wealth into the financial instruments the suitable investment strategy θ^* that maximizes the reward function $\mathscr{V}^{\theta}(t,y)$, i.e.

$$V(t,y) \equiv \sup_{\theta \in \Delta} \left\{ \mathscr{V}^{\theta}(t,y) \right\} = \mathscr{V}^{\theta^*}(t,y) \qquad (t,y) \in [0,T) \times \mathbb{R} \tag{3.31b}$$

where $\Delta_t^T \equiv \Delta$ denotes the set of all admissible controls at time $t \in [0,T]$ - we are interested in all strategies under which (3.31a) is finite.

The *Bellman's Principle of Optimality* (e.g. Bellman [5], Fletcher [26], Kirk [40], Smith [64] or Bertsekas [8]) states that for an arbitrary initial state (t,y) and initial investment strategy decision the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision and the assets are continuously traded. Therefore for any choice of referring time $t \in [0,T]$ and $s \to t^+$ we may rewrite

$$Y_t = y \,, \qquad Y_{s+t}^{\theta} = F(y,t,\theta) \,, \tag{3.31c}$$

where θ stands for the arbitrary investment strategy made at time t and the investor's wealth level y. Then the optimal wealth process follows the incremental footsteps on the time horizon $[t,T]$ for $dt \equiv \tau \to 0^+$ and the initial wealth y

$$V(t,y) = \sup_{\theta \in \Delta} \left\{ \mathbb{E}\left[V(t+\tau, F(Y_t,t,\lambda)) \mid Y_t = y \right] \right\}, \qquad 0 \le t < t+\tau \le T \,, \tag{3.31d}$$

$$V(T,y) = U(y) \,, \qquad y \in \mathbb{R} \,. \tag{3.31e}$$

Henceforth in order to obtain the continuous stochastic optimal control problem formulation, we need to perform some calculations. Applying Itô's calculus (see chapter 3.1.2) on $dZ_t^{\theta} = A_t(\theta)dt + B_t(\theta)dW_t$ where $Z_t^{\theta} \equiv \ln Y_t^{\theta}$ for which $dZ_t^{\theta} = \frac{dY_t^{\theta}}{Y_t^{\theta}}$ and A_t, B_t are evaluated at the referring time $t \in [0,T]$, leads to the next stochastic differential equation for $V \equiv V(t,y)$

$$dV(t,y) = \left\{ \partial_t V(t,y) + A_t(\theta) y \partial_y V(t,y) + \frac{1}{2} B_t(\theta) B_t^T(\theta) y^2 \partial_{yy} V(t,y) \right\} dt + B_t(\theta) y \partial_y V(t,y) dW_t \,, \tag{3.32a}$$

$$A_t \equiv A_t(\theta) = r_t + \theta_t^T [\mu_t - r_t \mathbf{1}] \,, \tag{3.32b}$$

$$B_t \equiv B_t(\theta) = \theta_t^T \sigma_t \,, \tag{3.32c}$$

where $\partial_t V$, $\partial_y V$, $\partial_{yy} V$ symbolize the time or space, respectively derivatives of the value function $V \equiv V(t,y)$.
Under the idea of optimality principle stated above, the optimal portfolio value process, driven accordingly to (3.31d)–(3.31e) must satisfy

$$\mathbb{E}\left[\frac{V(t+\tau,F(Y_t,t,\theta))-V(t,Y_t)}{\tau} \mid Y_t = y\right] = 0\,, \qquad 0 \le t < t+\tau \le T\ ,\ y \in \mathbb{R}. \tag{3.33a}$$

We aim your attention to the fact that the vector of Brownian motions is a zero mean normally distributed random vector, i.e. $W_t = \mathcal{N}(0,\sigma^T\sigma)$ and hence each $\mathbb{E}\,[dW_t^{[i]}] = 0$. Observing that $dV_t \equiv V(t+\tau,Y_{t+\tau}) - V(t,Y_t)$ for small $\tau \to 0^+$, by substituting (3.32a) to (3.33a) for $\tau = dt$ we achieve

$$\begin{aligned} 0 &= \sup_{\theta\in\Delta}\left\{\mathbb{E}\left[\frac{V(t+\tau,F(Y_t,t,\theta))-V(t,Y_t)}{\tau} \mid Y_t = y\right]\right\} \\ &= \sup_{\theta\in\Delta}\left\{V_t(t,y) + A_t(\theta)y\partial_y V(t,y) + \frac{1}{2}B_t(\theta)B_t^T(\theta)y^2\partial_{yy}V(t,y)\right\}. \end{aligned} \tag{3.33b}$$

Inasmuch as $V(t,y)$ is independent of chosen strategy at time t, one may formulate the continuous stochastic optimal control problem defined on time horizon $[t_0,T]$ using the idea of the Hamilton function as follows

$$\begin{aligned} \frac{\partial V}{\partial t}(t,y) + H\left(t,y,\frac{\partial V}{\partial y}(t,y),\frac{\partial^2 V}{\partial y^2}(t,y)\right) &= 0\,, && (t,y)\in[0,T)\times\mathbb{R}\,, \\ V(T,y) &= U(y)\,, && y \in \mathbb{R}\,, \end{aligned}$$

where

$$\begin{aligned} H \equiv H(s,z,p,q) &= \sup_{\theta\in\Delta}\{\mathcal{H}(s,z,p,q,\theta)\}\,, && (s,z,p,q)\in[0,T)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\,, \\ \mathcal{H} \equiv \mathcal{H}(s,z,p,q,\theta) &= A_s(\theta)zp + \frac{1}{2}[B_sB_s^T](\theta)z^2q\,, && (s,z,p,q,\theta)\in[0,T)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\,, \\ A_s \equiv A_s(\theta) &= r_s + \theta_s^T[\mu_s - r_s\mathbf{1}]\,, && B_s \equiv B_s(\theta) = \theta_s^T\sigma_s\,. \end{aligned}$$

Forasmuch as $\mathcal{H}$ is quadratic in θ, applying the first order necessary condition on the function maximum under the assumption of $q<0$ and $B_s \neq 0$ one can straightforward infer that the optimal trading strategy $\theta^* \equiv \theta^*(t,y)$ at time t and wealth level y for $p \equiv \frac{\partial V}{\partial y}(t,y)$ and $q \equiv \frac{\partial^2 V}{\partial y^2}(t,y)$ is attained when

$$\theta^* \equiv \theta^*(t,y) = -\frac{\frac{\partial V}{\partial y}(t,y)}{y\frac{\partial^2 V}{\partial y^2}(t,y)}[\sigma_t\sigma_t^T]^{-1}(\mu - r_t\mathbf{1}). \tag{3.34}$$

Therefore the resulting Hamilton–Jacobi–Bellman equation associated to concerned optimal investment strategy problem for $t_0 = 0$ takes these form:

$$\begin{aligned} \frac{\partial V}{\partial t}(t,y) + yr\frac{\partial V}{\partial y}(t,y) - \frac{1}{2}(\mu - r\mathbf{1})^T[\sigma\sigma^T]^{-1}(\mu - r\mathbf{1})\frac{\left[\frac{\partial V}{\partial y}(t,y)\right]^2}{\frac{\partial^2 V}{\partial y^2}(t,y)} &= 0\,, && (t,y)\in[0,T)\times\mathbb{R}\,, \\ V(T,y) &= U(y)\,, && y\in\mathbb{R}\,. \end{aligned}$$

3.3 Utility Functions

The references of individuals about having n goods in corresponding quantities $x_1, \ldots, x_n$ are often represented by a utility function $U(x_1, \ldots, x_n)$. For a detailed theory we refer to Fishburn [25] and Dupacova et al. [20].

Remark that the only relevant feature of a utility function is its ordinal character, not its absolute values. The most crucial thing here is the right choice of the utility function and its parameters, reflecting in particular investor's attitude to risk. If $N = 1$, U has the following properties:

- $U(x)$ is increasing in x, hence *more is always better*;
- $U(x)$ is concave in x, that is referred to as a *risk aversion property*. From this point of view we distinguish three types of investors: risk loving, risk neutral, and risk averse, investors with convex, affine and concave utility function, respectively. Notice that it is convenient to model the investor as risk-averse.

There are several ways how to express the concrete investor risk aversion. A risk aversion coefficient is a special measure reflecting the character and degree of investor's risk aversion. In order to avoid the sensitivity in the utility function change, we define the *Arrow–Pratt absolute risk aversion coefficient* in the following manner:

Definition 5. *(Dupacova et al. [20], Fishburn [25]) The absolute risk aversion coefficient at a point x pertaining to a utility function $U = U(x)$ is defined as*

$$\lambda_A(x) = -\frac{U''(x)}{U'(x)} \tag{3.35}$$

Utility functions with a constant absolute risk aversion coefficient are called CARA utility functions.

Remark that for the major part of investors the absolute risk aversion coefficient has a decreasing character. It can be easily deduced that the utility function U exhibits constant absolute risk aversion if the absolute risk aversion coefficient does not depend on the wealth, hence $\lambda_A'(x) = 0$ for all x. A typical example of the constant absolute risk aversion utility function is the negative exponential utility function of the form

$$U(x) = -e^{-\alpha x} \qquad \alpha > 0\,.$$

Definition 6. *(Dupacova et al. [20], Fishburn [25]) The relative risk aversion coefficient at a point x pertaining to a utility function $U = u(x)$ is defined as*

$$\lambda_R(x) = -x\frac{U''(x)}{U'(x)} \tag{3.36}$$

Utility functions with a constant absolute risk aversion coefficient are called CRRA utility functions.

Most often investors are assumed to have constant relative risk aversion. For more details on the topics concerning about the utility function the reader is recommended to see e.g. Dupacova et al. [20] and Fishburn [25].

The constant relative risk aversion (CRRA) utility function A constant relative risk aversion $C(y) \equiv d > 0$ for every $y > 0$ would imply that an investor tends to hold a constant proportion of his wealth in any class of risky assets as the wealth varies. A constant relative risk aversion $C(y) \equiv d > 0$ for every $y > 0$ would imply that an investor tends to hold a constant proportion of his wealth in any class of risky assets as the wealth varies. The reader is refereed to a vast economic literature addressing the problem of a proper choice of investor's utility function (see e.g. I. Friend and Blume [31], Pratt [60] and Young [70]). In the case of a constant relative risk aversion $C(y) \equiv d > 0$ an increasing utility function U is uniquely (up to an multiplicative and additive constant) given by

$$U(y) = -y^{1-d} \quad \text{if } d > 1\,, \quad U(y) = \ln(y) \quad \text{if } d = 1\,, \quad U(y) = y^{1-d} \quad \text{if } d < 1\,. \tag{3.37}$$

The coefficient d of relative risk aversion plays an important role in many fields of theoretical economics. There is a wide consensus that the value should be less than 10 (see e.g Mehra and Prescott [46]). In our numerical experiments we considered values of d close to 9. But it could be also lower for lower equity premium. It is worth to note that the CRRA function is a smooth, increasing and strictly concave function for $y > 0$.

For the purpose of our forthcoming analysis we consider the utility function $U(y)$ of the form

$$U(y) = -y^{1-d} \qquad \text{where} \qquad d > 1. \tag{3.38}$$

The function U is a smooth strictly increasing concave function. Now it should be obvious that the power like behaviour of the utility function $U(y) = -y^{1-d}$ leads to the constant initial condition i.e.

$$\psi(0,x) = \gamma d, \quad \text{for any } \; x \in \mathbb{R}\,. \tag{3.39}$$

3.4 Discrete simple model derivation

In this section we first recall a discrete dynamic stochastic optimization problem arising in optimal portfolio selection. The discrete version of this model has been derived in Macová and Ševčovič [44], Múčka [51] and Kilianová et al. [37]. It was applied for solving a problem of construction of an optimal stock to bond proportion in pension fund selection for the second pillar of the Slovak pension system. In what follows, we recall key steps in derivation of the discrete dynamic stochastic optimization pension savings model (see Macová and Ševčovič [44], Kilianová et al. [37]).

In further part of this work we shall generalize the model from its discrete version to a continuous and more complex one that includes unrestricted amount of assets traded on the market. It will be shown that the continuous model for solving a problem of optimal cumulative stock to bond proportion in pension fund selection can be reformulated in terms of a fully non–linear parabolic equation also referred to as the Hamilton–Jacobi–Bellman equation.

In the discrete optimal pension fund selection model due to Kilianová et al. [37],Melicherčík and Ševčovič [47], a future pensioner with the expected retirement time in T years transfers regularly once a year an ε-part of his yearly salary with the deterministic rate of growth β_t to the pension fund investing in financial market with the yearly stochastic return r_t.

More precisely, we denote by B_t his yearly salary at the year t. Then the budget constraint equation for the total accumulated sum Y_t in his pensioner's account reads as follows:

$$\begin{aligned} Y_{t+1} &= (1+r_t)Y_t + \varepsilon B_{t+1}\,, \qquad \text{for } t = 1,2,\dots T-1, \\ Y_1 &= \varepsilon B_1\,. \end{aligned} \tag{3.40}$$

Supposing the wage growth β_t is known, one can derive the relation between two consecutive yearly salaries of the following form:

$$B_{t+1} = (1+\beta_t)B_t\,.$$

Since a certain standard of living in retirement guarantee is highly demanded by the investor, at the time of retiring a future pensioner will aim at maintaining his living standards compared to the level of the last salary at the retirement time $t = T$.
Therefore the absolute value of the total saved sum Y_T at the time of retirement T does not represent the quantity a future pensioner will be taking care about. A possible indicator of this important information for the saver can expressed by a ratio of the cumulative saved sum Y_T and the yearly salary B_T, henceforth at each time t we introduce the proportion

$$y_t = Y_t/B_t\,.$$

In terms of the quantity y_t representing the number of yearly salaries already saved at time t, the budget–constraint equation can be reformulated recurrently as follows:

$$y_1 = \varepsilon y_0\,, \qquad \text{and} \qquad y_{t+1} = y_t\frac{1+r_t}{1+\beta_t} + \varepsilon\,, \quad \text{for } t = 1,2,\dots T-1\,. \tag{3.41}$$

For the sake of simplicity, we presume that the investment strategy of the pension fund at time t is given by the proportion $\theta \in [0,1]$ of stocks and $1-\theta$ of bonds and that the fund return r_t is normally distributed with the mean value $\mu_t(\theta)$ and dispersion $\sigma_t^2(\theta)$ for any choice of the stock to bond proportion θ. It means that

$$r_t(\theta) \sim N(\mu_t(\theta), \sigma_t^2(\theta)), \qquad \text{i.e.} \qquad r_t(\theta) = \mu_t(\theta) + \sigma_t(\theta)Z\,, \tag{3.42}$$

where $Z \sim \mathscr{N}(0,1)$ is a normally distributed random variable having the density function

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2})\,, \qquad \text{for all } z \in \mathbb{R}\,.$$

Both μ_t and σ_t^2 depend directly on the choice of parameter θ representing stock to bond proportion in the portfolio of the investor's pension fund. It is assumed to belong to the prescribed admissible set $\Delta_t^T \equiv \Delta = [0,1]$, i.e. we impose only ban on short positions.The admissible set Δ is subject to governmental regulations that may be imposed on the stock to bond proportion in a specific time $t \in [0,T]$. Thus obviously, as various legislative norms take place, one may restrict investment strategies such that $\Delta_t^T = [l_t, u_t] \subseteq [0,1]$ for any time $t \in [0,T]$. At each time t, the mean value and volatility of the fund return r_t can be expressed in terms of expected values of returns $\mu_t^{(s)}, \mu_t^{(b)}$ and volatilities $\sigma_t^{(s)}, \sigma_t^{(b)}$ of stocks and bonds as follows:

$$\begin{aligned} \mu_t(\theta) &= \theta\mu_t^{(s)} + (1-\theta)\mu_t^{(b)}, && (3.43)\\ \sigma_t^2(\theta) &= \theta^2[\sigma_t^{(s)}]^2 + (1-\theta)^2[\sigma_t^{(b)}]^2 + 2\theta(1-\theta)\sigma_t^{(s)}\sigma_t^{(b)}\rho_t, && (3.44)\end{aligned}$$

where $\rho_t \in [-1,1]$ is a correlation coefficient between the returns on stocks and bonds at time t and the time-independent values of the parameters $\mu^{(s)}$, $\mu^{(b)}$, $\sigma^{(s)}$ and $\sigma^{(b)}$ are known at time $t \in [0,T]$, they follow their relevant mutually independent Markov processes. In view of the stock to bond proportion θ, the formula for the expected return of the portfolio above can be regarded as for the weighted average of the expected returns of both financial instruments where θ plays the rôle of weight.

Thus the time-evolution of the number of allocated yearly salaries can be formulated by the following recurrent equation:

$$\begin{aligned} y_1 = \varepsilon, \qquad & y_{t+1} = G_t^1(y_t, r_t(\theta_t)), && t = 1,2,\dots,T-1,\\ & G_t^1(y, r_t) = \varepsilon + y\frac{1+r_t}{1+\beta_t}, && t = 1,2,\dots,T-1. \end{aligned} \tag{3.45}$$

Notice that $r_t(\theta)$ is the only stochastic variable appearing in the recurrent definition of the processes for the amount y_t of yearly saved salaries.

Our aim is to determine the optimal strategy, i.e. the optimal value of the weight θ_t at each time t that maximizes the contributor's utility from the terminal wealth allocated on their pension account, and so taking into account knowledge of the saver's utility function U, the problem of discrete stochastic dynamic programming can be formulated as

$$\max_{\Delta} \mathbb{E}(U(y_T)), \tag{3.46}$$

subject to the constraint (3.45) where the maximum in the stochastic dynamic problem is taken over all non-anticipative strategies, time sequences of $\{\theta\}_t^T$ stocks proportions lying in $\Delta_t^T = \Delta = \{\theta : [t,T] \times \mathbb{R}^+ \mapsto \mathbb{R},\ \theta \geq 0\}$. Under the Bellman's optimality principle for an arbitrary initial state (t,y) and initial investment strategy decision the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. Therefore the optimal strategy of the problem (3.46) subject to (3.45) is the solution to the Bellman equation of the dynamic programming

$$W(t,y) = \begin{cases} U(y), & t = T,\\ \max\limits_{\theta\in\Delta} \mathbb{E}_Z\left(W(t+1, F_t^1(\theta,y,Z))\right), & t = T-1,\dots,2,1, \end{cases} \tag{3.47}$$

where

$$F_t^1(\theta,y,z) \equiv G_t^1(y, \mu_t(\theta) + \sigma_t(\theta)z) = y\frac{1+\mu_t(\theta)+\sigma_t(\theta)z}{1+\beta_t} + \varepsilon. \tag{3.48}$$

The expectation is taken with respect to the normally distributed random variable Z present as an argument of F_t^1 and the maximum in the stochastic dynamic problem is taken over all non-anticipative strategies, time sequences of $\{\theta\}_t^T \in \Delta$ stocks proportions for $t = 1,\dots,T$.

In our work the above proposed time–discrete model formulated as the stochastic dynamic problem, is extended – we enlarge the problem dimensions in accordance with the natural requirement of more realistic market where more than two assets are traded and, consequently, the investor's pension fund portfolio may consists of large amount of particular stocks and bonds, each in the corresponding optimal proportion. Furthermore, the continuous version of the studied model is obtained by the transformation of the derived discrete one. The proposed model applied on the second pillar of the Slovak pension system has been tested in Macová and Ševčovič [44], Múčka [51], Kilianová et al. [37] and Macová [43].

Chapter 4

PROBLEM STATEMENT

This study is focused on approximative solution to a specific Hamilton–Jacobi–Bellman equation arising from stochastic dynamic programming for trading the optimal investment decision technique for an individual investor during accumulation of pension savings. Such an optimization problem often emerges in optimal dynamic portfolio selection and asset allocation policy for an investor who is concerned about the performance of a portfolio relative to the given benchmark.

Hence, consider the function $V = V(t,y)$ defined for $t \in [0,T]$ and $y > 0$ following the subsequent fully non–linear Hamilton–Jacobi–Bellman partial differential equation:

$$0 = \frac{\partial V}{\partial t} + \max_{\theta \in \Delta_t^T} \left\{ A_\varepsilon(\theta,t,y) \frac{\partial V}{\partial y} + \frac{1}{2} B^2(\theta,t,y) \left[\frac{\partial^2 V}{\partial y^2} - \lambda \left[\frac{\partial V}{\partial y} \right]^2 \right] \right\}, \tag{4.1a}$$

for all $0 \leq t < T$, $y > 0$ and the terminal condition at $t = T$,

$$V(T,y) = U(y), \quad y \in (0,\infty), \tag{4.1b}$$

where all $U = U(y)$, $A_\varepsilon \equiv A_\varepsilon(\theta,t,y)$ and $B \equiv B(\theta,t,y)$ are smooth and ε, λ are small parameters, $0 < \varepsilon, \lambda \ll 1$.

Observe the presence of the function $V(t,y)$ space derivative squared term, $[\partial_y V]^2$ in (4.1a) – this is not obvious in the standard formulation of the Hamilton–Jacobi-Bellman equation. This term arises here due to our special choice of the problem terminal condition (4.1b) – the utility function U that besides an usual evaluation of the expected terminal portfolio wealth takes into account also the terminal volatility of its return. The multiplier λ scales the level of significance portfolio return volatility. Hence the utility function U comes out as a linear combination of two auxiliary CRRA–like functions

$$U(y) = -y^{1-d} + \frac{\lambda}{2} y^{2(1-d)}, \qquad d \gg 1$$

and enter the utility criterion function $\mathscr{K}$ that comes out as the key objective of our study, that lies behind the Hamilton–Jacobi–Bellman equation (4.1a)–(4.1b),

$$\max_{\theta \in \Delta_t^T} \{ \mathscr{K}[y_T(\theta)) \,|\, y_t = y] \}, \qquad \mathscr{K}(y) = \mathbb{E}[U(y)] - \frac{\lambda}{2} \mathbb{D}[y].$$

Evidently, the criterion $\mathscr{K}$ considers both aspects of investment – expected utility $\mathbb{E}$ and the volatility $\mathbb{D}$ of returns. The utility function choice is explained in detail in Section 4.2 and its form and basic properties summarized in Proposition 3.

Furthermore, we suppose that the subsequent additional key requirement are met:

1. The control function governing the underlying stochastic process $\{y_t^\theta\}_{t=0}^T$ is constituted by the mapping $\theta = \theta(t,y) : [0,T) \times \mathbb{R}^+ \mapsto \mathbb{R}^N$. We introduce the set of admissible strategies
$$\Delta \equiv \Delta_t^T = \{\theta : [t,T] \times \mathbb{R}^+ \mapsto \mathbb{R}^N \ : \ \theta^T \mathbf{1} = 1 \, , \ \theta \geq 0\} \, . \tag{4.2}$$
construed as the compact $N-$dimensional simplex and restrict the choice of the equation control parameter θ to Δ.

2. The finite time horizon Ito's process $\{y_t^\theta\}_{t=0}^\infty \equiv \{y_t\}_{t=0}^\infty$ is driven by the stochastic differential equation below:
$$dy_t = A_\varepsilon(\theta,t,y_t)dt + B(\theta,t,y_t)dW_t, \tag{4.3}$$
where $\{W_t, 0 \leq t \leq T\}$ is the standard Wiener process.

3. The function $A \equiv A_\varepsilon(\theta,t,y)$ is increasing and (not necessarily strictly) concave function in the control parameter $\theta \in \Delta$ for all $y > 0$ and $t \in [0,T]$.

4. The function $B^2 \equiv B^2(\theta,t,y)$ is increasing and strictly convex function in the control parameter $\theta \in \Delta$ for all $y > 0$ and $t \in [0,T]$.

5. The function $U = U(y)$ is strictly increasing concave bounded function for all $y > 0$.

The above stated Hamilton–Jacobi–Bellman equation results from a dynamic stochastic optimization problem which objective is to maximize at any time t the portfolio terminal utility evaluated in terms of the criterion functional:

$$\max_{\theta \in \Delta_t^T|_{[0,T)}} \{\mathscr{K}[y_T^\theta \,|\, y_t^\theta = \bar{y}]\} \, . \tag{4.4}$$

where $\{y_t^\theta\}_{t=0}^\infty$ in the finite time horizon Ito's process, $\bar{y}$ a given initial state of $\{y_t^\theta\}$ evaluated at time t and $\mathscr{K}$ denotes a given criterion functional.

4.1 Problem Background

We presuppose that at any time $t \in [0,T]$ the arbitrage–free market consists of $N+1$ continuously traded assets with multivariate normally distributed returns

$$R_t^{(i)} \sim \mathscr{N}(\mu^{(i)}, (\sigma^{(i)})^2) \, , \qquad \text{for all} \quad i = 0, \ldots, N \, , \, t \in [0,T] \, . \tag{4.5}$$

The investment strategy at time t is given by the vector $\theta_t \in \mathbb{R}^{N+1}$ satisfying $\theta_t^T \cdot \mathbf{1} = 1$ where each $\theta_t^{(i)}$ symbolize the portion of invested wealth allocated in the ith traded asset. Furthermore we ban short positions and borrowings, so we require $\theta_t^{(i)} \geq 0$. Thus at time t the market portfolio return r_t is normally distributed, i.e.

$$\begin{aligned} r_t = \theta_t^T \cdot R_t \sim \mathcal{N}(\mu(\theta_t), \sigma^2(\theta_t))\,, \\ \text{where} \quad \mu(\theta_t) = \theta_t^T \mu \ \text{ and } \ \sigma^2(\theta_t) = \theta_t^T \Sigma \theta_t\,, \end{aligned} \tag{4.6}$$

and for any $t \in [0, T]$, Σ represents positive definite assets returns covariance matrix.

The investor, currently a working person and the second pillar participant, regularly, on short time intervals $[0,\tau], [\tau, 2\tau], \ldots, [T-\tau, T]$, where $0 < \tau \ll 1$ is a small time increment and T is known terminal time, deposits a small portion of his/her salary with a deterministic growth rate β, of size $\varepsilon\tau$-part to the pension fund investing on the market. The quantity of investor's yearly contribution rate ε plays a crucial rôle in our study and will be subject of our investigation in this text.

Denote $\Delta_t^{t+\tau}$ the set of all strategies allowed within time interval $[t, t+\tau)$. Furthermore, since the investment allocation policy θ_t decision is taken in the beginning of time interval $[t, t+\tau]$ and the assets returns $R_{t+\tau}^{(i)}$ are realized in its end, the considered portfolio return at time $t+\tau$ satisfies the subsequent relationship:

$$r_{t+\tau}(\theta_t) = \theta_t^T \cdot R_{t+\tau} \equiv \sum_{i=0}^{N} \theta_t^{(i)} R_{t+\tau}^{(i)}\,. \tag{4.7a}$$

Then, making use of (4.6), for any small time increment $0 < \tau \ll 1$ the *stochastic* change in portfolio return $dr_t(\theta_t)$ can be modelled utilizing a random variable $Z \sim \mathcal{N}(0,1)$, as:

$$dr_t(\theta_t) \equiv r_{t+\tau}(\theta_t) - r_t(\theta_t) = \mu(\theta_t)\tau + \sigma(\theta_t) Z \sqrt{\tau}, \quad Z \sim N(0,1)\,. \tag{4.7b}$$

Therefore, assuming that time-dependent investor's *wealth-to-salary ratio* y_t taken at time $t \leq T - \tau$ is known and utilizing the relationship for investor's transfers, for small enough time increment τ, $y_{t+\tau}$ is driven by the subsequent relationship (see Kwok [42])

$$y_{t+\tau} \equiv y_{t+\tau}(\theta_t) = F_t^{\tau}(\theta_t, y_t, Z)\,, \quad Z \sim N(0,1)\,, \tag{4.8}$$

$$F_t^{\tau}(\theta, y, z) = y \exp\{[\mu(\theta) - \beta - \tfrac{1}{2}\sigma^2(\theta)]\tau + \sigma(\theta) z \sqrt{\tau}\} + \varepsilon\tau. \tag{4.9}$$

Our aim is to determine the optimal strategy for any time t, i.e. the policy vector θ_t that maximizes the contributor's utility from the terminal *wealth-to-salary ratio* y_T allocated on their pension account. The contributor's utility from the investment process is represented by chosen utility criterion $\mathcal{K}$ that is to be specified in the following passages. Various alternatives of the established model have been analysed in Songzhe [66], Macová [43], Melicherčík and Ševčovič [47], Macová and Ševčovič [44], Ishimura and Mita [33], Kilianová et al. [37], Múčka [51] or Abe and Ishimura [1].

4.2 Portfolio Utility Criterion Choice

Our financial decisions have two tight-knit dimensions: a *value dimension*, typically expressed in terms of the investment return expectation $\mathbb{E}$; and a *risk dimension* measured by a suitable translation–invariant deviation risk functional $\mathbb{D}$, in our case variance functional. Markowitz in his portfolio theory (see Pflug and Romisch [59], Markowitz [45]) introduced the concept of the *efficient frontier* expressing the curve of all optimal solution to this problem, i.e. portfolios with maximal return and minimal risk where the relationship between portfolio return and its variance. Our modification lies in employing investor–specific utility function U to obtain the maximal expected utility portfolio having deviation not surpassing the considered investor's risk aversion attitude λ. Hence, we launch the wealth criterion by a functional $\mathscr{K}$ defined for random variable Y as follows:

$$\mathscr{K}(Y) \equiv \mathscr{K}_\lambda(Y) = \mathbb{E}[U(Y)] - \frac{\lambda}{2}\mathbb{D}[Y]\,. \tag{4.10}$$

Evidently, the criterion functional consistency is retained since the functional above equals identity for any deterministic argument. Such type of decision criteria is widely discussed in Bergman [7] or Sharpe [62]. Obviously, an arbitrary risk measure can be used in order to construct the efficient frontier (see Pflug and Romisch [59] for more details).

There are two key aspects of the *utility criterion* presented above: the utility function U modelling the concrete investor personality, behaviour and preferences; and the parameter $0 \le \lambda \ll 1$ illustrating the risk dimension consideration in terms of investor–distinctive risk aversion coefficient.

We must emphasize that the utility function may vary across investors as it represents their attitude to risk - the issue of its proper choice is deeply argued in a large amount of economic literature, e.g. Pratt [60] or Bergman [7]. Since we are aimed on including the individual investor's risk aversion coefficient λ in the utility function representation, our decision about the appropriate utility function proposal lies in its subsequent formulation in terms of composition of two constant relative risk aversion (denoted as CRRA) utility functions:

$$U(y) \equiv U_0(y) + \lambda U_1(y) = -y^{1-d} + \tfrac{\lambda}{2} y^{2(1-d)}\,, \qquad d \gg 1\,.$$

It must be accentuated that this choice of utility criterion is fully revealed later in this thesis and it is strictly conditioned by approximative unconstrained solution to the Hamilton–Jacobi–Bellman equation consistency requirement.
We remark that our choices of utility sub-functions U_0 and U_1 are in accordance with a common assumptions on average investor's utility function reflecting their tendency to hold a constant proportion of their wealth in any class of risky assets as the wealth varies constant relative risk aversion (Pratt [60]).

Evidently, since the highly demanded utility function $U = U(y)$ monotonous increasing and strict concavity properties are *not* automatically guaranteed for all $y > 0$, its domain definition must be revised.

Proposition 3. *Let* $0 \leq \lambda$, $d > 1$ *and define*

$$\begin{cases} U(y) = U_0(y) + \lambda U_1(y)\,; \\ U_0(y) = -y^{1-d}\,,\ U_1(y) = \frac{1}{2}y^{2(1-d)}\,. \end{cases} \tag{4.11}$$

The function $U = U(y)$ *is a well–defined utility function on the domain*

$$\mathscr{D} = \{y > 0 \,|\, y^{d-1} > \frac{2d-1}{d}\lambda\}\,. \tag{4.12}$$

Proof. The domain of utility function concavity and increasingness is defined correctly. Indeed, differentiating $U = U(y)$ established by the foregoing prescription implies the utility function monotonous increasingness providing that $y^{d-1} > \lambda$. Then, taking the second derivative and imposing the concavity requirement induces that in order to have U concave we restrict the domain such that $y^{d-1} > \lambda(2d-1)/d$. But then inasmuch as we choose $d > 1$, concavity implies increasingness. □

Key Objective of this Study: Reflecting the above presented portfolio utility criterion $\mathscr{K}$, at any time $t \in [0, T)$ we are aimed on maximizing the terminal time investor's utility generated by the portfolio and represented by the terminal *wealth–to–salary* ratio y_T, i.e. for known y,

$$\max_{\theta \in \Delta_t^T} \{\mathscr{K}[y_T(\theta)) \,|\, y_t = y]\}\,. \tag{4.13}$$

Henceforth, at time $t \in [0, T]$ we choose such admissible allocation policy for given level of *wealth–to–salary* ratio y, that would induce maximal terminal investment portfolio wealth with respect to established utility criterion.

4.3 Hamilton–Jacobi–Bellman Equation

4.3.1 Portfolio Value Function

One of the fundamental aspects scrutinized in our paper is the investor's terminal wealth arising from the portfolio designed by applying admissible investment strategies. Concretely, since the investor is allowed to consume the wealth generated by this portfolio not earlier than at terminal time T, the terminal portfolio value plays the key role in our study. For the purpose of dynamic programming approach utilized in our study is assumed that the utility criterion functional $\mathscr{K}$ satisfies *Tower Law* (see the Appendix), i.e. for any σ–fields $\mathscr{G}_1, \mathscr{G}_2$, random variable Y and parameter $\lambda \in \mathbb{R}$,

$$\mathscr{K}[\mathscr{K}(Y \,|\, \mathscr{G}_2) \,|\, \mathscr{G}_1] = \mathscr{K}[Y \,|\, \mathscr{G}_1]\,, \qquad \text{for all} \quad \mathscr{G}_1 \subseteq \mathscr{G}_2\,.$$

We launch the value function $V(t, y)$ embodying the maximal terminal portfolio value utility evaluated in terms of the utility criterion, arranged by applying the optimal strategy made at time t given the ratio y at time t, as

$$V(t,y) = \max_{\theta \in \Delta_t^T} \{ \mathscr{K}[V(T, y_T(\theta)) \,|\, y_t = y] \} \,. \tag{4.14}$$

Inasmuch as the investor's utility function $U = U(y)$ is known, the consistence requirement insists on the following obvious terminal condition:

$$V(T,y) = U(y) \,, \qquad \text{for all } y \in \mathscr{D} \,. \tag{4.15}$$

As a consequence, combining (4.14)–(4.15) and utilizing the properties of the utility criterion (4.10) applied on the terminal value of V, hence $\mathscr{K}(V(T,y)) = V(T,y)$, concretely, one may deduce the evident redefinition of (4.14) in the subsequent form:

$$V(t,y) = \begin{cases} U(y) \,, & t = T \,; \\ \max\limits_{\theta \in \Delta_t^T} \{ \mathscr{K}[U(y_T(\theta)) \,|\, y_t = y] \} \,, & 0 \le t < T \,. \end{cases} \tag{4.16}$$

In our pension planning model, at any time t by making a decision $\theta \in \Delta_t^T$ on a particular admissible policy, we are aimed on maximizing the uncertain terminal year T wealth $V(T,y)$. The investor is dealing with this dilemma repeatedly, with small $0 < \tau \ll 1$ time period. Thus recalling relation (4.8) and assuming that y_t is known, the portfolio value function $V(t,y)$ is driven by the process defined in incremental τ steps

$$V(t,y) = \max_{\theta \in \Delta_t^{t+\tau}} \{ \mathscr{K}[V(t+\tau, y_{t+\tau}(\theta)) \mid y_t = y] \} \,, \quad 0 \le t < t+\tau \le T \,, 0 < \tau \ll 1 \,. \tag{4.17}$$

Then, applying the Bellman's optimality principle (see Bellman [5], Fletcher [26] or Bertsekas [8]) the optimal strategy for the problem of stochastic dynamic programming for $0 < \tau \ll 1$ can be formulated as follows:

$$V(t,y) = \begin{cases} U(y) \,, & t = T \,; \\ \max\limits_{\theta \in \Delta_t^{t+\tau}} \{ \mathscr{K}[V(t+\tau, y_{t+\tau}(\theta)) \mid y_t = y] \} \,, & t < t+\tau \le T \,. \end{cases} \tag{4.18}$$

For now we will scrutinize the investor's criterion presented in (4.10) written in terms of value functional V at time $t+\tau$ with stochastic *wealth–to–salary ratio* $y_{t+\tau}$ based on time t-wealth allocation policy $\theta \in \Delta_t^{t+\tau}$ undertaken at time t. Hence utilizing thr Tower law (see e.g. [56], or [42]) we achieve

$$\mathscr{K}[V(t+\tau, y_{t+\tau}(\theta))] \equiv \mathbb{E}[V(t+\tau, y_{t+\tau}(\theta))] - \frac{\lambda}{2} \mathbb{D}[V(t+\tau, y_{t+\tau}(\theta))] \,, \tag{4.19}$$

where both $\mathbb{E}$ and $\mathbb{D}$ are measured with respect to the portfolio return r_t realization observed at time t. Notice, that since at time t, $V(t,y_t)$ is known, using basic properties of random variable mean and variance, for the incremental variance in the value function $V(t+\tau, y_{t+\tau}(\theta)) - V(t,y_t)$ we obtain

$$\begin{aligned} &\mathscr{K}[V(t+\tau, y_{t+\tau}(\theta_t)) - V(t,y_t) \,|\, V(t,y_t)] \\ &= \mathbb{E}[V(t+\tau, y_{t+\tau}(\theta_t)) - V(t,y_t)] - \frac{\lambda}{2} \mathbb{D}[V(t+\tau, y_{t+\tau}(\theta_t)) - V(t,y_t)] \\ &= \mathscr{K}[V(t+\tau, y_{t+\tau}(\theta_t)) \,|\, V(t,y_t)] - V(t,y_t) \,. \end{aligned}$$

Thus, rearranging terms in (4.17) and utilizing the relation for the marginal alteration in the value function above, for any small $0 < \tau \ll 1$ we attain the subsequent result:

$$0 = \max_{\theta \in \Delta_t^{t+\tau}} \left\{ \frac{\mathscr{K}[V(t+\tau, y_{t+\tau}(\theta)) - V(t, y_t)]}{\tau} \right\}, \qquad 0 \le t < t+\tau \le T . \tag{4.20}$$

4.3.2 Derivation of the Hamilton–Jacobi–Bellman Equation

In this section we concentrate our effort first of all on discrete to continuous transformation of the discrete saturating fluctuation in the value function $V(t+\tau, y_{t+\tau}(\theta)) - V(t, y_t)$. Taking infinitesimally small $\tau \equiv dt \ll 1$ we are allowed to introduce the differential $dV = dV_t^\theta$ as a continuous version of the incremental alternation in the value function, as follows:

$$dV_t^\theta = V(t+dt, y_{t+dt}(\theta)) - V(t, y_t), \qquad 0 < dt \ll 1 . \tag{4.21}$$

Hence, for $dt \to 0^+$, we identify $\Delta_t \equiv \Delta_t^t$, and evidently,

$$\mathscr{K}[dV_t^\theta] = \mathbb{E}[dV_t^\theta] - \frac{\lambda}{2}\mathbb{D}[dV_t^\theta] .$$

Consequently, the equation (4.20) reformulated in terms of the differential dV_t^θ takes the forthcoming form

$$\max_{\theta \in \Delta_t} \left\{ \frac{\mathscr{K}[dV_t^\theta]}{dt} \right\} \equiv \max_{\theta \in \Delta_t} \left\{ \frac{\mathbb{E}[dV_t^\theta] - \frac{\lambda}{2}\mathbb{D}[dV_t^\theta]}{dt} \right\} = 0 . \tag{4.22}$$

In general, we suppose that there exist functions $A_\varepsilon(\theta, t, y)$ and $B(\theta, t, y)$ such that the random process $y_t, t \in [0, T]$, is driven by the following stochastic differential equation

$$dy_t = A_\varepsilon(\theta, t, y_t)dt + B(\theta, t, y_t)dW_t, \tag{4.23}$$

where $\{W_t, 0 \le t \le T\}$ is the Wiener process. Then, by using Itô's lemma (see see Kwok [42], Oksendal [56], Epps [21] or Chiang [13]) we obtain the expression for the differential $dV_t^\theta = V(t+dt, y_{t+dt}(\theta)) - V(t, y_t)$ in the form of a function of two independent variables t and y where $V = V(t, y_t)$:

$$\begin{aligned} V(t+dt, y_{t+dt}(\theta)) - V(t, y_t) \\ &= \Big[\frac{\partial V}{\partial t}(t, y_t) + A_\varepsilon(\theta, t, y_t)\frac{\partial V}{\partial y}(t, y_t) + \frac{1}{2}B^2(\theta, t, y_t)\frac{\partial^2 V}{\partial y^2}(t, y_t)\Big]dt \\ &+ B(\theta, t, y_t)\frac{\partial V}{\partial y}(t, y_t)dW_t . \end{aligned}$$

Since stochastic variables dW_t, $B(\theta, y_t)\frac{\partial V}{\partial y}(t, y_t)$ are independent, $\mathbb{E}(dW_t) = 0$, we obtain

$$\begin{aligned} \frac{\mathbb{E}[dV_t^\theta]}{dt} &= \frac{\partial V}{\partial t}(t, y_t) + A_\varepsilon(\theta, t, y_t)\frac{\partial V}{\partial y}(t, y_t) + \frac{1}{2}B^2(\theta, t, y_t)\frac{\partial^2 V}{\partial y^2}(t, y_t), \\ \frac{\mathbb{D}[dV_t^\theta]}{dt} &= B^2(\theta, t, y_t)\left[\frac{\partial V}{\partial y}(t, y_t)\right]^2 . \end{aligned}$$

Hence, letting $dt \to 0^+$ and combining the results above, one can summarize the results derived for $V = V(t, y)$ to the forthcoming statement:

Proposition 4. *The function $V = V(t,y)$ satisfies the following Hamilton-Jacobi-Bellman equation:*

$$0 = \frac{\partial V}{\partial t} + \max_{\theta \in \Delta_t} \left\{ A_\varepsilon(\theta,t,y)\frac{\partial V}{\partial y} + \frac{1}{2}B^2(\theta,t,y)\left[\frac{\partial^2 V}{\partial y^2} - \lambda\left[\frac{\partial V}{\partial y}\right]^2\right]\right\}, \tag{4.24}$$

and the terminal condition $V(T,y) = U(y)$ for $y > 0$ where

$$A_\varepsilon(\theta,t,y) = \varepsilon + \left[\mu(\theta) - \beta\right]y, \qquad \textit{and} \qquad B(\theta,t,y) = \sigma(\theta)y.$$

Notice that the concrete form of the functions $A_\varepsilon(\theta,t,y)$ and $B(\theta,t,y)$ driven by the stochastic process for y_t, as stated in the Proposition 4 can be easily derived by applying Itô's lemma on the expression for the differential $dy_t = y_{t+dt} - y_t$ for $0 < \tau \equiv dt \ll 1$, to obtain:

$$dy_t = \varepsilon dt + y_t\left[(\mu(\theta) - \beta)dt + \sigma(\theta)dW_t\right]. \tag{4.25}$$

This way we have shown that the functions $A(\theta,t,y)$ and $B(\theta,t,y)$ take the form given by the Proposition 4.

Assumption 1 (Admissible Strategies)**.** *We assume that for any $t \in [0,T)$ the set of all admissible strategies is given as*

$$\Delta \equiv \Delta_t^T = \{\theta = (\theta_1,\dots,\theta_N)^T \in \mathbb{R}^N \,:\, \theta^T\mathbf{1} = 1\,,\ \theta_i \geq 1\,,\ \forall i = 1,\dots,N\}\,.$$

The set of all admissible strategies reflect two key facts - firstly, all resources must be used. Secondly, the natural government limitations posed on pension fund allocation policy – no short selling is allowed – is highly desired and so each component of the optimal investment policy obtained is non-negative. No other restrictions on pension fund composition take place.

Chapter 5

Convex Optimization Problem

Within this chapter we will widely use ideas borrowed from Kilianová and Ševčovič [38], Múčka [51], Macová and Ševčovič [44], and Macová [43].

First of all by employing the subsequent change of variables,

$$s = T - t\,, \qquad \text{and} \qquad x = \ln y\,, \tag{5.1}$$

we transform the original Hamilton–Jacobi–Bellman Equation (4.24) designed for the value function $V = V(t,y)$ into its equivalent for the unknown $\mathscr{V} = \mathscr{V}(s,x)$ as follows:

$$\frac{\partial \mathscr{V}}{\partial s} = \max_{\theta \in \Delta_t^T} \left\{ \left[\varepsilon e^{-x} - \beta + \mu(\theta) - \frac{1}{2}\sigma^2(\theta) \right] \frac{\partial \mathscr{V}}{\partial x} + \frac{1}{2}\sigma^2(\theta) \left[\frac{\partial^2 \mathscr{V}}{\partial x^2} - \lambda \left[\frac{\partial \mathscr{V}}{\partial x} \right]^2 \right] \right\}, \tag{5.2}$$

for $s \in (0,T]$, $\Delta_t^T \equiv \Delta$ prescribed by Assumption 1 and $x \in \mathbb{R}$ with the initial condition $\mathscr{V}(0,x) = U(e^x)$.

5.1 Quasi–linear problem

Recalling to Abe and Ishimura [1], Ishimura and Nakamura [34], Ishimura and Ševčovič [35], Macová and Ševčovič [44] and Múčka [51] we introduce the Riccati transformation

$$\varphi(s,x) = -\frac{\dfrac{\partial^2 \mathscr{V}(s,x)}{\partial x^2}}{\dfrac{\partial \mathscr{V}(s,x)}{\partial x}} \tag{5.3}$$

where φ refers to the intermediate value function $\mathscr{V}$ coefficient of absolute risk aversion. Next, we launch $\zeta = \zeta(\varphi)$ as below

$$\zeta(\varphi(s,x)) = 1 + \varphi(s,x) + \lambda\,\omega(\varphi(s,x))\,, \qquad \omega(\varphi(s,x)) = \frac{\mathscr{V}(s,x)}{\partial x}\,, \tag{5.4}$$

for all $x \in \mathbb{R}$ and $s \in [0,T]$. Now suppose for a while that $\zeta(s,x) > 0$ for all $s \in [0,T]$ and $x \in \mathbb{R}$. Providing that $\mathscr{V}' > 0$ one may define

$$\omega(\varphi(s,x)) = -\kappa e^{-\int_{x_0}^{x} \varphi(s,z)dz}, \qquad \text{and} \qquad \widetilde{\omega}(s,x) \equiv \omega(\varphi(s,x))\,, \tag{5.5}$$

for some $x_0 \in \mathbb{R}$ and $\kappa \equiv \mathscr{V}'(s,x_0)$ is finite.
Hence suppose for a while that $\varphi > 0$. Thus we can rewrite (5.2) subsequently

$$\frac{\partial \mathscr{V}}{\partial s} = \mathscr{G}(s,x)\frac{\partial \mathscr{V}}{\partial x}, \qquad \text{for} \qquad \mathscr{G}(s,x) \equiv \varepsilon e^{-x} - \beta - \phi(\zeta(\varphi(s,x))), \tag{5.6a}$$

with $\phi = \phi(\zeta(\varphi))$ the value function of the parametric optimization problem

$$\phi(\zeta) = \min_{\theta\in\Delta}\left\{-\mu(\theta) + \frac{1}{2}\sigma^2(\theta)\zeta\right\}. \tag{5.6b}$$

Observe that (5.6b) is a parametric convex optimization problem since by (4.6), $\mu(\cdot)$ is linear and $\sigma^2(\cdot)$ is strictly convex (see Bank et al. [3]). Then taking the time derivative of (5.3) gives us

$$\partial_s \varphi = -\left(\partial_x \mathscr{V}\right)^{-1}\left[\partial_{sxx}\mathscr{V} + \varphi\, \partial_{sx}\mathscr{V}\right].$$

On the other side, differentiating (5.6a) w.r.t x and reusing the relationship (5.3) leads to

$$\partial_{sx}\mathscr{V} = \left[\partial_x\mathscr{G} - \varphi\mathscr{G}\right]\partial_x\mathscr{V}, \qquad \text{and} \qquad \partial_{sxx}\mathscr{V} = \left[\partial_{xx}\mathscr{G} - \partial_x\left(\varphi\mathscr{G}\right) - \varphi\left(\partial_x\mathscr{G} - \varphi\mathscr{G}\right)\right]\partial_x\mathscr{V}.$$

So, when combined together we obtain the subsequent

$$\begin{aligned}\partial_s \varphi &= -\partial_x\left[\partial_x\mathscr{G} - \varphi\mathscr{G}\right]\\ &= \partial_x\left[\partial_x\phi(\zeta(\varphi(s,x))) + (1+\varphi(s,x))\left(\varepsilon e^{-x} - \beta\right) - \varphi(s,x)\zeta(\phi(\varphi(s,x)))\right].\end{aligned}$$

Therefore, φ satisfies

$$\partial_s\varphi - \partial_x\left[\phi'\partial_x\varphi\right] - \partial_x\left[(1+\varphi(s,x))\left(\varepsilon e^{-x} - \beta\right) - \varphi(s,x)\phi(\zeta(\varphi(s,x)))\right] = 0,$$

with the initial condition $\varphi(0,x) = -e^xU''(e^x)/U'(e^x)$ and providing that ϕ is strictly increasing in φ, the foregoing equation in a quasi–linear parabolic Cauchy–type PDE. So recalling Kilianová and Ševčovič [38], we proved the following statement.

Theorem 1. *Let $\varphi = -\partial_{xx}\mathscr{V}/\partial_x\mathscr{V}$, and for all $x \in \mathbb{R}$, $s \in [0,T]$ define the function $\zeta = \zeta(s,x)$ in terms of* (5.4)–(5.5). *Assume that the intermediate value function $\mathscr{V} = \mathscr{V}(s,x)$ satisfies*

$$\frac{\partial \mathscr{V}}{\partial s} = \mathscr{G}(s,x)\frac{\partial \mathscr{V}}{\partial x}, \qquad \textit{for} \qquad \mathscr{G}(s,x) \equiv \varepsilon e^{-x} - \beta - \phi(\zeta(\varphi(s,x))), \quad s \in \mathbb{R},\, t \in (0,T],$$

and $\mathscr{V}(s,x) = U(e^x)$. Then φ is a solution to the Cauchy–type quasi–linear parabolic equation

$$\begin{cases} \dfrac{\partial \varphi}{\partial s} = \dfrac{\partial^2\phi(\zeta(\varphi))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\varphi)\left(\varepsilon e^{-x} - \beta\right) - \varphi\,\phi(\zeta(\varphi))\right], & x \in \mathbb{R}, s \in (0,T],\\ \varphi(0,x) = -\dfrac{U''(e^x)}{U'(e^x)}e^x, & x \in \mathbb{R}. \end{cases} \tag{5.7}$$

Furthermore, the fully non–linear parabolic PDE (5.2) is equivalent to its quasi–linear counterpart (5.6a)–(5.6b) so that given the underlying model parameters one can prefer to find the the solution to the parametric optimization problem $\phi(\varphi)$ and use it (5.6a) to look for the solution to the fully non–linear parabolic PDE (5.2). This approach is particularly useful since we are interested in the investor's optimal strategy θ whereas the portfolio intermediate function is the solution *by–product* only.

Theorem 2. *Let $\varphi(s,x)$ be a solution to the quasi–linear initial value problem* (5.7)*. Then the function $\mathscr{V} = \mathscr{V}(s,x)$ satisfying*

$$\begin{cases} 0 = \partial_s \mathscr{V} - \mathscr{G}(s,x)\partial_x \mathscr{V}\,, & x \in \mathbb{R}\,,\ s \in (0,T]\,, \\ \mathscr{V}(0,x) = U(e^x)\,, & x \in \mathbb{R}\,. \end{cases} \tag{5.8}$$

solves the fully non–linear PDE (5.2) *and $\varphi(s,x) = -\partial_{xx}\mathscr{V}(s,x)/\partial_x \mathscr{V}(s,x)$.*

Proof. Truly, assume that φ satisfies (5.7) with the initial condition $\varphi(0,x) = -U''(e^x)/U'(e^x)$ and take $\mathscr{V}$ as the unique solution to (5.8) having $\mathscr{V}(0,s) = U(e^x)$. Next, if we define $\widetilde{\varphi} \equiv -\partial_{xx}\mathscr{V}/\partial_x\mathscr{V}$, then $\widetilde{\varphi}$ satisfies (5.7).
Hence for $\delta^{\varphi}(s,x) \equiv \varphi(s,x) - \widetilde{\varphi}(s,x)$, the difference between φ and $\widetilde{\varphi}$ it holds that $\partial_s \delta^{\varphi} = \partial_x \phi(\delta^{\varphi})$ with $\delta^{\varphi}(0,s) = 0$. Therefore, $\varphi = \widetilde{\varphi}$ for all $s \in [0,T]$ and $s \in \mathbb{R}$ induces that forthcoming fully non–linear parabolic Cauchy–type PDE, with monotonous principal has a unique solution, $\mathscr{V}$:

$$\partial_s \mathscr{V} - \left\{\varepsilon e^{-x} - \beta - \phi(\zeta(-\partial_{xx}\mathscr{V}/\partial_x\mathscr{V}))\right\}\partial_x \mathscr{V} = 0\,, \qquad \text{and} \qquad \mathscr{V}(s,x) = U(e^x)\,. \tag{5.9}$$

Simply, the intermediate value function $\mathscr{V}$ satisfies (5.6a). $\square$

5.2 Optimization problem

Recalling Section 4.1 the vector of investment strategies $\theta \in \mathbb{R}^N$ belongs to the set Δ of all admissible strategies characterized by the prohibition of borrowings / short positions, hence Δ is given as the N–dimensional simplex. Next, as stated in (4.6) portfolio consisting of N assets has return $\mu(\theta) = \mu^T\theta$ and variance $\sigma^2(\theta) = \theta^T\Sigma\theta$ where Σ is assumed to be a symmetric positive definite covariance matrix.
Hence providing that $\zeta > 0$ we transform (5.6b) into a parametric quadratic convex programming problem:

$$\phi(\zeta) = \min_{\theta\in\Delta}\left\{-\mu^T\theta + \frac{1}{2}\zeta\theta^T\Sigma\theta\right\}. \tag{5.10}$$

For now we follow the footsteps of Kilianová and Ševčovič [38]. The key properties of strictly convex function minimized over Δ, compact and convex set, imply continuity of the mapping

$$\zeta \mapsto \widehat{\theta}(\zeta) \in \Delta\,, \qquad \forall\zeta \in (0,\infty)\,.$$

Next we launch the subsequent notation for the objective function in the optimization problem (5.10):

$$\nu(\theta,\zeta) \equiv -\mu^T\theta + \frac{1}{2}\zeta\theta^T\Sigma\theta\,.$$

Then, owing to $|\partial_\zeta \nu(\theta,\zeta)|$ continuity on the compact set Δ, ν is bounded on Δ and from strict convexity of ν in variable θ we deduce that there must exist a unique minimizer (function of ζ) of (5.10), denoted as $\widehat{\theta} = \widehat{\theta}(\zeta)$. The continuity of $\widehat{\theta}$ in ζ holds also for $\partial_\zeta \nu(\widehat{\theta}(\zeta),\zeta)$ inasmuch as

$$\partial_\zeta \nu(\widehat{\theta}(\zeta),\zeta) = \frac{1}{2}\widehat{\theta}(\zeta)\Sigma\widehat{\theta}^T(\zeta)\,.$$

Hence, as all requirements of the general envelop theorem are satisfied, the function $\phi(\zeta)$ is differentiable for in $\zeta \in (0,\infty)$.
Evidently, from the prescription of v we see that $v = v(\theta,\zeta)$ is linear in ζ for any choice of $\theta \in \Delta$ and so it is absolutely continuous in ζ for any $\theta \in \Delta$. Therefore applying the general envelop theorem we achieve the subsequent:

$$\phi(\zeta) = \phi(0) + \int_0^\zeta \partial_\zeta v(\widehat{\theta}(\varpi),\varpi)\, d\varpi\,.$$

Hence, as a result, for positive definite Σ

$$\phi(\zeta) = \partial_\zeta v(\widehat{\theta}(\varpi),\varpi) = \frac{1}{2}\widehat{\theta}(\zeta)\Sigma\widehat{\theta}^T(\zeta)$$

is strictly positive on Δ implying the C^1 continuity and increasingness of the mapping $\zeta \mapsto \phi(\zeta)$ for any $\zeta > 0$. Hence applying results from Klate [41], $\phi(\zeta)$ is locally Lipschitz continuous.

Thus, referring to Kilianová and Ševčovič [38] we summarize our observations:

Proposition 5. *Assume that $\mu \in \mathbb{R}^n$ and Σ is positive definite matrix. Then the optimal value function*

$$\phi(\zeta) = \min_{\theta\in\Delta}\left\{-\mu^T\theta + \frac{1}{2}\zeta\theta^T\Sigma\theta\right\}. \tag{5.11}$$

is $C^{1,1}$ continuous. Furthermore, the mapping $\zeta \mapsto \phi(\zeta)$ is strictly increasing and it holds that

$$\phi'(\zeta) = \frac{1}{2}\hat{\theta}^T\Sigma\hat{\theta}\,, \tag{5.12}$$

for $\hat{\theta} = \hat{\theta}(\zeta) \in \Delta$ the unique minimizer of (5.11) under the assumption of $\zeta > 0$ and the mapping $\zeta \mapsto \hat{\theta}$ is locally Lipschitz continuous.

Furthermore, evidently $\phi'(\zeta)$ attains its minimum and maximum on Δ inasmuch as Δ is a compact N–dimensional simplex and (5.10) is quadratic with positive definite Σ.

Remark 2. *Let us remind the reader about the mapping $\zeta = \zeta(\varphi)$ defined for $\varphi = \varphi(s,x)$ by (5.4)–(5.5) correctly for any small parameter $0 \le \lambda \ll 1$.*
Denote φ^0 the unique root of the problem $\zeta(\varphi) = 0$ (uniqueness of such φ^0 is guaranteed as by the implicit function theorem the derivative $\zeta'(\varphi)$ is non–zero for any φ increasing in x and satisfying $\zeta(\varphi) = 0$). Then for arbitrarily chosen φ bounded from below by φ^0, the mapping $\zeta(\varphi)$ is Lipschitz continuous as $0 < e^{-\int \varphi\, dx} < e^{-\int \varphi^0\, dx}$ and evidently $\zeta(\varphi)$ is strictly increasing in φ as $\widetilde{\omega}(\varphi)$ is an increasing function of $\varphi > \varphi^0$.
Furthermore, under the assumption of $\partial_x\varphi$ positive, it is smooth and the first derivative (taken with respect to φ) is given as

$$\zeta'(\varphi) = 1 + \lambda\frac{\varphi}{\partial_x\varphi}\widetilde{\omega}(\varphi(s,x))\,.$$

Hence the mapping $\varphi \mapsto \zeta(\varphi) \mapsto \phi(\zeta)$ is strictly increasing and locally Lipschitz continuous $C^{1,1}$ function of φ.

5.3 Optimal Allocation Policy

Denote $\mathscr{I}_\emptyset$ the set of all $\zeta > 0$ for which the (unique) minimizer $\hat{\theta}(\zeta) \in \Delta$ of (5.10) has positive components only and for any subset S of $\{1,\dots,N\}$ the set $\mathscr{I}_S$ of all functions $\zeta > 0$ for which the index set of $\hat{\theta}(\zeta) \in \Delta$ zero components coincide with S:

$$\mathscr{I}_\emptyset = \left\{ \zeta > 0 \,|\, \hat{\theta}_i(\zeta) > 0\,,\, \forall i = 1,\dots,N \right\}, \quad \mathscr{I}_S = \left\{ \zeta > 0 \,|\, \hat{\theta}_i(\zeta) = 0 \iff i \in S \right\}.$$

Then $\mathscr{I}_\emptyset$ is an open set since the mapping $\zeta \mapsto \hat{\theta}(\zeta)$ is continuous, $\mathscr{I}_S$ is a closed set for any non-empty $S \subset \{1,\dots,N\}$ and

$$(0,\infty) = \mathscr{I}_\emptyset \cup \bigcup_{1 \le |S| \le N-1} \mathscr{I}_S\,.$$

Hence in order to determine the optimal investment strategy in our analysis we distinguish between two cases.

Case 1: $\zeta \in \mathscr{I}_\emptyset$. We employ the technique of the Lagrange function $\mathscr{L} = \mathscr{L}(\theta,k)$ on the non–linear constrained optimization problem (5.10). Let

$$\mathscr{L}(\theta,k) \equiv -\mu^T\theta + \frac{1}{2}\zeta\theta^T\Sigma\theta - k(\mathbf{1}^T\theta - 1)\,. \tag{5.13}$$

Then the optimal solution $\hat{\theta} = \hat{\theta}(\zeta)$ and the associated value function $\phi = \phi(\zeta)$ satisfy the subsequent:

$$\hat{\theta}(\zeta) = \frac{1}{a}\Sigma^{-1}\left\{\mathbf{1} + (a\mu - b\mathbf{1})\frac{1}{\zeta}\right\}, \quad \phi(\zeta) = \frac{\zeta}{2a} - \frac{b}{a} - \frac{ac-b^2}{2a}\zeta^{-1}\,, \tag{5.14}$$

where

$$a = \mathbf{1}^T\Sigma^{-1}\mathbf{1}, \qquad b = \mu^T\Sigma^{-1}\mathbf{1}, \qquad c = \mu^T\Sigma^{-1}\mu\,. \tag{5.15}$$

Observe that both a and c are positive as Σ is positive definite, and $\phi(\zeta)$ is C^∞ for any $\zeta > 0$. Moreover, the Cauchy–Schwarz inequality implies that $ac - b^2$ is non-negative – obviously, zero occurs in case of linear dependent vectors μ and $\mathbf{1}$.

Case 2: $\zeta \in \mathscr{I}_S\,,\, S \neq \emptyset$. Providing that $\zeta \in \mathscr{I}_S$ for some non–empty subset S then we may reduce the problem dimension to a lower $N - |S|$ dimensional simplex Δ_S, as the values of optimal strategy components with index belonging to S are *already known* as they all equal zero. Therefore one may nullify the corresponding rows and columns elements from the matrix Σ and vector μ to get projections Σ_S and μ_S.
Then, as $\phi(\zeta)$ is smooth on $int(\mathscr{I}_S)$ launching $a_S = \mathbf{1}^T\Sigma_S^{-1}\mathbf{1}$, $b_S = \mu_S^T\Sigma_S^{-1}\mathbf{1}$ and $c_S = \mu_S^T\Sigma_S^{-1}\mu_S$ we write the optimal investment strategy $\hat{\theta}(\zeta)$ and the corresponding value function $\phi(\zeta)$ as follows:

$$\hat{\theta}(\zeta) = \frac{1}{a_S}\Sigma_S^{-1}\left\{\mathbf{1} + (a_S\mu_S - b_S\mathbf{1})\frac{1}{\zeta}\right\}, \quad \phi(\zeta) = \frac{\zeta}{2a_S} - \frac{b_S}{a_S} - \frac{a_Sc_S - b_S^2}{2a_S}\zeta^{-1}\,. \tag{5.16}$$

Hence we derived the subsequent statement.

Theorem 3. *The function*

$$\phi(\zeta) = \min_{\theta\in\Delta}\left\{-\mu^T\theta + \frac{1}{2}\zeta\theta^T\Sigma\theta\right\}, \qquad \zeta > 0$$

is C^∞ *on the open set* $\mathscr{I}_\emptyset \cup \bigcup_{1\le|S|\le N-1} int\,(\mathscr{I}_S)$ *for any* $S\subset\{1,\ldots,N\}$ *and*

$$\phi(\zeta) = \begin{cases} \dfrac{\zeta}{2a} - \dfrac{b}{a} - \dfrac{ac-b^2}{2a}\zeta^{-1}, & \zeta\in\mathscr{I}_\emptyset, \\ \dfrac{\zeta}{2a_S} - \dfrac{b_S}{a_S} - \dfrac{a_Sc_S-b_S^2}{2a_S}\zeta^{-1}, & \zeta\in int(\mathscr{I}_S). \end{cases} \tag{5.17}$$

Explicit Solution for Two–Dimensional Problem: Firstly notice that for the case of $N=2$, the set of all admissible strategies Δ is represented by the line segment $[0,1]$. Henceforth we denote $\theta^{(s)}\in[0,1]$ so that any $\theta\in\Delta$ is given as $\theta\equiv(\theta^{(s)},1-\theta^{(s)})$ for some $\theta^s\in[0,1]$.

Recalling the investment portfolio parameters introduced in (4.6) and the parameter of our interest, *stock-to-bond* ratio θ, we rename the lower indices utilized in market parameters, thus we identify the first index $\theta^{(s)}$ with more risky stocks and the second index $\theta^{(b)} = 1-\theta^{(s)}$ with the safe bonds to get

$$\mu(\theta) = \theta^T\mu = \mu^{(b)}\theta^{(b)} + \mu^{(s)}\theta^{(s)} = \mu^{(b)} + \theta^{(s)}(\mu^{(s)}-\mu^{(b)}), \tag{5.18a}$$

$$\begin{aligned}\sigma^2(\theta) &= \theta^T\Sigma\theta = [\theta^{(b)}\sigma^{(b)}]^2 + [\theta^{(s)}\sigma^{(s)}]^2 + 2\rho\theta^{(b)}\theta^{(s)}\sigma^{(b)}\sigma^{(s)}, \\ &= \alpha_\sigma[\theta^{(s)}]^2 - 2\beta_\sigma\theta^{(s)} + [\sigma^{(b)}]^2\end{aligned} \tag{5.18b}$$

where

$$\alpha_\sigma = \left[\sigma^{(s)}\right]^2 - 2\rho\sigma^{s()}\sigma^{(b)} + \left[\sigma^{(b)}\right]^2, \qquad \beta_\sigma = \sigma^{(b)}\left[\sigma^{(b)} - \rho\sigma^{(s)}\right].$$

In the relationships above, $\mu = (\mu^{(s)},\mu^{(b)})^T$ is the vector of financial assets returns with their respective volatilities, $\sigma^{(s)}$, $\sigma^{(b)}$ and ρ stands for the correlation coefficient measured at time t between stock's and bond's return.

Let us remind you the notation for parameters a, b and c (see (5.15)). Henceforth for the case of $N=2$ these can be evaluated as follows:

$$\begin{aligned} a &= \frac{[\sigma^{(b)}]^2 + [\sigma^{(s)}]^2 - 2\rho\sigma^{(b)}\sigma^{(s)}}{[\sigma^{(b)}\sigma^{(s)}]^2[1-\rho^2]}, \\ b &= \frac{\mu^{(s)}[\sigma^{(b)}]^2 + \mu^{(b)}[\sigma^{(s)}]^2 - \rho\sigma^{(b)}\sigma^{(s)}[\mu^{(s)}+\mu^{(b)}]}{[\sigma^{(b)}\sigma^{(s)}]^2[1-\rho^2]}, \\ c &= \frac{[\mu^{(s)}\sigma^{(b)}]^2 + [\mu^{(b)}\sigma^{(s)}]^2 - 2\rho\sigma^{(b)}\sigma^{(s)}\mu^{(s)}\mu^{(b)}}{[\sigma^{(b)}\sigma^{(s)}]^2[1-\rho^2]}. \end{aligned}$$

Moreover, the subsequent structural assumption on bond and stock average yields and their standard deviations, naturally expected and obviously fulfilled in stable financial markets (c.f. Kilianová et al. [37], Melicherčík and Ševčovič [47]), guarantee that the parameters a, b and c are correctly defined:

Assumption 2 (Stable Financial Market Assumptions)**.** *Assume that for all* $0\le s\le T$,

1. $-1 < \rho < 0$,
2. $\Delta\mu \equiv \mu^{(s)} - \mu^{(b)} > 0$,
3. $\sigma^{(s)} > \sigma^{(b)} > 0$.

Then evidently, referring to (5.18a)–(5.18b) for any $\zeta > 0$, we can transform the optimization problem (5.10) as follows:

$$\phi(\zeta) = \min_{\theta^{(s)}\in[0,1]} \left\{ -\left[\mu^{(b)} + \Delta\mu\,\theta^{(s)}\right] + \frac{\zeta}{2}\left[\alpha_\sigma[\theta^{(s)}]^2 - 2\beta_\sigma\theta^{(s)} + [\sigma^{(b)}]^2\right]\right\}. \tag{5.19}$$

Firstly relax the binding $\theta^{(s)} \in [0,1]$ hence assume that $\theta^{(s)} \in \mathbb{R}$ can be chosen arbitrarily. Hence the unconstrained maximizer can be determined straightforwardly as

$$\widehat{\theta}^{free} = \frac{\Delta\mu}{\alpha_\sigma\zeta} + \frac{\beta_\sigma}{\alpha_\sigma}, \tag{5.20}$$

where regardless stable financial market assumptions $\alpha_\sigma \geq 0$, and providing that they hold all α_σ, β_σ, and $\Delta\mu$, and so θ^{free} are positive. As a consequence the optimal solution for the constrained problem, i.e. the optimal investment share of stocks in the stock–bond portfolio, $\widehat{\theta} = \widehat{\theta}(\zeta)$ for $\theta^{(s)} \in [0,1]$ satisfies the subsequent prescription

$$\widehat{\theta}(\zeta) = \min\left\{1, \frac{\Delta\mu}{\alpha_\sigma\zeta} + \frac{\beta_\sigma}{\alpha_\sigma}\right\}, \tag{5.21}$$

and so for (5.10) providing that $\sigma^{(b)} - \rho\sigma^{(s)} > 0$ (or, equivalently $\alpha_\sigma > \beta_\sigma$ which is automatically satisfied for negatively correlated returns of stocks and bonds, $\rho < 0$) it holds that

$$\phi(\zeta) = \begin{cases} -\mu^{(b)} - \dfrac{\beta_\sigma\Delta\mu}{\alpha_\sigma} - \dfrac{(\Delta\mu)^2}{2\alpha_\sigma}\dfrac{1}{\zeta} + \dfrac{\zeta}{2}(1-\rho^2)[\sigma^{(b)}]^2[\sigma^{(b)}]^2, & \zeta > \dfrac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}, \\ \dfrac{[\sigma^{(s)}]^2}{2}\zeta - \mu^{(s)}, & \zeta \leq \dfrac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}. \end{cases} \tag{5.22}$$

On the other hand side, in case of $\alpha_\sigma > \beta_\sigma$ the unconstrained solution is never attained and so the optimal policy is always driven by $\widehat{\theta} = 1$. Therefore, employing the terminology of sets $\mathscr{I}_s$ and $\mathscr{I}_\emptyset$ we see that the space of all solutions is generated by two sets - the one corresponding ton the optimal unconstrained solution and the second one defined by the *no borrowings* constraint applied on stock investment. Hence, $(0,\infty) = \mathscr{I}_\emptyset \cup \mathscr{I}_{\{1\}}$ we can easily deduce that

$$\begin{array}{lllll} \mathscr{I}_\emptyset = \left(\dfrac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}, \infty\right), & \mathscr{I}_1 = \left(0, \dfrac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}\right), & \iff & \alpha_\sigma > \beta_\sigma, \\ \mathscr{I}_\emptyset = \emptyset, & \mathscr{I}_1 = (0,\infty), & \iff & \alpha_\sigma \leq \beta_\sigma. \end{array} \tag{5.23}$$

Finally, $\phi(\zeta)$ is $C^{1,1}$ for any ζ positive. Furthermore, providing that $\alpha_\sigma > \beta_\sigma$ and

$$\mathscr{I}_{C^\infty} = (0,\infty) - \left\{\frac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}\right\},$$

denotes the positive half–line except of the breakpoint $\Delta\mu/(\alpha_\sigma - \beta_\sigma)$, then the mapping $\zeta \mapsto \phi(\zeta)$ is surely C^∞ smooth on $\mathscr{I}_{C^\infty}$.

5.4 Classical Solution and its Properties

In this part we shall derive effective lower and upper bounds of a solution to the initial value quasi–linear problem established in Theorem 1. For further details concerning the sub- and super-solution construction methods the reader this text is highly recommended on relevant books on partial differential equations, for instance Evans.
The idea behind the construction of suitable sub- and super-solution is rather simple – it consists in the solution ordering properties exploitation while taking into account the form of the initial value condition.

Hence recalling the problem

$$\begin{cases} \dfrac{\partial \varphi}{\partial s} = \dfrac{\partial^2 \phi(\zeta(\varphi))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\varphi)(\varepsilon e^{-x} - \beta) - \varphi\,\phi(\zeta(\varphi))\right], & x \in \mathbb{R}, s \in (0,T], \\ \varphi(0,x) = -\dfrac{U''(e^x)}{U'(e^x)} e^x, & x \in \mathbb{R}. \end{cases},$$

and employing the parabolic operator $\mathscr{H}$ the system above can be reformulated in terms of a fully non–linear parabolic PDE as follows:

$$\frac{\partial \varphi}{\partial s} = \mathscr{H}(s,x,\varphi,\partial_x \varphi, \partial_x^2 \varphi), \tag{5.24a}$$

$$\mathscr{H} = \frac{\partial^2 \xi(\varphi)}{\partial x^2} + \frac{\partial}{\partial x}\left[(1+\varphi)\left(\varepsilon e^{-x} - \beta\right) - \varphi\,\xi(\varphi)\right] \tag{5.24b}$$

where $\xi(\varphi) \equiv \phi(\zeta(\varphi))$ for $\zeta(\varphi) = 1 + \varphi - \lambda e^{-\int^x \varphi}$. Denote $\xi'(\varphi) = \phi'_\zeta \zeta'(\varphi)$. Furthermore, $\mathscr{H}$ is a strictly parabolic operator and for all $\varphi = \varphi(s,x)$ increasing in x and such that $\zeta(\varphi) > 0$, it satisfies:

$$0 < \tau_- \leq \partial_x \mathscr{H}(s,x,\varphi,p,q) \equiv \xi'(\varphi) \leq \tau_+ < \infty,$$

due to boundedness of ξ' as can be seen from (5.12) for any $\theta \in \Delta$.

Next, remark that for any $0 < \lambda \ll 1$ function $\zeta^0(x,u) = 0$ defined implicitly as $\zeta^0(x,u) = 1 + u - \lambda e^{-xu}$ has invertible derivative taken with respect to variable u and the unique mapping $\underline{\varphi}^*(x)$ defined such that

$$\{(x, \underline{\varphi}^*(x))\,|\, x \in \mathbb{R}\} = \{(x,u) \in \mathbb{R} \times (0,1) | \zeta^0(x,u) = 0\},$$

is increasing in $x \in \mathbb{R}$ and bounded, as $-1 < u(x) < 0$ for all $x \in \mathbb{R}$.

Thus we define the sub– and super–solutions that coincide with the unique root $u(x)$ of the problem $\zeta^0(x,u(x)) = 0$ and constant upper bound, respectively, as follows:

$$\underline{\varphi}(s,x) \equiv \underline{\varphi}^*(x), \qquad \text{and} \qquad \overline{\varphi}(s,x) \equiv \overline{\varphi}^*_+, \qquad (s,x) \in (0,T) \times \mathbb{R}.$$

Then evidently, $\mathscr{H}(s,x,\underline{\varphi},\partial_x\underline{\varphi},\partial_x^2\underline{\varphi}) \geq 0$ and $\mathscr{H}(s,x,\overline{\varphi},\partial_x\overline{\varphi},\partial_x^2\overline{\varphi}) = -(1+\overline{\varphi}^*)(\varepsilon e^{-x}) < 0$. Therefore, as

$$\partial_s\underline{\varphi} - \mathscr{H}(s,x,\underline{\varphi},\partial_x\underline{\varphi},\partial_x^2\underline{\varphi}) \leq 0 \qquad \text{and} \qquad \partial_s\overline{\varphi} - \mathscr{H}(s,x,\overline{\varphi},\partial_x\overline{\varphi},\partial_x^2\overline{\varphi}) \geq 0,$$

$\underline{\varphi}$ and $\overline{\varphi}$ are considered for sub- and super-solutions to the strictly parabolic non–linear PDE given by (5.24a)–(5.24b) (see e.g. [24]) satisfying the initial value inequality

$$\underline{\varphi}(0,x) < \varphi(0,x) < \overline{\varphi}(0,x) , \qquad x \in \mathbb{R} .$$

Finally, applying the parabolic comparison principle valid for strongly parabolic equations,

$$-1 < \underline{\varphi}^*(x) < \varphi(s,x) < \overline{\varphi}^* , \qquad \text{for all} \quad s \in (0,T) , \, x \in \mathbb{R} . \tag{5.25}$$

Remark 3. *Obviously, one can easily derive the useful upper and lower boundaries applicable for $\zeta = \zeta(\varphi)$. Indeed, using the definition of ζ* (5.5) *we obtain*

$$0 < \zeta(\varphi) < \zeta(\overline{\varphi}^*) . \tag{5.26}$$

5.5 Travelling Wave Solution

Taking the inspiration from the useful upper and lower boundaries of the solution to the quasi–linear problem (5.7) introduced in Theorem 1, our objective is to construct a semi–explicit travelling wave type solution to (5.7). From a practical point of view, such a special solution, though obtained under some restrictive assumptions on model parameters, is particularly useful to estimate boundaries of the solution to the quasi–linear problem (5.7). Furthermore, this travelling wave solution provides us valuable information about the numerical accuracy and the convergence rate in case that a numerical scheme is employed to approximate the solution to (5.7).

The character of the object of our study insists on the boundedness nature of the solution to (5.7) and hence this essential solution property cannot be infracted by the associated solution asymptotic expansion. Therefore there is an unavoidable assertion placed on the solution to (5.7) subject to some initial condition specified later – any smooth enough function φ satisfying (5.7) under some suitably designed initial condition simply must be either strictly positive or strictly negative.

Hence we reformulate the problem (5.7) under the simplifying assumptions of $\varepsilon = \lambda = \beta = 0$ and take for granted the positive definiteness of the covariance matrix Σ as follows:

$$\begin{cases} \dfrac{\partial \varphi}{\partial s} = \dfrac{\partial^2 \phi(\zeta(\varphi))}{\partial x^2} - \dfrac{\partial}{\partial x}\left[\varphi\, \phi(\zeta(\varphi))\right] , & x \in \mathbb{R}, s \in (0,T] , \\ \varphi(0,x) = -\dfrac{U''(e^x)}{U'(e^x)} e^x , & x \in \mathbb{R} . \end{cases} , \tag{5.27}$$

and as λ is set to zero, $\zeta(\varphi) \equiv 1 + \varphi$ is positive for any $\varphi > -1$. Therefore, recalling Section 5.2, the function $\xi = \phi(\zeta(\varphi)) = \phi(1+\varphi)$ is locally $C^{1,1}$ smooth and strictly increasing function.

Now, borrowing the ideas introduced by Ishimura and Ševčovič [35] and reproducing the procedure from Kilianová and Ševčovič [38] we construct a travelling wave solution possessing the subsequent form

$$\varphi(s,x) \equiv w(x+cs) , \qquad \text{for all} \quad x \in \mathbb{R} , \, s \in [0,T] , \tag{5.28}$$

where $w = w(z)$ represent the wave profile and the constant c modulates the speed of wave and evidently, the initial condition of the wave profile coincides with the one associated with φ as $\varphi(0,x) = w(x)$ at time $t = 0$. Then, plugging (5.28) into (5.27) gives us the subsequent one dimensional problem:

$$c\frac{dw}{dz}(z) = \frac{d}{dz}\left\{\frac{d}{dz}\left[\xi(w(z))\right] - w(z)\xi(w(z))\right\}.$$

Hence rearranging terms in the equation above leads to the following:

$$\frac{d}{dz}\left\{\frac{d}{dz}\left[\xi(w(z))\right] - cw(z) - w(z)\xi(w(z))\right\} = 0,$$

and so there must be a constant $\kappa_0 \in \mathbb{R}$ such that

$$\frac{d}{dz}\left[\xi(w(z))\right] = \mathscr{Q}(w(z)), \qquad \text{and} \qquad \mathscr{Q}(w) = \kappa_0 + cw + w\xi(w), \qquad \forall z \in \mathbb{R}. \tag{5.29}$$

Therefore we introduce $u \equiv \xi(w)$ and so employing u in the relationship above implies the forthcoming ODE:

$$\frac{du}{dz}(z) = \mathscr{P}(u(z)), \qquad \mathscr{P}(u) = \mathscr{Q}(\xi^{-1}(u)) = \kappa_0 + c\xi^{-1}(u) + u\xi^{-1}(u) \qquad \forall u \in \mathbb{R}. \tag{5.30}$$

Let us define $-1 < w_- < w_+ < \infty$ for w_-, w_+ established as

$$w_- \equiv \lim_{u\to\infty} w(u) \qquad \text{and} \qquad w_+ \equiv \lim_{u\to-\infty} w(u).$$

Therefore these constants are the roots of $\mathscr{Q}$ (in the long run, $\xi(w(z))$ remain constant), i.e. it holds that $\mathscr{Q}(w_-) = \mathscr{Q}(w_+) = 0$. Therefore plugging successively w_+ and w_- into (5.29) induce a system of linear equations for unknown wave speed c and integration constant κ_0 that can be solved directly for $-1 < w_- < w_+ < \infty$:

$$c = \frac{w_+\xi(w_+) - w_-\xi(w_-)}{w_+ - w_-}, \qquad \text{and} \qquad \kappa_0 = -cw_+ - w_+\xi(w_+). \tag{5.31}$$

Likewise, we launch the associated values $u_- \equiv \xi(w_-)$ and $u_+ \equiv \xi(w_+)$ – then evidently, $\mathscr{P}(u_-) = \mathscr{P}(u_+) = 0$. Recalling $\zeta(w) = 1 + w$ and $\xi(w) \equiv \phi(\zeta(w))$, Theorem 3, for any arbitrary choice of $\zeta \in \mathscr{I} \subset (0,\infty)$ the mapping $\zeta = 1 + w \mapsto \phi(\zeta)$ such that

$$\phi(\zeta) = \frac{\zeta}{2a} - \frac{b}{a} - \frac{ac - b^2}{2a}\zeta^{-1}, \qquad \text{and} \qquad \zeta \equiv w + 1,$$

for some constants $a > 0$, $b \in \mathbb{R}$ and $c \in \mathbb{R}$ such that $ac > b^2$ is C^∞ smooth. Next, denote $\mathscr{R}(w) \equiv w\xi(w) = w\phi(\zeta(w))$ and so

$$c = \frac{\mathscr{R}(w_+) - \mathscr{R}(w_-)}{w_+ - w_-}, \qquad \text{and} \qquad \mathscr{Q}(w) = \kappa_0 + cw + \mathscr{R}(w).$$

Let us pay attention to the function $\mathscr{Q} = \mathscr{Q}(w)$ and scrutinize its behaviour. First of all observe that $\mathscr{R} = \mathscr{R}(w)$ is convex since $\mathscr{R}''(w) = a^{-1}\left[1 + (ac - b^2)(w+1)^{-3}\right]$ is positive for $w > -1$

and $a > 0$, $ac > b^2$ – and so the convexity property of $\mathscr{R}$ is transmitted on $\mathscr{Q}$. Furthermore, forasmuch as

$$\mathscr{Q}'(w) = c + \mathscr{R}'(w) = \frac{\mathscr{R}(w_+) - \mathscr{R}(w_-)}{w_+ - w_-} + \mathscr{R}'(w)\,.$$

Therefore using the definition of convex function, $\mathscr{Q}'(w_-) < 0$ whereas $\mathscr{Q}'(w_+) < 0$ and as

$$\mathscr{Q}(w) = -\frac{\mathscr{R}(w_+) - \mathscr{R}(w_-)}{w_+ - w_-}(w_+ - w) + [\mathscr{R}(w_+) - \mathscr{R}(w)]\,,$$

using the definition of convex function one may deduce that $\mathscr{Q}(w)$ is negative if and only if $w \in (w_-, w_+)$. As $\zeta = 1 + w$, $\phi = \phi(\zeta)$ increases with ζ (and so $\xi = \phi(\zeta(w))$ increases in w) and $\mathscr{P}(u) \equiv \mathscr{Q}(\xi^{-1}(u))$, it holds that $\mathscr{P}(u_-) < 0 < \mathscr{P}(u_+)$. Hence, returning back to the initial value ODE (5.30) for unknown $u = u(z)$, $\mathscr{P}(u_-)$ is stable and $\mathscr{P}(u_+)$ is unstable stationary solution to (5.30), i.e. any choice of starting point $u(0) \in (u_-, u_+)$ the function $u = u(z)$ satisfies the subsequent,

$$\lim_{z\to\infty} u(z) = u_-\,, \qquad \text{and} \qquad \lim_{z\to-\infty} u(z) = u_+\,, \qquad \forall u(0) \in (u_-, u_+)\,.$$

We summarize the result obtained using in below (for the similar formulation we recommend the reader to see Kilianová and Ševčovič [38], Múčka [51], Ishimura and Ševčovič [35], Macová and Ševčovič [44], or Macová [43]).

Theorem 4. *Assume that $w_-, w_+ \in \mathscr{I}$ are boundary values such that $-1 < w_- < w_+$. Then up to a shift constant there exists a unique travelling wave solution $\varphi(s,x) = w(x + cs)$ such that*

$$\lim_{x\to-\infty} \varphi(s,x) = w_+\,, \qquad \textit{and} \qquad \lim_{x\to+\infty} \varphi(s,x) = w_-\,.$$

The travelling wave speed is prescribed by

$$c = \frac{w_+\xi(w_+) - w_-\xi(w_-)}{w_+ - w_-}$$

for the travelling wave profile $w(z)$ which is decreases with z and is given by

$$w(z) \equiv \xi^{-1}(u(z))\,,$$

where $u = u(z)$ is a solution to

$$\frac{du}{dz}(z) = \mathscr{P}(u(z))\,, \qquad \mathscr{P}(u) = \kappa_0 + c\xi^{-1}(u) + u\xi^{-1}(u) \qquad \forall u \in \mathbb{R}\,,$$

where $\kappa_0 = -cw_+ - w_+\xi(w_+)$.

Remark 4. *The intention of the travelling wave formulation* (5.28) *allows more versatile usage of the solution to* (5.27), *hence it can be considered for the test function in the numerical approach to solution determination which may help us to estimate the numerical method order of convergence. Then there is a reasonable assumption that the same convergence order remains for any suitable choice of the initial condition, thus it holds even if the initial condition is prescribed in the form of the proper choice of the utility function even though the initial condition to* (5.27) *can postulated more generally in terms of a given function $g = g(x)$ such that $\varphi(0,x) = g(x)$ for all $x \in \mathbb{R}$.*

Chapter 6

Optimal Strategy Approximation

Owing to Theorem 3 both the form of the convex optimization problem (5.10) value function $\phi = \phi(\zeta)$ and the optimal allocation policy $\hat{\theta}$ are known.

Assume that $\zeta \in \mathscr{I}_{\emptyset}$, i.e. each component of the optimal policy vector $\hat{\theta} = \hat{\theta}(\zeta)$ is positive. Therefore using (5.4) defining $\zeta = \zeta(\varphi)$ as a function of φ in terms of (5.4)–(5.5) one may look for the solution to the quasi–linear initial value problem (5.7) taking the subsequent form:

$$
\begin{cases}
\dfrac{\partial \varphi}{\partial s} = \dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{ \dfrac{\partial \varphi}{\partial x}\left[1 + \frac{1}{\gamma^2 \zeta^2(\varphi)}\right]\zeta'(\varphi) \right. \\
\qquad\qquad \left. + 2a(1+\varphi)\left(\varepsilon e^{-x} - \beta\right) - \varphi\left[\zeta(\varphi) - 2b - \dfrac{1}{\gamma^2 \zeta(\varphi)}\right]\right\}, \\
\varphi(0,x) = -e^x \dfrac{U''(e^x)}{U'(e^x)},
\end{cases}
\tag{6.1}
$$

for any $x \in \mathbb{R}$, $s \in (0,T]$ and correctly defined

$$
\gamma = \frac{1}{\sqrt{ac - b^2}}. \tag{6.2}
$$

Then, the solution to the unconstrained problem (6.1) above is actually the super-solution to the quasi–linear initial value problem (5.7).

Truly, let $\widetilde{\Delta}$, $\overline{\Delta} \subset \mathbb{R}^N$ be two admissible sets for some $N \in \mathbb{N}$ such that $\widetilde{\Delta} \subset \overline{\Delta}$ at any time $s \in [0,T]$. Then, employing the statement of the Theorem 5 the optimal value function $\phi = \phi(\zeta)$ prescribed by (5.10) satisfies the subsequent inequality:

$$
\phi_{\overline{\Delta}}(\zeta) \equiv \min_{\theta \in \overline{\Delta}} \left\{ -\mu^T \theta + \frac{1}{2}\zeta \theta^T \Sigma \theta \right\} \le \min_{\theta \in \widetilde{\Delta}} \left\{ -\mu^T \theta + \frac{1}{2}\zeta \theta^T \Sigma \theta \right\} \equiv \phi_{\widetilde{\Delta}}(\zeta)
$$

Next, assume that $\widetilde{\Delta} \subset \overline{\Delta}$ correspond to the following pair of quasi–linear problems:

$$
\begin{cases}
\dfrac{\partial \overline{\varphi}}{\partial s} = \dfrac{\partial^2 \phi_{\overline{\Delta}}(\zeta(\overline{\varphi}))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\overline{\varphi})\left(\varepsilon e^{-x} - \beta\right) - \overline{\varphi}\phi_{\overline{\Delta}}(\zeta(\overline{\varphi}))\right], & x \in \mathbb{R}, s \in (0,T], \\
\overline{\varphi}(0,x) = -\dfrac{U''(e^x)}{U'(e^x)} e^x, & x \in \mathbb{R}.
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial \widetilde{\varphi}}{\partial s} = \dfrac{\partial^2 \phi_{\widetilde{\Delta}}(\zeta(\widetilde{\varphi}))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\widetilde{\varphi})\left(\varepsilon e^{-x} - \beta\right) - \widetilde{\varphi}\phi_{\widetilde{\Delta}}(\zeta(\widetilde{\varphi}))\right], & x \in \mathbb{R}, s \in (0,T], \\
\widetilde{\varphi}(0,x) = -\dfrac{U''(e^x)}{U'(e^x)} e^x, & x \in \mathbb{R}.
\end{cases}
$$

Then, as the initial values of $\widetilde{\varphi}$ and $\overline{\varphi}$ coincide and $\widetilde{\Delta} \subset \overline{\Delta}$, using the parabolic comparison principle (see Evans [24], Fletcher [26], or Smith [64]) for the quasi–linear initial value problem (5.7) we conclude the inequality

$$\overline{\varphi}(s,x) \geq \widetilde{\varphi}(s,x)\,,$$

for any $s \in [0,T]$ and $x \in \mathbb{R}$ as claimed. The above inequality enables us to refer to an unconstrained solution φ of (6.1) obtained for $\theta \in \mathbb{R}^N$ that $\theta^T \mathbf{1} = 1$ (hence, when the zero lower bound condition is relaxed) to as a super-optimal solution to the original quasi–linear problem (5.7).

Hence we have just proven the following statement:

Proposition 6. *Let $\widetilde{\Delta}$, $\overline{\Delta} \subset \mathbb{R}^N$ be two admissible sets for some $N \in \mathbb{N}$ such that $\widetilde{\Delta} \subset \overline{\Delta}$ at any time $s \in [0,T]$. Let $\overline{\varphi}(s,x)$, $\widetilde{\varphi}(s,x)$ be solutions to the quasi–linear problems with the corresponding optimal control*

$$\begin{cases} \dfrac{\partial \overline{\varphi}}{\partial s} = \dfrac{\partial^2 \phi_{\overline{\Delta}}(\zeta(\overline{\varphi}))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\overline{\varphi})\left(\varepsilon e^{-x} - \beta\right) - \overline{\varphi}\phi_{\overline{\Delta}}(\zeta(\overline{\varphi}))\right], & x \in \mathbb{R}, s \in (0,T]\,, \\ \qquad\qquad \phi_{\overline{\Delta}}(\zeta) \equiv \min\limits_{\theta \in \overline{\Delta}} \left\{ -\mu^T \theta + \dfrac{1}{2} \zeta \theta^T \Sigma \theta \right\}, & \\ \overline{\varphi}(0,x) = -\dfrac{U''(e^x)}{U'(e^x)} e^x\,, & x \in \mathbb{R}\,, \end{cases} \tag{6.3a}$$

$$\begin{cases} \dfrac{\partial \widetilde{\varphi}}{\partial s} = \dfrac{\partial^2 \phi_{\widetilde{\Delta}}(\zeta(\widetilde{\varphi}))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\widetilde{\varphi})\left(\varepsilon e^{-x} - \beta\right) - \widetilde{\varphi}\phi_{\widetilde{\Delta}}(\zeta(\widetilde{\varphi}))\right], & x \in \mathbb{R}, s \in (0,T]\,, \\ \qquad\qquad \phi_{\widetilde{\Delta}}(\zeta) \equiv \min\limits_{\theta \in \widetilde{\Delta}} \left\{ -\mu^T \theta + \dfrac{1}{2} \zeta \theta^T \Sigma \theta \right\}, & \\ \widetilde{\varphi}(0,x) = -\dfrac{U''(e^x)}{U'(e^x)} e^x\,, & x \in \mathbb{R}\,. \end{cases} \tag{6.3b}$$

Then the solution to the problem (6.3b) *is super–optimal for* (6.3a), *i.e.*

$$\begin{cases} \dfrac{\partial \overline{\varphi}}{\partial s} \geq \dfrac{\partial^2 \phi_{\widetilde{\Delta}}(\zeta(\overline{\varphi}))}{\partial x^2} + \dfrac{\partial}{\partial x}\left[(1+\overline{\varphi})\left(\varepsilon e^{-x} - \beta\right) - \overline{\varphi}\phi_{\widetilde{\Delta}}(\zeta(\overline{\varphi}))\right], & x \in \mathbb{R}, s \in (0,T]\,, \\ \qquad\qquad \phi_{\overline{\Delta}}(\zeta) \equiv \min\limits_{\theta \in \overline{\Delta}} \left\{ -\mu^T \theta + \dfrac{1}{2} \zeta \theta^T \Sigma \theta \right\}. & \\ \widetilde{\varphi}(0,x) = -\dfrac{U''(e^x)}{U'(e^x)} e^x\,, & x \in \mathbb{R}\,. \end{cases} \tag{6.3c}$$

Moreover,

$$\overline{\varphi}(s,x) \geq \widetilde{\varphi}(s,x)\,,$$

for any $s \in [0,T]$ and $x \in \mathbb{R}$.

Hence, in the following text we aim our attention to the unconstrained problem (6.1) and so looking for the super–solution to the original quasi–linear equation (5.7).

Next in order to find analytically a good approximation of the solution φ to the problem above, let us remind you that both parameter ε and λ representing the regular contribution rate and risk aversion sensitivity parameter, respectively, are small, i.e. $0 \leq \varepsilon, \lambda \ll 1$. This allows us to approach the exact solution φ by taking double power series terms up to a certain order.

Remark 5. *Though the region on which the utility function U concavity and increasingness properties hold, is for $\lambda > 0$ a proper subset of $\mathbb{R}$ as due to* (4.12) *we require that*

$$x > \frac{1}{d-1} \ln \left[\lambda \frac{2d-1}{d} \right] ,$$

we derive the solution to the unconstrained problem (6.1) *regardless this condition.*

Therefore firstly we write φ and U in terms of their asymptotic expansions (see e.g. Holmes [30], Bender and Orszag [6], Hinch [29] or O'Malley [57]) with respect to parameter λ as follows

$$\varphi(s,x) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(s,x) , \qquad \text{and} \qquad U(e^x) = \sum_{n=0}^{\infty} \lambda^n U_n(e^x) , \tag{6.4}$$

for any $x \in \mathbb{R}$ and $s \in [0,T]$. Plugging (6.4) into (5.17) leads to the subsequent infinite series of sub–problems that can be solved recursively:

Problem 1 (Quasi–Linear Problem λ–Asymptotic Expansion)**.** *For $\varphi_n = \varphi(s,x)$ and $U_n = U_n(e^x)$ given by* (6.4)*,*

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n \frac{\partial \varphi_n}{\partial s} = \frac{1}{2a} \frac{\partial}{\partial x} \Bigg\{ & \sum_{n=0}^{\infty} \lambda^n \frac{\partial \varphi_n}{\partial x} \left[1 + \frac{1}{\gamma^2 \zeta^2 \left(\sum_{n=0}^{\infty} \lambda^n \varphi_n \right)} \right] \zeta'_\varphi \left(\sum_{n=0}^{\infty} \lambda^n \varphi_n \right) \\ & + 2a \left(\varepsilon e^{-x} - \beta \right) \left[1 + \sum_{n=0}^{\infty} \lambda^n \varphi_n \right] - \sum_{n=0}^{\infty} \lambda^n \varphi_n \left[\zeta \left(\sum_{n=0}^{\infty} \lambda^n \varphi_n \right) - 2b - \frac{1}{\gamma^2 \zeta (\sum_{n=0}^{\infty} \lambda^n \varphi_n)} \right] \Bigg\} \end{aligned} \tag{6.5}$$

for any $s \in (0,T]$, $x \in \mathbb{R}$ and

$$\sum_{n=0}^{\infty} \lambda^n \varphi_n(0,x) = -e^x \frac{\sum_{n=0}^{\infty} \lambda^n U''_n(e^x)}{\sum_{n=0}^{\infty} \lambda^n U'_n(e^x)} , \qquad \textit{for} \quad x \in \mathbb{R} . \tag{6.6}$$

Terminal Condition Asymptotic Expansion Firstly, in order to make easier the forthcoming derivation of (6.5) expansion with respect to λ we infer the corresponding expansion of the Problem 1 terminal condition (6.6). Therefore, recalling the form of our utility function as introduced in (4.11) one can simplify (6.6) as follows:

$$\varphi(0,x) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(0,x) = \frac{d - \lambda(2d-1)e^{-(d-1)x}}{1 - \lambda e^{-(d-1)x}} , \qquad x \in \mathbb{R} . \tag{6.7}$$

Inasmuch as $\partial_\lambda \varphi(0,x) = -(d-1)e^{-(d-1)x}(1-\lambda e^{-(d-1)x})^{-2}$ one can easily obtain the asymptotic expansion of $\varphi(0,x)$ performed with respect to λ given below:

$$\varphi(0,x) = d + (d-1) \sum_{n=1}^{\infty} (-1)^n \lambda^n e^{-(d-1)nx} . \tag{6.8}$$

6.1 Equations for the Absolute Risk Aversion

Referring to the relative risk aversion $\varphi = \varphi(s,x)$ asymptotic expansion (6.4) with respect to the small parameter λ and the associated reformulation of the studied quasi–linear equation (6.5)–(6.6), for the purpose of this article we relax from its exact solution determined recurrently up to an arbitrary term $\varphi_n = \varphi_n(s,x)$. So we estimate the problem (6.5)–(6.6) the solution by restricting our attention only to its linear approximation that leads to coupled terminal value problems as derived below. Next for the sake of simplicity we introduce that subsequent linear transform of φ with the associate asymptotic expansion with respect to the small parameter λ:

$$\psi(s,x) = \gamma(1+\varphi(s,x)), \qquad \psi(s,x) = \sum_{n=0}^{\infty} \lambda^n \psi_n(s,x) \qquad s \in (0,T], \quad x \in \mathbb{R}, \tag{6.9}$$

Zero Term Problem In order to determine the zero term of the problem (6.5)–(6.6) parameter $\lambda-$expansion observe that in case of $\lambda = 0$, $\zeta(\varphi) = \psi/\gamma$. Hence, employing (6.9) we achieve the following

$$\begin{aligned}\partial_s \psi_0 &= \frac{1}{2a}\partial_x \left\{ \left[1+\frac{1}{\psi_0^2}\right]\partial_x\psi_0 + 2a\left(\varepsilon e^{-x}-\beta\right)\psi_0 + \left(\psi_0 - \frac{1}{\psi_0}\right) - \frac{\psi_0^2}{\gamma} + 2b\psi_0 + \left(\frac{1}{\gamma} - 2b\gamma\right)\right\} \\ &= \frac{1}{2a}\partial_x \left\{ [1+\partial_x]\left[\psi_0 - \frac{1}{\psi_0}\right]\partial_x\psi_0 + 2a\left(\varepsilon e^{-x} + \frac{b}{a} - \beta\right)\psi_0 - \frac{\psi_0^2}{\gamma}\right\}.\end{aligned}$$

For the purpose of the following text let us set ε to zero. Then evidently any constant solves the problem above. Furthermore, as the solution must be consistent with the initial condition, this constant, in fact the solution to the problem above coincides with the initial condition. Therefore, providing that $\varepsilon = 0$, then

$$\psi_0(s,x) = \gamma d\,. \tag{6.10}$$

Hence obviously, $\partial_x\psi_0 = \partial_s\psi_0 = 0$. In fact, zero order solution constancy is a key fact utilized in order to determine the higher order terms of the λ asymptotic expansion, as discussed in the forthcoming text.

Linear Term Problem First of all observe that the constant and linear terms of the expression $\zeta'(\sum_{n=0}^{\infty}\lambda^n\varphi_n)$ can be found easily, as they arises from

$$\zeta'\left(\sum_{n=0}^{\infty}\lambda^n\varphi_n\right) = \sum_{n=0}^{\infty}\lambda^n + \frac{\sum\limits_{n=0}^{\infty}\lambda^n\varphi_n(s,x)}{\sum\limits_{n=0}^{\infty}\lambda^n\partial_x\varphi_{n+1}(s,x)} e^{-\sum\limits_{n=0}^{\infty}\lambda^n\int^x\varphi_n(s,z)dz} = \sum_{n=0}^{\infty}\lambda^n\varpi_n(s,x)\,. \tag{6.11}$$

Above we used the fact that φ_0 is constant. Therefore, the first two terms ϖ_0 and ϖ_1 satisfy:

$$\begin{aligned}\varpi_0(s,x) &= 1 + \frac{\varphi_0}{\partial_x\varphi_1(s,x)}e^{-\varphi_0 x}\,, \\ \varpi_1(s,x) &= 1 + \left[\frac{\varphi_1}{\partial_x\varphi_1(s,x)} - \frac{\varphi_0\,\partial_x\varphi_2(s,x)}{(\partial_x\varphi_1(s,x))^2} - \frac{\varphi_0}{\partial_x\varphi_1(s,x)}\int^x\varphi_1(s,z)dz\right]e^{-\varphi_0 x}.\end{aligned} \tag{6.12}$$

Next, the first triple of components of the λ–expansion linear term satisfies the subsequent

$$
\begin{aligned}
&\Big\{\sum_{n=0}^{\infty}\lambda^n\frac{\partial\varphi_n}{\partial x}\Big[1+\frac{1}{\gamma^2\zeta^2(\sum\limits_{n=0}^{\infty}\lambda^n\varphi_n)}\Big]\zeta'(\sum_{n=0}^{\infty}\lambda^n\varphi_n)\Big\}\Big|_{\lambda^1}\\
&\approx\left\{[\partial_x\varphi_0+\lambda\partial_x\varphi_1]\left[1+\frac{1}{\gamma^2\left(1+\varphi_0+\lambda(\varphi_1+\omega(\varphi_0))\right)^2}\right][\varpi_0+\lambda\varpi_1]\right\}\Big|_{\lambda^1}\\
&\approx\left\{[\partial_x\varphi_0+\lambda\partial_x\varphi_1]\left[1+\frac{1}{\gamma^2(1+\varphi_0)^2}-\frac{2(\varphi_1+\omega(\varphi_0))}{\gamma^2(\varphi_0+1)^3}\lambda\right][\varpi_0+\lambda\varpi_1]\right\}\Big|_{\lambda^1}\\
&\approx\left[1+\frac{1}{\gamma^2(1+\varphi_0)^2}\right]\left[\partial_x\varphi_1+\varphi_0e^{-\varphi_0x}\right].
\end{aligned}
$$

Despite the presence of $\partial_x\varphi_2$ in ϖ_1 (that is unknown in the first order approximation and is not supposed to be determined at λ–linear level) notice that the whole term is ignored as it is considered only when multiplied by $\partial_x\varphi_0=0$. This observation is widely used in the procedure of obtaining higher order terms of the asymptotic expansion of φ with respect to λ.

Finally, the last pair of components of the λ–expansion linear term is given as follows:

$$
\begin{aligned}
&\Big\{\sum_{n=0}^{\infty}\lambda^n\varphi_n\Big[\zeta(\sum_{n=0}^{\infty}\lambda^n\varphi_n)-2b-\frac{1}{\gamma^2\zeta(\sum\limits_{n=0}^{\infty}\lambda^n\varphi_n)}\Big]\Big\}\Big|_{\lambda^1}\\
&\qquad\approx\varphi_0\left[1+\frac{1}{\gamma^2(1+\varphi_0)^2}\right][\varphi_1+\omega(\varphi_0)]+\varphi_1\left[1+\varphi_0-2b-\frac{1}{\gamma^2(1+\varphi_0)}\right]
\end{aligned}
$$

Problem 2 (Approximative Problem Statement)**.** *For all $s\in[0,T],x\in\mathbb{R}$ we approximate the function $\psi=\psi(s,x)=\gamma(1+\varphi(s,x))$ as follows*

$$\psi(s,x)=\psi_0(s,x)+\lambda\,\psi_1(s,x)\,, \tag{6.13}$$

where ψ_0 and ψ_1 are solutions to the following coupled problems for:

$$
[\mathbf{P_0}]\quad\begin{cases}
\dfrac{\partial\psi_0}{\partial s}=\dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\left[1+\dfrac{\partial}{\partial x}\right]\left[\psi_0-\dfrac{1}{\psi_0}\right]\dfrac{\partial\psi_0}{\partial x}\right.\\
\qquad\qquad\left.+2a\,(\varepsilon e^{-x}+p_0)\,\psi_0-\dfrac{\psi_0^2}{\gamma}\right\}\,, & (0,T]\times\mathbb{R};\\
\psi_0(0,x)=\gamma d\,, & x\in\mathbb{R}.
\end{cases}\tag{6.14}
$$

$$
[\mathbf{P_1}]\quad\begin{cases}
\dfrac{\partial\psi_1}{\partial s}=\dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\left[1+\dfrac{1}{\psi_0^2}\right]\left[\dfrac{\partial\psi_1}{\partial x}-q_1\psi_1\right]+2a\,[\varepsilon e^{-x}+p_1]\,\psi_1\right.\\
\qquad\qquad\left.+2\left[1+\dfrac{1}{\psi_0^2}\right]\gamma q_1e^{-q_1x}\right\}, & (0,T]\times\mathbb{R};\\
\psi_1(0,x)=\gamma(1-d)e^{(1-d)x}, & x\in\mathbb{R};
\end{cases}\tag{6.15}
$$

where the coefficients are given as

$$p_0=\frac{b}{a}-\beta\,,\qquad p_1(s,x)=-\beta+\frac{b}{a}-\frac{1}{2a}\frac{\psi_0^2-1}{\gamma\psi_0}\,,\qquad q_1=\frac{\psi_0}{\gamma}-1\equiv\varphi_0\,. \tag{6.16}$$

6.2 Zero Risk Distinctive Case

First of all we need to determine the absolute term $\psi_{0,0}$ in the solution $\psi_0 = \psi_0(s,x)$ asymptotic expansion introduced above in Problem 1 and its approximation (Problem 2). Henceforth our effort is concentrated on problem $[\mathbf{P_0}]$ solution determination.

There are several key facts that must be emphasized: forasmuch as Σ is positive definite matrix, recalling the corresponding definition of constants a, b and c, we can easily deduce that both a and c are strictly positive and even $ac - b^2$ is so. Moreover, since in general the asset returns can be bounded from above by one, $a > c$ and for positive assets returns, $a > b > c$. Thus, by taking small enough value of investor's salary growth rate β, we are able to make the additional structural presuppositions on the coefficient p_0 positiveness. Notice that in this paper the foregoing statements are taken for granted, as summarized in the subsequent statement.

Assumption 3. *We assume that,*

$$p_0 = \frac{b}{a} - \beta > 0,. \tag{6.17}$$

In order to construct a solution ψ_0 to $[\mathbf{P_0}]$ above, rewrite $\psi_0(s,x)$ in terms of the asymptotic series with respect to ε:

$$\psi_0(s,x) = \sum_{n=0}^{\infty} \varepsilon^n \psi_{0,n}(s,x) \,, \qquad s \in [0,T] \,, \ x \in \mathscr{D}'. \tag{6.18}$$

Notice that by this act we actually perform double (λ, ε) asymptotic expansion of the value function. The ε-expansion of ψ_0 is considered for a regular perturbation since the problem character is retained for $\varepsilon \to 0$ (see O'Malley [57], Bender and Orszag [6], Hinch [29] or Holmes [30]). First of all we pay attention to $\psi_{0,0}(s,x)$. Recalling the constant character of the utility function zero term γd, we have achieved the solution constancy (see Macová and Ševčovič [44], Múčka [51], Macová [43] for further details), i.e.

$$\psi_{0,0}(s,x) = \gamma d \quad \text{for any } s \in [0,T],\ x \in \mathbb{R}. \tag{6.19}$$

To approximate the function $\psi_0(s,x)$ for small ε, we use both the constant and the linear terms corresponding to the ε-expansion of the zero term of λ-expansion to get

$$\psi_0(s,x) = d\gamma + \varepsilon \psi_{0,1}(s,x) + O(\varepsilon^2) \quad \text{as} \ \ \varepsilon \to 0^+ \,. \tag{6.20}$$

In (6.20), $\psi_{0,1}(s,x)$ is the solution to the subsequent Cauchy problem arising from (6.14)

$$\begin{cases} \dfrac{\partial \psi_{0,1}}{\partial s} = \dfrac{1}{2a}\left[1 + \dfrac{1}{\psi_{0,0}^2}\right] \dfrac{\partial^2 \psi_{0,1}}{\partial x^2} + \dfrac{1}{2a}\left[1 + \dfrac{1}{\psi_{0,0}^2} + 2a\delta\right] \dfrac{\partial \psi_{0,1}}{\partial x} - \psi_{0,0} e^{-x} \,, & (0,T] \times \mathbb{R} \,; \\ \psi_{0,1}(0,x) = 0 \,, & \mathbb{R} \,; \end{cases} \tag{6.21}$$

where $d\gamma$ is replaced by the constant $\psi_{0,0}$ and δ stands for the following expression:

$$\delta = p_0 - \frac{d}{a} = p_0 - 2dq_0 = \frac{b-d}{a} - \beta. \tag{6.22}$$

The unique solution of (6.21) can be found for all $s \in [0,T]$ and $x \in \mathbb{R}$ in a separable form:

$$\psi_{0,1}(s,x) = \Phi_{0,1}(s)e^{-x}, \tag{6.23}$$

for the unknown function $\Phi_{0,1} = \Phi_{0,1}(s)$ that has to be determined. Hence we transform the original partial differential equation (6.21) into the subsequent time dependent ordinary differential equation:

$$\begin{cases} \Phi'_{0,1}(s) = -\delta\Phi_{0,1}(s) - \psi_{0,0}, & s \in (0,T], \\ \Phi_{0,1}(0) = 0, & s = 0. \end{cases}$$

So,

$$\Phi_{0,1}(s) = d\gamma\frac{e^{-\delta s}-1}{\delta}, \qquad s \in [0,T]. \tag{6.24}$$

Therefore, combining (6.19), (6.23) and (6.20) leads to the subsequent linear approximation of the problem $[\mathbf{P_0}]$ defined by (6.14) solution as stated below:

Proposition 7. *The linear approximation to the solution of the problem* $[\mathbf{P_0}]$ *defined by* (6.14) *is given as*

$$\psi_0(s,x) = \gamma d\left[1 + \varepsilon\frac{e^{-\delta s}-1}{\delta}e^{-x}\right] + o(\varepsilon^2). \tag{6.25}$$

If the higher order terms in (6.20) are omitted, we have found the asymptotic solution (6.23) to $[\mathbf{P_0}]$ and the associated zero term of the unconstrained policy θ^* λ–expansion for small ε by

$$\theta^*(t,y) = \frac{1}{a}\Sigma^{-1}\left\{\mathbf{1} + \frac{1}{\psi(T-t,\ln y)}\frac{a\mu - b\mathbf{1}}{\sqrt{ac-b^2}}\right\}. \tag{6.26}$$

The foregoing formula is valid only for a concave and increasing $V(t,y)$ - the region definition is a part of this study. Even though the higher order terms of ε-expansion can be easily obtained, for the purpose of this work we are satisfied with its linear approximation.

6.3 Linear Term in Solution λ-Expansion

Now we pay our attention to the determination procedure of the linear term associated with the λ–asymptotic expansion. Therefore our aim is to find a solution to the problem $[\mathbf{P_1}]$ as stated in (6.15) providing that the solution $\psi_0 = \psi_0(s,x)$ of the problem $[\mathbf{P_0}]$ introduced by (6.14) is known. For the purpose of our analysis as we are interested in the function $\psi(s,x)$ double λ–ε up to its linear term we need to derive the term ψ_{10} – the one characterized as the λ–linear and ε–absolute term of the double expansion.

Firstly, in case of $\varepsilon = 0$ evidently both $p_1 = p_1(s,x)$ and $q_1 = q_1(s,x)$ introduced by (6.16) remain constant due to $\psi_{0,0}$ constancy

$$p_0 = \frac{b}{a} - \beta, \qquad p_1 = -\beta + \frac{b}{a} - \frac{1}{2a}\frac{(\gamma d)^2 - 1}{\gamma^2 d}, \qquad q_1 = d - 1.$$

Next, employing the assumption of $\varepsilon = 0$ from the definition (5.4)–(5.5) of $\omega = \omega(\varphi) = \omega(\psi/\gamma - 1)$ one may deduce straightforwardly that

$$\omega(q_1) = -e^{(1-d)x}.$$

Therefore, plugging the problem $[\mathbf{P_0}]$ ε–linear solution ψ_0 approximation (6.25) into problem $[\mathbf{P_1}]$ (6.15) and then setting $\varepsilon = 0$ in the resulting problem leads to the following initial value problem formulated for the unknown $\psi_{1,0}$ associated with the absolute component in terms of ε–asymptotic expansion of the λ–linear term problem that has to be solved

$$\begin{cases} \dfrac{\partial \psi_{1,0}}{\partial s} = \dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{\left[1+\dfrac{1}{\psi_{0,0}^2}\right]\dfrac{\partial \psi_{1,0}}{\partial x} + \left[1+\dfrac{1}{\psi_{0,0}^2}+2a\delta\right]\psi_{1,0}\right. \\ \qquad\qquad\qquad \left. +2\gamma(d-1)\left[1+\dfrac{1}{\psi_{0,0}^2}\right]e^{(1-d)x}\right\}, \quad (0,T]\times\mathbb{R}; \\ \psi_1(0,x) = \gamma(1-d)e^{(1-d)x}, \qquad x\in\mathbb{R}; \end{cases} \tag{6.27}$$

where ψ_0 stands for γd and the parameter δ is prescribed by (6.22). Similarly to the case of problem $[\mathbf{P_0}]$ the character of the system (6.27) posed above allows us to look for its unique solution in the form of the time–space separable function

$$\psi_{1,0} = \Phi_{1,0}(s)e^{(1-d)x}$$

and allows us to reduce the problem dimension. Hence we need to determine the unique solution to the non–homogeneous ordinary differential equation for $\Phi_{1,0} = \Phi_{1,0}(s)$:

$$\begin{cases} \Phi'_{1,0}(s) = \dfrac{d-1}{2a}\left[(d-2)\left[1+\dfrac{1}{\psi_{0,0}^2}\right]-2a\delta\right]\Phi_{1,0}(s) - \dfrac{(d-1)^2}{a}\gamma\left[1+\dfrac{1}{\psi_{0,0}^2}\right], & s\in(0,T], \\ \Phi_{1,0}(0) = -\gamma(d-1), & s=0. \end{cases}$$

Thus we can straightforwardly deduce the solution to the foregoing ordinary differential equation as below:

$$\Phi_{1,0}(s) = \gamma(d-1)\left\{-\left[1+\widetilde{\phi}\right]e^{\widetilde{\delta}s}+\widetilde{\phi}\right\}, \qquad s\in[0,T], \tag{6.28}$$

where coefficients $\widetilde{\delta}$ and $\widetilde{\phi}$ are given stated by the forthcoming proposition:

Proposition 8. *The zero approximation to the solution of the problem* $[\mathbf{P_1}]$ *defined by* (6.15) *is given as*

$$\psi_{1,0}(s,x) = \gamma(d-1)e^{(1-d)x}\left\{-\left[1+\widetilde{\phi}\right]e^{\widetilde{\delta}s}+\widetilde{\phi}\right\}+o(\varepsilon^2), \tag{6.29}$$

with the coefficients given as follows

$$\widetilde{\delta} = \frac{d-1}{2a}\left[(d-2)\left[1+\frac{1}{\psi_0^2}\right]-2a\delta\right], \qquad \widetilde{\phi} = \frac{2}{(d-2)-2a\delta\left[1+\dfrac{1}{\psi_0^2}\right]^{-1}}. \tag{6.30}$$

6.4 Approximative Optimal Policy

Combining (6.25) and (6.29) one may achieve the first order approximation of the solution of the problem (6.1) in the forthcoming form:

$$\psi(s,x) = \gamma d + \varepsilon\Phi_{0,1}(s)e^{-x} + \lambda\Phi_{1,0}(s)e^{(1-d)x} + o((\varepsilon+\lambda)^2), \tag{6.31}$$

where $\Phi_{0,1}$ and $\Phi_{1,0}$ are prescribed by (6.23) and (6.28), respectively. Firstly we remind you that the foregoing formula holds only if the requirements under which the solution was derived are met. Concretely, we want $\zeta = \zeta(\varphi)$ to be positive. Thus applying the linear transform (6.9) on the approximative solution ψ as given by (6.31) and plugging the resulting function φ into the definition of $\zeta = \zeta(\varphi)$ one may obtain straightforwardly its linear approximation shown below

$$\zeta(\varphi(s,x)) = d - \varepsilon\Phi_\varepsilon(s)e^{-x} - \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x} + o((\varepsilon+\lambda)^2), \tag{6.32a}$$

for

$$\Phi_\varepsilon(s) \equiv -\frac{1}{\gamma}\Phi_{0,1}(s) = d\frac{1-e^{-\delta s}}{\delta}, \qquad \Phi_\lambda(s) \equiv -\frac{1}{\gamma}\Phi_{1,0}(s) = (d-1)\left\{\left[1+\widetilde{\phi}\right]e^{\widetilde{\delta}s} - \widetilde{\phi}\right\}. \tag{6.32b}$$

Next, let us remind you the natural requirement of the monotonous increasingness and strict concavity properties that should satisfy the utility function U introduced in Section 4.2 by (4.11). Recalling the domain over which the utility function of our choice attain the desired characteristics, (4.12) and the change of variables (t,y) to (s,x) such that $y = e^x$ for any $x \in \mathbb{R}$ we restrict the space variable x and call for

$$x > \frac{1}{d-1}\ln\left[\lambda\frac{2d-1}{d}\right] \equiv \Lambda.$$

Therefore, one can easily deduce the region, where the solution φ can be accepted as

$$\left\{(s,x)\in[0,T]\times(\Lambda,\infty),\, d - \varepsilon\Phi_\varepsilon(s)e^{-x} - \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x} > 0\right\}. \tag{6.33}$$

It is inevitable to mention that the set of (s,x) for which $\varphi(s,x)$ remains positive as described above defines only the unconstrained optimal solution domain. This is because the short selling ban requirement has not been applied yet.

Remark 6. *Recall the two dimensional problem solution for optimal investment strategy for stock investment θ formula derived for $\zeta > 0$ by the formula* (5.21) *under the assumption of $\alpha_\sigma > \beta_\sigma$ as*

$$\widetilde{\theta}^{(s)}(\zeta) = \min\left\{1, \frac{\Delta\mu}{\alpha_\sigma\zeta} + \frac{\beta_\sigma}{\alpha_\sigma}\right\},$$

where

$$\alpha_\sigma = \left[\sigma^{(s)}\right]^2 - 2\rho\sigma^{(s)}\sigma^{(b)} + \left[\sigma^{(b)}\right]^2, \qquad \beta_\sigma = \sigma^{(b)}\left[\sigma^{(b)} - \rho\sigma^{(s)}\right].$$

Therefore, the condition posed on $\zeta = \zeta(s,x)$ defined by (6.32a)*, i.e. $\zeta > \Delta\mu/(\alpha_\sigma - \beta_\sigma)$ demarks the region on which the prescription of ζ,* (6.31) *takes place. Otherwise all financial resources already accumulated in the portfolio have to be allocated into stocks (so $\widetilde{\theta}^{(s)}(\zeta) = 1$).*

Hence, providing that we are concerned about the two dimensional problem, in case of unconstrained optimal solution we require $\zeta = \zeta(\varphi)$ *to be positive, i.e. define the domain of* $\varphi = \varphi(s,x)$ *by* (6.33). *On the other side, if the ban on borrowing constraint is active, in order to apply the formula* (6.32a) *to define* ζ *we claim* ζ *to exceed* $\Delta\mu/(\alpha_\sigma - \beta_\sigma)$, *otherwise for* $0 < \zeta \leq \Delta\mu/(\alpha_\sigma - \beta_\sigma)$ *the appropriate value of* ζ *is directly determined by the optimal value of the stock investment* $\widetilde{\theta}^{(s)}(\zeta) = 1$. *Therefore, in the constrained optimal solution as defined above (or by* (5.21)*) in we consider the optimal allocation policy to follow the subsequent rule defined using the prescription of* ζ *given by* (6.32a)

$$\widetilde{\theta}^{(s)}(s,x) = \begin{cases} \dfrac{1}{\alpha_\sigma}\left[\beta_\sigma + \dfrac{\Delta\mu}{\zeta(s,x)}\right], & \zeta(s,x) > \dfrac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}, \\ 1 & 0 < \zeta(s,x) \leq \dfrac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}. \end{cases}$$

Evidently the optimal proportion of bonds in the investment portfolio is then determined as $\widehat{\theta}^{(b)} = 1 - \widetilde{\theta}^{(s)}$.

Henceforth, the region on which in case of the two dimensional problem the first order asymptotic approximation of ζ *established by* (6.32a) *is applied is defined by the following prescription:*

$$\left\{(s,x) \in [0,T] \times (\Lambda,\infty),\ \varepsilon\Phi_\varepsilon(s)e^{-x} + \lambda\left[\Phi_\lambda(s) + 1\right]e^{-(d-1)x} < d - \frac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}\right\}. \tag{6.34}$$

Next, in order to write ζ in the following manner

$$\zeta(\varphi(s,x)) = 1 + \varphi(s,x) + \lambda\,\omega(\varphi(s,x)), \qquad \omega(\varphi(s,x)) = -\kappa e^{-\int_{x_0}^{x}\varphi(s,z)dz}, \tag{6.35}$$

we call for $\partial_x\mathscr{V}$ to be positive. Therefore our aim is to determine the region of $s \in [0,T]$ and $x \in \mathbb{R}$ where this claim is satisfied. Simply, integrate (5.3) with respect to variable x to achieve

$$\frac{\partial\mathscr{V}}{\partial x}(s,x) = \exp\left[\rho(s) - \int_{-\infty}^{x}\varphi(s,z)dz\right], \tag{6.36}$$

for a unique function $\rho = \rho(s)$ defined such that $\rho(0) = \lim_{x_0 \to -\infty} \ln U'(e^{x_0})$. Then differentiate the result above with respect to s leads to the subsequent

$$\frac{\partial^2\mathscr{V}}{\partial x\partial s}(s,x) = \left[\rho'(s) - \int_{-\infty}^{x}\frac{\partial\,\varphi(s,z)}{\partial s}dz\right]\frac{\partial\mathscr{V}}{\partial x}(s,x).$$

On the other side taking the x–derivative of (5.6a) gives us the following:

$$\frac{\partial\mathscr{V}}{\partial s\partial x}(s,x) = \left[\frac{\partial\mathscr{G}}{\partial x}(s,x) - \varphi(s,x)\mathscr{G}(s,x)\right]\frac{\partial\mathscr{V}}{\partial x}(s,x)$$

Hence when the foregoing results are combined together, the resulting problem for $\rho = \rho(s)$ solved and the solution placed back into (6.36) we get the desired outcome

$$\frac{\partial\mathscr{V}}{\partial x}(s,x) = \exp\left\{-\int_{-\infty}^{x}\varphi(0,z)dz + \int_{0}^{s}\left[\frac{\partial\mathscr{G}}{\partial x}(\xi,x) - \varphi(\xi,x)\mathscr{G}(\xi,x)d\xi\right]\right\} > 0. \tag{6.37}$$

Finally the product of our effort can be summarized as below:

Theorem 5. *The first order approximation of the unconstrained solution to the problem* (6.1) *with respect to small model parameters* ε *and* λ *satisfies*

$$\varphi(s,x) = d - 1 - \varepsilon\Phi_\varepsilon(s)e^{-x} - \lambda\Phi_\lambda(s)e^{(1-d)x}, \qquad \textit{for all} \quad (s,x) \in \Omega \tag{6.38}$$

where the region Ω *is defined as follows:*

$$\Omega \equiv \{(s,x) \in [0,T] \times (\Lambda,\infty),\, d - \varepsilon\Phi_\varepsilon(s)e^{-x} - \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x} > 0\}, \tag{6.39}$$

and the auxiliary functions Φ_ε *and* Φ_λ *are given by the prescriptions:*

$$\Phi_\varepsilon(s) \equiv d\frac{1-e^{-\delta s}}{\delta}, \qquad \textit{and} \qquad \Phi_\lambda(s) \equiv (d-1)\left\{\left[1+\widetilde{\phi}\right]e^{\widetilde{\delta}s} - \widetilde{\phi}\right\}$$

for δ, $\widetilde{\delta}$ *and* $\widetilde{\phi}$ *introduced by* (6.22) *and* (6.30)*, respectively, and*

$$\Lambda \equiv \frac{1}{d-1}\ln\left[\lambda\frac{2d-1}{d}\right]. \tag{6.40}$$

The optimal unconstrained investment strategy defined as

$$\theta^*(s,x) = \frac{\Sigma^{-1}}{a}\left[\mathbf{1} + (a\mu - b\mathbf{1})[\zeta(s,x)]^{-1}\right], \qquad \textit{where} \qquad \zeta(s,x) = d - \varepsilon\Phi_\varepsilon(s)e^{-x} - \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x}, \tag{6.41}$$

is correctly defined on Ω.

Especially in case of the two dimensional problem, the optimal constrained allocation policy for the stock investment is defined as follows

$$\widetilde{\theta}^{(s)}(s,x) = \begin{cases} \dfrac{1}{\alpha_\sigma}\left[\beta_\sigma + \dfrac{\Delta\mu}{\zeta(s,x)}\right], & \zeta(s,x) \in \Omega_2^*, \\ 1 & 0 < \zeta(s,x), \quad \zeta(s,x) \notin \Omega_2^*, \end{cases} \tag{6.42}$$

for the constants

$$\alpha_\sigma = \left[\sigma^{(s)}\right]^2 - 2\rho\sigma^{(s)}\sigma^{(b)} + \left[\sigma^{(b)}\right]^2, \qquad \beta_\sigma = \sigma^{(b)}\left[\sigma^{(b)} - \rho\sigma^{(s)}\right].$$

and the function $\zeta = \zeta(s,x)$ *follows the prescription* (6.41) *on the region*

$$\Omega_2^* \equiv \left\{(s,x) \in [0,T] \times (\Lambda,\infty),\, \varepsilon\Phi_\varepsilon(s)e^{-x} + \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x} < d - \frac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}\right\}. \tag{6.43}$$

The optimal weight of bonds in the portfolio is determined as $\widetilde{\theta}^{(b)} = 1 - \widetilde{\theta}^{(s)}$.

6.5 Second Order Approximation

In order to describe the approximative solution (6.38) more precisely and receive detailed information about its behaviour we approach it up to its second order terms in sense of assumed double (λ,ε) asymptotic expansion:

$$\psi = \psi_{0,0} + \varepsilon\psi_{0,1} + \lambda\psi_{1,0} + \varepsilon^2\psi_{0,2} + 2\varepsilon\lambda\psi_{1,1} + \lambda^2\psi_{2,0} + o\left((\varepsilon+\lambda)^3\right). \tag{6.44}$$

In the ψ expansion launched above, $\psi_{0,0}$ and $\psi_{0,1}$ are given by (6.25) whereas $\psi_{1,1}$ is prescribed by (6.29). Therefore the following text is dedicated to three main sub–problems: detection of the first (λ,ε) mixed term, and the second term derivation in case of missing either ε or λ.

6.5.1 The Mixed Second Derivative Term

So as to have a deeper knowledge of the $[\mathbf{P_0}]$–$[\mathbf{P_1}]$ coupled problems solution behaviour, we concentrate on finding the term corresponding to $\varepsilon^1\lambda^1$ multiplicative factor. Hence we need to compute the linear term in ε–asymptotic expansion of the Problem $[\mathbf{P_1}]$ launched by (6.15). So we introduce the power series expansion of $\psi_1 = \psi_1(s,x)$ with respect to the small parameter ε as follows:

$$\psi_1(s,x) = \sum_{n=0}^{\infty} \varepsilon^n \psi_{1,n}(s,x)\,. \tag{6.45}$$

Therefore, plugging back (6.45) into problem $[\mathbf{P_1}]$ (6.15) and collecting the terms associated with ε^1 results in the following initial value problem established for the function $\psi_{1,1}$ which is to be determined:

$$\begin{cases} \dfrac{\partial \psi_{1,1}}{\partial s} = \dfrac{1}{2a}\dfrac{\partial}{\partial x}\left\{ \left[1+\dfrac{1}{\psi_{0,0}^2}\right]\dfrac{\partial \psi_{1,1}}{\partial x} + \left[1+\dfrac{1}{\psi_{0,0}^2}+2a\delta\right]\psi_{1,0} \right. & \\ \qquad\qquad\qquad\qquad\qquad\left. +2ae^{-x}\psi_{1,0}(s,x)\right\}\,, & (0,T]\times\mathbb{R}; \\ \psi_{1,1}(0,x) = 0\,, & x\in\mathbb{R}; \end{cases} \tag{6.46}$$

where $\psi_{0,0}$ stands for γd and the parameter δ is prescribed by (6.22). Moreover, $\psi_{1,0}$ is the time–space solution to the problem $[\mathbf{P_1}]$ with $\varepsilon = 0$, already determined by (6.29) and solving the system (6.27).

Similarly to the case of problem $[\mathbf{P_1}]$ pondering the character of the system (6.46) posed above and the separability of the known $\psi_{1,0}(s,x) = \Phi_{1,0}(s)e^{-(d-1)x}$ induces the unique solution to (6.46) in the form of the time–space separable function

$$\psi_{1,1}(s,x) = \Phi_{1,1}(s)e^{-dx}\,, \tag{6.47}$$

and so to the problem dimension reduction. Hence we need to determine the unique solution to the non–homogeneous Cauchy–type ordinary differential equation for the unknown $\Phi_{1,1} = \Phi_{1,1}(s)$:

$$\begin{cases} \Phi_{1,1}'(s) = \dfrac{d}{2a}\left[(d-1)\left[1+\dfrac{1}{\psi_{0,0}^2}\right]-2a\delta\right]\Phi_{1,1}(s) - d\Phi_{1,0}(s)\,, & s\in(0,T]; \\ \Phi_{1,1}(0) = 0\,, & s=0; \end{cases} \tag{6.48}$$

with $\Phi_{1,0}$ given by (6.28). Thus as for $d \gg 1$

$$\frac{d}{2a}\left[(d-1))\left[1+\frac{1}{\psi_{0,0}^2}\right]-2a\delta\right] \neq \frac{d-1}{2a}\left[(d-2)\left[1+\frac{1}{\psi_0^2}\right]-2a\delta\right]$$

we can straightforwardly deduce the solution to the foregoing ordinary differential equation as below:

$$\Phi_{1,1}(s) = \gamma(d-1)\left[-\widetilde{\phi}_1 + \left(\widetilde{\phi}_1 - \widetilde{\phi}_2\right)e^{\widetilde{\delta}_1} + \widetilde{\phi}_2 e^{\widetilde{\delta}}\right], \tag{6.49}$$

where $\widetilde{\delta}$ was launched by (6.30) and the remaining parameters follow the subsequent prescriptions:

$$\widetilde{\delta}_1 = \frac{d}{2a}\left[(d-1))\left[1+\frac{1}{\psi_{0,0}^2}\right] - 2a\delta\right], \qquad \widetilde{\phi}_1 = -\frac{d}{\widetilde{\delta}_1}\widetilde{\phi}, \qquad \widetilde{\phi}_2 = \frac{d}{\widetilde{\delta}_1 - \widetilde{\delta}_0}\left[\frac{\widetilde{\delta}_0}{\widetilde{\delta}_1}\widetilde{\phi} + 1\right]. \tag{6.50}$$

Therefore, combining (6.47) and (6.49) implies the time–space separable form of the mixed $\varepsilon^1\lambda^1$ as below

$$\psi_{1,1}(s,x) = \gamma(d-1)\left[-\widetilde{\phi}_1 + \left(\widetilde{\phi}_1 - \widetilde{\phi}_2\right)e^{\widetilde{\delta}_1 s} + \widetilde{\phi}_2 e^{\widetilde{\delta} s}\right]e^{-dx}. \tag{6.51}$$

6.5.2 Quadratic Term in the ε Expansion

For the reason of better approximation of the function $\psi_0 = \psi_0(s,x)$ for small enough values of the parameter ε, now we make use the Taylor expansion (6.18) up to the second order term, so

$$\psi_0(s,x) = \psi_{0,0} + \varepsilon\psi_{0,1}(s,x) + \varepsilon^2\psi_{0,2}(s,x) + O(\varepsilon^3), \tag{6.52}$$

as $\varepsilon \to 0^+$ where both $\psi_{0,0}$ and $\psi_{0,1}$ has already been uncovered (see (6.25)). Inserting the quadratic approximation as stated above of the function $\psi_0(s,x)$ into equation (6.14), collecting and calculating all the terms of the order $O(\varepsilon^2)$ we conclude that the function $\psi_{0,2}$ is a solution to the following linear parabolic equation:

$$\begin{cases} \dfrac{\partial\psi_{0,2}}{\partial s}(s,x) = \dfrac{1}{2a}\left[1+\dfrac{1}{\psi_{0,0}^2}\right]\dfrac{\partial^2\psi_{0,2}}{\partial x^2}(s,x) \\ \qquad\qquad + \dfrac{1}{2a}\left[1+\dfrac{1}{\psi_{0,0}^2}+2a\delta\right]\psi_{0,2}(s,x) + e^{-2x}\xi_2(s), & x\in\mathbb{R}, s\in(0,T], \\ \psi_{0,2}(0,x) = 0, & x\in\mathbb{R}, \end{cases} \tag{6.53}$$

where $\xi_2 = \xi_2(s)$ solves the subsequent

$$\xi_2(s) = \frac{1}{a}\left[\frac{1}{\gamma} - \frac{1}{\psi_{0,0}^3}\right]\Phi_1^2(s) - 2\Phi_1(s). \tag{6.54}$$

Recalling the procedure employed for the case of the linear term in ε expansion, we seek the solution to the problem presented above in terms of the time–space separable function. Inspired by the foregoing auxiliary function $e^{-2x}\xi_2(s)$ form we presuppose that

$$\psi_{0,2}(s,x) = \Phi_{0,2}(s)e^{-2x}, \qquad s\in[0,T],\, x\in\mathbb{R},$$

for some unknown function $\Phi_{0,2}(s)$ satisfying $\Phi_{0,2}(0) = 0$. Hence, in order to determine $\Phi_{0,2}$ our aim is to solve the ODE problem formulated below:

$$\begin{cases} \Phi'_{0,2}(s) = \dfrac{1}{a}\left[\left(1+\dfrac{1}{\psi_{0,0}^2}\right) - 2a\delta\right]\Phi_{0,2}(s) - 2\xi_2(s), & s\in(0,T] \\ \Phi_{0,2}(0) = 0, & s = 0. \end{cases}$$

Therefore the explicit solution of the problem (6.53) can be written in a closed form:

$$\psi_{0,2}(s,x) = e^{-2x} \int_0^s \xi_2(s) e^{(1/a)(s-z)(1+1/\psi_{0,0}^2 - 2a\delta)} dz . \tag{6.55}$$

The integral appearing in the foregoing expression can be explicitly computed and it can be expressed as a linear combination of of three exponential functions in the s variable.

6.5.3 Quadratic Term in the λ Expansion

Let us introduce the function $\psi = \psi(s,x)$ quadratic term of the asymptotic expansion with respect to the parameter λ under the assumption of $\varepsilon = 0$ as follows:

$$\psi(s,x) = \psi_{0,0} + \lambda \psi_{1,0}(s,x) + \lambda^2 \psi_{2,0}(s,x) + O(\lambda^3) , \tag{6.56}$$

for known $\psi_{0,0} = \gamma d$ and $\psi_{1,0}$ established by (6.29). Hence we plug (6.56) into (6.5) and collect the terms associated with λ^2. Following procedure for the linear term problem the first triple of components in (6.5) has the subsequent λ–quadratic term:

$$\begin{aligned}
&\Big\{ \sum_{n=0}^{\infty} \lambda^n \frac{\partial \varphi_n}{\partial x} \Big[1 + \frac{1}{\gamma^2 \zeta^2 (\sum\limits_{n=0}^{\infty} \lambda^n \varphi_n)} \Big] \zeta' (\sum_{n=0}^{\infty} \lambda^n \varphi_n) \Big\} \Big|_{\lambda^2} \\
&\approx \Big\{ \big[\partial_x \varphi_0 + \lambda \partial_x \varphi_1 + \lambda^2 \partial_{x^2} \varphi_2 \big] \Big[\widetilde{\zeta}_0 + \lambda \widetilde{\zeta}_1 + \lambda^2 \widetilde{\zeta}_2 \Big] \big[\varpi_0 + \lambda \varpi_1 + \lambda^2 \varpi_2 \big] \Big\} \Big|_{\lambda^2} \\
&\approx \left[1 + \frac{1}{\gamma^2 d^2} \right] \varpi_0 \partial_x \varphi_2 + \left\{ \left[1 + \frac{1}{\gamma^2 (1+\varphi_0)^2} \right] \varpi_1 + \widetilde{\zeta}_1 \right\} \partial_x \varphi_1 , \\
&= \left[1 + \frac{1}{\gamma^2 (1+\varphi_0)^2} \right] \{ \partial_x \varphi_2 - (d-1) \varphi_1 \} + 2\Phi_\lambda - \frac{2}{\gamma^2 d^3} e^{-(d-1)x} \frac{[\Phi_\lambda - 1]^2}{\Phi_\lambda} ,
\end{aligned}$$

as we made use the knowledge of both φ_0 (which is constant)and φ_1 . Above the form of terms ϖ_0 and φ_1 is shown in (6.12) and $\widetilde{\zeta}_1 = 2(\varphi_1 + \omega(\varphi_0))/[\gamma^2(\varphi_0 + 1)^3]$. Finally, the last pair of components of the λ–expansion quadratic term is given as follows:

$$\begin{aligned}
&\Big\{ \sum_{n=0}^{\infty} \lambda^n \varphi_n \Big[\zeta (\sum_{n=0}^{\infty} \lambda^n \varphi_n) - 2b - \frac{1}{\gamma^2 \zeta (\sum\limits_{n=0}^{\infty} \lambda^n \varphi_n)} \Big] \Big\} \Big|_{\lambda^2} \approx -2b\varphi_2 + \sum_{k=0}^{2} \varphi_k \left[\zeta_{n-k} - \frac{\nu_{n-k}}{\gamma^2} \right] \\
&= -2b\varphi_2 + \varphi_2 \left[2d - 1 - \frac{1}{\gamma^2 d^2} \right] - e^{-(d-1)x} \left[(d-1) + \varphi_1 \left(1 - (\Phi_\lambda - 1) \left(1 + \frac{1}{d^2 \gamma^2} \right) \right) \right] \\
&\quad + 2\frac{d-1}{d} e^{-2(d-1)x} [\Phi_\lambda - 1]^2
\end{aligned}$$

Therefore, by putting all together and setting $\varepsilon = 0$ we obtain the forthcoming problem:

$$\begin{cases}
\dfrac{\partial \psi_2}{\partial s} = \dfrac{1}{2a} \dfrac{\partial}{\partial x} \left\{ \left[1 + \dfrac{1}{\gamma^2 d^2} \right] \left[\dfrac{\partial \psi_2}{\partial x} + \psi_2 \right] + 2a\delta \psi_2 \right. \\
\qquad \left. + e^{-(d-1)x} \eta_1(s) + e^{-2(d-1)x} \eta_2(s) \right\} , \quad x \in \mathbb{R} , \, s \in (0,T] , \\
\psi(0,x) = \gamma(d-1) e^{-2(d-1)x} , \qquad x \in \mathbb{R} , \, s = 0 ,
\end{cases} \tag{6.57}$$

where the time dependent ancillary functions are prescribed as follows:

$$\begin{aligned}\eta_1(s) &= \frac{d-1}{\gamma}\left[1-\left[1+\frac{1}{\gamma^2 d^2}\right]\Phi_\lambda(s)\right]-\frac{2}{\gamma^3 d^3}\frac{[\Phi_\lambda(s)-1]^2}{\Phi_\lambda(s)}\,,\\ \eta_2(s) &= \Phi_\lambda(s)-(\Phi_\lambda(s)-1)\left[\left[-1+\frac{1}{\gamma^2 d^2}+\frac{2}{d}\right]\Phi_\lambda(s)+2\frac{d-1}{d}\right].\end{aligned} \tag{6.58}$$

Thus we seek the solution to (6.57)–(6.58) possessing the form of linearly combined time–space separable functions:

$$\psi_2(s,x) = e^{-(d-1)x}u_1(s)+e^{-2(d-1)x}u_2(s)\,, \qquad u_1(0)=0\,, \quad u_2(0)=\gamma(d-1) \tag{6.59}$$

for some time–varying functions $u_1(s)$, $u_2(s)$ to be determined. Therefore, using our judgement (6.59) about the solution to the problem (6.57) form in (6.57) leads into the following pair of ordinary differential equations for the unknown $u_1(s)$ and $u_2(s)$:

$$\begin{cases} u_1'(s) = \dfrac{d-1}{2a}\left[(d-2)\left(1+\dfrac{1}{\gamma^2 d^2}\right)-2a\delta\right]u_1(s)-\dfrac{d-1}{2a}\eta_1(s)\,, & s\in(0,T]\,,\\ u_1(0)=0\,, & s=0\,,\end{cases} \tag{6.60a}$$

$$\begin{cases} u_2'(s) = \dfrac{d-1}{a}\left[(2d-3)\left(1+\dfrac{1}{\gamma^2 d^2}\right)-2a\delta\right]u_2(s)-\dfrac{d-1}{a}\eta_2(s)\,, & s\in(0,T]\,,\\ u_2(0)=\gamma(d-1)\,, & s=0\,,\end{cases} \tag{6.60b}$$

Then evidently

$$\begin{aligned} u_1(s) &= -\frac{d-1}{2a}\int_0^s \eta_1(\tau)e^{\widetilde{\delta}_1(s-\tau)}\,d\tau\,,\\ u_2(s) &= (d-1)e^{\widetilde{\delta}_2 s}\left[\gamma-\frac{1}{2a}\int_0^s \eta_2(\tau)e^{-\widetilde{\delta}_2\tau}\,d\tau\right],\end{aligned} \tag{6.61}$$

where

$$\widetilde{\delta}_1 = \frac{d-1}{2a}\left[(d-2)\left(1+\frac{1}{\gamma^2 d^2}\right)-2a\delta\right], \qquad \widetilde{\delta}_2 = \frac{d-1}{a}\left[(2d-3)\left(1+\frac{1}{\gamma^2 d^2}\right)-2a\delta\right].$$

Hence, the solution to (6.57)is given by (6.59) where u_1 and u_2 take the forms prescribed by (6.60a) and (6.60b), respectively.

Proposition 9. *The quadratic approximation of the unconstrained solution to the problem* (6.1) *with respect to model parameters ε and λ satisfy the subsequent prescriptions:*

$$\begin{aligned}\varphi(s,x) &= d-1-\varepsilon\Phi_\varepsilon(s)e^{-x}-\lambda\Phi_\lambda(s)e^{(1-d)x}\\ &\quad -\varepsilon^2\Phi_{\varepsilon^2}(s)e^{-2x}-2\varepsilon\lambda\Phi_{\varepsilon\lambda}(s)e^{-dx}+\lambda^2\left[\Phi_{\lambda^2,1}(s)e^{-(d-1)x}+\Phi_{\lambda^2,2}(s)e^{-2(d-1)x}\right],\end{aligned} \tag{6.62}$$

where the linear terms satisfy (6.38),

$$\Phi_{\varepsilon^2}(s) = \int_0^s \overline{\xi}_2(s)e^{(1/a)(s-z)(1+1/\psi_{0,0}^2-2a\delta)}\,dz\,, \qquad \overline{\xi}_2(s) = 2\Phi_\varepsilon(s)-\frac{1}{a}\left[1-\frac{1}{\gamma^2 d^3}\right]\Phi_\varepsilon^2(s)\,,$$

$$\Phi_{\varepsilon\lambda}(s) = (d-1)\left[-\widetilde{\phi}_1+\left(\widetilde{\phi}_1-\widetilde{\phi}_2\right)e^{\widetilde{\delta}_1 s}+\widetilde{\phi}_2 e^{\widetilde{\delta}s}\right],$$

$$\Phi_{\lambda^2,1}(s) = -\frac{d-1}{2a\gamma}\int_0^s \eta_1(\tau)e^{\widetilde{\delta}_1(s-\tau)}\,d\tau\,, \qquad \Phi_{\lambda^2,2}(s) = (d-1)e^{\widetilde{\delta}_2 s}\left[1-\frac{1}{2a\gamma}\int_0^s \eta_2(\tau)e^{-\widetilde{\delta}_2\tau}\,d\tau\right]$$

for $\widetilde{\phi}_1$, $\widetilde{\phi}_2$ and $\widetilde{\delta}_1$ given by (6.50) *and η_1, η_2 prescribed by* (6.58).

The reader interested in the structure of the double (λ, ε) asymptotic expansion is referred to Appendix B.

6.6 Sensitivity Analysis

Firstly it is inevitable to remark that our sensitivity analysis is aimed on the approximative solution to the unconstrained problem (6.1) and so we describe the qualitative properties of the *super–solution* to the original problem (5.7).

Within the following text we firstly describe main qualitative properties of the function ζ established by (6.41) using the approximative solution to the problem (6.1) given by (6.38). Though not only quadratic but even general terms of the solution approximation have been derived, will show that its first order approximation is also capable of capturing all interesting phenomena that are present in our dynamic stochastic optimization problem.

Regardless the dimension of optimal allocation policy problem, the function ζ established by (6.41) enters into the prescription for $\hat{\theta}$ in the form of its inverse. Hence we approximate ζ^{-1} as follows:

$$\begin{aligned} \zeta^{-1}(s,x) \approx \omega(s,x) &\equiv \frac{1}{d} + \varepsilon\frac{\Phi_\varepsilon(s)}{d^2}e^{-x} + \lambda\frac{\Phi_\lambda(s)+1}{d^2}e^{-(d-1)x} \\ &= \frac{1}{d}\left\{1 + \varepsilon\frac{1-e^{-\delta s}}{\delta}e^{-x} + \lambda\left[\frac{d-1}{d}\left((1+\widetilde{\phi})e^{\widetilde{\delta}s} - \widetilde{\phi}\right)+1\right]e^{-(d-1)x}\right\}. \end{aligned} \tag{6.63}$$

In the following text we are concentrated on scrutinization of the key properties of the function $\omega(s,x)$ and provide a full description of its behaviour depending on model parameters.

6.6.1 Optimal unconstrained policy properties

Our aim is to determine the optimal unconstrained investment policy (i.e. actually the super–optimal solution to the constrained problem) as a function of the time and space variables. Furthermore, in case of the two dimensional problem, this behaviour can be directly revealed whereas providing that the problem is not planar, we apply our observation on the function ω as an approach to the inverse of ζ prescribed by (6.41).

For the purpose of the forthcoming analysis it is worth to notice two basic observations:

- $\Phi_\varepsilon(s)$ is non–negative, monotonously increasing, and strictly concave for all $s \in [0, T]$;
- Φ_λ in positive, monotonously increasing, and strictly convex for all $s \in [0, T]$.

Thus, at first sight it is evident that ζ is monotonously increasing and strictly concave in variable $x \in \mathbb{R}$ as

$$\begin{aligned}\frac{\partial \zeta}{\partial x}(s,x) &= \varepsilon \Phi_\varepsilon(s) e^{-x} + \lambda (d-1)\left[\Phi_\lambda(s)+1\right] e^{(1-d)x} > 0\,,\\ \frac{\partial^2 \zeta}{\partial x^2}(s,x) &= -\left[\varepsilon \Phi_\varepsilon(s) e^{-x} + \lambda (d-1)^2\left[\Phi_\lambda(s)+1\right] e^{(1-d)x}\right] < 0\,,\end{aligned}$$

for all $(s,x) \in \Omega$. Next ζ decreases monotonously in variable $s \in [0,T]$ since

$$\frac{\partial \varphi}{\partial s}(s,x) = -\left[d e^{-\delta s} e^{-x} + (d-1+\widetilde{\phi})\widetilde{\delta} e^{\widetilde{\delta} s} e^{(1-d)x}\right] < 0\,.$$

Therefore, recalling the optimal investment strategy definition the reverse statements hold for ω i.e. the optimal policy sub–function in decreasing and strictly convex in variable x while it ascends with s increasing.

Especially, for the case of $N = 2$ let us remind the formula (6.42) describing for the optimal constrained allocation policy. In order to describe better its qualitative behaviour we switch from (s,x) to (t,y) coordinates using the change of variables launched by (5.1), i.e. $t = T - s$, $y = e^x$ and establish $\widetilde{\zeta}(t,y) \equiv \zeta(T-t, \ln x)$. Therefore the optimal allocation policy restrained (due to ban on short positions) by $[0,1]$ follows the subsequent:

$$\begin{aligned}\widehat{\theta}^{(s)}(t,y) &= \begin{cases} \widetilde{\theta}^{(s)}(t,y)\,, & \zeta(T-t,\ln y) \in \Omega_2^*\,,\\ 1 & 0 < \zeta(T-t,\ln y)\,, \quad \zeta(T-t,\ln y) \notin \Omega_2^*\,,\end{cases}\\ \text{for} \qquad \widetilde{\theta}^{(s)}(t,y) &\equiv \frac{1}{\alpha_\sigma}\left[\beta_\sigma + \frac{\Delta\mu}{\zeta(T-t,\ln y)}\right]\,,\end{aligned} \tag{6.64}$$

under the assumption of $\alpha_\sigma > \beta_\sigma$ for the constants

$$\alpha_\sigma = \left[\sigma^{(s)}\right]^2 - 2\rho\sigma^{(s)}\sigma^{(b)} + \left[\sigma^{(b)}\right]^2\,, \qquad \beta_\sigma = \sigma^{(b)}\left[\sigma^{(b)} - \rho\sigma^{(s)}\right]\,.$$

and the region on which $\zeta = \zeta(s,x)$ follows the prescription (6.41),

$$\begin{aligned}\widehat{\Omega}_2^* \equiv \Bigg\{ (t,y)\,|\, 0 \le t \le T\,,\; y^{d-1} > \lambda\frac{2d-1}{d}\,,&\\ \varepsilon\frac{\Phi_\varepsilon(T-t)}{y} + \lambda\frac{\Phi_\lambda(T-t)+1}{y^{d-1}} < d - \frac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}\Bigg\}\,.&\end{aligned} \tag{6.65}$$

It must be emphasized that $\widetilde{\theta}^{(s)}$ represents the *unconstrained* optimal weight of the stock (risky asset) in the portfolio of two securities and in fact coincides with $\widehat{\theta}^{free}$ launched by (5.20) and evidently

$$\widehat{\theta}^{(s)}(t,y) \equiv \widetilde{\theta}^{(s)}(t,y)\,, \qquad (t,y) \in \widehat{\Omega}_2^*\,.$$

Furthermore, recalling Theorem 3 from Section 5, we know that $\widetilde{\theta}^{(s)}$ is C^∞ as well as the constrained optimal policy $\widehat{\theta}^{(s)}$, but *wts* is not $C^{1,1}$ on $[0,T] \times \mathbb{R}^+$ as it is not differentiable on $\widehat{\Omega}_2^*$. Therefore, in the following text we will scrutinize the qualitative and quantitative properties of the *unconstrained* optimal policy $\widetilde{\theta}^{(s)}$ only – this relaxation will not cause heavy looses on our knowledge of the optimal allocation policy as it remains constant out of $\widehat{\Omega}_2^*$.

Under the assumptions of stable financial market, $\alpha_\sigma > \beta_\sigma$ and $\Delta\mu > 0$. Therefore as

$$\frac{d\widetilde{\theta}^{(s)}}{d\zeta}(t,y) = -\frac{\Delta\mu}{\alpha_\sigma}\frac{1}{\zeta^2(T-t,\ln y)} < 0\,, \qquad (t,y) \in \Omega_2^* \tag{6.66}$$

combined with the results obtained above for ζ it is obvious that

$$\frac{\partial \widetilde{\theta}^{(s)}}{\partial y}(t,y) < 0\,, \qquad \frac{\partial^2 \widetilde{\theta}^{(s)}}{\partial y^2}(t,y) > 0\,, \qquad \frac{\partial \widetilde{\theta}^{(s)}}{\partial t}(t,y) < 0\,, \qquad (t,y) \in \Omega_2^*\,.$$

Hence, the weight of stocks (risky assets) in the stock–bond portfolio descends as time approaches the retirement age and declines with even increasing speed as the *wealth–to–salary* (yearly saved salaries) grows.

This result is fully consistent with the observed reality. Indeed, a stabilization phase takes place during the last years of the accumulation period when a future pensioner is deliberate about the investment return certainty rather than a highly volatile investment promising markable outperform of given benchmark – as there is only a little time to wipe off possible heavy losses associated with such risky investment. On the other side, the latter option is more attractive when a saver needs to increase the portfolio wealth despite the existence of significant uncertainty of payoffs that accompanies expected higher returns – in the early periods of his/her active life there is still enough time to diminish occurred losses.

Evidently, any government restriction placed on the future pensioner investment strategies (e.g. lower and upper limitations on weights of securities with particular risk profiles) should take into consideration these properties.

In case of two–dimensional problem with the investment portfolio consisting of one stock and one bond and non–binding ban on short position constraint, under the assumptions of stable financial market, the share of stock in the portfolio declines as time approaches retirement date and drops with even accelerating speed with wealth already allocated:

$$\frac{\partial \widetilde{\theta}^{(s)}}{\partial y}(t,y) < 0\,, \qquad \frac{\partial^2 \widetilde{\theta}^{(s)}}{\partial y^2}(t,y) > 0\,, \qquad \frac{\partial \widetilde{\theta}^{(s)}}{\partial t}(t,y) < 0\,, \qquad \forall (t,y) \in \Omega_2^*\,.$$

Policy Implications & Recommendations: Do not prescribe any investment regulations forcing to raise the proportion of more risky financial instrument in the portfolio as time approaches retirement age or the wealth allocated on a saver's pension account increases.

A future pensioner is advised to be more aggressive in his/her investment decision in the beginning of the active life and as time approaches the planned retirement age and the amount of allocated wealth on his/her pension account raises, decline gradually the share of investment in risky assets while moving towards more safe financial market instruments. Hence, a typical saver should start with risky stocks (or stock indices) and then in very last years before retirement switch to highly rated bonds.

6.6.2 Influence of the contribution rate on the optimal unconstrained policy

First we consider the dependence of the optimal policy sub–function ω on the small parameter $0 < \varepsilon \ll 1$ representing the saver's contribution rate, i.e. the percentage of the transfer of yearly salary to pensioner's account. It follows from

$$\frac{\partial \omega}{\partial \varepsilon}(s,x) = \Phi_\varepsilon(s,x)e^{-x} > 0\,, \qquad (s,x) \in \Omega \tag{6.67}$$

that ω monotonically increases with ε (and vice–versa, ζ drops with ε).

Furthermore, in case of the two dimensional problem, recalling (6.66) we deduce that the optimal value $\widetilde{\theta}^{(s)}$ is an increasing function in the contribution rate ε when restrained on Ω_2^*. Taking into account the possible application in the dynamic accumulation pension saving model, we can conclude that the higher percentage ε of salary transferred each year to a pension fund would lead to higher optimal stock–to–bond proportion $\widetilde{\theta}^{(s)}$ and thus bring in much higher expected terminal return $\mathbb{E}[y_T]$. Therefore evidently, in order to heighten the expected future payoffs from the Second pillar of the Slovak pension system, raising the saver's contribution rate ε is a possible way how achieve it.

Next, assume that the percentage ε represents investor's net contributing ratio, i.e.

$$\varepsilon = (1-\kappa_\varepsilon)\widetilde{\varepsilon}$$

where κ_ε are managing costs, a regular fee charged by the the pension fund management institutions administering investor's private pension account and $\widetilde{\varepsilon}$ stands for the gross salary ratio of the financial transfer. Then evidently,

$$\frac{\partial \omega}{\partial \kappa_\varepsilon}(s,x) = \widetilde{\varepsilon}\Phi_\varepsilon(s,x)e^{-x} < 0 \tag{6.68}$$

and so $\partial\widetilde{\theta}^{(s)}/\partial\kappa_\varepsilon < 0$ which means that the increase in managing costs implies decrease in the unconstrained optimal stock share in the portfolio $\widetilde{\theta}^{(s)}$, as expected, and hence induces the decline in the expected terminal wealth allocated to a saver's pension account.

Alternatively, being below the optimal proportion of volatile stocks (accompanied with much smaller risk exposition) along with higher contribution rate ε can still induce the same portfolio terminal utility as the comparable portfolio with higher share of risky stocks and lower contribution rate.

In case of two–dimensional problem with the investment portfolio consisting of one stock and one bond and non–binding ban on short position constraint, under the assumptions of stable financial market, the share of stock in the portfolio raises with regular contribution rate ε and descends with managing fees charged by the PAMC:

$$\frac{\partial\widetilde{\theta}^{(s)}}{\partial\varepsilon}(t,y) > 0\,, \qquad \frac{\partial^2\widetilde{\theta}^{(s)}}{\partial\kappa_\varepsilon}(t,y) < 0\,, \qquad \forall (t,y) \in \Omega_2^*\,.$$

Policy Implications & Recommendations: In order to augment the expected future payoffs from the Second pillar of the Slovak pension system, raising the saver's contribution rate ε and/or decline managing fees κ_ε are possible ways to achieve it.

6.6.3 Impact of macro parameters of the unconstrained optimal policy

In the forthcoming text we scrutinize how the model macro parameters – the gross wage growth rate β and the length of the accumulation period T – affect to unconstrained optimal investment strategy $\widetilde{\theta}^{(s)}$.

Effects of changes in the gross wage growth rate on the super–optimal policy

Firstly observe that

$$\frac{\partial \Phi_\varepsilon}{\partial \delta}(s,x) = -d\frac{e^{-\delta s}}{\delta^2}\left[e^{\delta s} - 1 - \delta s\right] < 0\,, \qquad \text{and} \qquad \frac{\partial \delta}{\partial \beta} = -1$$

so that $\partial_\beta \Phi_\varepsilon > 0$ for any $(s,x) \in \Omega$. Similarly, as far as

$$\frac{\partial \widetilde{\delta}}{\partial \delta} = -(d-1) < 0\,, \qquad \text{and} \qquad \frac{\partial \widetilde{\phi}}{\partial \delta} = 4a\left[1+\frac{1}{\psi_0^2}\right]^{-1}\left[(d-2) - 2a\delta\left[1+\frac{1}{\psi_0^2}\right]^{-1}\right]^{-2} > 0\,,$$

we may deduce that $\partial_\beta \widetilde{\delta} = d-1 > 0$ and $\partial_\beta \widetilde{\phi} < 0$.
Then, $\partial_\beta \Phi_\lambda > 0$ as $(d-1)(1+\widetilde{\phi}) > \partial_\delta \widetilde{\phi}$. Therefore, combination of the foregoing results induces in the following:

$$\frac{\partial \omega}{\partial \beta} = \varepsilon\frac{\partial \Phi_\varepsilon}{\partial \beta}e^{-x} + \lambda\frac{\partial \Phi_\lambda}{\partial \beta}e^{-(d-1)x} > 0\,, \tag{6.69}$$

and so ω increases monotonously with β.

In case of the two dimensional model we conclude that the optimal proportion of stock investment $\widetilde{\theta}^{(s)}$ is also an increasing function with respect to the wage growth β when restricted on region Ω_2^*. This is an expected result as the higher acceleration in wage growth pushes to invest into assets with higher returns to keep the level of the *wealth–to–salary* ratio y since the terminal portfolio wealth will be measured with respect to much higher salary.

Simultaneously, owing to faster growing gross wage, the size of regular contributions also increases and hence allows to allocate more wealth on saver's private pension account.

So, providing that a saver's gross wage growth rate increased he/she should follow more dynamic investment strategy in order to preserve the pre–retirement living standard during his/her post–productive phase of live.

Influence of movements in pension age on the unconstrained optimal policy

Pension age defines the end of the time–horizon of length T considered for the accumulation period. Thus, in terms of policy implications, the length of accumulation period is equivalent to retirement age.
Recalling the relationship between s and T, namely $s = T - t$, our observations on behaviour

of $\zeta = \zeta(s,x)$ with respect to changes occurring in s can be reused for the movement of pension age T and the resulting effects on ζ. Therefore, as ζ declines monotonically in s it does so in T. Obviously, this behaviour is reversed for ω,

$$\frac{\partial \omega}{\partial T}(s,x) > 0\,, \qquad (s,x) \in \Omega\,. \tag{6.70}$$

In case of the two dimensional model for one representative stock and the one for bonds retirement age shifting forward causes increase in the unconstrained optimal stock proportion in the portfolio. This is an intuitive scenario as the return volatility accompanying stocks is spread over time while the portfolio value is expected to raise above the one with lower share of stocks.

Hence a more aggressive investment strategy is allowed as there is more time to wipe off possible losses associated with risky investment – therefore retirement age delay brings in higher expected terminal payoffs (cash flow from the private Second pillar of the Slovak pension system) for the investor.

Alternatively, being below the optimal proportion of volatile stocks along with retirement age prolongation can still induce the same portfolio terminal utility as the comparable portfolio with higher share of risky stocks and shorter accumulation period.

In case of two–dimensional problem with the investment portfolio consisting of one stock and one bond and non–binding ban on short position constraint, under the assumptions of stable financial market, the share of stock in the portfolio augments with both the length of the accumulation period in investment (retirement age) T and saver's gross wage growth rate β:

$$\frac{\partial \widetilde{\theta}^{(s)}}{\partial T}(t,y) > 0\,, \qquad \frac{\partial^2 \widetilde{\theta}^{(s)}}{\partial \beta}(t,y) > 0\,, \qquad \forall (t,y) \in \Omega_2^*\,.$$

Policy Implications & Recommendations: Elevate the retirement age to increase the expected future payoffs from the Second pillar of the Slovak pension system.
A typical saver whose gross wage growth rate increased should follow more dynamic investment strategy.

6.6.4 Micro–parameters and the optimal unconstrained investment policy

In the subsequent text we aim our attention on micro model parameters, namely a typical saver risk aversion relative coefficient d and the small portfolio volatility sensitivity parameter λ.

We will investigate their effects on the unconstrained optimal weight of stock in the investment portfolio.

Effect of the risk aversion related coefficient d

Apparently, in case that $\zeta(s,x)$ is given by (6.41) one may easily determine that the sign of the derivative of ω with respect to d coincides with the sign of

$$-d-\varepsilon e^{-x}\left[2\Phi_\varepsilon(s)-\Phi'_\varepsilon(s)\right]-\lambda e^{-(d-1)x}\left[3\left(\Phi_\lambda(s)-1\right)-\Phi'_\lambda(s)\right] .$$

Let us scrutinize the sign of $\omega_\varepsilon(s)\equiv 2\Phi_\varepsilon(s)-\Phi'_\varepsilon(s)$. Evidently, ω_ε is monotonously increasing and providing that we restrict our choice of d such that $d(2-1/(a\delta))\geq 1$, its minimum attained for $s=0$ as well as ω_ε itself are non–negative. The behaviour of $\omega_\lambda(s)\equiv 3(\Phi_\lambda(s)-1)-\Phi'_\lambda(s)$ can be uncovered in a similar way. Truly, ω_λ raises monotonously and its minimum is positive and so it holds for ω_λ. Therefore we claim that ω (the inverse of ζ) declines monotonously with $d\gg 1$. The similar result holds for the case of $N=2$ as ω and $\widetilde{\theta}^{(s)}$ move in the opposite direction on Ω_2^*. In other words, higher risk aversion leads to less amount of stocks in saver's portfolio, as expected. This is fully in accordance with observed recommendations about investment – indeed, highly risk averse investor following a motto *a bird in hand is worth two in the bush* is advised to allocate his/her wealth in securities with low volatility of returns and be better off with lower but more certain payoffs.

Dependence of the optimal unconstrained policy on risk sensitivity parameter λ

Now we observe the impact of the small parameter $0<\lambda\ll 1$ on the optimal policy sub–function ζ^{-1}. Remark that λ symbolises the saver's aversion against volatility in the portfolio returns and amplifies the negative effect of portfolio return variance on the overall utility as measured via utility criterion $\mathscr{K}$. Hence we may easily deduce that

$$\frac{\partial\omega}{\partial\lambda}(s,x)=-\Phi_\lambda(s,x)e^{-(d-1)x}<0\,,\qquad (s,x)\in\Omega \tag{6.71}$$

and so ω monotonically increases with λ. Therefore, in case of two dimensional model, increase in λ is accompanied with growing optimal stock investment $\widetilde{\theta}^{(s)}$.

In case of two–dimensional problem with the investment portfolio consisting of one stock and one bond and non–binding ban on short position constraint, under the assumptions of stable financial market, the share of stock in the portfolio increases with the small portfolio volatility sensitivity parameter λ and drops with the Arrow–Pratt related risk aversion parameter d:

$$\frac{\partial\widetilde{\theta}^{(s)}}{\partial\lambda}(t,y)>0\,,\qquad \frac{\partial^2\widetilde{\theta}^{(s)}}{\partial d}(t,y)<0\,,\qquad \forall(t,y)\in\Omega_2^*\,.$$

Policy Implications & Recommendations: Relax regulations and extend investment opportunities to create a large spectrum of portfolios with various risk profiles.
One strategy does not fits all – let saver to choose the investment strategy consider carefully his/her risk aversion attitude, so that very risk–aware investor should choose more conservative investment strategy with higher share of bonds in the investment portfolio.

We summarize the results obtained for the functions ζ and its inverse ω below:

Proposition 10. *The function $\zeta = \zeta(T-t, \ln y)$ defined by* (6.41) *on Ω exhibits the subsequent properties:*

- *raises monotonically in both wealth–to–salary y and time t and it is strictly convex in y,*
- *increases in risk aversion coefficient d,*
- *declines in both small model parameters: contribution rate ε and return volatility sensitivity parameter λ,*
- *descends in both gross wage growth rate β and retirement age T.*

The opposite statements hold for ω the first order approximation of ζ^{-1} taken with respect to both small parameters ε and λ.

6.6.5 Transmission of the financial market turbulences on the optimal unconstrained policy

In case of the two dimensional problem we may proceed further and provide a full analysis of the optimal stock proportion $\widetilde{\theta}^{(s)}$ behaviour from the prospective of the financial market turbulences. We restrain our analysis on the region Ω_2^* where the optimal constrained and optimal unconstrained policies coincide.

Firstly let us remind you about that

$$\frac{\partial \Phi_\varepsilon}{\partial \delta}(s,x) = -d\frac{e^{-\delta s}}{\delta^2}\left[e^{\delta s} - 1 - \delta s\right] < 0\,, \qquad \frac{\partial \widetilde{\delta}}{\partial \delta} = -(d-1) < 0\,,$$
$$\frac{\partial \widetilde{\phi}}{\partial \delta} = 4a\left[1+\frac{1}{\psi_0^2}\right]^{-1}\left[(d-2) - 2a\delta\left[1+\frac{1}{\psi_0^2}\right]^{-1}\right]^{-2} < 0\,,$$

and so we may deduce that $\partial_\delta \Phi_\lambda > 0$ as $(d-1)(1+\widetilde{\phi}) > \partial_\delta \widetilde{\phi}$. Next, observe that the impact of changes in stock returns on δ is positive in case of negative correlated returns of stocks and bonds as

$$\frac{\partial \delta}{\partial \mu^{(s)}} = \frac{1}{a}\frac{\partial b}{\partial \mu^{(s)}} = \frac{\sigma^{(b)}[\sigma^{(b)} - \rho\sigma^{(s)}]}{a} > 0\,.$$

Therefore, ω raises due to increase in stock returns $\mu^{(b)}$ and as neither α_σ nor β_σ are affected by any change in asset returns, recalling the definition of the constrained optimal stock allocation strategy (6.42) we claim that the weight of stocks in the portfolio $\widetilde{\theta}^{(s)}$ augments with their expected returns $\mu^{(s)}$.

Obviously it works in the opposite direction for changes in bond returns – any improvement in bond returns declines stock share in the portfolio as it is optimal to put more into bonds, so it holds the subsequent:

$$\frac{\partial \widetilde{\theta}^{(s)}}{\partial \mu^{(s)}}(t,y) > 0\,, \qquad \frac{\partial \widetilde{\theta}^{(s)}}{\partial \sigma^{(b)}}(t,y) < 0\,, \qquad (t,y) \in \Omega_2^*\,. \tag{6.72}$$

Next, let us discuss the effect of variation in stock returns volatility on their optimal weight in the pension fund portfolio. Using the results above, notice that

$$\frac{\partial \delta}{\partial \sigma^{(s)}} = \frac{\partial}{\partial \sigma^{(s)}} \left(\frac{b}{a}\right) < 0\,.$$

Furthermore,

$$\frac{\partial}{\partial \sigma^{(s)}} \left(\frac{\beta_\sigma}{\alpha_\sigma}\right) < -\frac{\sigma^{(b)}}{\alpha_\sigma^2} \left[-\rho \left[(\sigma^{(s)})^2 + (\sigma^{(b)})^2\right] + 2\sigma^{(s)}\sigma^{(b)}(1-\rho)^2\right], \tag{6.73}$$

under the assumption of negatively correlated returns of assets the relationship above is negative. Therefore, as $\partial \alpha_\sigma > 0$ and stock returns are assumed to surpass bond returns, we deduce that increase in stock returns volatility causes decline of their share in the portfolio.

Evidently, the reverse must be true for the affect of higher bond returns volatility on the optimal proportion of stocks in the investment portfolio on Ω_2^*. Hence,

$$\frac{\partial \widetilde{\theta}^{(s)}}{\partial \sigma^{(s)}}(t,y) < 0\,, \qquad \text{and} \qquad \frac{\partial \widetilde{\theta}^{(s)}}{\partial \sigma^{(b)}}(t,y) < 0\,, \qquad (t,y) \in \Omega_2^*\,. \tag{6.74}$$

Finally we pay attention to the influence of movements in the correlation between stocks and bonds on the optimal stock weight in the pension fund portfolio. Firstly, observe that under stable financial market assumptions

$$\frac{\partial}{\partial \rho}\left[\frac{\beta_\sigma}{\alpha_\sigma}\right] = -\frac{\sigma^{(b)}\sigma^{(s)}}{\alpha_\sigma}\left[(\sigma^{(s)})^2 - (\sigma^{(b)})^2\right] < 0\,.$$

On the other side,

$$\frac{\partial \delta}{\partial \rho} = \frac{\partial}{\partial \rho}\left[\frac{b}{a}\right] = \frac{\sigma^{(b)}\sigma^{(s)}\left(\mu^{(s)} + \mu^{(b)}\right)\left((\sigma^{(s)})^2 + (\sigma^{(b)})^2\right)}{a^2} > 0\,.$$

Therefore, ζ raises with ρ and so the unconstrained optimal stock weight $\widetilde{\theta}^{(s)}$ in the investment portfolio declines as the coefficient of correlation augments. Thus an increase in the tendency of stock and bond returns co-movements affect the descend the gap between their weights in the portfolio,

$$\frac{\partial \widetilde{\theta}^{(s)}}{\partial \rho}(t,y) < 0\,, \qquad (s,x) \in \Omega_2^*\,. \tag{6.75}$$

In case of two–dimensional problem with the investment portfolio consisting of one stock and one bond and non–binding ban on short position constraint, under the assumptions of stable financial market, the share of stock in the portfolio heighten with stock return $\mu^{(s)}$ and is brought down with both the stock return volatility $\sigma^{(s)}$ and the coefficient of correlation between stock and bond, ρ:

$$\frac{\partial \widetilde{\theta}^{(s)}}{\partial \mu^{(s)}}(t,y) > 0\,, \qquad \frac{\partial \widetilde{\theta}^{(s)}}{\partial \sigma^{(s)}}(t,y) < 0\,, \qquad \frac{\partial \widetilde{\theta}^{(s)}}{\partial \rho}(t,y) < 0\,, \qquad \forall (t,y) \in \Omega_2^*\,.$$

Policy Implications & Recommendations: Active portfolio management is crucial.

Proposition 11. *The optimal unconstrained stock–to–bond proportion* $\widetilde{\theta}^{(s)}$ *defined by* (6.64) *on* Ω_2^* *exhibits the subsequent properties:*

- *falls monotonically in both wealth–to–salary* y *and time* t *and it is strictly convex in* y*,*
- *descends in risk aversion coefficient* d*,*
- *raises in both small model parameters: contribution rate* ε *and return volatility sensitivity parameter* λ*,*
- *augments in both gross wage growth rate* β *and retirement age* T*,*
- *enlarges in stock returns* $\mu^{(s)}$ *and drops in bond returns* $\mu^{(b)}$*,*
- *decreases in both stock returns volatility* $\sigma^{(s)}$ *and the coefficient of correlation between the returns of stocks and bonds,* ρ*, while grows with bond returns volatility* $\sigma^{(b)}$ *.*

Chapter 7

APPLICATIONS AND RESULTS

The subsequent passages are dedicated to the concrete applications of our derived model – we apply the model on investor's decision taking problem of optimal resource allocation in various financial instruments represented by several pension funds and optimal fund composition.

Actually, our research was originally motivated by the pension system of Slovak Republic, composing of three complementary coexisting pension pillars. The traditional first, gradually downsized *pay–as–you–go* philosophy based public, unfunded and mandatory pillar represents a state–guaranteed pension insurance performed by the Social Insurance Company.
The mandatory second pillar, commercially supervised by the pension funds management companies, is fully funded from saver's regular contributions and introduces an alternative to save for a pension on an private pension account. Financial resources accumulated on the pension account possesses the ability of value appraising via subsidization allocation into the forthcoming predefined investment funds:

1. *Bond Fund*: investment strategies are restrained to highly rated short-term bonds and money instruments;
2. *Balanced Fund*: the portfolio is limited to be composed of at least 50% of bonds and money investments, up to 50% of stocks and up to 20% of precious metal investment instruments;
3. *Stock Fund*: the portfolio is formed by stocks (at most 80%), precious metal investments (not more than 20%) and up to 80% of the fund property by bonds and money investment instruments;
4. *Equity-Linked Index Fund*: benchmark of this passively managed fund tracks the performance of one or more selected equity indexes and there are no restrictions on exchange traded funds, assets or derivatives when replicating the benchmark formed initially.

The investment decision of the saver already registered in one of the pension fund management companies is made by selecting at most two of the funds mentioned above - providing

that two funds are chosen, one of them must be Bond Fund. On the other side, each pension fund management company as a part of their investment decision specifies a benchmark for each of the investments fund except the Bond Fund that would satisfy the prescribed restrictions imposed by government. It is evident that the pension fund management company implements their investment decision by constituting such portfolios that would copy or outperform in their return the corresponding benchmark - otherwise the fees charged on savers for management services provided by the company are lowered by the law.
The typical sign of the voluntary third pillar resides in its supplementary pension accounts financed by means of saver's regular transfers and managed by the supplementary pension companies. Similarly to the second pillar, the saver's resource allocation strategy (made by choosing one of given investment third-pillar funds) results in their pension account evaluation. For more detailed information the reader is advised to see Macová and Ševčovič [44], Macová [43], Múčka [51], Kilianová [36] or Kilianová et al. [37].

7.1 Problem Formulation

Considering the research motivation, our work is devoted to the problem of the second pillar investment decision. Even through these four funds are strictly predetermined in terms of their risky profiles and the concrete fund choice is curtailed by many factors, we pose the strategy decision questions in a different, more fundamental manner:

Q1 *If there was only one risk profile unrestricted investment fund what should be the optimal resource allocation strategy peculiar to a typical future pensioner of a certain age and wealth, that would ceteris paribus possibly generate the maximal income at retirement date, providing that the proportions of bonds and stocks in such investment portfolio are unrestrained. And how the optimal fund risk profile for such investor should be.*

Q2 *Given a typical future pensioner investing in a specific pension fund, what would be the optimal portfolio benchmark when compared with the fund-prescribed portfolio benchmark. We assume government limitations imposed on the fund investment strategies and possibility of dynamical portfolio construction.*

These two problems will be dealt deeply in the forthcoming text.

7.2 One-Stock-One-Bond Problem

The goal of first application presented in this paper is to establish the saver's optimal strategy in pension fund selection, conditioned primarily by their time to retirement, intermediate *wealth-to-salary ratio* and various model parameters.

Technically, our aim is to evaluate the approximative optimal investment strategy $\widetilde{\theta}^{(s)} = \widetilde{\theta}^{(s)}(t,y)$ introduced in sense of (6.42) and (6.63) in the forthcoming manner:

$$\widetilde{\theta}^{(s)}(t,y) = \begin{cases} \dfrac{1}{\alpha_\sigma}\left[\beta_\sigma + \dfrac{\Delta\mu}{\zeta(T-t,\ln y)}\right], & \zeta(T-t,\ln y) \in \Omega_2^*, \\ 1 & 0 < \zeta(T-t,\ln y), \quad \zeta(T-t,\ln y) \notin \Omega_2^*, \end{cases} \tag{7.1}$$

under the assumption of $\alpha_\sigma > \beta_\sigma$ for the constants

$$\alpha_\sigma = \left[\sigma^{(s)}\right]^2 - 2\rho\sigma^{(s)}\sigma^{(b)} + \left[\sigma^{(b)}\right]^2, \qquad \beta_\sigma = \sigma^{(b)}\left[\sigma^{(b)} - \rho\sigma^{(s)}\right].$$

and the region on which $\zeta = \zeta(s,x)$ follows the prescription (6.43),

$$\Omega_2^* \equiv \left\{(s,x) \in [0,T]\times\mathbb{R},\ \varepsilon\Phi_\varepsilon(s)e^{-x} + \lambda\left[\Phi_\lambda(s)+1\right]e^{-(d-1)x} < d - \frac{\Delta\mu}{\alpha_\sigma - \beta_\sigma}\right\}. \tag{7.2}$$

Furthermore, instead of ζ^{-1} we consider its first order approximation $\omega \approx \zeta^{-1}$ defined as follows:

$$\begin{aligned}\zeta^{-1}(s,x) \approx \omega(s,x) &\equiv \frac{1}{d} + \varepsilon\frac{\Phi_\varepsilon(s)}{d^2}e^{-x} + \lambda\frac{\Phi_\lambda(s)+1}{d^2}e^{-(d-1)x} \\ &= \frac{1}{d}\left\{1 + \varepsilon\frac{1-e^{-\delta s}}{\delta}e^{-x} + \frac{\lambda}{d}\left[(d-1)\left((1+\widetilde{\phi})e^{\widetilde{\delta}s} - \widetilde{\phi}\right)+1\right]e^{-(d-1)x}\right\},\end{aligned} \tag{7.3}$$

and (s,x) is subject to change of variables, $s = T-t$ and $x = \ln y$.

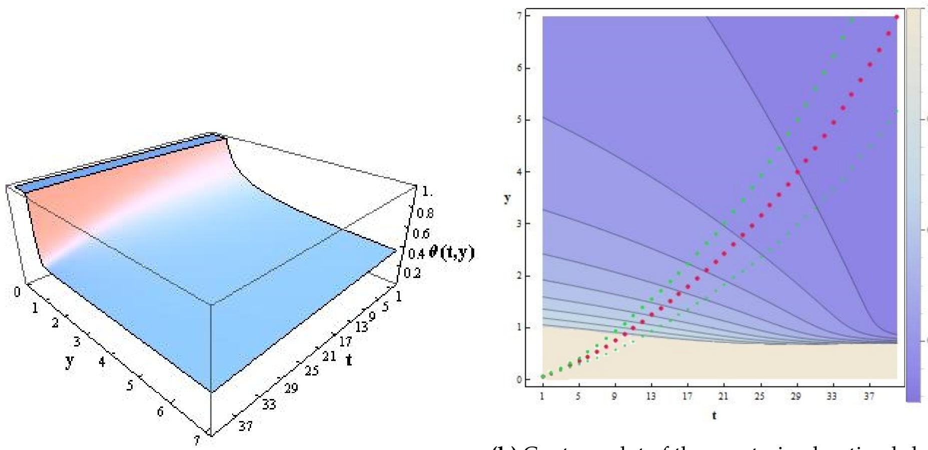
Referring to Slovak pension system presented earlier in this chapter, from the saver's point of view the investment decision essence lies in detecting the best fitting ratio between resources allocated to the *Equity–Linked Index Fund* (symbolizes stocks) and the *Bond Fund* (depicts bonds). In this sense, even the *Equity–Linked Index Fund* allocation strategy applied when replicating the performance of the benchmark prescribed by the pension fund management, is unlimited in the choice of stocks, financial derivatives or exchange traded funds, for the sake of simplicity we assume the fund investment decision is restricted in stocks only. Furthermore for the same intention we presuppose normally distributed returns of both funds thought it might be more convenient to employ the Cox–Ingersoll–Ross model to design the *Bond Fund* returns (see e.g. Cox et al. [19], Kwok [42] or Shreve [63]). Similarly, based on empirical observations, usage of the Normal Inverse Gaussian distribution (c.f. Andersen et al. [2], Barndorff-Nielsen [4], Corrado and Su [15], Onalan [58]) or the Heston model concept of stochastic volatility (see Cizeau et al. [14], Stein and Stein [67]) to model the *Equity–Linked Index Fund* returns is more suitable. Applying non-normal distribution when modelling assets returns is one of the objectives of our future research.
Moreover we remark that in this problem there exists an obvious restriction on the stocks and bonds proportions - naturally, both ratios must be non-negative, so that no short–selling is allowed.

The forthcoming text is devoted to the original problem restrained to *one-stock-one-bond* case, so with two opposite behaving assets with normally distributed returns. Even though it may seem to be too restrictive and simplified, this assumption admits us to understand better the value function nature, optimal *stock-to-bond* investment strategy and their dependence on various model parameters.

7.2.1 Model parameters calibration

We have tested the proposed model on the second pillar of the Slovak pension system. According to recently changed Slovak legislature, in September 2012 the regular contribution level of a private scheme participant dropped from their original value of 9% to 4% of his/her gross wage. This rate prescription is valid until 2017 and then gradually raises by 0.25 p.p. such that in 2024 it attains the value of 6%. Hence in the baseline scenario we set $\varepsilon = 0.06$. As ε plays the key role not only in this model, but in its actual application to Slovak pension system, we have tested several levels of ε to scrutinize the model outcomes for various ε values and study how its value affect both the portfolio component weights and the expected terminal *wealth–to–salary* payoffs. For the comparison purpose we consider the option of permanent decline in this rate to $\varepsilon = 0.04$ and also the alternative *2012 - no policy change* eventuality, i.e. $\varepsilon = 0.09$. Therefore ε can be considered as a small parameter. Moreover, since each private asset management company charges fund management fees defined as 1% of an investor's contribution, within our model we use the effective contribution rate in all scenarios.

We have assumed the overall time period $T = 40$ of saving of an individual pensioner and the value of their risk aversion attitude coefficient was estimated on 0.04, i.e. $\lambda = 0.04$. The recent data collected by the Slovak Statistical Office in the period 2008–2013 establish the average gross wage growth rate (annualized and seasonally adjusted quarterly based time series) on 2.76 we adopted the expert judgement taken from the Slovak Institute of Financial Policy macroeconomic forecast (see Ministry of Finance of the Slovak Republic [50]) and estimated (in average value) it as for 3.5% p.a., i.e. $\beta = 0.035$. Regarding market data, we pay



(a) 3D plot of the constrained optimal share of MSCI All Country World Index (2009–2012 data)

(b) Contour plot of the constrained optimal share of MSCI All Country World Index (2009–2012 data)

Figure 7.1: 3D plot and Contour plot of the constrained optimal share of MSCI World index in the portfolio of 10-Year Slovak Government Bonds and MSCI All Country World Index, based on data between 2009-2012

attention to the recent time periods: 2009–2012 and 2003–2012. Next, the investment portfolio consists of two securities: 10−Year zero coupon Slovak Government Bonds and *MSCI All Country World Index*. Our choice of these financial assets comes from real composition of

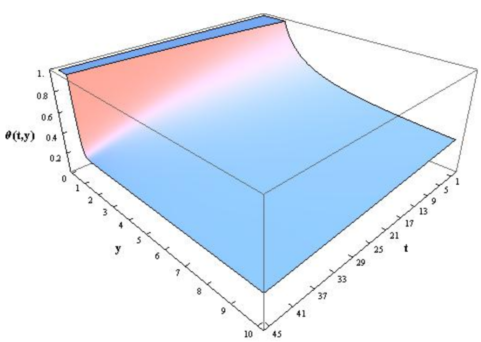
Asset	2009-2012 mean	2009-2012 st.deviation	2009-2012 correlation	2003-2012 mean	2003-2012 st.deviation	2003-2012 correlation
MSCI World	0.1053	0.1423	-0.8344	0.0763	0.2050	-0.1688
10-Y Slovak Bonds	0.0439	0.0036		0.0447	0.0047	
DAX	0.1328	0.1738	-0.1727	0.1286	0.2270	-0.0951
10-Y German Bunds	0.02552	0.0063		0.0335	0.0085	
S&P500	0.1242	0.0964	-0.1518	0.0667	0.1799	-0.0535
10-Y US treasuries	0.0276	0.0068		0.0367	0.0092	

Source: Bloomberg, MSCI, ECB, EuroStat, US Treasury

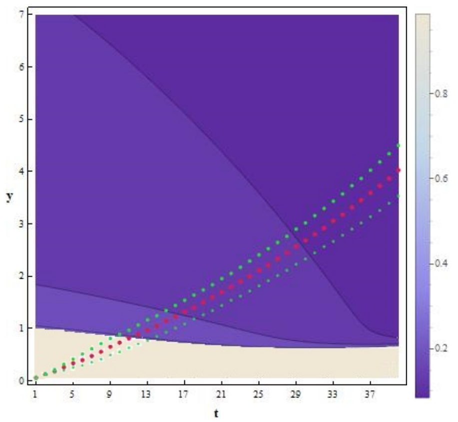
Table 7.1: Descriptive statistics of selected market data observed in periods 2009–2012 and 2003–2012

pension funds in Slovakia. For the comparison purpose, we provide another two pair of investment options, namely 10-Year US Treasury Bonds versus S&P500 index, and 10-Year German Bunds versus DAX index.

Within the period 2009–2012, the MSCI All Country World Index representing stocks yielded the average return $\mu^{(s)} = 0.1053$ with the standard deviation achieving $\sigma^{(s)} = 0.1423$, whereas assuming the longer period (2003–2012), the average stock return rapidly drops to the level of $\mu^{(s)} = 0.0763$ and the deviation raised at $\sigma^{(s)} = 0.2050$. Likewise, between 2009–2012 (2003–2012) DAX index exhibited returns of $\mu^{(s)} = 0.1328$ ($\mu^{(s)} = 0.1286$) with volatilities $\sigma^{(s)} = 0.1738$ ($\sigma^{(s)} = 0.2270$). Finally, on the shorter (longer) time period S&P500 brought returns at level $\mu^{(s)} = 0.1242$ ($\mu^{(s)} = 0.0667$) accompanied with volatilities $\sigma^{(s)} = 0.0964$ ($\sigma^{(s)} = 0.1799$). Notice that the statistics for the MSCI All Country World Index, and DAX index were taken from Bloomberg official web page S&P500 index data have been borrowed from S&P500 index official web page.



(a) 3D plot of the constrained optimal share of MSCI All Country World Index (2003–2012 data)

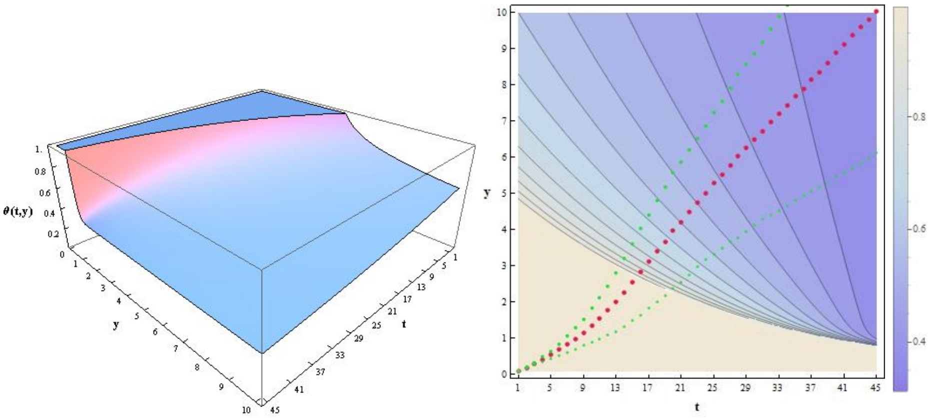


(b) Contour plot of the constrained optimal share of MSCI All Country World Index (2003–2012 data)

Figure 7.2: 3D plot and Contour plot of the constrained optimal share of MSCI World index in the portfolio of 10-Year Slovak Government Bonds and MSCI All Country World Index, based on data between 2003-2012

As the modelling of bond returns is concerned within the baseline scenario we have considered the 10–Year zero coupon Slovak Government Bond on both time horizons studied, i.e. 2009–2012 and 2003–2012. In the recent time period Slovak bond yields hit $\mu^{(b)} = 0.0439$ with volatility $\sigma^{(b)} = 0.0036$ whereas the average yield and associated volatility in the distant period is characterized by similar data, $\mu^{(b)} = 0.0447$ and $\sigma^{(b)} = 0.0047$. Data were taken from Eurostat official web site and Slovak Debt and Liquidity Management Agency (ARDAL) official web site.

The alternative strategies model their conservative element, bond investment in terms of 10–Year zero coupon German government Bund yielded $\mu^{(b)} = 0.0252$ between 2009–2012 and $\mu^{(b)} = 0.0335$ with volatilities $\sigma^{(b)} = 0.0073$ and $\sigma^{(b)} = 0.0086$, respectively. Data for German interest rates were taken from Eurostat web page and ECB official web site. Ultimately, the last investment strategy assumed 10-Year US Treasury Bonds taken the same time periods 2009–2012 and 2003–2012. Parameters of bond returns μ_b and their volatilities σ_b are available on US Treasury Department official web page. For the shorter time period (2009–2012) we considered the average bond yield $\mu^{(b)} = 0.0276$ with the standard deviation $\sigma^{(b)} = 0.0068$ p.a while during the longer period (2003–2012) bonds exhibit the higher average yield of $\mu^{(b)} = 0.0367$ p.a. with the higher standard deviation $\sigma^{(b)} = 0.0093$. The descriptive statistics



(a) 3D plot of the constrained optimal share of DAX Index (2009–2012)

(b) Contour plot of the constrained optimal share of DAX Index (2009–2012)

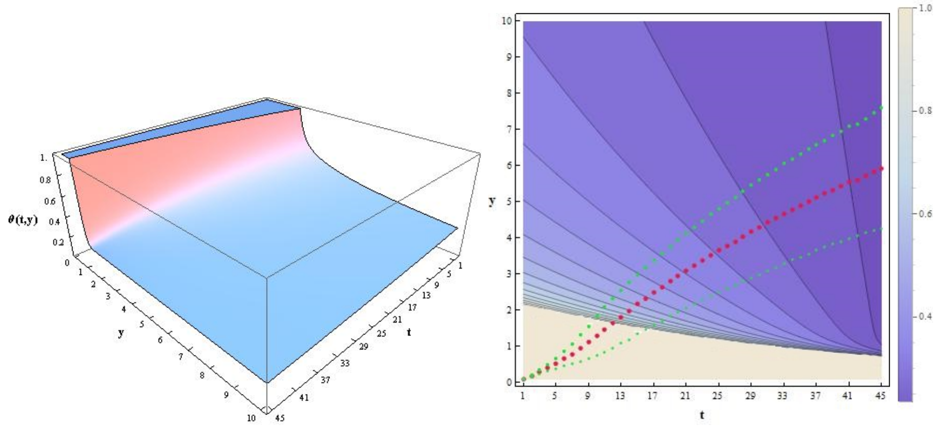
Figure 7.3: 3D plot and Contour plot of the constrained optimal share of DAX Index in the portfolio of 10-Year German Bunds and DAX World Index, based on data between 2009-2012

obtained are summarized in Table 7.1.

7.2.2 Results and Discussion

On Figures 7.1–7.2 we present the 3D plots as well as the contour plots of the constrained optimal share of assets (represented by the MSCI World index) in the pension fund portfolio consisting of 10-Year zero coupon Slovak Government Bonds and MSCI World index calculated based on financial market data in time periods 2009–2012 and 2003–2012, respectively. This constrained optimal share $\widetilde{\theta}^{(s)}$ is modelled as a function of time $t \in [0, T]$ and *wealth–to–salary ratio y*.

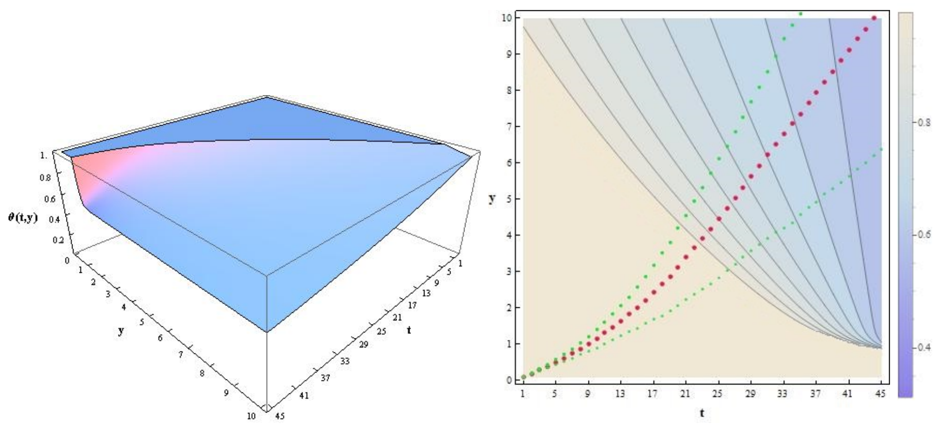
In order to compute $\widetilde{\theta}^{(s)}$ we used (7.1) with the double first order expansion (7.3) performed with respect to both small parameters – saver's contribution rate ε, and volatility sensitiv-



(a) 3D plot of the constrained optimal share of DAX Index (2003–2012) **(b)** Contour plot of the constrained optimal share of DAX Index (2003–2012)

Figure 7.4: 3D plot and Contour plot of the constrained optimal share of DAX Index in the portfolio of 10-Year German Bunds and DAX World Index, based on data between 2003–2012

ity λ and apply ether period 2009–2012 (see Figure 7.1) or 2003–2012 (Figure 7.2) financial market data. Furthermore, within this baseline scenario we considered the Arrow–Pratt risk aversion related coefficient $d = 10$ and the gross wage rate of growth $\beta = 0.035$. Both small parameters (λ and the contribution rate ε) were set at level 0.04. The optimal investment strategy is constrained as the share of assets cannot exceeds 100% since borrowings are forbidden.



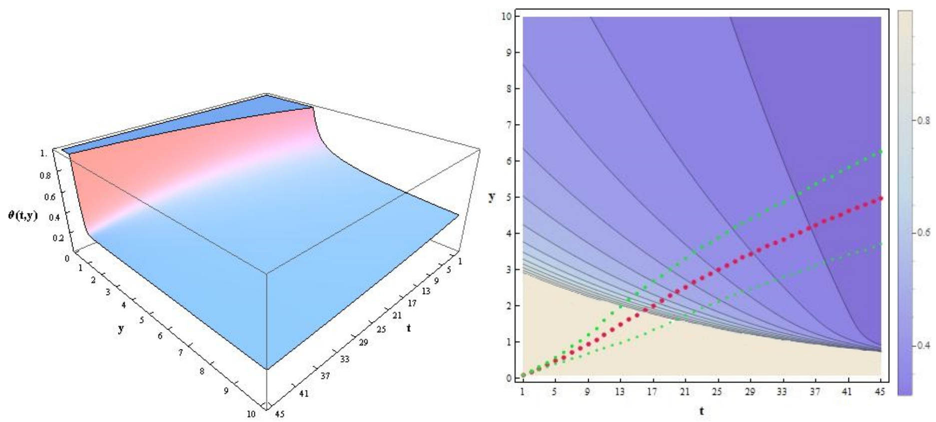
(a) 3D plot of the constrained optimal share of S&P500 Index (2009–2012) **(b)** Contour plot of the constrained optimal share of S&P500 Index (2009–2012)

Figure 7.5: 3D plot and Contour plot of the constrained optimal share of S&P500 Index in the portfolio of 10-Year US Treasuries and S&P500 Index, based on data between 2009-2012

On both contour plots (Figure 7.1b and Figure 7.2b) the mean portfolio wealth $\mathbb{E}[y_t]$ (red dot line) is obtained by performing 10000 Monte-Carlo simulations of random paths $\{y_t\}_{t=1}^{T}$ calculated according to the recurrent equation (4.8)–(4.9) with one year period ($\tau = 1$), so employing random variable $Z \sim \mathcal{N}(0,1)$ we get:

$$y_{t+1}(\theta_t) = F_t^1(\theta_t, y_t, Z)\,, \quad Z \sim N(0,1)\,,$$
$$F_t^1(\theta, y, z) = y\exp\{[\mu(\theta) - \beta - \frac{1}{2}\sigma^2(\theta)] + \sigma(\theta)z\} + \varepsilon\,.$$

The green dot lines depict the mean wealth plus/minus one standard deviation of the random variable. The simulations were attained employing the optimal share of stocks in the pension fund portfolio $\widetilde{\theta}^{(s)} = \widetilde{\theta}^{(s)}(t,y)$ depending on the value of simulated yearly accumulated wealth y_t at time t and at the terminal time $t = T$. Providing that financial market data from 2009–2012 were applied, we observe that at the end of simulation period, $t = T$, the average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T] \approx 7.05$ meaning that the future pensioner following the optimal investment strategy given by $\widetilde{\theta}^{(s)}$ has accumulated approximately 7.05 multiples of her/his last yearly salary. Considering the longer time period 2003–2012 the average accumulated wealth–to–salary $\mathbb{E}[y_T] \approx 4.10$ is quite lower and much higher share of wealth is held in bonds (approximately 90% in the last decade of the accumulation period) when compared to the recent short time period 2009–2012 due to worse performance and highly volatile of the MSCI All Country World index. Such result is in consistence with reality observed.



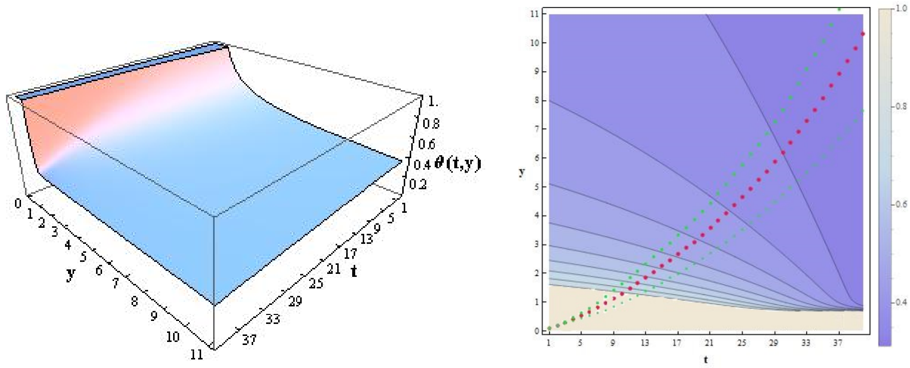
(a) 3D plot of the constrained optimal share of S&P500 Index (2003–2012)

(b) Contour plot of the constrained optimal share of S&P500 Index (2003–2012)

Figure 7.6: 3D plot and Contour plot of the constrained optimal share of S&P500 Index in the portfolio of 10-Year US Treasuries and S&P500 Index, based on data between 2013–2012

Furthermore as a result of confronting Figure 7.1a with Figure 7.2a we remark that the combination of MSCI All Country World index worse performance and high volatility with more or less unchanged Slovak Government Bond characteristics and significant shift in correlation of their returns on longer time period 2003–2012 induces comparably lower constrained optimal share of the MSCI All Country World index in the investment portfolio when the 10-year period for market data it taken.

For the comparison purpose we present similar plots for alternative investment strategies, namely DAX index versus 10–Year zero coupon German Bunds (Figures 7.3–7.4), and the S&P500 Index with 10–Year US Treasuries (Figures 7.5–7.6). We consider the baseline setting

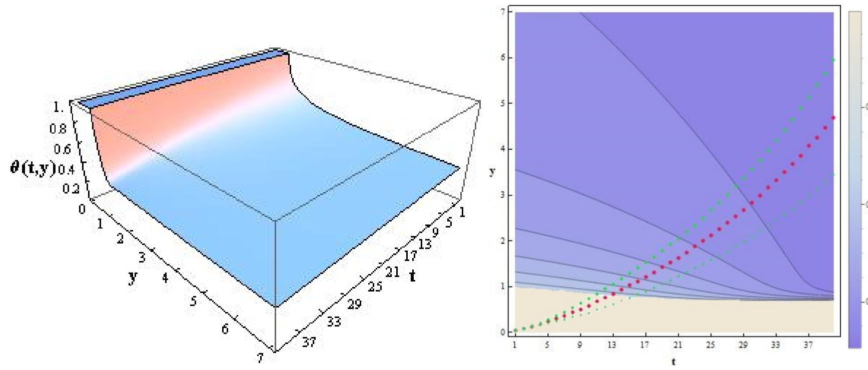


(a) 3D plot of the constrained optimal share of MSCI Index for $\varepsilon = 0.09$

(b) Contour plot of the constrained optimal share of MSCI Index for $\varepsilon = 0.09$

Figure 7.7: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the saver's regular contribution rate ε raises to 9%, based on financial market data from 2009-2012.

for all model parameters but the financial market data which are taken from Table 7.1. Similarly to the baseline scenario with Slovak Government Bonds and MSCI All Country World Index we evaluate the constrained optimal policies with financial market data observed between 2009–2012 and 2003–2012. Both alternative investment strategies exhibit behaviour similar to the baseline strategy characterized by higher shares of risky assets yielding higher terminal average accumulated *wealth–to–salary ratios* on the recent time period 2009–2012 than in the longer one.



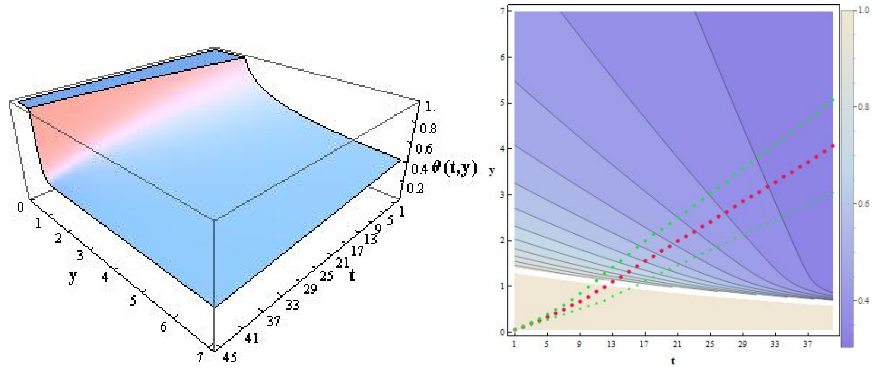
(a) 3D plot of the constrained optimal share of MSCI Index for $\varepsilon = 0.04$

(b) Contour plot of the constrained optimal share of MSCI Index for $\varepsilon = 0.04$

Figure 7.8: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the saver's regular contribution rate drops to 4%, based on financial market data from 2009-2012.

The subsequent text is devoted to provide the description of the effects of changes in key model parameters on the constrained optimal share of the MSCI All Country World index in the investment portfolio $\widetilde{\theta}^{(s)}$, and on the terminal average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T]$. Within the model we considered financial market data from the period 2009–2012. Furthermore, we aim our attention particularly on the consequences of fluctuations in prescribed contribution rate ε and retirement age T, thus the factors that policy makers can directly rule.

Saver's Contribution Rate ε. Firstly, on Figure 7.7 we propose the illustration of the optimal policy behaviour under the crucial model structural parameter variation – we ponder the *2012 - no policy change scenario* increase the saver's regular contribution rate ε from 6% to 9% per year and observe higher share of risky investment during the whole accumulation period in comparison with the case of $\varepsilon = 0.6$ (Figure 7.7a) and an essential rise in the terminal average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T] \approx 10.65$ (Figure 7.7b).

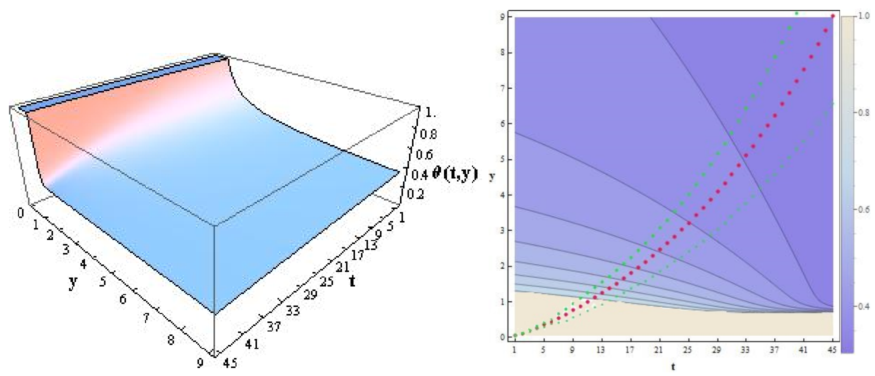


(a) 3D plot of the constrained optimal share of MSCI Index for $\beta = 0.061$

(b) Contour plot of the constrained optimal share of MSCI Index for $\beta = 0.061$

Figure 7.9: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI All Country World Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the gross wage growth rate β raises to 6.1% yearly, based on financial market data from 2009-2012.

Hence, assuming the saver's equal contribution to both mandatory pillars of the Slovak pension scheme generating the same expected future pay–offs, the future pensioner may expect to be able to cover the expenses during approximately 21 years of his/her retirement with the lower level of government implicit liabilities. Furthermore, his/her investment strategy is aimed more on risky assets in compare to the baseline scenario, as in the first half of the the accumulation period more than 3/4 of his wealth is stored in the MSCI Index – and even in the 10 years this share does not decline below 40% of the portfolio.

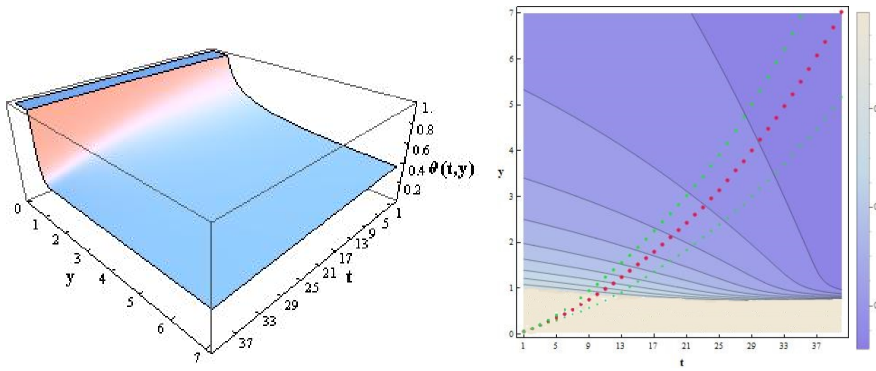


(a) 3D plot of the constrained optimal share of MSCI Index for $T = 45$

(b) Contour plot of the constrained optimal share of MSCI Index for $T = 45$

Figure 7.10: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI All Country World Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the accumulation period length T increases to 45 years, based on financial market data from 2009-2012.

On the other side, a drop of the contribution rate ε to 4% of yearly salary leads to a core conservative strategy in terms of a substantial fall in MSCI Index weight in the pension fund investment portfolio and a decline in the terminal average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T]$ to approximately 4.65 (see Figures 7.8a–7.8b). In our concrete application, 4% saver's regular contribution to the private pension scheme represent only 22% of his/her overall pension system payments. Then, assuming the proportional expected future pay–offs from both public and private mandatory schemes the future pensioner may expect to be able to cover the expenses during approximately 21 years of his/her retirement with the substantially higher level of government implicit liabilities. These results are in consistence with our



(a) 3D plot of the constrained optimal share of MSCI Index for $\lambda = 0.1$

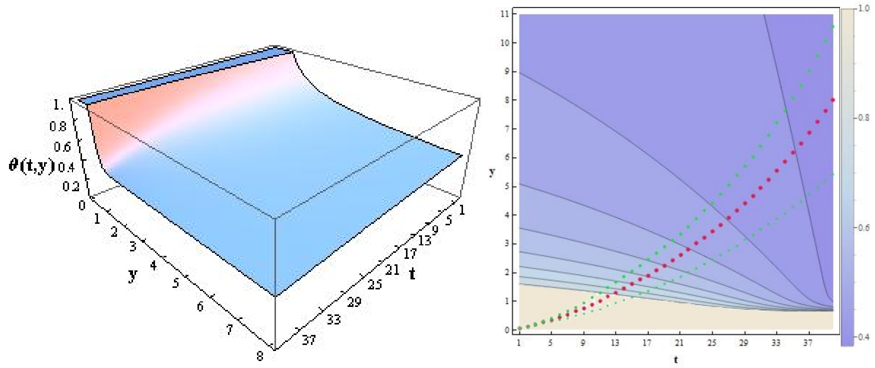
(b) Contour plot of the constrained optimal share of MSCI Index for $\lambda = 0.1$

Figure 7.11: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI All Country World Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the volatility sensitivity small model parameter λ augments to $\lambda = 0.1$, based on financial market data from 2009-2012.

sensitivity analysis and the observed reality.

Retirement Age and Wage Growth. Next, on Figures 7.9–7.10 we provide the depiction of the optimal policy behaviour under the changes in other structural model *macro* parameters: gross wage growth β, and accumulation period length T. The first alternative is represented by augmentation in the saver's gross wage rate of growth β from the originally assumed 3.5% to 6.1% while the second one is designed in terms of the accumulation period T prolongation by 5 years. The concrete value of the alternative scenario for the gross wage growth rate arises as an average for the time period 2003–2012 based on data reported by the Slovak Statistical Office. Expectations about the optimal policy behaviour arising from the sensitivity analysis performed are met (see Figure 7.9) as higher growth of gross wage causes lowering the optimal share of risky investment and obviously also due to noticeably higher final year salary the expected terminal wealth–to–salary ratio $\mathbb{E}[y_T] \approx 4.83$ (Figure 7.9b) is brought down.

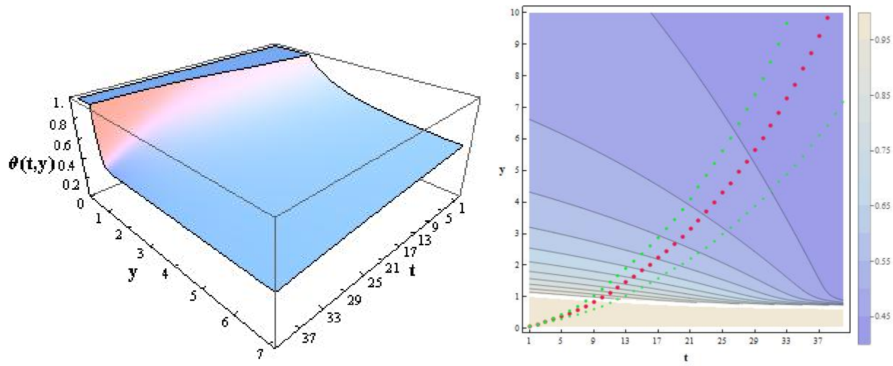
On the other side, extending working life (equivalent for accumulation period T prolongation) has an effect on raising share of risky asset in the portfolio (there is a slower shift towards less risky bond and even in the last decade of the accumulation period more than 40% of wealth is placed in the risky stock) and thus causes even larger expected terminal



(a) 3D plot of the constrained optimal share of MSCI Index for $d = 8$

(b) Contour plot of the constrained optimal share of MSCI Index for $d = 8$

Figure 7.12: 3D plot and Contour plot depicting changes in the constrained optimal share of MSCI All Country World Index in the portfolio of 10-Year zero coupon Slovak Government Bonds and MSCI All Country World Index provided that the risk aversion related coefficient d declines to 8, based on financial market data from 2009-2012.



(a) 3D plot of the constrained optimal share of MSCI Index for $\mu^{(s)} = 0.1253$

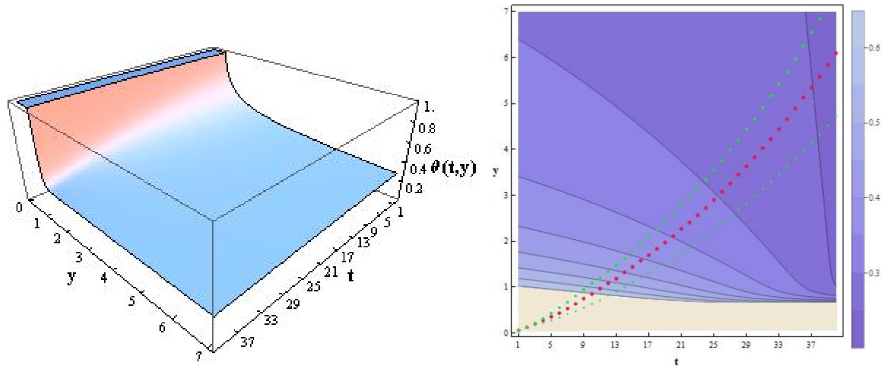
(b) Contour plot of the constrained optimal share of MSCI Index for $\mu^{(s)} = 0.1253$

Figure 7.13: 3D plot and Contour plot describing changes in the constrained optimal share of MSCI Index in the portfolio of Slovak Government Bonds and MSCI All Country World Index provided that MSCI return $\mu^{(s)}$ augments to $\lambda = 0.1253$, as based on financial market data from 2009-2012.

wealth–to–salary ratio $\mathbb{E}[y_T] \approx 9.05$.

Risk aversion coefficients. Moreover, on Figures 7.11–7.12 we provide the demonstration of the optimal policy behaviour under the changes in the model *micro* parameters: risk aversion parameter d and small asymptotic risk–sensitivity related parameter λ.

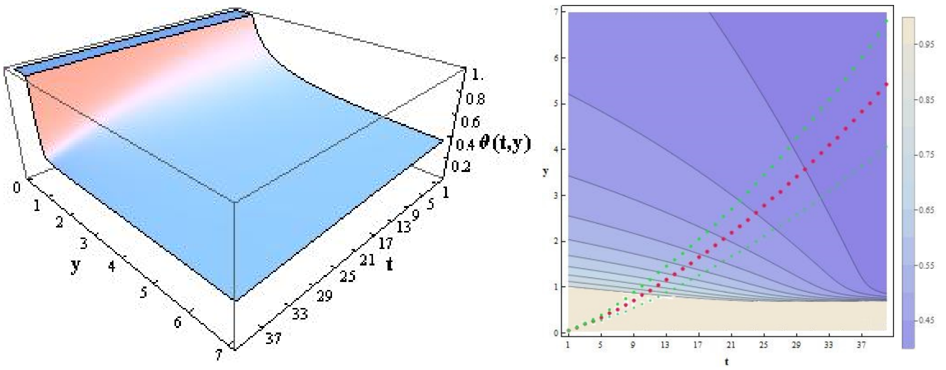
The impact of drop in risk aversion coefficient d from its initial value 10 to 8 and the effect of small volatility sensitivity parameter λ growth to 0.1 have result in similar movements in the constrained optimal share of the MSCI All Country World index in the investment portfolio $\widetilde{\theta}^{(s)}$, and the terminal average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T]$. Indeed, accordingly to our sensitivity analysis both changes cause moving investment shares towards risky asset causing higher amount of saved salaries on the retirement time as in case of $d = 8$, $\mathbb{E}[y_T] \approx 8.22$ while for $\lambda = 0.1$, $\mathbb{E}[y_T] \approx 7.05$.



(a) 3D plot of the constrained optimal share of MSCI Index for $\sigma^{(s)} = 0.1623$

(b) Contour plot of the constrained optimal share of MSCI Index for $\sigma^{(s)} = 0.1623$

Figure 7.14: 3D plot and Contour plot describing changes in the constrained optimal share of MSCI Index in the portfolio of Slovak Government Bonds and MSCI All Country World Index provided that the MSCI Index volatility $\sigma^{(s)}$ rises to $\sigma^{(s)} = 0.1623$, as based on financial market data from 2009-2012.



(a) 3D plot of the constrained optimal share of MSCI Index for $\rho = -0.4$

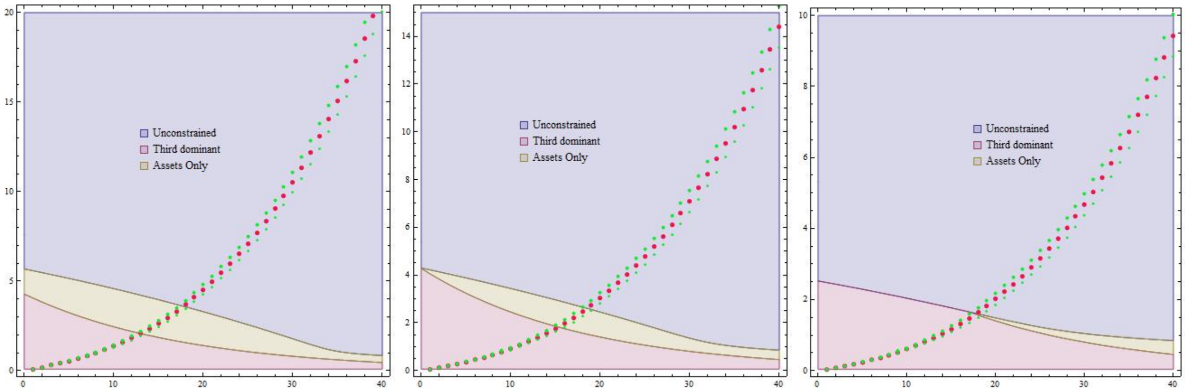
(b) Contour plot of the constrained optimal share of MSCI Index for $\rho = -0.4$

Figure 7.15: 3D plot and Contour plot describing changes in the constrained optimal share of MSCI Index in the portfolio of Slovak Government Bonds and MSCI All Country World Index provided that the correlation coefficient between the returns of MSCI Index and Slovak Bonds ρ increases to $\rho = -0.4$, as based on financial market data from 2009-2012.

Financial Market Parameters. At this stage our aim is to describe how movements on the financial market transmit to constrained optimal allocation of financial resources decision and hence, affect our expectations about the terminal average accumulated *wealth–to–salary ratio* $\mathbb{E}[y_T]$.

Firstly, raise in expected return of risky assets by 2p.p. at $\mu^{(s)} = 0.1253$ as illustrated on Figure 7.13 significantly changes the investment profile as it causes an increase in weight of stocks in the portfolio – even in the last decade of the accumulation period the investor holds about half of his/her wealth in the risky asset. This strategy thus leads to improvement in the wealth–to–salary ratio as it grows to $\mathbb{E}[y_T] = 11.15$ accompanied by obvious higher uncertainty about the returns (in the terminal year the the wealth–to–salary ratio is expected to be between 5.2 and 12.85 as a result of increasing share of volatile MSCI index) as it can be easily deduced from Figures 7.13a and 7.13b.

On the other side, an opposite behaviour is associated with risky assets volatility augmenta-

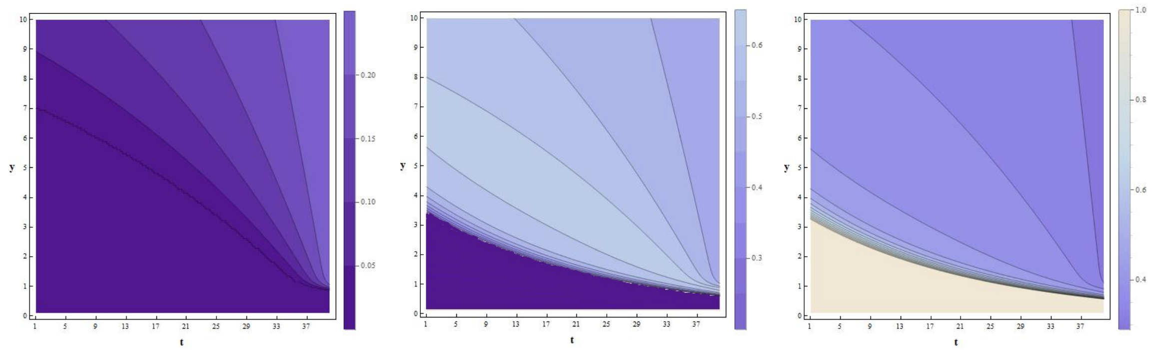


(a) Optimal allocation policy for the case of two assets and one bond if $\varepsilon = 0.09$

(b) Optimal allocation policy for the case of two assets and one bond if $\varepsilon = 0.06$

(c) Optimal allocation policy for the case of two assets and one bond if $\varepsilon = 0.04$

Figure 7.16: Regions of various prescriptions of the optimal allocation policy for the case of two assets (MSCI Index, DAX Index) and one bond (10Y Slovak Government Bond) for the case of either $\varepsilon = 0.09$ (left), $\varepsilon = 0.06$ (middle, default scenario) or $\varepsilon = 0.04$ (right): Purple colour marks the region, where the unconstrained policy applied; Grey colour indicates the region where it is optimal to divide the investment between exactly two assets and pink colour highlight the region with pure dominance of DAX Index.



(a) Contour plot for Slovak Bonds weight in the portfolio, for $\varepsilon = 0.06$

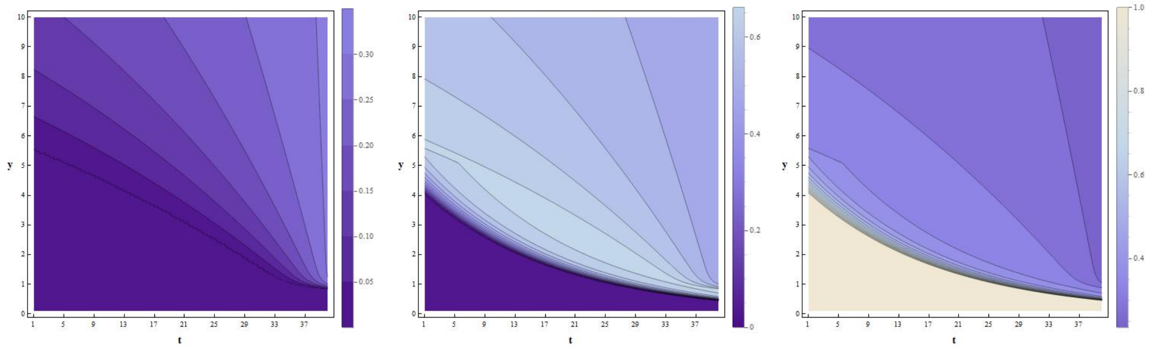
(b) Contour plot for MSCI Index weight in the investment portfolio, for $\varepsilon = 0.06$

(c) Contour plot for DAX Index weight in the investment portfolio, for $\varepsilon = 0.06$

Figure 7.17: Contour plots depicting the time–space evolution of shares of 10-Year Slovak Government Bonds, MSCI All Country World Index and DAX Index within the portfolio they form based on financial market data from 2009-2012 and baseline scenario for the *One–Bond–One–Stock Problem* parameters with the default saver's contribution rate $\varepsilon = 0.06$.

tion to $\sigma^{(s)} = 0.1623$ implying not only worsen the stock position in the portfolio but implies also an expected decline in the wealth–to–salary ratio as it grows to $\mathbb{E}[y_T] = 6.05$ as seen on Figure 7.14. Lower uncertainty about the terminal wealth–to–salary ratio is due to drop in the share of highly volatile portfolio component (see Figures 7.14a and 7.14b). Finally, consistently with our sensitivity analysis observe the decline in MSCI Index share in the portfolio as a result of increase (Figure 7.15) in the coefficient of correlation between the returns of the MSCI Index and Slovak Government Bonds, ρ, to -0.4.

We summarize the key observations, investment and policy recommendations below.



(a) Contour plot for Slovak Bonds weight in the portfolio for $\varepsilon = 0.09$

(b) Contour plot for MSCI Index weight in the investment portfolio for $\varepsilon = 0.09$

(c) Contour plot for DAX Index weight in the investment portfolio for $\varepsilon = 0.09$

Figure 7.18: Contour plots depicting the time–space evolution of shares of 10-Year Slovak Government Bonds, MSCI All Country World Index and DAX Index within the portfolio they form based on financial market data from 2009-2012 and baseline scenario for the *One–Bond–One–Stock Problem* parameters providing that the contribution rate ε increased to the *2012 no policy change* value of 9%.

Assume a typical participant of the Second pillar of Slovak pension system deciding how to split optimally the wealth already allocated on his/her pension account between the Index Fund (represented by MSCI All Country World Index) and Bond Fund (deputized by 10–year zero coupon Slovak government bonds).
Providing that $\varepsilon = 0.06$, $T = 40$ and 2009–2012 financial market data are taken, saver is advised to place more that 50% of his wealth in the Index Fund in the first 30 years of the accumulation period (more than 65% during the first half of the active life) and even in the last decade this proportion should not fall under 30%. Following such strategy would bring him/her approximately 7 yearly salaries. Furthermore, if the contribution rate increases to 9%, he/she might expect to earn around 10.65 yearly salaries emulating more aggressive strategy with more than 3/4 of investment allocated in the Index fund during the first half of his/her active life and more than 40% in the last decade. A similar effect can be observed when the retirement age is elevated – it is optimal for a future pensioner to choose more dynamic strategy with high share of wealth invested in the Index Fund and slower shift towards Bond Fund yielding in 9 yearly salaries saved.

7.3 Multiple Stock–Bond Problem

The goal of the second application presented in this paper is to establish the saver's optimal strategy in their private pension fund composition, assuming more bonds and stock can be selected. Referring to Slovak pension system, from the saver's point of view the investment decision essence lies in detecting the best fitting proportions in resources allocation problem. In this case, not only the *stock–to–bond ratio* is of concern to the investor, but he / she primarily creates the investment portfolio consisting of more bonds and more stocks (indices) in their proper weights – and the key thing is to design these weights optimally conditioned future pensioner time to retirement, intermediate *wealth-to-salary ratio*, various model parameters

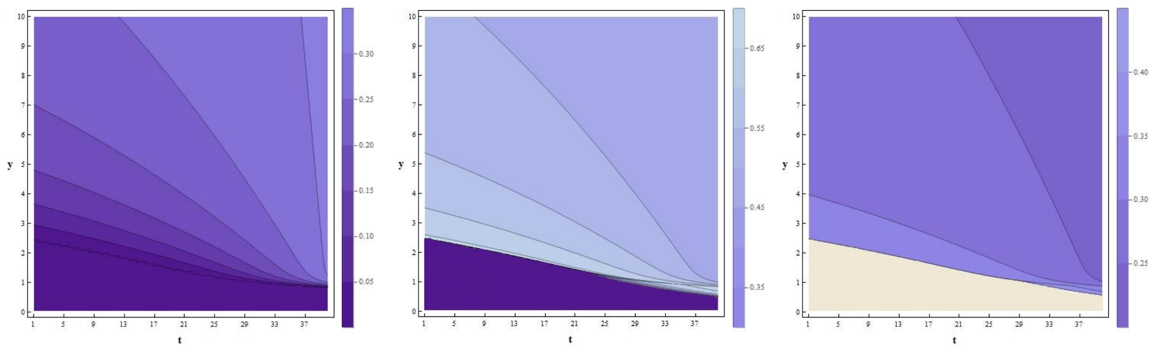
	Statistics				
Asset	Mean	Std. deviation	Correlation Matrix		
MSCI World	0.1053	0.1423	1	-0.8344	0.6875
10-Y Slovak Bonds	0.0439	0.0036	-0.8344	1	-0.8084
DAX	0.1328	0.1738	0.6875	-0.8084	1

Source: Bloomberg, MSCI, EuroStat

Table 7.2: Descriptive characteristics (mean, standard deviation and correlation) of selected market data observed in periods 2009–2012 for investment portfolio of two stocks and one bond

and restrictions. The investor's choice is restrained by the natural non-negativity limitations imposed on the portfolio weights corresponding to each of available assets. Evidently, the previously studied *stock–to–bond ratio* will arise as the by–product of this allocation problem.

Similarly to the *one-stock-one-bond* problem, even the *Equity–Linked Index Fund* (representing stocks) allocation strategy applied when replicating the performance of the benchmark prescribed by the pension fund management, is unlimited in the choice of stocks, financial derivatives or exchange traded funds, for the sake of simplicity we presuppose the fund investment decisions restricted in stocks only. Furthermore, for the same reason we assume normally distributed stock return, though usage of the Normal Inverse Gaussian distribution (c.f. Andersen et al. [2], Barndorff-Nielsen [4], Corrado and Su [15], Onalan [58]) is more convenient. Likewise, we model the bond returns employing normal distribution, howbeit applying Cox–Ingersoll–Ross model (see e.g. Cox et al. [19], Kwok [42] or Shreve [63]) seems more suitable. Assuming non–normal distributions when modelling assets returns is one of the objectives of our future research.



(a) Contour plot for Slovak Bonds weight in the portfolio for $\varepsilon = 0.04$

(b) Contour plot for MSCI Index weight in the investment portfolio for $\varepsilon = 0.04$

(c) Contour plot for DAX Index weight in the investment portfolio for $\varepsilon = 0.04$

Figure 7.19: Contour plots depicting the time–space evolution of shares of 10-Year Slovak Government Bonds, MSCI All Country World Index and DAX Index within the portfolio they form based on financial market data from 2009-2012 and baseline scenario for the *One–Bond–One–Stock Problem* parameters providing that the contribution rate ε permanently lowered to the value of 4%.

The optimal allocation policy restrained on $N--$dimensional simplex as a result of the natural ban on short positions, that should follow the future pensioner is given in sense of Section

Asset	**Statistics** Mean	Std. deviation	Correlation Matrix		
10-Y German Bunds	0.02552	0.0063	1	-0.2896	0.7899
S&P500	0.1242	0.0964	-0.2896	1	-0.1518
10-Y US treasuries	0.0276	0.0068	0.7899	-0.1518	1

Source: Bloomberg, ECB, US Treasury

Table 7.3: Descriptive characteristics (mean, standard deviation and correlation) of selected market data observed in periods 2009–2012 for investment portfolio of one stock and two bonds

5.3 and Theorem 3 as subsequently

$$\hat{\theta}(t,y) = \begin{cases} \dfrac{1}{a}\Sigma^{-1}\left\{\mathbf{1} + (a\mu - b\mathbf{1})\dfrac{1}{\zeta(T-t,\ln y)}\right\}, & \zeta \in \mathscr{I}_\emptyset, \\ \dfrac{1}{a_S}\Sigma_S^{-1}\left\{\mathbf{1} + (a_S\mu_S - b_S\mathbf{1})\dfrac{1}{\zeta(T-t,\ln y)}\right\}, & \zeta \in int(\mathscr{I}_S), \end{cases} \tag{7.4}$$

where $\mathscr{I}_\emptyset \cup \bigcup_{1\leq|S|\leq N-1} int\,(\mathscr{I}_S)$ for any $S \subset \{1,\ldots,N\}$ are the sets defined such that

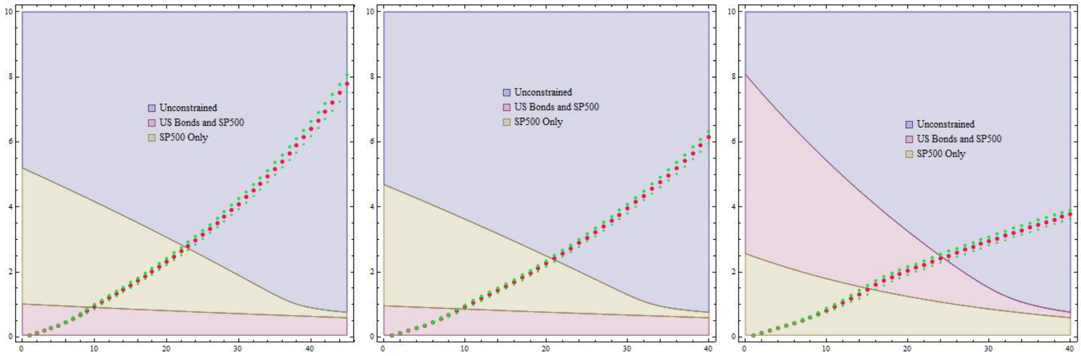
- $\mathscr{I}_\emptyset$ the set of all $\zeta > 0$ for which the (unique) minimizer $\hat{\theta}(\zeta) \in \Delta$ has positive components only,
$$\mathscr{I}_\emptyset = \{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) > 0,\ \forall i = 1,\ldots,N\},$$
- For any subset S of $\{1,\ldots,N\}$ the set $\mathscr{I}_S$ of all functions $\zeta > 0$ for which the index set of $\hat{\theta}(\zeta) \in \Delta$ zero components coincide with S;
$$\mathscr{I}_S = \{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) = 0 \iff i \in S\}.$$

Furthermore, $a = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$, $b = \mu^T\Sigma^{-1}\mathbf{1}$ and $c = \mu^T\Sigma^{-1}\mu$ Similarly, $a_S = \mathbf{1}^T\Sigma_S^{-1}\mathbf{1}$, $b_S = \mu_S^T\Sigma_S^{-1}\mathbf{1}$ and $c_S = \mu_S^T\Sigma_S^{-1}\mu_S$ are determined for the problem dimension reduced to lower $N-|S|$ dimensional simplex Δ_S with nullified rows and columns elements from the matrix Σ and vector μ corresponding to components with index belonging to S (already known as they all are zero) to get projections Σ_S and μ_S. Finally, instead of ζ itself we employ $\omega \approx \zeta^{-1}$ the first order approximation to the inverse of ζ introduced by (7.3).

7.3.1 Model parameters calibration

Regarding non–market data (investor characteristics, legislative norms) we employ the values introduced in Section 7.2.1 considered within the same time periods, shorter 2009–2012 and longer 2003–2012. Furthermore, in order to clarify the optimal strategy decision process in case of higher dimensional problem we present two examples:

1. Investment portfolio formed by two risky and one safe securities: MSCI All Country World Index, DAX Index and 10-Year zero coupon Slovak Government Bond.
2. Investment portfolio formed by one risky and two safe securities: S&P500 Index, 10-Year zero coupon German Bunds and 10-Year US Treasuries.



(a) Optimal allocation policy for the case of two assets and one bond if $T = 45$ and $\beta = 0.035$

(b) Optimal allocation policy for the case of two bonds and one stock if $T = 40$ and $\beta = 0.035$

(c) Optimal allocation policy for the case of two assets and one bond if $T = 40$ and $\beta = 0.061$

Figure 7.20: Regions of various prescriptions of the optimal allocation policy for the case of two bonds (German Bunds and US Treasuries) and one stock (S&P500 Index): unconstrained policy regions are marked purple; on grey region it is optimal to divide the investment between US Treasuries and S&P500 Index and pink highlight the region of pure dominance of S&P500 Index. We assume that either higher $T = 45$ (left), or the default scenario with $T = 40$ and $\beta = 0.035$ (middle), or higher $\beta = 0.061$ (right).

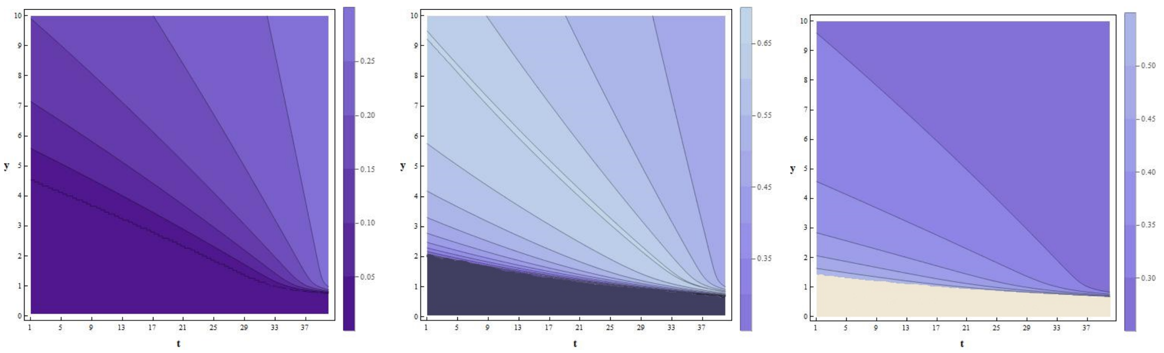
The descriptive statistics obtained are summarized in Table 7.2. Remaining model parameters are set to their initial values coming from the default scenario for the One–Stock–One–Bond Problem (Section 7.2). Therefore the gross wage growth rate $\beta = 0.035$, the saver's regular gross contribution rate $\varepsilon = 0.06$, accumulation period length $T = 40$, volatility sensitivity parameter $\lambda = 0.04$ and risk aversion coefficient $d = 10$.

7.3.2 Case Study I.: Two Stocks & One Bond Problem

The pension fund investment portfolio is composed of two risky assets represented by the MSCI All Country Index and DAX Index, and one safe security, 10-Year zero coupon Slovak Government Bonds. Financial market data are shown in Table 7.2.

In order to determine the optimal constrained investment strategy we employed the technique presented in (7.4). We observed that there are three regions over which a different investment technique must be applied: the unconstrained region where all three securities are active; the region where only risky assets are considered; and finally the region of pure dominance of only one asset, DAX Index, as depicted in detail on Figure 7.16b. The weight of securities forming the investment portfolio are illustrated on Figures 7.17a–7.17c.
Observe the large region over which investor allocates his/her resources into risky assets only – providing that the objective is to allocate approximately six yearly salaries he/she should put all the wealth into assets during the first 20 years of accumulation period and then only slowly decline assets share in the portfolio with still a large share of both indices in the remaining 20 years (see Figures 7.17a–7.17c). Following such strategy should bring the saver more than 14 yearly salaries, as $\mathbb{E}[y_T] \approx 14.25$. On Figure 7.16b the mean portfolio wealth $\mathbb{E}[y_t]$ (red dot line) is obtained by performing 10000 Monte-Carlo simulations of random paths $\{y_t\}_{t=1}^{T}$ calculated according to the recurrent equation (4.8)–(4.9) with one year period ($\tau = 1$). The green dot lines depict the mean wealth plus/minus one standard devi-

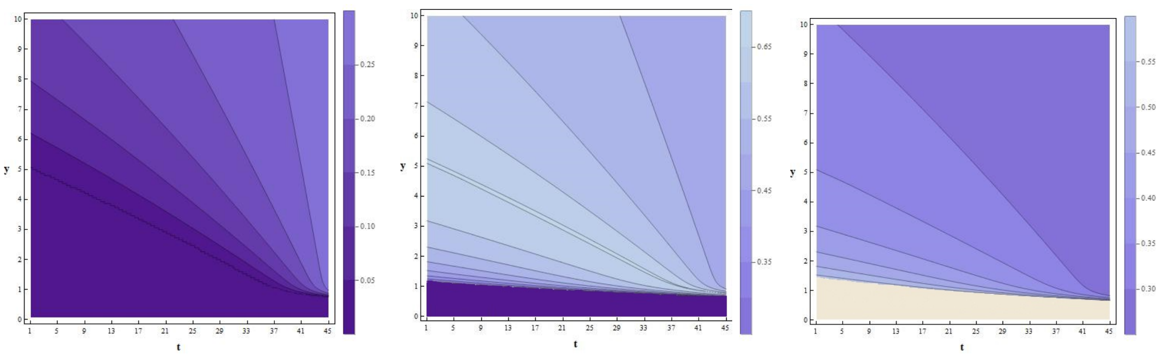
ation of the random variable. The simulations were attained employing the optimal share of stocks in the pension fund portfolio $\widetilde{\theta}^{(s)} = \widetilde{\theta}^{(s)}(t, y)$ depending on the value of simulated yearly accumulated wealth y_t at time t and at the terminal time $t = T$.



(a) Contour plot for German Bunds in the portfolio for $T = 40$ and $\beta = 0.035$ **(b)** Contour plot for US Treasuries in the portfolio for $T = 40$ and $\beta = 0.035$ **(c)** Contour plot for S&P500 Index in the portfolio for $T = 40$ and $\beta = 0.035$

Figure 7.21: Contour plots depicting the time–space evolution of shares of German Bunds, US Treasuries and the S&P500 Index within the portfolio they form based on financial market data from 2009-2012 and baseline scenario for the *One–Bond–One–Stock Problem* parameters with the default setting: the length of the accumulation period $T = 40$ and the gross wage growth rate $\beta = 0.035$.

Furthermore, consistently with our sensitivity analysis, the increase in the saver's regular contribution rate to the *2012 – no policy change* value $\varepsilon = 0.09$ implies a significant increase in accumulated yearly salaries earned to even approx. $\mathbb{E}[y_T] \approx 21.85$ (see Figure 7.16a). This amount is achieved via raise in the share of both risky securities in the investment portfolio over the time to (see Figure 7.18). On the other side, considering the temporal drop in the contribution rate to 4% for the persistent brings the future pensioner the expected terminal return of only $\mathbb{E}[y_T] \approx 7.55$ (see Figure 7.16c) obtained following more conservative investment strategy as demonstrated on Figure 7.19.



(a) Contour plot for German Bunds in the portfolio for $T = 45$ and $\beta = 0.035$ **(b)** Contour plot for US Treasuries in the portfolio for $T = 45$ and $\beta = 0.035$ **(c)** Contour plot for S&P500 Index in the portfolio for $T = 45$ and $\beta = 0.035$

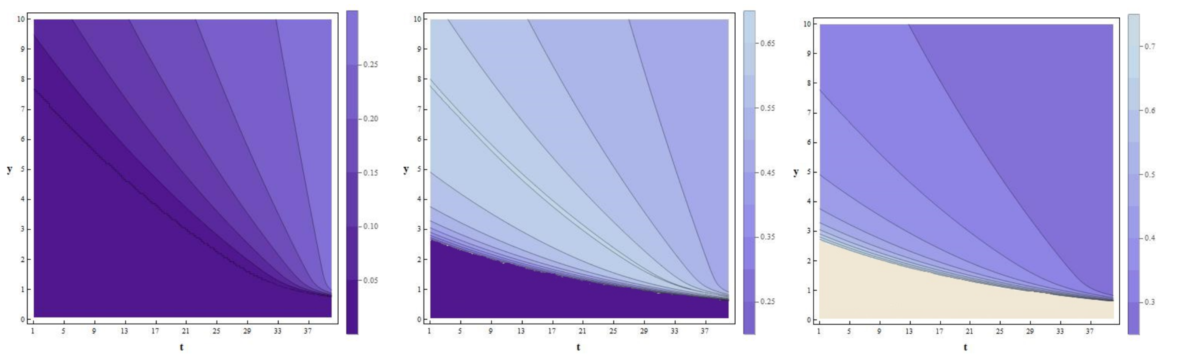
Figure 7.22: Contour plots for the time–space evolution of shares of German Bunds, US Treasuries and the S&P500 Index within the portfolio they form based on financial market data from 2009-2012 and baseline scenario for the *One–Bond–One–Stock Problem* parameters with higher pension age $T = 45$ and the default $\beta = 0.035$.

7.3.3 Case Study II.: One Asset & Two Bonds Problem

In this case the pension fund investment portfolio is created by of two safe securities represented by the 10–Year zero coupon German Bunds and 10-Year US Treasuries, whereas the risky part is generated by considering S&P500 Index. Financial market data for this portfolio are shown in Table 7.3.

Similarly to the previous example to determine the optimal constrained investment strategy we apply the mechanism launched in (7.4). We discovered that again there are three regions over which a different investment strategy works: the unconstrained region where all three securities are active; the region where only one bond (US Treasuries) along with the risky asset (SP&500) enters the decision; and finally the region of pure dominance of the only risky element of the pension fund portfolio, i.e. S&P500 Index, as illustrated in detail on Figure 7.20b. The shares of securities creating the pension fund investment portfolio are depicted on Figures 7.21a–7.21c. The region of purely risky investment shrank comparing to the previous case – an average saver allocates wealth into only risky security during the first 10 years and its share within the portfolio gradually decline: after approximately 22 years to 45 % and in the end of accumulation period it drops down to 27%. In compare to the previous example this strategy leads to lower expected terminal value of wealth–to–salary ratio, only as $\mathbb{E}[y_T] \approx 6.45$.

On Figure 7.20b the mean portfolio wealth $\mathbb{E}[y_t]$ (red dot line) is obtained by performing 10000 Monte-Carlo simulations of random paths $\{y_t\}_{t=1}^{T}$ calculated according to the recurrent equation (4.8)–(4.9) with one year period ($\tau = 1$). The green dot lines depict the mean wealth plus/minus one standard deviation of the random variable. The simulations were attained employing the optimal share of stocks in the pension fund portfolio $\widetilde{\theta}^{(s)} = \widetilde{\theta}^{(s)}(t, y)$ depending on the value of simulated yearly accumulated wealth y_t at time t and at the terminal time $t = T$. We depict the situation in which the pension age has been postponed, $T = 45$. This change in the model structural parameter leads to higher expected future returns $\mathbb{E}[y_t]$ associated with more risky investment strategies (see Figure 7.22).



(a) Contour plot for German Bunds in the portfolio for $T = 40$ and $\beta = 0.061$ **(b)** Contour plot for US Treasuries in the portfolio for $T = 40$ and $\beta = 0.061$ **(c)** Contour plot for S&P500 Index in the portfolio for $T = 40$ and $\beta = 0.061$

Figure 7.23: Contour plots for the time–space evolution of shares of German Bunds, US Treasuries and the S&P500 Index within the portfolio they form based on financial market data from 2009-2012 and baseline scenario for the *One–Bond–One–Stock Problem* parameters with default pension age $T = 40$ and raised wage growth rate $\beta = 0.061$.

CONCLUSION

The main objectives of this dissertation thesis were to formulate properly the continuous–time pension investment portfolio selection problem that encounters any participant of the Second pillar of the Slovak pension system properly, and find the relationship between optimal portfolio allocation policy and its intermediate value function.
Secondly, we were aimed to build a simple explicit analytic decision mechanism estimating a future pensioner's optimal portfolio selection strategy that based on a saver's time to retirement and already allocated wealth advice him/her how to allocate his/her wealth optimally between unlimited number of more or less risky securities. The decision formula derived in this thesis reflects individual characteristics of a risk–sensitive investor (risk aversion attitude, gross wage growth rate), existing government restrictions (retirement age, contribution rate) and financial market data.
Furthermore we concentrated our attention to provide a deep analysis of the optimal investment strategy decision tool from a qualitative and quantitative perspective on which basis we emphasized fundamental policy implications and recommendations. We calibrated the model on Slovak data and illustrate its behaviour on various examples.

Firstly, in Section 2 we sketched baseline structure and concept of the three–pillar Slovak pension system and focused on its private defined–contribution based Second pillar. We clarified its underlying processes and ideas,pension account management, private asset management companies, available investment strategies and legislative framework. Using several graphical schemes and figures we depicted the characteristics of current participants of the Second pillar and their actual investment decisions. We uncovered two ticking time bombs – bad demographic forecasts and public finance sustainability issue – representing serious reasons generating continuous pressure on policy–makers to undergo economical, social, and labour market reforms – inducing the reform of the Slovak pension system and all its three pillars. These argument were supported by many projections and calculations performed by e.g. OECD [55], [54] Council for Budget Responsibility [18], [17], Infostat [32] and European Commision [23],[22].

Next, in Section 3 we summarized theoretical background necessary to understand and derive the optimal investment strategy model (normal distribution and its properties, Itô calculus, utility function concept, Bellman's dynamic programming optimality principle, stochastic optimization, Lagrange multipliers, Hamilton–Jacobi–Bellman equation). Here we referred to several standard books e.g. Bellman [5], Bertsekas [8], Chiang [13], Evans [24], [25], Kirk [40], Kwok [42], Markowitz [45], Oksendal [56], Sharpe [62] or [63]. Furthermore, in order unfold the motivation and highlight certain interesting attributes of the studied model we demonstrated some practical examples presented simplest variant of our problem – two securities discrete–time stochastic dynamic model as it was presented in Kilianová et al. [37], Kilianová [36], and Macová and Ševčovič [44].

The core of this dissertation thesis was constituted by Sections 4–6.
In Section 4 we formulated the continuous–time pension investment portfolio selection problem: firstly by making time steps infinitely small we transformed the intuitive discrete–time model for unlimited number of traded securities into its continuous–time counterpart. Then, employing stochastic calculus we formulated this problem in terms of the fully non–linear Hamilton–Jacobi–Bellman equation. We also launched the concept of the investor's utility criterion employed in our model that captures both the expected terminal return of the investor's pension portfolio and the associated volatility (Pflug and Romisch [59], Markowitz [45], and Sharpe [62]). From this perspective we accounted for a natural investor's risk aversion attitude (see Pratt [60], Bielecki et al. [9]) and translated it into his/her criterion of utility. Next, in Section 5 we applied the Riccati transform (Abe and Ishimura [1], Macová and Ševčovič [44], Ishimura and Mita [33], Kilianová and Ševčovič [38], Ishimura and Nakamura [34], Múčka [51] or Ishimura and Ševčovič [35]) that enabled us to rewrite the original Hamilton–Jacobi–Bellman equation into twinned specific parametric convex optimization problem. Referring to Kilianová and Ševčovič [38] we showed the equivalence between a particular Cauchy–type quasi–linear parabolic equation and this convex optimization problem for which we proved the existence of a unique solution. This allowed us to determine the general C^∞–smooth formula for the constrained optimal portfolio allocation policy and for the sake of concreteness we demonstrated its usage for the case of *one–stock–one–bond* problem. At this stage the optimal investment strategy could not be determined directly as the optimal policy relationship was prescribed as a function of a transformed portfolio value function. Finally, derived effective lower and upper bounds of a solution to the initial value quasi–linear problem, presented its fundamental properties and constructed its semi–explicit travelling wave type solution (see Kilianová and Ševčovič [38] or Macová and Ševčovič [44]).
The key objective of Section 6 was to determine a simple formula that approximate the optimal investment strategy enough precisely. Therefore, we firstly substituted the implicit optimal allocation policy formula determined in Section 5 into the quasi–linear initial value equation. As the solution to resulting problem could not be find explicitly referring to Bender and Orszag [6], Hinch [29], Holmes [30], Macová and Ševčovič [44], Múčka [51] and O'Malley [57] we put in use the asymptotic expansion technique. The modified quasi–linear initial value equation was approximated up to its second order with respect to a pair (λ, ε) of small model parameters. A simple approximative prescription of the optimal allocation policy of a future pensioner was estimated explicitly as a function of his/her time to retirement and already allocated wealth (considered relatively to his/her salary). For the sake of simplicity here we used only absolute and linear terms of the solution asymptotic expansion though the general n–th order terms of the asymptotic expansion with respect to both small parameters was derived in the Appendix. The optimal allocation policy formula derived

accounted for investor's characteristics (gross wage growth rate, risk attitude), legislative framework (retirement age, contribution rate) and financial market performance. Finally, the obtained policy was analysed from a qualitative and quantitative perspective and resulting policy implications and recommendations applicable for policy–makers, pension fund managers and Second pillar participants were emphasized. We showed how both the optimal allocation policy and the expected terminal portfolio wealth are affected by varying model parameters and highlighted the effects of changes in fiscal policy parameters – prescribed retirement age and contribution rate.

Finally, in the application part of this dissertation thesis (Section 7) we brought the Section 6 model on Slovak data. We tested alternative setting of model key parameters and scrutinized changes in both the saver's optimal investment allocation policy and the terminal expected wealth allocated on investor's pension account implied by variations in model parameters. We were particularly concentrated on the effect caused by the prescribed contribution rate ε and retirement age T, thus the factors that policy makers could directly play with. The allocation strategy was illustrated through three types of situations studied – the simplest *One–Stock–One–Bond* problem, and in order to clarify the case of higher dimensional problem we presented the *Two–Stocks–One–Bond* problem and the *One–Stock–Two–Bonds* problem.

We certify that the fundamental objectives of this dissertation thesis stated in Section 1.1 were accomplished, the obtained results and policy implications are in consistence with intuition and reality observations.
Recalling the results obtained in Section 6.6 in order to increase the Second pillar retirement benefit of a future pensioner we recommend the policy–makers to increase regular contribution rate ε, elevate the retirement age and reduce fees charged by the private asset management companies. A future pensioner is advised to be more aggressive in his/her investment decision in the beginning of the active life and as time approaches the planned retirement age and the amount of allocated wealth on his/her pension account raises, decline gradually the share of investment in risky assets while moving towards more safe financial market instruments. Hence, a typical saver should start with risky stocks (or stock indices) and then in very last years before retirement switch to highly rated bonds. Furthermore we suggest a saver to ponder carefully his/her risk aversion attitude, so that very risk–aware investor should choose more conservative investment strategy with higher share of bonds in the investment portfolio. On the other side, providing that a saver's gross wage growth rate increased he/she should follow more dynamic investment strategy. Finally, due to volatile financial markets active portfolio management is essential – hence, the pension fund portfolio weight of the financial instrument which appreciates or its returns are getting more stable (i.e. returns are higher or less volatile) raises.
Assume a typical participant of the Second pillar of Slovak pension system deciding how to split optimally the wealth already allocated on his/her pension account between the Index Fund (represented by MSCI All Country World Index) and Bond Fund (deputized by 10–year zero coupon Slovak government bonds). Providing that the regular contribution rate ε is 6%, the accumulation period length $T = 40$ and 2009–2012 financial market data are taken, recalling our analysis from Section 7 saver is advised to place more that 50% of his wealth

in the Index Fund in the first 30 years of the accumulation period (more than 65% during the first half of the active life) and even in the last decade this proportion should not fall under 30%. Following such strategy would bring him/her approximately 7 yearly salaries. Furthermore, if the contribution rate increases to 9%, he/she might expect to earn around 10.65 yearly salaries emulating more aggressive strategy with more than 3/4 of investment allocated in the Index fund during the first half of active life and more than 40% in the last decade. A similar effect can be observed when the retirement age is elevated – it is optimal for a future pensioner to choose more dynamic strategy with high share of wealth invested in the Index Fund and slower shift towards Bond Fund yielding in 9 yearly salaries saved.

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Appendix A

TOWER LAW

Theorem 6. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space and take arbitrary* σ*–fields* $\mathscr{G}_1 \in \mathscr{F}$ *and* $\mathscr{G}_2 \in \mathscr{F}$ *such that* $\mathscr{G}_1 \subseteq \mathscr{G}_2$*. Then for any random variable* Y*,*

$$\mathscr{K}\left[\mathscr{K}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right] = \mathscr{K}\left[Y\,|\,\mathscr{G}_1\right], \qquad \mathscr{G}_1 \subseteq \mathscr{G}_2\,.$$

Proof. Making use of utility criterion $\mathscr{K}$ defined for any random variable Y and parameter $\lambda \in \mathbb{R}$ as

$$\mathscr{K}(Y) \equiv \underset{\lambda}{\mathscr{K}}(Y) = \mathbb{E}[Y] - \frac{\lambda}{2}\,\mathbb{D}[Y]\,,$$

and performing an easily calculation, we can straightforwardly derive that

$$\begin{aligned}\mathscr{K}\left[\mathscr{K}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right] = {} & \mathbb{E}\left[\mathbb{E}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right] - \tfrac{\lambda}{2}\left\{\mathbb{E}\left[\mathbb{D}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right] + \mathbb{D}\left[\mathbb{E}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right]\right\} \\ & + \tfrac{\lambda^2}{4}\,\mathbb{D}\left[\mathbb{D}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right].\end{aligned} \tag{A.1}$$

Obviously, the *Tower Law* holds for conditional expectation, hence

$$\mathbb{E}\left[\mathbb{E}(Y\,|\mathscr{G}_2)\,|\,\mathscr{G}_1\right] = \mathbb{E}\left[Y\,|\,\mathscr{G}_1\right], \qquad \mathscr{G}_1 \subseteq \mathscr{G}_2\,. \tag{A.2a}$$

Moreover, the *Law of Total Variance* conditional variant,

$$\mathbb{D}\left[Y\,|\,\mathscr{G}_1\right] = \mathbb{E}\left[\mathbb{D}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right] + \mathbb{D}\left[\mathbb{E}(Y\,|\mathscr{G}_2)\,|\,\mathscr{G}_1\right], \qquad \mathscr{G}_1 \subseteq \mathscr{G}_2\,. \tag{A.2b}$$

is valid since for all $\mathscr{G}_1 \subseteq \mathscr{G}_2$,

$$\begin{aligned}\mathbb{D}\left[Y\,|\,\mathscr{G}_1\right] &= \mathbb{E}\left[Y^2\,|\,\mathscr{G}_1\right] - \mathbb{E}\left[Y\,|\,\mathscr{G}_1\right]^2 = \mathbb{E}\left[\mathbb{E}(Y^2\,|\mathscr{G}_2)\,|\,\mathscr{G}_1\right] - \mathbb{E}\left[\mathbb{E}(Y\,|\mathscr{G}_2)\,|\,\mathscr{G}_1\right]^2 \\ &= \mathbb{E}\left[\mathbb{D}(Y\,|\,\mathscr{G}_2) + \mathbb{E}(Y\,|\mathscr{G}_2)^2\,|\,\mathscr{G}_1\right] - \mathbb{E}\left[\mathbb{E}(Y\,|\mathscr{G}_2)\,|\,\mathscr{G}_1\right]^2 \\ &= \mathbb{E}\left[\mathbb{D}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right] + \left\{\mathbb{E}\left[\mathbb{E}(Y\,|\mathscr{G}_2)^2\,|\,\mathscr{G}_1\right] - \mathbb{E}\left[\mathbb{E}(Y\,|\mathscr{G}_2)\,|\,\mathscr{G}_1\right]^2\right\} \\ &= \mathbb{E}\left[\mathbb{D}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right] + \mathbb{D}\left[\mathbb{E}(Y\,|\mathscr{G}_2)\,|\,\mathscr{G}_1\right].\end{aligned}$$

The last term in (A.1) identically equals zero, forasmuch as

$$\begin{aligned}\mathbb{D}\left[\mathbb{D}(Y\,|\,\mathscr{G}_2)\,|\,\mathscr{G}_1\right] &= \mathbb{E}\left[\left[\mathbb{D}(Y\,|\,\mathscr{G}_2) - \mathbb{E}(\mathbb{D}(Y\,|\,\mathscr{G}_2)\,|\mathscr{G}_2)\right]^2|\,\mathscr{G}_1\right] \\ &= \mathbb{E}\left[\left[\mathbb{D}(Y\,|\,\mathscr{G}_2) - \mathbb{D}(Y\,|\,\mathscr{G}_2)\right]^2|\,\mathscr{G}_1\right] = 0\,.\end{aligned} \tag{A.2c}$$

Henceforth, by applying outcomes (A.2a)–(A.2c) in equation (A.1), we can shown that the key *Tower Law* property holds for our utility criterion $\mathscr{K}$, since for any σ-fields $\mathscr{G}_1 \subseteq \mathscr{G}_2$ and random variable Y,

$$\mathscr{K}\left[\mathscr{K}\left(Y \,|\, \mathscr{G}_2\right) |\, \mathscr{G}_1\right] = \mathbb{E}\left[Y \,|\, \mathscr{G}_1\right] - \tfrac{\lambda}{2}\,\mathbb{D}\left[Y \,|\, \mathscr{G}_1\right] = \mathscr{K}\left[Y \,|\, \mathscr{G}_1\right]. \tag{A.3}$$

$\square$

Appendix B

HIGHER ORDER TERMS OF EXPANSION

The following text is devoted to detection of the general nth term derivation in case of missing either ε or λ.

B.1 Higher Order Terms in ε-Expansion

For the reason of better approximation of the function $\psi_0(s,x)$ for small enough values of the parameter ε now we make use the Taylor expansion (6.18) up to an arbitrary, general n-th term.

From the analysis of the ε-asymptotic expansion, we deduce not only the separability property of the terms with respect to both variables s and x but the exponential contribution of the variable x to the solution's nth term. Recall that we already have computed the first two terms $\psi_{0,0}$ and $\psi_{0,1}$ in the expansion. Namely, $\psi_{0,0}(s,x) = \gamma d$ is a constant and $\psi_{0,1}(s,x) = \Psi_{0,1}(s)e^{-x}$ for $\Psi_{0,1}(s)$ launched in (6.23) is time–space separable.

Hence, we are allowed to look for the solution to $[\mathbf{P_0}]$ acquiring the form of

$$\psi_0(s,x) = \sum_{n=0}^{\infty} \varepsilon^n \psi_{0,n}(s,x) = \sum_{n=0}^{\infty} \varepsilon^n \Phi_{0,n}(s) e^{-nx}\,, \qquad \text{for all } (s,x) \in \Omega\,, \tag{B.1}$$

as $\varepsilon \to 0^+$.

Furthermore we can approximate $\psi_0(s,x)$ arbitrarily precise employing the general n-th term $\psi_{0,n}(s,x)$ of ε-expansion derivation by solving recursively the associated first order ordinary differential equation for $\Phi_{0,n}(s)$ with $n \geq 1$:

$$[\mathbf{P_{0,n}}] \qquad \begin{cases} \Phi_n'(s) + \delta_n \Phi_n(s) - z_n(s) = 0\,, & s \in (0,T] \\ \Phi_n(s) = 0\,, & s = 0\,; \end{cases} \tag{B.2a}$$

$$\text{with} \qquad \delta_n = n\left[\delta - \frac{n-1}{2a}\left(1 + \frac{1}{\Phi_0^2}\right)\right], \tag{B.2b}$$

$$z_n(s) \quad = \quad \begin{cases} -\Phi_{0,0}\,, & n = 1; \\ a^{-1}[\gamma^{-1}\Phi_{0,1}(s) - \Phi_{0,0}^{-3} - 2a]\Phi_{0,1}(s)\,, & n = 2; \\ \frac{n}{a}\left[(\gamma^{-1} - (n-1)\Phi_{0,0}^{-3})\Phi_1(s) - a\right]\Phi_{0,n-1}(s) + \zeta_n(s)\,, & n > 2; \end{cases} \tag{B.2c}$$

where for all $n > 2$,

$$\begin{aligned}
\zeta_n(s) &= -\frac{n}{2a}\sum_{k=2}^{n-2}\Phi_{0,n-k}(s)\left\{(n-1)\left[\frac{\Phi_{0,1}(s)}{\Phi_{0,0}^2}\omega_{k-1}(s)-\omega_k(s)\right]-\frac{\Phi_k(s)}{\gamma}\right\},\\
\omega_n(s) &= \begin{cases}\Phi_{0,0}^{-1}, & n=0;\\ \Phi_{0,0}^{-1}\left[\Phi_{0,0}^{-1}\Phi_{0,n}(s)+\sum\limits_{k=1}^{n-1}\omega_k(s)\Phi_{0,n-k}(s)\right], & n>0.\end{cases}
\end{aligned}$$

Therefore, referring to the notation (B.2b)–(B.2c) introduced above, the solution to Problem $[\mathbf{P_{0,n}}]$ can be derived straightforwardly and attains the subsequent form:

$$\Phi_{0,n}(s) = \int_0^s z_n(u)e^{-\delta_n(s-u)}\,du\,, \qquad n>0\,,\ 0\le s\le T\,. \tag{B.3}$$

Notice that the above derived solution formula is consistent with those previous for the case $n = 1$ or $n = 2$.

B.2 Higher Order Terms in λ-Expansion

Now we pay attention to formulate the $[\mathbf{P_{n,0}}]$ problem associated with the general nth term of the original problem (6.5) asymptotic expansion taken with respect to parameter λ under the assumption of no regular contributions, i.e. $\varepsilon = 0$.

Firstly, let us define

$$\mathscr{F}_k \equiv \{f:[0,T]\times\mathbb{R}\mapsto\mathbb{R}\,|\, f(s,x)=\sum_{i=0}^{k}e^{-i(d-1)x}\alpha_{k,i}(s)\,,\alpha_{k,i}\in C^\infty\}\,. \tag{B.4}$$

Recalling the form of the quadratic term in of λ–expansion we assume that all the terms of the order lower than n can be written such that

$$\varphi_k = \varphi_k(s,x)\in\mathscr{F}_k\,, \qquad \forall 0\le k\le n-1\,. \tag{B.5}$$

Following the procedure of obtaining the quadratic term of the asymptotic expansion with respect to λ firstly, observe that for the general nth term it holds that the first triple of its components satisfies the subsequent

$$\begin{aligned}
&\Big\{\Big[\sum_{n=0}^{\infty}\lambda^n\frac{\partial\varphi_n}{\partial x}\Big]\Big[1+\frac{1}{\gamma^2\zeta^2(\sum\limits_{n=0}^{\infty}\lambda^n\varphi_n)}\Big]\zeta'(\sum_{n=0}^{\infty}\lambda^n\varphi_n)\Big\}\Big|_{\lambda^n}\\
&\approx\left\{\left[\sum_{k=0}^{n}\lambda^k\frac{\partial\varphi_k}{\partial x}\right]\left[\sum_{k=0}^{n}\lambda^k\widetilde{\zeta}_k(s,x)\right]\left[\sum_{k=0}^{n}\lambda^k\varpi_k(s,x)\right]\right\}\Big|_{\lambda^n}\\
&=\sum_{(i_\varphi,i_\zeta,i_\varpi)^T\succeq 0}^{i_\varphi+i_\zeta+i_\omega=n}\frac{\partial\varphi_{i_\varphi}}{\partial x}(s,x)\widetilde{\zeta}_{i_\zeta}(s,x)\varpi_{i_\varpi}(s,x)\,.
\end{aligned} \tag{B.6}$$

Next we recall that the term of the highest order associated with the λ–expansion of the function $\zeta'_\varphi(s,x)$ takes the form of

$$-\varphi_0\frac{\partial_x\varphi_{n+1}}{(\partial_x\varphi_1)^2}e^{-(d-1)x}\,.$$

Even though it contains the unknown term φ_{n+1} linked with the λ^{n+1}th order of expansion, the whole term ϖ_n cancels out as it is multiplied $\partial_x\varphi_0 \equiv 0$. Hence in order to capture φ_n–associated terms we consider $-(n-1)(d-1)e^{-(d-1)x}\partial_x\varphi_n/\varphi_1^n$ the term ϖ_{n-1} in the combination with $\partial_x\varphi_1$ and $\widetilde{\zeta}_0$. The remaining unknown part (for φ_n) is generated only by $\partial_x\varphi_n\widetilde{\zeta}_0\varpi_0$.

Hence, as $\varphi_1 = \Phi_\lambda(s)e^{-(d-1)x}$, and $\varpi_0 = 1 - e^{-(d-1)x}\varphi_0/\partial_x\varphi_1$ the relationship (B.6) can be rewritten as:

$$\begin{aligned} &\frac{\partial\varphi_n}{\partial x}\widetilde{\zeta}_0\left[\varpi_0 - \varphi_0 e^{-(d-1)x}(\partial_x\varphi_1)^{-1}\right] + \sum_{(i_\varphi,i_\zeta,i_\varpi)^T\in\mathscr{C}} \frac{\partial\,\varphi_{i_\varphi}}{\partial x}(s,x)\widetilde{\zeta}_{i_\zeta}(s,x)\varpi_{i_\varpi}(s,x) \\ &\quad = \left[1+\frac{1}{\gamma^2 d^2}\right]\frac{\partial\varphi_n}{\partial x} + \sum_{(i_\varphi,i_\zeta,i_\varpi)^T\in\mathscr{C}} \frac{\partial\,\varphi_{i_\varphi}}{\partial x}(s,x)\widetilde{\zeta}_{i_\zeta}(s,x)\varpi_{i_\varpi}(s,x)\,, \end{aligned} \tag{B.7}$$

where

$$\mathscr{C}_n = \{(i,j,k)\in\mathbb{N}^3 \mid i+j+k=n\,,\ 0\le i,j,k<n\,,\ k\ne n-1\}\,.$$

Observe that under the assumption on the form of terms of order lower than n, i.e. $\varphi_k \in \mathscr{F}_k$ if $k<n$, the sum of terms with multi-indices taken from $\mathscr{C}_n$ can be expressed as a linear combination of products of time–dependent functions with exponential functions of x. Hence, it has a nature of (B.5) and it is a member of $\mathscr{F}_n$.

Next the last pair of components of the λ–expansion linear term is given as follows:

$$\begin{aligned} &\Big\{\sum_{n=0}^{\infty}\lambda^n\varphi_n\Big[\zeta\big(\sum_{n=0}^{\infty}\lambda^n\varphi_n\big) - 2b - \frac{1}{\gamma^2\zeta\big(\sum_{n=0}^{\infty}\lambda^n\varphi_n\big)}\Big]\Big\}\Big|_{\lambda^n} \\ &\qquad\approx \left\{\left[\sum_{k=0}^{n}\lambda^k\varphi_k\right]\left[-2b+\sum_{k=0}^{n}\lambda^k\left[\zeta_k(s,x)-\gamma^{-2}\nu_k(s,x)\right]\right]\right\}\Big|_{\lambda^n} \\ &\qquad= -2b\varphi_n(s,x) + \sum_{k=0}^{n}\varphi_k(s,x)\left[\zeta_{n-k}(s,x)-\gamma^{-2}\nu_{n-k}(s,x)\right]\,, \end{aligned} \tag{B.8}$$

where

$$\nu_n(s,x) = \begin{cases} \zeta_0^{-1}\,, & n=0\,,\\ -\zeta_0^{-1}\sum_{j=0}^{n-1}\nu_j(s,x)\zeta_{n-j}(s,x)\,, & n>0\,. \end{cases}$$

Then, in order to extract the coefficients associated with the unknown function φ_n observe that

$$-\frac{\nu_n}{\gamma^2}(s,x) = \frac{\zeta_n(s,x)}{\gamma^2 d^2} + \frac{\sum_{j=1}^{n-1}\nu_j(s,x)\zeta_{n-j}(s,x)}{\gamma^2 d}\,,\qquad \zeta_n(s,x) = \varphi_n(s,x) - e^{-(d-1)x}\upsilon_{n-1}(s,x)\,,$$

where υ_n represents the nth term of the function $e^{-\int^x \nu f(s,z)\,dz-\varphi_0 x}$ asymptotic expansion performed with respect to λ. Hence, ζ_n is given as the sum of φ_n and lower order terms. Furthermore, the summation above is a function of φ_k, $1\le k\le n-1$ that have already been determined and assuming that each $\varphi_k\in\mathscr{F}_k$ if $k<n$, in the summation we deduce that the summation itself possesses that form but with a small difference – it also includes the term with $e^{-n(d-1)x}$ and so it belongs to $\mathscr{F}_n$.

Therefore in order to group the coefficients of (B.8) corresponding the unknown φ_n besides $-2b\varphi_n$ we pay attention only to the zero and last term in the summation expression present in (B.8). Hence rearrange (B.8) as subsequently:

$$\begin{aligned}\varphi_n(s,x)\left[-2b+d\left(1-\frac{1}{\gamma^2 d^2}\right)+\varphi_0\left(1+\frac{1}{\gamma^2 d^2}\right)\right]+\frac{\sum_{j=1}^{n-1}\nu_j(s,x)\zeta_{n-j}(s,x)}{\gamma^2 d}\\ -e^{-(d-1)x}\upsilon_{n-1}(s,x)+\sum_{k=1}^{n-1}\varphi_k(s,x)\left[\zeta_{n-k}(s,x)-\gamma^{-2}\nu_{n-k}(s,x)\right].\end{aligned} \tag{B.9}$$

Now let us assume that $\varphi_k(s,x)$ has already been determined for all $0\le k\le n-1$ and all $\varphi_k(s,x)$ can be written in terms of (B.5), hence all $\Phi_{k,0}$ are known. Therefore applying the knowledge of zero contribution rate ($\varepsilon=0$) and combining (B.7) and (B.9) one may formulate the problem $[\mathbf{P_{n,0}}]$ for the unknown $\psi_{n,0}=\psi_{n,0}(s,x)$ arising from the linear transform of the function φ to ψ (see the relationship (6.9)) in the forthcoming manner

$$\frac{\partial\psi_{n,0}}{\partial s}(s,x)=\frac{1}{2a}\frac{\partial}{\partial x}\left\{\left[1+\frac{1}{\psi_{0,0}^2}\right]\frac{\partial\psi_{n,0}}{\partial x}+\left[1+\frac{1}{\psi_{0,0}^2}+2a\delta\right]\psi_{n,0}(s,x)+\Omega_n(s,x)\right\} \tag{B.10}$$

where $\Omega_n(s,x)$ is known function of all sub–function associated with ψ_k of order lower than n having the subsequent form of a linear combination of $n+1$ known linearly independent time–space separable functions, so $\Omega_n(s,x)\in\mathscr{F}_n$ and

$$\Omega_n(s,x)=\sum_{i=0}^{n}e^{-i(d-1)x}\beta_{n,i}(s)$$

for some C^∞ smooth time dependent functions $\beta_{n,i}=\beta_{n,i}(s)$ defined for $s\in[0,T]$, $i=0,\dots,n$. Evidently, recalling (6.8) the prescription of the problem (B.10) satisfies the power–like initial condition

$$\psi_{n,0}(0,x)=(-1)^n\gamma(d-1)e^{-n(d-1)x},\qquad x\in\mathbb{R},\, n\ge 0. \tag{B.11}$$

Apparently, the nature of the components the problem for the general nth term as stated in (B.10) subject to the initial condition (B.11) inspires us to look for the solution to (B.10) in terms of a linear combination of $n+1$ linearly independent time–space separable functions and so w assume that $\psi_n=\psi_n(s,x)$ can be written in terms of $\mathscr{F}_n$, i.e. there exist a set of unknown C^∞ smooth time dependent functions $\alpha_{n,i}=\alpha_{n,i}(s)$ defined for $s\in[0,T]$, $i=0,\dots,n$ such that

$$\psi_n(s,x)=\sum_{i=0}^{n}e^{-i(d-1)x}\alpha_{n,i}(s). \tag{B.12}$$

Therefore, plugging (B.12) into (B.10)–(B.11) leads to the subsequent sequence of the Cauchy – type ordinary differential equations

$$\begin{cases}\alpha'_{n,k}(s)=\dfrac{k(d-1)}{2a}\left[\left(1+\dfrac{1}{\gamma^2 d^2}\right)[k(d-1)-1]-2a\delta\right]\alpha_{n,k}(s) & \\ \qquad\qquad -\dfrac{k(d-1)}{2a}\beta_{n,k}(s), & s\in(0,T],\ 0\le k\le n,\\ \alpha_{n,n}(0)=(-1)^n\gamma(d-1), & s=0,\ 0\le k=n,\\ \alpha_{n,k}(0)=0, & s=0,\ 0\le k<n.\end{cases} \tag{B.13}$$

Obviously, the solution to the problem above can be written as follows:

$$\begin{cases} \alpha_{n,k}(s) = -\dfrac{k(d-1)}{2a} \int_0^s \beta_{n,k}(\tau) e^{\widetilde{\delta}_k (s-\tau)} d\tau\,, & k < n\,, \\ \alpha_{n,n}(s) = -\dfrac{n(d-1)}{2a} e^{\widetilde{\delta}_n s} \left[\int_0^s \beta_{n,n}(\tau) e^{-\widetilde{\delta}_k \tau} d\tau + (-1)^{n+1} \dfrac{2a\gamma}{n} \right]\,, & k = n\,, \end{cases} \tag{B.14}$$

where the eigenvalues $\widetilde{\delta}_k$ satisfy the subsequent relationship

$$\widetilde{\delta}_k = \frac{k(d-1)}{2a} \left[\left(1 + \frac{1}{\gamma^2 d^2} \right) [k(d-1)-1] - 2a\delta \right]\,, \qquad 1 \leq k \leq n\,. \tag{B.15}$$

So combining (B.12) and (B.14) after the transformation of ψ_n to φ_n one may easily deduce the general nth term of the asymptotic expansion of φ. Notice that the above derived solution formula is consistent with those previous for the case $n = 1$ or $n = 2$.

Appendix C

MODEL IMPLEMENTATION

In order to demonstrate the graphical output of the studied model under various scenarios and problem dimensions considered we employed the software *Wolfram Mathematica 9.0*. Therefore all the key codes presented in the forthcoming text are written using this language.

We show the process of calculation the optimal investment policy for all the situations described in Section 7, namely the *One–Stock–One–Bond* problem, *Two–Stocks–One–Bond* problem and *One–Stock–Two–Bonds* problem. Furthermore we provide the major code snippets defining the procedure for the general *N*–dimensional problem.

C.1 One–Stock–One–Bond Case

Firstly referring to Section 7.2 we demonstrate how to determine the optimal policy providing that only one stock and one bond are considered.

Below we depict how to treat the model inputs and calculate its parameters later on used in the optimal strategy formula.

```
(* Input Data *)

(* returns *)
mu = {0.0439, 0.1053};
one = {1, 1};
deltaMu = mu.{0, 1} - mu.{1, 0};
(* volatilities and correlation of returns *)
s0 = 0.0036; s1 = 0.1423; rho = -0.8344;
s = Inverse[{{s0^2, rho s0 s1}, {rho s0 s1, s1^2}}];
(* coefficients *)
a = one . s . one;
b = one . s . mu;
c = mu  . s . mu;
(* other parameters *)
T = 40; betta = 0.035; d = 10;
(* small parameters *)
lambda = 0.04; epsilon = 0.06;

(* parameters used in the optimal solution formula *)
gamma = (a c - b^2)^(-0.5);
```

```
delta = (b - d) / a - betta;
tau = 1 + (gamma d)^(-2);
alpha = ((d - 1) / (2 a))((d - 2) tau - 2 a delta);
pi = 2 / ((d - 2) - 2 a delta / tau);
```

Recalling the relationship (7.3) we define $\omega = \omega(t,y)$ the approximation of the inverse of the function ζ in terms of `Omega[t,y]` as subsequently:

```
(* Functions and Subfunctions *)
VarphiEpsilon[t_, y_]:= (1 - Exp[- delta (T - t)]) / (y delta);
VarphiLambda[t_, y_]:= (d y^(d - 1))^(-1)
    * (1 + (d - 1)((1 + pi)Exp[alpha (T - t)] - pi));

(* Omega enters directly the optimal strategy formula and composes of two
   double (epsilon, lambda) asymptotic expansion related terms*)
Omega[t_, y_]:= 1 + epsilon  VarphiEpsilon[t, y]
              + lambda VarphiLambda[t, y];
```

Finally, the optimal policy `theta[t,y]` as a function of both time and space variables is compiled as follows:

```
alphaLimit = s0^2 + s1^2 - 2 rho s0 s1;
betaLimit = s0 (s0 - rho s1);

theta [t_, y_] := Min[1, (betaLimit + (deltaMu / d) Omega[t, y])
                          /(alphaLimit)];
```

C.2 Three Assets Case

The code presented in this text is applicable for both examples presented in Section 7.3: Two–Stocks–One–Bond problem (Section 7.3.2) and One–Stock–Two–Bonds problem (Section 7.3.3).

First of all remark that we generalize the same mechanism to calculate optimal strategy parameters and itself as introduced in Section C.1.
Hence, the procedure `CalculateCoefficients[index]` takes as an argument the set `index` of all active (i.e. strictly positive) indices.

```
CalculateCoefficients[index_]:= Do[
    (* financial market parameters *)
    (* Return *)
    muSubset = mu[[index]];
    (* Covariance matrix *)
    sigmaSubset = Sigma[[index,  index]];
    sigmaInvSubset = Inverse[sigmaSubset];
    cnt = Length[index];
    oneSubset = Table[1,{cnt}];
```

```
    (* coefficients *)
    a = oneSubset.sigmaInvSubset.oneSubset;
    b = oneSubset.sigmaInvSubset.muSubset;
    c = muSubset.sigmaInvSubset.muSubset;
    mult = a muSubset - b oneSubset;

    (* parameters used in the optimal solution formula *)
    gamma = (a c - b^2)^(-0.5);
    delta = (b - d) / a - betta;
    tau = 1 + (gamma d)^(-2);
    alpha = ((d - 1) / (2a))((d - 2) tau - 2 a delta);
    pi = 2 / ((d - 2) - 2 a delta / tau);
    ,
    {1}
];
```

Below in the function `OptimalUnconstrainedStrategy[indexList]` we demonstrate the code used to determine the optimal unconstrained strategies for each of the following situations: all indices active, one index inactive.

```
indexList = Range[3];
strategiesList = {};

OptimalUnconstrainedStrategy[indexList_]:= For[i = 0, i <=3,
    (* remove the unnecessary rows & columns from the return vector and
       covariance matrix, but not if all indices are active *)
    index := If[ i = 0, indexList, Drop[indexList,{4 - i}]];

    (* Calculate parameters for the solution given the currently used
       set of active indices *)
    CalculateCoefficients[index];

    (* functions *)
    theta[t, y]:= (1 / a) sigmaSubset . oneSubset
            + (Omega[t, y] / (a d)) sigmaSubset . mult;
    (* Two dimensional case must be handled in a special way as its
       optimal solution is cut by [0,1] *)
    If [cnt <= 2, theta[t, y] := {{Max[0,  Min[1, {1,0}.theta[t, y]]]},
     {Max[0, Min[1, {0,1}.theta[t, y]]]}}];


    (* Append the calculated solution for the given subset of indices to
       the list of all available strategies *)
    AppendTo[strategiesList, {index, theta[t, y]}];
    i++;
];
OptimalUnconstrainedStrategy[indexList];
```

Then we combine the solutions obtained for each of the case of active indices into one. The process of determination the optimal policy at point (t, y) can be illustrated by the forthcoming scheme, where x represents the unconstrained strategy with all indices active and `x3` , x2 , x1 stand for the optimal unconstrained strategies where the third, second, or first index, respectively, are do not play role:

```
theta[t_, y_]:= Which[
    (* If all indices are active i.e. positive then take the unconstrained
       solution *)
    x[t, y] [[1]]> 0 &&  x[t, y] [[2]]> 0 && x[t, y] [[3]]> 0,
        x[t, y],
     (* Otherwise if the third index is not active, take the 2D
        unconstrained solution x3[t,y] obtained for the for the first two
        indices. Then, cut it to be bounded by [0,1]. *)
        x[t, y] [[1]]<= 0 , Which[
            x3[t, y] >= 1, {0,0,1},
            x3[t, y] <= 0, {0,1,0},
            true,          {0, 1 - x3[t, y], x3[t, y]}
        ],
     (* Otherwise if the second index is not active, take the 2D
        unconstrained solution x2[t,y] obtained for the for the first and
        last index. Then, cut it to be bounded by [0,1]. *)
        x[t, y] [[2]] <=  0 , Which[
            x2[t, y] >= 1, {0,0,1},
            x2[t, y] <= 0, {1,0,0},
            true,          {1 - x2[t, y], 0, x2[t, y]}
        ],
     (* Finally providing that none from the cases above hold, the first
        index must be non-positive. Thus the 2D unconstrained solution
        x1[t,y] obtained for the last two indices and cut it to be
        bounded by [0,1]. *)
        true, Which[
            x1[t, y] >= 1, {0,1,0},
            x1[t, y] <= 0, {1,0,0},
            true,          {1 - x1[t, y], x1[t, y], 0}
        ]
]
```

Since in order to simplify the process of finding the optimal strategy we use here the special way in which the tree of all subsets and the corresponding unconstrained strategies is built and so that we can simplify the scheme launched above in the following manner.
Indeed in the procedure `CompileOptimalPolicy[t,y]` we ransack the list of all partial solutions that correspond to subsets of active indices to determine the one with all components positive:

```
(* Check whether all components of the first input parameter (vector) are
   nonnegative and even positive on the index set *)
IsPositive[vec_, indexSet_] := Function[{u, v}, Do[
    ind = 0;
    While[++ ind <= Length[ v ] + 1 & u[v[ind]] > 0];
    Return ind == 4 &  Min[ u ] >= 0 ;
    ,
    {1}
];
][vec, indexSet];
```

```
(* For each [t,y] determine the optimal policy *)
CompileOptimalPolicy[t_, y_]:= Do[

    ind = 0;
    While[IsNonnegative[ ++ ind <= 4
         & strategiesList[[ind, 2]][t, y]] == false];
    Return[strategiesList[[ind - 1, 2]]];
    ,
    {1}
];
```

C.3 General Case

For a given dimension N of the problem, generate the list of unconstrained optimal solution associated with the sets $\mathscr{I}_\emptyset \cup \bigcup_{1 \le |S| \le N-1} int\,(\mathscr{I}_S)$ for any $S \subset \{1,\ldots,N\}$ defined as

$$\mathscr{I}_\emptyset = \left\{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) > 0,\, \forall i = 1,\ldots,N\right\}, \qquad \mathscr{I}_S = \left\{\zeta > 0 \,|\, \hat{\theta}_i(\zeta) = 0 \iff i \in S\right\}.$$

```
(* define the index list *)
indexList = Range[N];

(* generate all subsets of the indexList containing at least two
   elements *)
allSubsets = Subsets[indexList, {2, N}];
strategiesList = {};

OptimalStrategy[strategies_, index_] := Do[

    cnt = Length[index];
    (* Extend the table of strategies associated with index subsets by
       the newly created pair: index subset and the corresponding
       unconstrained solution *)
    bound = Subsets[index, {len - 1}];
    AppendTo[strategies, {index,
        OptimalUnconstrainedStrategy[index]}];

    (* Use formula for two dimensional problem and then terminate this
       branch *)
    If [cnt <= 2, Return[{strategies,
        allSubsets[[Position[allSubsets, index] + 1]]}]];

    (* Recursive call: otherwise, distribute the current problem
       to the subproblems *)
    For[ind = 1, ind <= cnt, OptimalStrategy[strategies, bound[[ind++]]]];
    ,
    {1}
];
```

For a given subset $\mathscr{J} \in \mathscr{I}_{\emptyset} \cup \bigcup_{1 \leq |S| \leq N-1} int\,(\mathscr{I}_S)$ the objective is to find the optimal unconstrained strategy. The function `OptimalUnconstrainedStrategy` input parameter determines the set of active indices in the optimal policy. The process of calculating coefficients and the optimal unconstrained policy function for each subset of $\{1, \ldots, N\}$ of active indices remains the same as in Section C.2.

```
OptimalUnconstrainedStrategy[index_]:= Do[
    (* Calculate parameters for the solution given the currently used set
       of active indices *)
    CalculateCoefficients[index];

    (* functions *)
    theta[t, y]:= (1 / a) sigmaSubset . oneSubset
            + (Omega[t, y] / (a d)) sigmaSubset . mult;

    (* Two dimensional case must be handled in a special way as its
       optimal solution is cut by [0,1] *)
    If [cnt <= 2, theta[t, y] := {{Max[0,  Min[1, {1,0}.theta[t, y]]]},
        {Max[0, Min[1, {0,1}.theta[t, y]]]}}];

    (* return the optimal policy as a function of (t,y)*)
    Return[theta[t, y]];
    ,
    {1}
];
```

Finally, it is necessary to compile the derived branches of unconstrained optimal solutions associated with different subsets into one constrained in each component. This task is accomplished by the algorithm `CompileOptimalPolicy` which employs special construction of the double list `strategiesList`when searching for the correct prescription.

```
(* Check whether all components of the first input parameter (vector) are
   nonnegative and even positive on the index set *)
IsPositive[vec_, indexSet_] := Function[{u, v}, Do[
    ind = 0;
    While[++ ind <= Length[v] + 1 & u[v[ind]] > 0];
    Return ind ==  Length[u] + 1  &  Min[ u ] >= 0 ;
    ,
    {1}
    ];
][vec, indexSet];

(* For each [t,y] determine the optimal policy *)
CompileOptimalPolicy[t_, y_]:= Do[
    ind = 1;
    While[IsPositive[strategiesList[[ind ++, 2]][t, y]] == false,
        ind <= 2^(N - 1)];
    Return[strategiesList[[ind - 1, 2]]];
    ,
    {1}
];
```