Comenius University in Bratislave

Faculty of mathematics, physics and informatics

Július Pačuta

Self-report of the dissertation thesis

A priori estimates of solutions of superlinear elliptic and parabolic problems

for obtaining academic title philosophiae doctor

in branch of PhD. study:

9.1.9 Applied mathematics

Bratislava, 2015

Dizertačná práca bola vypracovaná v dennej forme doktorandského štúdia na Katedre aplikovanej matematiky a štatistiky Fakulty matematiky, fyziky a informatiky Univerzity Komenského v Bratislave.

Predkladateľ: Július Pačuta
KAMŠ, FMFI UK
Mlynská dolina
842 48 Bratislava

Školiteľ: prof. RNDr. Pavol Quittner, DrSc.
KAMŠ, FMFI UK
Mlynská dolina
842 48 Bratislava

Oponenti: ....
....
....
....
....
....

Obhajoba dizertačnej práce sa koná .... o .... h
pred komisiou pre obhajobu dizertačnej práce v odbore doktorandského štúdia vymenovanou predsedom odborovej komisie prof. RNDr. Marekom Filom, DrSc.

v študijnom odbore 9.1.9 Aplikovaná matematika

na Fakulte matematiky, fyziky a informatiky UK, Mlynská dolina, 842 48 Bratislava, miestnosť ……..

Predseda odborovej komisie:
prof. RNDr. Marek Fila, DrSc.
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského
Mlynská Dolina, 842 48 Bratislava

1 Introduction

Superlinear parabolic problems represent important mathematical models for various phenomena occurring in physics, chemistry or biology. Therefore such problems have been intensively studied by many authors. Beside solving the question of existence, uniqueness, regularity etc. significant effort has been made to obtain a priori estimates of solutions. A priori estimates are important in the study of global solutions (i.e. solutions which exist for all positive times) or blow-up solutions (i.e. solutions whose L^∞-norm becomes unbounded in finite time); superlinear parabolic problems may possess both of these types of solutions. Uniform a priori estimates also play a crucial role in the study of so-called threshold solutions, i.e. solutions lying on the borderline between global existence and blow-up.

Stationary solutions of parabolic problems are particular global solutions and their a priori estimates are of independent interest since they can be used to prove the existence and/or multiplicity of steady states, for example. The proofs of such estimates are usually much easier than the proofs of estimates of time-dependent solutions. On the other hand, the methods of the proofs of a priori estimates of stationary solutions can often be modified to yield a priori estimates of global time-dependent solutions.

In this thesis we will prove a priori estimates for positive solutions of two model problems. In both cases we study a system of two equations in a smoothly bounded domain $\Omega \subset \mathbb{R}^N$ complemented by the homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega$. The problems involve power nonlinearities and have been intensively studied in the past (see Section 2 for known results and precise formulation of our main results). Our approach is based on bootstrap in suitable weighted Lebesgue spaces.

In Section 4 we prove a priori estimates and existence of positive stationary solutions: We consider the elliptic problem

$$\left.\begin{array}{rcll} -\Delta u &=& a(x)|x|^{-\kappa}v^q, & x\in\Omega, \\ -\Delta v &=& b(x)|x|^{-\lambda}u^p, & x\in\Omega, \\ u &=& v \;=\; 0, & x\in\partial\Omega, \end{array}\right\} \tag{1}$$

where $a, b \in L^\infty(\Omega)$ are nonnegative, $\kappa, \lambda \in (0, 2)$, $p, q > 0$, $pq > 1$, and $0 \in \partial\Omega$. We deal with so-called very weak solutions and we find optimal conditions on the exponents κ, λ, p, q guaranteeing a priori estimates and existence of such solutions. These results have been published in [25].

In Section 5 we study global classical positive solutions of the problem

$$\left.\begin{array}{rcll} u_t-\Delta u &=& u^r v^p, & (x,t)\in\Omega\times(0,\infty), \\ v_t-\Delta v &=& u^q v^s, & (x,t)\in\Omega\times(0,\infty), \\ u(x,t) &=& v(x,t) \;=\; 0, & (x,t)\in\partial\Omega\times(0,\infty), \\ u(x,0) &=& u_0(x), & x\in\Omega, \\ v(x,0) &=& v_0(x), & x\in\Omega \end{array}\right\} \tag{2}$$

where $p, q, r, s \geq 0$. In this case, optimal conditions on the exponents p, q, r, s guaranteeing a priori estimates and existence of positive stationary very weak solutions have been obtained in [30], and we find sufficient conditions on the exponents guaranteeing uniform a priori estimates of global classical solutions. Our method is in some sense similar to that used in [30] (both methods are based on bootstrap in weighted Lebesgue spaces and estimates of auxiliary functions of the form $u^a v^{1-a}$) but our proofs are much more involved. In particular, we have to use precise estimates of the Dirichlet heat semigroup and several additional ad-hoc arguments. These difficulties cause that our sufficient conditions are quite technical and probably not optimal. On the other hand, our results are new and our approach is also new in the parabolic setting: Although the bootstrap in weighted Lebesgue spaces has been used many times in the case of superlinear elliptic problems (see the references in [30], for example), it has not yet been used to prove a priori estimates of global solutions of superlinear parabolic problems. In fact, the known methods for obtaining such estimates always require some special structure of the problem (see a more detailed discussion in Section 2) and cannot be used for system (2) in general. In addition, our method is quite robust: It can also be used if the problem is perturbed or if we replace the Dirichlet boundary conditions by the Neumann ones, for example.

This thesis is organized as follows. In Section 2 we discuss known results and methods of proofs of a priori estimates of stationary and time-dependent solutions of superlinear parabolic problems and we also formulate our main results. Section 3 contains preliminary lemmas and inequalities that we need in subsequent sections. In Section 4 we prove a priori estimates and existence of positive solutions of system (1). In Section 5 we prove a priori estimates of positive global solutions of problem (2).

2 Known and main results

Unless stated otherwise, in the whole section we assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and by a solution we mean a nonnegative classical solution.

2.1 Elliptic scalar case

One of the simplest examples of superlinear elliptic problems is the Dirichlet problem for the Lane-Emden (or Lane-Emden-Fowler) equation (see [21, 13, 17]):

$$\begin{aligned} -\Delta u &= u^p, \quad x \in \Omega, \\ u &= 0, \quad\ \ x \in \partial\Omega, \end{aligned} \tag{3}$$

where $p > 1$. The motivation for the study of this problem originates in astrophysics (see [21, 13]) but this problem and its modifications also play a crucial role in the study of the standing wave solutions of the nonlinear Schrödinger equation or in the differential geometry (the Yamabe problem). Of course, solutions of (3) are also steady states of the corresponding nonlinear heat (or wave) equation. Finally, problem (3) and its parabolic counterpart are very useful model problems: On one hand, they look very simple so that it is definitely easier to study their solutions than those of more complicated systems, and the methods of the proofs developed for these model problems can often be used for more complicated ones. On the other hand, the structure of these model problems is extremely rich and their study represents a great mathematical challenge: In spite of their intensive study (see [34] and the references therein), they still offer many open questions.

Let us mention some results about the existence and a priori estimates of solutions of problem (3). If Ω is starshaped then a positive solution of (3) exists if and only if $p < p_S$ where

$$p_S = \frac{N+2}{(N-2)^+}$$

is the so-called Sobolev exponent, see [1, 27]. The history of a priori estimates of positive solutions of (3) is quite long. They have been proved first in [37] if $N = 2$ and $p < 3$, and then in [24] if $p < N/(N-1)$, in [6] if $p < (N+1)/(N-1)$ and, finally, in [18, 10] if $p < p_S$. More precisely, the following theorem was proved in [18, 10]:

Theorem 2.1. *Let Ω be a smooth bounded domain and $p \in (1, p_S)$. Then there exists a constant C such that for all positive solutions u of* (3) *satisfy the estimate $\|u\|_\infty \le C$.*

The methods of the proof Theorem 2.1 in [18, 10] were quite different: The method in [18] was based on scaling arguments and the corresponding Liouville theorem from [19] (guaranteeing the nonexistence of positive solutions of (3) for $\Omega = \mathbb{R}^N$ and $1 < p < p_S$); the method in [10] was based on the method of moving planes and the Pohozaev identity (which requires a special structure of the problem).

Interestingly, the exponent $p = (N+1)/(N-1)$ is also critical for problem (3) in some sense. More precisely, so-called very weak solutions of problem (3) are known to be bounded (and, consequently, satisfy the a priori estimate in Theorem 2.1) if and only if $p < (N+1)/(N-1)$, see [36, 12]. In addition, the proof of the boundedness and a priori estimates of very weak solutions is quite easy (it is sufficient to use a relatively simple bootstrap argument in weighted Lebesgue spaces, see [33]), and can be used in a much more general situation, where both methods from [18, 10] fail (see [30] and the references therein).

2.2 Elliptic vector case

The method based on bootstrap in weighted Lebesgue spaces mentioned at the end of Subsection 2.1 has been successfully used for many elliptic systems, see [33, 30] and the references therein. Of course, each particular use of this method usually requires some extra ad-hoc arguments. In particular, in the case of stationary solutions of problem (2), one of such ad-hoc arguments was a universal bound of the auxiliary function $u^a v^{1-a}$ for suitable $a \in (0,1)$. Using this argument and the notation

$$\alpha := 2\frac{p+1-s}{pq-(1-r)(1-s)}, \quad \beta := 2\frac{q+1-r}{pq-(1-r)(1-s)},$$

the following theorem was proved in [30]:

Theorem 2.2. *Let Ω be a smooth bounded domain, $p, q, r, s \ge 0$ satisfy*

$$pq \ne (1-r)(1-s) \tag{4}$$

and

$$\left.\begin{array}{c} \min\{p+r, q+s\}, r, s < \frac{N+1}{N-1}, \\ \text{if } pq > (1-r)(1-s) \text{ then } \max\{\alpha, \beta\} > N-1. \end{array}\right\} \tag{5}$$

Then there exists a positive stationary solution of (2)*. In addition, there exists a positive constant C depending on Ω, N, p, q, r, s such that $\|u\|_\infty + \|v\|_\infty < C$ for any positive very weak stationary solution of* (2)*.*

The nondegeneracy condition (4) in Theorem 2.2 is also necessary for the existence and a priori estimates of (classical) positive stationary solutions of (2), and the subcriticality condition (5) is also optimal for the boundedness of very weak positive stationary solutions of (2) (it corresponds to the condition $p < (N+1)/(N-1)$ for the scalar problem (3)).

On the other hand, it is known that condition (5) is not necessary for the existence and a priori estimates of classical positive stationary solutions of (2): An optimal condition for general p, q, r, s does not seem to be known, see the discussion in [30]. We will use similar approach as in [30] in order to find sufficient conditions on p, q, r, s guaranteeing uniform a priori estimates of global (time-dependent) classical positive solutions of (2).

In this thesis we also use bootstrap in weighted Lebesgue spaces in order to prove a priori estimates and existence of positive very weak solutions of the non-homogeneous elliptic system (1), where

$$p, q > 0,\ pq > 1, \quad a, b \in L^\infty(\Omega),\ a, b \geq 0,\ a, b \not\equiv 0 \tag{6}$$

and some additional assumptions are satisfied. We say that (u, v) is a very weak solution of (1) if $u, v \in L^1(\Omega)$, the right-hand sides in (1) belong to the weighted Lebesgue space $L^1(\Omega; \operatorname{dist}(x, \partial\Omega)\, \mathrm{d}x)$ and

$$-\int_\Omega u\Delta\varphi\, \mathrm{d}x = \int_\Omega a(x)|x|^{-\kappa} v^q \varphi\, \mathrm{d}x, \quad -\int_\Omega v\Delta\varphi\, \mathrm{d}x = \int_\Omega b(x)|x|^{-\lambda} u^p \varphi\, \mathrm{d}x \tag{7}$$

for every $\varphi \in C^2(\overline{\Omega}),\ \varphi = 0$ on $\partial\Omega$.

Problem (1) with $\kappa = \lambda = 0$ has been widely studied. Concerning very weak solutions, necessary and sufficient conditions for their boundedness were found in [5], [33] and [36]. In those papers the existence of very weak solution was studied, as well.

Problem (1) with $a = b \equiv 1$, $0 \in \Omega$ and general $\kappa, \lambda \in \mathbb{R}$ has been studied by several authors, who were mainly interested in the existence of classical solutions (if $\max\{\kappa, \lambda\} \leq 0$) or solutions of the class $C^2(\Omega\backslash\{0\}) \cap C(\Omega)$ (if $\max\{\kappa, \lambda\} > 0$). If $\max\{\kappa, \lambda\} \geq 2$, then (1) has no positive solution in this class for any domain Ω containing the origin; see [3]. If $\max\{\kappa, \lambda\} < 2$, Ω is a bounded starshaped domain and some additional assumptions are satisfied, then (1) has a positive solution if and only if the following condition is satisfied

$$\frac{N-\kappa}{1+q} + \frac{N-\lambda}{1+p} > N-2; \tag{8}$$

see e.g. [7], [11], [15], [22] for details. If $\max\{\kappa, \lambda\} < 2$ and $\Omega = \mathbb{R}^N$, $N \geq 3$, then (1) has no positive radial solution if and only if (8) is true. The conjecture is, that if (8) holds, (1) has no positive nonradial solution for $\Omega = \mathbb{R}^N$; see [4]. This conjecture has been partially proved in e.g. [26].

We consider the case $0 \in \partial\Omega$ and $\kappa, \lambda \in (0, 2)$. Our main result guarantees a priori estimates of positive very weak solutions of (1) and its modifications whenever $\max\{\alpha, \beta\} > N - 1$, where

$$\alpha := \frac{(2-\lambda)q + 2 - \kappa}{pq - 1}, \quad \beta := \frac{(2-\kappa)p + 2 - \lambda}{pq - 1}. \tag{9}$$

These estimates enable us also to prove the following existence result.

Theorem 2.3. *Let Ω be a smooth bounded domain, $0 \in \partial\Omega$, $\kappa, \lambda \in (0, 2)$ and assume also* (6)*. Let α, β be defined by* (9)*.*

(i) Assume $\max\{\alpha, \beta\} > N - 1$. Then there exists a positive bounded very weak solution of problem (1) *and each positive very weak solution (u, v) of* (1) *is bounded and satisfies the estimate*

$$\|u\|_\infty + \|v\|_\infty \leq C(\Omega, a, b, p, q, \kappa, \lambda).$$

(ii) Assume $\max\{\alpha, \beta\} < N - 1$. Then there exist functions a, b satisfying (6) *and a positive very weak solution (u, v) of problem* (1) *such that $u, v \notin L^\infty(\Omega)$.*

2.3 Parabolic scalar case

Consider the model parabolic problem

$$\left.\begin{array}{rcll} u_t - \Delta u &=& u^p, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) &=& 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &=& u_0(x), & x \in \Omega \end{array}\right\} \tag{10}$$

where Ω is a bounded domain with smooth boundary, $p > 1$, and $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. It is known that under some restrictions on the exponent p, global positive solutions of (10) satisfy various uniform a priori estimates. Let us mention some of them:

i) a priori estimate depending on the initial data

$$\sup_\Omega u(.,t) \leq C(\Omega, p, u_0) \quad \text{for} \quad t \geq 0,$$

ii) uniform a priori estimate

$$\sup_\Omega u(.,t) \leq C(\Omega, p, \|u_0\|_\infty) \quad \text{for} \quad t \geq 0,$$

iii) universal a priori estimate

$$\sup_\Omega u(.,t) \leq C(\Omega, p, \tau) \quad \text{for} \quad t \geq \tau > 0,$$

where the constant C may explode as $\tau \to 0^+$,

iv) asymptotic a priori bound of the form

$$\limsup_{t\to\infty} \|u(.,t)\|_\infty \leq C(\Omega, p).$$

Estimate of type i) says that each global positive solution of (10) is bounded, uniformly with respect to $t \in (0,\infty)$. Such estimates have been first obtained in [23] for convex domains Ω under the assumption $p < \frac{N+2}{N}$ and then in [8] for general bounded domains and $p < p_S$. The proof in [8] heavily used the variational structure of problem (10). In [23] it was also proved that for $p \geq p_S$, there exist global unbounded weak (so-called L^1) solutions. In fact, it was proved much later that these unbounded weak solutions are classical if $p = p_S$ but they may blow up in finite time if $p > p_S$, see the references in [34].

The stronger estimate of type ii) was derived in [8] for global (not necessarily positive) solutions of problem (10) under the assumption $p < \frac{3N+8}{(3N-4)^+}$ and in [20] for global positive solutions under the optimal assumption $p < p_S$. The positivity assumption in [20] was removed in [29] (the nonlinearity u^p is unterstood as $|u|^{p-1}u$ in the case of sign-changing solutions). All proofs in [8, 20, 29] heavily used the variational structure of problem (10). Estimates of type ii) have several important applications, see [34]. In particular, they guarantee that all threshold solutions lying on the borderline between global existence and blow-up are global, bounded and their ω-limit sets consist of nontrivial steady states (such results cannot be proved by using the weaker estimate of type i)).

Universal estimate of type iii) for global positive solutions of (10) has firstly been obtained in [16] under the assumption $p < \frac{N+1}{N-1}$. The same estimate has then been proved in [31] for $p < p_S$ and $N \leq 3$ and in [35] for $p < p_S$ if $N \leq 4$ and $p < (N-1)/(N-3)^+$ if $N > 4$. Finally, the following quantitative version of estimate of type iii) was proved in [28] and [32].

Theorem 2.4. *Assume that* $p < \frac{N(N+2)}{(N-1)^2}$ *or* $N = 2$ *(or* $p < p_S$*,* Ω *is a ball and* u_0 *is radially symmetric). Then there exists a constant* $C(\Omega, p) > 0$ *such that all global positive classical solutions of* (10) *satisfy the estimate*

$$\sup_{\Omega} u(\cdot, t) \leq C(1 + t^{-1/(p-1)}), \qquad t > 0. \tag{11}$$

Estimate (11) is based on scaling, doubling arguments, and parabolic Liouville theorems for entire solutions of problem (10) in $\mathbb{R}^N \times (-\infty, \infty)$ and $\mathbb{R}^N_+ \times (-\infty, \infty)$ (where $\mathbb{R}^N_+$ is a halfspace). Similarly as in the case of estimates of type ii) and i), all proofs of estimate iii) used the special structure of problem (10). Notice also that estimate iii) implies estimate iv) and estimate iv) implies uniform estimate for stationary positive solutions of (10).

2.4 Parabolic vector case

As mentioned in Subsection 2.3, all proofs of (optimal) a priori estimates of global positive solutions of the scalar problem (10) heavily used the special structure of the problem. In fact, all of them either used directly the variational structure of (10) or the scaling invariance and the validity of suitable parabolic Liouville theorems (which are known due to the special structure of (10)).

Recall that we are interested in problem (2) which, in general, does not have variational structure. In addition, the known parabolic Liouville theorems for (2) in [14] are just of Fujita-type (hence require severe restrictions on the exponents) and the nonexistence of entire solutions is only guaranteed for solutions (u, v) with both components being positive. In fact, if $p, s > 0$, for example, then problem (2) in $\mathbb{R}^N \times (-\infty, \infty)$ always possesses semi-trivial solutions of the form $(u, v) = (C, 0)$, where C is a positive constant, so that standard scaling arguments yielding a priori estimates cannot be used. Due to these facts, there are no results on a priori estimates of global positive solutions of (2) in the general (superlinear) case, even if the global existence and blow-up for (2) have been intensively studied in such general situation. Of course, for some very special choices of exponents p, q, r, s problem (2) does have variational structure and then some of the methods mentioned in Subsection 2.3 can be used. Similarly, if $r = s = 0$, for example, then the semi-trivial solutions mentioned above do not exist, so that one can use the corresponding parabolic Liouville theorems.

Since we wish to prove uniform a priori estimates of global positive solutions of (2) and one of the main applications of such estimates is the proof of global existence and boundedness of threshold solutions lying on the borderline between global existence and blow-up, let us first mention conditions on p, q, r, s guaranteeing that both global and blow-up solutions (hence also threshold solutions) of (2) exist. The following theorem was proved in [2, 38] (see also [9, 39, 40] for other results on blow-up of positive solutions of (2)).

Theorem 2.5. *Let* Ω *be smooth and bounded,* $p, q, r, s \geq 0$*,* $p + r > 0$*,* $q + s > 0$ *and let the initial data* $u_0, v_0 \in C(\bar{\Omega})$ *be nonnegative and satisfy the compatibility conditions.*

(i) Assume that

$$r \leq 1, s \leq 1 \quad \text{and} \quad pq < (1 - r)(1 - s). \tag{12}$$

Then all solutions of (2) *exist globally.*

(ii) Assume that

$$r > 1,\ p > 0,\ q = 0,\ s = 1,\ \lambda_1 < 1,\ r \le 1 + p\frac{1-\lambda_1}{\lambda_1} \tag{13}$$

or

$$s > 1,\ q > 0,\ p = 0,\ r = 1,\ \lambda_1 < 1,\ s \le 1 + q\frac{1-\lambda_1}{\lambda_1}, \tag{14}$$

where λ_1 is the least eigenvalue of the negative Dirichlet Laplacian in Ω. Then, for any initial data $u_0, v_0 \ge 0$, $u_0, v_0 \not\equiv 0$, the solution of (2) blows up in finite time.

(iii) If (12), (13) and (14) do not hold then the solution of (2) exists globally for small initial data, and blows up in finite time for large initial data.

Next we present our main results concerning problem (2). We will assume that

$$\Omega \text{ is smooth and bounded, } u_0, v_0 \in L^\infty(\Omega) \text{ are nonnegative,} \tag{15}$$

and

$$p, q, r, s \ge 0; \qquad \text{if } q = 0 \text{ then either } r > 1 \text{ or } s \le 1. \tag{16}$$

Theorem 2.6. *Assume (15), (16) and $pq > (r-1)(s-1)$. Assume also that either $r > 1$, $p > 0$, $p + r < \frac{N+3}{N+1}$, $s + \frac{2}{N+1}\frac{r-1}{p+r-1} < \frac{N+3}{N+1}$ or $r \le 1$, $0 < p < \frac{2}{N+1}$, $s < \frac{N+3}{N+1}$. Let (u, v) be a global nonegative solution of problem (2). Then there exists $C = C(p, q, r, s, \Omega, \|u(\tau)\|_\infty, \|v(\tau)\|_\infty)$ such that*

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_\infty + \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_\infty \le C \tag{17}$$

for every $T, \tau \ge 0$.

Theorem 2.7. *Assume (15), (16) and either $\max\{r, s\} > 1$ or $pq > (r-1)(s-1)$. Assume also $p \ge 1$, $p + r < \frac{N+3}{N+1}$, $s \le 1$,*

$$(p + r)\left(p - \frac{2}{N+1}\right) + r < 1$$

and

$$0 < q < \frac{1-r}{p - \frac{2}{N+1}}\left(1 - \frac{N-1}{N+1}s\right).$$

Let (u, v) be a global nonnegative solution of problem (2). Then, given $\tau > 0$, there exists $C = C(p, q, r, s, \Omega, \tau, \|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$ such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \le C, \quad t \ge \tau.$$

Remark. The constant C in Theorem 2.7 may explode if $\tau \to 0^+$, and is bounded for $\|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta}$ bounded. By $\|\cdot\|_{1,\delta}$ we denote the norm in the weighted Lebesgue space $L^1(\Omega; \operatorname{dist}(x, \partial\Omega)\,\mathrm{d}x)$.

As already mentioned, the proofs of Theorems 2.6 and 2.7 are mainly based on bootstrap in weighted Lebesgue spaces, universal estimates of auxiliary functions of the form

$u^a v^{1-a}$ and precise estimates of the Dirichlet heat kernel. Our approach can also be used, for example, for the following problem with Neumann boundary conditions

$$\left.\begin{array}{rcll} u_t - \Delta u &=& u^r v^p - \lambda u, & (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta v &=& u^q v^s - \lambda v, & (x,t) \in \Omega \times (0,\infty), \\ u_\nu(x,t) &=& v_\nu(x,t) \;=\; 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &=& u_0(x), & x \in \Omega, \\ v(x,0) &=& v_0(x), & x \in \Omega \end{array}\right\} \tag{18}$$

where Ω, p, q, r, s and u_0, v_0 are as above, $\lambda > 0$ and ν is the outer unit normal on the boundary $\partial\Omega$. The terms $-\lambda u, -\lambda v$ with $\lambda > 0$ are needed in (18), since otherwise (18) cannot admit both global and blow-up positive solutions. Let us also note that in this case one has to work in standard (and not weighted) Lebesgue spaces and that the restrictions on the exponents p, q, r, s are less severe than in the case of Dirichlet boundary conditions: Roughly speaking, one can replace N with $N-1$ in those restrictions (in particular, the condition $p + r < \frac{N+3}{N+1}$ becomes $p + r < \frac{N+2}{N}$ in this case).

If $r = s = 0$ and $p, q > 1$ then the following universal estimate of solutions of problem (2) was proved in [16].

Theorem 2.8. *Assume $r = s = 0$, $1 < p, q < \frac{N+3}{N+1}$ and let $\tau > 0$. There exists a constant $C(\Omega, p, q, \tau) > 0$, such that all nonnegative global classical solutions of (2) satisfy the estimate*

$$\sup_\Omega u(.,t) + \sup_\Omega v(.,t) \leq C(\Omega, p, q, \tau) \quad \textit{for} \quad t \geq \tau. \tag{19}$$

Let us also note that if $r = s = 0$ and $p, q > 1$ then a very easy argument in [16] yields a universal estimate of $\|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta}$ for all $\tau \geq 0$, hence Theorem 2.7 also guarantees estimate (19) in this case and the assumptions on p, q are different from those in Theorem 2.8. In particular, q need not satisfy the condition $q < \frac{N+3}{N+1}$. Of course, if $r = s = 0$ then (as mentioned above) one could also use the parabolic Liouville theorems in [14] together with scaling and doubling arguments to prove quantitative universal estimates. The main advantage of our results and proofs is the fact that we do not need the assumption $r = s = 0$.

Conclusion

The aim of this thesis is to obtain a priori estimates for positive global solutions of problem

$$\left.\begin{array}{rcll} u_t - \Delta u &=& u^r v^p, & (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta v &=& u^q v^s, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) &=& v(x,t) \;=\; 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &=& u_0(x), & x \in \Omega, \\ v(x,0) &=& v_0(x), & x \in \Omega, \end{array}\right\} \tag{20}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $u_0, v_0 \in L^\infty(\Omega)$ are nonnegative functions and $p, q, r, s \geq 0$. For general p, q, r, s, usual methods fail. It turns out that the method from [30] used for an elliptic problem can be modified to yield the desired results. The modification is nontrivial and requires several technical restrictions on the exponents p, q, r, s. Despite these restrictions, our theorems still can be used for several interesting problems studied by other authors.

Beside modifications of the ideas in [30], we also heavily used estimates of Dirichlet heat semigroup in weighted Lebesgue spaces and the variation-of-constants formula. Our method is suitable for many perturbations or modifications of problem (20) and also for problem (18) with homogeneous Neumann boundary conditions.

In the thesis, we also present our results form [25] for the following elliptic problem

$$\left.\begin{array}{rcll} -\Delta u &=& a(x)|x|^{-\kappa}v^q, & x\in\Omega, \\ -\Delta v &=& b(x)|x|^{-\lambda}u^p, & x\in\Omega, \\ u &=& v \;=\; 0, & x\in\partial\Omega, \end{array}\right\} \tag{21}$$

where Ω is a bounded domain with smooth boundary, $p,q>0$, $pq>1$, $a,b\in L^\infty(\Omega)$, $a,b\geq 0$, $a,b\not\equiv 0$, $\kappa,\lambda\in\mathbb{R}$. Using bootstrap in weighted Lebesgue spaces, we proved a priori estimate of nonnegative very weak solutions, and using these estimates and topological degree arguments we also proved the existence of positive very weak solution of (21).

References

[1] Ambrosetti A., Rabinowitz P.H.: *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349–381.

[2] Bai X.: *Finite time blow-up for a reaction-diffusion system in bounded domain*, Z. angew. Math. Phys. 65 (2014), 135–138.

[3] Bidaut-Véron M.F.: *Local behaviour of the solutions of a class of nonlinear elliptic systems*, Adv. Differential Equations 5 (2000), 147–192.

[4] Bidaut-Véron M.F., Giacomini H.: *A new dynamical approach of Emden-Fowler equations and systems*, Adv. Differential Equations 15 (2010), 1033–1082.

[5] Bidaut-Véron M.F., Yarur C.: *Semilinear elliptic equations and systems with measure data: existence and a priori estimates*, Adv. Differential Equations 7 (2002), 257–296.

[6] Brézis H., Turner R.E.L.: *On a class of superlinear elliptic problems*, Commun. Partial Differ. Equations 2 (1977), 601–614.

[7] Calanchi M., Ruf B.: *Radial and non radial solutions for Hardy-Hénon type elliptic systems*, Calc. Var. 38 (2010), 111–133.

[8] Cazenave T., Lions P.L.: *Solutions globales déquations de la chaleur semilinq'eaires*, Commun. Partial Differ. Equations 9 (1984), 955–978.

[9] Chen H.W.: *Global existence and blow-up for a nonlinear reaction-diffusion system*, J. Math. Anal. Appl. 212 (1997), 481–492.

[10] De Figueiredo D.G., Lions P.-L., Nussbaum R.D.: *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. pures et appl. 61 (1982), 41–63.

[11] De Figueiredo D.G., Peral I., Rossi J.D.: *The critical hyperbola for a Hamiltonian elliptic system with weights*, Ann. Mat. Pura Appl. 187 (2008), 531–545.

[12] Del Pino M., Musso M., Pacard F.: *Boundary singularities for weak solutions of semilinear elliptic problems*, J. Funct. Anal. 253 (2007), 241–272.

[13] Emden R.: *Gaskugeln: Anwemdungen der mechanischen Waermetheorie auf kosmologische und meteorologische Probleme*, Teubner, Leipzig (1907).

[14] Escobedo M., Levine, H.A.: *Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations*, Arch. Rational Mech. Anal. 129 (1995), 47–100.

[15] Fazly M.: *Liouville type theorems for stable solutions of certain elliptic systems*, Advanced Nonlinear Studies 12 (2012), 1–17.

[16] Fila M., Souplet Ph., Weissler F.B.: *Linear and nonlinear heat equations in L^q_δ spaces and universal bounds for global solutions*, Math. Ann. 320 (2001), 87–113.

[17] Fowler R.H.: *Further studies of Emden's and similar differential equations*, Quart. Journal Math 2 (1931), 259–288.

[18] Gidas B., Spruck J.: *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations 6 (1981), 883–901.

[19] Gidas B., Spruck J.: *Global and local behavior of positive solutions of nonlinear elliptic equations*, Commun. Pure Appl. Math. 34 (1981), 525–598.

[20] Giga Y.: *A bound for global solutions of semi-linear heat equations*, Comm. Math. Phys. 103 (1986), 415–421.

[21] Lane J.H.: *On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment*, The American Journal of Science and Arts, 2nd ser. 50 (1870), 57–74.

[22] Liu F., Yang J.: *Nontrivial solutions of Hardy-Hénon type elliptic systems*, Acta Math. Sci. Ser. B Engl. Ed. 27 (2007), 673–688.

[23] Ni W.M, Sacks P.E., Tavantzis J.: *On the asymptotic behavior of solutions of certain quasi-linear equations of parabolic type*, J. Differ. Equations 54 (1984), 97–120 .

[24] Nussbaum R.D.: *Positive solutions of nonlinear elliptic boundary value problems*, J. Math. Anal. Appl. 51 (1975), 461–482.

[25] Pačuta J.: *Existence and a priori estimates for semilinear elliptic systems of Hardy type*, Acta Math. Univ. Comenianae 83 (2014), 321–330.

[26] Phan Q.H.: *Liouville-type theorems and bounds of solutions for Hardy-Hénon elliptic systems*, Adv. Differential Equations 17 (2012), 605–634.

[27] Pohozaev S.I.: *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$* , Soviet. Math. Dokl. 5 (1965), 1408–1411.

[28] Poláčik P., Quittner P., Souplet Ph.: *Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part II: parabolic equations*, Indiana Univ. Math. J. 56 (2007), 879–908.

[29] Quittner P.: *A priori bounds for global solutions of a semilinear parabolic problem*, Acta Math. Univ. Comenianae 68 (1999), 195–203.

[30] Quittner P.: *A priori estimates, existence and Liouville theorems for semilinear elliptic systems with power nonlinearities*, Nonlinear Analysis TM&A 102 (2014), 144–158.

[31] Quittner P.: *Universal bound for global positive solutions of a superlinear parabolic problem*, Math. Ann. 320 (2001), 299–305.

[32] Quittner P.: *Liouville theorems for scaling invariant superlinear parabolic problems with gradient structure*, Math. Ann., to appear.

[33] Quittner P., Souplet Ph.: *A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces*, Arch. Rational Mech. Anal. 174 (2004), 49–81.

[34] Quittner P., Souplet Ph.: *Superlinear Parabolic Problems*, Birkhäuser Verlag AG, Basel (2007).

[35] Quittner P., Souplet Ph., Winkler M.: *Initial blow-up rates and universal bounds for nonlinear heat equations*, J. Differ. Equations 196 (2004), 316–339.

[36] Souplet Ph.: *Optimal regularity conditions for elliptic problems via $L^p_\delta(\Omega)$-spaces*, Duke Math. J. 127 (2005), 175–192.

[37] Turner R.E.L.: *A priori bounds for positive solutions of nonlinear elliptic equations in two variables*, Duke Math. J. 41 (1974), 759–774.

[38] Wang M.: *Global existence and finite time blow up for a reaction-diffusion system*, Z. angew. Math. Phys. 51 (2000), 160–167.

[39] Xu X., Ye Z.: *Life span of solutions with large initial data for a class of coupled parabolic systems*, Z. angew. Math. Phys. 64 (2013), 705–717.

[40] Zou H.: *Blow-up rates for semi-linear reaction-diffusion systems*, J. Differential Equations 257 (2014), 843–867.