Univerzita Komenského v Bratislave

Fakulta matematiky, fyziky a informatiky

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Autoreferát dizertačnej práce

SOLVABILITY OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH NON-LINEAR BOUNDARY CONDITIONS

na získanie akademického titulu *philosophiae doctor*
v odbore doktorandského štúdia 9.1.9 Aplikovaná matematika

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pred komisiou pre obhajobu dizertačnej práce v odbore doktorandského štúdia vymenovanou predsedom odborovej komisie dňa ....

v študijnom odbore 9.1.9 Aplikovaná matematika

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1 Introduction

We investigate a boundary value problem containing non-linearities both in the equation and the boundary conditions. The problem has the form

$$\begin{cases} u''(x) = a|u(x)|^{p-1}u(x), & x \in (-l,l), \\ u'(\pm l) = \pm|u(\pm l)|^{q-1}u(\pm l), \end{cases} \tag{1}$$

Here a and l can take any positive value, while the conditions on p and q will be specified later. As one can see, the boundary conditions are symmetric, and both of the non-linearities are of power type. Our aim is to determine the number of classical solutions for as large set of values of the parameters as possible.

Most of this thesis concernes positive solutions, which solve the simpler-looking problem

$$\begin{cases} u''(x) = au^p(x), & x \in (-l,l), \\ u'(\pm l) = \pm u^q(\pm l), \end{cases}$$

while p and q can be arbitrary real numbers. On the other hand, if one is interested in the existence and multiplicity of sign-changing solutions, only $p > 0$, $q \in \mathbb{R}$ can be considered. We present results for $p > -1$, $q \geq 0$ and $p = -1$, $q = 0$ regarding positive solutions, and for $p = 1$, $q \in (0,1)$ and $p > 1$, $q \in [\frac{1}{2}, \frac{p+1}{2})$ regarding sign-changing solutions.

The first systematic study of positive solutions of (1) was done by M. Chipot, M. Fila and P. Quittner in [5]. They also studied the N-dimensional version of (1), but they were interested mainly in global existence and boundedness or blow-up of positive solutions of the corresponding N-dimensional parabolic problem

$$\begin{cases} u_t = \Delta u - a|u|^{p-1}u & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = |u|^{q-1}u & \text{in } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \overline{\Omega}, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, n is the unit outer normal vector to $\partial\Omega$, $u_0 : \overline{\Omega} \to [0,\infty)$, $p, q > 1$ and $a > 0$. The cited article provides a complete answer for the question of the existence and number of positive symmetric (i. e. even) solutions of (1) for $p, q > 1$. However, only partial results were presented in it regarding positive non-symmetric solutions, the study of which is much more complicated.

Let us remark that positive symmetric solutions of (1) (and also solutions of (2) for $N = 1$) were independently studied in [12].

Sign-changing solutions of (1) were systematically investigated for the first time in [6] by M. Chipot and P. Quittner, considering $p \geq 1$ and $q > 1$. The number of sign-changing antisymmetric (i. e. odd) solutions was determined for all these values of p and q, but again, only partial results were achieved concerning sign-changing non-antisymmetric solutions.

The results from [5] have been generalised in many other directions: In [15] the behaviour of positive solutions of (2) was examined for all $p, q > 1$. Positive solutions of the elliptic problem with $-\lambda u + u^p$ on the right-hand side of the equation were dealt with in [13] for $\lambda \in \mathbb{R}$, $p, q > 1$, and later in [10] for $\lambda \in \mathbb{R}$, $p, q > 0$, $(p, q) \notin (0, 1)^2$. In [11] and [16], positive and sign-changing solutions of the parabolic problem with more general non-linearities $f(u)$, $g(u)$ instead of $a|u|^{p-1}u$, $|u|^{q-1}u$ were studied, while $f(x, u)$, $g(x, u)$ were considered in [2]. Many results concerning elliptic problems with non-linear boundary conditions were summarised in [17]. Further extensions of the results from [5] can be found in [1, 3, 4, 7, 8, 9].

However, this thesis focuses only on (1), and extends results known from [5] and [6] to larger sets of parameters.

We apply the so-called shooting method, which was also used in the the mentioned articles. Its substance is to express the solutions of the given boundary value problem by means of the solutions of the same differential equation subject to appropriate initial conditions, leading to the definition of some functions called time maps, the properties of which directly determine the number of solutions of the considered boundary value problem. Thus, we will need only the tools of real analysis. On the other hand, it is not so easy to examine the properties of the time maps, because they are given by a formula containing an improper integral, which can be calculated only for some special values of p, and the upper limit of which is given only implicitly.

2 Goals

- To determine the number of positive non-symmetric solutions of (1) for all $p, q > 1$.
- To determine the number of sign-changing non-antisymmetric solutions of (1) for all $p, q > 1$.
- To determine the number of positive solutions of (1) for as large set of values of p and q as possible.

3 Results

3.1 The shooting method for positive solutions of (1)

Let $p, q \in \mathbb{R}$, $a, l > 0$. If u is a positive solution of (1), then $u'(-l) < 0 < u'(l)$, therefore u has a stationary point $x_0 \in (-l, l)$. So the function $u(\cdot + x_0)$ solves

$$\begin{cases} u'' = au^p, \\ u(0) = m, \\ u'(0) = 0 \end{cases} \tag{3}$$

for some $m > 0$. Since $u \mapsto au^p$ is locally Lipschitz continuous on $(0, \infty)$, (3) has a unique maximal solution, which is apparently even and strictly convex. We will denote it by $u_{m,p,a}$ and its domain by $(-\Lambda_{m,p,a}, \Lambda_{m,p,a})$.

Let us also introduce the notation $\mathcal{S}(l) = \mathcal{S}(l; p, q, a)$ and $\mathcal{N}(l) = \mathcal{N}^+(l; p, q, a)$ for the set of all positive symmetric (i. e. even) and positive nonsymmetric solutions of (1) respectively.

3.1 Remark
Assume $p, q \in \mathbb{R}$, $a, l > 0$. Obviously, $\mathcal{S}(l)$ consists of all such functions $u_{m,p,a}|_{[-l,l]}$ that $0 < l < \Lambda_{m,p,a}$ and $u'_{m,p,a}(l) = u^q_{m,p,a}(l)$. On the other hand, if $l_1 \neq l_2$ are such numbers that $0 < l_i < \Lambda_{m,p,a}$, $u'_{m,p,a}(l_i) = u^q_{m,p,a}(l_i)$ for $i = 1, 2$ and $l_1 + l_2 = 2l$, then $u_{m,p,a}(\cdot - (l_1 - l_2)/2)|_{[-l,l]} \in \mathcal{N}(l)$.

3.2 Lemma
Let $p, q \in \mathbb{R}$, $a > 0$. Then the following statements are equivalent for arbitrary $m, l > 0$:

(i) $l < \Lambda_{m,p,a}$ *and* $u'_{m,p,a}(l) = u^q_{m,p,a}(l)$,

(ii) *the equation*

$$0 = \mathcal{F}(m,x) := \mathcal{F}_{p,q,a}(m,x) := \begin{cases} \dfrac{x^{2q}}{2a} - \dfrac{x^{p+1}}{p+1} + \dfrac{m^{p+1}}{p+1} & \text{if } p \neq -1, \\ \dfrac{x^{2q}}{2a} - \ln x + \ln m & \text{if } p = -1 \end{cases} \tag{4}$$

with the unknown $x > 0$ has some solution $R > m$, and

$$l = \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{R}{m}\right),$$

where

$$I_p(y) := \int_1^y \sqrt{\frac{p+1}{V^{p+1} - 1}}\, \mathrm{d}V, \quad y \geq 1.$$

One can see that $\mathcal{F}(m, \cdot)$ has different behaviour for $p > -1$, $p = -1$ and $p < -1$ as well as for $q > 0$, $q = 0$ and $q < 0$. It also matters which of the exponents $2q$, $p + 1$ is greater. So we have to distinguish thirteen cases shown in Figure 1.

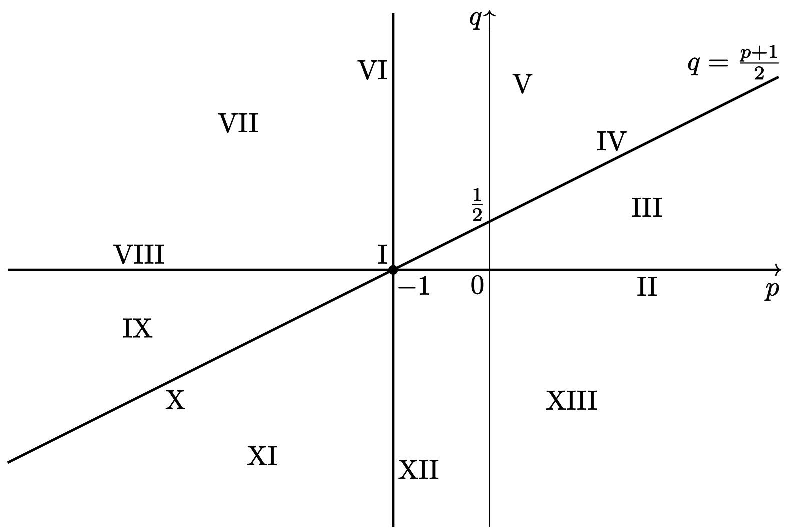

Figure 1: Cases I to XIII.

Let us notice that the set of parameters $p, q > 1$, which was investigated in [5], forms only part of cases III–V, and we will see that more complicated and interesting things happen outside it.

3.3 Lemma

Let $p, q \in \mathbb{R}$, $a, m > 0$. Function $\mathcal{F}(m, \cdot)$ has at most two zeros, and both lie in (m, ∞). We denote them $R_{p,q,a}(m) =: R(m)$ if there is only one zero, and $R_{1;p,q,a}(m) =: R_1(m)$ and $R_{2;p,q,a}(m) =: R_2(m)$ if there are two, while $R_1(m) < R_2(m)$.

Let us also introduce

$$M := M_{p,q,a} := \begin{cases} \left(\dfrac{2q-p-1}{2q}\right)^{\frac{1}{p+1}} \left(\dfrac{a}{q}\right)^{\frac{1}{2q-p-1}} & \text{if } p \neq -1,\ q > 0,\ q > \frac{p+1}{2} \text{ (V, VII)}, \\ \left(\dfrac{a}{\mathrm{e}q}\right)^{\frac{1}{2q}} & \text{if } p = -1,\ q > 0 \text{ (VI)}, \\ \left(-\dfrac{p+1}{2a}\right)^{\frac{1}{p+1}} & \text{if } p < -1,\ q = 0 \text{ (VIII)}. \end{cases}$$

The following holds for the number of zeros:

(i) *If $q < 0$ or $q < \frac{p+1}{2}$ or $p = -1$, $q = 0$ (cases I–III, IX–XIII), then $\mathcal{F}(m, \cdot)$ has exactly one zero for arbitrary $m > 0$. Moreover, for $p > -1$, $0 < q < \frac{p+1}{2}$ (case III) we have*

$$R(m) > \left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}. \tag{5}$$

(ii) *If $p > -1$, $q = \frac{p+1}{2}$ (case IV), then $\mathcal{F}(m, \cdot)$ has one zero for $q < a$ and none for $q \geq a$.*

(iii) *If $p < -1$, $q = 0$ (case VIII), then $\mathcal{F}(m, \cdot)$ has one zero for $m < M$ and none for $m \geq M$.*

(iv) *If $q > 0$ and $q > \frac{p+1}{2}$ (cases V–VII), then $\mathcal{F}(m, \cdot)$ has two zeros for $m < M$, one for $m = M$ and none for $m > M$. Meanwhile,*

$$R_1(m) < \underbrace{\left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}}_{=R(M)} < R_2(m). \tag{6}$$

Now, as a simple consequence of Lemma 3.3, we formulate a non-existence result related to (1), and afterwards we introduce the notion of the time map.

3.4 Theorem

Let $p \in \mathbb{R}$, $a > 0$.

(i) *If $q \leq 0$ or $q \leq \frac{p+1}{2}$ (cases I–IV and VIII–XIII), then $\mathcal{N}(l) = \emptyset$ for all $l > 0$.*

(ii) *If $p > -1$, $q = \frac{p+1}{2} \geq a$ (case IV), then $\mathcal{S}(l) = \emptyset$ for all $l > 0$.*

3.5 Definiton
Let $p, q \in \mathbb{R}$, $a > 0$ and

$$L(m) := L_{p,q,a}(m) := \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{R_{p,q,a}(m)}{m}\right)$$

for all such m that $R_{p,q,a}(m)$ is defined. We introduce $L_{1;p,q,a}(m) =: L_1(m)$ and $L_{2;p,q,a}(m) =: L_2(m)$ analogously. Functions L, L_1 and L_2 will be called **time maps** (associated with (3)).

Using Lemmata 3.2 and 3.3, we can reformulate the statement of Remark 3.1 in the following way:

3.6 Lemma
For all $p, q \in \mathbb{R}$, $a, l > 0$:

$$\mathcal{S}(l) = \left\{ u_{m,p,a}\big|_{[-l,l]} \ : \ L(m) = l \text{ or } L_1(m) = l \text{ or } L_2(m) = l \right\},$$

$$\mathcal{N}(l) = \begin{cases} \left\{ u_{m,p,a}\left(\cdot \pm \frac{L_2(m)-L_1(m)}{2}\right)\Big|_{[-l,l]} \ : \ L_1(m)+L_2(m)=2l \right\} & \text{if } q > 0 \text{ and } q > \frac{p+1}{2} \text{ (V–VII)}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, to determine the number of positive symmetric solutions of (1) for given $p, q \in \mathbb{R}$, $a, l > 0$, we need to calculate the limits of functions L, L_1, L_2 at the endpoints of their domains, to find the intervals where the functions are monotone and finally to estimate their possible relative extrema. For non-symmetric solutions we execute the same with $L_1 + L_2$ if $q > 0$ a $q > \frac{p+1}{2}$ (cases V–VII).

3.2 Case I ($p = -1$, $q = 0$)

This case is the simplest one since

$$L(m) = \frac{m}{\sqrt{2a}} I_{-1}\left(\mathrm{e}^{\frac{1}{2a}}\right), \quad m > 0.$$

Thus, the time map, which determines the relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$, is linear. So substituting into Lemma 3.6, we obtain the following theorem:

3.7 Theorem
Assume $p = -1$, $q = 0$, $a > 0$. Then for arbitrary $l > 0$:

$$\mathcal{S}(l) = \left\{ u_{m,-1,a}\big|_{[-l,l]} \ : \ m = \frac{\sqrt{2a}}{I_{-1}\left(\mathrm{e}^{\frac{1}{2a}}\right)} l \right\},$$

$$\mathcal{N}(l) = \emptyset.$$

3.3 Case II ($p > -1$, $q = 0$)

Figure 2 shows the properties of L in case II depending on p. Let us mention that

$$L(0) := L_{p,q,a}(0) := \frac{2}{1-p}\left(\frac{p+1}{2a}\right)^{\frac{1}{p+1}}, \qquad p \in (-1, 1).$$

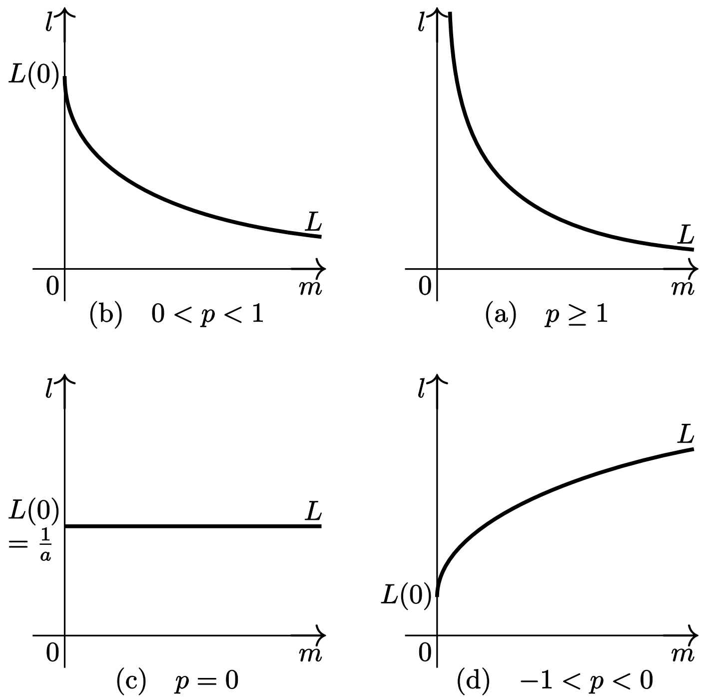

Figure 2: The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case II.

From these results, applying Lemma 3.6, we obtain the following statement:

3.8 Theorem
Assume $p > -1$, $q = 0$ and $a, l > 0$. Then $\mathcal{N}(l) = \emptyset$, and the following holds for positive symmetric solutions of (1):

If $p \geq 1$, then $|\mathcal{S}(l)| = 1$, and L is decreasing. (Recall that $L(u(0)) = l$ for any $u \in \mathcal{S}(l)$.)

If $p = 0$, then (1) has a solution only for $l = \frac{1}{a}$, namely

$$\mathcal{S}\left(\frac{1}{a}\right) = \left\{x \mapsto \frac{a}{2}x^2 + m,\ x \in [-l, l]\ :\ m > 0\right\}.$$

If $p < 1$ and $p \neq 0$, then

$$|\mathcal{S}(l)| = \begin{cases} 1 & \text{if } l \text{ is between } L(0) \text{ and } \lim_{m\to\infty} L(m), \\ 0 & \text{otherwise,} \end{cases}$$

and L is strictly monotone.

3.4 Case III ($p > -1$, $0 < q < \frac{p+1}{2}$)

The properties of L in case III are summarised in Figure 3, which shows all the possible graphs of L with the corresponding sets of parameters in the (p, q)-plane, distinguished by colours. Here,

$$L(0) := L_{p,q,a}(0) := \frac{2}{1-p}\left(\frac{p+1}{2a}\right)^{\frac{1}{p+1}}, \qquad p < 1,$$

and $m_0 = m_{0;p,q,a}$ is a stationary point of L, not given analytically.

Using Lemma 3.6, we can state the main result of this section. Recall that $L(u(0)) = l$ for any $u \in \mathcal{S}(l)$.

3.9 Theorem
Assume $p > -1$, $0 < q < \frac{p+1}{2}$ and $a, l > 0$. Then $\mathcal{N}(l) = \emptyset$, and the following holds for the positive symmetric solutions of (1):

If $p > 0$ and $q > p$, then

$$|\mathcal{S}(l)| = \begin{cases} 2 & \text{if } l \in (L(m_0), L(0)), \\ 1 & \text{if } l \in \{L(m_0)\} \cup [L(0), \infty), \\ 0 & \text{otherwise,} \end{cases}$$

and L decreases on $(0, m_0]$ and increases on $[m_0, \infty)$, see Figure 3.

In all the other cases,

$$|\mathcal{S}(l)| = \begin{cases} 1 & \text{if } l \text{ is between } L(0) \text{ and } \lim_{m\to\infty} L(m), \\ 0 & \text{otherwise,} \end{cases}$$

and L is strictly monotone, see Figure 3.

3.5 Case IV ($p > -1$, $q = \frac{p+1}{2}$)

In this case the time map is defined only for $q < a$, and is given by

$$L(m) = \frac{1}{\sqrt{2a}}\, I_p\Bigg(\underbrace{\left(\frac{a}{a-q}\right)^{\frac{1}{2q}}}_{=:r_{q,a}}\Bigg)\, m^{\frac{1-p}{2}}, \quad m > 0.$$

As a consequence, we have the following result:

3.10 Theorem
Let $p > -1$, $q = \frac{p+1}{2}$, $a > 0$. Then for arbitrary $l > 0$:

$$\mathcal{S}(l) = \begin{cases} \left\{ u_{m,p,a}\big|_{[-l,l]} \;:\; m = \left(\frac{\sqrt{2a}}{I_p(r_{q,a})}l\right)^{\frac{2}{1-p}} \right\} & \text{if } p \neq 1,\ q < a, \\ \{x \mapsto m\,\mathrm{ch}(\sqrt{a}x), x \in [-l,l] \;:\; m > 0\} & \text{if } p = 1,\ a > 1,\ l = \frac{1}{2\sqrt{a}}\ln\frac{\sqrt{a}+1}{\sqrt{a}-1}, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\mathcal{N}(l) = \emptyset.$$

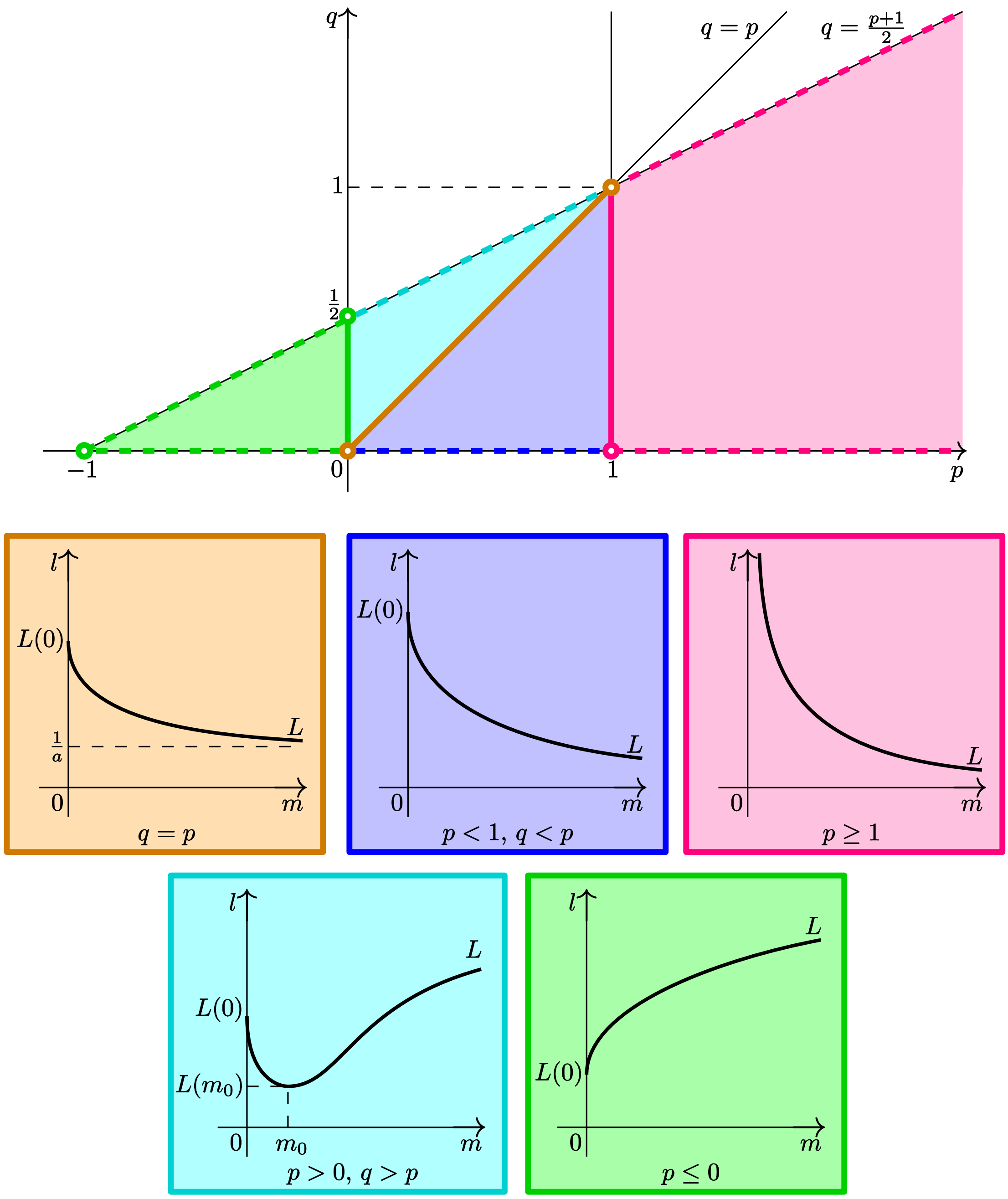

Figure 3: The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case III.

3.6 Case V ($p > -1$, $q > \frac{p+1}{2}$), symmetric solutions

Recall that due to Lemma 3.3 (iv), we have the following time maps in case V: $L_1 < L_2$ defined on $(0, M)$ and L defined on $\{M\}$. Figure 4 shows all their possible graphs and the corresponding sets of (p, q).

Let us mention that

$$L_2(0) := L_{2;p,q,a}(0) := \frac{2}{1-p}\left(\frac{p+1}{2a}\right)^{\frac{1}{p+1}}, \qquad p \in (-1, 1).$$

Furthermore, $m_0 = m_{0;p,q,a}$, $m_1 = m_{1;p,q,a}$ and $m_2 = m_{2;p,q,a}$ are stationary points of L_1 or L_2 for certain values of p and q, not given analytically. They fulfil

$$0 < m_1 < \overline{m} < m_2 < M,$$

where

$$\overline{m} := \overline{m}_{p,q,a} := \left(\frac{(p+q)(2q-p-1)}{2q(q-1)}\right)^{\frac{1}{p+1}}\left(\frac{a(p-q)}{q(q-1)}\right)^{\frac{1}{2q-p-1}}, \qquad q < |p|$$

Finally, $q^* : (-1, -\frac{1}{2}) \to \mathbb{R}$ is a continuous function, while $q = q^*(p)$ is given as the only solution of the equation

$$I_p(g(p,q)) - \frac{1}{1-p}\sqrt{\frac{2(q-p)(1-q)}{q}}\,\overset{*}{g}^{\frac{1-p}{2}}(p,q) =: f^*(p,q) = 0$$

in $(\frac{p+1}{2}, -p)$, where

$$\overset{*}{g}(p,q) = \left(\frac{2q(q-1)}{(2q-p-1)(p+q)}\right)^{\frac{1}{p+1}}.$$

In addition, $\lim_{p\to -1/2} q^*(p) = \frac{1}{2}$ and $\lim_{p\to -1} q^*(p) \in (0,1)$.

The results summarised in Figure 4 are sufficient to determine the number of the symmetric solutions of (1) in case V depending on p, q, a, l (see Lemma 3.6) except for $p < -\frac{1}{2}$, $q^*(p) < q < -p$ because it is required to investigate, for which p, q is $L_2(0) > L_2(m_2)$. It can be expected that this domain is divided by a continuous curve into three sets where $L_2(0) = L_2(m_2)$ for (p,q) lying on the curve, $L_2(0) < L_2(m_2)$ above it, and $L_2(0) > L_2(m_2)$ under it. This hypothesis is also consistent with numerical calculations.

So let us state the main result of this section.

3.11 Theorem
Suppose $p > -1$, $q > \frac{p+1}{2}$ and $a > 0$.

(a) *If $q < p$, then*

$$\big\{|\mathcal{S}(l)| \ :\ l > 0\big\} = \big\{0, 1, 2\big\}.$$

(b) *If $q = p$, then*

$$\big\{|\mathcal{S}(l)| \ :\ l > 0\big\} = \big\{0, 1\big\}.$$

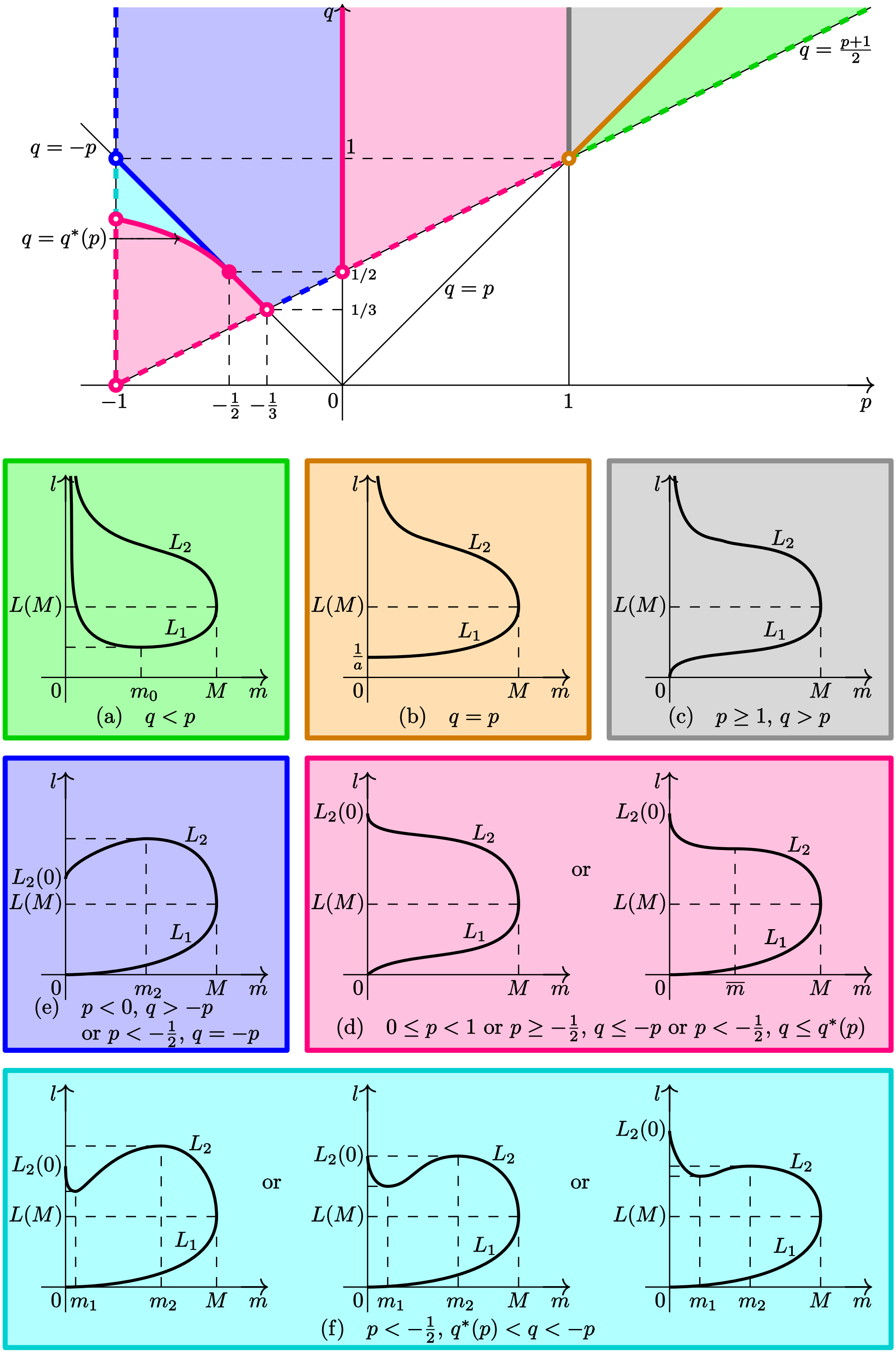

Figure 4: The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case V.

(c) *If* $p \geq 1$ *and* $q > p$, *then*

$$|\mathcal{S}(l)| = 1 \quad \text{for } l > 0.$$

(d) *If* $0 \leq p < 1$ *or* $p \geq -\frac{1}{2}$, $q \leq -p$ *or* $p < -\frac{1}{2}$, $q \leq q^*(p)$, *then*

$$\big\{|\mathcal{S}(l)| \;:\; l > 0\big\} = \big\{0, 1\big\}.$$

(e) *If* $p < 0$, $q > -p$ *or* $p < -\frac{1}{2}$, $q = -p$, *then*

$$\big\{|\mathcal{S}(l)| \;:\; l > 0\big\} = \big\{0, 1, 2\big\}.$$

(f) *If* $p < -\frac{1}{2}$ *and* $q^*(p) < q < -p$, *then*

$$\big\{|\mathcal{S}(l)| \;:\; l > 0\big\} = \big\{0, 1, 2, 3\big\}.$$

The exact dependence of $|\mathcal{S}(l)|$ *on* l *as well as the monotonicity properties of* L_1 *and* L_2 *are indicated in Figure 4. (Recall Lemma 3.6.)*

3.7 Case V ($p > -1$, $q > \frac{p+1}{2}$), non-symmetric solutions

Assume

$$p > -1,\; q > \frac{p+1}{2},\; a > 0 \tag{7}$$

and $l > 0$. Then, following from Lemmata 3.3 (iv) and 3.6, (1) can possess positive non-symmetric solutions, and their number is determined by the properties of $L_1 + L_2$. We already know that

$$\begin{aligned} \lim_{m\to 0}(L_1 + L_2)(m) &= \begin{cases} \infty & \text{if } p \geq 1, \\ L_2(0) & \text{if } p \in (-1, 1), \end{cases} \\ \lim_{m\to M}(L_1 + L_2)(m) &= 2L(M). \end{aligned} \tag{8}$$

In this section the question of the monotonicity of $L_1 + L_2$ will be examined.

It was shown in [5, Theorems 34], that if (7) holds, then

$$1 < p \leq 4 \quad \text{or} \quad p > 4,\; q \geq p - 1 - \frac{1}{p-2} \tag{9}$$

is a sufficient condition for the decrease of $L_1 + L_2$. However, our result is that:

3.12 Lemma
If (7) *holds with* $p \geq 1$, *then* $(L_1 + L_2)' < 0$.

3.13 Remark
The first step of the proof of Lemma 3.12 is a sufficient condition for $(L_1+L_2)' < 0$, which was motivated by [6, Remark 5.3], where a similarly looking condition, sufficient for $(\overline{L}_1 + \overline{L}_2)' < 0$ ($\overline{L}_1$ and $\overline{L}_2$ being the time maps associated with (11)), had been derived.

Lemma 3.12—together with (8) and Lemma 3.6—leads to this result:

3.14 Theorem
If (7) *holds with* $p \geq 1$, *then*

$$|\mathcal{N}(l)| = \begin{cases} 2 & \text{if } l > L(M), \\ 0 & \text{if } l \leq L(M). \end{cases}$$

The case of $p < 1$ is much more complicated, except these two special cases:

3.15 Lemma

(i) *If* $p = 0$, $q = 1$, $a > 0$, *then*

$$L_1 + L_2 \equiv 2 \text{ on } (0, M) = \left(0, \frac{a}{2}\right).$$

(ii) *If* $p = -\frac{1}{2}$, $q = \frac{1}{2}$, $a > 0$, *then*

$$L_1 + L_2 \equiv \frac{16a}{3} \text{ on } (0, M) = \left(0, a^2\right).$$

For other p, q we have succeeded only in describing the behaviour of $L_1 + L_2$ near 0 and M, except $p \in (-1, 0) \cup (0, 1)$, $q = \widehat{q}(p)$ and $p \in (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, -\frac{1}{7})$, $q = \overline{q}(p)$, for which we have no information at all.

Let us mention that $\overline{q} : (-1, -\frac{1}{7}) \to \mathbb{R}$ is a continuous function fulfilling $\lim_{p\to-1} \widehat{q}(p) \in (1, \infty)$, $\widehat{q} > 1$ on $(-1, 0)$, $\widehat{q}(-\frac{1}{2}) = \frac{3}{2}$, $\widehat{q}(0) = 1$, $\widehat{q} < 1$ on $(0, 1)$, and $\lim_{p\to1} \widehat{q}(p) = 1$. Furthermore, $\widehat{q} : (-1, 1) \to \mathbb{R}$ is a continuous function fulfilling $\lim_{p\to-1} \overline{q}(p) \in (0, 1)$, $\overline{q}(p) < -p$ for $p \in (-1, -\frac{1}{2})$, $\overline{q}(-\frac{1}{2}) = \frac{1}{2}$, $\overline{q}(p) > -p$ for $p \in (-\frac{1}{2}, -\frac{1}{7})$, and $\lim_{p\to-1/7} \overline{q}(p) = \frac{3}{7}$.

More specifically, for all $p \in [-\frac{1}{7}, 1)$, $q = \widehat{q}(p)$ is given as the only solution of

$$\frac{\sqrt{2}(q - p + 2)}{3\sqrt{q}} g^{\frac{1-p}{2}}(p, q) + (p - 1) I_p\big(g(p, q)\big) =: f(p, q) = 0 \tag{10}$$

in $(\frac{p+1}{2}, \infty)$, where

$$g(p, q) = \left(\frac{2q}{2q - p - 1}\right)^{\frac{1}{p+1}}.$$

Similarly, for all $p \in (-1, -\frac{1}{7})$, $q = \overline{q}(p)$ and $q = \widehat{q}(p)$ are the only solutions of (10) in $[p + \sqrt{2p(p-1)}, \infty)$ and $(\frac{p+1}{2}, p + \sqrt{2p(p-1)}]$ respectively.

Figure 5 shows the graphs of $L_1 + L_2$ and the corresponding sets of (p, q).

Using numerical calculations, one can observe that $L_1 + L_2$ has probably at most one strict relative extremum for any $p \in (-1, 1)$, $q > \frac{p+1}{2}$. If it is true, the behaviour of $L_1 + L_2$ on the whole interval $(0, M)$ is clear for all $p \in (-1, 1)$, $q \notin \{\widehat{q}(p), \overline{q}(p)\}$, and due to the continuous dependence of $L_{1;p,q,a}(m)$ and $L_{2;p,q,a}(m)$ on p and q, even for $q = \widehat{q}(p)$ and $q = \overline{q}(p)$.

3.8 Sign-changing non-antisymmetric solutions of (1)

Let $p \geq 1$, $q \in \mathbb{R}$. The modification of the shooting method can also be used for the study of the sign-changing solutions of (1). Unlike the shooting method used

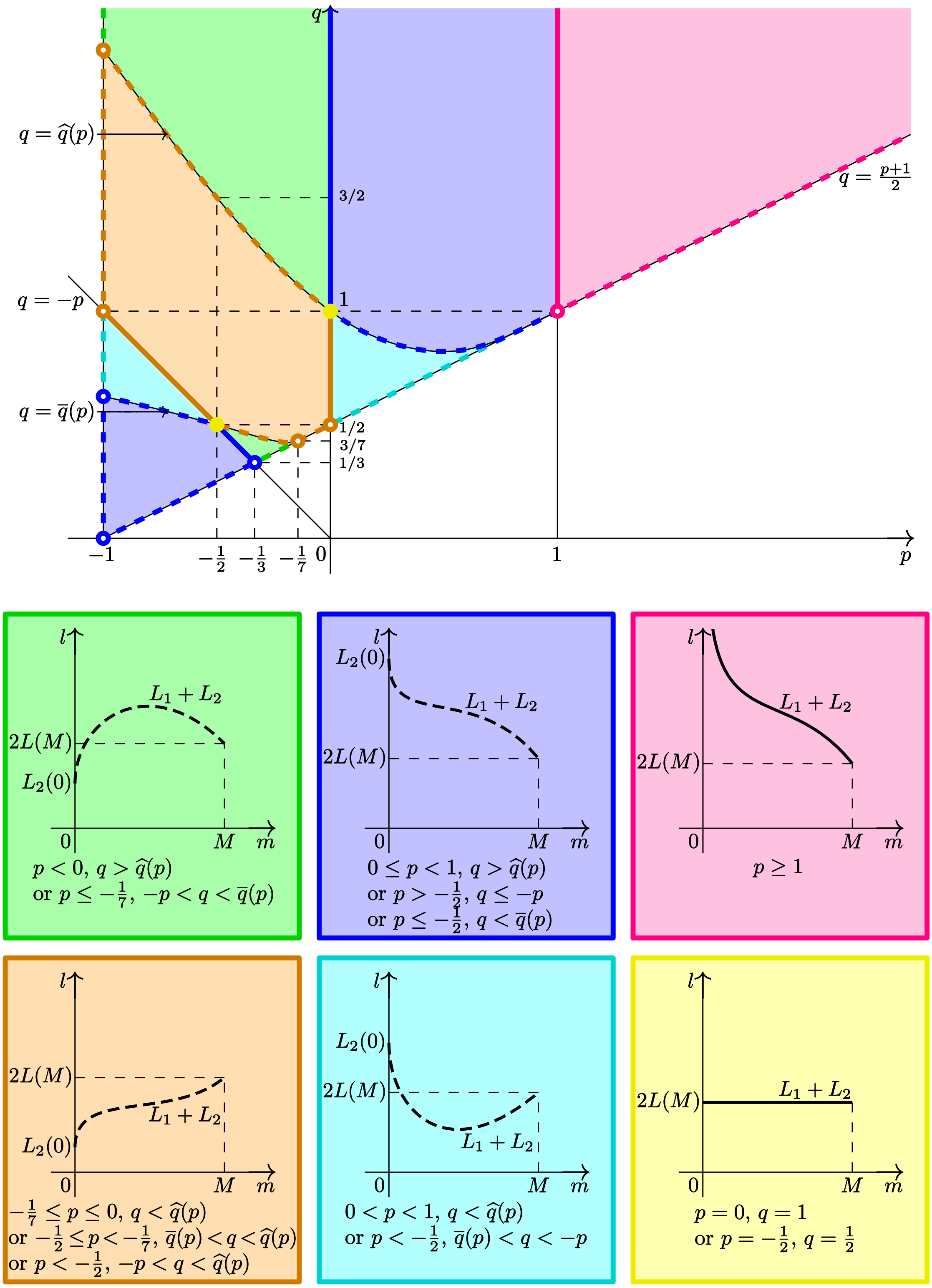

Figure 5: The behaviour of $L_1 + L_2$ in case V.
The dashed graphs mean that for those values of p and q the behaviour of $L_1 + L_2$ has been examined only near 0 and M, and the graph has been plotted assuming that $L_1 + L_2$ has at most one stationary point. (This assumption is consistent with numerical calculations.)

for the study of positive solutions of (1), for sign-changing solutions one consideres

$$\begin{cases} u'' = a|u|^{p-1}u, \\ u(0) = 0, \\ u'(0) = \theta \end{cases} \tag{11}$$

instead of (3). However, we do not explain the details.

Let us notice that (1) has sense for any $p > 0$, $q \in \mathbb{R}$, but we do not consider $p \in (0, 1)$, because in that case (11) has infinitely many solutions for $\theta = 0$, which causes difficulties for the study of (1).

We introduce the notation $\mathcal{N}^{\pm}(l) = \mathcal{N}^{\pm}(l; p, q, a)$ for the set of all sign-changing non-antisymmetric (i. e. not odd) solutions of (1).

If $q \notin (0, \frac{p+1}{2})$, then $\mathcal{N}^{\pm}(l) = \emptyset$ for. Therefore, let $q \in (0, \frac{p+1}{2})$. According to [6, Theorem 1.3 (iii)], if $q > 1$ and

$$(p - q)(2q + 1 - p)(p + 1) \geq 2q(p - 1)$$

or equivalently,

$$q > \frac{p(p-1)}{p+1}, \tag{12}$$

then there exist such a number $\overline{L}(\Theta)$ depending on p, q, a, given analytically that

$$\left|\mathcal{N}^{\pm}(l)\right| = \begin{cases} 4 & \text{if } l > \overline{L}(\Theta), \\ 0 & \text{if } l \leq \overline{L}(\Theta). \end{cases} \tag{13}$$

However, our result is that this property holds even without assuming (12), and also for some $q \leq 1$:

3.16 Theorem
If $a, l > 0$ and either $p = 1$, $q \in (0, 1)$ or $p > 1$, $q \in [\frac{1}{2}, \frac{p+1}{2})$, then (13) holds.

Let us remark that numerical calculations suggest that if $p > 1$ is big enough and $q \in (0, \frac{1}{2})$ is small enough, then

$$\left\{|\mathcal{N}^{\pm}(l)| \ : \ l > 0\right\} = \{0, 4, 8\}.$$

4 Summary

In this thesis we got familiar with the shooting method, which made it possible to simplify the question of the solvability of (1) to the question of the properties of the time maps, which are real functions of one real variable. Examining their properties, we were able to determine the number of positive symmetric solutions of (1) for $p > -1$, $q \geq 0$ and $p = -1$, $q = 0$, the number of its positive non-symmetric solutions for $p \geq 1$, $q > \frac{p+1}{2}$ with some partial results for $p \in (-1, 1)$, $q > \frac{p+1}{2}$, and the number of its sign-changing non-antisymmetric solutions for $p = 1$, $q \in (0, 1)$ and $p > 1$, $q \in [\frac{1}{2}, \frac{p+1}{2})$, while the number of its sign-changing antisymmetric solutions for $p \geq 1$, $q > 1$ is known from [6].

The predominant majority of the results mentioned above are new results achieved by the author. Theorems 3.14 and 3.16 provide the answers for two long-standing open questions arising in [5] and [6], while the other statements deal with values of parameters not considered before.

The given topic has not been exhausted by this thesis at all. There remains to verify analytically the numerically predicted properties of q^* seen in Figure 4, and that of $\widehat{q}$ and $\overline{q}$ seen in Figure 5, as well as to determine the sign of $L_2(0) - L_2(m_2)$ in case V for $p < -\frac{1}{2}$, $q \in (q^*(p), -p)$ in dependence on p, q (see the second paragraph above Theorem 3.11), and to investigate the so far unknown properties of $L_1 + L_2$ in case V for $p < 1$ (see the last paragraph of Subsection 3.7). And naturally, a further goal can be to determine the number of positive solutions of (1) in cases VI–XIII, the number of its sign-changing antisymmetric solutions for $p \geq 1$, $q \leq 1$, and the number of its sign-changing non-antisymmetric solutions for $p > 1$, $q \in (0, \frac{1}{2})$. Moreover, one could also study the sign-changing solutions of (1) for $p \in (0, 1)$, $q \in \mathbb{R}$.

Throughout this whole thesis, we could get by only using the knowledge of real analysis (except for the use of Picard's existence theorem), but in spite of this, this topic cannot be called too simple or uninteresting. On the contrary, the author consideres it especially nice and hopes that the reader has acquired a similar impression.

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- Študentská vedecká konferencia FMFI UK, Bratislava, april 2011