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SOLVABILITY OF A NONLINEAR BOUNDARY VALUE PROBLEM

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Abstract. We study the existence and multiplicity of positive solutions of a nonlinear second order ordinary differential equation with symmetric nonlinear boundary conditions where both of the nonlinearities are of power type.

1. Introduction

We deal with the existence and number of positive solutions of the following class of boundary value problems:

$$\begin{cases} u''(x) = au^p(x), & x \in (-l, l), \\ u'(\pm l) = \pm u^q(\pm l) \end{cases} \tag{1}$$

where $p, q \in \mathbb{R}$ a $a, l > 0$ are parameters.

Our principal reference is [**5**] where M. Chipot, M. Fila and P. Quittner studied also the N-dimensional version of (1):

$$\begin{cases} \Delta u(x) = au^p(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial n}(x) = u^q(x), & x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, n is the unit outer normal vector to $\partial\Omega$, $N \in \mathbb{N}$. First of all, they were interested in global existence and boundedness or blow-up of positive solutions of the corresponding parabolic problem

$$\begin{cases} u_t = \Delta u - au^p & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = u^q & \text{in } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \overline{\Omega} \end{cases} \tag{2}$$

where $u_0 : \overline{\Omega} \to [0, \infty)$ but they restricted their investigation to $p, q > 1$. The same problem was independently studied in [**12**] for $N = 1$.

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The results from [**5**] have been generalised in many directions. In [**14**], the behaviour of positive solutions of (2) was examined for all $p, q > 1$ while sign changing solutions were considered in [**6**] for $p, q > 1$—in that case, u^p and u^q are replaced by $|u|^{p-1}u$ and $|u|^{q-1}u$, respectively. Positive solutions of the elliptic problem with $-\lambda u + u^p$ on the right-hand side of the equation were dealt with in [**13**] for $\lambda \in \mathbb{R}$, $p, q > 1$ and later in [**10**] for $\lambda \in \mathbb{R}$, $p, q > 0$, $(p, q) \notin (0, 1)^2$. In [**11**] and [**15**], positive and sign changing solutions of the parabolic problem with more general nonlinearities $f(u)$, $g(u)$ instead of au^p, u^q have been studied while $f(x, u)$, $g(x, u)$ were considered in [**2**]. Further extensions of results from [**5**] can be found in [**1, 3, 4, 7, 8, 9**]. Finally we mention [**16**], which was devoted to elliptic problems with nonlinear boundary conditions.

In this paper, we focus only on (1) and we extend the results known for $p, q > 1$ to a larger set of parameters, namely to $p > -1$, $q \geq 0$ and $p = -1$, $q = 0$. The main results are included in Theorems 2.6 (a nonexistence result), 4.1 ($p = -1$, $q = 0$), 5.4 ($p > -1$, $q = 0$), 6.6 ($p > -1$, $0 < q < \frac{p+1}{2}$), 7.1 ($p > -1$, $q = \frac{p+1}{2}$) and 8.9 ($p > -1$, $q > \frac{p+1}{2}$). However, in case of $p > -1$, $q > \frac{p+1}{2}$ only symmetric solutions are concerned and some small questions are left open (see the text above Theorem 8.9). Our aim is to answer these questions in the future as well as to examine the number of nonsymmetric solutions for $p > -1$, $q > \frac{p+1}{2}$ and the solvability of (1) for the values of p and q not considered in this paper.

We use the method included in Section 3 (dealing with the case $N = 1$) of [**5**]: After considering an appropriate initial value problem, we introduce a function L or functions L_1 and L_2, the so-called time maps, the graphs of which directly determine the number of solutions of (1), so we will need only the tools of real analysis. On the other hand, it is not so easy to examine the properties of L, L_1 and L_2 because they are given by a formula that contains an improper integral, with an upper limit, which is given only implicitly.

2. The initial value problem and the time maps

If u is a positive solution of (1), then $u'(-l) < 0 < u'(l)$, therefore u has a stationary point $x_0 \in (-l, l)$. So the function $u(\cdot + x_0)$ solves

$$\begin{cases} u'' = au^p, \\ u(0) = m, \\ u'(0) = 0 \end{cases} \tag{3}$$

for some $m > 0$. In the following theorem we summarise the facts known about the solvability of this problem. The proof for $p, q > 1$ can be found in [**5**], for other p, q it is done analogously.

Theorem 2.1 (for $p, q > 1$ see [**5**, pp. 53–54])**.** *Suppose $m, a > 0$, $p \in \mathbb{R}$. Then* (3) *has a unique maximal solution. We will denote it by $u_{m,p,a}$ and its domain by $(-\Lambda_{m,p,a}, \Lambda_{m,p,a})$. Function $u_{m,p,a}$ is even, strictly convex, unbounded from above* *and fulfils*

$$|x| = \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{u_{m,p,a}(x)}{m}\right), \quad x \in (-\Lambda_{m,p,a}, \Lambda_{m,p,a}) \tag{4}$$

where $I_p : [1,\infty) \to [0,\infty)$ *is given as*

$$I_p(y) = \begin{cases} \displaystyle\int_1^y \sqrt{\frac{p+1}{V^{p+1}-1}}\,\mathrm{d}V & \text{if } p \neq -1, \\ \displaystyle\int_1^y \frac{\mathrm{d}V}{\sqrt{\ln V}} & \text{if } p = -1 \end{cases}$$

and

$$\Lambda_{m,p,a} = \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} \lim_{y\to\infty} I_p(y) \begin{cases} < \infty & \text{if } p > 1, \\ = \infty & \text{if } p \leq 1. \end{cases} \tag{5}$$

Finally, for $x \in (-\Lambda_{m,p,a}, \Lambda_{m,p,a})$ *we have:*

$$|u'_{m,p,a}(x)| = \begin{cases} \sqrt{\dfrac{2a}{p+1}\big(u^{p+1}_{m,p,a}(x) - m^{p+1}\big)} & \text{if } p \neq -1, \\ \sqrt{2a\big(\ln u_{m,p,a}(x) - \ln m\big)} & \text{if } p = -1. \end{cases} \tag{6}$$

Definition 2.2. For given $p, q \in \mathbb{R}$, $a, l > 0$ denote the set of all positive symmetric (i. e. even) and positive nonsymmetric solutions of (1) by $\mathcal{S}(l) = \mathcal{S}(l; p, q, a)$ and $\mathcal{N}(l) = \mathcal{N}(l; p, q, a)$, respectively.

Remark 2.3 ([**5**, pp. 53–54])**.** Assume $p, q \in \mathbb{R}$, $a, l > 0$. Obviously, $\mathcal{S}(l)$ consists of all such functions $u_{m,p,a}|_{[-l,l]}$ that $0 < l < \Lambda_{m,p,a}$ and $u'_{m,p,a}(l) = u^q_{m,p,a}(l)$. On the other hand, if $l_1 \neq l_2$ are such numbers that $0 < l_i < \Lambda_{m,p,a}$, $u'_{m,p,a}(l_i) = u^q_{m,p,a}(l_i)$ for $i = 1, 2$ and $l_1 + l_2 = 2l$, then $u_{m,p,a}(\cdot - (l_1 - l_2)/2)|_{[-l,l]} \in \mathcal{N}(l)$.

Lemma 2.4 (for $p, q > 1$ see [**5**, pp. 54–55])**.** *Let* $p, q \in \mathbb{R}$, $a > 0$. *Then the following statements are equivalent for arbitrary* $m, l > 0$*:*

(i) $l < \Lambda_{m,p,a}$ *and* $u'_{m,p,a}(l) = u^q_{m,p,a}(l)$,

(ii) *the equation*

$$0 = \mathcal{F}(m, x) := \mathcal{F}_{p,q,a}(m, x) := \begin{cases} \dfrac{x^{2q}}{2a} - \dfrac{x^{p+1}}{p+1} + \dfrac{m^{p+1}}{p+1} & \text{if } p \neq -1, \\ \dfrac{x^{2q}}{2a} - \ln x + \ln m & \text{if } p = -1 \end{cases} \tag{7}$$

with the unknown $x > 0$ *has some solution* $R > m$ *and*

$$l = \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{R}{m}\right).$$

Proof. In order to derive (ii) from (i), it suffices to use (6), denote $u_{m,p,a}(l) =: R > m$ and realise (4) for $x = l$. The reversed implication is proved essentially in the same way. ∎

Function $\mathcal{F}(m,\cdot)$ has obviously different behaviour for $p > -1$, $p = -1$ and $p < -1$ as well as for $q > 0$, $q = 0$ and $q < 0$. It also matters which of the exponents $2q$, $p+1$ is greater. So we have to distinguish thirteen cases shown in Figure 1.

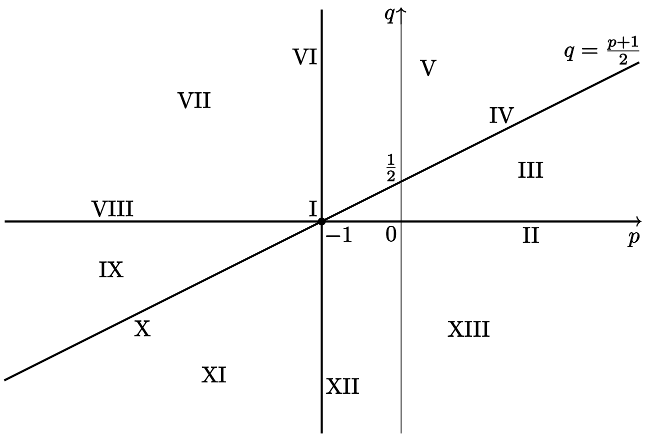

Figure 1. Cases I to XIII.

Lemma 2.5 (for $p, q > 1$ see [**5**, proofs of Lemma 3.1 and 3.2 with pp. 57–58])**.** *Let $p, q \in \mathbb{R}$, $a, m > 0$. Function $\mathcal{F}(m,\cdot)$ has at most two zeros and both lie in (m,∞). We denote them $R_{p,q,a}(m) =: R(m)$ if there is only one zero and $R_{1;p,q,a}(m) =: R_1(m)$ and $R_{2;p,q,a}(m) =: R_2(m)$ if there are two while $R_1(m) < R_2(m)$.*

Let us also introduce

$$M := M_{p,q,a} := \begin{cases} \left(\dfrac{2q-p-1}{2q}\right)^{\frac{1}{p+1}} \left(\dfrac{a}{q}\right)^{\frac{1}{2q-p-1}} & \text{if } p \neq -1,\ q > 0,\ q > \frac{p+1}{2} \text{ (V, VII)}, \\ \left(\dfrac{a}{\mathrm{e}q}\right)^{\frac{1}{2q}} & \text{if } p = -1,\ q > 0 \text{ (VI)}, \\ \left(-\dfrac{p+1}{2a}\right)^{\frac{1}{p+1}} & \text{if } p < -1,\ q = 0 \text{ (VIII)}. \end{cases}$$

The following holds for the number of zeros:

(i) *If $q < 0$ or $q < \frac{p+1}{2}$ or $p = -1$, $q = 0$ (cases I–III, IX–XIII), then $\mathcal{F}(m,\cdot)$ has exactly one zero for arbitrary $m > 0$. Moreover, for $p > -1$, $0 < q < \frac{p+1}{2}$ (case III) we have*

$$R(m) > \left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}. \tag{8}$$

(ii) *If* $p > -1$, $q = \frac{p+1}{2}$ *(case IV), then* $\mathcal{F}(m,\cdot)$ *has one zero for* $q < a$ *and none for* $q \geq a$.

(iii) *If* $p < -1$, $q = 0$ *(case VIII), then* $\mathcal{F}(m,\cdot)$ *has one zero for* $m < M$ *and none for* $m \geq M$.

(iv) *If* $q > 0$ *and* $q > \frac{p+1}{2}$ *(cases V–VII), then* $\mathcal{F}(m,\cdot)$ *has two zeros for* $m < M$, *one for* $m = M$ *and none for* $m > M$. *Meanwhile,*

$$R_1(m) < \underbrace{\left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}}_{=R(M)} < R_2(m). \tag{9}$$

Moreover,

$$R(m) = \begin{cases} e^{\frac{1}{2a}} m & \text{if } p = -1,\ q = 0 \text{ (I)}, \\ \left(m^{p+1} + \dfrac{p+1}{2a}\right)^{\frac{1}{p+1}} & \text{if } p > -1,\ q = 0 \text{ (II)} \\ & \text{or } p < -1,\ q = 0,\ m < M \text{ (VIII)}, \\ \left(\dfrac{a}{a-q}\right)^{\frac{1}{2q}} m & \text{if } p > -1,\ q = \frac{p+1}{2} < a \text{ (IV)} \\ & \text{or } p < -1,\ q = \frac{p+1}{2} \text{ (X)}. \end{cases}$$

Proof. Investigating the behaviour of $\mathcal{F}(m,\cdot)$, we obtain the facts collected in Table 1. They are sufficient to determine the number of zeros of $\mathcal{F}(m,\cdot)$ in cases I–IV and VIII–XIII as well as to verify (8).

In cases V–VII, $\mathcal{F}(m,\cdot)$ has exactly one relative minimum, the value of which can be easily calculated. So there exist two zeros if and only if this minimum is negative, what happens just for $m < M$. Further, for $m = M$ there is only one zero and for $m > M$ there is none. The validity of (9) is apparent.

Now let us prove that each zero of $\mathcal{F}(m,\cdot)$ is greater than m. In cases I–IV and VIII–XIII it is guaranteed by the simple fact that $\mathcal{F}(m,m) = m^{2q}/2a > 0$ for $p, q \in \mathbb{R}$, $a, m > 0$. In cases V and VII for $m \leq M$, we need to consider

$$m \leq M < \left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}$$

too, similarly in case VI.

Finally, equation (7) is linear in $\ln x$ and x^{p+1} in cases I and II, VIII, IV, X respectively, so explicit solutions can be found. □

Let us notice that the set of parameters $p, q > 1$, which was investigated in [**5**], forms only part of cases III–V and we will see that more complicated and interesting things happen outside it.

Although there is no difference in the properties of $\mathcal{F}(m,\cdot)$ summarised in Table 1 between cases IX, X and XI, it is not clear whether or not different results hold for (1) in these cases. For this reason we have not merged them into one case.

Now, as a simple consequence of Lemma 2.5, we formulate a nonexistence result related to (1), and afterwards we introduce the notion of the time map.

<table>
<tr><th></th><th>$\lim\limits_{x\to 0} \mathcal{F}(m,x)$</th><th>monotonicity on $(0,\infty)$</th><th>$\lim\limits_{x\to\infty} \mathcal{F}(m,x)$</th></tr>
<tr><td>I. $p=-1$, $q=0$</td><td>∞</td><td rowspan="2">decreases</td><td rowspan="3">$-\infty$</td></tr>
<tr><td>II. $p>-1$, $q=0$</td><td>$\frac{1}{2a}+\frac{m^{p+1}}{p+1}>0$</td></tr>
<tr><td>III. $p>-1$, $0<q<\frac{p+1}{2}$</td><td rowspan="3">$\frac{m^{p+1}}{p+1}>0$</td><td>increases on $\big(0,(a/q)^{1/(2q-p-1)}\big]$, decreases on $\big[(a/q)^{1/(2q-p-1)},\infty\big)$</td></tr>
<tr><td>IV. $p>-1$, $q=\frac{p+1}{2}$</td><td>decreases if $q<a$, is constant if $q=a$, increases if $q>a$</td><td>$-\infty$ if $q<a$, $\frac{m^{p+1}}{p+1}>0$ if $q=a$, ∞ if $q>a$</td></tr>
<tr><td>V. $p>-1$, $q>\frac{p+1}{2}$</td><td rowspan="3">decreases on $\big(0,(a/q)^{1/(2q-p-1)}\big]$, increases on $\big[(a/q)^{1/(2q-p-1)},\infty\big)$</td><td rowspan="3">$\infty$</td></tr>
<tr><td>VI. $p=-1$, $q>0$</td><td rowspan="8">∞</td></tr>
<tr><td>VII. $p<-1$, $q>0$</td></tr>
<tr><td>VIII. $p<-1$, $q=0$</td><td rowspan="6">decreases</td><td>$\frac{1}{2a}+\frac{m^{p+1}}{p+1}$ >0 if $m>M$, $=0$ if $m=M$, <0 if $m<M$</td></tr>
<tr><td>IX. $p<-1$, $\frac{p+1}{2}<q<0$</td><td rowspan="3">$\frac{m^{p+1}}{p+1}<0$</td></tr>
<tr><td>X. $p<-1$, $q=\frac{p+1}{2}$</td></tr>
<tr><td>XI. $p<-1$, $q<\frac{p+1}{2}$</td></tr>
<tr><td>XII. $p=-1$, $q<0$</td><td rowspan="2">$-\infty$</td></tr>
<tr><td>XIII. $p>-1$, $q<0$</td></tr>
</table>

Table 1. The properties of $\mathcal{F}(m,\cdot)$.

Theorem 2.6. *Let $p \in \mathbb{R}$, $a > 0$.*

(i) *If $q \le 0$ or $q \le \frac{p+1}{2}$ (cases I–IV and VIII–XIII), then $\mathcal{N}(l) = \emptyset$ for all $l > 0$.*

(ii) *If $p > -1$, $q = \frac{p+1}{2} \ge a$ (case IV), then $\mathcal{S}(l) = \emptyset$ for all $l > 0$.*

Definition 2.7. Let $p, q \in \mathbb{R}$, $a > 0$ and

$$L(m) := L_{p,q,a}(m) := \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{R_{p,q,a}(m)}{m}\right)$$

for all such m that $R_{p,q,a}(m)$ is defined. We introduce $L_{1;p,q,a}(m) =: L_1(m)$ and $L_{2;p,q,a}(m) =: L_2(m)$ analogously. We call functions L, L_1 and L_2 *time maps.*

Using Lemmata 2.4 and 2.5, we can reformulate the statement of Remark 2.3 in the following way:

Lemma 2.8. *For all $p, q \in \mathbb{R}$, $a, l > 0$:*

$$\mathcal{S}(l) = \Big\{ u_{m,p,a}\big|_{[-l,l]} \;:\; L(m) = l \text{ or } L_1(m) = l \text{ or } L_2(m) = l \Big\},$$

$$\mathcal{N}(l) = \begin{cases} \Big\{ u_{m,p,a}\Big(\cdot \pm \frac{L_2(m)-L_1(m)}{2} \Big)\Big|_{[-l,l]} \;:\; L_1(m)+L_2(m)=2l \Big\} & \text{if } q>0 \text{ and } q > \frac{p+1}{2} \text{ (V–VII)}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, to determine the number of positive symmetric solutions of (1) for given $p, q \in \mathbb{R}$, $a, l > 0$, we need to calculate the limits of functions L, L_1, L_2 at the endpoints of their domains, to find the intervals where the functions are monotone and finally to estimate their possible relative extrema. For nonsymmetric solutions we execute the same with $L_1 + L_2$ if $q > 0$ a $q > \frac{p+1}{2}$ (cases V–VII). Therefore, we now derive formulae for the derivatives of the time map and other functions we will need in the rest of this article.

Lemma 2.9 (for $p, q > 1$ see [**5**, proofs of Theorem 3.1 and Lemma 3.5])**.** *Assume $p, q \in \mathbb{R}$, $a > 0$. Let $\mathcal{R}$ be one of the functions R, R_1, R_2 and suppose that its domain is an interval, denote it by I. Let $\mathcal{L} \in \{L, L_1, L_2\}$ be the corresponding time map. Then $\mathcal{R}, \mathcal{L} \in C^\infty(I)$ and the following formulae hold for $m \in I$:*

$$\mathcal{R}'(m) = \left(\frac{m}{\mathcal{R}(m)}\right)^p \frac{1}{1 - \frac{q}{a}\mathcal{R}^{2q-p-1}(m)}, \tag{10}$$

$$\left(\frac{\mathcal{R}(m)}{m}\right)' = \frac{2q-p-1}{2am^{p+2}}\mathcal{R}^{2q}(m)\mathcal{R}'(m), \tag{11}$$

$$\left(I_p\left(\frac{\mathcal{R}(m)}{m}\right)\right)' = \frac{2q-p-1}{\sqrt{2a}} m^{\frac{p-3}{2}} \frac{\mathcal{R}^{q-p}(m)}{1 - \frac{q}{a}\mathcal{R}^{2q-p-1}(m)}, \tag{12}$$

$$\mathcal{L}'(m) = \frac{1-p}{2m}\mathcal{L}(m) + \frac{2q-p-1}{2am^{p+1}}\mathcal{R}^q(m)\mathcal{R}'(m), \tag{13}$$

$$\begin{aligned} \mathcal{L}''(m) = &-\frac{p+1}{2m}\mathcal{L}'(m) + \frac{2q-p-1}{2am^{2p+1}} \\ &\cdot \Big((q-1)\frac{q}{a}\mathcal{R}^{2q-p-1}(m) + q - p\Big)\mathcal{R}^{p+q-1}(m)(\mathcal{R}'(m))^3. \end{aligned} \tag{14}$$

Proof. The C^∞-smoothness of $\mathcal{R}$ and the formula for its derivative follows from the implicit function theorem due to Lemma 2.5. If $\mathcal{R} \in \{R_1, R_2\}$ (cases V–VII), then (9) is used as well. The other formulae can be derived from (10) in such a way as it is done in [**5**] for $p > 1$. □

Now we introduce some further functions, the relation of which to the time maps will be seen from the subsequent lemma. They will be used in the proofs of Lemmata 6.5 and 8.6.

Definition 2.10. Let $p, q \in \mathbb{R}$, $p \neq 1$, $a > 0$ and

$$K(m) := K_{p,q,a}(m) := \frac{2q-p-1}{(p-1)a} \frac{R_{p,q,a}^{q-p}(m)}{1 - \frac{q}{a}R_{p,q,a}^{2q-p-1}(m)}$$

for all such m that $R_{p,q,a}(m)$ is defined. We introduce $K_{1;p,q,a}(m) =: K_1(m)$ and $K_{2;p,q,a}(m) =: K_2(m)$ analogously.

Lemma 2.11. *Assume $p, q \in \mathbb{R}$, $p \neq 1$, $a > 0$. Let $\mathcal{R}$ be one of functions R, R_1, R_2 and suppose that its domain is an interval, denote it by I. Let $\mathcal{L} \in \{L, L_1, L_2\}$ and $\mathcal{K} \in \{K, K_1, K_2\}$ be the corresponding functions. Then $\mathcal{K} \in C^\infty(I)$ and the following holds for all $m \in I$:*

$$\mathcal{L}'(m) = 0 \iff \mathcal{L}(m) = \mathcal{K}(m),$$

$$\mathcal{K}'(m) = \frac{2q-p-1}{(p-1)am^{2p}} \left((q-1)\frac{q}{a}\mathcal{R}^{2q-p-1}(m) + q - p \right) \mathcal{R}^{p+q-1}(m)(\mathcal{R}'(m))^3.$$

Proof. Both of the assertions can be proved using Lemma 2.9. □

Remark 2.12. Let $p, q \in \mathbb{R}$, $a > 0$ and let $\mathcal{R}$, $\mathcal{L}$ and I have the same meaning as in Lemma 2.11. It follows from (10) that $\mathcal{R}$ has no stationary point. So it can be seen from (13) that if $p = 1$ (the case not dealt with in Lemma 2.11), then either $\mathcal{L}' \equiv 0$ (for $q = 1$) or $\mathcal{L}$ has no stationary point (for $q \neq 1$).

In the subsequent sections we will look for extrema of $\mathcal{L}$, among other things. So assume now only $p \neq 1$. If $m \in I$ is a stationary point of $\mathcal{L}$, then $\mathcal{L}''(m) = 0$ (the case when it is more difficult to determine whether there is an extremum) if and only if

$$q = \frac{p+1}{2} \quad \text{or} \quad (q-1)q\mathcal{R}^{2q-p-1}(m) = (p-q)a. \tag{15}$$

Let us notice that it is also a necessary and sufficient condition under that $\mathcal{K}'(m) = 0$ holds. Thus:

(i) If $q = \frac{p+1}{2}$ or $p = q = 0$, then $\mathcal{K}' \equiv 0$.

(ii) If $q = 0$, $p \neq 0, -1$ or $q = 1$, then $\mathcal{K}$ has no stationary point.

(iii) If $q \neq 0, 1, \frac{p+1}{2}$, then (15) is equivalent to

$$\mathcal{R}^{2q-p-1}(m) = \frac{(p-q)a}{(q-1)q},$$

which can hold for at most one $m \in I$ due to the strict monotonicity of $\mathcal{R}$. Therefore, if (p, q) does not belong to cases V–VII, then $\mathcal{K} = K$ has at most one stationary point, which will be denoted by $\overline{m} = \overline{m}_{p,q,a}$ (see Lemma 6.5). On the other hand, if $q > 0, \frac{p+1}{2}$ (cases V–VII), then R_1 and R_2 have disjoint ranges (due to (9)), so at most one of K_1 and K_2 can have a stationary point, which will be denoted by $\overline{m} = \overline{m}_{p,q,a}$ as well (see Definition 8.2 and Lemmata 8.3 (ii), 8.6, 8.7).

3. Properties of function I_p

The first lemma introduces the first two terms of the asymptotic expansion of $I_p(y)$ (see Theorem 2.1) for $y \to 1$. In the next theorem we show explicit formulae of I_p for special values of p. However, the most important statement of this section is Lemma 3.4, which gives the asymptotic expansion of $I_p(y)$ for $y \to \infty$, $p > -1$.

It is essential for investigating the behaviour of the time maps in many cases but was not needed in [**5**] for $p, q > 1$. Afterwards we also derive the corresponding asymptotic expansion for $p = -1$.

We will use standard asymptotic notations: If f, g are functions defined in some punctured neighbourhood of a point $a \in \mathbb{R} \cup \{\pm\infty\}$, then

$$f(x) \sim g(x),\ x \to a \quad \text{means} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = 1,$$
$$f(x) = o(g(x)),\ x \to a \quad \text{means} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = 0,$$
$$f(x) = O(g(x)),\ x \to a \quad \text{means} \quad \limsup_{x\to a} \left|\frac{f(x)}{g(x)}\right| < \infty.$$

Lemma 3.1. *For arbitrary $p \in \mathbb{R}$ we have*

$$I_p(y) = 2\sqrt{y-1}\Big(1 - \frac{p}{12}(y-1) + o(y-1)\Big), \quad y \to 1.$$

Proof. Suppose $p \neq -1$. Then

$$I_p(y) = \int_0^{y-1} f_p(x)\,\mathrm{d}x$$

where

$$f_p(x) = \sqrt{\frac{p+1}{(1+x)^{p+1}-1}} = \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1+\frac{p}{2}x + o(x)}} = \frac{1}{\sqrt{x}} - \frac{p}{4}\sqrt{x} + o(\sqrt{x}), \quad x \to 0.$$

(We used the Maclaurin polynomial of $y \mapsto (1+y)^\alpha$ for $\alpha = p+1$ and $\alpha = -\frac{1}{2}$.) So it suffices to integrate the obtained asymptotic expansion from 0 to $y-1$.

The case $p = -1$ is analogous. □

Definition 3.2. For all $s \geq 0$ set

$$p_s := -\frac{2s-1}{2s+1}.$$

Thus,

$$\{p_n\}_{n=0}^{\infty} = \big(1, -\tfrac{1}{3}, -\tfrac{3}{5}, -\tfrac{5}{7}, \dots\big),$$
$$\Big\{p_{n+\frac{1}{2}}\Big\}_{n=0}^{\infty} = \big(0, -\tfrac{1}{2}, -\tfrac{2}{3}, -\tfrac{3}{4}, \dots\big).$$

The integral I_p can be explicitly calculated for these values.

Theorem 3.3. *Let $n \in \mathbb{N} \cup \{0\}$. Then*

(16) $$I_{p_{n+1/2}}(y) = 2\sqrt{n+1}\, \widetilde{I}_n\Big(y^{\frac{1}{n+1}} - 1\Big), \quad y \geq 1$$

where

$$\widetilde{I}_n(z) = \sqrt{z} \sum_{k=0}^{n} \frac{1}{2k+1} \binom{n}{k} z^k, \quad z \geq 0$$

and

(17) $$I_{p_n}(y) = \sqrt{2(2n+1)}\, \widehat{I}_n\Big(y^{\frac{2}{2n+1}}\Big), \quad y \geq 1$$

where

$$\widehat{I}_n(z) = \frac{(2n-1)!!}{(2n)!!}\left(\ln\left(\sqrt{z}+\sqrt{z-1}\right) + \sqrt{1-\frac{1}{z}}\sum_{k=1}^{n}\frac{(2k-2)!!}{(2k-1)!!}\,z^k\right), \quad z \geq 1.$$

(We set $(-1)!! := 1$*.)*

Proof. Using the substitution

$$\sqrt{V^{p_{n+1/2}}-1} = \sqrt{V^{\frac{1}{n+1}}-1} =: u$$

and denoting

$$\int_0^{\sqrt{z}} \left(u^2+1\right)^n \mathrm{d}u =: \widetilde{I}_n(z),$$

we obtain (16). The integral $\widetilde{I}_n(z)$ can be calculated by the binomial theorem.

By the substitutions

$$V^{p_n+1} = V^{\frac{2}{2n+1}} =: \frac{1}{\cos^2 v}, \quad v \in \left[0, \tfrac{\pi}{2}\right), \qquad \sin v =: u$$

we obtain (17) with

$$\widehat{I}_n(z) = \int_0^{\sqrt{1-\frac{1}{z}}} \frac{\mathrm{d}u}{(1-u^2)^{n+1}}.$$

Integrating $\widehat{I}_n(z)$ by parts, we can derive the recurrent relation

$$\widehat{I}_n(z) = \frac{2n-1}{2n}\left(\widehat{I}_{n-1}(z) + \frac{1}{2n-1}\sqrt{1-\frac{1}{z}}\,z^n\right),$$

from which the formula in the theorem follows. □

We will also use the following special cases of (17) and (16):

$$\begin{aligned} I_1(y) &= \sqrt{2}\ln\left(y+\sqrt{y^2-1}\right),\\ I_0(y) &= 2\sqrt{y-1},\\ I_{-1/2}(y) &= \frac{2\sqrt{2}}{3}\sqrt{\sqrt{y}-1}\left(\sqrt{y}+2\right). \end{aligned} \tag{18}$$

Lemma 3.4. *For* $k \in \mathbb{N}\cup\{0\}$ *and* $p \in (-1,\infty)\setminus\{p_k\}$ *put*

$$b_k(p) := \frac{(2k-1)!!}{(2k)!!}\,\frac{2}{(2k+1)(p-p_k)} = \frac{(2k-1)!!}{(2k)!!}\,\frac{1}{\frac{p-1}{2}+k(p+1)}$$

and for $p > -1$ *set*

$$B_p := \sum_{\substack{k\in\mathbb{N}\cup\{0\}\\ p_k\neq p}} b_k(p) \in \mathbb{R}.$$

Then the following holds for $y \to \infty$*:*

(i) *If* $p > 1$*, then*

$$\frac{I_p(y)}{\sqrt{p+1}} = B_p + o(1).$$

(ii) *If* $p_{n+1} < p < p_n$ *for some* $n \in \mathbb{N} \cup \{0\}$, *then*

$$\frac{I_p(y)}{\sqrt{p+1}} = \sum_{k=0}^{n} \underbrace{(-b_k(p))}_{>0} y^{\overbrace{\tfrac{1-p}{2}-k(p+1)}^{}}_{} + B_p + o(1).$$

$$\frac{I_p(y)}{\sqrt{p+1}} = \sum_{k=0}^{n} \underbrace{(-b_k(p))}_{>0}\, y^{\underbrace{\frac{1-p}{2}-k(p+1)}_{>0}} + B_p + o(1).$$

(iii) *If* $p = p_n$ *for some* $n \in \mathbb{N} \cup \{0\}$, *then*

$$\frac{I_p(y)}{\sqrt{p+1}} = \sum_{k=0}^{n-1} \underbrace{(-b_k(p))}_{>0}\, y^{\underbrace{\frac{1-p}{2}-k(p+1)}_{>0}} + \frac{(2n-1)!!}{(2n)!!} \ln y + B_p + o(1).$$

Furthermore, the function $p \mapsto B_p$ *belongs to* C^∞ *on each of intervals* (p_0, ∞), (p_1, p_0), (p_2, p_1), $\ldots$ *and decreases on each of them while*

$$\lim_{p \to p_0+} B_p = \infty, \quad \lim_{p\to\infty} B_p = 0$$

and for all $n \in \mathbb{N}$ *we have:*

$$\lim_{p \to p_{n+1}+} B_p = \infty, \quad B_{p_{n+1/2}} = 0, \quad \lim_{p\to p_n-} B_p = -\infty.$$

Proof. It consists of

1. expressing $I_p(y)$ as the sum of a series (see (19)),
2. proving the finiteness of B_p and verifying statements (i), (ii), (iii)
3. and examining the properties of the function $p \mapsto B_p$.

1. Let $p > -1$ and $y \geq 1$. The substitution $V := x^{-1/(p+1)}$ gives:

$$\frac{I_p(y)}{\sqrt{p+1}} = \frac{1}{p+1} \int_{1/y^{p+1}}^{1} \frac{1}{\sqrt{1-x}} x^{-\frac{1}{2}-\frac{1}{p+1}}\, \mathrm{d}x.$$

Using the Maclaurin series of the function $x \mapsto 1/\sqrt{1-x}$, we get that

$$\frac{I_p(y)}{\sqrt{p+1}} = \frac{1}{p+1} \int_{1/y^{p+1}}^{1} \left(\sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} x^{k-\frac{1}{2}-\frac{1}{p+1}} \right) \mathrm{d}x.$$

Levi's monotone convergence theorem allows us to exchange the order of integration and summation, resulting in

$$\frac{I_p(y)}{\sqrt{p+1}} = \sum_{k=0}^{\infty} a_{k,p}(y) \tag{19}$$

where

$$a_{k,p}(y) = \begin{cases} b_k(p)\Big(1 - y^{\frac{1-p}{2}-k(p+1)}\Big) & \text{if } p \neq p_k, \\ \dfrac{(2k-1)!!}{(2k)!!} \ln y & \text{if } p = p_k. \end{cases}$$

2. It is obvious that for all $k \in \mathbb{N} \cup \{0\}$ and $p > -1$, $a_{k,p}$ is increasing, positive on $(1, \infty)$ and

$$\lim_{y\to\infty} a_{k,p}(y) = \begin{cases} b_k(p) & \text{if } p > p_k, \\ \infty & \text{if } p \le p_k. \end{cases} \tag{20}$$

Now let $m \in \mathbb{N} \cup \{0\}$ and $p > p_m$. Stirling's formula ($n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n$ for $n \to \infty$) implies that

$$b_k(p) \sim \frac{1}{\sqrt{\pi}(p+1)k^{3/2}}, \quad k \to \infty,$$

which guarantees the convergence of $\sum_{k=m}^{\infty} b_k(p)$ (and also the finiteness of B_p). We are going to prove that

$$\lim_{y\to\infty} \sum_{k=m}^{\infty} a_{k,p}(y) = \sum_{k=m}^{\infty} b_k(p) \tag{21}$$

because statement (i) follows from (19) and (21) with $m = 0$ while statements (ii), (iii) from (19) and (21) with $m = n + 1$.

The inequality "$\le$" in (21) is clear from (20) and the increase of $a_{k,p}$. In order to prove the opposite inequality, let us choose any $\varepsilon > 0$. We have that

$$\sum_{k=m}^{n_0} b_k(p) > \sum_{k=m}^{\infty} b_k(p) - \frac{\varepsilon}{2}$$

for some $n_0 \ge m$. The positivity of $a_{k,p}$ on $(1, \infty)$ together with (20) yields that there exists a number $K > 1$ such that

$$\sum_{k=m}^{\infty} a_{k,p}(y) > \sum_{k=m}^{n_0} a_{k,p}(y) > \sum_{k=m}^{n_0} b_k(p) - \frac{\varepsilon}{2}$$

for all $y > K$. Joining the last two inequalities, we obtain (21).

3. The decrease of $p \mapsto B_p$ on intervals (p_0, ∞), (p_1, p_0), (p_2, p_1), ... follows immediately from the decrease of functions b_k on these intervals.

Let us now prove that $(p \mapsto B_p) \in C^\infty((-1, \infty) \setminus \bigcup_{n=0}^{\infty}\{p_n\})$. We will use the C^∞-smoothness of functions b_k. If we choose arbitrary $m, n \in \mathbb{N} \cup \{0\}$ and $[\alpha, \beta] \subseteq (p_n, \infty)$, then applying the Weierstraß criterion, we can verify that $\sum_{k=n}^{\infty}(b_k)^{(m)}$ converges uniformly on $[\alpha, \beta]$, therefore we can differentiate it term by term. So the sum of $\sum_{k=n}^{\infty} b_k$ belongs to $C^\infty([\alpha, \beta])$, thus also to $C^\infty((p_n, \infty))$, from which the C^∞-smoothness of the function $p \mapsto B_p$ on $(-1, \infty) \setminus \bigcup_{n=0}^{\infty}\{p_n\}$ follows.

The one-sided limits of $p \mapsto B_p$ in p_0, p_1, ... are found easily. They—together with its continuity and decrease on (p_{n+1}, p_n)—guarantee the existence of a unique point $p_n^* \in (p_{n+1}, p_n)$ such that $B_{p_n^*} = 0$. Statement (ii) gives the expansion

$$I_{p_{n+\frac{1}{2}}}(y) = 2\sqrt{n+1} \sum_{k=0}^{n} \frac{1}{2n-2k+1} \frac{(2k-1)!!}{(2k)!!} \left(y^{\frac{1}{n+1}}\right)^{\frac{1}{2}+n-k} + \frac{B_{p_{n+1/2}}}{\sqrt{n+1}} + o(1)$$

for $y \to \infty$. On the other hand, from (16), using the binomial theorem and the Maclaurin polynomial of $x \mapsto \sqrt{1+x}$ of degree n, we obtain that

$$I_{p_{n+\frac{1}{2}}}(y) = \sqrt{z} \cdot 2\sqrt{n+1}\sqrt{1-\frac{1}{z}\sum_{i=0}^{n}\frac{1}{2i+1}\binom{n}{i}(z-1)^i}$$
$$= \sum_{k=0}^{n} c_{n,k} z^{\frac{1}{2}+n-k} + O\left(\frac{1}{\sqrt{z}}\right)$$

for $z = y^{1/(n+1)} \to \infty$ and some constants $c_{n,k}$, $k = 0, 1, \ldots n$. Consequently, $p_n^* = p_{n+1/2}$.

Finally, in order to find $\lim_{p\to\infty} B_p$, we employ the uniform convergence of $\sum_{k=0}^{\infty} b_k$ on (α, ∞) for $\alpha > 1$, and so we exchange the order of the limit and the sum. □

The following assertions will be needed only in the proofs of Lemmata 8.7 and 8.8.

Theorem 3.5. *The mapping $(y, p) \mapsto I_p(y)$ is continuous on $[1, \infty) \times \mathbb{R}$. Furthermore, $p \mapsto I_p(y)$ is decreasing on $\mathbb{R}$ for any $y > 1$.*

Proof. Let us express $I_p(y)$ as

$$I_p(y) = \int_1^y f(V, p)\,\mathrm{d}V$$

where

$$f(V, p) = \begin{cases} \sqrt{\dfrac{p+1}{V^{p+1}-1}} & \text{if } p \neq -1,\ V > 1, \\ \dfrac{1}{\sqrt{\ln V}} & \text{if } p = -1,\ V > 1. \end{cases}$$

Function f is continuous in both variables and is decreasing in V, consequently it is continuous (on $(1, \infty) \times \mathbb{R}$). Similarly, if we prove the continuity of $p \mapsto I_p(y)$ for all $y > 1$ (for $y = 1$ it is evident), then using the continuity and increase of I_p for any $p \in \mathbb{R}$, we will have that $(y, p) \mapsto I_p(y)$ is continuous.

For this purpose, it will be important to know the behaviour of $f(V, \cdot)$. We can derive that for any $p \neq -1$ and $V > 1$:

$$\frac{\partial}{\partial p}\frac{1}{f^2(V,p)} > 0 \quad \Longleftrightarrow \quad \ln V^{p+1} + \frac{1}{V^{p+1}} - 1 > 0,$$

which can be equivalently written as $\ln x < x - 1$ for $x := 1/V^{p+1} \in (0, 1) \cup (1, \infty)$. Thus, $1/^2(V, \cdot)$ is increasing on $\mathbb{R}$, therefore $f(V, \cdot)$ is decreasing and the second assertion of the lemma holds.

Now choose arbitrary $y > 1$, $p_0 \in \mathbb{R}$. Since $f(\cdot, p_0)$ is an integrable majorant of $\{f(\cdot, p)\}_{p \geq p_0}$ and $f(V, \cdot)$ is continuous, we have the continuity of $p \mapsto I_p(y)$ on $[p_0, \infty)$. □

Lemma 3.6. *For every $y > 1$, $n \in \mathbb{N}$:*

$$I_{-1}(y) = \sum_{k=0}^{n-1} \frac{(2k-1)!!}{2^k} \frac{y}{\ln^{k+1/2} y} + O\left(\frac{y}{\ln^{n+1/2} y}\right), \qquad y \to \infty.$$

Proof. Set

$$\overline{I}_n(y) := \int_{\mathrm{e}}^{y} \frac{\mathrm{d}V}{\ln^{n+1/2} V}$$

for all $N \in \mathbb{N} \cup \{0\}$ and $y > 1$. Integrating by parts, we can derive the recurrent relation

$$\overline{I}_n(y) = \frac{y}{\ln^{n+1/2} y} - \mathrm{e} + \frac{2n+1}{2} \overline{I}_{n+1}(y).$$

Using it n times, we obtain

$$I_{-1}(y) = \overline{I}_0(y) + \int_1^{\mathrm{e}} \frac{\mathrm{d}V}{\sqrt{\ln V}} = \sum_{k=0}^{n-1} \frac{(2k-1)!!}{2^k} \frac{y}{\ln^{k+1/2} y} + R_n(y)$$

where

$$R_n(y) = \int_1^{\mathrm{e}} \frac{\mathrm{d}V}{\sqrt{\ln V}} - \sum_{k=0}^{n-1} \frac{(2k-1)!!}{2^k} \mathrm{e} + \frac{(2n-1)!!}{2^n} \overline{I}_n(y) \sim \frac{(2n-1)!!}{2^n} \frac{y}{\ln^{n+1/2} y}$$

for $y \to \infty$, which can be proved using l'Hôpital's rule. □

Notice that although Lemma 3.6 gives an asymptotic expansion, the corresponding series

$$\sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k} \frac{y}{\ln^{k+1/2} y}$$

diverges for all $y > 1$.

4. Case I ($p = -1$, $q = 0$)

This case is the simplest one since from Lemma 2.5 it directly follows that

$$L(m) = \frac{m}{\sqrt{2a}} I_{-1}\left(\mathrm{e}^{\frac{1}{2a}}\right), \quad m > 0.$$

Thus, the time map, which determines the relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$, is linear. So substituting into Lemma 2.8, we obtain the following theorem:

Theorem 4.1. *Assume $p = -1$, $q = 0$, $a > 0$. Then for arbitrary $l > 0$:*

$$\mathcal{S}(l) = \left\{ u_{m,-1,a}\big|_{[-l,l]} \ : \ m = \frac{\sqrt{2a}}{I_{-1}\left(\mathrm{e}^{\frac{1}{2a}}\right)} l \right\},$$
$$\mathcal{N}(l) = \emptyset.$$

5. Case II ($p > -1,\ q = 0$)

In this section we answer the question of the solvability of (1) for

$$p > -1,\ q = 0,\ a > 0 \tag{22}$$

finding $\lim_{m\to 0} L(m)$, $\lim_{m\to\infty} L(m)$ and proving the monotonicity of L. However, let us first summarise the properties of R that will be used in the subsequent lemmata.

Lemma 5.1. *Let* (22) *hold. Then* $R' > 0$ *and*

$$\lim_{m\to 0} R(m) = \left(\frac{p+1}{2a}\right)^{\frac{1}{p+1}},$$

$$R(m) = m\left(1 + \frac{1}{2am^{p+1}} + o\left(\frac{1}{m^{p+1}}\right)\right), \quad m \to \infty.$$

Proof. It suffices to use the explicit formula for $R(m)$ given by Lemma 2.5. □

Lemma 5.2. *Assume* (22). *Then*

$$\lim_{m\to 0} L(m) = \begin{cases} \infty & \text{if } p \ge 1,\\ \dfrac{2}{1-p}\left(\dfrac{p+1}{2a}\right)^{\frac{1}{p+1}} =: L_{p,0,a}(0) =: L(0) & \text{if } p \in (-1,1),\end{cases}$$

$$\lim_{m\to\infty} L(m) = \begin{cases} 0 & \text{if } p > 0,\\ \dfrac{1}{a} & \text{if } p = 0,\\ \infty & \text{if } p \in (-1,0).\end{cases}$$

Proof. For $p > 1$ and $p = 1$, $\lim_{m\to 0} L(m)$ is easily found using Lemma 5.1 and (5). In the case of $p \in (-1,1)$, it is of type $\frac{\infty}{\infty}$:

$$\lim_{m\to 0} L(m) = \lim_{m\to 0} \frac{I_p\left(\frac{R(m)}{m}\right)}{\sqrt{2a}m^{\frac{p-1}{2}}}$$

and we calculate it by l'Hôpital's rule, (12) and Lemma 5.1.

According to Lemmata 5.1 and 3.1:

$$L(m) \sim \sqrt{\frac{2}{a}}\, m^{\frac{1-p}{2}} \sqrt{\frac{R(m)}{m} - 1}, \quad m \to \infty$$

while

$$\frac{R(m)}{m} - 1 \sim \frac{1}{2am^{p+1}}, \quad m \to \infty.$$

Connecting these two expansions, we obtain that $L(m) \sim \frac{1}{am^p}$ for $m \to \infty$ and the second assertion follows. □

Lemma 5.3. *Let* (22) *hold. Then:*

(i) *If* $p > 0$, *then* $L' < 0$.

(ii) *If* $p = 0$, *then* $L \equiv \frac{1}{a}$.

(iii) *If* $-1 < p < 0$, *then* $L' > 0$.

Proof.

(i) Firstly, let us consider $p > 0$. Due to (13), the case $p \geq 1$ is clear. So let $0 < p < 1$. If L has a stationary point $m_0 > 0$, then $L''(m_0) > 0$ according to (14) and Lemma 5.1, thus it is a point of strict relative minimum. Therefore, either L has no stationary point or it has exactly one, which is a point of global minimum. However, the second possibility contradicts the fact that $\lim_{m\to\infty} L(m) = 0$ (Lemma 5.2).

(ii) For $p = 0$, Lemma 2.5 gives the formula $R(m) = m + \frac{1}{2a}$, so $L(m) = \frac{1}{a}$ according to (18).

(iii) Finally, let us have $p \in (-1, 0)$ and let us proceed as for $p \in (0, 1)$. Now L attains a strict relative maximum in each of its stationary points. On the other hand, $\lim_{m\to\infty} L(m) = \infty$ so the only possibility is that $L' > 0$ on $(0, \infty)$. □

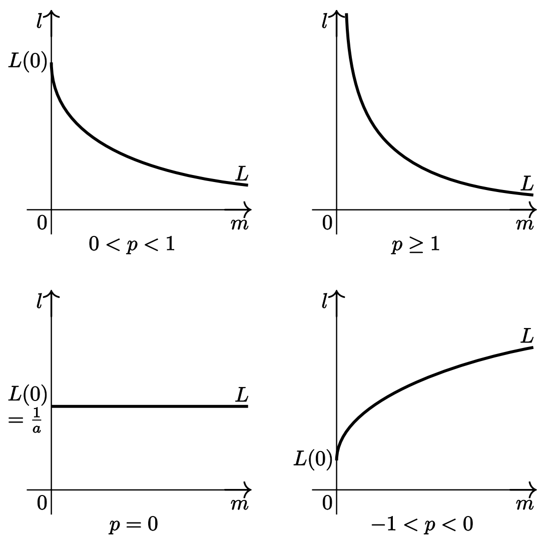

Figure 2. The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case II ($p > -1$, $q = 0$, $a > 0$) according to Lemmata 2.8, 5.2 and 5.3. See also Theorem 5.4.

From the results of the last two lemmata (which are summarised in Figure 2), applying Lemma 2.8, we obtain the main statement of this section:

Theorem 5.4. *Assume* (22) *and* $l > 0$. *Then* $\mathcal{N}(l) = \emptyset$ *and the following holds for positive symmetric solutions of* (1)*:*

If $p \geq 1$, *then* $|\mathcal{S}(l)| = 1$ *and* L *is decreasing. (Recall that* $L(u(0)) = l$ *for any* $u \in \mathcal{S}(l)$*.)*

If $p=0$, then (1) *has a solution only for $l=\frac{1}{a}$, namely*

$$\mathcal{S}\left(\frac{1}{a}\right)=\left\{x\mapsto\frac{a}{2}x^2+m,\ x\in[-l,l]\ :\ m>0\right\}.$$

If $p<1$ and $p\neq 0$, then

$$|\mathcal{S}(l)|=\begin{cases}1 & \textit{if } l \textit{ is between } L(0) \textit{ and } \lim_{m\to\infty}L(m),\\ 0 & \textit{otherwise}\end{cases}$$

and L is strictly monotone. (See Lemma 5.2 about $L(0)$ and $\lim_{m\to\infty}L(m)$.)

The last question we will answer in this section is whether $L_{\cdot,0,a}(0)$ is monotone.

Lemma 5.5. *Suppose that* (22) *holds, let $\overline{p}$ be the unique solution of the equation $p^3-7p-2=0$ in $(-1,0)$ and set*

$$\overline{a}:=\frac{\overline{p}+1}{2}\mathrm{e}^{\frac{2}{3-\overline{p}}-2}\in\left(\frac{1}{2\mathrm{e}^2},\frac{1}{\mathrm{e}}\right).$$

Then:

(i) *If $a>\overline{a}$, then $\frac{\partial}{\partial p}L_{p,0,a}(0)>0$ for $p\in(-1,1)$.*

(ii) *If $a=\overline{a}$, then $\frac{\partial}{\partial p}L_{p,0,a}(0)>0$ for $p\in(-1,1)\setminus\{\overline{p}\}$ and $\frac{\partial}{\partial p}L_{p,0,a}(0)|_{p=\overline{p}}=0$.*

(iii) *If $0<a<\overline{a}$, then $p\mapsto L_{p,0,a}(0)$ has two stationary points: $p_1=p_1(a)\in(-1,\overline{p})$ and $p_2=p_2(a)\in(\overline{p},1)$ while $\frac{\partial}{\partial p}L_{p,0,a}(0)>0$ for $p\in(-1,p_1)\cup(p_2,1)$ and $\frac{\partial}{\partial p}L_{p,0,a}(0)<0$ for $p\in(p_1,p_2)$.*

Furthermore, for all $a>0$ we have

$$\lim_{p\to-1+}L_{p,0,a}(0)=0,\qquad \lim_{p\to1-}L_{p,0,a}(0)=\infty.$$

Proof. The limits of $L_{p,0,a}(0)$ can be easily calculated. We also have that

$$\frac{\partial}{\partial p}L_{p,0,a}(0)>0\quad\Longleftrightarrow\quad \ln\frac{p+1}{2a}-\frac{(p+1)^2}{1-p}-1=:\psi_a(p)<0.$$

So we need to examine the properties of ψ_a. It is not difficult to derive that

$$\psi_a'(p)>0\quad\Longleftrightarrow\quad p^3-7p-2=:\omega(p)>0.$$

Since ω is decreasing on $(-1,1)$ and $\omega(0)<0<\lim_{p\to-1}\omega(p)$, it has a unique zero $\overline{p}\in(-1,0)$. It means that ψ_a increases on $(-1,\overline{p}]$ and decreases on $[\overline{p},1)$. However, $\lim_{p\to-1+}\psi_a(p)=\lim_{p\to1-}\psi_a(p)=-\infty$, thus $L_{\cdot,0,a}(0)$ has the properties from parts (i), (ii) or (iii) if $\psi_a(\overline{p})<0$, $\psi_a(\overline{p})=0$ or $\psi_a(\overline{p})>0$, respectively.

Using that $\omega(\overline{p})=0$, we obtain:

$$\psi_a(\overline{p})=\ln\frac{\overline{p}+1}{2a}+\frac{2}{3-\overline{p}}-2=0\quad\Longleftrightarrow\quad a=\overline{a}.$$

Furthermore, $a\mapsto\psi_a(\overline{p})$ is decreasing, so really $\psi_a(\overline{p})<0$ for $a>\overline{a}$ and $\psi_a(\overline{p})>0$ for $a\in(0,\overline{a})$. It remains to check that $\overline{a}\in(\frac{1}{2\mathrm{e}^2},\frac{1}{\mathrm{e}})$. However, it can be directly proved that $\psi_a<0$ for $a\geq\frac{1}{\mathrm{e}}$, so $\overline{a}<\frac{1}{\mathrm{e}}$ and $\psi_a(0)\geq0$ and consequently $\psi_a(\overline{p})>0$ for $a\leq\frac{1}{2\mathrm{e}^2}$, so $\overline{a}>\frac{1}{2\mathrm{e}^2}$. □

Let us mention that $\overline{p} \approx -0.289$ and using Cardano's formula one can also derive that

$$\overline{p} = 2\sqrt{\frac{7}{3}} \cos \frac{\arccos \frac{3\sqrt{3}}{7\sqrt{7}} - 2\pi}{3}.$$

6. Case III ($p > -1,\ 0 < q < \frac{p+1}{2}$)

A part of case III was already examined in [**5**] (see Lemma 6.2). For the rest we will need the asymptotic expansions of $R(m)$ for $m \to 0$ and $m \to \infty$ (Lemma 6.1) and also Lemma 3.4. We will deal only with

$$p > -1,\ 0 < q < \frac{p+1}{2},\ a > 0. \tag{23}$$

Lemma 6.1. *Let* (23) *hold. Then* $R' > 0$ *and*

$$\frac{R(m)}{R(0)} = 1 - \frac{m^{p+1}}{(2q-p-1)R^{p+1}(0)} + o\big(m^{p+1}\big), \qquad m \to 0$$

$$\frac{R(m)}{m} = 1 + \frac{1}{2a}m^{2q-p-1} + \frac{4q-p}{8a^2}m^{2(2q-p-1)} + o\big(m^{2(2q-p-1)}\big), \qquad m \to \infty$$

where

$$R(0) = R_{p,q,a}(0) = \lim_{m\to 0} R(m) = \left(\frac{2a}{p+1}\right)^{\frac{1}{2q-p-1}}.$$

Proof. It is clear from (10) and Lemma 2.5 (i) that $R' > 0$, so R has a positive and finite limit (denoted by $R(0)$) at 0, the value of which can be obtained from the equality

$$0 = \lim_{m\to 0} \mathcal{F}(m, R(m)) = \frac{R^{p+1}(0)}{2a}\left(R^{2q-p-1}(0) - \frac{2a}{p+1}\right).$$

Now we will look for such $c, d > 0$ that

$$\frac{R(m)}{R(0)} - 1 \sim cm^d, \quad m \to 0.$$

So let us calculate the following limit using l'Hôpital's rule and (10):

$$\lim_{m\to 0} \frac{\frac{R(m)}{R(0)} - 1}{m^d} = -\frac{p+1}{(2q-p-1)dR^{p+1}(0)} \lim_{m\to 0} m^{p+1-d}.$$

It should be positive and finite, determining the value of c. Therefore, we have $d = p+1$ and c is also given as in the lemma.

The decrease of $m \mapsto R(m)/m \geq 1$ (see (11)) guarantees the existence of its positive and finite limit at ∞. So we can use l'Hôpital's rule and (10) to derive that

$$A := \lim_{m\to\infty} \frac{R(m)}{m} = \lim_{m\to\infty} \left(\frac{m}{R(m)}\right)^p = \frac{1}{A^p}.$$

Consequently, $A = 1$. The asymptotic expansion of $R(m)/m$ for $m \to \infty$ can be also found by the method of undetermined coefficients, which we used for $m \to 0$. However, let us show an iterative method borrowed from [**5**, proof of Lemma 3.3]:

Multiplying the equality $\mathcal{F}(m, R(m)) = 0$ (see (7)) by $(p+1)/m^{p+1}$ and expressing $R(m)/m$ from it, we obtain:

$$\frac{R(m)}{m} = \left(1 + \frac{p+1}{2a} m^{2q-p-1} \left(\frac{R(m)}{m}\right)^{2q}\right)^{\frac{1}{p+1}}. \tag{24}$$

The expression $(R(m)/m)^{2q}$ on the right-hand side can be replaced by $1 + o(1)$, so

$$\frac{R(m)}{m} = \left(1 + \frac{p+1}{2a} m^{2q-p-1} + o\big(m^{2q-p-1}\big)\right)^{\frac{1}{p+1}} = 1 + \frac{1}{2a} m^{2q-p-1} + o\big(m^{2q-p-1}\big)$$

(We used the Maclaurin polynomial of $x \mapsto (1+x)^{1/(p+1)}$.) Now let us insert the asymptotic expansion we have just obtained in the right-hand side of (24) again. It yields

$$\frac{R(m)}{m} = \left(1 + \frac{p+1}{2a} m^{2q-p-1} + \frac{(p+1)q}{2a^2} m^{2(2q-p-1)} + o\big(m^{2(2q-p-1)}\big)\right)^{\frac{1}{p+1}},$$

which can be rewritten in the form from the lemma.

Let us remark that we could use this iterative method in the case of $m \to 0$ as well. We only would replace (24) by

$$R(m) = R(0)\left(1 - \frac{m^{p+1}}{R^{p+1}(m)}\right)^{\frac{1}{2q-p-1}},$$

which can be derived from the equality $\mathcal{F}(m, R(m)) = 0$ multiplying it by $(p+1)/R^{p+1}(m)$. ☐

Lemma 6.2 (for $p, q > 1$ see [**5**, Theorem 3.1])**.** *If* (23) *holds and* $p \geq 1$*, then*

$$\lim_{m \to 0} L(m) = \infty, \qquad L' < 0 \text{ on } (0, \infty), \qquad \lim_{m \to \infty} L(m) = 0.$$

Proof. The proof from [**5**] for $p, q > 1$ is also valid for $p > 1$ and the case $p = 1$ is similar. ☐

In the next two lemmata we find the limits of L—denoted by $L(0)$ and $L(\infty)$—for $p < 1$. For the proof of Lemma 6.5 it is also necessary to know the sign of $L - L(0)$ and $L - L(\infty)$ near 0 and ∞, respectively, for certain values of p, q.

Lemma 6.3. *Assume* (23) *and* $p < 1$*. Then*

$$\lim_{m \to 0} L(m) = \frac{2}{1-p} \left(\frac{p+1}{2a}\right)^{\frac{q-1}{2q-p-1}} =: L_{p,q,a}(0) =: L(0)$$

and furthermore, $L > L(0)$ *in some neighbourhood of* 0 *for* $-\frac{1}{3} < p \leq 0$ *and* $L < L(0)$ *in some neighbourhood of* 0 *for* $0 < p < 1$.

Proof. The $\lim_{m \to 0} L(m)$ is found the in same way as in Lemma 5.2. So choose any $p \in (-\frac{1}{3}, 1)$ and let us calculate the second term of the asymptotic expansion of

$L(m)$ for $m \to 0$, which will allow us to determine whether $L < L(0)$ or $L > L(0)$ near 0. Lemma 6.1 yields:

$$R(m) = R(0)\big(1 + O\big(m^{p+1}\big)\big) = R(0)\Big(1 + o\Big(m^{\frac{1-p}{2}}\Big)\Big).$$

Joining it with the expansion of $I_p(y)$ from Lemma 3.4, we obtain:

$$L(m) = L(0) + \sqrt{\frac{p+1}{2a}} B_p m^{\frac{1-p}{2}} + o\Big(m^{\frac{1-p}{2}}\Big).$$

As we know, $B_p > 0$ for $p \in (-\frac{1}{3}, 0)$ and $B_p < 0$ for $p \in (0,1)$, guaranteeing the validity of the statement of the lemma for these values of p.

It remains to examine $p = 0$. In that case we can use (18). So

$$L(m) = L(0)\sqrt{1 + \frac{2q}{1-2q}(2a)^{\frac{1}{1-2q}} m + o(m)} = L(0) + \underbrace{\frac{2q}{1-2q}(2a)^{\frac{q}{1-2q}}}_{>0} m + o(m)$$

due to Lemma 6.1. □

Lemma 6.4. *If* (23) *holds and* $p < 1$, *then*

$$\lim_{m\to\infty} L(m) = \begin{cases} 0 & \text{if } q < p, \\ \frac{1}{a} & \text{if } q = p, \\ \infty & \text{if } q > p \end{cases}$$

and furthermore, $L > \frac{1}{a}$ *in some neighbourhood of* ∞ *for* $q = p$.

Proof. The proof of the first statement does not differ from that of Lemma 5.2. So let $q = p$ and join the expansions of Lemmata 3.1 and 6.1 for $m \to \infty$:

$$\begin{aligned} L(m) &= \frac{1}{a}\sqrt{1 + \frac{3p}{4a} m^{p-1} + o\big(m^{p-1}\big)}\Big(1 - \frac{p}{24a} m^{p-1} + o\big(m^{p-1}\big)\Big) \\ &= \frac{1}{a} + \frac{p}{3a^2} m^{p-1} + o\big(m^{p-1}\big). \end{aligned}$$

Since $p \in (0,1)$ and hence $\frac{p}{3a^2} > 0$, $L > \frac{1}{a}$ near ∞ indeed. □

Lemma 6.5. *Suppose that* (23) *holds and for* $q > |p|$ *set*

$$\overline{m} := \overline{m}_{p,q,a} := \left(\frac{(p+q)(2q-p-1)}{2q(q-1)}\right)^{\frac{1}{p+1}} \left(\frac{a(q-p)}{q(1-q)}\right)^{\frac{1}{2q-p-1}}.$$

(i) *If* $p < 1$, $q \le p$, *then* $L' < 0$ *on* $(0,\infty)$.

(ii) *If* $p > 0$, $q > p$, *then* L *has a stationary point* $m_{0;p,q,a} =: m_0 \in (0, \overline{m}]$ *while* $L' < 0$ *on* $(0, m_0)$, $L' > 0$ *on* (m_0, ∞).

(iii) *If* $p \le 0$, $q > -p$, *then* $L' > 0$ *on* $(0,\infty) \setminus \{\overline{m}\}$.

(iv) *If* $q \le -p$, *then* $L' > 0$ *on* $(0,\infty)$.

Proof. It is similar to the proof of Lemma 5.3. So suppose that $m_0 > 0$ is a stationary point of L. From (14) it is clear that $L''(m_0)$ has the same sign as

$$(1-q)\frac{q}{a}R^{2q-p-1}(m_0) + p - q =: \varrho_{p,q,a}(m_0) =: \varrho(m_0).$$

Therefore, if $q \le p$, then L has at most one stationary point and if it has some, then it attains a strict relative minimum there. However, L cannot increase near ∞ (see Lemma 6.4), thus statement (i) holds.

In the rest of the proof we will deal with $q > p$. We have

$$L''(m_0) > 0 \iff R(m_0) < \left(\frac{a(q-p)}{q(1-q)}\right)^{\frac{1}{2q-p-1}} =: \overline{R}_{p,q,a} =: \overline{R}$$

and

$$\overline{R} > R(0) \iff (2q-p-1)(p+q) < 0 \iff q > -p.$$

Since $(R(0), \infty)$ is the range of R, each stationary point of L is a point of strict relative maximum for $q \le -p$ and statement (iv) follows due to Lemma 6.4.

We will suppose $q > -p$ from now on (together with $q > p$), thus $-\frac{1}{3} < p < 1$. Consequently,

$$L''(m_0) > 0 \iff m_0 < R^{-1}(\overline{R}) = \overline{R}\left(1 - \frac{p+1}{2a}\overline{R}^{2q-p-1}\right)^{\frac{1}{p+1}} = \overline{m}.$$

So Lemma 6.4 guarantees that L does not attain any relative extremum in $(\overline{m}, \infty)$. Furthermore, if $p \le 0$, then no point of relative extremum lies in $(0, \overline{m})$ as well (see Lemma 6.3), as it is stated in (iii). On the other hand, if $p > 0$, then a similar consideration shows that L has exactly one relative extremum, which is a global minimum attained at some point $m_0 \in (0, \overline{m}]$ and in case of $m_0 < \overline{m}$, $\overline{m}$ may be a stationary point of L as well. In order to complete the verification of statement (ii), let us show that L cannot have two stationary points for $0 < p < 1$, $q > p$: From Lemma 2.11 we see that $K'(m)$ has the opposite sign to $\varrho(m)$ for any $m > 0$. Consequently K decreases on $(0, \overline{m}]$. However, if L had a relative minimum at some point $m_0 \in (0, \overline{m})$ and $\overline{m}$ were another stationary point of L, we would have $K(m_0) = L(m_0) < L(\overline{m}) = K(\overline{m})$ (see Lemma 2.11), a contradiction to $K(m_0) > K(\overline{m})$. □

The properties of L ascertained in this section are summarised in Figure 3, which shows all the possible graphs of L with the corresponding sets of parameters in the (p, q)-plane, distinguished by colours. (Note that although we have not ruled out in Lemma 6.5 the possibility that $\overline{m}$ is a stationary point of L for $p \le 0$, $q > -p$, it has no influence on the number of solutions of (1).) Using Lemma 2.8 , we can state the main result of this section. Recall that $L(u(0)) = l$ for any $u \in \mathcal{S}(l)$ and see also Lemmata 6.2, 6.3, 6.4 and 6.5 concerning $L(0)$, $\lim_{m\to\infty} L(m)$ and m_0.

Theorem 6.6. *Assume* (23) *and* $l > 0$*. Then* $\mathcal{N}(l) = \emptyset$ *and the following holds for the positive symmetric solutions of* (1)*:*

If $p > 0$ and $q > p$, then

$$|\mathcal{S}(l)| = \begin{cases} 2 & \text{if } l \in (L(m_0), L(0)), \\ 1 & \text{if } l \in \{L(m_0)\} \cup [L(0), \infty), \\ 0 & \text{otherwise} \end{cases}$$

and L decreases on $(0, m_0]$ and increases on $[m_0, \infty)$, see Figure 3.

In all the other cases,

$$|\mathcal{S}(l)| = \begin{cases} 1 & \text{if } l \text{ is between } L(0) \text{ and } \lim_{m\to\infty} L(m), \\ 0 & \text{otherwise} \end{cases}$$

and L is strictly monotone, see Figure 3.

7. Case IV ($p > -1$, $q = \frac{p+1}{2}$)

In this case we have from Lemma 2.5 that the time map is defined only for $q < a$ and is given by

$$L(m) = \frac{1}{\sqrt{2a}} I_p\Bigg(\underbrace{\left(\frac{a}{a-q}\right)^{\frac{1}{2q}}}_{=:r_{q,a}}\Bigg) m^{\frac{1-p}{2}}, \quad m > 0.$$

Thus, it is a bijection of $(0, \infty)$ onto $(0, \infty)$ for $p \neq 1$ and a constant function for $p = 1$. Namely, we can use (18) to derive that

$$L_{1,1,a}(m) = \frac{1}{\sqrt{a}} \ln \frac{\sqrt{a}+1}{\sqrt{a-1}} = \frac{1}{2\sqrt{a}} \ln \left(\frac{\sqrt{a}+1}{\sqrt{a-1}}\right)^2 = \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a}+1}{\sqrt{a}-1}.$$

Furthermore, solving (3) for $p = 1$, we obtain that $u_{m,1,a}(x) = m\,\mathrm{ch}(\sqrt{a}x)$. So according to Lemma 2.8, we can state the following:

Theorem 7.1. *Let $p > -1$, $q = \frac{p+1}{2}$, $a > 0$. Then for arbitrary $l > 0$:*

$$\mathcal{S}(l) = \begin{cases} \left\{ u_{m,p,a}\big|_{[-l,l]} \;:\; m = \left(\dfrac{\sqrt{2a}}{I_p(r_{q,a})} l\right)^{\frac{2}{1-p}} \right\} & \text{if } p \neq 1,\ q < a, \\ \{x \mapsto m\,\mathrm{ch}(\sqrt{a}x), x \in [-l,l] \;:\; m > 0\} & \text{if } p = 1,\ a > 1,\ l = \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a}+1}{\sqrt{a}-1}, \\ \emptyset & \text{otherwise}, \end{cases}$$

$$\mathcal{N}(l) = \emptyset.$$

8. Case V ($p > -1$, $q > \frac{p+1}{2}$), symmetric solutions

Recall that due to Lemma 2.5, we have the following time maps in case V: $L_1 < L_2$ defined on $(0, M)$ and L defined on $\{M\}$. In this section we describe their behaviour for

$$p > -1,\ q > \frac{p+1}{2},\ a > 0. \tag{25}$$

Lemma 8.1 (for $p > 1$ see [**5**, p. 57 and Lemma 3.3])**.** *Assume* (25)*. Then* $R_1' > 0$ *while*

$$\lim_{m\to 0} \frac{R_1(m)}{m} = 1, \qquad \lim_{m\to M} R_1(m) = R(M) = \left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}$$

and $R_2' < 0$ *while*

$$\lim_{m\to 0} R_2(m) = \left(\frac{2a}{p+1}\right)^{\frac{1}{2q-p-1}} =: R_{2;p,q,a}(0) =: R_2(0), \qquad \lim_{m\to M} R_2(m) = R(M).$$

Moreover,

$$\frac{R_2(m)}{R_2(0)} = 1 - \frac{m^{p+1}}{(2q-p-1)R_2^{p+1}(0)} - \frac{2q+p}{2(2q-p-1)^2 R_2^{2(p+1)}(0)} m^{2(p+1)} + o\big(m^{2(p+1)}\big)$$

for $m \to 0$*.*

Proof. It is clear from Lemma 2.5 (iv) and (10) that $R_1' > 0$ and $R_2' < 0$. The limits of $R_1(m)$, $R_1(m)/m$ and $R_2(m)$ can be calculated in the same way as in [**5**] for $p > 1$ and the derivation of the asymptotic expansion of $R_2(m)$ for $m \to 0$ does not differ from that of $R(m)$ for $m \to 0$ and $m \to \infty$ in the proof of Lemma 6.1. □

Definition 8.2. For p, q, a satisfying (25) and $q < |p|$ set

$$\overline{m} := \overline{m}_{p,q,a} := \left(\frac{(p+q)(2q-p-1)}{2q(q-1)}\right)^{\frac{1}{p+1}} \left(\frac{a(p-q)}{q(q-1)}\right)^{\frac{1}{2q-p-1}}.$$

Lemma 8.3 (for $p > 1$ see [**5**, Lemmata 3.1, 3.4, 3.3, 3.2 and 3.5])**.** *If* (25) *holds, then*

$$\lim_{m\to M} L_1(m) = L(M), \qquad \lim_{m\to M} L_1'(m) = \infty,$$

$$\lim_{m\to 0} L_1(m) = \begin{cases} 0 & \text{if } q > p, \\ \frac{1}{a} & \text{if } q = p, \\ \infty & \text{if } q < p \end{cases}$$

and the following holds concerning the monotonicity of L_1*:*

(i) *If* $q \geq p$*, then* $L_1' > 0$*.*

(ii) *If* $q < p$*, then there exists such a point* $m_{0;p,q,a} =: m_0 \in [\overline{m}, M)$ *that*

$$L_1' < 0 \text{ on } (0, m_0), \quad L_1' > 0 \text{ on } (m_0, M).$$

Proof. It does not differ from the proof that can be found in [**5**] for $p, q > 1$. □

Lemma 8.4 (for $p > 1$ see [**5**, Lemmata 3.1, 3.4 and 3.3])**.** *If* (25) *holds, then*

$$\lim_{m\to M} L_2(m) = L(M), \qquad \lim_{m\to M} L_2'(m) = -\infty,$$

$$\lim_{m\to 0} L_2(m) = \begin{cases} \infty & \text{if } p \geq 1, \\ \dfrac{2}{1-p}\left(\dfrac{p+1}{2a}\right)^{\frac{q-1}{2q-p-1}} =: L_{2;p,q,a}(0) =: L_2(0) & \text{if } p \in (-1,1). \end{cases}$$

Proof. The limits at M can be calculated in the same way as it was done in [**5**] for $p, q > 1$ while the proof of the second part of the lemma is essentially the same as that of Lemma 5.2. □

Lemma 8.5. *Suppose that* (25) *holds. Then*

(i) *if* $0 \le p < 1$ *or* $q < -p$ *or* $p \ge -\frac{1}{2}$, $q = -p$, *then* $L_2 < L_2(0)$ *in some neighbourhood of* 0

(ii) *and if* $p < 0$, $q > -p$ *or* $p < -\frac{1}{2}$, $q = -p$, *then* $L_2 > L_2(0)$ *in some neighbourhood of* 0.

(We recommend the reader to draw a picture about these two sets in the (p, q)*-plane.)*

Proof. We use the asymptotic expansions of $I_p(y)$ and $R_2(m)$ from Lemmata 3.4 and 8.1, respectively and our goal is to find the second term of the asymptotic expansion of $L_2(m)$ for $m \to 0$ and to determine its sign. However, as we will see, it has eight different forms depending on the value of p and q.

All the asymptotic expansions in this proof will concern $y \to \infty$ and $m \to 0$.

1. For $-\frac{1}{3} < p < 1$ the expansion of $L_2(m)$ looks like that of $L(m)$ and is derived in the same way as in the proof of Lemma 6.3.

2. If $p = -\frac{1}{3}$, then writing $B_p + o(1)$ as $O(1)$ and $R_2(m)$ as $R_2(0)(1+O(m^{2/3}))$, we obtain:

$$\begin{aligned} L_2(m) &= \frac{1}{2}\sqrt{\frac{3}{a}} R_2^{2/3}(m) + \frac{1}{2\sqrt{3a}} m^{2/3} \ln \frac{R_2(m)}{m} + O(m^{2/3}) \\ &= L_2(0) + \frac{1}{2\sqrt{3a}} m^{2/3} \ln \frac{1}{m} + O\big(m^{2/3}\big). \end{aligned}$$

3. Now let $-1 < p < -\frac{1}{3}$. In general, we have the expansion

$$\frac{I_p(y)}{\sqrt{p+1}} = \frac{2}{1-p} y^{\frac{1-p}{2}} - \frac{1}{3p+1} y^{-\frac{3p+1}{2}} + \varrho_p(y)$$

for some function ϱ_p, which is given by different formulae depending on p and will be specified later. It can be derived from Lemma 8.1 that

$$\begin{aligned} R_2^{\frac{1-p}{2}}(m) = R_2^{\frac{1-p}{2}}(0)\Bigg(1 &- \frac{1-p}{2(2q-p-1)R_2^{p+1}(0)} m^{p+1} \\ &- \frac{(1-p)(4q+3p+1)}{8(2q-p-1)^2 R_2^{2(p+1)}(0)} m^{2(p+1)} + o\big(m^{2(p+1)}\big)\Bigg) \end{aligned}$$

and

$$R_2^{-\frac{3p+1}{2}}(m) = R_2^{-\frac{3p+1}{2}}(0)\left(1 + \frac{3p+1}{2(2q-p-1)R_2^{p+1}(0)} m^{p+1} + o\big(m^{p+1}\big)\right),$$

which yield:

$$L_2(m) = L_2(0) + C_{p,q,a} m^{p+1} + D_{p,q,a} m^{2(p+1)} + \sqrt{\frac{p+1}{2a}} m^{\frac{1-p}{2}} \varrho_p\left(\frac{R_2(m)}{m}\right) + o\big(m^{2(p+1)}\big) \tag{26}$$

where

$$C_{p,q,a} = -\frac{2(p+q)}{(3p+1)(2q-p-1)R^{p+q}_{2;p,q,a}(0)} \begin{cases} > 0 & \text{if } q > -p, \\ = 0 & \text{if } q = -p, \\ < 0 & \text{if } q < -p, \end{cases}$$

$$D_{p,q,a} = -\frac{8q+p-1}{4(2q-p-1)^2 R^{q+2p+1}_{2;p,q,a}(0)}.$$

Using that $\varrho_p(y) = o(y^{-(3p+1)/2})$ and $R_2(m) = O(1)$, we can rewrite (26) in the form

$$L_2(m) = L_2(0) + C_{p,q,a} m^{p+1} + o\big(m^{p+1}\big),$$

thus further calculation are needed for $q = -p$.

(a) Let us consider $-q = p \in (-\frac{3}{5}, -\frac{1}{3})$. Since $\varrho_p(y) = B_p + o(1)$ and $O(m^{2(p+1)}) = o(m^{(1-p)/2})$, we have

$$L_2(m) = L_2(0) + \sqrt{\frac{p+1}{2a}} B_p m^{\frac{1-p}{2}} + o\Big(m^{\frac{1-p}{2}}\Big)$$

from (26). According to Lemma 3.4, $B_p < 0$ for $p \in (-\frac{1}{2}, -\frac{1}{3})$ and $B_p > 0$ for $p \in (-\frac{3}{5}, -\frac{1}{2})$. In the case $p = -\frac{1}{2}$ the expansion from Lemma 3.4 does not suffice for us but we can use (18) together with

$$\sqrt{R_2(m)} = 4a - \sqrt{m} - \frac{m}{4a} + o(m)$$

to derive that

$$\begin{aligned} L_2(m) &= \frac{16a}{3}\sqrt{1 - \frac{\sqrt{m}}{2a} - \frac{m}{16a^2} + o(m)}\left(1 + \frac{\sqrt{m}}{4a} - \frac{m}{16a^2} + o(m)\right) \\ &= L_2(0) - \frac{m}{a} + o(m). \end{aligned}$$

(b) If $-q = p = -\frac{3}{5}$, then inserting $\varrho_p(y) = \frac{3}{8}\ln y + O(1)$ and $R_2(m) = O(1)$ in (26), we obtain that

$$L_2(m) = L_2(0) + \frac{3}{8\sqrt{5a}} m^{4/5} \ln\frac{1}{m} + O\big(m^{4/5}\big).$$

(c) Finally, for $-q = p \in (-1, -\frac{3}{5})$ we have

$$\varrho_p(y) = -\frac{3}{4(5p+3)} y^{-\frac{5p+3}{2}} + o\Big(y^{-\frac{5p+3}{2}}\Big),$$

which together with $R_2(m) = R_2(0) + o(1)$ and (26) yields

$$L_2(m) = L_2(0) + \underbrace{\frac{2p(p+1)}{(5p+3)(3p+1)^2 R_2^{p+1}(0)}}_{>0} m^{2(p+1)} + o\big(m^{2(p+1)}\big).$$

□

The next three lemmata deal with the monotonicity and the stationary points of L_2.

Lemma 8.6. *Assume* (25). *The following holds:*

(i) *If* $p \geq 0$ *or* $p \geq -\frac{1}{2}$, $q = -p$, *then*

$$L_2' < 0 \text{ on } (0, M).$$

(ii) *If* $p < 0$, $q > -p$ *or* $p < -\frac{1}{2}$, $q = -p$, *then* L_2 *has a unique stationary point* $m_{0;p,q,a} =: m_0 \in (0, M)$ *while*

$$L_2' > 0 \text{ on } (0, m_0), \quad L_2' < 0 \text{ on } (m_0, M).$$

(iii) *If* $q < -p$, *then one of the following holds:*

A: $L_2' < 0$ *on* $(0, M)$,

B: $L_2' < 0$ *on* $(0, \overline{m})$, $L_2'(\overline{m}) = 0$ *and* $L_2' < 0$ *on* $(\overline{m}, M)$,

C: $L_2' < 0$ *on* $(0, m_1)$, $L_2' > 0$ *on* (m_1, m_2) *and* $L_2' < 0$ *on* (m_2, M) *for some* $m_1 = m_{1;p,q,a} \in (0, \overline{m})$, $m_2 = m_{2;p,q,a} \in [\overline{m}, M)$.

Proof. The case $p \geq 1$ is trivial, so let $p < 1$ and suppose that $m_0 \in (0, M)$ is a stationary point of L_2. Recall that $L_2' < 0$ near M due to Lemma 8.4.

Firstly, let us consider $q \geq 1$. Then $L_2''(m_0) < 0$, so there are only two possibilities: Either $L_2' < 0$ on $(0, M)$ or L_2 has a unique stationary point, which is a point of strict relative maximum. Lemma 8.5 guarantees that the first one holds for $p \geq 0$ and the second one for $p < 0$.

Now let $q < 1$. Consequently:

$$L_2''(m_0) < 0 \iff R_2(m_0) < \left(\frac{a(q-p)}{q(1-q)}\right)^{\frac{1}{2q-p-1}} =: \overline{R}_{2;p,q,a} =: \overline{R}_2. \tag{27}$$

Recall that $(R(M), R_2(0))$ is the range of R_2. The inequality $\overline{R}_2 > R(M)$ holds always while $\overline{R}_2 < R_2(0)$ only for $q < -p$. (In the latter case, we have $R_2(\overline{m}) = \overline{R}_2$.) So if $q \geq -p$, then each stationary point of L_2 is a point of strict relative maximum and by means of Lemma 8.5 we have again that $L_2' < 0$ for $p \geq 0$ and for $-q = p \in [-\frac{1}{2}, -\frac{1}{3})$ and L_2 has a unique stationary point for $p < 0$, $q > -p$ and for $p < -\frac{1}{2}$, $q = -p$.

From now on we will consider only $q < -p$ (thus, $-1 < p < -\frac{1}{3}$ and $q < 1$). So we have

$$L_2''(m_0) < 0 \iff m_0 > R_2^{-1}\big(\overline{R}_2\big) = \overline{m}.$$

It means that L_2 has at most one stationary point (a point of strict relative minimum) in $(0, \overline{m})$, at most one (a point of strict relative maximum) in $(\overline{m}, M)$ and $\overline{m}$

may be a stationary point as well. Suppose that $\overline{m}$ and some $m_2 > \overline{m}$ are both stationary points of L_2, thus L_2 increases on $[\overline{m}, m_2]$. Since K_2 decreases on $[\overline{m}, M)$, we have $L_2(\overline{m}) = K_2(\overline{m}) > K_2(m_2) = L_2(m_2)$ (see Lemma 2.11), a contradiction. Therefore, L_2 has at most one stationary point in $[\overline{m}, M)$. Furthermore, due to Lemma 8.5 only A, B or C can hold. □

Lemma 8.7. *Assume* (25) *and* $q < -p$. *There exists a continuous function* $q^* : (-1, -\frac{1}{2}) \to \mathbb{R}$ *such that* $\frac{p+1}{2} < q^*(p) < -p$ *for* $p \in (-1, -\frac{1}{2})$, $\lim_{p\to -1/2} q^*(p) = \frac{1}{2}$ *and the following holds:*

(i) *If* $p \geq -\frac{1}{2}$, $q < -p$ *or* $p < -\frac{1}{2}$, $q < q^*(p)$, *then*

$$L_2' < 0 \text{ on } (0, M).$$

(ii) *If* $p < -\frac{1}{2}$ *and* $q = q^*(p)$, *then* $\overline{m}$ *is a stationary point of* L_2 *while*

$$L_2' < 0 \text{ on } (0, \overline{m}), \quad L_2' < 0 \text{ on } (\overline{m}, M).$$

(iii) *If* $p < -\frac{1}{2}$ *and* $q^*(p) < q < -p$, *then* L_2 *has two stationary points* $m_{1;p,q,a} =: m_1$, $m_{2;p,q,a} =: m_2$ *while* $m_1 < \overline{m} < m_2$ *and*

$$L_2' < 0 \text{ on } (0, m_1), \quad L_2' > 0 \text{ on } (m_1, m_2), \quad L_2' < 0 \text{ on } (m_2, M).$$

For all $p \in (-1, -\frac{1}{2})$, $q = q^*(p)$ *is the only solution of the equation*

$$I_p(g(p,q)) - \underbrace{\frac{1}{1-p}\sqrt{\frac{2(q-p)(1-q)}{q}}\, g^{\frac{1-p}{2}}(p,q)}_{=:G(p,q)} =: f(p,q) = 0$$

in $(\frac{p+1}{2}, -p)$ *where*

$$g(p,q) = \left(\frac{2q(q-1)}{(2q-p-1)(p+q)}\right)^{\frac{1}{p+1}}.$$

Proof. From Lemma 8.6 we already know that only A, B or C can hold for $q < -p$. Let us notice the crucial role of the sign of $L_2'(\overline{m})$: If it is $+$, then C holds, if 0, then B or C occurs and if $-$, then A holds. So we derive the following condition:

$$L'_{2;p,q,a}(\overline{m}_{p,q,a}) > 0 \iff L_2(\overline{m}) - \frac{(1-q)\overline{R}_2^{\frac{2q-p-1}{2}}}{a(1-p)}\overline{R}_2^{\frac{1-p}{2}} > 0 \iff f(p,q) > 0$$

(see (27) for the definition of $\overline{R}_2$) and in the sequel we

1. find $\lim_{q\to(p+1)/2} f(p,q)$
2. and $\lim_{q\to -p} f(p,q)$
3. and investigate the monotonicity of $f(p,\cdot)$.

Afterwards we will be able to describe the sets where f (or equivalently $L_2'(\overline{m})$) is positive, zero and negative, resp.

1. Since $\lim_{q\to(p+1)/2} g(p,q) = \infty$, using the first term of the asymptotic expansion of $I_p(y)$ for $y\to\infty$ (see Lemma 3.4), we obtain:

$$\lim_{q\to\frac{p+1}{2}} \frac{f(p,q)}{g^{\frac{1-p}{2}}(p,q)} = \frac{3p+1}{(1-p)\sqrt{p+1}} < 0,$$

thus

$$\lim_{q\to\frac{p+1}{2}} f(p,q) = -\infty.$$

2. We are going to find $\lim_{q\to -p} f(p,q)$, so we denote $-q-p =: r$ for the sake of simplicity. All the asymptotic expansions in this step will concern $r\to 0+$ or $y\to\infty$. We will see that the first two terms of the asymptotic expansions of $I_p(g(p,q))$ and $G(p,q)$ are identical, therefore we need to calculate the first three. We have:

$$\begin{aligned} G(p,q) &= \frac{2\sqrt{p+1}}{1-p}\sqrt{\frac{1+\frac{3p+1}{2p(p+1)}r+\frac{1}{2p(p+1)}r^2}{1+\frac{r}{p}}}\, g^{\frac{1-p}{2}}(p,q) \\ &= \frac{2\sqrt{p+1}}{1-p}\sqrt{1+\frac{p-1}{2p(p+1)}r+\frac{1}{2p^2(p+1)}r^2+O(r^3)}\; g^{\frac{1-p}{2}}(p,q) \\ &= \frac{2\sqrt{p+1}}{1-p}\left(1+\frac{p-1}{4p(p+1)}r-\frac{p^2-10p-7}{32p^2(p+1)^2}r^2+O\big(r^3\big)\right) g^{\frac{1-p}{2}}(p,q). \end{aligned}$$

It will be useful to write the asymptotic expansion of $I_p(y)$ in the form

$$\frac{I_p(y)}{\sqrt{p+1}} = \frac{2}{1-p}\left(1+\frac{p-1}{2(3p+1)}\frac{1}{y^{p+1}}\right) y^{\frac{1-p}{2}} + \varrho_p(y)$$

where function ϱ_p will be specified later. Joining the last formula with

$$(28)\qquad \begin{aligned} \frac{1}{g^{p+1}(p,q)} &= \frac{3p+1}{2p(p+1)} r \frac{1+\frac{2}{3p+1}r}{1+\frac{2p+1}{p(p+1)}r+\frac{1}{p(p+1)}r^2} \\ &= \frac{3p+1}{2p(p+1)} r\left(1-\frac{4p^2+3p+1}{p(p+1)(3p+1)}r+O\big(r^2\big)\right), \end{aligned}$$

we obtain that

$$\begin{aligned} I_p(g(p,q)) = \frac{2\sqrt{p+1}}{1-p}&\left(1+\frac{p-1}{4p(p+1)}r+\frac{(1-p)(4p^2+3p+1)}{4p^2(p+1)^2(3p+1)}r^2+O\big(r^3\big)\right) \\ &\cdot g^{\frac{1-p}{2}}(p,q)+\sqrt{p+1}\,\varrho_p(g(p,q)), \end{aligned}$$

consequently

$$(29)\qquad \begin{aligned} f(p,q) = &\left(\frac{\sqrt{p+1}(29p^3+21p^2+15p-1)}{16p^2(p+1)^2(3p+1)(p-1)}r^2+O\big(r^3\big)\right) g^{\frac{1-p}{2}}(p,q) \\ &+\sqrt{p+1}\,\varrho_p(g(p,q)). \end{aligned}$$

(a) Let $-\frac{3}{5} < p < -\frac{1}{3}$, thus $\varrho_p(y) = B_p + o(1)$. Since

$$g^{\frac{1-p}{2}}(p,q) = O\Big(r^{\frac{p-1}{2(p+1)}}\Big) = o\Big(\frac{1}{r^2}\Big),$$

we have

$$f(p,q) = \sqrt{p+1}\; B_p + o(1).$$

So $\lim_{q\to -p} f(p,q)$ is negative for $p \in (-\frac{1}{2}, -\frac{1}{3})$, zero for $p = -\frac{1}{2}$ and positive for $p \in (-\frac{3}{5}, -\frac{1}{2})$ due to Lemma 3.4.

(b) If $p = -\frac{3}{5}$, then inserting $\varrho_p(y) = \frac{3}{8}\ln y + O(1)$ and $g^{\frac{1-p}{2}}(p,q) = O(\frac{1}{r^2})$ in (29), we obtain that

$$f(p,q) = \frac{3\sqrt{5}}{8\sqrt{2}} \ln\frac{1}{r} + O(1) \;\longrightarrow\; \infty.$$

(c) For $p \in (-1, -\frac{3}{5})$ we have

$$\varrho_p(y) = \Big(-\frac{3}{4(5p+3)}\frac{1}{y^{2(p+1)}} + o\Big(\frac{1}{y^{2(p+1)}}\Big)\Big) y^{\frac{1-p}{2}},$$

hence (29) yields

$$f(p,q) = \Big(\frac{4(p+1)^{3/2}}{p(3p+1)(5p+3)(p-1)} r^2 + o\big(r^2\big)\Big) g^{\frac{1-p}{2}}(p,q) \;\longrightarrow\; \infty.$$

(See (28).)

So we have derived that

$$\lim_{q\to -p} f(p,q) \begin{cases} < 0 & \text{if } -\frac{1}{2} < p < -\frac{1}{3}, \\ = 0 & \text{if } p = -\frac{1}{2}, \\ > 0 & \text{if } -1 < p < -\frac{1}{2}. \end{cases}$$

3. The increase of $f(p,\cdot)$ can be proved using

$$\begin{aligned} \frac{\partial f}{\partial q}(p,q) &= \left(\sqrt{\frac{p+1}{g^{p+1}(p,q)-1}} - \frac{\sqrt{(2q-p-1)(p-q)(p+q)}}{2q}\right)\frac{\partial g}{\partial q}(p,q) \\ &\quad + \frac{q^2-p}{(1-p)q\sqrt{2q(q-p)(1-q)}}\,\frac{1}{\sqrt{g^{p+1}(p,q)}}\, g(p,q) \\ &= \frac{1}{2q}\sqrt{\frac{(2q-p-1)(p+q)}{p-q}}\left(\frac{q^2-p}{q(1-q)(1-p)} g(p,q) + (p+q)\frac{\partial g}{\partial q}(p,q)\right) \end{aligned}$$

and

$$\frac{\partial g}{\partial q}(p,q) = -\frac{q^2-2pq+p}{q(1-q)(2q-p-1)(p+q)} g(p,q),$$

which yield

$$\frac{\partial f}{\partial q}(p,q) = \frac{p+q}{q^2(p-1)}\sqrt{\frac{(p+q)(p-q)}{2q-p-1}}g(p,q) > 0.$$

From *1.*, *2.* and *3.* we can see that if $p \in [-\frac{1}{2}, -\frac{1}{3})$, $q \in (\frac{p+1}{2}, -p)$, then $f(p,q) < 0$, i. e. $L_2' < 0$. Moreover, $f(p,\cdot)$ has a unique zero—denote it by $q^*(p)$—for all $p \in (-1, -\frac{1}{2})$ and

- if $\frac{p+1}{2} < q < q^*(p)$, then $L_2'(\overline{m}) < 0$, so A holds,
- if $q^*(p) < q < -p$, then $L_2'(\overline{m}) > 0$, so C holds with $m_2 > \overline{m}$
- and if $q = q^*(p)$, then $L_2'(\overline{m}) = 0$, so either B holds or C with $m_2 = \overline{m}$. Nevertheless, we prove that only B can hold for $q = q^*(p)$: So suppose that C holds for some $p = p_0 \in (-1, -\frac{1}{2})$ and $q = q^*(p_0)$, thus $L'_{2;p_0,q^*(p_0),a}(\widetilde{m}) > 0$ for some $\widetilde{m} \in (0, M)$. From the definition of R_2 and the implicit function theorem it follows that $R_{2;p_0,\cdot,a}(\widetilde{m})$ is continuous, which together with (13), (10) and Theorem 3.5 guarantees the continuity of $L'_{2;p_0,\cdot,a}(\widetilde{m})$. Hence, $L'_{2;p_0,q^*(p_0)-\varepsilon,a}(\widetilde{m}) > 0$ if $\varepsilon > 0$ is small enough, giving a contradiction.

At this moment, assertions (i)–(iii) have been proved. Since f is continuous due to Theorem 3.5, from the implicit function theorem we have the continuity of q^* as well. So there only remains to find its limit at $-\frac{1}{2}$. Recall that $\lim_{q\to 1/2} f(-\frac{1}{2}, q) = 0$ and choose arbitrary $\varepsilon \in (0, \frac{1}{2})$. From the increase of $f(-\frac{1}{2}, \cdot)$ we have $f(-\frac{1}{2}, \frac{1}{2} - \varepsilon) < 0$, therefore $f(p, \frac{1}{2} - \varepsilon) < 0$ for all $p \in (-\frac{1}{2} - \delta, -\frac{1}{2})$ and some suitable $\delta \in (0, \frac{1}{2})$ and the increase of $f(p,\cdot)$ yields that $\frac{1}{2} - \varepsilon < q^*(p) < -p$ for $p \in (-\frac{1}{2} - \delta, -\frac{1}{2})$. So we conclude that $\lim_{p\to -1/2} q^*(p) = \frac{1}{2}$. □

Lemma 8.8. *There exists*

$$\lim_{p\to -1} q^*(p) =: q^*(-1) \in (0,1).$$

Proof. An easy calculation yields that

$$\lim_{p\to -1} g(p,q) = \mathrm{e}^{\frac{q+1}{2q(1-q)}} =: \psi(q)$$

and

$$\lim_{p\to -1} f(p,q) = I_{-1}(\psi(q)) - \sqrt{\frac{1-q^2}{2q}}\psi(q) =: \varphi(q)$$

for all $q \in (0,1)$. In the sequel we examine the behaviour of φ.

Since $\lim_{q\to 0} \psi(q) = \infty$ and $I_{-1}(y) = o(y)$ for $y \to \infty$ (see Lemma 3.6),

$$\varphi(q) = -\frac{1}{\sqrt{2q}}\big(1 + o(1)\big)\psi(q) \longrightarrow -\infty, \quad q \longrightarrow 0.$$

Set $r := 1-q$ and consider $r \to 0+$. Using Lemma 3.6 with $n = 4$ and the formulae

$$\begin{aligned}
\frac{1}{\sqrt{\ln\psi(q)}} &= \sqrt{r}\Big(1-\frac{r}{4}-\frac{5}{32}r^2-\frac{13}{128}r^3+O\big(r^4\big)\Big),\\
\frac{1}{\ln\psi(q)} &= r\Big(1-\frac{r}{2}-\frac{r^2}{4}+O\big(r^3\big)\Big),\\
\frac{1}{\ln^2\psi(q)} &= r^2\big(1-r+O\big(r^2\big)\big),\\
\frac{1}{\ln^3\psi(q)} &= r^3\big(1+O(r)\big),
\end{aligned}$$

we obtain that

$$I_{-1}(\psi(q)) = \sqrt{r}\Big(1+\frac{r}{4}+\frac{7}{32}r^2+\frac{89}{128}r^3+O\big(r^4\big)\Big)\psi(q).$$

On the other hand,

$$\sqrt{\frac{1-q^2}{2q}} = \sqrt{r}\Big(1-\frac{r}{2}\Big)^{1/2}(1-r)^{-1/2} = \sqrt{r}\Big(1+\frac{r}{4}+\frac{7}{32}r^2+\frac{25}{128}r^3+O\big(r^4\big)\Big),$$

hence

$$\varphi(q) = \frac{r^{7/2}}{2}\psi(1-r)\big(1+O(r)\big) = \frac{r^{7/2}}{2}\mathrm{e}^{\frac{1}{r}+\frac{1}{2}}\big(1+O(r)\big) \longrightarrow \infty.$$

It is not hard to derive that

$$\psi'(q) = \frac{q^2+2q-1}{2q^2(1-q)^2}\psi(q)$$

and

$$\begin{aligned}
\varphi'(q) &= \left(\frac{1}{\sqrt{\ln\psi(q)}}-\sqrt{\frac{1-q^2}{2q}}\right)\psi'(q)+\sqrt{\frac{2q}{1-q^2}}\,\frac{q^2+1}{4q^2}\psi(q)\\
&= \frac{1-q}{2q^2}\sqrt{\frac{1-q^2}{2q}}\psi(q) > 0.
\end{aligned}$$

So we conclude that φ has a unique zero $q_0 \in (0,1)$. Since φ increases and $\lim_{p\to -1} f(p,q) = \varphi(q)$, we have that for arbitrary $\varepsilon \in (0, \min\{q_0, 1-q_0\})$ there exists such $\delta > 0$ that

$$\forall p \in (-1,-1+\delta)\colon\ f(p,q_0-\varepsilon) < 0 < f(p,q_0+\varepsilon),$$

hence

$$\forall p \in (-1,-1+\delta)\colon\ q_0-\varepsilon < q^*(p) < q_0+\varepsilon$$

and therefore $\lim_{p\to -1} q^*(p) = q_0$. □

Numerical calculations indicate that q^* is probably decreasing, concave, its graph touches the graph of $q = -p$ in $-\frac{1}{2}$, and $q^*(-1) \approx 0.730$. We would like to prove some of these observations analytically in the future.

We append Figure 4 with all the possible graphs of L_1 and L_2 and the corresponding sets of (p, q), based on the lemmata of this section. These results are sufficient to determine the number of the symmetric solutions of (1) in case V depending on p, q, a, l (see Lemma 2.8) except for $-1 < p < -\frac{1}{2}$, $q^*(p) < q < -p$ because it is required to investigate, for which p, q is $L_2(0) > L_2(m_2)$. In view of Lemmata 8.6 (ii) and 8.7 (ii), it can be expected that this domain is divided by a continuous curve into three sets where $L_2(0) = L_2(m_2)$ for (p, q) lying on the curve, $L_2(0) < L_2(m_2)$ above it and $L_2(0) > L_2(m_2)$ under it. This hypothesis is also consistent with numerical calculations and will be an object of further research.

So let us state the main result of this section.

Theorem 8.9. *Suppose* (25).

(a) *If* $q < p$, *then*

$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1, 2\}.$$

(b) *If* $q = p$, *then*

$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1\}.$$

(c) *If* $p \geq 1$ *and* $q > p$, *then*

$$|\mathcal{S}(l)| = 1 \quad \textit{for } l > 0.$$

(d) *If* $0 \leq p < 1$ *or* $p \geq -\frac{1}{2}$, $q \leq -p$ *or* $p < -\frac{1}{2}$, $q \leq q^*(p)$, *then*

$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1\}.$$

(e) *If* $p < 0$, $q > -p$ *or* $p < -\frac{1}{2}$, $q = -p$, *then*

$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1, 2\}.$$

(f) *If* $p < -\frac{1}{2}$ *and* $q^*(p) < q < -p$, *then*

$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1, 2, 3\}.$$

The exact dependence of $|\mathcal{S}(l)|$ *on* l *as well as the monotonicity properties of* L *are indicated in Figure 4. (Recall that* $L(u(0)) = l$ *for any* $u \in \mathcal{S}(l)$*.)*

In this paper, we have not dealt with the monotonicity of $L_1 + L_2$, which is related to the number of nonsymmetric solutions of (1). It was proved in [**5**] that $(L_1 + L_2)' < 0$ for $1 < p \leq 4$ and for $p > 4$, $q \geq p - 1 - \frac{1}{p-2}$. Our future goal is to examine the behaviour of $L_1 + L_2$ for the rest of case V and to study cases VI–XIII.

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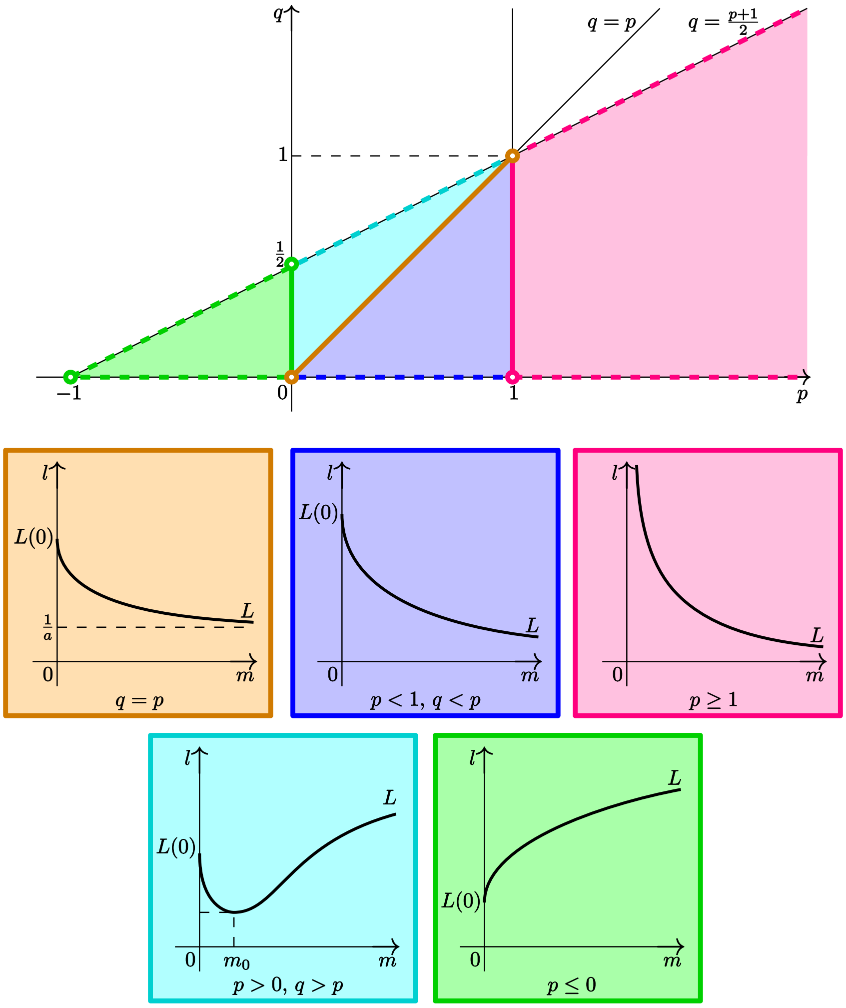

Figure 3. The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case III ($p > -1$, $0 < q < \frac{p+1}{2}$, $a > 0$) according to Lemmata 2.8, 6.2, 6.3, 6.4 and 6.5. See also Theorem 6.6.

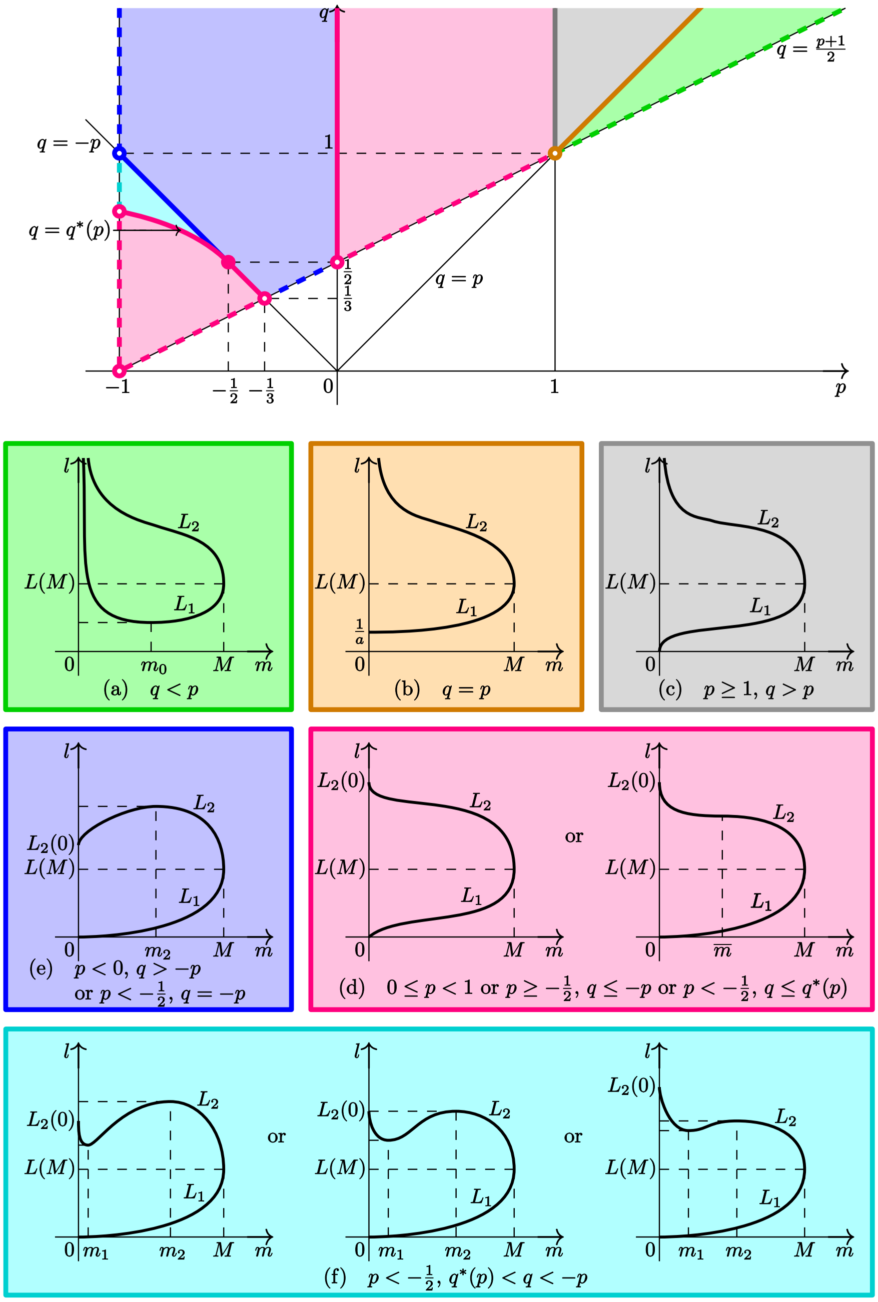

Figure 4. The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case V ($p > -1$, $q > \frac{p+1}{2}$, $a > 0$) according to Lemmata 2.8, 8.3, 8.4, 8.6, 8.7 and 8.8. See also Theorem 8.9.

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