COMENIUS UNIVERSITY IN BRATISLAVA

FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

A PRIORI ESTIMATES OF SOLUTIONS OF SUPERLINEAR ELLIPTIC AND PARABOLIC PROBLEMS

Dissertation thesis

2015

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Dissertation thesis

Branch of study: Applied mathematics
Study program: 9.1.9 Applied mathematics
Department: Department of applied mathematics and statistics
Tutor: prof. RNDr. Pavol Quittner, DrSc.

Bratislava, 2015 Július Pačuta

Acknowledgement

I would like to thank my tutor prof. RNDr. Pavol Quittner, DrSc. for his valuable help and advices that improved this dissertation thesis.

Thesis title: A priori estimates of solutions of superlinear elliptic and parabolic problems

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Date: 20.04.2015

Abstract: We consider the parabolic system $u_t - \Delta u = u^r v^p$, $v_t - \Delta v = u^q v^s$ in $\Omega \times (0, \infty)$, complemented by the homogeneous Dirichlet boundary conditions and the initial conditions $(u, v)(\cdot, 0) = (u_0, v_0)$ in Ω, where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $u_0, v_0 \in L^\infty(\Omega)$ are nonnegative functions. We find conditions on p, q, r, s guaranteeing a priori estimates of nonnegative classical global solutions. More precisely every such solution is bounded by a constant depending on suitable norm of the initial data. Our proofs are based on bootstrap in weighted Lebesgue spaces, universal estimates of auxiliary functions and estimates of the Dirichlet heat kernel. We also present results from [29] on the elliptic system $-\Delta u = a(x)|x|^{-\kappa} v^q$, $-\Delta v = b(x)|x|^{-\lambda} u^p$, $x \in \Omega$, complemented by the homogeneous Dirichlet boundary condition, where Ω is a smooth bounded domain, $a, b \in L^\infty(\Omega)$ are nonnegative and not identically zero. Under some assumptions on p, q, κ, λ we prove a priori estimates and existence of positive very weak solutions of the system.

Keywords: parabolic system, elliptic system, a priori estimates, bootstrap

Názov práce:	Apriórne odhady riešení superlineárnych eliptických a parabolických úloh
Pracovisko:	Katedra aplikovanej matematiky a štatistiky, FMFI UK Bratislava
Autor:	Július Pačuta
Školiteľ:	Prof. RNDr. Pavol Quittner, DrSc.
Dátum:	20.04.2015

Abstrakt: Uvažujme parabolický systém $u_t - \Delta u = u^r v^p$, $v_t - \Delta v = u^q v^s$, $(x,t) \in \Omega \times (0,\infty)$, doplnený homogénnymi Dirichletovými okrajovými podmienkami a riešenie (u,v) spĺňa počiatočnú podmienku $(u,v)(\cdot,0) = (u_0,v_0)(x)$, $x \in \Omega$, kde Ω je hladká ohraničená oblasť v $\mathbb{R}^N$, $u_0, v_0 \in L^\infty(\Omega)$ sú nezáporné funkcie. Nájdeme podmienky na p,q,r,s, ktoré zaručujú apriórne odhady nezáporných klasických globálnych riešení. Presnejšie, každé také riešenie je ohraničené konštantou, ktorá závisí na vhodnej norme počiatočných dát. Naše dôkazy sú založené na bootstrape vo vahových Lebesgueových priestoroch, univerzálnych odhadoch pomocných funkcií a odhadoch Dirichletovho tepelného jadra. Taktiež predkladáme výsledky z [29] pre eliptický systém $-\Delta u = a(x)|x|^{-\kappa} v^q$, $-\Delta v = b(x)|x|^{-\lambda} u^p$, $x \in \Omega$, $(x,t) \in \Omega \times (0,\infty)$, doplnený homogénnymi Dirichletovými okrajovými podmienkami, kde Ω je hladká ohraničená oblasť, $a, b \in L^\infty(\Omega)$ sú nezáporné a nie sú identicky rovné nule. Za nejakých podmienok na p,q,κ,λ dokážeme apriórne odhady a existenciu kladných veľmi slabých riešení systému.

Kľúčové slová: parabolický systém, eliptický systém, apriórne odhady, bootstrap

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1 Introduction

Superlinear parabolic problems represent important mathematical models for various phenomena occurring in physics, chemistry or biology. Therefore such problems have been intensively studied by many authors. Beside solving the question of existence, uniqueness, regularity etc. significant effort has been made to obtain a priori estimates of solutions. A priori estimates are important in the study of global solutions (i.e. solutions which exist for all positive times) or blow-up solutions (i.e. solutions whose L^∞-norm becomes unbounded in finite time); superlinear parabolic problems may possess both of these types of solutions. Uniform a priori estimates also play a crucial role in the study of so-called threshold solutions, i.e. solutions lying on the borderline between global existence and blow-up.

Stationary solutions of parabolic problems are particular global solutions and their a priori estimates are of independent interest since they can be used to prove the existence and/or multiplicity of steady states, for example. The proofs of such estimates are usually much easier than the proofs of estimates of time-dependent solutions. On the other hand, the methods of the proofs of a priori estimates of stationary solutions can often be modified to yield a priori estimates of global time-dependent solutions.

In this thesis we will prove a priori estimates for positive solutions of two model problems. In both cases we study a system of two equations in a smoothly bounded domain $\Omega \subset \mathbb{R}^N$ complemented by the homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega$. The problems involve power nonlinearities and have been intensively studied in the past (see Section 2 for known results and precise formulation of our main results). Our approach is based on bootstrap in suitable weighted Lebesgue spaces.

In Section 4 we prove a priori estimates and existence of positive stationary solutions: We consider the elliptic problem

$$\left.\begin{aligned} -\Delta u &= a(x)|x|^{-\kappa}v^q, && x \in \Omega, \\ -\Delta v &= b(x)|x|^{-\lambda}u^p, && x \in \Omega, \\ u &= v = 0, && x \in \partial\Omega, \end{aligned}\right\} \tag{1}$$

where $a, b \in L^\infty(\Omega)$ are nonnegative, $\kappa, \lambda \in (0,2)$, $p, q > 0$, $pq > 1$, and $0 \in \partial\Omega$. We deal with so-called very weak solutions and we find optimal conditions on the exponents κ, λ, p, q guaranteeing a priori estimates and existence of such solutions. These results have been published in [29].

In Section 5 we study global classical positive solutions of the problem

$$\left.\begin{array}{rcll} u_t - \Delta u &=& u^r v^p, & (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta v &=& u^q v^s, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) &=& v(x,t) \;=\; 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &=& u_0(x), & x \in \Omega, \\ v(x,0) &=& v_0(x), & x \in \Omega \end{array}\right\} \tag{2}$$

where $p, q, r, s \geq 0$. In this case, optimal conditions on the exponents p, q, r, s guaranteeing a priori estimates and existence of positive stationary very weak solutions have been obtained in [34], and we find sufficient conditions on the exponents guaranteeing uniform a priori estimates of global classical solutions. Our method is in some sense similar to that used in [34] (both methods are based on bootstrap in weighted Lebesgue spaces and estimates of auxiliary functions of the form $u^a v^{1-a}$) but our proofs are much more involved. In particular, we have to use precise estimates of the Dirichlet heat semigroup and several additional ad-hoc arguments. These difficulties cause that our sufficient conditions are quite technical and probably not optimal. On the other hand, our results are new and our approach is also new in the parabolic setting: Although the bootstrap in weighted Lebesgue spaces has been used many times in the case of superlinear elliptic problems (see the references in [34], for example), it has not yet been used to prove a priori estimates of global solutions of superlinear parabolic problems. In fact, the known methods for obtaining such estimates always require some special structure of the problem (see a more detailed discussion in Section 2) and cannot be used for system (2) in general. In addition, our method is quite robust: It can also be used if the problem is perturbed or if we replace the Dirichlet boundary conditions by the Neumann ones, for example.

This thesis is organized as follows. In Section 2 we discuss known results and methods of proofs of a priori estimates of stationary and time-dependent solutions of superlinear parabolic problems and we also formulate our main results. Section 3 contains preliminary lemmas and inequalities that we need in subsequent sections. In Section 4 we prove a priori estimates and existence of positive solutions of system (1). In Section 5 we prove a priori estimates of positive global solutions of problem (2).

2 Known and main results

Unless stated otherwise, in the whole section we assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and by a solution we mean a nonnegative classical solution.

2.1 Elliptic scalar case

One of the simplest examples of superlinear elliptic problems is the Dirichlet problem for the Lane-Emden (or Lane-Emden-Fowler) equation (see [24, 14, 18]):

$$\begin{aligned} -\Delta u &= u^p, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{3}$$

where $p > 1$. The motivation for the study of this problem originates in astrophysics (see [24, 14]) but this problem and its modifications also play a crucial role in the study of the standing wave solutions of the nonlinear Schrödinger equation or in the differential geometry (the Yamabe problem). Of course, solutions of (3) are also steady states of the corresponding nonlinear heat (or wave) equation. Finally, problem (3) and its parabolic counterpart are very useful model problems: On one hand, they look very simple so that it is definitely easier to study their solutions than those of more complicated systems, and the methods of the proofs developed for these model problems can often be used for more complicated ones. On the other hand, the structure of these model problems is extremely rich and their study represents a great mathematical challenge: In spite of their intensive study (see [38] and the references therein), they still offer many open questions.

Let us mention some results about the existence and a priori estimates of solutions of problem (3). If Ω is starshaped then a positive solution of (3) exists if and only if $p < p_S$ where

$$p_S = \frac{N+2}{(N-2)^+}$$

is the so-called Sobolev exponent, see [1, 31]. The history of a priori estimates of positive solutions of (3) is quite long. They have been proved first in [41] if $N = 2$ and $p < 3$, and then in [27] if $p < N/(N-1)$, in [6] if $p < (N+1)/(N-1)$ and, finally, in [19, 11] if $p < p_S$. More precisely, the following theorem was proved in [19, 11]:

Theorem 2.1. *Let Ω be a smooth bounded domain and $p \in (1, p_S)$. Then there exists a constant C such that for all positive solutions u of* (3) *satisfy the estimate $\|u\|_\infty \le C$.*

The methods of the proof Theorem 2.1 in [19, 11] were quite different: The method in [19] was based on scaling arguments and the corresponding Liouville theorem from [20] (guaranteeing the nonexistence of positive solutions of (3) for $\Omega = \mathbb{R}^N$ and $1 < p < p_S$); the method in [11] was based on the method of moving planes and the Pohozaev identity (which requires a special structure of the problem).

Interestingly, the exponent $p = (N+1)/(N-1)$ is also critical for problem (3) in some sense. More precisely, so-called very weak solutions of problem (3) are known to be bounded (and, consequently, satisfy the a priori estimate in Theorem 2.1) if and only if $p < (N+1)/(N-1)$, see [40, 13]. In addition, the proof of the boundedness and a priori estimates of very weak solutions is quite easy (it is sufficient to use a relatively simple bootstrap argument in weighted Lebesgue spaces, see [37]), and can be used in a much more general situation, where both methods from [19, 11] fail (see [34] and the references therein).

2.2 Elliptic vector case

The method based on bootstrap in weighted Lebesgue spaces mentioned at the end of Subsection 2.1 has been successfully used for many elliptic systems, see [37, 34] and the references therein. Of course, each particular use of this method usually requires some extra ad-hoc arguments. In particular, in the case of stationary solutions of problem (2), one of such ad-hoc arguments was a universal bound of the auxiliary function $u^a v^{1-a}$ for suitable $a \in (0,1)$. Using this argument and the notation

$$\alpha := 2\frac{p+1-s}{pq-(1-r)(1-s)}, \quad \beta := 2\frac{q+1-r}{pq-(1-r)(1-s)},$$

the following theorem was proved in [34]:

Theorem 2.2. *Let Ω be a smooth bounded domain, $p,q,r,s \geq 0$ satisfy*

$$pq \neq (1-r)(1-s) \tag{4}$$

and

$$\left.\begin{array}{r} \min\{p+r,q+s\}, r,s < \frac{N+1}{N-1}, \\ \text{if } pq > (1-r)(1-s) \text{ then } \max\{\alpha,\beta\} > N-1. \end{array}\right\} \tag{5}$$

Then there exists a positive stationary solution of (2). *In addition, there exists a positive constant C depending on Ω, N, p, q, r, s such that $\|u\|_\infty + \|v\|_\infty < C$ for any positive very weak stationary solution of* (2).

The nondegeneracy condition (4) in Theorem 2.2 is also necessary for the existence and a priori estimates of (classical) positive stationary solutions of (2), and the subcriticality condition (5) is also optimal for the boundedness of very weak positive stationary solutions of (2) (it corresponds to the condition $p < (N+1)/(N-1)$ for the scalar problem (3)). On the other hand, it is known that condition (5) is not necessary for the existence and a priori estimates of classical positive stationary solutions of (2): An optimal condition for general p,q,r,s does not seem to be known, see the discussion in [34]. We will use similar approach as in [34] in order to find sufficient conditions on p,q,r,s guaranteeing uniform a priori estimates of global (time-dependent) classical positive solutions of (2).

In this thesis we also use bootstrap in weighted Lebesgue spaces in order to prove a priori estimates and existence of positive very weak solutions of the non-homogeneous elliptic system (1), where

$$p, q > 0,\ pq > 1, \quad a, b \in L^\infty(\Omega),\ a, b \geq 0,\ a, b \not\equiv 0 \tag{6}$$

and some additional assumptions are satisfied. We say that (u, v) is a very weak solution of (1) if $u, v \in L^1(\Omega)$, the right-hand sides in (1) belong to the weighted Lebesgue space $L^1(\Omega; \operatorname{dist}(x, \partial\Omega)\, \mathrm{d}x)$ and

$$-\int_\Omega u\Delta\varphi\, \mathrm{d}x = \int_\Omega a(x)|x|^{-\kappa} v^q \varphi\, \mathrm{d}x, \quad -\int_\Omega v\Delta\varphi\, \mathrm{d}x = \int_\Omega b(x)|x|^{-\lambda} u^p \varphi\, \mathrm{d}x \tag{7}$$

for every $\varphi \in C^2(\overline{\Omega}),\ \varphi = 0$ on $\partial\Omega$.

Problem (1) with $\kappa = \lambda = 0$ has been widely studied. Concerning very weak solutions, necessary and sufficient conditions for their boundedness were found in [5], [37] and [40]. In those papers the existence of very weak solution was studied, as well.

Problem (1) with $a = b \equiv 1$, $0 \in \Omega$ and general $\kappa, \lambda \in \mathbb{R}$ has been studied by several authors, who were mainly interested in the existence of classical solutions (if $\max\{\kappa, \lambda\} \leq 0$) or solutions of the class $C^2(\Omega\backslash\{0\}) \cap C(\Omega)$ (if $\max\{\kappa, \lambda\} > 0$). If $\max\{\kappa, \lambda\} \geq 2$, then (48) has no positive solution in this class for any domain Ω containing the origin; see [3]. If $\max\{\kappa, \lambda\} < 2$, Ω is a bounded starshaped domain and some additional assumptions are satisfied, then (1) has a positive solution if and only if the following condition is satisfied

$$\frac{N-\kappa}{1+q} + \frac{N-\lambda}{1+p} > N-2; \tag{8}$$

see e.g. [7], [12], [16], [25] for details. If $\max\{\kappa, \lambda\} < 2$ and $\Omega = \mathbb{R}^N$, $N \geq 3$, then (1) has no positive radial solution if and only if (8) is true. The conjecture is, that if (8) holds, (1) has no positive nonradial solution for $\Omega = \mathbb{R}^N$; see [4]. This conjecture has been partially proved in e.g. [30].

We consider the case $0 \in \partial\Omega$ and $\kappa, \lambda \in (0, 2)$. Our main result guarantees a priori estimates of positive very weak solutions of (1) and its modifications whenever $\max\{\alpha, \beta\} > N - 1$, where

$$\alpha := \frac{(2-\lambda)q + 2 - \kappa}{pq - 1}, \quad \beta := \frac{(2-\kappa)p + 2 - \lambda}{pq - 1}, \tag{9}$$

see Theorem 4.1. These estimates enable us also to prove the following existence result.

Theorem 2.3. *Let Ω be a smooth bounded domain, $0 \in \partial\Omega$, $\kappa, \lambda \in (0, 2)$ and assume also (6). Let α, β be defined by (9).*

(i) Assume $\max\{\alpha,\beta\} > N-1$. *Then there exists a positive bounded very weak solution of problem* (1) *and each positive very weak solution* (u,v) *of* (1) *is bounded and satisfies the estimate*

$$\|u\|_\infty + \|v\|_\infty \le C(\Omega, a, b, p, q, \kappa, \lambda).$$

(ii) Assume $\max\{\alpha,\beta\} < N-1$. *Then there exist functions* a, b *satisfying* (6) *and a positive very weak solution* (u,v) *of problem* (1) *such that* $u, v \notin L^\infty(\Omega)$.

2.3 Parabolic scalar case

Consider the model parabolic problem

$$\left.\begin{array}{rcll} u_t - \Delta u &=& u^p, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) &=& 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &=& u_0(x), & x \in \Omega \end{array}\right\} \tag{10}$$

where Ω is a bounded domain with smooth boundary, $p > 1$, and $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$. It is known that under some restrictions on the exponent p, global positive solutions of (10) satisfy various uniform a priori estimates. Let us mention some of them:

i) a priori estimate depending on the initial data

$$\sup_\Omega u(.,t) \le C(\Omega, p, u_0) \quad \text{for} \quad t \ge 0,$$

ii) uniform a priori estimate

$$\sup_\Omega u(.,t) \le C(\Omega, p, \|u_0\|_\infty) \quad \text{for} \quad t \ge 0,$$

iii) universal a priori estimate

$$\sup_\Omega u(.,t) \le C(\Omega, p, \tau) \quad \text{for} \quad t \ge \tau > 0,$$

where the constant C may explode as $\tau \to 0^+$,

iv) asymptotic a priori bound of the form

$$\limsup_{t\to\infty} \|u(.,t)\|_\infty \le C(\Omega, p).$$

Estimate of type i) says that each global positive solution of (10) is bounded, uniformly with respect to $t \in (0,\infty)$. Such estimates have been first obtained in [26] for convex domains Ω under the assumption $p < \frac{N+2}{N}$ and then in [8] for general bounded domains and $p < p_S$. The proof in [8] heavily used the variational structure of problem (10). In [26] it was also proved that for $p \geq p_S$, there exist global unbounded weak (so-called L^1) solutions. In fact, it was proved much later that these unbounded weak solutions are classical if $p = p_S$ but they may blow up in finite time if $p > p_S$, see the references in [38].

The stronger estimate of type ii) was derived in [8] for global (not necessarily positive) solutions of problem (10) under the assumption $p < \frac{3N+8}{(3N-4)^+}$ and in [21] for global positive solutions under the optimal assumption $p < p_S$. The positivity assumption in [21] was removed in [33] (the nonlinearity u^p is unterstood as $|u|^{p-1}u$ in the case of sign-changing solutions). All proofs in [8, 21, 33] heavily used the variational structure of problem (10). Estimates of type ii) have several important applications, see [38]. In particular, they guarantee that all threshold solutions lying on the borderline between global existence and blow-up are global, bounded and their ω-limit sets consist of nontrivial steady states (such results cannot be proved by using the weaker estimate of type i)).

Universal estimate of type iii) for global positive solutions of (10) has firstly been obtained in [17] under the assumption $p < \frac{N+1}{N-1}$. The same estimate has then been proved in [35] for $p < p_S$ and $N \leq 3$ and in [39] for $p < p_S$ if $N \leq 4$ and $p < (N-1)/(N-3)^+$ if $N > 4$. Finally, the following quantitative version of estimate of type iii) was proved in [32] and [36].

Theorem 2.4. *Assume that* $p < \frac{N(N+2)}{(N-1)^2}$ *or* $N = 2$ *(or* $p < p_S$*,* Ω *is a ball and* u_0 *is radially symmetric). Then there exists a constant* $C(\Omega, p) > 0$ *such that all global positive classical solutions of* (10) *satisfy the estimate*

$$\sup_{\Omega} u(\cdot, t) \leq C(1 + t^{-1/(p-1)}), \qquad t > 0. \tag{11}$$

Estimate (11) is based on scaling, doubling arguments, and parabolic Liouville theorems for entire solutions of problem (10) in $\mathbb{R}^N \times (-\infty, \infty)$ and $\mathbb{R}^N_+ \times (-\infty, \infty)$ (where $\mathbb{R}^N_+$ is a halfspace). Similarly as in the case of estimates of type ii) and i), all proofs of estimate iii) used the special structure of problem (10). Notice also that estimate iii) implies estimate iv) and estimate iv) implies uniform estimate for stationary positive solutions of (10).

2.4 Parabolic vector case

As mentioned in Subsection 2.3, all proofs of (optimal) a priori estimates of global positive solutions of the scalar problem (10) heavily used the special structure of the problem. In fact, all of them either used directly the variational structure of (10) or the scaling invariance and the validity of suitable parabolic Liouville theorems (which are known due to the special structure of (10)).

Recall that we are interested in problem (2) which, in general, does not have variational structure. In addition, the known parabolic Liouville theorems for (2) in [15] are just of Fujita-type (hence require severe restrictions on the exponents) and the nonexistence of entire solutions is only guaranteed for solutions (u,v) with both components being positive. In fact, if $p,s>0$, for example, then problem (2) in $\mathbb{R}^N\times(-\infty,\infty)$ always possesses semi-trivial solutions of the form $(u,v)=(C,0)$, where C is a positive constant, so that standard scaling arguments yielding a priori estimates cannot be used. Due to these facts, there are no results on a priori estimates of global positive solutions of (2) in the general (superlinear) case, even if the global existence and blow-up for (2) have been intensively studied in such general situation. Of course, for some very special choices of exponents p,q,r,s problem (2) does have variational structure and then some of the methods mentioned in Subsection 2.3 can be used. Similarly, if $r=s=0$, for example, then the semi-trivial solutions mentioned above do not exist, so that one can use the corresponding parabolic Liouville theorems.

Since we wish to prove uniform a priori estimates of global positive solutions of (2) and one of the main applications of such estimates is the proof of global existence and boundedness of threshold solutions lying on the borderline between global existence and blow-up, let us first mention conditions on p,q,r,s guaranteeing that both global and blow-up solutions (hence also threshold solutions) of (2) exist. The following theorem was proved in [2, 42] (see also [9, 43, 44] for other results on blow-up of positive solutions of (2)).

Theorem 2.5. *Let Ω be smooth and bounded, $p,q,r,s\geq 0$, $p+r>0$, $q+s>0$ and let the initial data $u_0,v_0\in C(\bar\Omega)$ be nonnegative and satisfy the compatibility conditions.*

(i) Assume that

$$r\leq 1, s\leq 1 \quad \text{and} \quad pq<(1-r)(1-s). \tag{12}$$

Then all solutions of (2) exist globally.

(ii) Assume that

$$r>1,\ p>0,\ q=0,\ s=1,\ \lambda_1<1,\ r\leq 1+p\frac{1-\lambda_1}{\lambda_1} \tag{13}$$

or

$$s>1,\ q>0,\ p=0,\ r=1,\ \lambda_1<1,\ s\leq 1+q\frac{1-\lambda_1}{\lambda_1}, \tag{14}$$

where λ_1 is the least eigenvalue of the negative Dirichlet Laplacian in Ω. Then, for any initial data $u_0,v_0\geq 0$, $u_0,v_0\not\equiv 0$, the solution of (2) blows up in finite time.

(iii) If (12), (13) and (14) do not hold then the solution of (2) exists globally for small initial data, and blows up in finite time for large initial data.

Next we present our main results concerning problem (2). We will assume that

$$\Omega \text{ is smooth and bounded, } u_0, v_0 \in L^\infty(\Omega) \text{ are nonnegative,} \tag{15}$$

and

$$p, q, r, s \geq 0; \qquad \text{if } q = 0 \text{ then either } r > 1 \text{ or } s \leq 1. \tag{16}$$

Theorem 2.6. *Assume* (15), (16) *and* $pq > (r-1)(s-1)$. *Assume also that either* $r > 1$, $p > 0$, $p + r < \frac{N+3}{N+1}$, $s + \frac{2}{N+1}\frac{r-1}{p+r-1} < \frac{N+3}{N+1}$ *or* $r \leq 1$, $0 < p < \frac{2}{N+1}$, $s < \frac{N+3}{N+1}$. *Let* (u, v) *be a global nonnegative solution of problem* (2). *Then there exists* $C = C(p, q, r, s, \Omega, \|u(\tau)\|_\infty, \|v(\tau)\|_\infty)$ *such that*

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_\infty + \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_\infty \leq C \tag{17}$$

for every $T, \tau \geq 0$.

Theorem 2.7. *Assume* (15), (16) *and either* $\max\{r, s\} > 1$ *or* $pq > (r-1)(s-1)$. *Assume also* $p \geq 1$, $p + r < \frac{N+3}{N+1}$, $s \leq 1$,

$$(p + r)\left(p - \frac{2}{N+1}\right) + r < 1$$

and

$$0 < q < \frac{1-r}{p - \frac{2}{N+1}}\left(1 - \frac{N-1}{N+1}s\right).$$

Let (u, v) *be a global nonnegative solution of problem* (2). *Then, given* $\tau > 0$, *there exists* $C = C(p, q, r, s, \Omega, \tau, \|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$ *such that*

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C, \quad t \geq \tau.$$

Remark. The constant C in Theorem 2.7 may explode if $\tau \to 0^+$, and is bounded for $\|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta}$ bounded. By $\|\cdot\|_{1,\delta}$ we denote the norm in the weighted Lebesgue space $L^1(\Omega; \operatorname{dist}(x, \partial\Omega)\,\mathrm{d}x)$.

As already mentioned, the proofs of Theorems 2.6 and 2.7 are mainly based on bootstrap in weighted Lebesgue spaces, universal estimates of auxiliary functions of the form $u^a v^{1-a}$ and precise estimates of the Dirichlet heat kernel. Our approach can also be used, for example, for the following problem with Neumann boundary conditions

$$\left.\begin{array}{rcll} u_t - \Delta u &=& u^r v^p - \lambda u, & (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta v &=& u^q v^s - \lambda v, & (x,t) \in \Omega \times (0,\infty), \\ u_\nu(x,t) &=& v_\nu(x,t) \ = \ 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &=& u_0(x), & x \in \Omega, \\ v(x,0) &=& v_0(x), & x \in \Omega \end{array}\right\} \tag{18}$$

where Ω, p, q, r, s and u_0, v_0 are as above, $\lambda > 0$ and ν is the outer unit normal on the boundary $\partial\Omega$. The terms $-\lambda u, -\lambda v$ with $\lambda > 0$ are needed in (18), since otherwise (18) cannot admit both global and blow-up positive solutions. Let us also note that in this case one has to work in standard (and not weighted) Lebesgue spaces and that the restrictions on the exponents p, q, r, s are less severe than in the case of Dirichlet boundary conditions: Roughly speaking, one can replace N with $N-1$ in those restrictions (in particular, the condition $p + r < \frac{N+3}{N+1}$ becomes $p + r < \frac{N+2}{N}$ in this case).

If $r = s = 0$ and $p, q > 1$ then the following universal estimate of solutions of problem (2) was proved in [17].

Theorem 2.8. *Assume $r = s = 0$, $1 < p, q < \frac{N+3}{N+1}$ and let $\tau > 0$. There exists a constant $C(\Omega, p, q, \tau) > 0$, such that all nonnegative global classical solutions of* (2) *satisfy the estimate*

$$\sup_\Omega u(.,t) + \sup_\Omega v(.,t) \leq C(\Omega, p, q, \tau) \quad \textit{for} \quad t \geq \tau. \tag{19}$$

Let us also note that if $r = s = 0$ and $p, q > 1$ then a very easy argument in [17] yields a universal estimate of $\|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta}$ for all $\tau \geq 0$, hence Theorem 2.7 also guarantees estimate (19) in this case and the assumptions on p, q are different from those in Theorem 2.8. In particular, q need not satisfy the condition $q < \frac{N+3}{N+1}$. Of course, if $r = s = 0$ then (as mentioned above) one could also use the parabolic Liouville theorems in [15] together with scaling and doubling arguments to prove quantitative universal estimates. The main advantage of our results and proofs is the fact that we do not need the assumption $r = s = 0$.

3 Preliminaries

We introduce some notation we will use frequently. Denote

$$\delta(x) = \mathrm{dist}(x, \partial\Omega) \quad \text{for} \quad x \in \Omega,$$

and for $1 \leq p \leq \infty$ define the weighted Lebesgue spaces $L^p_\delta = L^p_\delta(\Omega) := L^p(\Omega; \delta(x)\,\mathrm{d}x)$. If $1 \leq p < \infty$, then the norm in L^p_δ is defined by

$$\|u\|_{p,\delta} = \left(\int_\Omega |u(x)|^p \delta(x)\,\mathrm{d}x \right)^{1/p}.$$

Recall that $L^\infty_\delta = L^\infty(\Omega; \mathrm{d}x)$ with $\|u\|_{\infty,\delta} = \|u\|_\infty$. We will use the notation $\|\cdot\|_p$ for the norm in $L^p(\Omega)$ for $p \in [1, \infty)$, as well.

Let λ_1 be the first eigenvalue of the problem

$$\left.\begin{array}{rcl} -\Delta\phi &=& \lambda\phi, \quad x\in\Omega, \\ \phi &=& 0, \qquad x\in\partial\Omega, \end{array}\right\}$$

and φ_1 to be the corresponding positive eigenfunction satisfying $\|\varphi_1\|_2 = 1$. There holds

$$C(\Omega)\delta(x) \le \varphi_1(x) \le C'(\Omega)\delta(x) \text{ for all } x\in\Omega. \tag{20}$$

Therefore the norm

$$\|u\|_{p,\varphi_1} = \left(\int_\Omega |u(x)|^p\varphi_1(x)\,\mathrm{d}x\right)^{1/p}$$

is equivalent to the norm $\|u\|_{p,\delta}$ in $L^p_\delta(\Omega)$ for $1\le p<\infty$.

Let (u,v) be a solution of system (2). Then (u,v) solves the system of integral equations

$$\begin{aligned} u(\tau+t) &= e^{t\Delta}u(\tau) + \int_\tau^{\tau+t} e^{(\tau+t-s')\Delta}u^r v^p(s')\,\mathrm{d}s', \\ v(\tau+t) &= e^{t\Delta}v(\tau) + \int_\tau^{\tau+t} e^{(\tau+t-s')\Delta}u^q v^s(s')\,\mathrm{d}s' \end{aligned} \tag{21}$$

where $\tau,t\ge 0$ and $\left(e^{t\Delta}\right)_{t\ge 0}$ is the Dirichlet heat semigroup in Ω. In the following lemma we recall some basic properties of the semigroup $\left(e^{t\Delta}\right)_{t\ge 0}$, which we will use often. The corresponding proofs can be found e.g. in [17].

Lemma 3.1. *Let Ω be arbitrary bounded domain.*

i) If $\phi\in L^1_\delta(\Omega)$, $\phi\ge 0$ then $e^{t\Delta}\phi\ge 0$.

ii) $\|e^{t\Delta}\phi\|_{1,\varphi_1} = e^{-\lambda_1 t}\|\phi\|_{1,\varphi_1}$ for $t\ge 0$, $\phi\in L^1_\delta(\Omega)$.

iii) If $p\in(1,\infty)$ then $\|e^{t\Delta}\phi\|_{p,\varphi_1}\le C(\Omega)e^{-\lambda_1 t}\|\phi\|_{p,\varphi_1}$ for $t\ge 0$, $\phi\in L^p_\delta(\Omega)$.

iv) $\|e^{t\Delta}\phi\|_\infty \le C(\Omega)e^{-\lambda_1 t}\|\phi\|_\infty$ for $t\ge 0$, $\phi\in L^\infty(\Omega)$.

v) Let Ω be of the class C^2. For $1\le p<q<\infty$, there exists constant $C=C(\Omega)$ such that, for all $\phi\in L^p_\delta(\Omega)$, it holds

$$\|e^{t\Delta}\phi\|_{q,\delta}\le C(\Omega)t^{-\frac{N+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_{p,\delta}, \quad t>0.$$

vi) Let Ω be of the class C^2. For $1\le p<\infty$, there exists constant $C=C(\Omega)$ such that, for all $\phi\in L^p_\delta(\Omega)$, it holds

$$\|e^{t\Delta}\phi\|_\infty\le C(\Omega)t^{-\frac{N+1}{2p}}\|\phi\|_{p,\delta}, \quad t>0.$$

Assertions iii) and iv) from Lemma 3.1 for $1 \le p < q < \infty$, $t > 0$ and $\varepsilon \in (0,1)$ imply

$$\begin{aligned} \|e^{t\Delta}\phi\|_{q,\delta} &= \|e^{\varepsilon t\Delta}(e^{(1-\varepsilon)t\Delta}\phi)\|_{q,\delta} \le C(\Omega)e^{-\lambda_1\varepsilon t}\|e^{(1-\varepsilon)t\Delta}\phi\|_{q,\delta} \\ &\le C(\Omega)e^{-\lambda_1\varepsilon t}((1-\varepsilon)t)^{-\frac{N+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_{p,\delta}, \quad \phi \in L^p_\delta(\Omega). \end{aligned} \tag{22}$$

Assertions iv) and vi) from Lemma 3.1 for $1 \le p < \infty$, $t > 0$ and $\varepsilon \in (0,1)$ imply

$$\begin{aligned} \|e^{t\Delta}\phi\|_{\infty} &= \|e^{\varepsilon t\Delta}(e^{(1-\varepsilon)t\Delta}\phi)\|_{\infty} \le C(\Omega)e^{-\lambda_1\varepsilon t}\|e^{(1-\varepsilon)t\Delta}\phi\|_{\infty} \\ &\le C(\Omega)e^{-\lambda_1\varepsilon t}((1-\varepsilon)t)^{-\frac{N+1}{2}\frac{1}{p}}\|\phi\|_{p,\delta}, \quad \phi \in L^p_\delta(\Omega). \end{aligned} \tag{23}$$

If we multiply the equations in (21) by φ_1 and integrate on Ω then assertions i) and ii) from Lemma 3.1 imply

$$\begin{aligned} \int_\Omega u(\tau+t)\varphi_1 \,\mathrm{d}x \ge e^{-\lambda_1 t}\int_\Omega u(\tau)\varphi_1 \,\mathrm{d}x, \\ \int_\Omega v(\tau+t)\varphi_1 \,\mathrm{d}x \ge e^{-\lambda_1 t}\int_\Omega v(\tau)\varphi_1 \,\mathrm{d}x. \end{aligned} \tag{24}$$

Let (u,v) be a solution of system (18). Then (u,v) solves the system of integral equations

$$\begin{aligned} u(\tau+t) = e^{tL}u(\tau) + \int_\tau^{\tau+t} e^{(\tau+t-s')L}(u^r v^p)(s') \,\mathrm{d}s', \\ v(\tau+t) = e^{tL}v(\tau) + \int_\tau^{\tau+t} e^{(\tau+t-s')L}(u^q v^s)(s') \,\mathrm{d}s' \end{aligned} \tag{25}$$

where $\tau, t \ge 0$, $e^{tL} := e^{-\lambda t}e^{t\Delta_N}$ is the semigroup corresponding to operator

$$L := \Delta - \lambda$$

with homogeneous Neumann boundary condition and $\left(e^{t\Delta_N}\right)_{t\ge0}$ is the Neumann heat semigroup in Ω. For the Neumann semigroup, the following estimates are true.

Lemma 3.2. *Let Ω be a smoothly bounded domain.*

i) For all $\phi \in L^1(\Omega)$, it holds

$$\|e^{t\Delta_N}\phi\|_1 = \|\phi\|_1, \quad t \ge 0.$$

ii) For $1 \le p < q < \infty$, there exists a constant $C = C(\Omega)$ such that, for all $\phi \in L^p(\Omega)$, it holds

$$\|e^{t\Delta_N}\phi\|_q \le C(\Omega)\left(\min\{1,t\}\right)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_p, \quad t > 0.$$

iii) For $1 \le p < \infty$, there exists a constant $C = C(\Omega)$ such that, for all $\phi \in L^p(\Omega)$, it holds

$$\|e^{t\Delta_N}\phi\|_\infty \le C(\Omega)\,(\min\{1,t\})^{-\frac{N}{2p}}\,\|\phi\|_p, \quad t > 0.$$

The assertions in Lemma 3.2 are proved in [10, 28]. From i) we have

$$\|e^{tL}\phi\|_1 = e^{-\lambda t}\|\phi\|_1$$

and hence we obtain inequalities similar to (24) with φ_1 replaced by 1 and λ_1 replaced by λ.

In the following we will use the notation from [34]. We set

$$A := \begin{cases} [a_r, a_s] \cap (0,1) & \text{if } pq \ge (r-1)(s-1) \text{ or } \min\{r,s\} \le 1, \\ [a_s, a_r] \cap (0,1) & \text{if } pq < (r-1)(s-1) \text{ and } r, s > 1 \end{cases}$$

where

$$a_r := \begin{cases} \frac{r-1}{p+r-1} & \text{if} \quad r > 1, \\ 0 & \text{if} \quad r \le 1, \end{cases} \qquad a_s := \begin{cases} \frac{q}{q+s-1} & \text{if} \quad s > 1, \\ 1 & \text{if} \quad s \le 1. \end{cases}$$

Note that the set A is nonempty provided there holds

$$\left.\begin{array}{l} \text{if } p = 0 \text{ then either } s > 1 \text{ or } r \le 1, \\ \text{if } q = 0 \text{ then either } r > 1 \text{ or } s \le 1. \end{array}\right\} \tag{26}$$

The following lemma is an adaptation of [34, Lemma 7] to systems (2) and (18):

Lemma 3.3. *Assume $p, q, r, s \ge 0$, $pq \ne (1-r)(1-s)$ and (26). For given $a \in A$, there exists $\kappa' \ge 0$ and $C = C(p,q,r,s,a)$ such that any global nonnegative solution of (2) satisfies*

$$(u^a v^{1-a})_t - \Delta(u^a v^{1-a}) \ge F_a(u,v) \ge C(u^a v^{1-a})^{\kappa'}, \quad t \in (0,\infty) \tag{27}$$

where

$$\begin{aligned} F_a(u,v) &:= a u^{a-1} v^{1-a}(u_t - \Delta u) + (1-a) u^a v^{-a}(v_t - \Delta v) \\ &= a u^{r+a-1} v^{p+1-a} + (1-a) u^{q+a} v^{s-a}, \quad t \in (0,\infty). \end{aligned} \tag{28}$$

For any global nonnegative solution (18), there holds

$$(u^a v^{1-a})_t - \Delta(u^a v^{1-a}) + \lambda(u^a v^{1-a}) \ge G_a(u,v) \ge C(u^a v^{1-a})^{\kappa'}, \quad t \in (0,\infty) \tag{29}$$

where

$$\begin{aligned} G_a(u,v) &:= a u^{a-1} v^{1-a}(u_t - \Delta u + \lambda u) + (1-a) u^a v^{-a}(v_t - \Delta v + \lambda v) \\ &= a u^{r+a-1} v^{p+1-a} + (1-a) u^{q+a} v^{s-a}, \quad t \in (0,\infty). \end{aligned} \tag{30}$$

If

$$\max\{r,s\} > 1 \quad \text{or} \quad pq > (r-1)(s-1) \tag{31}$$

then $\kappa' > 1$.

Proof. Let (u, v) be a solution of (2). A direct computation shows

$$(u^a v^{1-a})_{x_i} = au^{a-1}v^{1-a}u_{x_i} + (1-a)u^a v^{-a}v_{x_i}$$

and

$$\begin{aligned}(u^a v^{1-a})_t - \Delta(u^a v^{1-a}) = \quad & au^{a-1}v^{1-a}(u_t - \Delta u) + (1-a)u^a v^{-a}(v_t - \Delta v) \\ + \quad & a(1-a)[u^{a-2}v^{1-a}|\nabla u|^2 + u^a v^{-1-a}|\nabla v|^2 \\ - \quad & 2u^{a-1}v^{-a}\nabla u \cdot \nabla v].\end{aligned}$$

We use the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}(u^a v^{1-a})_t - \Delta(u^a v^{1-a}) \geq \quad & au^{a-1}v^{1-a}(u_t - \Delta u) + (1-a)u^a v^{-a}(v_t - \Delta v) \\ + \quad & a(1-a)[u^{a-2}v^{1-a}|\nabla u|^2 + u^a v^{-1-a}|\nabla v|^2 \\ - \quad & 2u^{a-1}v^{-a}|\nabla u||\nabla v|] \\ = \quad & au^{a-1}v^{1-a}(u_t - \Delta u) + (1-a)u^a v^{-a}(v_t - \Delta v) \\ + \quad & a(1-a)u^{a-2}v^{-1-a}[v|\nabla u| - u|\nabla v|]^2 \\ \geq \quad & F_a(u, v) = au^{r+a-1}v^{p+1-a} + (1-a)u^{q+a}v^{s-a}.\end{aligned}$$

Thus we proved the first inequality in (27). Now we prove the second one. It is sufficient to find $\theta \in [0, 1]$ such that

$$\begin{gathered}\theta(q + a) + (1 - \theta)(r + a - 1) = a\kappa', \\ \theta(s - a) + (1 - \theta)(p + 1 - a) = (1 - a)\kappa'\end{gathered}$$

or equivalently

$$\begin{gathered}r - 1 + \theta(q - r + 1) = a(\kappa' - 1), \\ p - \theta(p - s + 1) = (1 - a)(\kappa' - 1).\end{gathered} \tag{32}$$

Set

$$\begin{gathered}D_\xi := \{\theta \in [0, 1] : \ (r - 1 + \theta(q - r + 1))(p - \theta(p - s + 1)) > 0\}, \\ R_\xi := \{\xi(\theta) : \ \theta \in D_\xi\}\end{gathered}$$

where

$$\xi : D_\xi \to (0, \infty); \quad \xi(\theta) := \frac{r - 1 + \theta(q - r + 1)}{p - \theta(p - s + 1)}.$$

We obtain the derivative of ξ

$$\xi'(\theta) = \frac{pq - (1 - r)(1 - s)}{(p - \theta(p - s + 1))^2}.$$

We see that ξ is monotone and $\xi' > 0$ if and only if $pq > (1 - r)(1 - s)$. Observe that $a \in (0, 1)$ is a solution of (32) with some $\kappa' \neq 1$ and $\theta \in [0, 1]$ if and only if $\frac{a}{1-a} = \xi(\theta)$

for some $\theta \in D_\xi$. Consequently, the second inequality in (27) holds with some $\kappa' \neq 1$, if $a \in A_\xi$, where

$$A_\xi := \left\{ a \in (0,1) : \frac{a}{1-a} \in R_\xi \right\}.$$

If $r, s > 1$ and $pq > (1-r)(1-s)$ then ξ is increasing. Since there holds $\xi(0) = \frac{r-1}{p} > 0$ and $\frac{p}{p-s+1} > 1$ if $p+1 > s$, the function ξ is positive and finite on interval $[0,1]$. Hence it holds

$$D_\xi = [0,1], \quad R_\xi = \left[\frac{r-1}{p}, \frac{q}{s-1}\right], \quad A_\xi = [a_r, a_s], \quad \kappa' > 1.$$

If $r, s > 1$ and $pq < (1-r)(1-s)$ then similarly we obtain

$$A_\xi = [a_s, a_r] \cap (0,1), \quad \kappa' > 1.$$

If $r > 1 \geq s$ then due to (26), $p > 0$ and

$$D_\xi = \left[0, \frac{p}{p-s+1}\right), \quad R_\xi = \left[\frac{r-1}{p}, \infty\right), \quad A_\xi = [a_r, 1), \quad \kappa' > 1.$$

If $s > 1 \geq r$ then $q > 0$ and

$$D_\xi = \left(\frac{1-r}{q-r+1}, 1\right], \quad R_\xi = \left(0, \frac{q}{s-1}\right], \quad A_\xi = (0, a_s], \quad \kappa' > 1.$$

If $r, s \leq 1$ and $pq > (1-r)(1-s)$ then

$$D_\xi = \left(\frac{1-r}{q-r+1}, \frac{p}{p-s+1}\right), \quad R_\xi = (0, \infty), \quad A_\xi = (0,1), \quad \kappa' > 1.$$

If $r, s \leq 1$ and $pq < (1-r)(1-s)$ then

$$D_\xi = \left(\frac{p}{p-s+1}, \frac{1-r}{q-r+1}\right), \quad R_\xi = (0, \infty), \quad A_\xi = (0,1), \quad \kappa' < 1.$$

Now we can write

$$u^{\theta(q+a)+(1-\theta)(r+a-1)} v^{\theta(s-a)+(1-\theta)(p+1-a)} = (u^a v^{1-a})^{\kappa'}.$$

Let $\theta \in D_\xi$. If $\theta \in \{0,1\}$ then there holds (27). If $\theta \in (0,1)$ then we use Young's inequality to obtain

$$(u^{q+a} v^{s-a})^\theta (u^{r+a-1} v^{p+1-a})^{1-\theta} \leq \theta u^{q+a} v^{s-a} + (1-\theta) u^{r+a-1} v^{p+1-a}.$$

Hence (27) is true if $\theta \in [0,1]$.

If (u, v) is a solution of (18) then we have

$$\begin{aligned}(u^a v^{1-a})_t - \Delta(u^a v^{1-a}) + \lambda(u^a v^{1-a}) = \quad & a u^{a-1} v^{1-a}(u_t - \Delta u + \lambda u) \\ + \quad & (1-a) u^a v^{-a}(v_t - \Delta v + \lambda v) \\ + \quad & a(1-a)[u^{a-2} v^{1-a} |\nabla u|^2 \\ + \quad & u^a v^{-1-a} |\nabla v|^2 - 2u^{a-1} v^{-a} \nabla u \cdot \nabla v].\end{aligned}$$

The rest of the proof is similar to the proof of (27).

□

Let (u, v) be a global nonnegative solution of system (2). Denote

$$w = w(t) := \int_\Omega u^a v^{1-a}(t) \varphi_1 \, \mathrm{d}x.$$

The following estimatea are based on ideas from [23]. Let $a \in A$ and condition (31) be true (then $\kappa' > 1$). Then due to Lemma 3.3 and due to Jensen's inequality, it holds

$$w_t + \lambda_1 w \geq C \int_\Omega u^{a\kappa'} v^{(1-a)\kappa'} \varphi_1 \, \mathrm{d}x \geq C w^{\kappa'}, \quad t \in (0, \infty) \tag{33}$$

where $C = C(\Omega, p, q, r, s, a)$ is independent of w. Since w is global and satisfies the inequality (33) for all $t > 0$, it holds

$$w(t) = \int_\Omega u^a v^{1-a} \varphi_1 \, \mathrm{d}x \leq \left(\frac{\lambda_1}{C}\right)^{\frac{1}{\kappa'-1}} \qquad \text{for all } t \geq 0 \text{ and } a \in A. \tag{34}$$

Indeed, assume contradiction to (34). Let there exists $t_0 > 0$ such that $w(t_0) > \left(\frac{\lambda_1}{C}\right)^{\frac{1}{\kappa'-1}}$. Hence

$$w(t) \geq \left(\frac{\lambda_1 + \varepsilon}{C}\right)^{\frac{1}{\kappa'-1}} \tag{35}$$

for $t = t_0$ and some $\varepsilon > 0$. Then (33) implies $w_t(t_0) > 0$, hence $w_t > 0$ on interval $[t_0, t')$ where

$$t' := \sup\{t > t_0 : \ w_t > 0 \text{ on } [t_0, t)\}.$$

If $t' < \infty$ then $w_t(t') \leq 0$. Then (33) implies $w(t') \leq \left(\frac{\lambda_1}{C}\right)^{\frac{1}{\kappa'-1}}$. This is not possible due to our definition of t' and (35) (for $t = t_0$). Hence $t' = \infty$ and (35) is true for $t \geq t_0$. Then there holds $\frac{C\lambda_1}{\varepsilon + \lambda_1} w^{\kappa'}(t) \geq \lambda_1 w(t)$ and (33) yields

$$w_t \geq C w^{\kappa'} - \lambda_1 w \geq \frac{C\varepsilon}{\varepsilon + \lambda_1} w^{\kappa'}, \quad t \geq t_0.$$

Thus w cannot exist globally and this proves (33).

Lemma 3.3 also implies

$$w_t(s') + \lambda_1 w(s') \geq C \int_\Omega u^{r+a-1} v^{p+1-a}(s') \varphi_1 \, \mathrm{d}x, \quad s' \in (0, \infty). \tag{36}$$

Multiplying inequality (36) by $e^{\lambda_1 s'}$ and integrating on interval $[\tau, \tau+t]$ with respect to s' we get

$$e^{\lambda_1(\tau+t)}w(\tau+t) - e^{\lambda_1\tau}w(\tau) \geq C\int_\tau^{\tau+t} e^{\lambda_1 s'}\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s'. \tag{37}$$

Since $0 \leq w \leq C$, from (37) we deduce that

$$\int_\tau^{\tau+t} e^{-\lambda_1(\tau+t-s')}\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s' \leq C. \tag{38}$$

Since there holds $e^{-\lambda_1(\tau+t-s')} \geq e^{-\lambda_1 t}$ for $s' \in [\tau, \tau+t]$, the ineqality (38) implies

$$\int_\tau^{\tau+t}\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s' \leq Ce^{\lambda_1 t} \leq C' \tag{39}$$

where $C' = C'(\Omega, p, q, r, s, a, t)$. Similarly we obtain

$$\int_\tau^{\tau+t} e^{-\lambda_1(\tau+t-s')}\int_\Omega u^{q+a}v^{s-a}(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s' \leq C \tag{40}$$

and

$$\int_\tau^{\tau+t}\int_\Omega u^{q+a}v^{s-a}(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s' \leq Ce^{\lambda_1 t} \leq C'. \tag{41}$$

Let (u, v) be a global nonnegative solution of system (18). Since (u, v) satisfies homogeneous Neumann boundary conditions, so does $u^a v^{1-a}$ and hence Green's formula implies

$$\int_\Omega \Delta(u^a v^{1-a}(t)) \,\mathrm{d}x = 0$$

for $t \geq 0$ and $a \in A$. Denote

$$z = z(t) := \int_\Omega u^a v^{1-a}(t) \,\mathrm{d}x.$$

Lemma 3.3 and Jensen's inequality imply

$$z_t + \lambda z \geq Cz^{\kappa'}, \quad t \in (0, \infty). \tag{42}$$

Since z is global and there holds $\lambda > 0$, similarly as in the proof of (34) we have

$$z(t) = \int_\Omega u^a v^{1-a}(t) \,\mathrm{d}x \leq \left(\frac{\lambda}{C}\right)^{\frac{1}{\kappa'-1}} \qquad \text{for all } t \geq 0 \text{ and } a \in A \tag{43}$$

if (31) is true. Again, we obtain estimates similar to (38)-(41) with φ_1 replaced by 1 in (38)-(41) and with λ_1 replaced by λ in (38), (40).

Beside Hölder's, Young's and Jensen's inequalities we will also use so-called interpolation inequality. We formulate it in the following lemma.

Lemma 3.4. *Let Ω be arbitrary domain in $\mathbb{R}^N$ and $1 \le p < r < q < \infty$. If $f \in L^p_\delta(\Omega) \cap L^q_\delta(\Omega)$ then $f \in L^r_\delta(\Omega)$ and there holds*

$$\|f\|_{r,\delta} \le \|f\|_{p,\delta}^{\frac{p}{r}\frac{q-r}{q-p}} \|f\|_{q,\delta}^{\frac{q}{r}\frac{r-p}{q-p}}.$$

If $f \in L^p_\delta(\Omega) \cap L^\infty(\Omega)$ then $f \in L^r_\delta(\Omega)$ and there holds

$$\|f\|_{r,\delta} \le \|f\|_{p,\delta}^{\frac{p}{r}} \|f\|_{\infty}^{\frac{r-p}{r}}.$$

Proof. Let $q < \infty$. Assume $f \in L^p_\delta(\Omega) \cap L^q_\delta(\Omega)$. We have

$$\|f\|_{r,\delta}^r = \int_\Omega |f|^r \delta(x)\,\mathrm{d}x = \int_\Omega \left[(|f|^p \delta(x))^{\frac{q-r}{q-p}}\right]\left[(|f|^q \delta(x))^{\frac{r-p}{q-p}}\right]\,\mathrm{d}x.$$

Now we use Hölder's inequality to estimate

$$\|f\|_{r,\delta}^r \le \left(\int_\Omega |f|^p \delta(x)\,\mathrm{d}x\right)^{\frac{q-r}{q-p}} \left(\int_\Omega |f|^q \delta(x)\,\mathrm{d}x\right)^{\frac{r-p}{q-p}} = \|f\|_{p,\delta}^{p\frac{q-r}{q-p}} \|f\|_{q,\delta}^{q\frac{r-p}{q-p}}. \tag{44}$$

Since the right-hand side in (44) if finite, the left-hand side is finite.

Now let $q = \infty$. Assume $f \in L^\infty(\Omega) \cap L^p_\delta(\Omega)$. Then we obtain

$$\|f\|_{r,\delta}^r = \int_\Omega |f|^r \delta(x)\,\mathrm{d}x = \int_\Omega \left[|f|^p \delta(x)\right]\left[|f|^{r-p}\right]\,\mathrm{d}x \le \|f\|_{p,\delta}^p \|f\|_\infty^{r-p}.$$

This finishes the proof.

$\square$

For $F : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ we denote $F^{(0)}(x) = x$ and $F^{(j)}(x) = F(F^{(j-1)}(x))$ ($j \in \mathbb{N}$), the j-th iteration of F.

Lemma 3.5. *Let $F : [a,b) \to \mathbb{R}$ be a continuous function ($b \le \infty$) and*

$$F(x) > x \quad \forall x \in [a,b). \tag{45}$$

Then

$$\forall Q \in (a,b) \quad \exists j \in \mathbb{N} \quad F^{(j)}(a) > Q.$$

Proof. The function F is continuous on the compact interval $[a,Q]$. The inequality (45) implies the existence of $\mu = \mu(Q) > 0$ such that for every $x \in [a,Q]$ we have

$$F(x) \ge \mu + x.$$

This implies $F^{(j)}(a) \ge j\mu + a$ for all $j \in \mathbb{N}$ such that $F^{(j-1)}(a) \le Q$.

$\square$

Now we state lemma similar to Lemma 3.5.

Lemma 3.6. *Let $G : [a,b) \to \mathbb{R}$, $g : [a,b) \to \mathbb{R}$ be continuous functions ($b \leq \infty$), g increasing in $[a,b)$ and*

$$G(x) > g(x) \quad \forall x \in [a,b). \tag{46}$$

Then

$$\forall Q \in (g(a), \lim_{x\to b^-} g(x)) \quad \exists j \in \mathbb{N} \quad (G \circ g^{-1})^{(j)}(g(a)) > Q.$$

Proof. We define $F : [g(a), \lim_{x\to b^-} g(x)) \to \mathbb{R}$, $F = G \circ g^{-1}$ and use Lemma 3.5 for this F.

□

We consider the problem

$$\left.\begin{array}{rcl} -\Delta u &=& f, \quad x \in \Omega, \\ u &=& 0, \quad x \in \partial\Omega. \end{array}\right\} \tag{47}$$

We say that u is a very weak solution of (47) if $u \in L^1(\Omega)$, the right-hand side f in (47) belongs to $L^1_\delta(\Omega)$ and

$$-\int_\Omega u\Delta\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x$$

for every $\varphi \in C^2(\overline{\Omega})$, $\varphi = 0$ on $\partial\Omega$.

Lemma 3.7. [38, Theorem 49.1, Theorem 49.2(i)] *Let Ω be a bounded domain of class $C^{2+\gamma}$ for some $\gamma \in (0,1)$. Assume that $1 \leq p \leq q \leq \infty$ satisfy*

$$\frac{1}{p} - \frac{1}{q} < \frac{2}{N+1}.$$

Let $f \in L^1_\delta(\Omega)$. Then there exists a unique very weak solution u of (47). If $f \in L^p_\delta(\Omega)$, then $u \in L^q_\delta(\Omega)$ and

$$\|u\|_{q,\delta} \leq C(p,q,\Omega)\|f\|_{p,\delta}.$$

Lemma 3.8. [38, Remark 49.12(i)] *Let $f \in L^1_\delta(\Omega)$ satisfy $f \geq 0$ a.e. Then the very weak solution of (47) satisfies*

$$u(x) \geq C(\Omega)\|f\|_{1,\delta}\delta(x), \quad x \in \Omega.$$

Lemma 3.9. [40] *Let $N \geq 2$ and let Ω be a smooth bounded domain. Assume that $0 \in \partial\Omega$. Let $-2 < \gamma < N-1$. Then there exist $R > 0$ and a revolution cone Σ_1 of the vertex 0, with $\Sigma := \Sigma_1 \cap \{x \in \mathbb{R}^n;\ |x| < R\} \subset \Omega \cup \{0\}$, such that the function*

$$\phi := |x|^{-(\gamma+2)}\chi_\Sigma$$

belongs to $L^1_\delta(\Omega)$ and the very weak solution $u > 0$ of the problem

$$\left.\begin{array}{rcl} -\Delta u &=& \phi, \quad x \in \Omega, \\ u &=& 0, \quad x \in \partial\Omega \end{array}\right\}$$

satisfies the estimate

$$u \geq C|x|^{-\gamma}\chi_\Sigma.$$

4 Results for elliptic system

In this section we will assume that Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$, $0 \in \partial\Omega$, $p, q > 0$, $pq > 1$, $a, b \in L^\infty(\Omega)$, $a, b \geq 0$, $a, b \not\equiv 0$, $\kappa, \lambda \in (0, 2)$. We will prove that there exists a positive very weak solution (see the definition (7)) of the problem

$$\left.\begin{array}{rcll} -\Delta u &=& a(x)|x|^{-\kappa}v^q, & x \in \Omega, \\ -\Delta v &=& b(x)|x|^{-\lambda}u^p, & x \in \Omega, \\ u &=& v \;=\; 0, & x \in \partial\Omega. \end{array}\right\} \tag{48}$$

To do this we will prove a priori estimates for the problem

$$\left.\begin{array}{rcll} -\Delta u &=& a(x)|x|^{-\kappa}v^q + t(u+\varphi_1), & x \in \Omega, \\ -\Delta v &=& b(x)|x|^{-\lambda}u^p, & x \in \Omega, \\ u &=& v \;=\; 0, & x \in \partial\Omega \end{array}\right\} \tag{49}$$

if $q \geq 1$, $p > 0$, and for the problem

$$\left.\begin{array}{rcll} -\Delta u &=& a(x)|x|^{-\kappa}v^q, & x \in \Omega, \\ -\Delta v &=& b(x)|x|^{-\lambda}u^p + t(v+\varphi_1), & x \in \Omega, \\ u &=& v \;=\; 0, & x \in \partial\Omega \end{array}\right\} \tag{50}$$

if $q < 1$, $p > 1$. In both cases we will assume $t \geq 0$. The terms $t(u + \varphi_1)$ in (49) or $t(v + \varphi_1)$ in (50) are needed to use the topological degree in the proof of the existence of solutions of (48). Denote

$$\alpha := \frac{(2-\lambda)q + 2 - \kappa}{pq - 1}, \quad \beta := \frac{(2-\kappa)p + 2 - \lambda}{pq - 1}. \tag{51}$$

We will prove the following results:

Theorem 4.1. *Assume* $\max\{\alpha, \beta\} > N - 1$. *If* $q \geq 1$, $p > 0$, *then for every nonnegative very weak solution of problem* (49) *with* $t \geq 0$ *we have* $u, v \in L^\infty(\Omega)$ *and there exists constant* $C(\Omega, a, b, p, q, \kappa, \lambda) > 0$ *such that*

$$t + \|u\|_\infty + \|v\|_\infty \leq C(\Omega, a, b, p, q, \kappa, \lambda).$$

If $q < 1$, $p > 1$, *then the same result holds for nonnegative very weak solutions of problem* (50) *with* $t \geq 0$.

Theorem 4.2. *Assume* $\max\{\alpha, \beta\} > N - 1$. *Then there exists a positive bounded very weak solution of problem* (48).

Theorem 4.3. *Assume* $\max\{\alpha, \beta\} < N - 1$. *Then there exist functions* $a, b \in L^\infty(\Omega)$, $a, b \geq 0$, $a, b \not\equiv 0$ *and a positive very weak solution* (u, v) *of problem* (48) *such that* $u, v \notin L^\infty(\Omega)$.

Theorem 4.1 will be proved by a bootstrap method in weighted Lebesgue spaces used in [5], [37], for example. Although [37, Theorem 2.1] also implies the assertion of Theorem 4.1, the corresponding assumptions on p, q, κ, λ are more restrictive than our condition $\max\{\alpha, \beta\} > N-1$. Theorem 4.3 is based on a modification of the proof in [40].

Proof of Theorem 4.1. In the proof we will use C or C' to denote constants which can vary from step to step. We will not emphasize the dependence of these constants on $\Omega, a, b, p, q, \kappa, \lambda$.

Observe that α, β defined by (51) satisfy

$$\begin{aligned} \alpha p + \lambda &= \beta + 2, \\ \beta q + \kappa &= \alpha + 2. \end{aligned} \tag{52}$$

Suppose first $\alpha \geq \beta$, so $\alpha > N-1$. Using these conditions and (52) we obtain

$$p < \frac{N+1-\lambda}{N-1}, \quad q > 1. \tag{53}$$

Thus we will deal with system (49) in the following. The case $\beta \geq \alpha$ can be treated similarly dealing with system (50).

Denote $f(x,v) = a(x)|x|^{-\kappa}v^q + t(u+\varphi_1)$, $g(x,u) = b(x)|x|^{-\lambda}u^p$. Let (u,v) be a very weak solution of (49), $u, v \geq 0$. By definition of a very weak solution we have $u, v \in L^1(\Omega)$, $f, g \in L^1_\delta(\Omega)$ and for $\varphi = \varphi_1$ it holds

$$\begin{gathered} \lambda_1 \int_\Omega u\varphi_1 \,\mathrm{d}x = \int_\Omega u(-\Delta\varphi_1)\,\mathrm{d}x = \int_\Omega f\varphi_1\,\mathrm{d}x, \\ \lambda_1 \int_\Omega v\varphi_1\,\mathrm{d}x = \int_\Omega g\varphi_1\,\mathrm{d}x, \end{gathered} \tag{54}$$

where λ_1 is the first eigenvalue of the problem

$$\left.\begin{aligned} -\Delta\phi &= \lambda\phi, \quad x \in \Omega, \\ \phi &= 0, \qquad x \in \partial\Omega \end{aligned}\right\}$$

and φ_1 is the corresponding positive eigenfunction satisfying $\|\varphi_1\|_2 = 1$. Using (54) we have

$$(\lambda_1 - t)\int_\Omega u\varphi_1\,\mathrm{d}x = \int_\Omega a|x|^{-\kappa}v^q\varphi_1\,\mathrm{d}x + t \geq 0, \tag{55}$$

therefore $t \leq \lambda_1$ for $u \not\equiv 0$. The equality in (55) further implies that $(0,v)$ is not a solution of problem (49) for any nonnegative $v \in L^1(\Omega)$ and $t > 0$. Hence in both cases we have $t \leq C$.

Using (54) and (20) we get

$$C\|f\|_{1,\delta} \leq \|u\|_{1,\delta} \leq C'\|f\|_{1,\delta}, \quad C\|g\|_{1,\delta} \leq \|v\|_{1,\delta} \leq C'\|g\|_{1,\delta}. \tag{56}$$

In this part of the proof we estimate $\int_\Omega f^r\delta\,\mathrm{d}x$, $\int_\Omega g^s\delta\,\mathrm{d}x$ for $r,s\geq 1$. Let (u,v) be a very weak solution of (49), $u\in L^k_\delta(\Omega)$, $v\in L^l_\delta(\Omega)$ for $k,l\geq 1$, $u,v\geq 0$. Then it holds

$$\begin{aligned}\int_\Omega f^r\delta\,\mathrm{d}x &\leq C(r)\left(\int_\Omega a^r|x|^{-\kappa r}v^{qr}\delta\,\mathrm{d}x+\int_\Omega((t\,u)^r+(t\,\varphi_1)^r)\delta\,\mathrm{d}x\right)\\ &\leq C(r,\theta_1)\left(1+\int_\Omega |x|^{-\frac{\kappa r}{\theta_1}+1}\,\mathrm{d}x+\int_\Omega(v^{\frac{qr}{1-\theta_1}}+u^r)\delta\,\mathrm{d}x\right),\end{aligned}\tag{57}$$

$\theta_1\in(0,1)$, where we successively used boundedness of function a, the Young inequality, boundedness of t and the assumption $0\in\partial\Omega$ (then it holds $\delta(x)\leq|x|$). Similarly it holds

$$\int_\Omega g^s\delta\,\mathrm{d}x \leq C(s,\theta_2)\left(\int_\Omega|x|^{-\frac{\lambda s}{\theta_2}+1}\,\mathrm{d}x+\int_\Omega u^{\frac{ps}{1-\theta_2}}\delta\,\mathrm{d}x\right),\quad \theta_2\in(0,1).\tag{58}$$

We will show that if k,l are large enough, then the right-hand sides in (57), (58) can be estimated by $\|u\|_{k,\delta}$, $\|v\|_{l,\delta}$ for some $r,s\geq 1$.

Now we determine the dependence r,s on k,l. If

$$r<\widetilde{r}(l):=\frac{(N+1)l}{\kappa l+(N+1)q},$$

then there exists $\theta_1\in(0,1)$ such that

$$-\frac{\kappa r}{\theta_1}+1>-N,\quad \frac{qr}{1-\theta_1}\leq l.$$

If moreover $r\leq k$, estimate (57) then implies $f\in L^r_\delta(\Omega)$. Thus

$$\|f\|_{r,\delta}\leq C(r,\|u\|_{k,\delta},\|v\|_{l,\delta})\quad\text{if}\quad r<\min\{\widetilde{r}(l),k\}.\tag{59}$$

Similarly

$$s<\widetilde{s}(k):=\frac{(N+1)k}{\lambda k+(N+1)p}$$

implies the existence of $\theta_2\in(0,1)$ such that

$$-\frac{\lambda s}{\theta_2}+1>-N,\quad \frac{ps}{1-\theta_2}\leq k.$$

Estimate (58) then implies $g\in L^s_\delta(\Omega)$. Thus

$$\|g\|_{s,\delta}\leq C(s,\|u\|_{k,\delta})\quad\text{if}\quad s<\widetilde{s}(k).\tag{60}$$

On the other hand, Lemma 3.7 gives us estimates for $\|u\|_{k,\delta}$, $\|v\|_{l,\delta}$, $k,l\geq 1$. If $f\in L^r_\delta(\Omega)$, then $u\in L^k_\delta(\Omega)$ and it holds

$$\|u\|_{k,\delta}\leq C(k,r)\|f\|_{r,\delta},\tag{61}$$

where $1 \le r \le k \le \infty$ satisfy $\frac{1}{r} - \frac{1}{k} < \frac{2}{N+1}$. In particular we can take

$$k < \widetilde{k}(r) := \frac{(N+1)r}{N+1-2r} \quad \text{if} \quad r \in \left[1, \frac{N+1}{2}\right).$$

If $r = \frac{N+1}{2}$, $1 \le k < \infty$ can be chosen arbitrarily and if $r > \frac{N+1}{2}$, then we can take $k = \infty$. Similarly, if $g \in L^s_\delta(\Omega)$, then $v \in L^l_\delta(\Omega)$ and it holds

$$\|v\|_{l,\delta} \le C(l,s)\|g\|_{s,\delta}, \tag{62}$$

where $1 \le s \le l \le \infty$ satisfy

$$l < \widetilde{l}(s) := \frac{(N+1)s}{N+1-2s} \quad \text{if} \quad s \in \left[1, \frac{N+1}{2}\right).$$

If $s = \frac{N+1}{2}$, $1 \le l < \infty$ can be chosen arbitrarily and if $s > \frac{N+1}{2}$, then we can take $l = \infty$.

We know that $f \in L^1_\delta(\Omega)$. Estimate (61) implies $u \in L^k_\delta(\Omega)$ for $1 < k < k_0$ where $k_0 := \frac{N+1}{N-1} = \widetilde{k}(1)$. Given $s < \widetilde{s}(k_0) = \frac{N+1}{\lambda+(N-1)p}$, the continuity and the monotonicity of $\widetilde{s}$ assures existence of $k < k_0$ such that $s < \widetilde{s}(k) < \widetilde{s}(k_0)$. Hence $g \in L^s_\delta(\Omega)$ for $s \in \left(1, \frac{N+1}{\lambda+(N-1)p}\right)$ (inequality (53) implies $\frac{N+1}{\lambda+(N-1)p} > 1$). If $p > \frac{2-\lambda}{N-1}$, then $v \in L^l_\delta(\Omega)$ for $l < l_0 := \widetilde{l}(\widetilde{s}(k_0)) = \frac{N+1}{\lambda-2+(N-1)p}$. Finally we have $f \in L^r_\delta(\Omega)$ for $r < \min\left\{\widetilde{r}\left(\frac{N+1}{\lambda-2+(N-1)p}\right), k_0\right\} = \min\left\{\frac{N+1}{\kappa+(\lambda+(N-1)p-2)q}, \frac{N+1}{N-1}\right\} =: r_0$. Then $r_0 > 1$ due to the assumption $\alpha > N-1$. If $p \le \frac{2-\lambda}{N-1}$, then $\frac{N+1}{\lambda+(N-1)p} \ge \frac{N+1}{2}$ and due to the continuity and the monotonicity of $\widetilde{l}$ we have $v \in L^l_\delta(\Omega)$ for all $l < \infty$. Thus $f \in L^r_\delta(\Omega)$ for $r < \min\left\{\frac{N+1}{\kappa}, \frac{N+1}{N-1}\right\} =: r_0'$. The preceding computations show that if $k \le k_0$ ($l \le l_0$) is close enough to k_0 (l_0) or larger, then the right-hand sides in (57), (58) can be estimated by $\|u\|_{k,\delta}$, $\|v\|_{l,\delta}$ for some $r, s \ge 1$.

We have shown that if $f \in L^1_\delta(\Omega)$, then $f \in L^r_\delta(\Omega)$ for $r < r_0$ ($r < r_0'$) if $p > \frac{2-\lambda}{N-1}$ ($p \le \frac{2-\lambda}{N-1}$). We claim that there holds

$$\text{if } f \in L^r_\delta(\Omega) \text{ for some } r \in \left[1, \frac{N+1}{\kappa}\right), \text{ then } f \in L^{F(r)}_\delta(\Omega) \tag{63}$$

for some continuous function $F : \left[1, \frac{N+1}{\kappa}\right) \to \mathbb{R}$ satisfying (45). In the following we will give expression of such function F. For $p > \frac{2-\lambda}{N-1}$ denote

$$\widetilde{F}(r) := \begin{cases} \min\{\widetilde{r}(\widetilde{l}(\widetilde{s}(\widetilde{k}(r)))), \widetilde{k}(r)\} = \\ \min\left\{\dfrac{N+1}{\kappa + \left(\lambda + \left(\frac{N+1}{r} - 2\right)p - 2\right)q}, \dfrac{(N+1)r}{N+1-2r}\right\}, & r \in \left[1, \dfrac{(N+1)p}{2p+2-\lambda}\right), \\ \min\left\{\dfrac{N+1}{\kappa}, \dfrac{(N+1)r}{N+1-2r}\right\}, & r \in \left[\dfrac{(N+1)p}{2p+2-\lambda}, \dfrac{N+1}{2}\right), \\ \dfrac{N+1}{\kappa}, & r \in \left[\dfrac{N+1}{2}, \dfrac{N+1}{\kappa}\right) \end{cases}$$

(for such p there holds $\frac{(N+1)p}{2p+2-\lambda} > 1$). For $p \le \frac{2-\lambda}{N-1}$ denote

$$\widetilde{F}(r) := \begin{cases} \dfrac{(N+1)r}{N+1-2r}, & r \in \left[1, \dfrac{N+1}{2+\kappa}\right) \text{ if } \dfrac{N+1}{2+\kappa} > 1, \\ \dfrac{N+1}{\kappa}, & r \in \left[\max\left\{1, \dfrac{N+1}{2+\kappa}\right\}, \dfrac{N+1}{\kappa}\right). \end{cases}$$

Function $\widetilde{F} : \left[1, \frac{N+1}{\kappa}\right) \to \mathbb{R}$ is continuous and due to the assumption $\alpha > N-1$ there holds (45). Define $F(r) := \frac{\widetilde{F}(r)+r}{2}$, then $r < F(r) < \widetilde{F}(r)$ for all $r \in \left[1, \frac{N+1}{\kappa}\right)$. Observe that $\widetilde{F}(1) = r_0$ ($\widetilde{F}(1) = r_0'$) for $p > \frac{2-\lambda}{N-1}$ ($p \le \frac{2-\lambda}{N-1}$), hence claim (63) has already been proved for $r = 1$. For $r > 1$ fixed the same monotonicity and continuity argument will be used. If $p > \frac{2-\lambda}{N-1}$ and $r < \frac{(N+1)p}{2p+2-\lambda}$, then $u \in L^k_\delta(\Omega)$ for $k < \widetilde{k}(r)$ due to (61). Consequently from (60) we get $g \in L^s_\delta(\Omega)$ for $s < \widetilde{s}(\widetilde{k}(r))$ and then (62) implies $v \in L^l_\delta(\Omega)$ for $l < \widetilde{l}(\widetilde{s}(\widetilde{k}(r)))$. Finally (59) implies $f \in L^{r'}_\delta(\Omega)$ for $r' < \min\{\widetilde{r}(\widetilde{l}(\widetilde{s}(\widetilde{k}(r)))), \widetilde{k}(r)\} = \widetilde{F}(r)$, hence $f \in L^{F(r)}_\delta(\Omega)$. Claim (63) in the remaining cases can be proved similarly.

The assumptions of Lemma 3.5 are satisfied for F, hence

$$\exists\, \overline{j} \in \mathbb{N} \quad F^{(\overline{j})}(1) > \frac{N+1}{2} + \epsilon \tag{64}$$

for $\epsilon > 0$ small. Using (63) $\overline{j}$-times we get $f \in L^{F^{(\overline{j})}(1)}_\delta(\Omega)$, thus $f \in L^{\frac{N+1}{2}+\epsilon}_\delta(\Omega)$ from (64). Lemma 3.7 then implies $u \in L^\infty(\Omega)$, from (37) we get $g \in L^{\frac{N+1}{2}+\epsilon}_\delta(\Omega)$ and consequently $v \in L^\infty(\Omega)$.

Now we prove

$$\|u\|_\infty + \|v\|_\infty \le C(\|u\|_{1,\delta}, \|v\|_{1,\delta}), \tag{65}$$

where the constant C is bounded for $\|u\|_{1,\delta}, \|v\|_{1,\delta}$ bounded. Using (59), (60), (61), (62) we have

$$\|f\|_{F(r),\delta} \le C(k, l, r, s, \|f\|_{r,\delta}, \|g\|_{s,\delta}). \tag{66}$$

Iterating (66) $\overline{j}$-times and using (64), (56) we have

$$\|f\|_{\frac{N+1}{2}+\epsilon,\delta} \le C\|f\|_{F^{(\overline{j})}(1),\delta} \le C(\|u\|_{1,\delta}, \|v\|_{1,\delta}).$$

Lemma 3.7 and (60) then imply assertion (65).

Now we turn to prove uniform boundedness of $\|u\|_{1,\delta}$ and $\|v\|_{1,\delta}$. Due to Lemma 3.8 there holds

$$\begin{aligned} u &\ge C\,\delta \int_\Omega a|x|^{-\kappa} v^q \delta + t(u + \varphi_1)\delta \,\mathrm{d}x, \\ v &\ge C\,\delta \int_\Omega b|x|^{-\lambda} u^p \delta \,\mathrm{d}x. \end{aligned}$$

This implies

$$\begin{aligned}\int_\Omega a|x|^{-\kappa}v^q\delta + t(u+\varphi_1)\delta\,\mathrm{d}x &\geq C\int_\Omega a|x|^{-\kappa}\delta^{q+1}\,\mathrm{d}x\left(\int_\Omega b|x|^{-\lambda}u^p\delta\,\mathrm{d}x\right)^q \\ &\geq C\left(\int_\Omega b|x|^{-\lambda}u^p\delta\,\mathrm{d}x\right)^q\end{aligned} \tag{67}$$

and

$$\int_\Omega b|x|^{-\lambda}u^p\delta\,\mathrm{d}x \geq C\left(\int_\Omega a|x|^{-\kappa}v^q\delta + t(u+\varphi_1)\delta\,\mathrm{d}x\right)^p. \tag{68}$$

Using (67), (68) and the assumption $pq > 1$ we get $\|f\|_{1,\delta} + \|g\|_{1,\delta} \leq C$. The estimate $\|u\|_{1,\delta}+\|v\|_{1,\delta} \leq C$ then follows from (56). Inequality (65) then implies the last assertion of the theorem.

□

Proof of Theorem 4.2. Suppose first $\alpha \geq \beta$. As in proof of Theorem 4.1 it is enough to deal with system (49) in the following. Again, the case $\beta \geq \alpha$ can be treated similarly dealing with system (50).

Denote now $f(x,v) = a(x)|x|^{-\kappa}|v|^q$, $g(x,u) = b(x)|x|^{-\lambda}|u|^p$. Set $X := L^\infty(\Omega) \times L^\infty(\Omega)$. Given $(u,v) \in X$ and $t \geq 0$, let $S_t(u,v) = (w,w')$ be the unique solution of the linear problem

$$\left.\begin{aligned} -\Delta w &= f + t(|u| + \varphi_1), && x \in \Omega, \\ -\Delta w' &= g, && x \in \Omega, \\ w &= w' = 0, && x \in \partial\Omega. \end{aligned}\right\} \tag{69}$$

We will prove that there exists a nontrivial fixed point of operator S_0. Since $f \in L^k(\Omega)$ for $k < \frac{N}{\kappa}$ and $g \in L^l(\Omega)$ for $l < \frac{N}{\lambda}$, we have $S_t(u,v) \in W^{2,r}(\Omega) \times W^{2,r}(\Omega)$ for $r \in (\frac{N}{2}, \min\{\frac{N}{\kappa}, \frac{N}{\lambda}\})$. Therefore, $S_t : X \to X$ is compact. Observe that the right-hand sides in (69) are nonnegative for every $(u,v) \in X$, hence w, w' are nonnegative. Thus S_t has no fixed point beyond the nonnegative cone $K = \{(u',v') \in X;\ u',v' \geq 0\}$ for any $t \geq 0$.

Let $\|(u,v)\|_X = \epsilon$ for $\epsilon > 0$ small, $\theta \in [0,1]$. Assume $(u,v) = \theta S_0(u,v)$. Using L^p-estimates (see [22, Chapter 9]) we have

$$\|u\|_\infty \leq C\|u\|_{2,r} \leq C\|f\|_r \leq C\|a|x|^{-\kappa}\|_r\|v\|_\infty^q \leq C\|v\|_\infty^q,$$

where $\|\,.\,\|_{2,r}$ denotes the norm in $W^{2,r}(\Omega)$. Similarly we obtain $\|v\|_\infty \leq C\|u\|_\infty^p$. Combining the last two estimates we have

$$\|u\|_\infty \leq C\|u\|_\infty^{pq} \leq C\epsilon^{pq-1}\|u\|_\infty.$$

This is a contradiction for ϵ sufficiently small due to the assumption $pq > 1$. Hence $(u, v) \neq \theta S_0(u, v)$ and the homotopy invariance of the topological degree implies

$$\deg(I - S_0, 0, B_\epsilon) = \deg(I, 0, B_\epsilon) = 1, \tag{70}$$

where I denotes the identity and $B_\epsilon := \{(u, v) \in X : \|(u, v)\|_X < \epsilon\}$.

Theorem 4.1 immediately implies $S_T(u, v) \neq (u, v)$ for T large and $(u, v) \in \overline{B_R} \cap K$ and $S_t(u, v) \neq (u, v)$ for $t \in [0, T]$ and $(u, v) \in (\overline{B_R} \backslash B_R) \cap K$ (where $R > 0$ is large enough), hence we have

$$\deg(I - S_0, 0, B_R) = \deg(I - S_T, 0, B_R) = 0. \tag{71}$$

Equalities (70) and (71) imply $\deg(I - S_0, 0, B_R \backslash \overline{B_\epsilon}) = -1$, hence there exist $u, v \in (B_R \backslash \overline{B_\epsilon}) \cap K$ such that $S_0(u, v) = (u, v)$. Finally, the maximum principle implies the positivity of u, v.

$\square$

Proof of Theorem 4.3. Basic ideas used in the proof are from [40]. Lemma 3.9 assures the existence of sets Σ_ϕ, Σ_ψ such that $\phi := \chi_{\Sigma_\phi} |x|^{-(\alpha+2)}$, $\psi := \chi_{\Sigma_\psi} |x|^{-(\beta+2)}$ belong to $L^1_\delta(\Omega)$, where α, β are defined by (51). Let (u, v) be the (positive) very weak solution of

$$\left.\begin{array}{rcll} -\Delta u &=& \phi, & x \in \Omega, \\ -\Delta v &=& \psi, & x \in \Omega, \\ u &=& v \;=\; 0, & x \in \partial\Omega. \end{array}\right\}$$

Lemma 3.9 then implies

$$u \geq C|x|^{-\alpha} \chi_{\Sigma_\phi}, \quad v \geq C|x|^{-\beta} \chi_{\Sigma_\psi}, \tag{72}$$

hence $u, v \notin L^\infty(\Omega)$. Observe that (72) and (52) imply $a', b' \in L^\infty(\Omega)$, where $a' := \frac{|x|^\kappa \phi}{v^q}$, $b' := \frac{|x|^\lambda \psi}{u^p}$ are nonnegative functions and (u, v) is a very weak solution of (48) with $a = a'$, $b = b'$.

$\square$

5 Results for parabolic system

In the following proofs, every constant may depend on Ω, p, q, r, s, however we do not denote this dependence. The constants may vary from step to step.

For $0 < p < \frac{2}{N+1},\ r \leq 1$ denote

$$
\begin{aligned}
K &: \left[1, \tfrac{p+1}{p}\right) \longrightarrow \mathbb{R} \cup \{\infty\}, \\
K(M) &= \begin{cases} \frac{M(p+1)(N+1)}{(p+1)(N+1)-2M}, & M \in \left[1, \frac{(p+1)(N+1)}{2}\right), \\ \infty, & M \in \left[\frac{(p+1)(N+1)}{2}, \frac{p+1}{p}\right), \end{cases} \\
k &: \left[1, \tfrac{p+1}{p}\right) \longrightarrow \mathbb{R}, \\
k(M) &= \tfrac{M(p+r)}{M-(M-1)(p+1)}.
\end{aligned} \tag{73}
$$

For $r > 1,\ p + r < \frac{N+3}{N+1}$ denote

$$
\begin{aligned}
K' &: \left[1, \tfrac{p+r}{p+r-1}\right) \longrightarrow \mathbb{R} \cup \{\infty\}, \\
K'(M) &= \begin{cases} \frac{M(p+r)(N+1)}{(p+r)(N+1)-2M}, & M \in \left[1, \frac{(p+r)(N+1)}{2}\right), \\ \infty, & M \in \left[\frac{(p+r)(N+1)}{2}, \frac{p+r}{p+r-1}\right), \end{cases} \\
k' &: \left[1, \tfrac{p+r}{p+r-1}\right) \longrightarrow \mathbb{R}, \\
k'(M) &= \tfrac{M(p+r)}{M-(M-1)(p+r)}.
\end{aligned} \tag{74}
$$

Observe that

$$K(M) > \max\{M, k(M)\} \quad \text{for all} \quad M \in \left[1, \frac{p+1}{p}\right), \tag{75}$$

since $p < \frac{2}{N+1}$ and

$$K'(M) > k'(M) > M \quad \text{for all} \quad M \in \left[1, \frac{p+r}{p+r-1}\right), \tag{76}$$

since $p + r < \frac{N+3}{N+1}$.

Lemma 5.1. *Let $p + r < \frac{N+3}{N+1},\ p > 0$ and conditions* (26), (31) *be true. Let (u, v) be a global nonnegative solution of problem* (2).

i) Assume $r > 1$. Then for $\gamma \in [p+r, \infty]$ and $T \geq 0$, there exists $C = C(p, q, r, s, \Omega, T)$ such that

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{\gamma,\delta} \leq C\|u(\tau)\|_{\gamma,\delta}, \quad \tau \geq 0. \tag{77}$$

ii) Assume $r > 1,\ pq > (r-1)(s-1)$ or $r \leq 1,\ p < \frac{2}{N+1}$. Then for $\gamma \in \left[\max\{1, p+r\}, \frac{N+3}{N+1}\right)$, there exists $C = C(p, q, r, s, \Omega)$ such that

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{\gamma,\delta} \leq C(1 + \|u(\tau)\|_{\gamma,\delta}), \quad \tau, T \geq 0. \tag{78}$$

iii) Assume $r \leq 1$, $p < \frac{2}{N+1}$. Then for $\gamma \in [\max\{1, p+r\}, \infty]$ and $T \geq 0$, there exists $C = C(p, q, r, s, \Omega, T)$ such that

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{\gamma,\delta} \leq C(1 + \|u(\tau)\|_{\gamma,\delta}), \quad \tau \geq 0. \tag{79}$$

Remark. In the assertion i) of Lemma 5.1, the constant C is bounded for T bounded. **Proof of Lemma 5.1** Let $\gamma \in \left[\max\{1, p+r\}, \frac{N+3}{N+1}\right)$, $a \in A$ and $\varepsilon \in \left(0, 1 - \frac{p}{p+1-a}\right)$. For $\tau, T \geq 0, t \in [0, T]$ we estimate

$$\begin{aligned}
&\|u(\tau+t)\|_{\gamma,\delta} \leq C \Bigg[\|u(\tau)\|_{\gamma,\delta} \\
&+ \int_{\tau}^{\tau+t} e^{-\lambda_1\left(\frac{p}{p+1-a}+\varepsilon\right)(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \|u^r v^p(s')\|_{1,\delta}\, \mathrm{d}s' \Bigg] \\
&\leq C \Bigg[\|u(\tau)\|_{\gamma,\delta} + \int_{\tau}^{\tau+t} \int_{\Omega} \left[e^{-\lambda_1\left(\frac{p}{p+1-a}\right)(\tau+t-s')} u^{r-\frac{(p+r)(1-a)}{p+1-a}} v^p(s') \right] \times \\
&\times \left[e^{-\lambda_1 \varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} u^{\frac{(p+r)(1-a)}{p+1-a}}(s') \right] \varphi_1 \,\mathrm{d}x\, \mathrm{d}s' \Bigg].
\end{aligned} \tag{80}$$

Here we used (22) and (21) and the assertions iii) (ii), if $\gamma = 1$) and v) from Lemma 3.1. Now, using Hölder's inequality in the last term in (80) we obtain

$$\begin{aligned}
\|u(\tau+t)\|_{\gamma,\delta} \leq\ & C \Bigg[\|u(\tau)\|_{\gamma,\delta} \\
&+ \left(\int_{\tau}^{\tau+t} e^{-\lambda_1(\tau+t-s')} \int_{\Omega} u^{r+a-1} v^{p+1-a}(s') \varphi_1 \,\mathrm{d}x\, \mathrm{d}s' \right)^{\frac{p}{p+1-a}} \times \\
&\times \left(\int_{\tau}^{\tau+t} e^{-\lambda_1 \frac{p+1-a}{1-a}\varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{p+1-a}{1-a}} \times \right. \\
&\times \left. \int_{\Omega} u^{p+r}(s') \varphi_1 \,\mathrm{d}x\, \mathrm{d}s' \right)^{\frac{1-a}{p+1-a}} \Bigg].
\end{aligned} \tag{81}$$

Now we use (38) in (81) to estimate

$$\|u(\tau+t)\|_{\gamma,\delta} \leq C \left[\|u(\tau)\|_{\gamma,\delta} + (I(t))^{\frac{1-a}{p+1-a}} \left(\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{\gamma,\delta} \right)^{\kappa} \right] \tag{82}$$

where

$$I(t) := \int_{\tau}^{\tau+t} e^{-\lambda_1 \frac{p+1-a}{1-a}\varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{p+1-a}{1-a}}\, \mathrm{d}s' \tag{83}$$

and

$$\kappa := \frac{(p+r)(1-a)}{p+1-a}. \tag{84}$$

We prove that the function I is finite in $[0,\infty)$, i.e. due to our assumptions on p,q,r,s, there holds

$$\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{p+1-a}{1-a}<1 \tag{85}$$

for some $a\in A$.

In fact, in the following proof we will choose

$$a=\frac{r-1}{p+r-1} \qquad \text{in case i)}, \tag{86}$$

$$a>\frac{r-1}{p+r-1} \quad \text{sufficiently close to} \quad \frac{r-1}{p+r-1} \qquad \text{in case ii) for } r>1, \tag{87}$$

$$a>0 \quad \text{sufficiently small} \qquad \text{in case iii) or ii) for } r\le 1. \tag{88}$$

The choice (87) is possible, since due to the assumptions $pq>(r-1)(s-1)$ and $p>0$ we have $a\in A$. The choices (87) and (88) of a will be specified more precisely during the proof.

If a is defined by (86) or (87) then $\frac{p+1-a}{1-a}$ is close to $p+r$ and condition $p+r<\frac{N+3}{N+1}$ implies the inequality (85).

If a is defined by (88) then $\frac{p+1-a}{1-a}$ is close to $p+1$ and condition $p<\frac{2}{N+1}$ implies the inequality (85).

Note that the function I defined by (83) is continuous, increasing, $I(0)=0$, and I is bounded by a constant independent of τ,T.

First we prove ii). In the estimate (82) we choose a defined by (87), if $r>1$, or by (88), if $r\le 1$. In both cases we have $\kappa<1$ (where κ is defined by (84)), hence we can use Young's inequality to obtain

$$\|u(\tau+t)\|_{\gamma,\delta}\le C\left[\|u(\tau)\|_{\gamma,\delta}+\varepsilon\left(\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{\gamma,\delta}\right)+C_\varepsilon\right]. \tag{89}$$

If we choose $\varepsilon=\frac{1}{2C}$ then the assertion ii) follows.

Assertion iii) for $\gamma\in\left[\max\{1,p+r\},\frac{N+3}{N+1}\right)$ follows from assertion ii).

To prove i) for $\gamma\in\left[p+r,\frac{N+3}{N+1}\right)$ we choose a defined by (86) in estimate (82). Then $\kappa=1$ and the assertion i) for $\gamma\in\left[p+r,\frac{N+3}{N+1}\right)$ and T_0 small enough follows from the estimate (82).

Till now we obtained the estimate

$$\sup_{s'\in[\tau,\tau+T_0]}\|u(s')\|_{\gamma,\delta}\le C(T_0)\|u(\tau)\|_{\gamma,\delta}, \quad \gamma\in\left[p+r,\frac{N+3}{N+1}\right),\ \tau\ge 0 \tag{90}$$

where T_0 is sufficiently small and is independent of τ. We check that the estimate (90) actually holds for every $T\ge 0$ with a constant $C(T)$.

For $T>0$ fixed there exists $n\in\mathbb{N}\cup\{0\}$ such that $T\in(nT_0,(n+1)T_0]$. Then for $\gamma\in\left[p+r,\frac{N+3}{N+1}\right)$ there holds

$$\begin{aligned}\sup_{s'\in[\tau+nT_0,\tau+T]}\|u(s')\|_{\gamma,\delta} &\le \sup_{s'\in[\tau+nT_0,\tau+(n+1)T_0]}\|u(s')\|_{\gamma,\delta}\\ &\le C(T_0)\|u(\tau+nT_0)\|_{\gamma,\delta},\quad \tau\ge 0\end{aligned}\tag{91}$$

where we used (90) with τ replaced by $\tau+nT_0$. For $l\in\{0,\dots,n-1\}$ we have

$$\begin{aligned}\|u(\tau+(l+1)T_0)\|_{\gamma,\delta} &\le \sup_{s'\in[\tau+lT_0,\tau+(l+1)T_0]}\|u(s')\|_{\gamma,\delta}\\ &\le C(T_0)\|u(\tau+lT_0)\|_{\gamma,\delta},\quad \tau\ge 0.\end{aligned}\tag{92}$$

We choose $t'\in[0,T]$. For such t' there exists $k\in\mathbb{N}\cup\{0\}$, $k\le n$ such that $t'\in(kT_0,(k+1)T_0]$. Using (92) k-times (if $t'>\tau+nT_0$ then we also use (91)) we obtain

$$\begin{aligned}\|u(\tau+t')\|_{\gamma,\delta} &\le \sup_{s'\in[\tau+kT_0,\tau+(k+1)T_0]}\|u(s')\|_{\gamma,\delta}\le C\|u(\tau+kT_0)\|_{\gamma,\delta}\\ &\le C^2\|u(\tau+(k-1)T_0)\|_{\gamma,\delta}\le\ldots\le C^{k+1}\|u(\tau)\|_{\gamma,\delta}\le C^{n+1}\|u(\tau)\|_{\gamma,\delta},\end{aligned}$$

since $C=C(T_0)\ge 1$ due to the inequality (90). Thus (90) is true for all $T_0\ge 0$.

Now we prove the assertion i) for $\gamma\in\left[\frac{N+3}{N+1},\infty\right]$. Fix $K\in\left[\frac{N+3}{N+1},\infty\right)$. Then there exists $M\in\left[1,\frac{(p+r)(N+1)}{2}\right]$ such that $K'(M)>K>k=k'(M)$ (where functions K', k' are defined by (74)). For $\tau\ge 0$, $t\in[0,T]$ and a defined by (86) we estimate

$$\begin{aligned}\|u(\tau+t)\|_{K,\delta} &\le C\left[\|u(\tau)\|_{K,\delta}+\int_\tau^{\tau+t}(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\|u^r v^p(s')\|_{M,\delta}\,\mathrm{d}s'\right]\\ &\le C\left[\|u(\tau)\|_{K,\delta}+\int_\tau^{\tau+t}(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\times\right.\\ &\quad\times\left.\left(\int_\Omega\left[u^{pM\frac{r+a-1}{p+1-a}}v^{pM}(s')\right]\left[u^{M\frac{(p+r)(1-a)}{p+1-a}}(s')\right]\varphi_1\,\mathrm{d}x\right)^{\frac{1}{M}}\mathrm{d}s'\right].\end{aligned}\tag{93}$$

Here we used Lemma 3.1 iii) and v). Observe that $M<\frac{p+1-a}{p}$, since $M\le\frac{(p+r)(N+1)}{2}<\frac{p+r}{p+r-1}$ (the last inequality is true due to the assumption $p+r<\frac{N+3}{N+1}$). Hence we can use Hölder's inequality in the spatial integral to obtain

$$\begin{aligned}&\|u(\tau+t)\|_{K,\delta}\le C\left[\|u(\tau)\|_{K,\delta}+\int_\tau^{\tau+t}(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\times\right.\\ &\times\left.\left(\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p}{p+1-a}}\left(\int_\Omega u^{M\frac{(p+r)(1-a)}{p+1-a-pM}}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p+1-a-pM}{M(p+1-a)}}\mathrm{d}s'\right]\\ &\le C\left[\|u(\tau)\|_{K,\delta}+\sup_{s'\in[\tau,\tau+T]}\|u(s')\|^{\kappa}_{M\frac{(p+r)(1-a)}{p+1-a-pM},\delta}\int_\tau^{\tau+t}\left[(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\right]\times\right.\\ &\times\left.\left(\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p}{p+1-a}}\mathrm{d}s'\right]\end{aligned}\tag{94}$$

where κ is defined by (84). Notice that $\kappa = 1$ and $k = M\frac{(p+r)(1-a)}{p+1-a-pM}$ due to our choice of a. We use Hölder's inequality in the time integral in (94) to have

$$\begin{aligned} \|u(\tau+t)\|_{K,\delta} \leq C \Bigg[& \|u(\tau)\|_{K,\delta} + \sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{k,\delta} \times \\ & \times \left(\int_\tau^{\tau+t} \int_\Omega u^{r+a-1} v^{p+1-a}(s') \varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \right)^{\frac{p}{p+1-a}} \times \\ & \times \left(\int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)\frac{p+1-a}{1-a}} \, \mathrm{d}s' \right)^{\frac{1-a}{p+1-a}} \Bigg]. \end{aligned} \tag{95}$$

Define

$$I_0(t) := \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)\frac{p+1-a}{1-a}} \, \mathrm{d}s'. \tag{96}$$

We use (39) to obtain

$$\|u(\tau+t)\|_{K,\delta} \leq C \left[\|u(\tau)\|_{K,\delta} + C(T) \sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{k,\delta} \left(I_0(T)\right)^{\frac{1-a}{p+1-a}} \right]$$

where I_0 is defined by (96). Observe that I_0 is finite on $[0, \infty)$, since

$$\frac{N+1}{2} \left(\frac{1}{M} - \frac{1}{K} \right) \frac{p+1-a}{1-a} < 1. \tag{97}$$

This follows from the definition (74) of function K' and our choice of K. The function I_0 is continuous, increasing, $I_0(0) = 0$, however I_0 is unbounded on $[0, \infty)$. Since $k < K$, we have

$$\|u(\tau+t)\|_{K,\delta} \leq C \left[\|u(\tau)\|_{K,\delta} + C(T_0) \left(I_0(T)\right)^{\frac{1-a}{p+1-a}} \sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{K,\delta} \right]. \tag{98}$$

For T_0 sufficiently small, there holds $C(T_0) \left(I_0(T_0)\right)^{\frac{1-a}{p+1-a}} < \frac{1}{2C}$ and the inequality (98) implies

$$\sup_{s' \in [\tau, \tau+T_0]} \|u(s')\|_{K,\delta} \leq C(T_0) \|u(\tau)\|_{K,\delta} \tag{99}$$

for $K \in \left[\frac{N+3}{N+1}, \infty\right)$. Again, as in the proof of i) for $\gamma \in \left[p+r, \frac{N+3}{N+1}\right)$ this estimate is true for every $T \geq 0$.

If $M \in \left(\frac{(p+r)(N+1)}{2}, \frac{p+r}{p+r-1} \right)$ then we can choose $K = \infty$ and $k \in (k'(M), \infty)$. Using estimates similar to (93)-(98) (instead of iii) and v) in Lemma 3.1 used to prove (93) we use iv) and vi))we have

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_\infty \leq C(T) \|u(\tau)\|_\infty.$$

Now we prove the assertion iii) for $\gamma \in \left[\frac{N+3}{N+1}, \infty\right]$. Fix $K \in \left[\frac{N+3}{N+1}, \infty\right)$. Then there exists $M \in \left[1, \frac{(p+1)(N+1)}{2}\right]$ such that $K(M) > K > \max\{1, k(M)\}$ (where functions K, k are defined by (73)). Now fix $k \in (\max\{1, k(M)\}, K)$. Since $k(1) = p + r$ and possibly $p + r < 1$, we need also to assume $k > 1$. For $\tau \geq 0$, $t \in [0, T]$ and a defined by (88) we estimate

$$
\begin{aligned}
\|u(\tau+t)\|_{K,\delta} \leq\ & C\left[\|u(\tau)\|_{K,\delta} + \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u^r v^p(s')\|_{M,\delta}\,\mathrm{d}s'\right] \\
\leq\ & C\left[\|u(\tau)\|_{K,\delta} + \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \times\right. \\
\times\ & \left.\left(\int_\Omega \left[u^{pM\frac{r+a-1}{p+1-a}} v^{pM}(s')\right]\left[u^{M\frac{(p+r)(1-a)}{p+1-a}}(s')\right]\varphi_1\,\mathrm{d}x\right)^{\frac{1}{M}} \mathrm{d}s'\right].
\end{aligned}
\tag{100}
$$

Here we used the assertions iii) and v) from Lemma 3.1. Observe that

$$
M \leq \frac{(p+1)(N+1)}{2} < \frac{p+1}{p} \tag{101}
$$

yields $M < \frac{p+1-a}{p}$. In fact, this is possible, if we choose a sufficiently small depending on fixed M. The last inequality in (101) is true due to the assumption $p < \frac{2}{N+1}$. Hence we can use Hölder's inequality in the spatial integral in (100) to obtain

$$
\begin{aligned}
&\|u(\tau+t)\|_{K,\delta} \leq C\left[\|u(\tau)\|_{K,\delta} + \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \times\right. \\
&\times \left.\left(\int_\Omega u^{r+a-1} v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p}{p+1-a}} \left(\int_\Omega u^{M\frac{(p+r)(1-a)}{p+1-a-pM}}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p+1-a-pM}{M(p+1-a)}} \mathrm{d}s'\right] \\
&\leq C\left[\|u(\tau)\|_{K,\delta} + \sup_{s'\in[\tau,\tau+T]} \|u(s')\|^\kappa_{M\frac{(p+r)(1-a)}{p+1-a-pM},\delta} \int_\tau^{\tau+t} \left[(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\right] \times\right. \\
&\times \left.\left(\int_\Omega u^{r+a-1} v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p}{p+1-a}} \mathrm{d}s'\right]
\end{aligned}
\tag{102}
$$

where κ is defined by (84). Notice that $\kappa < 1$ and $M\frac{(p+r)(1-a)}{p+1-a-pM}$ is close to $k(M)$ and $M\frac{(p+r)(1-a)}{p+1-a-pM} < k$, if we choose a sufficiently small; this choice depends on k, M fixed. We use Hölder's inequality in the time integral in (102) to have

$$
\begin{aligned}
\|u(\tau+t)\|_{K,\delta} \leq\ & C\left[\|u(\tau)\|_{K,\delta} + \sup_{s'\in[\tau,\tau+T]} \|u(s')\|^\kappa_{k,\delta} \times\right. \\
& \times \left(\int_\tau^{\tau+t}\int_\Omega u^{r+a-1} v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right)^{\frac{p}{p+1-a}} \times \\
& \times \left.\left(\int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)\frac{p+1-a}{1-a}}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right].
\end{aligned}
\tag{103}
$$

Now we use (39) to obtain

$$\begin{aligned}
\|u(\tau+t)\|_{K,\delta} \leq\ & C\Bigg[\|u(\tau)\|_{K,\delta}+C(T)\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{k,\delta}^{\kappa}\times \\
&\times\ \left(\int_{\tau}^{\tau+t}(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)\frac{p+1-a}{1-a}}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\Bigg] \\
\leq\ & C\left[\|u(\tau)\|_{K,\delta}+C(T)\left(I_0(T)\right)^{\frac{1-a}{p+1-a}}\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{k,\delta}^{\kappa}\right],
\end{aligned} \tag{104}$$

where the function I_0 is defined by (96). The inequality (97) is true for a sufficiently small. Hence the function I_0 is finite on $[0,\infty)$. Since $k<K$, for arbitrary $T\geq 0$ and $K\in\left[\frac{N+3}{N+1},\infty\right)$ we have

$$\|u(\tau+t)\|_{K,\delta}\leq C(T)\left[\|u(\tau)\|_{K,\delta}+\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{K,\delta}^{\kappa}\right]. \tag{105}$$

Since $\kappa<1$, we use Young's inequality to deduce

$$\|u(\tau+t)\|_{K,\delta}\leq C(T)\left[\|u(\tau)\|_{K,\delta}+C_\varepsilon+\varepsilon\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{K,\delta}\right]. \tag{106}$$

Setting $\varepsilon=\frac{1}{2C(T)}$ we finally obtain desired estimate.

If $M\in\left(\frac{(p+1)(N+1)}{2},\frac{p+1}{p}\right)$ then we can choose $K=\infty$ and $k\in(k(M),\infty)$. Using estimates similar to (100)-(106) (instead of iii) and v) in Lemma 3.1 used to prove (100) we use iv) and vi)) we have

$$\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_\infty\leq C(T)\left(1+\|u(\tau)\|_\infty\right).$$

$\square$

Lemma 5.2. *Let $p+r<\frac{N+3}{N+1}$, $p>0$ and conditions (26), (31) be true. Let (u,v) be a global nonnegative solution of problem (2).*

i) Assume $r>1$. Then for $\gamma\in\left(1,\frac{1}{2-(p+r)}\right]$ and $T\geq 0$, there exists $C=C(p,q,r,s,\Omega,T)$ such that

$$\int_\tau^{\tau+T}\|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'\leq C\|u(\tau+T)\|_{1,\delta},\quad \tau\geq 0. \tag{107}$$

ii) Assume $r\leq 1$, $p+r>1$. Then for $\gamma\in\left(1,\frac{1}{2-(p+r)}\right]$ and $T\geq 0$, there exists $C=C(p,q,r,s,\Omega,T)$ such that

$$\int_\tau^{\tau+T}\|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'\leq C(1+\|u(\tau+T)\|_{1,\delta}),\quad \tau\geq 0. \tag{108}$$

If $p+r\leq 1$ then the estimate (108) is true for $\gamma\in\left[1,\frac{N+1}{N-1}\right)$.

Proof of Lemma 5.2 We define exponent

$$\gamma = \frac{1}{2-(p+r)}. \tag{109}$$

The conditions $1 < p+r < \frac{N+3}{N+1}$ imply $p+r < \gamma < \frac{N+1}{N-1}$. For $\tau, T \geq 0$ and $t \in (\tau, \tau+T]$ we estimate

$$\|u(t)\|_{\gamma,\delta} \leq C \left[(t-\tau)^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \|u(\tau)\|_{1,\delta} + \int_\tau^t (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \|u^r v^p(s')\|_{1,\delta} \, \mathrm{d}s' \right] \tag{110}$$

where we used Lemma 3.1 iii) and v). Integrating (110) on interval $[\tau, \tau+T]$ with respect to t we have

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta} \, \mathrm{d}t \leq & \ C \left[\|u(\tau)\|_{1,\delta} \int_\tau^{\tau+T} (t-\tau)^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \, \mathrm{d}t \right. \\ & + \left. \int_\tau^{\tau+T} \int_\tau^t (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \|u^r v^p(s')\|_{1,\delta} \, \mathrm{d}s' \, \mathrm{d}t \right]. \end{aligned} \tag{111}$$

Now we use Fubini's theorem in the last term in (111) to obtain

$$\begin{aligned} & \int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta} \, \mathrm{d}t \leq C \left[T^{1-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \|u(\tau)\|_{1,\delta} \right. \\ & + \left. \int_\tau^{\tau+T} \left(\int_{s'}^{\tau+T} (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \, \mathrm{d}t \right) \|u^r v^p(s')\|_{1,\delta} \, \mathrm{d}s' \right]. \end{aligned} \tag{112}$$

Since $s' \in [\tau, \tau+T]$, we can estimate

$$\int_{s'}^{\tau+T} (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \, \mathrm{d}t \leq C T^{1-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}.$$

Note that

$$\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right) < 1,$$

since $\gamma < \frac{N+1}{N-1}$.

Using (112) we have

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta} \, \mathrm{d}t & \leq C T^{1-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \left[\|u(\tau)\|_{1,\delta} + \int_\tau^{\tau+T} \|u^r v^p(s')\|_{1,\delta} \, \mathrm{d}s' \right] \\ & \leq C \left[\|u(\tau)\|_{1,\delta} + \int_\tau^{\tau+T} \int_\Omega \left[u^{p\frac{r+a-1}{p+1-a}} v^p(s') \right] \left[u^{\frac{(p+r)(1-a)}{p+1-a}}(s') \right] \varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \right]. \end{aligned} \tag{113}$$

Now Hölder's inequality in the last term in (113) implies

$$\begin{aligned} & \int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta} \, \mathrm{d}t \leq C \left[\|u(\tau)\|_{1,\delta} + \left(\int_\tau^{\tau+T} \int_\Omega u^{r+a-1} v^{p+1-a}(s') \varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \right)^{\frac{p}{p+1-a}} \times \right. \\ & \times \left. \left(\int_\tau^{\tau+T} \|u(s')\|_{p+r,\delta}^{p+r} \, \mathrm{d}s' \right)^{\frac{1-a}{p+1-a}} \right]. \end{aligned} \tag{114}$$

As in the proof of Lemma 5.1 we use (39) to obtain

$$\int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C\left[\|u(\tau)\|_{1,\delta} + C(T)\left(\int_\tau^{\tau+T} \|u(s')\|_{p+r,\delta}^{p+r}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right]. \tag{115}$$

In the last term in (115) we use the interpolation inequality (Lemma 3.4)

$$\|u(s')\|_{p+r,\delta}^{p+r} \le \|u(s')\|_{1,\delta}^{\frac{\gamma-(p+r)}{\gamma-1}} \|u(s')\|_{\gamma,\delta}^{\frac{\gamma(p+r-1)}{\gamma-1}}, \quad s' \in [\tau, \tau+T]. \tag{116}$$

From the definition (109) of γ we see that $\frac{\gamma(p+r-1)}{\gamma-1} = 1$. Hence

$$\begin{aligned}\int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta}\,\mathrm{d}t &\le C(T)\left[\|u(\tau)\|_{1,\delta} + \left(\int_\tau^{\tau+T} \|u(s')\|_{1,\delta}^{\frac{\gamma-(p+r)}{\gamma-1}} \|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right]\\ &\le C(T)\left[\|u(\tau)\|_{1,\delta} + \left(\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{1,\delta}\right)^{\frac{\gamma-(p+r)}{\gamma-1}\frac{1-a}{p+1-a}} \left(\int_\tau^{\tau+T} \|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right].\end{aligned} \tag{117}$$

Using Young's inequality we have

$$\begin{aligned}\int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta}\,\mathrm{d}t \le\ & C(T)\Bigg[\|u(\tau)\|_{1,\delta}\\ &+ C_\varepsilon\left(\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{1,\delta}\right)^{\beta} + \varepsilon\left(\int_\tau^{\tau+T} \|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'\right)\Bigg]\end{aligned} \tag{118}$$

where $\beta = \frac{\gamma-(p+r)}{\gamma-1}\frac{1-a}{p}$. For ε sufficiently small in (118) we have

$$\int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C(T)\left[\|u(\tau)\|_{1,\delta} + \left(\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{1,\delta}\right)^{\beta}\right]. \tag{119}$$

Using the estimate (119) we are ready to prove the assertions of the Lemma. First we prove the assertion i). If $r > 1$ then we choose $a = \frac{r-1}{p+r-1}$ in the definition of β, hence $\beta = 1$. We use (24) to obtain

$$\begin{aligned}\int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta}\,\mathrm{d}t \le\ & C(T)\left[e^{\lambda_1 T}\|u(\tau+T)\|_{1,\delta} + \sup_{s'\in[\tau,\tau+T]} e^{\lambda_1(\tau+T-s')}\|u(\tau+T)\|_{1,\delta}\right]\\ \le\ & C(T)e^{\lambda_1 T}\|u(\tau+T)\|_{1,\delta}\end{aligned}$$

and i) then follows.

Now we prove the assertion ii) for $p + r > 1$. We choose arbitrary $a \in A$ in the definition of β, hence $\beta < 1$. In this case we again use Young's inequality in the last term in (119) to obtain

$$\int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C(T)\left[\|u(\tau)\|_{1,\delta} + 1 + \sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{1,\delta}\right]. \tag{120}$$

Using (24) the assertion ii) for $p+r>1$ follows.

If $p+r \le 1$ then for $\gamma \in \left[1, \frac{N+1}{N-1}\right)$ we obtain

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{\gamma,\delta}\, \mathrm{d}t \le \;& C\left[\|u(\tau)\|_{1,\delta} + C(T)\left(\int_\tau^{\tau+T}\|u(s')\|_{1,\delta}^{p+r}\, \mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right] \\ \le \;& C(T)\left[\|u(\tau)\|_{1,\delta} + \sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{1,\delta}^{\frac{(p+r)(1-a)}{p+1-a}}\right] \end{aligned}$$

in the same way as we obtained (115). Since $\frac{(p+r)(1-a)}{p+1-a} < 1$, we can use Young's inequality to obtain (120). Using (24) the assertion ii) for $p+r\le 1$ follows. The proof of Lemma 5.2 is complete.

□

In Lemma 5.3 we will use the following notation. For $r \le 1$ and $0 < p < \frac{2}{N+1}$ denote

$$\begin{aligned} K_0 \;&:\; \left[1, \tfrac{p+1}{p}\right) \longrightarrow \mathbb{R}\cup\{\infty\}, \\ K_0(M) \;&=\; \begin{cases} \frac{M(N+1)}{(N+1)-2M}, & M \in \left[1, \frac{N+1}{2}\right), \\ \infty, & M \in \left[\frac{N+1}{2}, \frac{p+1}{p}\right). \end{cases} \end{aligned} \tag{121}$$

For $r>1$ denote

$$\begin{aligned} K_0' \;&:\; \left[1, \tfrac{p+r}{p+r-1}\right) \longrightarrow \mathbb{R}\cup\{\infty\}, \\ K_0'(M) \;&=\; \begin{cases} \frac{M(N+1)}{(N+1)-2M}, & M \in \left[1, \frac{N+1}{2}\right), \\ \infty, & M \in \left[\frac{N+1}{2}, \frac{p+r}{p+r-1}\right). \end{cases} \end{aligned} \tag{122}$$

Lemma 5.3. *Let $p+r < \frac{N+3}{N+1}$, $p>0$ and conditions (26), (31) be true. Let (u,v) be a global nonnegative solution of problem (2).*

i) Assume $r>1$. Then for $T \ge 0$, there exists $C = C(p,q,r,s,\Omega,T)$ such that

$$\int_\tau^{\tau+T} \|u(s')\|_{K,\delta}\, \mathrm{d}s' \le C\|u(\tau)\|_{k,\delta}, \quad \tau \ge 0 \tag{123}$$

for $K_0'(M) > K > k = k'(M)$, $M \in \left[1, \frac{N+1}{2}\right]$. If $M \in \left(\frac{N+1}{2}, \frac{p+r}{p+r-1}\right)$ then we can take $K=\infty$.

ii) Assume $r \le 1$, $\frac{2}{N+1} > p$. Then for $T\ge 0$, there exists $C = C(p,q,r,s,\Omega,T)$ such that

$$\int_\tau^{\tau+T} \|u(s')\|_{K,\delta}\, \mathrm{d}s' \le C(1+\|u(\tau)\|_{\max\{M,k\},\delta}), \quad \tau \ge 0 \tag{124}$$

for $K_0(M) > K > k > k(M)$, $k \ge 1$, $M \in \left[1, \frac{N+1}{2}\right]$. If $M \in \left(\frac{N+1}{2}, \frac{p+1}{p}\right)$ then we can take $K=\infty$.

Proof of Lemma 5.3 We choose a as follows

$$a = \frac{r-1}{p+r-1} \qquad \text{for part i),} \tag{125}$$

$$a > 0 \quad \text{sufficiently close to } 0 \qquad \text{for part ii).} \tag{126}$$

The choice (126) of a will be specified more precisely during the proof.

First we prove i). Observe that $\frac{N+1}{2} < \frac{p+r}{p+r-1}$ and $K_0'(M) > K'(M)$ for every $M \in \left[1, \frac{p+r}{p+r-1}\right)$ due to conditions $1 < p+r < \frac{N+3}{N+1}$ (see the definition (74) of functions K', k' and the definition (122) of K_0'). Hence (76) implies

$$K_0'(M) > k'(M) > M \quad \text{for all} \quad M \in \left[1, \frac{p+r}{p+r-1}\right). \tag{127}$$

For $K_0'(M) > K > k = k'(M)$, $M \in \left[1, \frac{N+1}{2}\right]$, $\tau, T \geq 0$ and $t \in (\tau, \tau+T]$ we estimate

$$\begin{aligned} \|u(t)\|_{K,\delta} \;\leq\; & C \Bigg[(t-\tau)^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u(\tau)\|_{M,\delta} \\ & + \int_\tau^t (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u^r v^p(s')\|_{M,\delta} \, \mathrm{d}s' \Bigg]. \end{aligned} \tag{128}$$

where we used Lemma 3.1 iii) and v). Integrating (128) on interval $[\tau, \tau+T]$ with respect to t we have

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{K,\delta} \, \mathrm{d}t \leq\; & C \Bigg[\|u(\tau)\|_{M,\delta} \int_\tau^{\tau+T} (t-\tau)^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \, \mathrm{d}t \\ & + \int_\tau^{\tau+T} \int_\tau^t (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u^r v^p(s')\|_{M,\delta} \, \mathrm{d}s' \, \mathrm{d}t \Bigg]. \end{aligned} \tag{129}$$

Observe that for such K, M there holds

$$\frac{N+1}{2}\left(\frac{1}{M} - \frac{1}{K}\right) < 1.$$

Now we use Fubini's theorem in the last term in (129) to obtain

$$\begin{aligned} & \int_\tau^{\tau+T} \|u(t)\|_{K,\delta} \, \mathrm{d}t \leq C \Bigg[T^{1-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u(\tau)\|_{M,\delta} \\ & + \int_\tau^{\tau+T} \left(\int_{s'}^{\tau+T} (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \, \mathrm{d}t \right) \|u^r v^p(s')\|_{M,\delta} \, \mathrm{d}s' \Bigg]. \end{aligned} \tag{130}$$

Since $s' \in [\tau, \tau+T]$, we can estimate

$$\int_{s'}^{\tau+T} (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \, \mathrm{d}t \leq C T^{1-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}.$$

As in the proof of Lemma 5.2 we rewrite (130) into the estimate

$$
\begin{aligned}
\int_\tau^{\tau+T} \|u(t)\|_{K,\delta}\,\mathrm{d}t \le\ & CT^{1-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\left[\|u(\tau)\|_{M,\delta}+\int_\tau^{\tau+T}\|u^r v^p(s')\|_{M,\delta}\,\mathrm{d}s'\right]\\
\le\ & C(T)\left[\|u(\tau)\|_{M,\delta}+\int_\tau^{\tau+T}\left(\int_\Omega\left[u^{pM\frac{r+a-1}{p+1-a}}v^{pM}(s')\right]\times\right.\right. \qquad (131)\\
\times\ & \left.\left.\left[u^{M\frac{(p+r)(1-a)}{p+1-a}}(s')\right]\varphi_1\,\mathrm{d}x\right)^{\frac{1}{M}}\mathrm{d}s'\right]
\end{aligned}
$$

where a is defined by (125). Observe that $M < \frac{p+1-a}{p}$, since $M < \frac{p+r}{p+r-1}$. Therefore we can use Hölder's inequality in the spatial integral in the last term in (131) to estimate

$$
\begin{aligned}
\int_\tau^{\tau+T}\|u(t)\|_{K,\delta}\,\mathrm{d}t \le\ & C(T)\left[\|u(\tau)\|_{M,\delta}+\int_\tau^{\tau+T}\left(\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p}{p+1-a}}\times\right.\\
\times\ & \left.\left(\int_\Omega u^{M\frac{(p+r)(1-a)}{p+1-a-pM}}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p+1-a-pM}{M(p+1-a)}}\mathrm{d}s'\right].
\end{aligned}
\qquad (132)
$$

Using Hölder's inequality in the time integral we have

$$
\begin{aligned}
\int_\tau^{\tau+T}\|u(t)\|_{K,\delta}\,\mathrm{d}t \le C(T)&\left[\|u(\tau)\|_{M,\delta}+\left(\int_\tau^{\tau+T}\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right)^{\frac{p}{p+1-a}}\times\right.\\
\times&\left.\left(\int_\tau^{\tau+T}\|u(s')\|^{p+r}_{M\frac{(p+r)(1-a)}{p+1-a-pM},\delta}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right].
\end{aligned}
\qquad (133)
$$

Now the inequality (39) implies

$$
\int_\tau^{\tau+T}\|u(t)\|_{K,\delta}\,\mathrm{d}t \le C(T)\left[\|u(\tau)\|_{M,\delta}+C(T)\left(\int_\tau^{\tau+T}\|u(s')\|^{p+r}_{M\frac{(p+r)(1-a)}{p+1-a-pM},\delta}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right]. \qquad (134)
$$

Note that $\frac{(p+r)(1-a)}{p+1-a} = 1$ and $M\frac{(p+r)(1-a)}{p+1-a-pM} = k$ for our choice (125) of a. Hence using Lemma 5.1 i) we have

$$
\int_\tau^{\tau+T}\|u(t)\|_{K,\delta}\,\mathrm{d}t \le C(T)\left(\|u(\tau)\|_{M,\delta}+\|u(\tau)\|_{k,\delta}\right) \le C(T)\|u(\tau)\|_{k,\delta}, \qquad (135)
$$

since $k > M$.

If $M \in \left(\frac{N+1}{2}, \frac{p+r}{p+r-1}\right)$ then we can choose $K = \infty$ and $k'(M) < k < \infty$. Hence the estimates (128)-(135) imply

$$
\int_\tau^{\tau+T}\|u(t)\|_\infty\,\mathrm{d}t \le C(T)\|u(\tau)\|_{k,\delta}, \qquad (136)
$$

thus we proved i).

Now we prove ii). Observe that $\frac{N+1}{2} < \frac{p+1}{p}$ and $K_0(M) > K(M)$ for every $M \in \left[1, \frac{p+1}{p}\right)$ due to conditions $0 < p < \frac{2}{N+1}$ (see the definition (73) of functions K, k and the definition (121) of K_0). Hence (75) implies

$$K_0(M) > \max\{k(M), M\} \quad \text{for all} \quad M \in \left[1, \frac{p+1}{p}\right). \tag{137}$$

Choose $K_0(M) > K > k > k(M)$, $k \geq 1$, $M \in \left[1, \frac{N+1}{2}\right]$. We need $k \geq 1$, since $k(1) = p + r$ and possibly $p + r < 1$. For $\tau, T \geq 0$ and $t \in (\tau, \tau + T]$ we estimate

$$\begin{aligned} \|u(t)\|_{K,\delta} \ \leq \ & C \Bigg[(t-\tau)^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u(\tau)\|_{M,\delta} \\ & + \int_\tau^t (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u^r v^p(s')\|_{M,\delta} \, \mathrm{d}s' \Bigg]. \end{aligned} \tag{138}$$

where we used Lemma 3.1 iii) and v). Integrating (138) on interval $[\tau, \tau + T]$ with respect to t we have

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{K,\delta} \, \mathrm{d}t \leq \ & C \Bigg[\|u(\tau)\|_{M,\delta} \int_\tau^{\tau+T} (t-\tau)^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \, \mathrm{d}t \\ & + \int_\tau^{\tau+T} \int_\tau^t (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u^r v^p(s')\|_{M,\delta} \, \mathrm{d}s' \, \mathrm{d}t \Bigg]. \end{aligned} \tag{139}$$

Observe that for such K, M there holds

$$\frac{N+1}{2}\left(\frac{1}{M} - \frac{1}{K}\right) < 1.$$

Now we use Fubini's theorem in the last term in (139) to obtain

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{K,\delta} \, \mathrm{d}t \leq \ & C \Bigg[T^{1-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \|u(\tau)\|_{M,\delta} \\ & + \int_\tau^{\tau+T} \left(\int_{s'}^{\tau+T} (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \, \mathrm{d}t \right) \|u^r v^p(s')\|_{M,\delta} \, \mathrm{d}s' \Bigg] \end{aligned} \tag{140}$$

Since $s' \in [\tau, \tau + T]$, we can estimate

$$\int_{s'}^{\tau+T} (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \, \mathrm{d}t \leq C T^{1-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}.$$

We rewrite (140) into the estimate

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{K,\delta} \, \mathrm{d}t \leq \ & C T^{1-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \left[\|u(\tau)\|_{M,\delta} + \int_\tau^{\tau+T} \|u^r v^p(s')\|_{M,\delta} \, \mathrm{d}s' \right] \\ \leq \ & C(T) \Bigg[\|u(\tau)\|_{M,\delta} + \int_\tau^{\tau+T} \left(\int_\Omega \left[u^{pM\frac{r+a-1}{p+1-a}} v^{pM}(s') \right] \times \right. \\ \times \ & \left. \left[u^{M\frac{(p+r)(1-a)}{p+1-a}}(s') \right] \varphi_1 \, \mathrm{d}x \right)^{\frac{1}{M}} \mathrm{d}s' \Bigg], \end{aligned} \tag{141}$$

where a is defined by (126). Observe that

$$M < \frac{p+1-a}{p}, \tag{142}$$

since $M < \frac{p+1}{p}$: the inequality (142) is true, if for M fixed we choose a sufficiently small. Hence we can use Hölder's inequality in the spatial integral in the last term in (141) to estimate

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{K,\delta}\, \mathrm{d}t \leq\ & C(T) \left[\|u(\tau)\|_{M,\delta} + \int_\tau^{\tau+T} \left(\int_\Omega u^{r+a-1} v^{p+1-a}(s')\varphi_1\, \mathrm{d}x \right)^{\frac{p}{p+1-a}} \times \right. \\ & \left. \times \left(\int_\Omega u^{\widetilde{k}(M)}(s')\varphi_1\, \mathrm{d}x \right)^{\frac{p+1-a-pM}{M(p+1-a)}} \mathrm{d}s' \right] \end{aligned} \tag{143}$$

where $\widetilde{k}(M) = M\frac{(p+r)(1-a)}{p+1-a-pM}$. Using Hölder's inequality in the time integral in (143) we have

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{K,\delta}\, \mathrm{d}t \leq\ & C(T) \left[\|u(\tau)\|_{M,\delta} + \left(\int_\tau^{\tau+T} \int_\Omega u^{r+a-1} v^{p+1-a}(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s' \right)^{\frac{p}{p+1-a}} \times \right. \\ & \left. \times \left(\int_\tau^{\tau+T} \|u(s')\|^{p+r}_{\max\{1,\widetilde{k}(M)\},\delta}\, \mathrm{d}s' \right)^{\frac{1-a}{p+1-a}} \right]. \end{aligned}$$

Now the inequality (39) implies

$$\int_\tau^{\tau+T} \|u(t)\|_{K,\delta}\, \mathrm{d}t \leq C(T) \left[\|u(\tau)\|_{M,\delta} + C(T) \left(\int_\tau^{\tau+T} \|u(s')\|^{p+r}_{\max\{1,\widetilde{k}(M)\},\delta}\, \mathrm{d}s' \right)^{\frac{1-a}{p+1-a}} \right]. \tag{144}$$

Note that $\kappa' = \frac{(p+r)(1-a)}{p+1-a} < 1$ and $\widetilde{k}(M)$ is close to $k(M)$ and thus $\widetilde{k}(M) < k$ for a sufficiently small. If $p+r<1$ then $\widetilde{k}(1) < 1$ for a small. Hence using Lemma 5.1 iii) we have

$$\int_\tau^{\tau+T} \|u(t)\|_{K,\delta}\, \mathrm{d}t \leq C(T) \left[\|u(\tau)\|_{M,\delta} + (1 + \|u(\tau)\|_{k,\delta})^{\kappa'} \right].$$

Using Young's inequality in the last term we obtain

$$\int_\tau^{\tau+T} \|u(t)\|_{K,\delta}\, \mathrm{d}t \leq C(T)\left(1 + \|u(\tau)\|_{M,\delta} + \|u(\tau)\|_{k,\delta}\right) \leq C(T)\left(1 + \|u(\tau)\|_{\max\{k,M\},\delta}\right). \tag{145}$$

If $M \in \left(\frac{N+1}{2}, \frac{p+1}{p}\right)$ is sufficiently close to $\frac{p+1}{p}$ then $k(M) > M$ and we can choose $K = \infty$ and $k(M) < k < \infty$. Hence the estimates (138)-(145) imply

$$\int_\tau^{\tau+T} \|u(t)\|_\infty\, \mathrm{d}t \leq C(T)\left(1 + \|u(\tau)\|_{k,\delta}\right),$$

thus we proved ii).

□

Theorem 5.1. *Let $p+r<\frac{N+3}{N+1}$, $p>0$ and conditions (26), (31) be true. Let (u,v) be a global nonnegative solution of problem (2).*

i) Assume $r>1$. Then for every $\tau>0$, there exists $C=C(p,q,r,s,\Omega,\tau)$ such that

$$\|u(t)\|_\infty \le C\|u(t)\|_{1,\delta} \tag{146}$$

for every $t\ge\tau$.

ii) Assume $r\le 1, p<\frac{2}{N+1}$. Then for every $\tau>0$, there exists $C=C(p,q,r,s,\Omega,\tau)$ such that

$$\|u(t)\|_\infty \le C(1+\|u(t)\|_{1,\delta}) \tag{147}$$

for every $t\ge\tau$.

Remark. The constant C from both assertions of Theorem 5.1 may explode if $\tau\to 0^+$.

Proof of Theorem 5.1 First we prove i). Let

$$\gamma=\frac{1}{2-(p+r)}.$$

Conditions $1<p+r<\frac{N+3}{N+1}$ imply $p+r<\gamma<\frac{N+1}{N-1}$. Fix $1>\tau_0>0$ and let $t>0$ be arbitrary. Note that there exists $\tau'\in[\tau_0+t,2\tau_0+t]$ such that

$$\|u(\tau')\|_{\gamma,\delta}=\tau_0^{-1}\int_{\tau_0+t}^{2\tau_0+t}\|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'. \tag{148}$$

Obviously, this τ' may depend on t and u. Since $2\tau_0+t\in[\tau',\tau'+\tau_0]$, there holds

$$\|u(2\tau_0+t)\|_{\gamma,\delta}\le \sup_{s'\in[\tau',\tau'+\tau_0]}\|u(s')\|_{\gamma,\delta}. \tag{149}$$

We use Lemma 5.1 i) with τ replaced by τ' to obtain

$$\sup_{s'\in[\tau',\tau'+\tau_0]}\|u(s')\|_{\gamma,\delta}\le C\|u(\tau')\|_{\gamma,\delta} \tag{150}$$

where C does not depend on τ',τ_0. Lemma 5.2 i) implies

$$\int_{\tau_0+t}^{2\tau_0+t}\|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'\le C\|u(2\tau_0+t)\|_{1,\delta}. \tag{151}$$

Finally, the equality (148) and the estimates (149)-(151) imply

$$\begin{aligned}\|u(2\tau_0+t)\|_{\gamma,\delta}\le{}& \sup_{s'\in[\tau',\tau'+\tau_0]}\|u(s')\|_{\gamma,\delta}\le C\|u(\tau')\|_{\gamma,\delta}\\ ={}& C\tau_0^{-1}\int_{\tau_0+t}^{2\tau_0+t}\|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'\le C\tau_0^{-1}\|u(2\tau_0+t)\|_{1,\delta},\end{aligned}$$

thus

$$\|u(2\tau_0+t)\|_{\gamma,\delta} \le C(\tau_0)\|u(2\tau_0+t)\|_{1,\delta}. \tag{152}$$

Now fix $l \in \mathbb{N}, l > 1$ and K, k such that $K_0'(M) > K > k = k'(M),\ M \in \left(1, \frac{N+1}{2}\right]$ (see the definition (74), (122) of function k', K_0', respectively). This choice is possible due to inequality (127). Again, there exists $\tau' \in [l\tau_0+t, (l+1)\tau_0+t]$ such that

$$\|u(\tau')\|_{K,\delta} = \tau_0^{-1}\int_{l\tau_0+t}^{(l+1)\tau_0+t} \|u(s')\|_{K,\delta}\,\mathrm{d}s'. \tag{153}$$

Since $(l+1)\tau_0+t \in [\tau', \tau'+\tau_0]$, there holds

$$\|u((l+1)\tau_0+t)\|_{K,\delta} \le \sup_{s'\in[\tau',\tau'+\tau_0]} \|u(s')\|_{K,\delta}. \tag{154}$$

We use Lemma 5.1 i) with τ replaced by τ' to obtain

$$\sup_{s'\in[\tau',\tau'+\tau_0]} \|u(s')\|_{K,\delta} \le C\|u(\tau')\|_{K,\delta} \tag{155}$$

where C does not depend on τ'. Lemma 5.3 i) implies

$$\int_{l\tau_0+t}^{(l+1)\tau_0+t} \|u(s')\|_{K,\delta}\,\mathrm{d}s' \le C\|u(l\tau_0+t)\|_{k,\delta}. \tag{156}$$

Finally, the equality (153) and the estimates (154)-(156) imply

$$\begin{aligned}\|u((l+1)\tau_0+t)\|_{K,\delta} \le & \sup_{s'\in[\tau',\tau'+\tau_0]} \|u(s')\|_{K,\delta} \le C\|u(\tau')\|_{K,\delta} \\ = & \ \tau_0^{-1}\int_{l\tau_0+t}^{(l+1)\tau_0+t} \|u(s')\|_{K,\delta}\,\mathrm{d}s' \le C\tau_0^{-1}\|u(l\tau_0+t)\|_{k,\delta}\end{aligned}$$

where C does not depend on τ', τ_0 and t. Thus

$$\|u((l+1)\tau_0+t)\|_{K,\delta} \le C(\tau_0)\|u(l\tau_0+t)\|_{k,\delta}. \tag{157}$$

If we choose $\frac{p+r}{p+r-1} > M_0 > \frac{N+1}{2}$ then in (157), we can take $K = \infty$ and $k = k_0$ for some

$$\infty > k_0 > k'(M_0). \tag{158}$$

Hence we have

$$\|u((l+1)\tau_0+t)\|_{\infty} \le C(\tau_0)\|u(l\tau_0+t)\|_{k_0,\delta} \tag{159}$$

for all $l \in \mathbb{N},\ l > 1$. Now we apply bootstrap argument on (157): Since $K_0'(M) > k'(M)$ for $M \in \left[1, \frac{N+1}{2}\right)$ and $K_0'(\frac{N+1}{2}) = \infty$, there exists small enough $\varepsilon > 0$ such that

$$\widetilde{K} := \min\left\{\frac{2\gamma}{p+r}k', (1-\varepsilon)K_0'\right\} > (1+\varepsilon)k' =: \widetilde{k} \quad \text{on} \quad \left[1, \frac{p+r}{p+r-1}\right) \tag{160}$$

and $\widetilde{K}(1) > \gamma > \widetilde{k}(1) > p + r$. This follows from continuity of functions K_0', k'. For $M \in \left[1, \frac{p+r}{p+r-1}\right)$ given, there holds $K_0'(M) > \widetilde{K}(M) > \widetilde{k}(M) > k = k'(M)$ and $\widetilde{K} < \infty$. Hence (157) yields

$$\|u((l+1)\tau_0 + t)\|_{\widetilde{K}(M),\delta} \leq C(\tau_0)\|u(l\tau_0 + t)\|_{\widetilde{k}(M),\delta}. \tag{161}$$

Let $m_1 = 1$ and for $i \in \mathbb{N}$ denote $m_{i+1} = (\widetilde{k})^{-1}(\widetilde{K}(m_i))$. Observe that $\widetilde{K}(m_i) > \widetilde{k}(m_i)$, $m_i < \frac{p+r}{p+r-1}$ and $m_{i+1} > m_i$. The estimate (161) implies

$$\|u((l+1)\tau_0 + t)\|_{\widetilde{K}(m_i),\delta} \leq C(\tau_0)\|u(l\tau_0 + t)\|_{\widetilde{k}(m_i),\delta}, \quad i, l \in \mathbb{N}. \tag{162}$$

Due to Lemma 3.6 there exists $l_0 = l_0(p, q, r, s) \in \mathbb{N}$ such that $\widetilde{K}(m_{l_0}) > k_0$ (where k_0 is chosen by (158)). Therefore (159) implies

$$\|u((l_0 + 3)\tau_0 + t)\|_\infty \leq C(\tau_0)\|u((l_0 + 2)\tau_0 + t)\|_{\widetilde{K}(m_{l_0}),\delta}. \tag{163}$$

Using (162) l_0-times (for $l = 2, \ldots, l_0 + 1$ and $i = 1, \ldots, l_0$) we have

$$\|u((l_0 + 2)\tau_0 + t)\|_{\widetilde{K}(m_{l_0}),\delta} \leq C(\tau_0)\|u(2\tau_0 + t)\|_{\widetilde{k}(m_1),\delta}.$$

This inequality, $\gamma > \widetilde{k}(m_1)$, (163) and (152) imply

$$\|u((l_0{+}3)\tau_0{+}t)\|_\infty \leq C(\tau_0)\|u(2\tau_0{+}t)\|_{\widetilde{k}(m_1),\delta} \leq C(\tau_0)\|u(2\tau_0{+}t)\|_{\gamma,\delta} \leq C(\tau_0)\|u(2\tau_0{+}t)\|_{1,\delta}.$$

Finally, (24) yields

$$\|u((l_0 + 3)\tau_0 + t)\|_\infty \leq C(\tau_0)\|u(2\tau_0 + t)\|_{1,\delta} \leq C(\tau_0)e^{\lambda_1(l_0+1)\tau_0}\|u((l_0 + 3)\tau_0 + t)\|_{1,\delta}.$$

Letting $\tau_0 = \frac{\tau}{l_0+3}$ implies the assertion i).

Now we prove ii). We choose

$$\gamma = \frac{1}{2 - (p + r)}, \quad \text{if } p + r > 1$$

or

$$\gamma \in \left(1, \frac{N+1}{N-1}\right), \quad \text{if } p + r \leq 1,$$

and t, τ_0 are as at the beginning of the proof of i). There exists $\tau' \in [\tau_0 + t, 2\tau_0 + t]$ such that

$$\|u(\tau')\|_{\gamma,\delta} = \tau_0^{-1}\int_{\tau_0+t}^{2\tau_0+t} \|u(s')\|_{\gamma,\delta}\, \mathrm{d}s'. \tag{164}$$

Obviously, this τ' may depend on t and u. Since $2\tau_0 + t \in [\tau', \tau' + \tau_0]$, there holds

$$\|u(2\tau_0 + t)\|_{\gamma,\delta} \leq \sup_{s' \in [\tau', \tau'+\tau_0]} \|u(s')\|_{\gamma,\delta}. \tag{165}$$

We use Lemma 5.1 iii) with τ replaced by τ' to obtain

$$\sup_{s'\in[\tau',\tau'+\tau_0]} \|u(s')\|_{\gamma,\delta} \le C\,(1+\|u(\tau')\|_{\gamma,\delta}) \tag{166}$$

where C does not depend on τ',τ_0. Lemma 5.2 ii) implies

$$\int_{\tau_0+t}^{2\tau_0+t} \|u(s')\|_{\gamma,\delta}\,\mathrm{d}s' \le C\,(1+\|u(2\tau_0+t)\|_{1,\delta})\,. \tag{167}$$

Finally, the equality (164) and the estimates (165)-(167) imply

$$\begin{aligned}\|u(2\tau_0+t)\|_{\gamma,\delta} \le\;& C\,(1+\|u(\tau')\|_{\gamma,\delta}) = C\left(1+\tau_0^{-1}\int_{\tau_0+t}^{2\tau_0+t}\|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'\right)\\ \le\;& C(\tau_0)\,(1+\|u(2\tau_0+t)\|_{1,\delta})\,.\end{aligned} \tag{168}$$

Now fix $l\in\mathbb{N}, l>1$ and K such that $K_0(M)>K>\max\{k(M),1\}$, $M\in\left(1,\frac{N+1}{2}\right]$ (see the definition (73), (121) of function k, K_0, respectively). Then fix $k\in(K,\max\{k(M),1\})$. These choices are possible due to inequality (137). Again, there exists $\tau'\in[l\tau_0+t,(l+1)\tau_0+t]$ such that

$$\|u(\tau')\|_{K,\delta} = \tau_0^{-1}\int_{l\tau_0+t}^{(l+1)\tau_0+t}\|u(s')\|_{K,\delta}\,\mathrm{d}s'. \tag{169}$$

Since $(l+1)\tau_0+t\in[\tau',\tau'+\tau_0]$, there holds

$$\|u((l+1)\tau_0+t)\|_{K,\delta} \le \sup_{s'\in[\tau',\tau'+\tau_0]}\|u(s')\|_{K,\delta}. \tag{170}$$

We use Lemma 5.1 iii) with τ replaced by τ' to obtain

$$\sup_{s'\in[\tau',\tau'+\tau_0]} \|u(s')\|_{K,\delta} \le C\,(1+\|u(\tau')\|_{K,\delta}) \tag{171}$$

where C does not depend on τ',τ_0. Lemma 5.3 ii) implies

$$\int_{l\tau_0+t}^{(l+1)\tau_0+t}\|u(s')\|_{K,\delta}\,\mathrm{d}s' \le C\,(1+\|u(l\tau_0+t)\|_{k,\delta})\,. \tag{172}$$

Finally, the equality (169) and the estimates (170)-(172) imply

$$\begin{aligned}\|u((l+1)\tau_0+t)\|_{K,\delta} \le\;& C\,(1+\|u(\tau')\|_{K,\delta})\\ \le\;& C(\tau_0)\left(1+\int_{l\tau_0+t}^{(l+1)\tau_0+t}\|u(s')\|_{K,\delta}\,\mathrm{d}s'\right)\\ \le\;& C(\tau_0)\,(1+\|u(l\tau_0+t)\|_{k,\delta})\end{aligned} \tag{173}$$

where C does not depend on τ' and t.

If we choose $M_1 > \frac{N+1}{2}$ then in (173), we can take $K = \infty$ and $\infty > k_1 > k'(M_1)$ such that there holds

$$\|u((l+1)\tau_0 + t)\|_\infty \le C(\tau_0)\,(1 + \|u(l\tau_0 + t)\|_{k_1,\delta}) \tag{174}$$

for all $l \in \mathbb{N}$, $l > 1$. As in the proof of i) we apply bootstrap argument on (173): using (173) l_1-times (for $l = 2, \ldots, l_1 + 1$; $l_1 = l_1(p,q,r,s)$) we have

$$\|u((l_1+2)\tau_0 + t)\|_{K_1,\delta} \le C(\tau_0)\,(1 + \|u(2\tau_0 + t)\|_{\gamma,\delta}) \tag{175}$$

for some $K_1 > k_1$ (note that $p + r = k(1)$ and $\gamma > \max\{1, p+r\}$). This inequality and (175) imply

$$\|u((l_1+3)\tau_0 + t)\|_\infty \le C(\tau_0)(1 + \|u(2\tau_0 + t)\|_{\gamma,\delta}).$$

It is possible to use (168) and (24) to obtain

$$\begin{aligned}\|u((l_1+3)\tau_0 + t)\|_\infty \le\ & C(\tau_0)\,(1 + \|u(2\tau_0 + t)\|_{\gamma,\delta}) \le C(\tau_0)\,(1 + \|u(2\tau_0 + t)\|_{1,\delta}) \\ \le\ & C(\tau_0)\left(1 + e^{\lambda_1(l_1+1)\tau_0}\|u((l_1+3)\tau_0 + t)\|_{1,\delta}\right).\end{aligned}$$

Letting $\tau_0 = \frac{\tau}{l_1+3}$ implies the assertion of the theorem.

$\square$

Corollary 1. *Assume $p + r < \frac{N+3}{N+1}$, $\frac{2}{N+1} > p > 0$ and let conditions (26), (31) be true. If $r > 1$ then assume also $pq > (1-r)(1-s)$. Let (u,v) be a global nonnegative solution of problem (2). Then for $\tau > 0$, there exists $C = C(p,q,r,s,\tau,\Omega)$ such that*

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_\infty \le C(1 + \|u(\tau)\|_\infty), \quad T \ge 0. \tag{176}$$

Proof. This follows from (146) (if $r > 1$) or (147) (if $r \le 1$) and Lemma 5.1 ii).

$\square$

Lemma 5.4. *Assume $p + r < \frac{N+3}{N+1}$, $\frac{2}{N+1} > p > 0$, $s < \frac{N+3}{N+1}$, (26) and (31). If $r > 1$ then assume also $pq > (1-r)(1-s)$. Let (u,v) be a global nonnegative solution of problem (2). Then there exists $C = C(p,q,r,s,\Omega,\|u(\tau)\|_\infty,\|v(\tau)\|_\infty, \sup_{s' \in [\tau,\tau+T]} \|v(s')\|_{1,\delta})$ such that*

$$\sup_{s' \in [\tau, \tau+T]} \|v(s')\|_\infty \le C, \quad T, \tau \ge 0. \tag{177}$$

Proof of Lemma 5.4 Due to Corollary 1 we can write $u(x,t) \le C(\|u(\tau)\|_\infty)$ for $(x,t) \in \Omega \times [\tau, \infty)$ (note that the constant C in (176) is independent of T). Then v satisfies

$$v_t - \Delta v \le C(\|u(\tau)\|_\infty)^q v^s, \quad (x,t) \in \Omega \times [\tau, \infty) \tag{178}$$

where $s < \frac{N+3}{N+1}$.

Assume $s > 1$. We choose arbitrary γ such that

$$\frac{1}{2-s} < \gamma < \frac{N+1}{N-1}. \tag{179}$$

Note that $\frac{1}{2-s} < \frac{N+1}{N-1}$, since $s < \frac{N+3}{N+1}$.

For fixed $T > 0$ and $t \in [0, T]$ we estimate

$$\begin{aligned} \|v(\tau+t)\|_{\gamma,\delta} \leq C(\|u(\tau)\|_\infty) \Bigg[&\|v(\tau)\|_{\gamma,\delta} \\ + \int_\tau^{\tau+t} &e^{-\frac{\lambda_1}{2}(t+\tau-s')}(t+\tau-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \int_\Omega v^s(s')\varphi_1 \,\mathrm{d}x \,\mathrm{d}s' \Bigg]. \end{aligned} \tag{180}$$

In (180) we estimate term $\|v(\tau)\|_{\gamma,\delta}$ by a constant $C(\|v(\tau)\|_\infty)$ to obtain

$$\begin{aligned} \|v(\tau+t)\|_{\gamma,\delta} \leq C(\|u(\tau)\|_\infty) \Bigg[&C(\|v(\tau)\|_\infty) + \sup_{s' \in [\tau,\tau+T]} \|v(s')\|_{s,\delta}^s \times \\ \times \int_\tau^{\tau+t} &e^{-\frac{\lambda_1}{2}(t+\tau-s')}(t+\tau-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \,\mathrm{d}s' \Bigg]. \end{aligned} \tag{181}$$

Note that the integral in (181) is bounded by a constant independent of t, hence

$$\|v(\tau+t)\|_{\gamma,\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \left[1 + \sup_{s' \in [\tau,\tau+T]} \|v(s')\|_{s,\delta}^s \right]. \tag{182}$$

As in the proof of Lemma 5.2 we use the interpolation inequality and Young's inequality to obtain

$$\|v(s')\|_{s,\delta}^s \leq \|v(s')\|_{1,\delta}^{\frac{\gamma-s}{\gamma-1}} \|v(s')\|_{\gamma,\delta}^{\frac{\gamma(s-1)}{\gamma-1}} \leq C_\varepsilon \|v(s')\|_{1,\delta}^{\frac{\gamma-s}{(\gamma-1)(1-\theta)}} + \varepsilon \|v(s')\|_{\gamma,\delta}^{\frac{\gamma(s-1)}{(\gamma-1)\theta}} \tag{183}$$

where $\theta \in (0,1)$. Due to our choice (179) of γ there holds $\frac{\gamma(s-1)}{\gamma-1} < 1$, hence there exists $\theta \in (0,1)$ such that

$$\frac{\gamma(s-1)}{(\gamma-1)\theta} = 1.$$

Finally, (183) implies

$$\begin{aligned} &\|v(\tau+t)\|_{\gamma,\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \times \\ &\times \left[1 + C_\varepsilon \left(\sup_{s' \in [\tau,\tau+T]} \|v(s')\|_{1,\delta} \right)^{\frac{\gamma-s}{(\gamma-1)(1-\theta)}} + \varepsilon \sup_{s' \in [\tau,\tau+T]} \|v(s')\|_{\gamma,\delta} \right]. \end{aligned}$$

Choosing ε sufficiently small yields

$$\sup_{s' \in [\tau,\tau+T]} \|v(s')\|_{\gamma,\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty, \sup_{s' \in [\tau,\tau+T]} \|v(s')\|_{1,\delta}), \quad \gamma \in \left(1, \frac{N+1}{N-1}\right). \tag{184}$$

For $0 < s \le 1$ we choose arbitrary γ such that

$$1 < \gamma < \frac{N+1}{N-1}.$$

For fixed $T > 0$ and $t \in [0, T]$ we estimate

$$\begin{aligned}
\|v(\tau+t)\|_{\gamma,\delta} \le\ & C(\|u(\tau)\|_\infty) \Bigg[\|v(\tau)\|_{\gamma,\delta} \\
+ & \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(t+\tau-s')} (t+\tau-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \|v(s')\|_{s,\delta}^{s} \, \mathrm{d}s' \Bigg] \\
\le\ & C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \left[1 + \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta} \right] \\
\le\ & C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty, \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta}),
\end{aligned}$$

hence the estimate (184) follows.

For $s = 0$ the assertion (184) follows from estimate analogous to (178).

Now we prove that (184) holds for $\gamma = \infty$. For $s \ge \frac{2}{N+1}$ we denote

$$\begin{aligned}
K_1 &: [1, \infty) \longrightarrow \mathbb{R} \cup \{\infty\}, \\
K_1(Q) &= \begin{cases} \frac{Q(N+1)}{s(N+1)-2Q}, & Q \in \left[1, \frac{s(N+1)}{2}\right), \\ \infty, & Q \ge \frac{s(N+1)}{2}. \end{cases}
\end{aligned} \tag{185}$$

Assume $\sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta} \le C$. Our goal is to prove

$$\text{if } s \ge \frac{2}{N+1}, \ \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{Q,\delta} \le C, \ \ Q \in \left[1, \frac{s(N+1)}{2}\right] \quad \text{then } \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{K,\delta} \le C \tag{186}$$

for

$$K < K_1(Q) \tag{187}$$

and

$$\text{if } \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{Q,\delta} \le C, \ \ Q > \max\left\{1, \frac{s(N+1)}{2}\right\} \quad \text{then } \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_\infty \le C. \tag{188}$$

In (186) and (188) the constant C may depend on (u, v), more precisely $C = C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty, \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta})$.

We estimate

$$\begin{aligned}
&\|v(\tau+t)\|_{K,\delta} \le C(\|u(\tau)\|_\infty) \Bigg[\|v(\tau)\|_{K,\delta} \\
&+ \int_\tau^{\tau+t} e^{\frac{-\lambda_1}{2}(t+\tau-s')} (t+\tau-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \left(\int_\Omega v(s')^{Ms} \varphi_1 \, \mathrm{d}x \right)^{\frac{1}{M}} \mathrm{d}s' \Bigg].
\end{aligned} \tag{189}$$

In (189) we estimate term $\|v(\tau)\|_{K,\delta}$ by a constant $C(\|v(\tau)\|_\infty)$ to obtain

$$\begin{aligned}\|v(\tau+t)\|_{K,\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \Bigg[1 + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|^s_{Ms,\delta} \times \\ \times \int_\tau^{\tau+t} e^{\frac{-\lambda_1}{2}(t+\tau-s')}(t+\tau-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\,\mathrm{d}s'\Bigg].\end{aligned} \tag{190}$$

To prove the assertion (186) we choose

$$M = \frac{Q}{s}.$$

Due to this definition of M and our choice (187) of K there holds

$$\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right) < 1.$$

Hence the integral in (190) is bounded by a constant independent of t and we can estimate

$$\|v(\tau+t)\|_{K,\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \left[1 + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|^s_{Ms,\delta}\right]. \tag{191}$$

Therefore (186) follows from (191).

Now we prove (188). Similar estimates to (189) and (190) (with $K=\infty$) are true in case of $Q > \frac{s(N+1)}{2}$. In (190) we set $M = \frac{Q}{s}$. Then we use the assertions iv) and vi) in Lemma 3.1 instead of iii) and v) in the estimate (189). We obtain

$$\|v(\tau+t)\|_\infty \leq C\left[1 + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|^s_{Q,\delta}\right]. \tag{192}$$

Thus (188) holds.

Let first $s \geq \frac{2}{N+1}$. Since $s < \frac{N+3}{N+1}$, there holds

$$K_1(Q) > Q \quad \text{for } Q \geq 1 \tag{193}$$

and to finish the proof for $s \geq \frac{2}{N+1}$ we use bootstrap argument: Due to the inequality (193) there exists $\varepsilon > 0$ such that $\widetilde{K}(Q) := \min\{(1+\varepsilon)Q, (1-\varepsilon)K_1(Q)\} > Q$ for $Q \geq 1$. Denote $Q_1 = 1$ and $Q_{i+1} = \widetilde{K}(Q_i)$ for $i \in \mathbb{N}$. Due to Lemma 3.5 there exists $i_0 \in \mathbb{N}$ such that $Q_{i_0} > \frac{s(N+1)}{2}$. Since we proved (186), there holds

$$\sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{Q_{i_0},\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty, \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{1,\delta}).$$

The assertion (188) then implies

$$\sup_{s'\in[\tau,\tau+T]} \|v(s')\|_\infty \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty, \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{1,\delta}).$$

Thus the proof for $s \geq \frac{2}{N+1}$ is finished.

If $s < \frac{2}{N+1}$ then the assertion of the Lemma follows immediately from the estimate (192). Indeed, since the estimate (184) is true, then there holds

$$\sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{Q,\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty, \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta}).$$

for some $Q > 1$. Then (188) finishes the proof.

□

Proof of Theorem 2.6 First assume $s \geq 1$. For $\eta \in [0, 1-a)$ denote

$$\begin{aligned} \gamma'(\eta) &:= \frac{s-a}{1-\eta-a}, \\ \varepsilon(\eta) &:= \frac{(q+a)(s-1+\eta)}{s-a}. \end{aligned} \tag{194}$$

The assumption $s \geq 1$ guarantees that $\varepsilon(\eta) > 0$ for all $\eta \in (0, 1-a)$.

In the following proof we will choose

$$a = \frac{r-1}{p+r-1} \qquad \text{in case } r > 1, \tag{195}$$

$$a > 0 \quad \text{sufficiently small} \qquad \text{in case } r \leq 1. \tag{196}$$

If a is defined by (195) then the condition

$$pq > (r-1)(s-1)$$

implies $\varepsilon(0) < q$ and the condition

$$s + \frac{2}{N+1}\frac{r-1}{p+r-1} < \frac{N+3}{N+1} \tag{197}$$

implies $\gamma'(0) < \frac{N+3}{N+1}$. Hence

$$1 < \gamma'(\eta) < \frac{N+3}{N+1}, \quad \varepsilon(\eta) < q \tag{198}$$

for $\eta > 0$ sufficiently small.

If a is chosen by (196) then there holds (198) for small $\eta > 0$. The choice of a may vary from step to step.

Now we choose η such that in the both cases (195) and (196) there holds (198) and for the rest of the proof denote

$$\gamma' := \gamma'(\eta), \quad \varepsilon' := \varepsilon(\eta). \tag{199}$$

For $t \in [0,T]$ and $\varepsilon \in \left(0, \frac{1}{\gamma'}\right)$ we estimate

$$\begin{aligned} \|v(\tau+t)\|_{\gamma',\delta} \leq\ & C\Bigg[\|v(\tau)\|_{\gamma',\delta} \\ +\ & \int_\tau^{\tau+t} e^{-\lambda_1\left(1-\frac{1}{\gamma'}+\varepsilon\right)(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)}\|u^q v^s(s')\|_{1,\delta}\,\mathrm{d}s'\Bigg] \\ \leq\ & C\Bigg[\|v(\tau)\|_{\gamma',\delta} + \int_\tau^{\tau+t} e^{-\lambda_1\left(1-\frac{1}{\gamma'}+\varepsilon\right)(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \times \\ \times\ & \int_\Omega u^{\varepsilon'} v^s u^{q-\varepsilon'}(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\Bigg]. \end{aligned} \tag{200}$$

The term $u^{q-\varepsilon'}$ can be estimated by a constant depending on $\|u(\tau)\|_\infty$ due to Corollary 1. The term $\|v(\tau)\|_{\gamma',\delta}$ can be estimated by $C\|v(\tau)\|_\infty$. Hence it holds

$$\begin{aligned} \|v(\tau+t)\|_{\gamma',\delta} \leq\ & C(\|v(\tau)\|_\infty)\Bigg[1 + (1+\|u(\tau)\|_\infty)^{q-\varepsilon'} \times \\ \times\ & \int_\tau^{\tau+t} e^{-\lambda_1\left(1-\frac{1}{\gamma'}+\varepsilon\right)(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \times \\ \times\ & \int_\Omega u^{\varepsilon'} v^s(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\Bigg]. \end{aligned} \tag{201}$$

We rewrite the estimate (201) to obtain

$$\begin{aligned} \|v(\tau+t)\|_{\gamma',\delta} \leq\ & C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty)\Bigg[1 + \int_\tau^{\tau+t} e^{-\lambda_1\left(1-\frac{1}{\gamma'}+\varepsilon\right)(\tau+t-s')} \times \\ \times\ & (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \int_\Omega [u^{\varepsilon'} v^{s-1+\eta}(s')][v^{1-\eta}(s')]\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\Bigg]. \end{aligned} \tag{202}$$

Since $s \geq 1$ and $\eta \in (0, 1-a)$, we have $0 < \frac{s-1+\eta}{s-a} < 1$. Hence we can use Hölder's inequality in (202) to obtain

$$\begin{aligned} \|v(\tau+t)\|_{\gamma',\delta} \leq\ & C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty)\Bigg[1 + \int_\tau^{\tau+t} e^{-\lambda_1\left(1-\frac{1}{\gamma'}+\varepsilon\right)(\tau+t-s')} \times \\ \times\ & (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \left(\int_\Omega u^{q+a} v^{s-a}(s')\varphi_1\,\mathrm{d}x\right)^{1-\frac{1}{\gamma'}} \times \\ \times\ & \left(\int_\Omega v^{(1-\eta)\gamma'}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{1}{\gamma'}}\,\mathrm{d}s'\Bigg]. \end{aligned} \tag{203}$$

Now we can write

$$\begin{aligned} \|v(\tau+t)\|_{\gamma',\delta} \leq\ & C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty)\Bigg[1 + \sup_{s' \in [\tau, \tau+T]} \|v(s')\|^{1-\eta}_{(1-\eta)\gamma',\delta} \times \\ \times\ & \int_\tau^{\tau+t} \left[e^{-\lambda_1\varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)}\right] \times \\ \times\ & \left(e^{-\lambda_1(\tau+t-s')}\int_\Omega u^{q+a} v^{s-a}\varphi_1\,\mathrm{d}x\right)^{1-\frac{1}{\gamma'}}\,\mathrm{d}s'\Bigg]. \end{aligned} \tag{204}$$

Hölder's inequality in the time integral in (204) implies

$$\begin{aligned}\|v(\tau+t)\|_{\gamma',\delta} \leq\ & C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty)\left[1 + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|^{1-\eta}_{(1-\eta)\gamma',\delta} \times\right. \\ \times\ & \left(\int_\tau^{\tau+t} e^{-\lambda_1\gamma'\varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}(\gamma'-1)}\,\mathrm{d}s'\right)^{\frac{1}{\gamma'}} \times \\ \times\ & \left.\left(\int_\tau^{\tau+t} e^{-\lambda_1(\tau+t-s')}\int_\Omega u^{q+a}v^{s-a}\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right)^{1-\frac{1}{\gamma'}}\right].\end{aligned} \tag{205}$$

We apply the inequality (40) to the estimate (205) to get

$$\begin{aligned}\|v(\tau+t)\|_{\gamma',\delta} \leq\ & C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty)\left[1 + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|^{1-\eta}_{(1-\eta)\gamma',\delta} \times\right. \\ \times\ & \left.\left(\int_\tau^{\tau+t} e^{-\lambda_1\gamma'\varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}(\gamma'-1)}\,\mathrm{d}s'\right)^{\frac{1}{\gamma'}}\right].\end{aligned} \tag{206}$$

The integral in (206) is uniformly bounded with respect to $t, \tau \geq 0$, since

$$\frac{N+1}{2}(\gamma'-1) < 1 \tag{207}$$

due to (198). Hence

$$\|v(\tau+t)\|_{\gamma',\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty)\left[1 + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|^{1-\eta}_{(1-\eta)\gamma',\delta}\right]. \tag{208}$$

Using Young's inequality in (208) we deduce that

$$\|v(\tau+t)\|_{\gamma',\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty)\left[1 + C_\varepsilon + \varepsilon \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{(1-\eta)\gamma',\delta}\right]. \tag{209}$$

Recall that $\eta \in (0,1)$. We set $\varepsilon > 0$ small enough in (209) and thus there holds

$$\sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{\gamma',\delta} \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty).$$

Corollary 1 and Lemma 5.4 now imply

$$\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_\infty + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_\infty \leq C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty). \tag{210}$$

Assume $0 \leq s < 1$. Similarly as in (200) (here we can choose arbitrary $\varepsilon \in (0,1)$) we estimate

$$\begin{aligned}&\|v(\tau+t)\|_{1,\delta} \leq C\left[\|v(\tau)\|_{1,\delta} + \int_\tau^{\tau+t} e^{-\lambda_1\varepsilon(\tau+t-s')}\int_\Omega u^q v^s(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right] \\ &\leq C\left[\|v(\tau)\|_{1,\delta} + \left(\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_\infty\right)^q \int_\tau^{\tau+t} e^{-\lambda_1\varepsilon(\tau+t-s')}\int_\Omega v^s(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right].\end{aligned}$$

Corollary 1 implies

$$\|v(\tau+t)\|_{1,\delta} \le C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \left[1 + \int_\tau^{\tau+t} e^{-\lambda_1 \varepsilon(\tau+t-s')} \int_\Omega v^s(s')\varphi_1 \,\mathrm{d}x \,\mathrm{d}s'\right]. \tag{211}$$

From (211) we obtain

$$\|v(\tau+t)\|_{1,\delta} \le C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \left[1 + \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta}^s \int_\tau^{\tau+t} e^{-\lambda_1 \varepsilon(\tau+t-s')} \,\mathrm{d}s'\right]. \tag{212}$$

Since the integral in the estimate (212) is bounded with respect to τ, t, we can write

$$\|v(\tau+t)\|_{1,\delta} \le C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \left[1 + \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta}^s\right]. \tag{213}$$

If $s = 0$ then we are done and (210) holds. If $s > 0$ then using Young's inequality in (213) we deduce that

$$\|v(\tau+t)\|_{1,\delta} \le C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty) \left(1 + C_\varepsilon + \varepsilon \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta}^s\right).$$

For $\varepsilon > 0$ small we finally come to

$$\sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{1,\delta} \le C(\|u(\tau)\|_\infty, \|v(\tau)\|_\infty), \tag{214}$$

hence (210) is true due to Lemma 5.4 and the proof of the Theorem is complete.

□

Theorem 5.2. *Assume p, q, r, s as in Theorem 2.6 and $s > 1$. Let (u, v) be a global nonnegative solution of problem (2). Then for $\tau_0 > 0$ and $T \ge 0$, there exists $C = C(p, q, r, s, \Omega, \tau_0, T, \|u(\tau_0)\|_\infty)$ such that*

$$\|v(t+T)\|_{s,\delta} \le C\|v(t)\|_{1,\delta} \tag{215}$$

for every $t \ge \tau_0$.

Remark. The constant C from Theorem 5.2 may explode if $\tau_0 \to 0^+$, and may be large for T large.

Proof of Theorem 5.2. In the proof we will use the following notations

$$a = \frac{r-1}{p+r-1} \qquad \text{in case } r > 1, \tag{216}$$

$$a > 0 \quad \text{sufficiently small} \qquad \text{in case } r \le 1 \tag{217}$$

and

$$\gamma' = \frac{s-a}{1-a}, \quad \varepsilon' = \frac{(q+a)(s-1)}{s-a}. \tag{218}$$

First we prove the following estimate

$$\sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{\gamma',\delta} \le C(T, \|u(\tau_0)\|_\infty)\|v(\tau)\|_{\gamma',\delta}, \quad \tau \ge \tau_0 \tag{219}$$

where the constant C is bounded for T bounded. Note that $s > 1$ implies $s < \gamma'$ and due to the assumption

$$s + \frac{2}{N+1}\max\left\{0, \frac{r-1}{p+r-1}\right\} < \frac{N+3}{N+1}$$

and any of definitions (216), (217) there holds

$$\gamma' < \frac{N+3}{N+1}. \tag{220}$$

For $t \in [0,T]$ and $\varepsilon \in \left(0, \frac{1}{\gamma'}\right)$ we estimate

$$\begin{aligned} \|v(\tau+t)\|_{\gamma',\delta} &\le C\left[\|v(\tau)\|_{\gamma',\delta} + \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \|u^q v^s(s')\|_{1,\delta}\, \mathrm{d}s'\right] \\ &\le C\left[\|v(\tau)\|_{\gamma',\delta} + \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \int_\Omega u^{\varepsilon'} v^s u^{q-\varepsilon'}(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s'\right]. \end{aligned} \tag{221}$$

Corollary 1 implies

$$\begin{aligned} &\|v(\tau+t)\|_{\gamma',\delta} \le C\left[\|v(\tau)\|_{\gamma',\delta} + \left(\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_\infty\right)^{q-\varepsilon'} \times\right. \\ &\left.\times \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \int_\Omega u^{\varepsilon'} v^s u(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s'\right] \\ &\le C\left[\|v(\tau)\|_{\gamma',\delta} + (1+\|u(\tau_0)\|_\infty)^{q-\varepsilon'} \times\right. \\ &\left.\times \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \int_\Omega u^{\varepsilon'} v^s(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s'\right] \\ &\le C(\|u(\tau_0)\|_\infty)\left[\|v(\tau)\|_{\gamma',\delta} + \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \int_\Omega u^{\varepsilon'} v^s(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s'\right]. \end{aligned} \tag{222}$$

We rewrite (222) and use Hölder's inequality (note that $0 < \frac{s-1}{s-a} < 1$) to obtain

$$\|v(\tau+t)\|_{\gamma',\delta} \leq C(\|u(\tau_0)\|_\infty) \left[\|v(\tau)\|_{\gamma',\delta} + \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \times \right.$$
$$\left. \times \int_\Omega [u^{\varepsilon'} v^{s-1}(s')][v(s')]\varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \right]$$
$$\leq C(\|u(\tau_0)\|_\infty) \left[\|v(\tau)\|_{\gamma',\delta} + \int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \times \right.$$
$$\left. \times \left(\int_\Omega u^{q+a} v^{s-a}(s')\varphi_1 \, \mathrm{d}x \right)^{1-\frac{1}{\gamma'}} \left(\int_\Omega v^{\gamma'}(s')\varphi_1 \, \mathrm{d}x \right)^{\frac{1}{\gamma'}} \mathrm{d}s' \right].$$

Now we can write

$$\|v(\tau+t)\|_{\gamma',\delta} \leq C(\|u(\tau_0)\|_\infty) \left[\|v(\tau)\|_{\gamma',\delta} + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{\gamma',\delta} \times \right.$$
$$\left. \times \int_\tau^{\tau+t} \left[(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \right] \left(\int_\Omega u^{q+a} v^{s-a}(s')\varphi_1 \, \mathrm{d}x \right)^{1-\frac{1}{\gamma'}} \mathrm{d}s' \right]. \tag{223}$$

Hölder's inequality in the time integral in (223) implies

$$\|v(\tau+t)\|_{\gamma',\delta} \leq C(\|u(\tau_0)\|_\infty) \left[\|v(\tau)\|_{\gamma',\delta} + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{\gamma',\delta} \times \right.$$
$$\left. \times \left(\int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}(\gamma'-1)} \, \mathrm{d}s' \right)^{\frac{1}{\gamma'}} \left(\int_\tau^{\tau+t} \int_\Omega u^{q+a} v^{s-a}(s')\varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \right)^{1-\frac{1}{\gamma'}} \right]. \tag{224}$$

We apply the inequality (41) to the estimate (224) to get

$$\|v(\tau+t)\|_{\gamma',\delta} \leq C(T, \|u(\tau_0)\|_\infty) \left[\|v(\tau)\|_{\gamma',\delta} + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{\gamma',\delta} \times \right.$$
$$\left. \times \left(\int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}(\gamma'-1)} \, \mathrm{d}s' \right)^{\frac{1}{\gamma'}} \right] \tag{225}$$

where the constant C is bounded for T bounded. For T_0 sufficiently small there holds

$$\int_\tau^{\tau+t} (\tau+t-s')^{-\frac{N+1}{2}(\gamma'-1)} \, \mathrm{d}s' < \frac{1}{2C(T_0, \|u(\tau_0)\|_\infty)} \quad \text{for } t \in (0, T_0],$$

since the condition $\frac{N+1}{2}(\gamma'-1) < 1$ is true due to (220). Hence

$$\sup_{s'\in[\tau,\tau+T_0]} \|u(s')\|_{\gamma',\delta} \leq C(T_0, \|u(\tau_0)\|_\infty) \|v(\tau)\|_{\gamma',\delta}.$$

As in the proof of Lemma 5.1 i), this estimate holds for every $T_0 \geq 0$.

Next for $\tau \geq \tau_0,\ T \geq 0,\ t \in (\tau, \tau + T]$, we prove

$$\int_t^{t+T} \|v(s')\|_{\gamma,\delta}\, \mathrm{d}s' \leq C(T, \|u(\tau)\|_\infty)\|v(t+T)\|_{1,\delta}, \quad \gamma = \frac{1}{2-\gamma'}. \tag{226}$$

We estimate

$$\|v(t)\|_{\gamma,\delta} \leq C\left[(t-\tau)^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\|v(\tau)\|_{1,\delta} + \int_\tau^t (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\|u^q v^s(s')\|_{1,\delta}\, \mathrm{d}s'\right]. \tag{227}$$

The conditions $1 < \gamma' < \frac{N+3}{N+1}$ imply $\gamma' < \gamma < \frac{N+1}{N-1}$, hence

$$\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right) < 1,$$

Integrating (227) on interval $[\tau, \tau+T]$ with respect to t we have

$$\begin{aligned}\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\, \mathrm{d}t \leq C\Bigg[\|v(\tau)\|_{1,\delta}\int_\tau^{\tau+T}(t-\tau)^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\, \mathrm{d}t \\ + \int_\tau^{\tau+T}\int_\tau^t (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\|u^q v^s(s')\|_{1,\delta}\, \mathrm{d}s'\, \mathrm{d}t\Bigg]\end{aligned} \tag{228}$$

Now we use Fubini's theorem in the last term in (228) to obtain

$$\begin{aligned}\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\, \mathrm{d}t \leq C\Bigg[T^{1-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\|v(\tau)\|_{1,\delta} \\ + \int_\tau^{\tau+T}\left(\int_{s'}^{\tau+T}(t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\, \mathrm{d}t\right)\|u^q v^s(s')\|_{1,\delta}\, \mathrm{d}s'\Bigg]\end{aligned} \tag{229}$$

Since $s' \in [\tau, \tau+T]$, we can estimate

$$\int_{s'}^{\tau+T}(t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\, \mathrm{d}t \leq CT^{1-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}.$$

Due to (229) we have

$$\begin{aligned}&\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\, \mathrm{d}t \leq CT^{1-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\left[\|v(\tau)\|_{1,\delta} + \int_\tau^{\tau+T}\|u^q v^s(s')\|_{1,\delta}\, \mathrm{d}s'\right] \\ &\leq C(T)\left[\|v(\tau)\|_{1,\delta} + \left(\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_\infty\right)^{q-\varepsilon'}\int_\tau^{\tau+T}\int_\Omega [u^{\varepsilon'}v^{s-1}(s')][v(s')]\varphi_1\, \mathrm{d}x\, \mathrm{d}s'\right].\end{aligned} \tag{230}$$

Corollary 1 yields

$$\begin{aligned}&\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\, \mathrm{d}t \\ &\leq C(T)\left[\|v(\tau)\|_{1,\delta} + (1+\|u(\tau_0)\|_\infty)^{q-\varepsilon'}\int_\tau^{\tau+T}\int_\Omega [u^{\varepsilon'}v^{s-1}(s')][v(s')]\varphi_1\, \mathrm{d}x\, \mathrm{d}s'\right].\end{aligned} \tag{231}$$

Now Hölder's inequality in the last term in (231) implies

$$\begin{aligned}\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C(T,\|u(\tau)\|_\infty)\Bigg[&\|v(\tau)\|_{1,\delta} \\ &+\left(\int_\tau^{\tau+T}\int_\Omega u^{q+a}v^{s-a}(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right)^{1-\frac{1}{\gamma'}}\left(\int_\tau^{\tau+T}\int_\Omega v^{\gamma'}(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right)^{\frac{1}{\gamma'}}\Bigg].\end{aligned} \tag{232}$$

We use (41) to obtain

$$\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C(T,\|u(\tau)\|_\infty)\left[\|v(\tau)\|_{1,\delta}+\left(\int_\tau^{\tau+T}\|v(s')\|_{\gamma',\delta}^{\gamma'}\,\mathrm{d}s'\right)^{\frac{1}{\gamma'}}\right]. \tag{233}$$

In the last term in (233) we use the interpolation inequality

$$\|v(s')\|_{\gamma',\delta}^{\gamma'} \le \|v(s')\|_{1,\delta}^{\frac{\gamma-\gamma'}{\gamma-1}}\|v(s')\|_{\gamma,\delta}^{\frac{\gamma(\gamma'-1)}{\gamma-1}}, \quad s'\in[\tau,\tau+T].$$

From the definition (226) of γ we see that $\frac{\gamma(\gamma'-1)}{\gamma-1}=1$. Hence

$$\begin{aligned}\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\,\mathrm{d}t &\le C(T,\|u(\tau)\|_\infty)\left[\|v(\tau)\|_{1,\delta}+\left(\int_\tau^{\tau+T}\|v(s')\|_{1,\delta}^{\frac{\gamma-\gamma'}{\gamma-1}}\|v(s')\|_{\gamma,\delta}\,\mathrm{d}s'\right)^{\frac{1}{\gamma'}}\right] \\ &\le C(T,\|u(\tau)\|_\infty)\left[\|v(\tau)\|_{1,\delta}+\left(\sup_{s'\in[\tau,\tau+T]}\|v(s')\|_{1,\delta}\right)^{\frac{\gamma-\gamma'}{\gamma-1}\frac{1}{\gamma'}}\left(\int_\tau^{\tau+T}\|v(s')\|_{\gamma,\delta}\,\mathrm{d}s'\right)^{\frac{1}{\gamma'}}\right].\end{aligned} \tag{234}$$

Using Young's inequality we have

$$\begin{aligned}\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C(T,\|u(\tau)\|_\infty)\Bigg[&\|v(\tau)\|_{1,\delta} \\ &+C_\varepsilon\left(\sup_{s'\in[\tau,\tau+T]}\|v(s')\|_{1,\delta}\right)^{\frac{\gamma-\gamma'}{(\gamma-1)(\gamma'-1)}}+\varepsilon\left(\int_\tau^{\tau+T}\|v(s')\|_{\gamma,\delta}\,\mathrm{d}s'\right)\Bigg].\end{aligned} \tag{235}$$

Note that $\frac{\gamma-\gamma'}{(\gamma-1)(\gamma'-1)}=1$. For ε sufficiently small in (235) we have

$$\int_\tau^{\tau+T} \|v(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C(T,\|u(\tau)\|_\infty)\left[\|v(\tau)\|_{1,\delta}+\sup_{s'\in[\tau,\tau+T]}\|v(s')\|_{1,\delta}\right],$$

hence (24) yields the assertion (226).

Using both estimates (219) and (226) we are ready to prove the assertion of the theorem. We use similar estimates as in the proof of Theorem 5.1. We choose $t,T\ge 0$. Note that there exists $\tau'\in[t,\tau_0+t]$ such that

$$\|v(\tau')\|_{\gamma,\delta} = \tau_0^{-1}\int_t^{\tau_0+t}\|v(s')\|_{\gamma,\delta}\,\mathrm{d}s'. \tag{236}$$

Again, this τ' may depend on t and v. Since $\tau_0 + t + T \in [\tau', \tau' + \tau_0 + T]$, there holds

$$\|v(\tau_0 + t + T)\|_{\gamma',\delta} \le \sup_{s' \in [\tau', \tau' + \tau_0 + T]} \|v(s')\|_{\gamma',\delta}. \tag{237}$$

We use the estimate (219) with τ, T replaced by τ', $\tau_0 + T$, respectively, to obtain

$$\sup_{s' \in [\tau', \tau' + \tau_0 + T]} \|v(s')\|_{\gamma',\delta} \le C(T, \|u(\tau_0)\|_\infty)\|v(\tau')\|_{\gamma',\delta} \le C(T, \|u(\tau_0)\|_\infty)\|v(\tau')\|_{\gamma,\delta} \tag{238}$$

where C does not depend on τ', since the constant in the estimate (219) is independent of τ. The estimate (226) with τ_0 instead of T implies

$$\int_t^{\tau_0+t} \|v(s')\|_{\gamma,\delta} \, \mathrm{d}s' \le C(T, \|u(\tau_0)\|_\infty)\|v(\tau_0 + t)\|_{1,\delta}. \tag{239}$$

Finally, the equality (236) and the estimates (237)-(239) imply

$$\|v(\tau_0 + t + T)\|_{\gamma',\delta} \le C(\tau_0, T, \|u(\tau_0)\|_\infty)\|v(\tau_0 + t)\|_{1,\delta} \quad \text{for } t \ge 0.$$

Since $s < \gamma'$, finally we obtain

$$\|v(t + T)\|_{s,\delta} \le C(\tau_0, T, \|u(\tau_0)\|_\infty)\|v(t)\|_{1,\delta}$$

for $t \ge \tau_0$ and $T \ge 0$.

$\square$

Lemma 5.5. *Let $p \ge 1$, $p + r < \frac{N+3}{N+1}$ and conditions (26), (31) be true. Let (u, v) be a global nonnegative solution of problem (2). Moreover assume*

$$(p + r)\left(p - \frac{2}{N+1}\right) + r < 1. \tag{240}$$

Then for

$$\gamma \in \left(p + r, \frac{1 - r}{p - \frac{2}{N+1}}\right)$$

and $\tau \ge 0$ there exists $C = C(p, q, r, s, \Omega)$ such that

$$\|u(t)\|_{\gamma,\delta} \le C\left(1 + \|u(\tau)\|_{\gamma,\delta}\right) \quad \textit{for } t \ge \tau.$$

Proof of Lemma 5.5 We choose

$$\gamma \in \left(p + r, \frac{1 - r}{p - \frac{2}{N+1}}\right). \tag{241}$$

This choice is possible due to the assumption (240). Next observe that the assumption $p + r < \frac{N+3}{N+1}$ implies $\frac{1-r}{p - \frac{2}{N+1}} > 1$. We will choose

$$a \in A \quad \text{sufficiently close to } 0. \tag{242}$$

The constant a will be specified more precisely during the proof.

Now we introduce the following exponents $\alpha_1, \alpha_2, \alpha_3$ satisfying conditions

$$\alpha_1, \alpha_2, \alpha_3 > 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1, \tag{243}$$

$$(1-a)\alpha_1 + (p+1-a)\alpha_2 = p, \tag{244}$$

$$\frac{r+\gamma(p-1)}{\gamma p - (1-r)} < \alpha_2 < \frac{p}{p+1}, \quad \alpha_2 \text{ sufficiently close to } \frac{r+\gamma(p-1)}{\gamma p-(1-r)}. \tag{245}$$

The condition $\gamma > p + r$ implies $\frac{r+\gamma(p-1)}{\gamma p-(1-r)} < \frac{p}{p+1}$.

We observe that there exist $\alpha_1, \alpha_2, \alpha_3$ such that the conditions (243)-(245) are true. Indeed, we choose α_2 to satisfy the condition (245). We set

$$\alpha_1 = \frac{p - \alpha_2(p+1-a)}{1-a}, \tag{246}$$

hence (244) is true. Since $\alpha_2 < \frac{p}{p+1}$, we have $p - \alpha_2(p+1) > 0$. For every $a \in A$ there holds $\alpha_1 > 0$. The inequalities

$$\frac{r+\gamma(p-1)}{\gamma p-(1-r)} \geq \frac{\gamma(p-1)}{\gamma p-(1-r)} \geq \frac{p-1}{p}$$

imply $\alpha_2 > \frac{p-1}{p}$, hence $p(1-\alpha_2) < 1$. Therefore

$$\alpha_1 + \alpha_2 = \frac{p(1-\alpha_2)}{1-a} < 1 \tag{247}$$

for $a < 1 - p(1-\alpha_2)$. Finally we set

$$\alpha_3 = 1 - \alpha_1 - \alpha_2 \tag{248}$$

and this with (247) proves (243).

We define exponent

$$\kappa = r - a\alpha_1 - (r+a-1)\alpha_2.$$

For a small there holds

$$0 < \kappa < 1. \tag{249}$$

For $\varepsilon \in (0, 1-\alpha_2)$, $\tau, T \geq 0, t \in [0,T]$ we estimate

$$\begin{aligned}
\|u(\tau+t)\|_{\gamma,\delta} \;&\leq C \Bigg[\|u(\tau)\|_{\gamma,\delta} \\
&+ \int_\tau^{\tau+t} e^{-\lambda_1(\alpha_2+\varepsilon)(\tau+t-s')} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \|u^r v^p(s')\|_{1,\delta} \, \mathrm{d}s' \Bigg] \\
&= C \Bigg[\|u(\tau)\|_{\gamma,\delta} + \int_\tau^{\tau+t} e^{-\lambda_1\varepsilon(\tau+t-s')} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \times \\
&\times \int_\Omega \left[u^a v^{1-a}\right]^{\alpha_1} \left[e^{-\lambda_1(\tau+t-s')} u^{r+a-1} v^{p+1-a}(s')\right]^{\alpha_2} \times \\
&\times \left[u^{\frac{\kappa}{\alpha_3}}(s')\right]^{\alpha_3} \varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \Bigg].
\end{aligned} \tag{250}$$

Here we used (22), (21), the assertions iii) and v) from Lemma 3.1 and the equality (244). Now, using Hölder's inequality in the last term in (250) we obtain

$$\begin{aligned}\|u(\tau+t)\|_{\gamma,\delta} &\leq C\Bigg[\|u(\tau)\|_{\gamma,\delta}+\int_\tau^{\tau+t} e^{-\lambda_1\varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\times \\ &\times\left(\int_\Omega u^a v^{1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\alpha_1}\left(\int_\Omega e^{-\lambda_1(\tau+t-s')}u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\alpha_2}\times \\ &\times\left(\int_\Omega u^{\frac{\kappa}{\alpha_3}}(s')\varphi_1\,\mathrm{d}x\right)^{\alpha_3}\,\mathrm{d}s'\Bigg].\end{aligned}$$

Due to the estimate (34), there holds

$$\begin{aligned}\|u(\tau+t)\|_{\gamma,\delta} &\leq C\Bigg[\|u(\tau)\|_{\gamma,\delta}+\int_\tau^{\tau+t} e^{-\lambda_1\varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\times \\ &\times\left(\int_\Omega e^{-\lambda_1(\tau+t-s')}u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\alpha_2}\times \\ &\times\left(\int_\Omega u^{\frac{\kappa}{\alpha_3}}(s')\varphi_1\,\mathrm{d}x\right)^{\alpha_3}\,\mathrm{d}s'\Bigg].\end{aligned} \tag{251}$$

Now we prove

$$\frac{\kappa}{\alpha_3}<\gamma. \tag{252}$$

Due to the equality (244), $\alpha_1+(p+1)\alpha_2$ is close to p and so $\gamma(1+p\alpha_2-p)$ is close to $\gamma(1-\alpha_1-\alpha_2)=\gamma\alpha_3$. The condition $\alpha_2>\frac{r+\gamma(p-1)}{\gamma p-(1-r)}$ implies

$$r+\alpha_2(1-r)<\gamma(1+p\alpha_2-p),$$

hence $\frac{r+\alpha_2(1-r)}{\alpha_3}<\gamma$. Thus we proved (252) for a small.

The inequality (252) and the estimate (251) yield

$$\begin{aligned}\|u(\tau+t)\|_{\gamma,\delta} &\leq C\Bigg[\|u(\tau)\|_{\gamma,\delta}+\sup_{s'\in[\tau,\tau+T]}\|u(s')\|^{\kappa}_{\gamma,\delta}\times \\ &\times\int_\tau^{\tau+t} e^{-\lambda_1\varepsilon(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\times \\ &\times\left(\int_\Omega e^{-\lambda_1(\tau+t-s')}u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\alpha_2}\,\mathrm{d}s'\Bigg].\end{aligned} \tag{253}$$

Using Hölder's inequality in (253) we obtain

$$\begin{aligned}\|u(\tau+t)\|_{\gamma,\delta} &\leq C\Bigg[\|u(\tau)\|_{\gamma,\delta}+\sup_{s'\in[\tau,\tau+T]}\|u(s')\|^{\kappa}_{\gamma,\delta}\times \\ &\times\left(\int_\tau^{\tau+t} e^{-\lambda_1(\tau+t-s')}\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right)^{\alpha_2}\times \\ &\times\left(\int_\tau^{\tau+t} e^{-\frac{\lambda_1\varepsilon}{1-\alpha_2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{1-\alpha_2}}\,\mathrm{d}s'\right)^{1-\alpha_2}\Bigg].\end{aligned} \tag{254}$$

Now we use (38) in (254) to estimate

$$\begin{aligned}\|u(\tau+t)\|_{\gamma,\delta} \;&\leq C\Bigg[\|u(\tau)\|_{\gamma,\delta} + \sup_{s'\in[\tau,\tau+T]} \|u(s')\|^{\kappa}_{\gamma,\delta} \times \\ &\times \left(\int_{\tau}^{\tau+t} e^{-\frac{\lambda_1\varepsilon}{1-\alpha_2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{1-\alpha_2}}\,\mathrm{d}s'\right)^{1-\alpha_2}\Bigg].\end{aligned} \tag{255}$$

We prove that the integral in (255) is uniformly bounded with respect to τ, t, i.e. for some α_2 close to $\frac{r+\gamma(p-1)}{\gamma p-(1-r)}$ there holds

$$\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{1-\alpha_2} < 1. \tag{256}$$

For α_2 sufficiently close to $\frac{r+\gamma(p-1)}{\gamma p-(1-r)}$, there holds

$$\left(1-\frac{1}{\gamma}\right)\frac{1}{1-\alpha_2} \quad \text{is sufficiently close to} \quad \frac{\gamma p-(1-r)}{\gamma}.$$

Our choice (241) of γ implies

$$\frac{\gamma p-(1-r)}{\gamma} < \frac{2}{N+1},$$

hence the inequality (256) is true.

We use the estimate (255) to obtain

$$\|u(\tau+t)\|_{\gamma,\delta} \leq C\left[\|u(\tau)\|_{\gamma,\delta} + \sup_{s'\in[\tau,\tau+T]} \|u(s')\|^{\kappa}_{\gamma,\delta}\right]. \tag{257}$$

Since the inequality (249) is true, we use Young's inquality in (257) to obtain

$$\|u(\tau+t)\|_{\gamma,\delta} \leq C\left[\|u(\tau)\|_{\gamma,\delta} + C_\varepsilon + \varepsilon \sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{\gamma,\delta}\right].$$

Thus for $\varepsilon > 0$ small the assertion follows.

□

Lemma 5.6. *Let $p \geq 1$, $p + r < \frac{N+3}{N+1}$ and conditions (26), (31) be true. Let (u, v) be a global nonnegative solution of problem (2). Then there holds*

$$\int_{\tau}^{\tau+T} \|u(t)\|_{\gamma',\delta}\,\mathrm{d}t \leq C\left(1 + \|u(\tau+T)\|_{1,\delta}\right) \quad \textit{for } \gamma' \in \left[1, \frac{N+1}{N-1}\right) \tag{258}$$

where $C = C(p, q, r, s, \Omega, T)$.

Proof of Lemma 5.6. We choose

$$\frac{1}{2-(p+r)} < \gamma' < \frac{N+1}{N-1}.$$

Note that $p+r \leq \frac{1}{2-(p+r)} < \frac{N+1}{N-1}$, since $1 \leq p+r < \frac{N+3}{N+1}$.

We introduce the following exponents $\alpha_1, \alpha_2, \alpha_3$ satisfying conditions (243), (244) with $a \in A$ and

$$\frac{r+\gamma'(p-1)}{\gamma' p-(1-r)} < \alpha_2 < \frac{1-r}{2-r} \leq \frac{p}{p+1}. \tag{259}$$

Note that the condition $\gamma' > \frac{1}{2-(p+r)}$ implies $\frac{r+\gamma'(p-1)}{\gamma' p-(1-r)} < \frac{1-r}{2-r}$ and $p+r \geq 1$ implies $\frac{1-r}{2-r} \leq \frac{p}{p+1}$. Observe that there exist $\alpha_1, \alpha_2, \alpha_3$ such that the conditions (243), (244) and (259) are true due to similar arguments as in the proof of Lemma 5.5 (we only replace γ with γ').

For $\tau, T \geq 0, t \in (\tau, \tau+T]$ we estimate

$$\begin{aligned} \|u(t)\|_{\gamma',\delta} &\leq C \Bigg[(t-\tau)^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \|u(\tau)\|_{1,\delta} \\ &+ \int_\tau^t (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \|u^r v^p(s')\|_{1,\delta} \, \mathrm{d}s' \Bigg] \end{aligned}$$

where we used Lemma 3.1 iii) and v). As in the proof of Lemma 5.2 we obtain

$$\int_\tau^{\tau+T} \|u(t)\|_{\gamma',\delta} \, \mathrm{d}t \leq CT^{1-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)} \left[\|u(\tau)\|_{1,\delta} + \int_\tau^{\tau+T} \|u^r v^p(s')\|_{1,\delta} \, \mathrm{d}s' \right], \tag{260}$$

since

$$\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right) < 1.$$

Using (260) we have

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{\gamma',\delta} \, \mathrm{d}t \leq \; & C(T) \Bigg[\|u(\tau)\|_{1,\delta} + \int_\tau^{\tau+T} \int_\Omega \left[u^a v^{1-a}(s')\right]^{\alpha_1} \times \\ \times \; & \left[u^{r+a-1} v^{p+1-a}(s')\right]^{\alpha_2} \left[u^{\frac{\kappa}{\alpha_3}}(s')\right]^{\alpha_3} \varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \Bigg] \end{aligned} \tag{261}$$

where $a \in A$ and $\kappa = r - a\alpha_1 - (r+a-1)\alpha_2$. Due to the condition (243), we can use Hölder's inequality in the last term in (261) to have

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{\gamma',\delta} \, \mathrm{d}t \leq \; & C \Bigg[\|u(\tau)\|_{1,\delta} + \int_\tau^{\tau+T} \left[\int_\Omega u^a v^{1-a}(s') \varphi_1 \, \mathrm{d}x\right]^{\alpha_1} \times \\ \times \; & \left[\int_\Omega u^{r+a-1} v^{p+1-a}(s') \varphi_1 \, \mathrm{d}x\right]^{\alpha_2} \left[\int_\Omega u^{\frac{\kappa}{\alpha_3}}(s') \varphi_1 \, \mathrm{d}x\right]^{\alpha_3} \mathrm{d}s' \Bigg]. \end{aligned}$$

We use (34) to obtain

$$\begin{aligned} \int_\tau^{\tau+T} \|u(t)\|_{\gamma',\delta} \, \mathrm{d}t \leq \; & C \Bigg[\|u(\tau)\|_{1,\delta} + \int_\tau^{\tau+T} \left[\int_\Omega u^{r+a-1} v^{p+1-a}(s') \varphi_1 \, \mathrm{d}x\right]^{\alpha_2} \times \\ \times \; & \left[\int_\Omega u^{\frac{\kappa}{\alpha_3}}(s') \varphi_1 \, \mathrm{d}x\right]^{\alpha_3} \mathrm{d}s' \Bigg]. \end{aligned} \tag{262}$$

We use Hölder's inequality in the time integral in (262) to have

$$\begin{aligned}\int_{\tau}^{\tau+T} \|u(t)\|_{\gamma',\delta}\, \mathrm{d}t \leq\ & C\left[\|u(\tau)\|_{1,\delta} + \left(\int_{\tau}^{\tau+T}\int_{\Omega} u^{r+a-1}v^{p+1-a}(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s'\right)^{\alpha_2} \times\right.\\ & \times\ \left.\left(\int_{\tau}^{\tau+T} \|u(s')\|_{\frac{\kappa}{\alpha_3},\delta}^{\frac{\kappa}{1-\alpha_2}}\, \mathrm{d}s'\right)^{1-\alpha_2}\right].\end{aligned} \tag{263}$$

It is easy to see that $\frac{\kappa}{\alpha_3} \geq 1$ for a sufficiently small. If $\frac{\kappa}{\alpha_3} > 1$ then in the last term in (263) we use the interpolation inequality (Lemma 3.4)

$$\|u(s')\|_{\frac{\kappa}{\alpha_3},\delta}^{\frac{\kappa}{1-\alpha_2}} \leq \|u(s')\|_{1,\delta}^{\frac{\alpha_3}{1-\alpha_2}\frac{\gamma'-\frac{\kappa}{\alpha_3}}{\gamma'-1}} \|u(s')\|_{\gamma',\delta}^{\frac{\alpha_3}{1-\alpha_2}\frac{\gamma'(\frac{\kappa}{\alpha_3}-1)}{\gamma'-1}}, \quad s' \in [\tau, \tau+T]. \tag{264}$$

Observe that the inequality $\alpha_2 > \frac{r+\gamma'(p-1)}{\gamma' p-(1-r)}$ implies $\frac{\kappa}{\alpha_3} < \gamma'$ (cf. the proof of inequality (252)) and $\alpha_2 < \frac{1-r}{2-r}$ implies $\frac{\kappa}{1-\alpha_2} < 1$. Hence there holds

$$\frac{\alpha_3}{1-\alpha_2}\frac{\gamma'(\frac{\kappa}{\alpha_3}-1)}{\gamma'-1} = \frac{\kappa}{1-\alpha_2}\frac{\gamma'(1-\frac{\alpha_3}{\kappa})}{\gamma'-1} < \frac{\gamma'(1-\frac{1}{\gamma'})}{\gamma'-1} = 1. \tag{265}$$

We use the inequalities (264) and (263) to obtain

$$\begin{aligned}\int_{\tau}^{\tau+T} \|u(t)\|_{\gamma',\delta}\, \mathrm{d}t \leq\ & C(T)\left[\|u(\tau)\|_{1,\delta}\right.\\ & +\ \left.\left(\int_{\tau}^{\tau+T} \|u(s')\|_{1,\delta}^{\frac{\alpha_3}{1-\alpha_2}\frac{\gamma'-\frac{\kappa}{\alpha_3}}{\gamma'-1}} \|u(s')\|_{\gamma',\delta}^{\frac{\alpha_3}{1-\alpha_2}\frac{\gamma'(\frac{\kappa}{\alpha_3}-1)}{\gamma'-1}}\, \mathrm{d}s'\right)^{1-\alpha_2}\right]\\ \leq\ & C(T)\left[\|u(\tau)\|_{1,\delta} + \left(\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{1,\delta}\right)^{\alpha_3\frac{\gamma'-\frac{\kappa}{\alpha_3}}{\gamma'-1}} \times\right.\\ & \times\ \left.\left(\int_{\tau}^{\tau+T} \|u(s')\|_{\gamma',\delta}^{\frac{\alpha_3}{1-\alpha_2}\frac{\gamma'(\frac{\kappa}{\alpha_3}-1)}{\gamma'-1}}\, \mathrm{d}s'\right)^{1-\alpha_2}\right].\end{aligned}$$

Due to (265) we can write

$$\begin{aligned}\int_{\tau}^{\tau+T} \|u(t)\|_{\gamma',\delta}\, \mathrm{d}t \leq\ & C(T)\left[\|u(\tau)\|_{1,\delta} + \left(\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{1,\delta}\right)^{\alpha_3\frac{\gamma'-\frac{\kappa}{\alpha_3}}{\gamma'-1}} \times\right.\\ & \times\ \left.\left(\int_{\tau}^{\tau+T} \|u(s')\|_{\gamma',\delta}\, \mathrm{d}s'\right)^{\frac{\alpha_3\gamma'(\frac{\kappa}{\alpha_3}-1)}{\gamma'-1}}\right].\end{aligned}$$

Using Young's inequality we have

$$\int_{\tau}^{\tau+T} \|u(t)\|_{\gamma',\delta}\, \mathrm{d}t \leq \; C(T) \Bigg[\|u(\tau)\|_{1,\delta} + \left(\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{1,\delta} \right)^{\kappa} + \left(\int_{\tau}^{\tau+T} \|u(s')\|_{\gamma',\delta}\, \mathrm{d}s' \right)^{\kappa} \Bigg].$$

Note that $\kappa < 1$ for a small. Hence Young's inequality and (24) imply

$$\int_{\tau}^{\tau+T} \|u(t)\|_{\gamma',\delta}\, \mathrm{d}t \leq C(T) \left[1 + \|u(\tau)\|_{1,\delta} + \sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{1,\delta} \right] + \frac{1}{2} \int_{\tau}^{\tau+T} \|u(t)\|_{\gamma',\delta}\, \mathrm{d}t.$$

This proves the Lemma.

□

Lemma 5.7. *Let $p \geq 1$, $p + r < \frac{N+3}{N+1}$, $s \leq 1$ and condition (31) be true. Let (u, v) be a global nonnegative solution of problem (2). Moreover assume (240) and*

$$0 < q < \frac{1-r}{p - \frac{2}{N+1}} \left(1 - \frac{N-1}{N+1} s \right) \tag{266}$$

Then for $\tau, T \geq 0$ there exists $C = C(p, q, r, s, \Omega, \tau, \|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$ such that

$$\|u(t)\|_{k,\delta} + \|v(t)\|_{k,\delta} \leq C \quad \textit{for } k \in \left[1, \frac{N+1}{N-1} \right), \; t \geq \tau.$$

Remark. The constant C from Lemma 5.7 may explode if $\tau \to 0^+$, and is bounded for $\|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta}$ bounded.

Proof of Lemma 5.7. We use Lemmas 5.5 and 5.6 and arguments as in the proof of Theorem 5.1 to obtain

$$\begin{aligned} \sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{\gamma,\delta} \leq & \sup_{s'\in[\tau',\tau'+\tau+T]} \|u(s')\|_{\gamma,\delta} \leq C\left(1 + \|u(\tau')\|_{\gamma,\delta}\right) \\ = \; & C \left(1 + \tfrac{2}{\tau} \int_{\frac{\tau}{2}}^{\tau} \|u(t)\|_{\gamma,\delta}\, \mathrm{d}t \right) \leq C(\tau)\left(1 + \|u(\tau)\|_{1,\delta}\right) \leq C_0 \end{aligned} \tag{267}$$

for $\tau \in (0, 1)$, γ chosen by (241), $C_0 = C_0(\tau, \|u(\tau)\|_{1,\delta})$ and some $\tau' \in \left[\frac{\tau}{2}, \tau\right]$. C_0 may vary from step to step, but always depends on parameters in brackets. The constant in (267) may explode if $\tau \to 0^+$. We prove the following assertion

$$\sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{k,\delta} \leq C_0\left(1 + \|v(\tau)\|_{k,\delta}\right), \quad T \geq 0 \tag{268}$$

for $k < \frac{N+1}{N-1}$ close to $\frac{N+1}{N-1}$.

For $\tau > 0, T \geq 0, t \in [0,T]$ we estimate

$$\begin{aligned} \|v(\tau+t)\|_{k,\delta} &\leq C\Bigg[\|v(\tau)\|_{k,\delta} \\ &+ \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \int_\Omega u^q v^s(s')\varphi_1 \,\mathrm{d}x \,\mathrm{d}s'\Bigg]. \end{aligned} \tag{269}$$

Assume $s \in (0,1)$. We use Hölder's inequality in the spatial integral in (269) to obtain

$$\begin{aligned} \|v(\tau+t)\|_{k,\delta} \leq\ & C\Bigg[\|v(\tau)\|_{k,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \times \\ &\times \left(\int_\Omega u^{\frac{q}{\theta}}(s')\varphi_1 \,\mathrm{d}x\right)^\theta \left(\int_\Omega v^{\frac{s}{1-\theta}}(s')\varphi_1 \,\mathrm{d}x\right)^{1-\theta} \mathrm{d}s'\Bigg] \end{aligned} \tag{270}$$

where $\theta = 1 - \frac{s}{k} \in \left(0, 1 - \frac{N-1}{N+1}s\right)$. Due to the assumption (266) for $k < \frac{N+1}{N-1}$ sufficiently close to $\frac{N+1}{N-1}$ there holds

$$q < \frac{(1-r)(1-\frac{s}{k})}{p - \frac{2}{N+1}},$$

hence

$$\frac{q}{\theta} < \frac{1-r}{p-\frac{2}{N+1}}. \tag{271}$$

Thus there exists $\gamma > \frac{q}{\theta}$ satisfying the condition (241) and we can use Lemma 5.5, (267) and (270) to estimate

$$\begin{aligned} \|v(\tau+t)\|_{k,\delta} \leq\ & C_0\Bigg[\|v(\tau)\|_{k,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \times \\ &\times \left(\int_\Omega v^{\frac{s}{1-\theta}}(s')\varphi_1 \,\mathrm{d}x\right)^{1-\theta} \mathrm{d}s'\Bigg]. \end{aligned}$$

Since $\frac{s}{1-\theta} = k$, we can write

$$\begin{aligned} \|v(\tau+t)\|_{k,\delta} \leq\ & C_0\Bigg[\|v(\tau)\|_{k,\delta} + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{k,\delta}^s \times \\ &\times \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \mathrm{d}s'\Bigg]. \end{aligned} \tag{272}$$

For $k < \frac{N+1}{N-1}$ close to $\frac{N+1}{N-1}$ the integral is finite and uniformly bounded with respect to τ, t, hence the estimate (272) implies

$$\|v(\tau+t)\|_{k,\delta} \leq C_0\left[\|v(\tau)\|_{k,\delta} + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{k,\delta}^s\right]. \tag{273}$$

In (273) we use Young's inequality to obtain

$$\|v(\tau+t)\|_{k,\delta} \le C_0 \left[\|v(\tau)\|_{k,\delta} + C_\varepsilon + \varepsilon \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{k,\delta} \right].$$

This proves the assertion (268).

If $s = 1$ then from (269) we deduce

$$\begin{aligned} \|v(\tau+t)\|_{k,\delta} \le \; & C \Bigg[\|v(\tau)\|_{k,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \times \\ & \times \int_\Omega \left[u^a v^{1-a}(s')\right]^\varepsilon \left[u^{\frac{q-a\varepsilon}{\theta}}(s')\right]^\theta \left[v^{\frac{1-(1-a)\varepsilon}{1-\varepsilon-\theta}}(s')\right]^{1-\varepsilon-\theta} \varphi_1 \,\mathrm{d}x \,\mathrm{d}s' \Bigg] \end{aligned} \tag{274}$$

where $a \in A$ and $0 < \varepsilon < \varepsilon' < 1$ for ε' such that $\theta := \theta(\varepsilon') = 1 - \varepsilon' - \frac{1-(1-a)\varepsilon'}{k} \in (0,1)$ (this is possible, if $k > 1$). Note that $\frac{q-a\varepsilon}{\theta} > 0$ for $\varepsilon > 0$ small, since $q > 0$. Hence there holds $\frac{1-(1-a)\varepsilon}{1-\varepsilon-\theta} < k$. We use Hölder's inequality in the spatial integral in (274) to obtain

$$\begin{aligned} \|v(\tau+t)\|_{k,\delta} \le \; & C \Bigg[\|v(\tau)\|_{k,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \times \\ & \times \left(\int_\Omega u^a v^{1-a}(s') \varphi_1 \,\mathrm{d}x \right)^\varepsilon \left(\int_\Omega u^{\frac{q-a\varepsilon}{\theta}}(s') \varphi_1 \,\mathrm{d}x \right)^\theta \times \\ & \times \left(\int_\Omega v^{\frac{1-(1-a)\varepsilon}{1-\varepsilon-\theta}}(s') \varphi_1 \,\mathrm{d}x \right)^{1-\varepsilon-\theta} \mathrm{d}s' \Bigg]. \end{aligned}$$

Using (34) we have

$$\begin{aligned} \|v(\tau+t)\|_{k,\delta} \le \; & C \Bigg[\|v(\tau)\|_{k,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')} (\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \times \\ & \times \left(\int_\Omega u^{\frac{q-a\varepsilon}{\theta}}(s') \varphi_1 \,\mathrm{d}x \right)^\theta \|v(s')\|_{k,\delta}^{1-(1-a)\varepsilon} \,\mathrm{d}s' \Bigg]. \end{aligned}$$

Due to the assumption (266) for $k < \frac{N+1}{N-1}$ sufficiently close to $\frac{N+1}{N-1}$ and ε' small there holds

$$q < \frac{(1-r)(1-\frac{1}{k})}{p - \frac{2}{N+1}},$$

hence

$$\frac{q-a\varepsilon}{\theta(\varepsilon')} < \frac{1-r}{p-\frac{2}{N+1}}$$

for ε' sufficiently small. Thus there exists $\gamma > \frac{q-a\varepsilon'}{\theta(\varepsilon)}$ satisfying the condition (241) and we can use Lemma 5.5 and (267) to estimate

$$\begin{aligned}\|v(\tau+t)\|_{k,\delta} \le\ & C_0 \Bigg[\|v(\tau)\|_{k,\delta} \\ & + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \|v(s')\|_{k,\delta}^{1-(1-a)\varepsilon}\, \mathrm{d}s' \Bigg].\end{aligned}$$

Hence we have

$$\|v(\tau+t)\|_{k,\delta} \le C_0 \left[\|v(\tau)\|_{k,\delta} + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{k,\delta}^{1-(1-a)\varepsilon}\, \mathrm{d}s' \right].$$

Since $1-(1-a)\varepsilon < 1$, we can use Young's inequality to obtain

$$\|v(\tau+t)\|_{k,\delta} \le C_0 \left(1 + \|v(\tau)\|_{k,\delta}\right)$$

for $k < \frac{N+1}{N-1}$ close to $\frac{N+1}{N-1}$.

It remains to prove the assertion (268) for $s=0$. In this case we estimate

$$\begin{aligned}\|v(\tau+t)\|_{k,\delta} \le\ & C \Bigg[\|v(\tau)\|_{k,\delta} \\ & + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \int_\Omega u^q(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s' \Bigg] \\ \le\ & C \Bigg[\|v(\tau)\|_{k,\delta} + \sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{q,\delta}^q \times \\ & \times \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)}\, \mathrm{d}s' \Bigg].\end{aligned} \tag{275}$$

Finally, we use Lemma 5.5 with any $\gamma > q$ satisfying the condition (241) to obtain the assertion (268).

We prove

$$\int_\tau^{\tau+T} \|v(t)\|_{k,\delta}\, \mathrm{d}t \le C(T,C_0)\left(1 + \|v(\tau+T)\|_{1,\delta}\right) \tag{276}$$

for $k \in \left[1, \frac{N+1}{N-1}\right)$. For $\tau > 0, T \ge 0, t \in (\tau, T+\tau]$ we estimate

$$\begin{aligned}\|v(t)\|_{k,\delta} &\le C \Bigg[(t-\tau)^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \|v(\tau)\|_{1,\delta} \\ &+ \int_\tau^t (\tau-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \int_\Omega u^q v^s(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s' \Bigg].\end{aligned} \tag{277}$$

As in the proof of Lemma 5.2 we use (277) to obtain

$$\int_\tau^{\tau+T} \|v(t)\|_{k,\delta}\, \mathrm{d}t \le CT^{1-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \left[\|v(\tau)\|_{1,\delta} + \int_\tau^{\tau+T} \int_\Omega u^q v^s(s')\varphi_1\, \mathrm{d}x\, \mathrm{d}s' \right]. \tag{278}$$

Let $s \in (0,1]$. Then Hölder's inequality implies

$$\begin{aligned}\int_\tau^{\tau+T} \|v(t)\|_{k,\delta}\,\mathrm{d}t \leq\ & CT^{1-\frac{N+1}{2}\left(1-\frac{1}{k}\right)}\Bigg[\|v(\tau)\|_{1,\delta} \\ &+ \int_\tau^{\tau+T}\left(\int_\Omega u^{\frac{q}{\theta}}(s')\varphi_1\,\mathrm{d}x\right)^\theta\left(\int_\Omega v^{\frac{s}{1-\theta}}(s')\varphi_1\,\mathrm{d}x\right)^{1-\theta}\mathrm{d}s'\Bigg]\end{aligned}$$

where $\theta = 1-\frac{s}{k} \in \left(0, 1-\frac{N-1}{N+1}s\right)$. There holds (271) for $k < \frac{N+1}{N-1}$ close to $\frac{N+1}{N-1}$ and there exists $\gamma > \frac{q}{\theta}$ satisfying (241). Thus due to Lemma 5.5, (267) and the definition of θ we have

$$\int_\tau^{\tau+T} \|v(t)\|_{k,\delta}\,\mathrm{d}t \leq C_0 T^{1-\frac{N+1}{2}\left(1-\frac{1}{k}\right)}\left[\|v(\tau)\|_{1,\delta} + \int_\tau^{\tau+T}\|v(s')\|_{k,\delta}^s\,\mathrm{d}s'\right]. \tag{279}$$

If $s \in (0,1)$ then (279) implies

$$\int_\tau^{\tau+T} \|v(t)\|_{k,\delta}\,\mathrm{d}t \leq C(T, C_0)\left[\|v(\tau)\|_{1,\delta} + \left(\int_\tau^{\tau+T}\|v(s')\|_{k,\delta}\,\mathrm{d}s'\right)^s\right].$$

Young's inequality and (24) then yield the assertion (276). If $s=1$ then (279) implies

$$\int_\tau^{\tau+T} \|v(t)\|_{k,\delta}\,\mathrm{d}t \leq C_0 T^{1-\frac{N+1}{2}\left(1-\frac{1}{k}\right)}\left[\|v(\tau)\|_{1,\delta} + \int_\tau^{\tau+T}\|v(s')\|_{k,\delta}\,\mathrm{d}s'\right].$$

For T_0 sufficiently small with help of (24) we deduce

$$\int_\tau^{\tau+T_0} \|v(t)\|_{k,\delta}\,\mathrm{d}t \leq C(T_0, C_0)\|v(\tau+T_0)\|_{1,\delta}.$$

This estimate is actually true for every $T \geq 0$ fixed, hence the assertion (276) is true also for $s=1$.

If $s=0$ then the assertion (276) follows from Lemma 5.5, (267) and (278).

Combining (268) and (276) (cf. the proof of (276)) we obtain

$$\sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{k,\delta} \leq C_0(1+\|v(\tau)\|_{1,\delta}) \leq C_1 \tag{280}$$

for $k \in \left[1, \frac{N+1}{N-1}\right)$, $C_1 = C_1(\tau, \|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$. The constant C_1 may vary from step to step, but always depends on parameters in brackets. Now we prove

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{k,\delta} \leq C_1\left(1+\|u(\tau)\|_{k,\delta}\right) \tag{281}$$

for $k < \frac{N+1}{N-1}$ sufficiently close to $\frac{N+1}{N-1}$.

For $r > 0$ we estimate

$$\begin{aligned}\|u(\tau+t)\|_{k,\delta}\ \leq\ & C\Bigg[\|u(\tau)\|_{k,\delta} \\ &+ \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)}\int_\Omega u^r v^p(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\Bigg].\end{aligned} \tag{282}$$

We use Hölder's inequality in the spatial integral in (282) to obtain

$$\begin{aligned}\|u(\tau+t)\|_{k,\delta} \leq\ & C\Bigg[\|u(\tau)\|_{k,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \times \\ & \times \left(\int_\Omega u^{\frac{r}{1-\theta}}(s')\varphi_1 \,\mathrm{d}x\right)^{1-\theta}\left(\int_\Omega v^{\frac{p}{\theta}}(s')\varphi_1 \,\mathrm{d}x\right)^{\theta} \mathrm{d}s'\Bigg]\end{aligned} \tag{283}$$

where $\theta = 1-\frac{r}{k} \in \left(0, 1-\frac{N-1}{N+1}r\right)$. Observe that $p+r < \frac{N+3}{N+1} < \frac{N+1}{N-1}$ implies $\frac{p}{1-\frac{N-1}{N+1}r} < \frac{N+1}{N-1}$. For $k < \frac{N+1}{N-1}$ sufficiently close to $\frac{N+1}{N-1}$, there holds

$$\frac{p}{\theta} < \frac{N+1}{N-1}$$

and we use the estimate (280) to obtain

$$\begin{aligned}\|u(\tau+t)\|_{k,\delta} \leq\ & C_1\Bigg[\|u(\tau)\|_{k,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \times \\ & \times \left(\int_\Omega u^{\frac{r}{1-\theta}}(s')\varphi_1 \,\mathrm{d}x\right)^{1-\theta} \mathrm{d}s'\Bigg] \\ \leq\ & C_1\Bigg[\|u(\tau)\|_{k,\delta} + \sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{k,\delta}^r \times \\ & \times \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \,\mathrm{d}s'\Bigg] \\ \leq\ & C_1\Bigg[\|u(\tau)\|_{k,\delta} + \sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{k,\delta}^r\Bigg].\end{aligned} \tag{284}$$

Since $r<1$ in (284) we can use Young's inequality to obtain

$$\|u(\tau+t)\|_{k,\delta} \leq C_1\Bigg[\|u(\tau)\|_{k,\delta} + C_\varepsilon + \varepsilon \sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{k,\delta}\Bigg].$$

This proves the assertion (281).

Now we prove (281) for $r=0$. We estimate

$$\begin{aligned}\|u(\tau+t)\|_{k,\delta} \leq\ & C\Bigg[\|u(\tau)\|_{k,\delta} \\ & + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \int_\Omega v^p(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s'\Bigg] \\ \leq\ & C\Bigg[\|u(\tau)\|_{k,\delta} + \sup_{s'\in[\tau,\tau+T]}\|v(s')\|_{p,\delta}^p \times \\ & \times \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{k}\right)} \,\mathrm{d}s'\Bigg].\end{aligned} \tag{285}$$

Finally, we use (280) to obtain the assertion (281).

We prove

$$\int_{\tau}^{\tau+T} \|u(t)\|_{k,\delta}\ \mathrm{d}t \le C(T,C_1)\,(1+\|u(\tau+T)\|_{1,\delta}) \tag{286}$$

for $k \in \left[1, \frac{N+1}{N-1}\right)$. For $\tau > 0, T \ge 0, t \in [\tau, \tau+T]$ we estimate

$$\begin{aligned} \|u(t)\|_{k,\delta} \ &\le C \Bigg[(t-\tau)^{-\frac{N+1}{2}(1-\frac{1}{k})} \|u(\tau)\|_{1,\delta} \\ &+ \int_{\tau}^{t} (t-s')^{-\frac{N+1}{2}(1-\frac{1}{k})} \int_{\Omega} u^r v^p(s')\varphi_1 \ \mathrm{d}x\ \mathrm{d}s' \Bigg]. \end{aligned} \tag{287}$$

As in the proof of Lemma 5.2 we use (287) to obtain

$$\int_{\tau}^{\tau+T} \|u(t)\|_{k,\delta}\ \mathrm{d}t \le C T^{1-\frac{N+1}{2}(1-\frac{1}{k})} \left[\|u(\tau)\|_{1,\delta} + \int_{\tau}^{\tau+T} \int_{\Omega} u^r v^p(s')\varphi_1 \ \mathrm{d}x\ \mathrm{d}s' \right]. \tag{288}$$

If $r > 0$ then Hölder's inequality implies

$$\begin{aligned} \int_{\tau}^{\tau+T} \|u(t)\|_{k,\delta}\ \mathrm{d}t \le \ & C(T) \Bigg[\|u(\tau)\|_{1,\delta} \\ &+ \int_{\tau}^{\tau+T} \left(\int_{\Omega} u^{\frac{r}{1-\theta}}(s')\varphi_1 \ \mathrm{d}x \right)^{1-\theta} \left(\int_{\Omega} v^{\frac{p}{\theta}}(s')\varphi_1\ \mathrm{d}x \right)^{\theta} \mathrm{d}s' \Bigg] \end{aligned}$$

where $\theta = 1 - \frac{r}{k} \in \left(0, 1 - \frac{N-1}{N+1} r\right)$. Observe that $p + r < \frac{N+1}{N-1}$ implies $\frac{p}{1-\frac{N-1}{N+1}r} < \frac{N+1}{N-1}$. For $k < \frac{N+1}{N-1}$ sufficiently close to $\frac{N+1}{N-1}$, there holds $\frac{p}{\theta} < \frac{N+1}{N-1}$. Hence using (280) we have

$$\int_{\tau}^{\tau+T} \|u(t)\|_{k,\delta}\ \mathrm{d}t \le C(T,C_1) \left[\|u(\tau)\|_{1,\delta} + \int_{\tau}^{\tau+T} \|u(s')\|_{k,\delta}^r \ \mathrm{d}s' \right]. \tag{289}$$

Finally, (289) and Jensen's inequality imply

$$\int_{\tau}^{\tau+T} \|u(t)\|_{k,\delta}\ \mathrm{d}t \le C(T,C_1) \left[\|u(\tau)\|_{1,\delta} + \left(\int_{\tau}^{\tau+T} \|u(s')\|_{k,\delta}\ \mathrm{d}s' \right)^{r} \right].$$

Young's inequality and (24) then yield the assertion (286). If $r = 0$ then the assertion (286) follows from (288) and (280).

Combining (281), (286) (cf. the proof of (276)) we finally get the assertion of the Lemma.

□

Proof of Theorem 2.7. Denote $C_0 = C_0(\tau, \|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$. C_0 may vary from step to step, but always depends on parameters in brackets. Let $\tau > 0, T \ge 0$. Assume that there holds

$$\sup_{s' \in [\tau,\tau+T]} \|u(s')\|_{k,\delta} + \sup_{s' \in [\tau,\tau+T]} \|v(s')\|_{k,\delta} \le C_0, \quad \text{for some } k \in \left(\frac{N+1}{N-1} - \varepsilon_0, \infty \right) \tag{290}$$

where $\varepsilon_0 > 0$ is sufficiently small. In Lemma 5.7 we proved (290) for $k \in \left[1, \frac{N+1}{N-1}\right)$. For the whole proof we choose

$$M = \frac{k}{p+r}, \quad M' = \frac{k}{q+s}.$$

For k chosen in (290), it holds $M, M' > 1$, since $\max\{p+r, q+s\} < \frac{N+1}{N-1}$. This is true, since $q+s < \frac{(1-r)(1-\frac{N-1}{N+1}s)}{p-\frac{2}{N+1}} + s \leq \frac{N+1}{N-1}(1-\frac{N-1}{N+1}s) + s = \frac{N+1}{N-1}$.

If $r > 0$ then we use Hölder's inequality to obtain

$$\left(\int_\Omega u^{Mr} v^{Mp}(s')\varphi_1 \, \mathrm{d}x\right)^{\frac{1}{M}} \leq \left(\int_\Omega u^{\frac{Mr}{1-\eta}}(s')\varphi_1 \, \mathrm{d}x\right)^{\frac{1-\eta}{M}} \left(\int_\Omega v^{\frac{Mp}{\eta}}(s')\varphi_1 \, \mathrm{d}x\right)^{\frac{\eta}{M}} \tag{291}$$

for $s' \in [\tau, \tau+T]$ and $\eta \in (0,1)$. If we choose $\eta = \frac{Mp}{k} = \frac{p}{p+r}$ then (291) and (290) imply

$$\left(\int_\Omega u^{Mr} v^{Mp}(s')\varphi_1 \, \mathrm{d}x\right)^{\frac{1}{M}} \leq \|u(s')\|_{k,\delta}^r \|v(s')\|_{k,\delta}^p \leq C_0. \tag{292}$$

If $s > 0$ then similarly we obtain

$$\left(\int_\Omega u^{M'q} v^{M's}(s')\varphi_1 \, \mathrm{d}x\right)^{\frac{1}{M'}} \leq \|u(s')\|_{k,\delta}^q \|v(s')\|_{k,\delta}^s \leq C_0. \tag{293}$$

Assume $r > 0$. For $1 < K \leq \infty$ satisfying

$$\frac{N+1}{2}\left(\frac{1}{M} - \frac{1}{K}\right) < 1$$

and $t \in [0, T]$ we estimate

$$\begin{aligned} \|u(\tau+t)\|_{K,\delta} \leq \; & C\Bigg[\|u(\tau)\|_{K,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \times \\ & \times \left(\int_\Omega u^{Mr} v^{Mp}(s')\varphi_1 \, \mathrm{d}x\right)^{\frac{1}{M}} \mathrm{d}s'\Bigg]. \end{aligned} \tag{294}$$

In particular, we can take

$$K < k_1(M) := \begin{cases} \dfrac{N+1}{\frac{N+1}{M} - 2}, & M \in \left[1, \frac{N+1}{2}\right), \\ \infty, & M \geq \frac{N+1}{2} \end{cases} \tag{295}$$

if $M \leq \frac{N+1}{2}$ and $K = \infty$ for $M > \frac{N+1}{2}$. We use (292) and (294) to obtain

$$\|u(\tau+t)\|_{K,\delta} \leq C_0\left[\|u(\tau)\|_{K,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \mathrm{d}s'\right].$$

Hence we have

$$\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{K,\delta} \le C_0(1+\|u(\tau)\|_{K,\delta}). \tag{296}$$

If $r=0$ then for $t\in[0,T]$ we estimate

$$\begin{aligned}\|u(\tau+t)\|_{K,\delta} \le\ & C\left[\|u(\tau)\|_{K,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\times\right.\\ &\times \left.\left(\int_\Omega v^{Mp}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{1}{M}}\,\mathrm{d}s'\right].\end{aligned}$$

Now the assertion (296) follows from the definition of M and the assumption (290).

For $1<K\le\infty$ satisfying

$$\frac{N+1}{2}\left(\frac{1}{M'}-\frac{1}{K}\right)<1$$

and $t\in[0,T]$ we estimate

$$\begin{aligned}\|v(\tau+t)\|_{K,\delta} \le\ & C\left[\|v(\tau)\|_{K,\delta} + \int_\tau^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}\left(\frac{1}{M'}-\frac{1}{K}\right)}\times\right.\\ &\times \left.\left(\int_\Omega u^{M'q}v^{M's}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{1}{M'}}\,\mathrm{d}s'\right].\end{aligned} \tag{297}$$

In particular, we can take $K<k_1(M')$. If $s>0$ then we use this estimate with (293) to obtain

$$\sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{K,\delta} \le C_0(1+\|v(\tau)\|_{K,\delta}). \tag{298}$$

If $s=0$ then the assertion (298) follows from (297).

For $k_1(M)>K>M$ (function k_1 is defined in (295)), $t\in(\tau,\tau+T]$ we estimate

$$\begin{aligned}\|u(t)\|_{K,\delta} \ \le\ & C\left[(t-\tau)^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\|u(\tau)\|_{M,\delta}\right.\\ &+\left.\int_\tau^t (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\|u^r v^p(s')\|_{M,\delta}\,\mathrm{d}s'\right].\end{aligned} \tag{299}$$

As in the proof of Lemma 5.2 we use (299) to obtain

$$\int_\tau^{\tau+T}\|u(t)\|_{K,\delta}\,\mathrm{d}t \le C(T)\left[\|u(\tau)\|_{M,\delta} + \int_\tau^{\tau+T}\left(\int_\Omega u^{Mr}v^{Mp}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{1}{M}}\,\mathrm{d}s'\right]. \tag{300}$$

If $r>0$ then we use (292) to get

$$\int_\tau^{\tau+T}\|u(t)\|_{K,\delta}\,\mathrm{d}t \le C(T,C_0)\left(1+\|u(\tau)\|_{M,\delta}\right) \le C(T,C_0)\left(1+\|u(\tau)\|_{k,\delta}\right), \tag{301}$$

since $k > M$.

If $r = 0$ then the assertion (301) follows from (300).

For $k_1(M') > K > M'$ (function k_1 is defined in (295)), $t \in (\tau, \tau + T]$ we estimate

$$\begin{aligned} \|v(t)\|_{K,\delta} &\leq C\left[(t-\tau)^{-\frac{N+1}{2}\left(\frac{1}{M'}-\frac{1}{K}\right)}\|v(\tau)\|_{M',\delta}\right. \\ &+ \left.\int_\tau^t (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M'}-\frac{1}{K}\right)}\|u^q v^s(s')\|_{M',\delta}\,\mathrm{d}s'\right]. \end{aligned} \tag{302}$$

As in the proof of Lemma 5.2 we use (302) to obtain

$$\int_\tau^{\tau+T}\|v(t)\|_{K,\delta}\,\mathrm{d}t \leq C(T)\left[\|v(\tau)\|_{M',\delta} + \int_\tau^{\tau+T}\left(\int_\Omega u^{M'q}v^{M's}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{1}{M'}}\,\mathrm{d}s'\right]. \tag{303}$$

If $s > 0$ then we use (293) to get

$$\int_\tau^{\tau+T}\|v(t)\|_{K,\delta}\,\mathrm{d}t \leq C(T, C_0)\left(1 + \|v(\tau)\|_{\max\{k,M'\},\delta}\right). \tag{304}$$

If $s = 0$ then the assertion (304) follows from (303).

As in (267) we use the estimates (296) with (301) and (298) with (304) to obtain

$$\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{K,\delta} + \sup_{s'\in[\tau,\tau+T]}\|v(s')\|_{K,\delta} \leq C_0$$

for all $K < k_1\left(\frac{k}{\max\{p+r,q+s\}}\right) =: k_2(k)$. Note that $k_2(k) = \infty$ for $k \geq \frac{(\max\{p+r,q+s\})(N+1)}{2}$ and we can take $K = \infty$ for $k > \frac{(\max\{p+r,q+s\})(N+1)}{2}$. As in the proof of Lemma 5.4 we use bootstrap argument: Due to the inequality $k_2(k) > k$ for $k \geq \frac{N+1}{N-1} - \varepsilon_0$ with ε_0 sufficiently small, there exists $\varepsilon > 0$ such that $\widetilde{K}(k) := \min\{(1+\varepsilon)k, (1-\varepsilon)k_2(k)\} > k$. Hence if

$$\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{k,\delta} + \sup_{s'\in[\tau,\tau+T]}\|v(s')\|_{k,\delta} \leq C_0$$

then

$$\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_{\widetilde{K}(k),\delta} + \sup_{s'\in[\tau,\tau+T]}\|v(s')\|_{\widetilde{K}(k),\delta} \leq C_0.$$

Denote $k_1 = \frac{N+1}{N-1} - \varepsilon_0$ and $k_{i+1} = \widetilde{K}(k_i)$ for $i \in \mathbb{N}$. Due to Lemma 3.5 there exists $i_0 \in \mathbb{N}$ such that $k_{i_0} > \frac{(\max\{p+r,q+s\})(N+1)}{2}$. Hence there holds

$$\sup_{s'\in[\tau,\tau+T]}\|u(s')\|_\infty + \sup_{s'\in[\tau,\tau+T]}\|v(s')\|_\infty \leq C_0.$$

Thus the proof of theorem is finished.

$\square$

Corollary 2. Let p, q, r, s be as in Theorem 2.7 with $r = s = 0$ and $p, q > 1$. Then for every $\tau > 0$, there exists $C = C(\Omega, p, q, \tau)$ (the constant C may explode if $\tau \to 0^+$) such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C, \quad t \geq \tau$$

for every global nonnegative solution (u, v) of problem (2).

Proof. In [17, Proposition 4.1], it was proved that

$$\|u(t)\|_{1,\delta} + \|v(t)\|_{1,\delta} \leq C, \quad t \geq 0,$$

hence Theorem 2.7 implies the assertion.

□

Theorem 5.3. *Consider problem*

$$\left.\begin{array}{rcl} u_t - \Delta u & = & uv - b_1 u, \qquad (x,t) \in \Omega \times (0, \infty), \\ v_t - \Delta v & = & b_2 u, \qquad (x,t) \in \Omega \times (0, \infty), \\ u(x,t) & = & v(x,t) \; = \; 0, \quad (x,t) \in \partial\Omega \times (0, \infty), \\ u(x,0) & = & u_0(x), \qquad x \in \Omega, \\ v(x,0) & = & v_0(x), \qquad x \in \Omega \end{array}\right\} \tag{305}$$

where Ω is a bounded domain with smooth boundary, $N \leq 2$, $b_1 = 0$, $b_2 > 0$ and $u_0, v_0 \in L^\infty(\Omega)$. Then there exists $C = C(\Omega, b_2)$ such that

$$\limsup_{t \to \infty} \left(\|u(t)\|_\infty + \|v(t)\|_\infty \right) \leq C$$

for every global nonnegative solution (u, v) of problem (305).

Remark. The proof of Theorem 5.3 can be very easily modified for system (305) with $b_1 > 0$, however also the proof of Theorem 2.7 has to be (very easily) modified.

Proof of Theorem 5.3. The constants in this proof may depend on Ω, b_2, however we will not emphasize this dependence. Observe that for problem (305), it holds $A = (0, 1)$, since $r = p = q = 1$ and $s = 0$ (in sense of the problem (2)). Lemma 3.3 implies

$$\int_\Omega u^a v^{1-a}(s') \varphi_1 \, \mathrm{d}x \leq C, \quad a \in (0, 1). \tag{306}$$

Thus there holds

$$\int_\tau^{\tau+t} e^{-\lambda_1(\tau+t-s')} \int_\Omega \left(u^a v^{2-a}(s') + u^{1+a'} v^{-a'}(s') \right) \varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \leq C \tag{307}$$

for $a, a' \in (0, 1)$. A direct computation shows

$$u^{\frac{2+a}{2}} = \left[u^{\frac{a^2}{2}} v^{\frac{(2-a)a}{2}} \right] \left[u^{\frac{2+a-a^2}{2}} v^{-\frac{(2-a)a}{2}} \right].$$

We use Young's inequality to obtain

$$u^{\frac{2+a}{2}} \leq \frac{a}{2} u^a v^{2-a} + \frac{2-a}{2} u^{1+a} v^{-a}.$$

We use this inequality and (307) to deduce

$$\int_\tau^{\tau+T} e^{-\lambda_1(\tau+T-s')} \int_\Omega u^{\frac{2+a}{2}}(s')\varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \leq C \tag{308}$$

or

$$\int_\tau^{\tau+T} \int_\Omega u^{\alpha_0}(s')\varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \leq C(T), \quad \alpha_0 \in \left[1, \frac{3}{2}\right).$$

Hölder's inequality then implies

$$\begin{aligned}\int_\tau^{\tau+T} \int_\Omega u(s')\varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \leq & \left(\int_\tau^{\tau+T} \int_\Omega u^{\frac{2+a}{2}}(s')\varphi_1 \, \mathrm{d}x \, \mathrm{d}s'\right)^{\frac{2}{2+a}} \left(\int_\tau^{\tau+T} \int_\Omega \varphi_1 \, \mathrm{d}x \, \mathrm{d}s'\right)^{\frac{a}{2+a}} \\ \leq & \ (C(T))^{\frac{2}{2+a}} T^{\frac{a}{2+a}}.\end{aligned}$$

In particular, we have

$$\int_\tau^{\tau+1} \int_\Omega u(s')\varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \leq C_1 \tag{309}$$

where C_1 is independent of u and τ. The constants C_i, $i \in \mathbb{N}$ will be fixed during the proof (where C_i, $i > 1$ will appear below).

Now we prove that there exists $t_0 \geq 0$ possibly depending on v, such that

$$\int_\Omega v(t_0)\varphi_1 \, \mathrm{d}x \leq \frac{4}{\lambda_1} b_2 C_1. \tag{310}$$

To prove (310) we multiply the second equation in (305) by φ_1 and integrate on $\Omega \times (\tau, \tau+1)$ for $\tau \geq 0$. Thus using (309) we have

$$\begin{aligned}\int_\Omega v(\tau+1)\varphi_1 \, \mathrm{d}x \ &+ \ \lambda_1 \int_\tau^{\tau+1} \int_\Omega v(s')\varphi_1 \, \mathrm{d}x \, \mathrm{d}s' \\ &= \ b_2 \int_\tau^{\tau+1} \int_\Omega u(s')\varphi_1 \, \mathrm{d}x \, \mathrm{d}s' + \int_\Omega v(\tau)\varphi_1 \, \mathrm{d}x \\ &\leq \ b_2 C_1 + \int_\Omega v(\tau)\varphi_1 \, \mathrm{d}x.\end{aligned} \tag{311}$$

Denote $C_2 := \int_\Omega v(0)\varphi_1 \, \mathrm{d}x$. If there holds

$$C_2 \leq \frac{4}{\lambda_1} b_2 C_1$$

then (310) is true with $t_0 = 0$. If there holds

$$C_2 > \frac{4}{\lambda_1} b_2 C_1$$

then necessarily

$$\int_\Omega v(t_1)\varphi_1 \,\mathrm{d}x < \frac{1+\frac{\lambda_1}{2}}{\lambda_1+1} C_2 \tag{312}$$

for some $t_1 \in (0,1)$. Indeed, if $\displaystyle\int_\Omega v(t)\varphi_1 \,\mathrm{d}x \geq \frac{1+\frac{\lambda_1}{2}}{\lambda_1+1} C_2$ for all $t \in (0,1)$ then (311) for $\tau = 0$ implies

$$\begin{aligned}
\left(1+\frac{\lambda_1}{2}\right) C_2 &= \tfrac{1+\frac{\lambda_1}{2}}{\lambda_1+1} C_2 + \tfrac{1+\frac{\lambda_1}{2}}{\lambda_1+1} \lambda_1 C_2 \\
&\leq \int_\Omega v(1)\varphi_1 \,\mathrm{d}x + \lambda_1 \int_0^1 \int_\Omega v(s')\varphi_1 \,\mathrm{d}x \,\mathrm{d}s' \\
&\leq b_2 C_1 + \int_\Omega v(0)\varphi_1 \,\mathrm{d}x = b_2 C_1 + C_2 < \left(1+\frac{\lambda_1}{4}\right) C_2,
\end{aligned}$$

a contradiction.

Denote $C_3 := \displaystyle\int_\Omega v(t_1)\varphi_1 \,\mathrm{d}x$. If there holds

$$C_3 \leq \frac{4}{\lambda_1} b_2 C_1$$

then (310) is true with $t_0 = t_1$. If there holds

$$C_3 > \frac{4}{\lambda_1} b_2 C_1$$

then using the same argument as for (312) we obtain

$$\int_\Omega v(t_2)\varphi_1 \,\mathrm{d}x < \frac{1+\frac{\lambda_1}{2}}{\lambda_1+1} C_3 < \left(\frac{1+\frac{\lambda_1}{2}}{\lambda_1+1}\right)^2 C_2$$

for some $t_2 \in (t_1, t_1+1)$.

In n-th such step we obtain

$$\int_\Omega v(t_n)\varphi_1 \,\mathrm{d}x < \frac{1+\frac{\lambda_1}{2}}{\lambda_1+1} C_{n+1} < \left(\frac{1+\frac{\lambda_1}{2}}{\lambda_1+1}\right)^n C_2$$

for some $t_n \in (t_{n-1}, t_{n-1}+1)$ if

$$C_2, C_3, \ldots, C_{n+1} > \frac{4}{\lambda_1} b_2 C_1. \tag{313}$$

Note that there exists $n_0 = n_0(C_2)$ such that

$$\int_\Omega v(t_{n_0})\varphi_1 \,\mathrm{d}x < \left(\frac{1+\frac{\lambda_1}{2}}{\lambda_1+1}\right)^{n_0} C_2 \leq \frac{4}{\lambda_1} b_2 C_1.$$

Hence we proved (310) with $t_0 = t_{n_0}$ if there holds (313) with n replaced by n_0.

For $t \geq 0$, $a \in (0,1)$, $\varepsilon \in (0,a)$ $\gamma \in \left[1, \frac{N+1}{N-1}\right)$ and t_0 from (310) we estimate

$$\begin{aligned}\|v(1+t_0+t)\|_{\gamma,\delta} \leq C\Bigg[&(t+1)^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\|v(t_0)\|_{1,\delta}\\ &+\int_{t_0}^{1+t_0+t} e^{-\lambda_1(\varepsilon+1-a)(1+t_0+t-s')}(1+t_0+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\times\\ &\times\int_\Omega u(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s'\Bigg].\end{aligned} \tag{314}$$

Observe that there holds $u = (u^a v^{1-a})^a \left(u^{1+a}v^{-a}\right)^{1-a}$. We use this identity in (314) and Hölder's inequality to obtain

$$\begin{aligned}\|v(1+t_0+t)\|_{\gamma,\delta} \leq C\Bigg[&\|v(t_0)\|_{1,\delta}\\ &+\int_{t_0}^{1+t_0+t} e^{-\lambda_1(\varepsilon+1-a)(1+t_0+t-s')}(1+t_0+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\times\\ &\times\left(\int_\Omega u^a v^{1-a}(s')\varphi_1\,\mathrm{d}x\right)^a\left(\int_\Omega u^{1+a}v^{-a}(s')\varphi_1\,\mathrm{d}x\right)^{1-a}\mathrm{d}s'\Bigg].\end{aligned}$$

The estimate (306) yields

$$\begin{aligned}\|v(1+t_0+t)\|_{\gamma,\delta} \leq C\Bigg[&\|v(t_0)\|_{1,\delta}\\ &+\int_{t_0}^{1+t_0+t} e^{-\lambda_1\varepsilon(1+t_0+t-s')}(1+t_0+t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\times\\ &\times\left(e^{-\lambda_1(1+t_0+t-s')}\int_\Omega u^{1+a}v^{-a}(s')\varphi_1\,\mathrm{d}x\right)^{1-a}\mathrm{d}s'\Bigg].\end{aligned}$$

For $a < 1$ close to 1 we have

$$\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{a} < 1,$$

hence using Hölder's inequality we have

$$\|v(1+t_0+t)\|_{\gamma,\delta} \leq C\left[\|v(t_0)\|_{1,\delta} + \int_{t_0}^{1+t_0+t} e^{-\lambda_1(1+t_0+t-s')}\int_\Omega u^{1+a}v^{-a}(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right].$$

Finally, we apply (310) and (307) to obtain

$$\|v(t)\|_{\gamma,\delta} \leq C, \quad t \geq T \tag{315}$$

for some $T = T(v)$ large and $\gamma \in \left[1, \frac{N+1}{N-1}\right)$. The estimate (309) with $\tau = T$ implies $\|u(t')\|_{1,\delta} \leq C$ for some $t' \in (T, T+1)$. Finally, (315) yields

$$\|u(t')\|_{1,\delta} + \|v(t')\|_{1,\delta} \leq C$$

and we use Theorem 2.7 (where u, p, r is interchanged with v, q, s, respectively) to conclude the proof.

$\square$

Conclusion

The aim of this thesis is to obtain a priori estimates for positive global solutions of problem

$$\left.\begin{array}{rcl} u_t - \Delta u &=& u^r v^p, \qquad\qquad (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta v &=& u^q v^s, \qquad\qquad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) &=& v(x,t) \;=\; 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &=& u_0(x), \qquad\qquad x \in \Omega, \\ v(x,0) &=& v_0(x), \qquad\qquad x \in \Omega, \end{array}\right\} \tag{316}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $u_0, v_0 \in L^\infty(\Omega)$ are nonnegative functions and $p, q, r, s \geq 0$. For general p, q, r, s, usual methods fail. It turns out that the method from [34] used for an elliptic problem can be modified to yield the desired results. The modification is nontrivial and requires several technical restrictions on the exponents p, q, r, s. Despite these restrictions, our theorems still can be used for several interesting problems studied by other authors: See Theorem 5.3 or the case $r = s = 0$.

Beside modifications of the ideas in [34], we also heavily used estimates of Dirichlet heat semigroup in weighted Lebesgue spaces and the variation-of-constants formula. Our method is suitable for many perturbations or modifications of problem (316) and also for problem (18) with homogeneous Neumann boundary conditions.

In the thesis, we also present our results form [29] for the following elliptic problem

$$\left.\begin{array}{rcl} -\Delta u &=& a(x)|x|^{-\kappa} v^q, \quad x \in \Omega, \\ -\Delta v &=& b(x)|x|^{-\lambda} u^p, \quad x \in \Omega, \\ u &=& v \;=\; 0, \qquad\quad x \in \partial\Omega, \end{array}\right\} \tag{317}$$

where Ω is a bounded domain with smooth boundary, $p, q > 0$, $pq > 1$, $a, b \in L^\infty(\Omega)$, $a, b \geq 0$, $a, b \not\equiv 0$, $\kappa, \lambda \in \mathbb{R}$. Using bootstrap in weighted Lebesgue spaces, we proved a priori estimate of nonnegative very weak solutions, and using these estimates and topological degree arguments we also proved the existence of positive very weak solution of (317).

Resumé

V tejto dizertačnej práci sa venujeme apriórnym odhadom kladných globálnych riešení parabolickej úlohy (316). Pre všeobecné p, q, r, s klasické metódy zlyhávajú. Ukazuje sa, že metóda z [34] použitá pre eliptický problem môže byť modifikovaná tak, aby dávala želané výsledky. Táto modifikácia je netriviálna a vyžaduje viaceré technické obmedzenia na exponenty p, q, r, s. Napriek týmto obmedzeniam, môžu byť naše vety pre viaceré zaujímavé úlohy študované inými autormi: Pozri Vetu 5.3 alebo prípad $r = s = 0$.

Okrem modifikácií myšlienok z [34] takisto sme podstatne využili odhady Dirichletovej tepelnej semigrupy vo váhových Lebesgueových priestoroch a formulu variácie konštánt. Naša metóda je vhodná pre viaceré perturbácie alebo modifikácie úlohy (316) a tiež pre úlohu (18) s homogennými Neumannovými hraničnými podmienkami.

V tejto dizertačnej práci tiež predkladáme naše výsledky z [29] pre eliptický systém (317). Použitím bootstrapu vo váhových Lebesgueových priestoroch sme dokázali apriórny odhad nezáporných veľmi slabých riešení a použitím týchto odhadov a metódy topologického stupňa sme tiež dokázali existenciu kladného veľmi slabého riešenia úlohy (317).

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