Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Solvability of Second Order Ordinary Differential Equations with Non-linear Boundary Conditions

Dissertation thesis

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Supervisor: Prof. RNDr. Marek Fila, DrSc.

Bratislava, 2013 **Mgr. Sámuel Peres**

Univerzita Komenského v Bratislave
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Riešiteľnosť obyčajných diferenciálnych rovníc druhého rádu s nelineárnymi okrajovými podmienkami

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Abstract

PERES, SÁMUEL: *Solvability of second order ordinary differential equations with non-linear boundary conditions* [dissertation thesis]. Comenius University in Bratislava; Faculty of Mathematics, Physics and Informatics; Department of Applied Mathematics and Statistics. Supervisor: Prof. RNDr. Marek Fila, DrSc. Bratislava, 2013. 77 pp.

This thesis deals with the existence and multiplicity of positive and sign-changing solutions of a non-linear second order ordinary differential equation with symmetric non-linear boundary conditions, where both of the non-linearities are of power type. It extends known results to a larger set of parameters, as well as provides answers to two long-standing open questions. The main tool is the shooting method.

Keywords: second order ordinary differential equation, non-linear boundary condition, existence and multiplicity of solutions, shooting method, time map.

Abstrakt

PERES, SÁMUEL: *Riešiteľnosť obyčajných diferenciálnych rovníc druhého rádu s nelineárnymi okrajovými podmienkami* [dizertačná práca]. Univerzita Komenského v Bratislave; Fakulta matematiky, fyziky a informatiky; Katedra aplikovanej matematiky a štatistiky. Školiteľ: Prof. RNDr. Marek Fila, DrSc. Bratislava, 2013. 77 s.

Práca sa zaoberá existenciou a multiplicitou kladných riešení a riešení meniacich znamienko istej nelineárnej obyčajnej diferenciálnej rovnice druhého rádu so symetrickými nelineárnymi okrajovými podmienkami, pričom obidve nelinarity sú mocninové. Rozširuje predtým známe výsledky na väčšiu množinu parametrov a taktiež dáva odpoveď na dve dlho otvorené otázky. Hlavným nástrojom je metóda streľby.

Kľúčové slová: obyčajná diferenciálna rovnica druhého rádu, nelineárna okrajová podmienka, existencia a multiplicita riešení, metóda streľby, zobrazenie dostrelu.

Foreword

Differential equations are an indispensable tool for all the branches of physics, having also countless applications in other sciences such as biology and economics. They have been studied for more than three centuries, and they are a virtually inexhaustible source of mathematical problems.

For most of differential equations no explicit formulae giving their solutions can be derived. In that case, one can only examine the existence and number of solutions, develop approximate methods for finding them, and investigate their qualitative properties, such as dependence on the parameters occuring in the equation and the initial or boundary conditions, smoothness, positivity or number of zeros, symmetry, monotonicity, periodicity, boundedness or asymptotic behaviour, a priori estimates, stability and mutual position of solutions.

The difficulties with studying differential equations are often caused by non-linearities. In this dissertation thesis we investigate a boundary value problem containing non-linearities both in the equation and the boundary conditions. The problem has the form

$$\begin{cases} u''(x) = a|u(x)|^{p-1}u(x), & x \in (-l, l), \\ u'(\pm l) = \pm|u(\pm l)|^{q-1}u(\pm l). \end{cases}$$

Here a and l can take any positive value, while the conditions on p and q will be specified later. As one can see, the boundary conditions are symmetric, and both of the non-linearities are of power type. Our aim is to determine the number of classical solutions for as large set of values of the parameters as possible.

Most of this thesis concernes positive solutions, which solve the simpler-looking problem

$$\begin{cases} u''(x) = au^p(x), & x \in (-l, l), \\ u'(\pm l) = \pm u^q(\pm l), \end{cases}$$

while p and q can be arbitrary real numbers. On the other hand, if one is interested in the existence and multiplicity of sign-changing solutions, only $p > 0$, $q \in \mathbb{R}$ can be considered. We present results for $p > -1$, $q \geq 0$ and $p = -1$, $q = 0$ regarding positive solutions, and for $p = 1$, $q \in (0, 1)$ and $p > 1$, $q \in [\frac{1}{2}, \frac{p+1}{2})$ regarding sign-changing solutions.

Our principal references are [5] and [6]. In these articles the solvability of the discussed problem was examined for $p, q > 1$ in the class of positive solutions and for $p \geq 1$, $q > 1$ in the class of sign-changing solutions respectively. However, both of them left a question partially open. (Namely, the question of the existence and multiplicity of positive non-symmetric and sign-changing non-antisymmetric solutions.) The answers will be given in this work, together with results concerning some of the values of p and q not considered in the articles mentioned above.

We apply the so-called shooting method, which was also used in the cited articles. Its substance is to express the solutions of the given boundary value problem by means of the solutions of the same differential equation subject to appropriate initial conditions, leading to the definition of some functions called time maps, the properties of which directly determine the number of solutions of the considered boundary value problem. Thus, we will need only the tools of real analysis. On the other hand, it is not so easy to examine the properties of the time maps, because they are given by a formula containing an improper integral, which can be calculated only for some special values of p, and the upper limit of which is given only implicitly.

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Introduction

Consider the problem

$$\begin{cases} u''(x) = a|u(x)|^{p-1}u(x), & x \in (-l, l), \\ u'(\pm l) = \pm|u(\pm l)|^{q-1}u(\pm l), \end{cases} \tag{1}$$

where $a, l > 0$, and $p, q \in \mathbb{R}$ in the case of positive solutions, while $p > 0$, $q \in \mathbb{R}$ in the case of sign-changing solutions.

The first systematic study of positive solutions of (1) was done by M. Chipot, M. Fila and P. Quittner in [5]. They also studied the N-dimensional version of (1), but they were interested mainly in global existence and boundedness or blow-up of positive solutions of the corresponding N-dimensional parabolic problem

$$\begin{cases} u_t = \Delta u - a|u|^{p-1}u & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = |u|^{q-1}u & \text{in } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \overline{\Omega}, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, n is the unit outer normal vector to $\partial\Omega$, $u_0 : \overline{\Omega} \to [0, \infty)$, $p, q > 1$ and $a > 0$. The cited article provides a complete answer for the question of the existence and number of positive symmetric (i. e. even) solutions of (1) for $p, q > 1$. However, only partial results were presented in it regarding positive non-symmetric solutions, the study of which is much more complicated.

Let us remark that positive symmetric solutions of (1) (and also solutions of (2) for $N = 1$) were independently studied in [12].

Sign-changing solutions of (1) were systematically investigated for the first time in [6] by M. Chipot and P. Quittner, considering $p \geq 1$ and $q > 1$. The number of sign-changing antisymmetric (i. e. odd) solutions was determined for all these values of p and q, but again, only partial results were achieved concerning sign-changing non-antisymmetric solutions.

The results from [5] have been generalised in many other directions: In [15] the behaviour of positive solutions of (2) was examined for all $p, q > 1$. Positive solutions of the elliptic problem with $-\lambda u + u^p$ on the right-hand side of the equation were dealt with in [13] for $\lambda \in \mathbb{R}$, $p, q > 1$, and later in [10] for $\lambda \in \mathbb{R}$, $p, q > 0$, $(p, q) \notin (0, 1)^2$. In [11] and [16], positive and sign-changing solutions of the parabolic problem with more general non-linearities $f(u)$, $g(u)$ instead of $a|u|^{p-1}u$, $|u|^{q-1}u$ were studied, while $f(x, u)$, $g(x, u)$ were considered in [2]. Many results concerning elliptic problems with non-linear boundary conditions were summarised in [17]. Further extensions of the results from [5] can be found in [1, 3, 4, 7, 8, 9].

However, this thesis focuses only on (1), and extends known results to larger sets of parameters. It is divided into two chapters, the first dealing with positive solutions of (1), and the second with its sign-changing solutions and with the Cauchy problem for $u'' = au^{-1/2}$, which can be explicitely solved. The chapters are divided into sections.

Section 1 explains the shooting method: Clearly, all the positive solutions of (1) for given $l > 0$ can be obtained from the solutions of the same differential equation subject to the initial conditions $u(0) = m$ and $u'(0) = 0$, choosing appropriate values of $m > 0$. This initial value problem possesses a unique solution for arbitrary $m > 0$. The connection between m and l is given by some functions called time maps, for which a formula will be derived, showing the need of studying positive symmetric and positive non-symmetric solutions separately. Furthermore, thirteen cases—numbered I to XIII—regarding the values of p and q should be distinguished. This thesis discusses the first five of them, which embrace $p > -1$, $q \geq 0$ and $p = -1$, $q = 0$, as opposed to $p, q > 1$ from [5]. We will see that the properties of the time maps are much more diverse and more difficult to examine outside the set $p, q > 1$.

In Section 2 an improper parametric integral as a function of its upper limit will be examined in detail. This integral is contained in the time map formula, and its properties will be used in the subsequent sections, in which the behaviour of the time maps will be determined in the individual cases.

Cases I–IV will be studied successively in Sections 3–6. Together they cover $p = -1$, $q = 0$ and $p > -1$, $0 \leq q \leq \frac{p+1}{2}$, and for these values all the positive solutions of (1) are symmetric.

Conversely, in case V ($p > -1$, $q > \frac{p+1}{2}$), (1) possesses both positive symmetric and positive non-symmetric solutions, which will be dealt with in Sections 7 and 8. The number of positive non-symmetric solutions of (1) will be determined for all $p \geq 1$, completing the results of [5]. However, their study for $p \in (0, 1)$, which seems to be even more complicated, remains unfinished.

Section 9 deals with sign-changing solutions of (1), assuming $p \geq 1$. (Although (1) has sense for any $p > 0$, $q \in \mathbb{R}$, we do not consider $p \in (0, 1)$, because in that case the initial value problem with $u(0) = u'(0) = 0$ has infinitely many solutions, which causes difficulties for the study of (1).) More specifically, we will investigate only sign-changing non-antisymmetric solutions, for the existence of which it is neccessary to suppose $q \in (0, \frac{p+1}{2})$. We extend the results of [6] to $p = 1$, $q \in (0, 1)$ and $p > 1$, $q \in [\frac{1}{2}, \frac{p+1}{2})$.

Finally, in Section 10 we explicitely solve the Cauchy problem for $u'' = au^{-1/2}$ with $a > 0$, using some formulae from Section 1 as well as Cardano's formula.

Chapter I

Positive solutions

1 The shooting method and the time maps

If u is a positive solution of (1), then $u'(-l) < 0 < u'(l)$, therefore u has a stationary point $x_0 \in (-l, l)$. So the function $u(\cdot + x_0)$ solves

$$\begin{cases} u'' = au^p, \\ u(0) = m, \\ u'(0) = 0 \end{cases} \tag{I.1}$$

for some $m > 0$. In the following theorem we summarise the facts known about the solvability of this problem. The proof for $p, q > 1$ can be found in [5], for other p, q it is done analogously.

1.1 Theorem (for $p, q > 1$ see [5, pp. 53–54])
Suppose $m, a > 0$, $p \in \mathbb{R}$. Then (I.1) *has a unique maximal solution. We will denote it by $u_{m,p,a}$ and its domain by $(-\Lambda_{m,p,a}, \Lambda_{m,p,a})$. Function $u_{m,p,a}$ is even, strictly convex, unbounded from above and fulfils*

$$|x| = \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{u_{m,p,a}(x)}{m}\right), \qquad x \in (-\Lambda_{m,p,a}, \Lambda_{m,p,a}), \tag{I.2}$$

where $I_p : [1, \infty) \to [0, \infty)$ is given as

$$I_p(y) = \begin{cases} \displaystyle\int_1^y \sqrt{\frac{p+1}{V^{p+1} - 1}}\, \mathrm{d}V & \text{if } p \neq -1, \\ \displaystyle\int_1^y \frac{\mathrm{d}V}{\sqrt{\ln V}} & \text{if } p = -1 \end{cases}$$

and

$$\Lambda_{m,p,a} = \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} \lim_{y \to \infty} I_p(y) \begin{cases} < \infty & \text{if } p > 1, \\ = \infty & \text{if } p \leq 1. \end{cases} \tag{I.3}$$

Finally, for $x \in (-\Lambda_{m,p,a}, \Lambda_{m,p,a})$ we have:

$$|u'_{m,p,a}(x)| = \begin{cases} \sqrt{\dfrac{2a}{p+1}\big(u_{m,p,a}^{p+1}(x) - m^{p+1}\big)} & \text{if } p \neq -1, \\ \sqrt{2a\big(\ln u_{m,p,a}(x) - \ln m\big)} & \text{if } p = -1. \end{cases} \tag{I.4}$$

1.2 Definiton
For given $p, q \in \mathbb{R}$, $a, l > 0$ denote the set of all positive symmetric (i. e. even) and positive non-symmetric solutions of (1) by $\mathcal{S}(l) = \mathcal{S}(l; p, q, a)$ and $\mathcal{N}(l) = \mathcal{N}(l; p, q, a)$ respectively.

1.3 Remark ([5, pp. 53–54])
Assume $p, q \in \mathbb{R}$, $a, l > 0$. Obviously, $\mathcal{S}(l)$ consists of all such functions $u_{m,p,a}|_{[-l,l]}$ that $0 < l < \Lambda_{m,p,a}$ and $u'_{m,p,a}(l) = u^q_{m,p,a}(l)$. On the other hand, if $l_1 \neq l_2$ are such numbers that $0 < l_i < \Lambda_{m,p,a}$, $u'_{m,p,a}(l_i) = u^q_{m,p,a}(l_i)$ for $i = 1, 2$ and $l_1 + l_2 = 2l$, then $u_{m,p,a}(\cdot - (l_1 - l_2)/2)|_{[-l,l]} \in \mathcal{N}(l)$.

1.4 Lemma (for $p, q > 1$ see [5, pp. 54–55])
Let $p, q \in \mathbb{R}$, $a > 0$. Then the following statements are equivalent for arbitrary $m, l > 0$:

(i) *$l < \Lambda_{m,p,a}$ and $u'_{m,p,a}(l) = u^q_{m,p,a}(l)$,*

(ii) *the equation*

$$0 = \mathcal{F}(m, x) := \mathcal{F}_{p,q,a}(m, x) := \begin{cases} \dfrac{x^{2q}}{2a} - \dfrac{x^{p+1}}{p+1} + \dfrac{m^{p+1}}{p+1} & \text{if } p \neq -1, \\ \dfrac{x^{2q}}{2a} - \ln x + \ln m & \text{if } p = -1 \end{cases} \tag{I.5}$$

with the unknown $x > 0$ has some solution $R > m$, and

$$l = \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{R}{m}\right).$$

Proof: In order to derive (ii) from (i), it suffices to use (I.4), denote $u_{m,p,a}(l) =: R > m$ and realise (I.2) for $x = l$. The reversed implication is proved essentially in the same way. $\square$

Function $\mathcal{F}(m, \cdot)$ has obviously different behaviour for $p > -1$, $p = -1$ and $p < -1$ as well as for $q > 0$, $q = 0$ and $q < 0$. It also matters which of the exponents $2q$, $p + 1$ is greater. So we have to distinguish thirteen cases shown in Figure 1.

1.5 Lemma (for $p, q > 1$ see [5, proofs of Lemma 3.1 and 3.2 with pp. 57–58])
Let $p, q \in \mathbb{R}$, $a, m > 0$. Function $\mathcal{F}(m, \cdot)$ has at most two zeros, and both lie in (m, ∞). We denote them $R_{p,q,a}(m) =: R(m)$ if there is only one zero, and $R_{1;p,q,a}(m) =: R_1(m)$ and $R_{2;p,q,a}(m) =: R_2(m)$ if there are two, while $R_1(m) < R_2(m)$.

Let us also introduce

$$M := M_{p,q,a} := \begin{cases} \left(\dfrac{2q - p - 1}{2q}\right)^{\frac{1}{p+1}} \left(\dfrac{a}{q}\right)^{\frac{1}{2q-p-1}} & \text{if } p \neq -1,\ q > 0,\ q > \frac{p+1}{2} \text{ (V, VII)}, \\ \left(\dfrac{a}{\mathrm{e}q}\right)^{\frac{1}{2q}} & \text{if } p = -1,\ q > 0 \text{ (VI)}, \\ \left(-\dfrac{p+1}{2a}\right)^{\frac{1}{p+1}} & \text{if } p < -1,\ q = 0 \text{ (VIII)}. \end{cases}$$

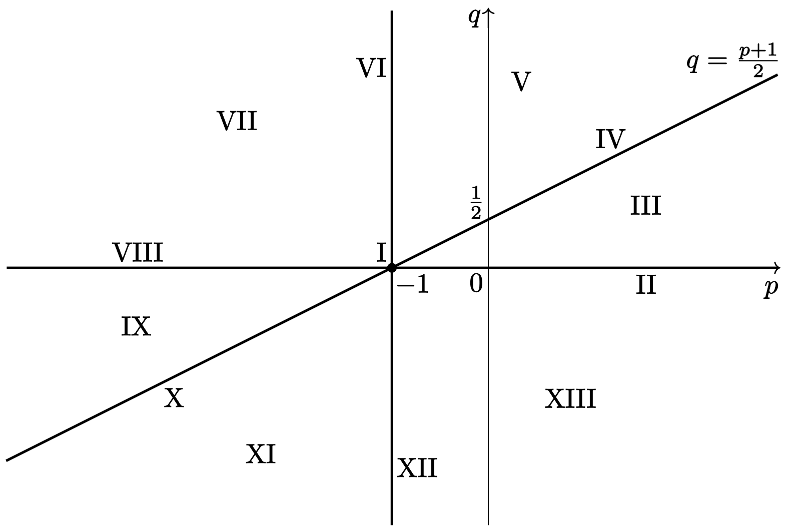

Figure 1: Cases I to XIII.

The following holds for the number of zeros:

(i) *If* $q < 0$ *or* $q < \frac{p+1}{2}$ *or* $p = -1$, $q = 0$ *(cases I–III, IX–XIII), then* $\mathcal{F}(m,\cdot)$ *has exactly one zero for arbitrary* $m > 0$. *Moreover, for* $p > -1$, $0 < q < \frac{p+1}{2}$ *(case III) we have*

$$R(m) > \left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}. \tag{I.6}$$

(ii) *If* $p > -1$, $q = \frac{p+1}{2}$ *(case IV), then* $\mathcal{F}(m,\cdot)$ *has one zero for* $q < a$ *and none for* $q \geq a$.

(iii) *If* $p < -1$, $q = 0$ *(case VIII), then* $\mathcal{F}(m,\cdot)$ *has one zero for* $m < M$ *and none for* $m \geq M$.

(iv) *If* $q > 0$ *and* $q > \frac{p+1}{2}$ *(cases V–VII), then* $\mathcal{F}(m,\cdot)$ *has two zeros for* $m < M$, *one for* $m = M$ *and none for* $m > M$. *Meanwhile,*

$$R_1(m) < \underbrace{\left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}}_{=R(M)} < R_2(m). \tag{I.7}$$

Moreover,

$$R(m) = \begin{cases} e^{\frac{1}{2a}} m & \text{if } p = -1,\ q = 0 \text{ (I)}, \\ \left(m^{p+1} + \dfrac{p+1}{2a}\right)^{\frac{1}{p+1}} & \text{if } p > -1,\ q = 0 \text{ (II)} \\ & \text{or } p < -1,\ q = 0,\ m < M \text{ (VIII)}, \\ \left(\dfrac{a}{a-q}\right)^{\frac{1}{2q}} m & \text{if } p > -1,\ q = \frac{p+1}{2} < a \text{ (IV)} \\ & \text{or } p < -1,\ q = \frac{p+1}{2} \text{ (X)}. \end{cases}$$

Proof: Investigating the behaviour of $\mathcal{F}(m,\cdot)$, we obtain the facts collected in Table 1. They are sufficient to determine the number of zeros of $\mathcal{F}(m,\cdot)$ in cases I–IV and VIII–XIII as well as to verify (I.6).

<table>
<tr><th></th><th>$\lim\limits_{x\to 0} \mathcal{F}(m,x)$</th><th>monotonicity on $(0,\infty)$</th><th>$\lim\limits_{x\to\infty} \mathcal{F}(m,x)$</th></tr>
<tr><td>I. $p=-1$, $q=0$</td><td>∞</td><td rowspan="2">decreases</td><td rowspan="3">$-\infty$</td></tr>
<tr><td>II. $p>-1$, $q=0$</td><td>$\frac{1}{2a}+\frac{m^{p+1}}{p+1}>0$</td></tr>
<tr><td>III. $p>-1$, $0<q<\frac{p+1}{2}$</td><td rowspan="3">$\frac{m^{p+1}}{p+1}>0$</td><td>increases on $\big(0,(a/q)^{1/(2q-p-1)}\big]$,
decreases on $\big[(a/q)^{1/(2q-p-1)},\infty\big)$</td></tr>
<tr><td>IV. $p>-1$, $q=\frac{p+1}{2}$</td><td>decreases if $q<a$,
is constant if $q=a$,
increases if $q>a$</td><td>$-\infty$ if $q<a$,
$\frac{m^{p+1}}{p+1}>0$ if $q=a$,
∞ if $q>a$</td></tr>
<tr><td>V. $p>-1$, $q>\frac{p+1}{2}$</td><td rowspan="3">decreases on $\big(0,(a/q)^{1/(2q-p-1)}\big]$,
increases on $\big[(a/q)^{1/(2q-p-1)},\infty\big)$</td><td rowspan="3">$\infty$</td></tr>
<tr><td>VI. $p=-1$, $q>0$</td><td rowspan="8">∞</td></tr>
<tr><td>VII. $p<-1$, $q>0$</td></tr>
<tr><td>VIII. $p<-1$, $q=0$</td><td rowspan="6">decreases</td><td>$\frac{1}{2a}+\frac{m^{p+1}}{p+1}$
>0 if $m>M$,
$=0$ if $m=M$,
<0 if $m<M$</td></tr>
<tr><td>IX. $p<-1$, $\frac{p+1}{2}<q<0$</td><td rowspan="3">$\frac{m^{p+1}}{p+1}<0$</td></tr>
<tr><td>X. $p<-1$, $q=\frac{p+1}{2}$</td></tr>
<tr><td>XI. $p<-1$, $q<\frac{p+1}{2}$</td></tr>
<tr><td>XII. $p=-1$, $q<0$</td><td rowspan="2">$-\infty$</td></tr>
<tr><td>XIII. $p>-1$, $q<0$</td></tr>
</table>

Table 1: The properties of $\mathcal{F}(m,\cdot)$

In cases V–VII, $\mathcal{F}(m,\cdot)$ has exactly one relative minimum, the value of which can be easily calculated. So there exist two zeros if and only if this minimum is negative, what happens just for $m < M$. Further, for $m = M$ there is only one zero and for $m > M$ there is none. The validity of (I.7) is apparent.

Now let us prove that each zero of $\mathcal{F}(m,\cdot)$ is greater than m. In cases I–IV and VIII–XIII it is guaranteed by the simple fact that $\mathcal{F}(m,m) = m^{2q}/2a > 0$ for $p, q \in \mathbb{R}$, $a, m > 0$. In cases V and VII for $m \le M$, we need to consider

$$m \le M < \left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}$$

too, similarly in case VI.

Finally, equation (I.5) is linear in $\ln x$ and x^{p+1} in cases I and II, VIII, IV, X respectively, so explicit solutions can be found. □

Let us notice that the set of parameters $p, q > 1$, which was investigated in [5], forms only part of cases III–V, and we will see that more complicated and interesting things happen outside it.

Although there is no difference in the properties of $\mathcal{F}(m,\cdot)$ summarised in Table 1 between cases IX, X and XI, it is not clear whether or not different results hold for (1) in these cases. For this reason we have not merged them into one case.

Now, as a simple consequence of Lemma 1.5, we formulate a non-existence result related to (1), and afterwards we introduce the notion of the time map.

1.6 Theorem
Let $p \in \mathbb{R}$, $a > 0$.

(i) *If $q \le 0$ or $q \le \frac{p+1}{2}$ (cases I–IV and VIII–XIII), then $\mathcal{N}(l) = \emptyset$ for all $l > 0$.*

(ii) *If $p > -1$, $q = \frac{p+1}{2} \ge a$ (case IV), then $\mathcal{S}(l) = \emptyset$ for all $l > 0$.*

1.7 Definiton
Let $p, q \in \mathbb{R}$, $a > 0$ and

$$L(m) := L_{p,q,a}(m) := \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{R_{p,q,a}(m)}{m}\right)$$

for all such m that $R_{p,q,a}(m)$ is defined. We introduce $L_{1;p,q,a}(m) =: L_1(m)$ and $L_{2;p,q,a}(m) =: L_2(m)$ analogously. Functions L, L_1 and L_2 will be called **time maps** (associated with (I.1)).

Using Lemmata 1.4 and 1.5, we can reformulate the statement of Remark 1.3 in the following way:

1.8 Lemma
For all $p, q \in \mathbb{R}$, $a, l > 0$:

$$\mathcal{S}(l) = \left\{ u_{m,p,a}\big|_{[-l,l]} \;:\; L(m) = l \text{ or } L_1(m) = l \text{ or } L_2(m) = l \right\},$$

$$\mathcal{N}(l) = \begin{cases} \left\{ u_{m,p,a}\left(\cdot \pm \frac{L_2(m)-L_1(m)}{2}\right)\Big|_{[-l,l]} \;:\; L_1(m)+L_2(m)=2l \right\} & \text{if } q > 0 \text{ and } q > \frac{p+1}{2} \text{ (V–VII)}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, to determine the number of positive symmetric solutions of (1) for given $p, q \in \mathbb{R}$, $a, l > 0$, we need to calculate the limits of functions L, L_1, L_2 at the endpoints of their domains, to find the intervals where the functions are monotone and finally to estimate their possible relative extrema. For non-symmetric solutions we execute the same with $L_1 + L_2$ if $q > 0$ a $q > \frac{p+1}{2}$ (cases V–VII). Therefore, we now derive formulae for the derivatives of the time map and other functions we will need in the rest of this article.

1.9 Lemma (for $p, q > 1$ see [5, proofs of Theorem 3.1 and Lemma 3.5])
Assume $p, q \in \mathbb{R}$, $a > 0$. Let $\mathcal{R}$ be one of the functions R, R_1, R_2, and suppose that its domain is an interval, denote it by I. Let $\mathcal{L} \in \{L, L_1, L_2\}$ be the corresponding

time map. Then $\mathcal{R}, \mathcal{L} \in C^\infty(I)$, and the following formulae hold for $m \in I$:

$$\mathcal{R}'(m) = \left(\frac{m}{\mathcal{R}(m)}\right)^p \frac{1}{1 - \frac{q}{a}\mathcal{R}^{2q-p-1}(m)}, \tag{I.8}$$

$$\left(\frac{\mathcal{R}(m)}{m}\right)' = \frac{2q-p-1}{2am^{p+2}}\mathcal{R}^{2q}(m)\mathcal{R}'(m), \tag{I.9}$$

$$\left(I_p\left(\frac{\mathcal{R}(m)}{m}\right)\right)' = \frac{2q-p-1}{\sqrt{2a}} m^{\frac{p-3}{2}} \frac{\mathcal{R}^{q-p}(m)}{1 - \frac{q}{a}\mathcal{R}^{2q-p-1}(m)}, \tag{I.10}$$

$$\mathcal{L}'(m) = \frac{1-p}{2m}\mathcal{L}(m) + \frac{2q-p-1}{2am^{p+1}}\mathcal{R}^q(m)\mathcal{R}'(m), \tag{I.11}$$

$$\begin{aligned}\mathcal{L}''(m) = & -\frac{p+1}{2m}\mathcal{L}'(m) + \frac{2q-p-1}{2am^{2p+1}} \\ & \cdot \left((q-1)\frac{q}{a}\mathcal{R}^{2q-p-1}(m) + q - p\right)\mathcal{R}^{p+q-1}(m)(\mathcal{R}'(m))^3.\end{aligned} \tag{I.12}$$

Proof: The C^∞-smoothness of $\mathcal{R}$ and the formula for its derivative follows from the implicit function theorem due to Lemma 1.5. If $\mathcal{R} \in \{R_1, R_2\}$ (cases V–VII), then (I.7) is used as well. The other formulae can be derived from (I.8) in such a way as it is done in [5] for $p > 1$. □

Now we introduce some further functions, the relation of which to the time maps will be seen from the subsequent lemma. They will be used in the proofs of Lemmata 5.5 and 7.6.

1.10 Definiton
Let $p, q \in \mathbb{R}$, $p \neq 1$, $a > 0$ and

$$K(m) := K_{p,q,a}(m) := \frac{2q-p-1}{(p-1)a} \frac{R_{p,q,a}^{q-p}(m)}{1 - \frac{q}{a}R_{p,q,a}^{2q-p-1}(m)}$$

for all such m that $R_{p,q,a}(m)$ is defined. We introduce $K_{1;p,q,a}(m) =: K_1(m)$ and $K_{2;p,q,a}(m) =: K_2(m)$ analogously.

1.11 Lemma
Assume $p, q \in \mathbb{R}$, $p \neq 1$, $a > 0$. Let $\mathcal{R}$ be one of functions R, R_1, R_2, and suppose that its domain is an interval, denote it by I. Let $\mathcal{L} \in \{L, L_1, L_2\}$ and $\mathcal{K} \in \{K, K_1, K_2\}$ be the corresponding functions. Then $\mathcal{K} \in C^\infty(I)$, and the following holds for all $m \in I$:

$$\mathcal{L}'(m) = 0 \iff \mathcal{L}(m) = \mathcal{K}(m),$$

$$\mathcal{K}'(m) = \frac{2q-p-1}{(p-1)am^{2p}}\left((q-1)\frac{q}{a}\mathcal{R}^{2q-p-1}(m) + q - p\right)\mathcal{R}^{p+q-1}(m)(\mathcal{R}'(m))^3.$$

Proof: Both of the assertions can be proved using Lemma 1.9. □

1.12 Remark
Let $p, q \in \mathbb{R}$, $a > 0$ and let $\mathcal{R}$, $\mathcal{L}$ and I have the same meaning as in Lemma 1.11. It follows from (I.8) that $\mathcal{R}$ has no stationary point. So it can be seen from (I.11)

that if $p = 1$ (the case not dealt with in Lemma 1.11), then either $\mathcal{L}' \equiv 0$ (for $q = 1$) or $\mathcal{L}$ has no stationary point (for $q \neq 1$).

In the subsequent sections we will look for extrema of $\mathcal{L}$, among other things. So assume now only $p \neq 1$. If $m \in I$ is a stationary point of $\mathcal{L}$, then $\mathcal{L}''(m) = 0$ (the case when it is more difficult to determine whether there is an extremum) if and only if

$$q = \frac{p+1}{2} \quad \text{or} \quad (q-1)q\mathcal{R}^{2q-p-1}(m) = (p-q)a. \tag{I.13}$$

Let us notice that it is also a necessary and sufficient condition under that $\mathcal{K}'(m) = 0$ holds. Thus:

(i) If $q = \frac{p+1}{2}$ or $p = q = 0$, then $\mathcal{K}' \equiv 0$.

(ii) If $q = 0$, $p \neq 0, -1$ or $q = 1$, then $\mathcal{K}$ has no stationary point.

(iii) If $q \neq 0, 1, \frac{p+1}{2}$, then (I.13) is equivalent to

$$\mathcal{R}^{2q-p-1}(m) = \frac{(p-q)a}{(q-1)q},$$

which can hold for at most one $m \in I$ due to the strict monotonicity of $\mathcal{R}$. Therefore, if (p, q) does not belong to cases V–VII, then $\mathcal{K} = K$ has at most one stationary point, which will be denoted by $\overline{m} = \overline{m}_{p,q,a}$ (see Lemma 5.5). On the other hand, if $q > 0, \frac{p+1}{2}$ (cases V–VII), then R_1 and R_2 have disjoint ranges (due to (I.7)), so at most one of K_1 and K_2 can have a stationary point, which will be denoted by $\overline{m} = \overline{m}_{p,q,a}$ as well (see Definition 7.2 and Lemmata 7.3 (ii), 7.6, 7.7).

2 Properties of function I_p

In this section we collect statements about I_p (see Theorem 1.1 for its definition) needed for later investigation of the time maps. More specifically, asymptotic expansions at both 1 and ∞, explicit formulae for special values of p, and continuity and differentiability results will be provided.

We will use standard asymptotic notations: If f, g are functions defined in some punctured neighbourhood of a point $a \in \mathbb{R} \cup \{\pm\infty\}$, then

$$\begin{aligned} f(x) \sim g(x),\ x \to a \quad &\text{means} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = 1, \\ f(x) = o(g(x)),\ x \to a \quad &\text{means} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = 0, \\ f(x) = O(g(x)),\ x \to a \quad &\text{means} \quad \limsup_{x\to a} \left|\frac{f(x)}{g(x)}\right| < \infty. \end{aligned}$$

2.1 Lemma
For arbitrary $p \in \mathbb{R}$ we have

$$I_p(y) = 2\sqrt{y-1}\Big(1 - \frac{p}{12}(y-1) + o(y-1)\Big), \quad y \to 1.$$

Proof: Suppose $p \neq -1$. Then

$$I_p(y) = \int_0^{y-1} f_p(x)\,\mathrm{d}x,$$

where

$$f_p(x) = \sqrt{\frac{p+1}{(1+x)^{p+1}-1}} = \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1+\frac{p}{2}x + o(x)}} = \frac{1}{\sqrt{x}} - \frac{p}{4}\sqrt{x} + o(\sqrt{x}), \quad x \to 0.$$

(We used the Maclaurin polynomial of $y \mapsto (1+y)^\alpha$ for $\alpha = p+1$ and $\alpha = -\frac{1}{2}$.) So it suffices to integrate the obtained asymptotic expansion from 0 to $y-1$.

The case $p = -1$ is analogous. □

2.2 Definiton
For all $s \geq 0$ set

$$p_s := -\frac{2s-1}{2s+1}.$$

Thus,

$$\begin{aligned} \{p_n\}_{n=0}^{\infty} &= \big(1, -\tfrac{1}{3}, -\tfrac{3}{5}, -\tfrac{5}{7}, \ldots\big), \\ \Big\{p_{n+\frac{1}{2}}\Big\}_{n=0}^{\infty} &= \big(0, -\tfrac{1}{2}, -\tfrac{2}{3}, -\tfrac{3}{4}, \ldots\big). \end{aligned}$$

The integral I_p can be explicitly calculated for these values.

2.3 Theorem
Let $n \in \mathbb{N} \cup \{0\}$. Then

$$I_{p_{n+1/2}}(y) = 2\sqrt{n+1}\,\widetilde{I}_n\Big(y^{\frac{1}{n+1}} - 1\Big), \quad y \geq 1, \tag{I.14}$$

where

$$\widetilde{I}_n(z) = \sqrt{z}\sum_{k=0}^{n}\frac{1}{2k+1}\binom{n}{k}z^k, \quad z \geq 0$$

and

$$I_{p_n}(y) = \sqrt{2(2n+1)}\,\widehat{I}_n\Big(y^{\frac{2}{2n+1}}\Big), \quad y \geq 1, \tag{I.15}$$

where

$$\widehat{I}_n(z) = \frac{(2n-1)!!}{(2n)!!}\left(\ln\big(\sqrt{z}+\sqrt{z-1}\big) + \sqrt{1-\frac{1}{z}}\sum_{k=1}^{n}\frac{(2k-2)!!}{(2k-1)!!}\,z^k\right), \quad z \geq 1.$$

(We set $(-1)!! := 1$*.)*

Proof: Using the substitution

$$\sqrt{V^{p_{n+1/2}}-1} = \sqrt{V^{\frac{1}{n+1}}-1} =: u$$

and denoting

$$\int_0^{\sqrt{z}}\big(u^2+1\big)^n\,\mathrm{d}u =: \widetilde{I}_n(z),$$

we obtain (I.14). The integral $\widetilde{I}_n(z)$ can be calculated by the binomial theorem.
By the substitutions

$$V^{p_n+1} = V^{\frac{2}{2n+1}} =: \frac{1}{\cos^2 v}, \quad v \in \big[0, \tfrac{\pi}{2}\big), \qquad \sin v =: u$$

we obtain (I.15) with

$$\widehat{I}_n(z) = \int_0^{\sqrt{1-\frac{1}{z}}}\frac{\mathrm{d}u}{(1-u^2)^{n+1}}.$$

Integrating $\widehat{I}_n(z)$ by parts, we can derive the recurrent relation

$$\widehat{I}_n(z) = \frac{2n-1}{2n}\left(\widehat{I}_{n-1}(z) + \frac{1}{2n-1}\sqrt{1-\frac{1}{z}}\,z^n\right),$$

from which the formula in the theorem follows. □

We will also use the following special cases of (I.15) and (I.14):

$$I_1(y) = \sqrt{2}\ln\big(y+\sqrt{y^2-1}\big), \tag{I.16}$$

$$I_0(y) = 2\sqrt{y-1}, \tag{I.17}$$

$$I_{-1/2}(y) = \frac{2\sqrt{2}}{3}\sqrt{\sqrt{y}-1}\big(\sqrt{y}+2\big). \tag{I.18}$$

Now the most important statement of this section follows, yielding the asymptotic expansion of $I_p(y)$ for $y \to \infty$, $p > -1$. It is essential for investigating the behaviour of the time maps in many cases, but was not needed in [5] for $p, q > 1$.

2.4 Lemma
For $k \in \mathbb{N} \cup \{0\}$ and $p \in (-1, \infty) \setminus \{p_k\}$ put

$$b_k(p) := \frac{(2k-1)!!}{(2k)!!} \frac{2}{(2k+1)(p-p_k)} = \frac{(2k-1)!!}{(2k)!!} \frac{1}{\frac{p-1}{2} + k(p+1)},$$

and for $p > -1$ set

$$B_p := \sum_{\substack{k \in \mathbb{N} \cup \{0\} \\ p_k \neq p}} b_k(p) \in \mathbb{R}.$$

Then the following holds for $y \to \infty$:

(i) *If $p > 1$, then*

$$\frac{I_p(y)}{\sqrt{p+1}} = B_p + o(1).$$

(ii) *If $p_{n+1} < p < p_n$ for some $n \in \mathbb{N} \cup \{0\}$, then*

$$\frac{I_p(y)}{\sqrt{p+1}} = \sum_{k=0}^{n} \underbrace{(-b_k(p))}_{>0} y^{\overbrace{\frac{1-p}{2} - k(p+1)}^{}}\underbrace{}_{>0} + B_p + o(1).$$

(iii) *If $p = p_n$ for some $n \in \mathbb{N} \cup \{0\}$, then*

$$\frac{I_p(y)}{\sqrt{p+1}} = \sum_{k=0}^{n-1} \underbrace{(-b_k(p))}_{>0} y^{\frac{1-p}{2} - k(p+1)}\underbrace{}_{>0} + \frac{(2n-1)!!}{(2n)!!} \ln y + B_p + o(1).$$

Furthermore, $p \mapsto B_p$ belongs to C^∞ on each of intervals (p_0, ∞), (p_1, p_0), (p_2, p_1), …, and decreases on each of them, while

$$\lim_{p \to p_0+} B_p = \infty, \quad \lim_{p \to \infty} B_p = 0,$$

and for all $n \in \mathbb{N}$ we have:

$$\lim_{p \to p_{n+1}+} B_p = \infty, \quad B_{p_{n+1/2}} = 0, \quad \lim_{p \to p_n-} B_p = -\infty.$$

Proof: It consists of

1. expressing $I_p(y)$ as the sum of a series (see (I.19)),
2. proving the finiteness of B_p and verifying statements (i), (ii), (iii),
3. and examining the properties of the function $p \mapsto B_p$.

1. Let $p > -1$ and $y \geq 1$. The substitution $V := x^{-1/(p+1)}$ gives:

$$\frac{I_p(y)}{\sqrt{p+1}} = \frac{1}{p+1} \int_{1/y^{p+1}}^{1} \frac{1}{\sqrt{1-x}} x^{-\frac{1}{2} - \frac{1}{p+1}} \, dx.$$

Using the Maclaurin series of the function $x \mapsto 1/\sqrt{1-x}$, we get that

$$\frac{I_p(y)}{\sqrt{p+1}} = \frac{1}{p+1} \int_{1/y^{p+1}}^{1} \left(\sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} x^{k - \frac{1}{2} - \frac{1}{p+1}} \right) dx.$$

Levi's monotone convergence theorem allows us to exchange the order of integration and summation, resulting in

$$\frac{I_p(y)}{\sqrt{p+1}} = \sum_{k=0}^{\infty} a_{k,p}(y), \tag{I.19}$$

where

$$a_{k,p}(y) = \begin{cases} b_k(p)\Big(1 - y^{\frac{1-p}{2} - k(p+1)}\Big) & \text{if } p \neq p_k, \\ \dfrac{(2k-1)!!}{(2k)!!} \ln y & \text{if } p = p_k. \end{cases}$$

2. It is obvious that for all $k \in \mathbb{N} \cup \{0\}$ and $p > -1$, $a_{k,p}$ is increasing, positive on $(1, \infty)$, and

$$\lim_{y \to \infty} a_{k,p}(y) = \begin{cases} b_k(p) & \text{if } p > p_k, \\ \infty & \text{if } p \leq p_k. \end{cases} \tag{I.20}$$

Now let $m \in \mathbb{N} \cup \{0\}$ and $p > p_m$. Stirling's formula ($n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n$ for $n \to \infty$) implies that

$$b_k(p) \sim \frac{1}{\sqrt{\pi}(p+1)k^{3/2}}, \quad k \to \infty,$$

which guarantees the convergence of $\sum_{k=m}^{\infty} b_k(p)$ (and also the finiteness of B_p). We are going to prove that

$$\lim_{y \to \infty} \sum_{k=m}^{\infty} a_{k,p}(y) = \sum_{k=m}^{\infty} b_k(p) \tag{I.21}$$

because statement (i) follows from (I.19) and (I.21) with $m = 0$, while statements (ii), (iii) from (I.19) and (I.21) with $m = n + 1$.

The inequality "$\leq$" in (I.21) is clear from (I.20) and the increase of $a_{k,p}$. In order to prove the opposite inequality, let us choose any $\varepsilon > 0$. We have that

$$\sum_{k=m}^{n_0} b_k(p) > \sum_{k=m}^{\infty} b_k(p) - \frac{\varepsilon}{2}$$

for some $n_0 \geq m$. The positivity of $a_{k,p}$ on $(1, \infty)$ together with (I.20) yields that there exists a number $K > 1$ such that

$$\sum_{k=m}^{\infty} a_{k,p}(y) > \sum_{k=m}^{n_0} a_{k,p}(y) > \sum_{k=m}^{n_0} b_k(p) - \frac{\varepsilon}{2}$$

for all $y > K$. Joining the last two inequalities, we obtain (I.21).

3. The decrease of $p \mapsto B_p$ on intervals (p_0, ∞), (p_1, p_0), (p_2, p_1), $\ldots$ follows immediately from the decrease of functions b_k on these intervals.

Let us now prove that $(p \mapsto B_p) \in C^\infty((-1, \infty) \smallsetminus \bigcup_{n=0}^{\infty}\{p_n\})$. We will use the C^∞-smoothness of functions b_k. If we choose arbitrary $m, n \in \mathbb{N} \cup \{0\}$ and $[\alpha, \beta] \subseteq (p_n, \infty)$, then applying the Weierstraß criterion, we can verify that $\sum_{k=n}^{\infty} (b_k)^{(m)}$ converges uniformly on $[\alpha, \beta]$, therefore we can differentiate it term by term. So the sum of $\sum_{k=n}^{\infty} b_k$ belongs to $C^\infty([\alpha,\beta])$, thus also to $C^\infty((p_n,\infty))$, from which the C^∞-smoothness of the function $p \mapsto B_p$ on $(-1,\infty) \setminus \bigcup_{n=0}^{\infty}\{p_n\}$ follows.

The one-sided limits of $p \mapsto B_p$ in p_0, p_1, ... are found easily. They—together with its continuity and decrease on (p_{n+1}, p_n)—guarantee the existence of a unique point $p_n^* \in (p_{n+1}, p_n)$ such that $B_{p_n^*} = 0$. Statement (ii) gives the expansion

$$I_{p_{n+\frac{1}{2}}}(y) = 2\sqrt{n+1}\sum_{k=0}^{n} \frac{1}{2n-2k+1}\,\frac{(2k-1)!!}{(2k)!!}\left(y^{\frac{1}{n+1}}\right)^{\frac{1}{2}+n-k} + \frac{B_{p_{n+1/2}}}{\sqrt{n+1}} + o(1)$$

for $y \to \infty$. On the other hand, from (I.14), using the binomial theorem and the Maclaurin polynomial of $x \mapsto \sqrt{1+x}$ of degree n, we obtain that

$$\begin{aligned} I_{p_{n+\frac{1}{2}}}(y) &= \sqrt{z}\cdot 2\sqrt{n+1}\sqrt{1-\frac{1}{z}}\sum_{i=0}^{n}\frac{1}{2i+1}\binom{n}{i}(z-1)^i \\ &= \sum_{k=0}^{n} c_{n,k} z^{\frac{1}{2}+n-k} + O\left(\frac{1}{\sqrt{z}}\right) \end{aligned}$$

for $z = y^{1/(n+1)} \to \infty$ and some constants $c_{n,k}$, $k = 0, 1, \ldots n$. Consequently, $p_n^* = p_{n+1/2}$.

Finally, in order to find $\lim_{p\to\infty} B_p$, we employ the uniform convergence of $\sum_{k=0}^{\infty} b_k$ on (α,∞) for $\alpha > 1$, and so we exchange the order of the limit and the sum. □

The asymptotic expansion of I_{-1} can be derived much easier.

2.5 Lemma
For every $y > 1$, $n \in \mathbb{N}$:

$$I_{-1}(y) = \sum_{k=0}^{n-1}\frac{(2k-1)!!}{2^k}\,\frac{y}{\ln^{k+1/2} y} + O\left(\frac{y}{\ln^{n+1/2} y}\right), \qquad y \to \infty.$$

Proof: Set

$$\overline{I}_n(y) := \int_{\mathrm{e}}^{y} \frac{\mathrm{d}V}{\ln^{n+1/2} V}$$

for all $N \in \mathbb{N} \cup \{0\}$ and $y > 1$. Integrating by parts, we can derive the recurrent relation

$$\overline{I}_n(y) = \frac{y}{\ln^{n+1/2} y} - \mathrm{e} + \frac{2n+1}{2}\overline{I}_{n+1}(y).$$

Using it n times, we obtain

$$I_{-1}(y) = \overline{I}_0(y) + \int_1^{\mathrm{e}} \frac{\mathrm{d}V}{\sqrt{\ln V}} = \sum_{k=0}^{n-1}\frac{(2k-1)!!}{2^k}\,\frac{y}{\ln^{k+1/2} y} + R_n(y),$$

where

$$R_n(y) = \int_1^{\mathrm{e}} \frac{\mathrm{d}V}{\sqrt{\ln V}} - \sum_{k=0}^{n-1}\frac{(2k-1)!!}{2^k}\mathrm{e} + \frac{(2n-1)!!}{2^n}\overline{I}_n(y) \sim \frac{(2n-1)!!}{2^n}\,\frac{y}{\ln^{n+1/2} y}$$

for $y \to \infty$, which can be proved using l'Hôpital's rule. $\square$

Notice that although Lemma 2.5 gives an asymptotic expansion, the corresponding series

$$\sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k} \frac{y}{\ln^{k+1/2} y}$$

diverges for all $y > 1$.

The last two assertions concern the continuity and differentiability of $I_p(y)$ as a function of two variables.

2.6 Theorem
The function $(y, p) \mapsto I_p(y)$ is continuous on $[1, \infty) \times \mathbb{R}$. Furthermore, $p \mapsto I_p(y)$ is decreasing on $\mathbb{R}$ for any $y > 1$.

Proof: Let us express $I_p(y)$ as

$$I_p(y) = \int_1^y \lambda(V, p) \, \mathrm{d}V,$$

where

$$\lambda(V, p) = \begin{cases} \sqrt{\dfrac{p+1}{V^{p+1} - 1}} & \text{if } p \neq -1,\ V > 1, \\ \dfrac{1}{\sqrt{\ln V}} & \text{if } p = -1,\ V > 1. \end{cases}$$

Obviously, λ is continuous in both of its variables, and is decreasing in V. Consequently, it is continuous (on $(1, \infty) \times \mathbb{R}$). Similarly, if we prove the continuity of $p \mapsto I_p(y)$ for all $y > 1$ (for $y = 1$ it is evident), then using the continuity and increase of I_p for any $p \in \mathbb{R}$, we will have that $(y, p) \mapsto I_p(y)$ is continuous.

For this purpose, it will be important to know the behaviour of $\lambda(V, \cdot)$. We can derive that for any $p \neq -1$ and $V > 1$:

$$\frac{\partial}{\partial p} \frac{1}{\lambda^2(V, p)} > 0 \quad \Longleftrightarrow \quad \ln V^{p+1} + \frac{1}{V^{p+1}} - 1 > 0,$$

which can be equivalently written as $\ln x < x - 1$ for $x := 1/V^{p+1} \in (0, 1) \cup (1, \infty)$. Thus, $1/\lambda^2(V, \cdot)$ is increasing on $\mathbb{R}$, therefore $\lambda(V, \cdot)$ is decreasing, and the second assertion of the lemma holds.

Now choose arbitrary $y > 1$, $p_0 \in \mathbb{R}$. Since $\lambda(\cdot, p_0)$ is an integrable majorant of $\{\lambda(\cdot, p)\}_{p \geq p_0}$ on $(1, y)$, and $\lambda(V, \cdot)$ is continuous, we have the continuity of $p \mapsto I_p(y)$ on $[p_0, \infty)$. $\square$

2.7 Theorem
The function $(y, p) \mapsto I_p(y)$ is continuously differentiable on $(1, \infty) \times (-1, \infty)$, while

$$\frac{\partial}{\partial p} \frac{I_p(y)}{\sqrt{p+1}} = -\frac{1}{2} \int_1^y \frac{V^{p+1} \ln V}{(V^{p+1} - 1)^{3/2}} \, \mathrm{d}V =: J_p(y) \tag{I.22}$$

for all $y > 1$, $p > -1$.

Proof: Firstly, we prove that $p \mapsto I_p(y)/\sqrt{p+1}$ is continuously differentiable on $(-1,\infty)$ for any $y > -1$, and fulfils (I.22). So chose arbitrary $y > 1$ and $p_0 > -1$. We have

$$\frac{I_p(y)}{\sqrt{p+1}} = \int_1^y \underbrace{\frac{1}{\sqrt{V^{p+1}-1}}}_{=:\mu(V,p)} \, \mathrm{d}V, \qquad p \geq p_0$$

with

$$\frac{\partial \mu}{\partial p}(V,p) = -\frac{V^{p+1}\ln V}{2(V^{p+1}-1)^{3/2}} < 0, \qquad V \in (1,y), \ p \geq p_0.$$

Since

$$\frac{\partial^2 \mu}{\partial p^2}(V,p) = \frac{V^{p+1}\big(V^{p+1}+2\big)\ln^2 V}{4(V^{p+1}-1)^{5/2}} > 0, \qquad V \in (1,y), \ p \geq p_0,$$

$-\frac{\partial \mu}{\partial p}(\cdot, p_0)$ is a majorant of $\{\frac{\partial \mu}{\partial p}(\cdot, p)\}_{p \geq p_0}$. And it is also integrable because

$$\frac{\partial \mu}{\partial p}(V,p_0) = -\frac{V^{p_0+1}\ln V}{2(V^{p_0+1}-1)^{3/2}} \sim \frac{1}{2(p_0+1)\sqrt{V-1}}, \qquad V \to 1$$

(Taylor polynomials can be used). Consequently, $p \mapsto I_p(y)/\sqrt{p+1}$ is differentiable on (p_0,∞), and (I.22) holds. Moreover, $p \mapsto J_p(y)$ is continuous on (p_0,∞) due to the continuity of $\frac{\partial \mu}{\partial p}(V,\cdot)$ for all $V \in (1,y)$.

In order to obtain the continuous differentiability of $(y,p) \mapsto I_p(y)/\sqrt{p+1}$ (or equivalently of $(y,p) \mapsto I_p(y)$), we have to validate the continuity of its partial derivatives: Since $J_p(y)$ is continuous in p, and is apparently continuous and decreasing in y, it is indeed continuous. And the continuity of

$$\frac{\partial}{\partial y}\frac{I_p(y)}{\sqrt{p+1}} = \frac{1}{\sqrt{y^{p+1}-1}}$$

is obvious. □

3 Case I ($p=-1$, $q=0$)

This case is the simplest one since from Lemma 1.5 it directly follows that

$$L(m) = \frac{m}{\sqrt{2a}}\, I_{-1}\Big(\mathrm{e}^{\frac{1}{2a}}\Big), \quad m > 0.$$

Thus, the time map, which determines the relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$, is linear. So substituting into Lemma 1.8, we obtain the following theorem:

3.1 Theorem
Assume $p=-1$, $q=0$, $a>0$. Then for arbitrary $l>0$:

$$\mathcal{S}(l) = \left\{ u_{m,-1,a}\big|_{[-l,l]} \;:\; m = \frac{\sqrt{2a}}{I_{-1}\big(\mathrm{e}^{\frac{1}{2a}}\big)}\, l \right\},$$
$$\mathcal{N}(l) = \emptyset.$$

4 Case II ($p > -1$, $q = 0$)

In this section we answer the question of the solvability of (1) for

$$p > -1, \ q = 0, \ a > 0 \tag{I.23}$$

finding $\lim_{m\to 0} L(m)$ and $\lim_{m\to\infty} L(m)$, and proving the monotonicity of L. However, let us first summarise the properties of R that will be used in the subsequent lemmata.

4.1 Lemma
Let (I.23) *hold. Then* $R' > 0$, *and*

$$\lim_{m\to 0} R(m) = \left(\frac{p+1}{2a}\right)^{\frac{1}{p+1}},$$

$$R(m) = m\left(1 + \frac{1}{2am^{p+1}} + o\left(\frac{1}{m^{p+1}}\right)\right), \quad m \to \infty.$$

Proof: It suffices to use the explicit formula for $R(m)$ given by Lemma 1.5. □

4.2 Lemma
Assume (I.23). *Then*

$$\lim_{m\to 0} L(m) = \begin{cases} \infty & \text{if } p \geq 1, \\ \dfrac{2}{1-p}\left(\dfrac{p+1}{2a}\right)^{\frac{1}{p+1}} =: L_{p,0,a}(0) =: L(0) & \text{if } p \in (-1,1), \end{cases}$$

$$\lim_{m\to\infty} L(m) = \begin{cases} 0 & \text{if } p > 0, \\ \frac{1}{a} & \text{if } p = 0, \\ \infty & \text{if } p \in (-1,0). \end{cases}$$

Proof: For $p > 1$ and $p = 1$, $\lim_{m\to 0} L(m)$ is easily found using Lemma 4.1 and (I.3). In the case of $p \in (-1,1)$, it is of type $\frac{\infty}{\infty}$:

$$\lim_{m\to 0} L(m) = \lim_{m\to 0} \frac{I_p\left(\frac{R(m)}{m}\right)}{\sqrt{2a}m^{\frac{p-1}{2}}},$$

and we calculate it by l'Hôpital's rule, (I.10) and Lemma 4.1.

According to Lemmata 4.1 and 2.1:

$$L(m) \sim \sqrt{\frac{2}{a}}\, m^{\frac{1-p}{2}} \sqrt{\frac{R(m)}{m} - 1}, \quad m \to \infty,$$

while

$$\frac{R(m)}{m} - 1 \sim \frac{1}{2am^{p+1}}, \quad m \to \infty.$$

Connecting these two expansions, we obtain that $L(m) \sim \frac{1}{am^p}$ for $m \to \infty$, and the second assertion follows. □

4.3 Lemma
Let (I.23) *hold. Then:*

(i) *If* $p > 0$, *then* $L' < 0$.

(ii) *If* $p = 0$, *then* $L \equiv \frac{1}{a}$.

(iii) *If* $-1 < p < 0$, *then* $L' > 0$.

Proof:

(i) Firstly, let us consider $p > 0$. Due to (I.11), the case $p \geq 1$ is clear. So let $0 < p < 1$. If L has a stationary point $m_0 > 0$, then $L''(m_0) > 0$ according to (I.12) and Lemma 4.1, thus it is a point of strict relative minimum. Therefore, either L has no stationary point or it has exactly one, which is a point of global minimum. However, the second possibility contradicts the fact that $\lim_{m\to\infty} L(m) = 0$ (Lemma 4.2).

(ii) For $p = 0$, Lemma 1.5 gives the formula $R(m) = m + \frac{1}{2a}$, so $L(m) = \frac{1}{a}$ according to (I.17).

(iii) Finally, let us have $p \in (-1, 0)$, and let us proceed as for $p \in (0, 1)$. Now L attains a strict relative maximum in each of its stationary points. On the other hand, $\lim_{m\to\infty} L(m) = \infty$ so the only possibility is that $L' > 0$ on $(0, \infty)$. □

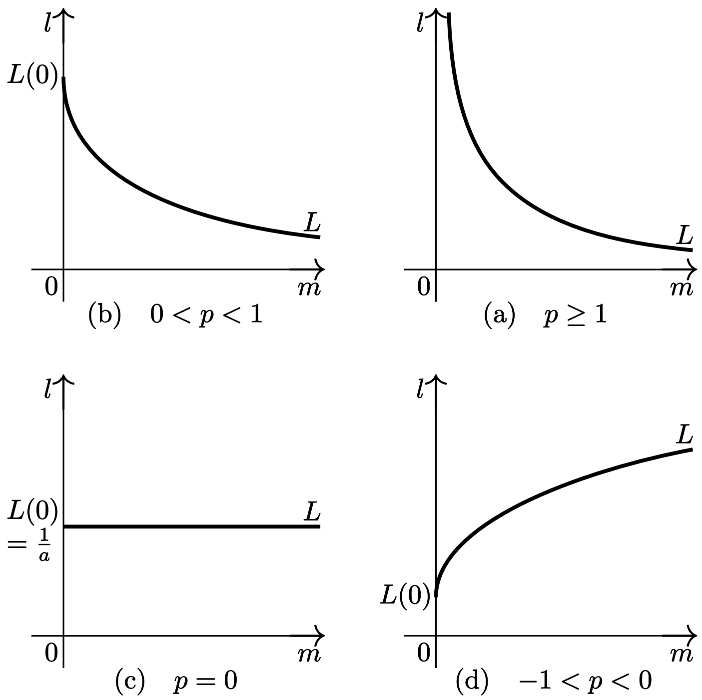

Figure 2: The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case II ($p > -1$, $q = 0$, $a > 0$) according to Lemmata 1.8, 4.2 and 4.3. See also Theorem 4.4.

From the results of the last two lemmata (which are summarised in Figure 2), applying Lemma 1.8, we obtain the main statement of this section:

4.4 Theorem
Assume (I.23) *and* $l > 0$. *Then* $\mathcal{N}(l) = \emptyset$, *and the following holds for positive symmetric solutions of* (1)*:*

If $p \geq 1$, *then* $|\mathcal{S}(l)| = 1$, *and* L *is decreasing. (Recall that* $L(u(0)) = l$ *for any* $u \in \mathcal{S}(l)$*.)*

If $p = 0$, *then* (1) *has a solution only for* $l = \frac{1}{a}$, *namely*

$$\mathcal{S}\left(\frac{1}{a}\right) = \left\{x \mapsto \frac{a}{2}x^2 + m,\ x \in [-l, l]\ :\ m > 0\right\}.$$

If $p < 1$ *and* $p \neq 0$, *then*

$$|\mathcal{S}(l)| = \begin{cases} 1 & \text{if } l \text{ is between } L(0) \text{ and } \lim\limits_{m\to\infty} L(m), \\ 0 & \text{otherwise,} \end{cases}$$

and L *is strictly monotone. (See Lemma 4.2 about* $L(0)$ *and* $\lim_{m\to\infty} L(m)$*.)*

The last question we will answer in this section is whether $L_{\cdot,0,a}(0)$ is monotone.

4.5 Lemma
Suppose that (I.23) *holds, let* $\overline{p}$ *be the unique solution of the equation* $p^3 - 7p - 2 = 0$ *in* $(-1, 0)$, *and set*

$$\overline{a} := \frac{\overline{p}+1}{2}\mathrm{e}^{\frac{2}{3-\overline{p}}-2} \in \left(\frac{1}{2\mathrm{e}^2}, \frac{1}{\mathrm{e}}\right).$$

Then:

(i) *If* $a > \overline{a}$, *then* $\frac{\partial}{\partial p}L_{p,0,a}(0) > 0$ *for* $p \in (-1, 1)$.

(ii) *If* $a = \overline{a}$, *then* $\frac{\partial}{\partial p}L_{p,0,a}(0) > 0$ *for* $p \in (-1, 1) \setminus \{\overline{p}\}$, *and* $\frac{\partial}{\partial p}L_{p,0,a}(0)|_{p=\overline{p}} = 0$.

(iii) *If* $0 < a < \overline{a}$, *then* $p \mapsto L_{p,0,a}(0)$ *has two stationary points:* $p_1 = p_1(a) \in (-1, \overline{p})$ *and* $p_2 = p_2(a) \in (\overline{p}, 1)$, *while* $\frac{\partial}{\partial p}L_{p,0,a}(0) > 0$ *for* $p \in (-1, p_1) \cup (p_2, 1)$, *and* $\frac{\partial}{\partial p}L_{p,0,a}(0) < 0$ *for* $p \in (p_1, p_2)$.

Furthermore, for all $a > 0$ *we have*

$$\lim_{p\to -1+} L_{p,0,a}(0) = 0, \qquad \lim_{p\to 1-} L_{p,0,a}(0) = \infty.$$

Proof: The limits of $L_{p,0,a}(0)$ can be easily calculated. We also have that

$$\frac{\partial}{\partial p}L_{p,0,a}(0) > 0 \quad \Longleftrightarrow \quad \ln\frac{p+1}{2a} - \frac{(p+1)^2}{1-p} - 1 =: \psi_a(p) < 0.$$

So we need to examine the properties of ψ_a. It is not difficult to derive that

$$\psi_a'(p) > 0 \quad \Longleftrightarrow \quad p^3 - 7p - 2 =: \varrho(p) > 0.$$

Since ϱ is decreasing on $(-1, 1)$, and $\varrho(0) < 0 < \lim_{p\to -1}\varrho(p)$, it has a unique zero $\overline{p} \in (-1, 0)$. It means that ψ_a increases on $(-1, \overline{p}]$, and decreases on $[\overline{p}, 1)$. However, $\lim_{p\to -1+}\psi_a(p) = \lim_{p\to 1-}\psi_a(p) = -\infty$, thus $L_{\cdot,0,a}(0)$ has the properties from parts (i), (ii) or (iii) if $\psi_a(\overline{p}) < 0$, $\psi_a(\overline{p}) = 0$ or $\psi_a(\overline{p}) > 0$ respectively.

Using that $\varrho(\overline{p}) = 0$, we obtain:

$$\psi_a(\overline{p}) = \ln\frac{\overline{p}+1}{2a} + \frac{2}{3-\overline{p}} - 2 = 0 \quad\iff\quad a = \overline{a}.$$

Furthermore, $a \mapsto \psi_a(\overline{p})$ is decreasing, so $\psi_a(\overline{p}) < 0$ indeed for $a > \overline{a}$, and $\psi_a(\overline{p}) > 0$ for $a \in (0, \overline{a})$. It remains to check that $\overline{a} \in (\frac{1}{2\mathrm{e}^2}, \frac{1}{\mathrm{e}})$. However, it can be directly proved that $\psi_a < 0$ for $a \geq \frac{1}{\mathrm{e}}$, so $\overline{a} < \frac{1}{\mathrm{e}}$ and $\psi_a(0) \geq 0$, and consequently, $\psi_a(\overline{p}) > 0$ for $a \leq \frac{1}{2\mathrm{e}^2}$, so $\overline{a} > \frac{1}{2\mathrm{e}^2}$. □

Let us mention that $\overline{p} \approx -0.289$, $\overline{a} \approx 0.088$, and using Cardano's formula one can also derive that

$$\overline{p} = 2\sqrt{\frac{7}{3}}\cos\frac{\arccos\frac{3\sqrt{3}}{7\sqrt{7}} - 2\pi}{3}.$$

5 Case III ($p > -1$, $0 < q < \frac{p+1}{2}$)

A part of case III was already examined in [5] (see Lemma 5.2). For the rest we will need the asymptotic expansions of $R(m)$ for $m \to 0$ and $m \to \infty$ (Lemma 5.1), and also Lemma 2.4. We will deal only with

$$p > -1,\ 0 < q < \frac{p+1}{2},\ a > 0. \tag{I.24}$$

5.1 Lemma
Let (I.24) *hold. Then* $R' > 0$*, and*

$$\frac{R(m)}{R(0)} = 1 - \frac{m^{p+1}}{(2q-p-1)R^{p+1}(0)} + o\big(m^{p+1}\big), \qquad m \to 0$$
$$\frac{R(m)}{m} = 1 + \frac{1}{2a}m^{2q-p-1} + \frac{4q-p}{8a^2}m^{2(2q-p-1)} + o\big(m^{2(2q-p-1)}\big), \qquad m \to \infty,$$

where

$$R(0) = R_{p,q,a}(0) = \lim_{m\to 0} R(m) = \left(\frac{2a}{p+1}\right)^{\frac{1}{2q-p-1}}.$$

Proof: It is clear from (I.8) and Lemma 1.5 (i) that $R' > 0$, so R has a positive and finite limit (denoted by $R(0)$) at 0, the value of which can be obtained from the equality

$$0 = \lim_{m\to 0} \mathcal{F}(m, R(m)) = \frac{R^{p+1}(0)}{2a}\left(R^{2q-p-1}(0) - \frac{2a}{p+1}\right).$$

Now we will look for such $c, d > 0$ that

$$\frac{R(m)}{R(0)} - 1 \sim cm^d, \quad m \to 0.$$

So let us calculate the following limit using l'Hôpital's rule and (I.8):

$$\lim_{m\to 0} \frac{\frac{R(m)}{R(0)} - 1}{m^d} = -\frac{p+1}{(2q-p-1)dR^{p+1}(0)} \lim_{m\to 0} m^{p+1-d}.$$

It should be positive and finite, determining the value of c. Therefore, we have $d = p+1$, and c is also given as in the lemma.

The decrease of $m \mapsto R(m)/m \geq 1$ (see (I.9)) guarantees the existence of its positive and finite limit at ∞. So we can use l'Hôpital's rule and (I.8) to derive that

$$A := \lim_{m\to\infty} \frac{R(m)}{m} = \lim_{m\to\infty} \left(\frac{m}{R(m)}\right)^p = \frac{1}{A^p}.$$

Consequently, $A = 1$. The asymptotic expansion of $R(m)/m$ for $m \to \infty$ can be also found by the method of undetermined coefficients, which we used for $m \to 0$. However, let us show an iterative method borrowed from [5, proof of Lemma 3.3]:

Multiplying the equality $\mathcal{F}(m, R(m)) = 0$ (see (I.5)) by $(p+1)/m^{p+1}$, and expressing $R(m)/m$ from it, we obtain:

$$\frac{R(m)}{m} = \left(1 + \frac{p+1}{2a} m^{2q-p-1} \left(\frac{R(m)}{m}\right)^{2q}\right)^{\frac{1}{p+1}}. \tag{I.25}$$

The expression $(R(m)/m)^{2q}$ on the right-hand side can be replaced by $1 + o(1)$, so

$$\frac{R(m)}{m} = \left(1 + \frac{p+1}{2a} m^{2q-p-1} + o\big(m^{2q-p-1}\big)\right)^{\frac{1}{p+1}} = 1 + \frac{1}{2a} m^{2q-p-1} + o\big(m^{2q-p-1}\big)$$

(We used the Maclaurin polynomial of $x \mapsto (1+x)^{1/(p+1)}$.) Now let us insert the asymptotic expansion we have just obtained in the right-hand side of (I.25) again. It yields

$$\frac{R(m)}{m} = \left(1 + \frac{p+1}{2a} m^{2q-p-1} + \frac{(p+1)q}{2a^2} m^{2(2q-p-1)} + o\big(m^{2(2q-p-1)}\big)\right)^{\frac{1}{p+1}},$$

which can be rewritten in the form from the lemma.

Let us remark that we could use this iterative method in the case of $m \to 0$ as well. We only would replace (I.25) by

$$R(m) = R(0)\left(1 - \frac{m^{p+1}}{R^{p+1}(m)}\right)^{\frac{1}{2q-p-1}},$$

which can be derived from the equality $\mathcal{F}(m, R(m)) = 0$ multiplying it by $(p+1)/R^{p+1}(m)$. □

5.2 Lemma (for $p, q > 1$ see [5, Theorem 3.1])
If (I.24) *holds and* $p \geq 1$*, then*

$$\lim_{m \to 0} L(m) = \infty, \qquad L' < 0 \text{ on } (0, \infty), \qquad \lim_{m \to \infty} L(m) = 0.$$

Proof: The proof from [5] for $p, q > 1$ is also valid for $p > 1$, and the case $p = 1$ is similar. □

In the next two lemmata we find the limits of L—denoted by $L(0)$ and $L(\infty)$—for $p < 1$. For the proof of Lemma 5.5 it is also necessary to know the sign of $L - L(0)$ and $L - L(\infty)$ near 0 and ∞ respectively, for certain values of p, q.

5.3 Lemma
Assume (I.24) *and* $p < 1$*. Then*

$$\lim_{m \to 0} L(m) = \frac{2}{1-p} \left(\frac{p+1}{2a}\right)^{\frac{q-1}{2q-p-1}} =: L_{p,q,a}(0) =: L(0)$$

and furthermore, $L > L(0)$ *in some neighbourhood of* 0 *for* $-\frac{1}{3} < p \leq 0$*, and* $L < L(0)$ *in some neighbourhood of* 0 *for* $0 < p < 1$*.*

Proof: The $\lim_{m\to 0} L(m)$ is found in the same way as in Lemma 4.2. So choose any $p \in (-\frac{1}{3}, 1)$, and let us calculate the second term of the asymptotic expansion of $L(m)$ for $m \to 0$, which will allow us to determine whether $L < L(0)$ or $L > L(0)$ near 0. Lemma 5.1 yields:

$$R(m) = R(0)\big(1 + O(m^{p+1})\big) = R(0)\Big(1 + o\Big(m^{\frac{1-p}{2}}\Big)\Big).$$

Joining it with the expansion of $I_p(y)$ from Lemma 2.4, we obtain:

$$L(m) = L(0) + \sqrt{\frac{p+1}{2a}} B_p m^{\frac{1-p}{2}} + o\Big(m^{\frac{1-p}{2}}\Big).$$

As we know, $B_p > 0$ for $p \in (-\frac{1}{3}, 0)$, and $B_p < 0$ for $p \in (0, 1)$, guaranteeing the validity of the statement of the lemma for these values of p.

It remains to examine $p = 0$. In that case we can use (I.17). So

$$L(m) = L(0)\sqrt{1 + \frac{2q}{1-2q}(2a)^{\frac{1}{1-2q}} m + o(m)} = L(0) + \underbrace{\frac{2q}{1-2q}(2a)^{\frac{q}{1-2q}}}_{>0} m + o(m)$$

due to Lemma 5.1. □

5.4 Lemma
If (I.24) *holds and* $p < 1$, *then*

$$\lim_{m\to\infty} L(m) = \begin{cases} 0 & \text{if } q < p, \\ \frac{1}{a} & \text{if } q = p, \\ \infty & \text{if } q > p \end{cases}$$

and furthermore, $L > \frac{1}{a}$ *in some neighbourhood of* ∞ *for* $q = p$.

Proof: The proof of the first statement does not differ from that of Lemma 4.2. So let $q = p$, and join the expansions of Lemmata 2.1 and 5.1 for $m \to \infty$:

$$\begin{aligned} L(m) &= \frac{1}{a}\sqrt{1 + \frac{3p}{4a}m^{p-1} + o\big(m^{p-1}\big)}\Big(1 - \frac{p}{24a}m^{p-1} + o\big(m^{p-1}\big)\Big) \\ &= \frac{1}{a} + \frac{p}{3a^2}m^{p-1} + o\big(m^{p-1}\big). \end{aligned}$$

Since $p \in (0, 1)$ and thus $\frac{p}{3a^2} > 0$, $L > \frac{1}{a}$ indeed near ∞. □

5.5 Lemma
Suppose that (I.24) *holds, and for* $q > |p|$ *set*

$$\overline{m} := \overline{m}_{p,q,a} := \left(\frac{(p+q)(2q-p-1)}{2q(q-1)}\right)^{\frac{1}{p+1}} \left(\frac{a(q-p)}{q(1-q)}\right)^{\frac{1}{2q-p-1}}.$$

(i) *If* $p < 1$, $q \le p$, *then* $L' < 0$ *on* $(0, \infty)$.

(ii) *If $p > 0$, $q > p$, then L has a stationary point $m_{0;p,q,a} =: m_0 \in (0, \overline{m}]$, while $L' < 0$ on $(0, m_0)$, $L' > 0$ on (m_0, ∞).*

(iii) *If $p \leq 0$, $q > -p$, then $L' > 0$ on $(0, \infty) \setminus \{\overline{m}\}$.*

(iv) *If $q \leq -p$, then $L' > 0$ on $(0, \infty)$.*

Proof: It is similar to the proof of Lemma 4.3. So suppose that $m_0 > 0$ is a stationary point of L. From (I.12) it is clear that $L''(m_0)$ has the same sign as

$$(1-q)\frac{q}{a}R^{2q-p-1}(m_0) + p - q =: \varrho_{p,q,a}(m_0) =: \varrho(m_0).$$

Therefore, if $q \leq p$, then L has at most one stationary point, and if it has some, then it attains a strict relative minimum there. However, L cannot increase near ∞ (see Lemma 5.4), thus statement (i) holds.

In the rest of the proof we will deal with $q > p$. We have

$$L''(m_0) > 0 \iff R(m_0) < \left(\frac{a(q-p)}{q(1-q)}\right)^{\frac{1}{2q-p-1}} =: \overline{R}_{p,q,a} =: \overline{R}$$

and

$$\overline{R} > R(0) \iff (2q-p-1)(p+q) < 0 \iff q > -p.$$

Since $(R(0), \infty)$ is the range of R, each stationary point of L is a point of strict relative maximum for $q \leq -p$, and statement (iv) follows due to Lemma 5.4.

We will suppose $q > -p$ from now on (together with $q > p$), thus $-\frac{1}{3} < p < 1$. Consequently,

$$L''(m_0) > 0 \iff m_0 < R^{-1}(\overline{R}) = \overline{R}\left(1 - \frac{p+1}{2a}\overline{R}^{2q-p-1}\right)^{\frac{1}{p+1}} = \overline{m}.$$

So Lemma 5.4 guarantees that L does not attain any relative extremum in $(\overline{m}, \infty)$. Furthermore, if $p \leq 0$, then no point of relative extremum lies in $(0, \overline{m})$ as well (see Lemma 5.3), as it is stated in (iii). On the other hand, if $p > 0$, then a similar consideration shows that L has exactly one relative extremum, which is a global minimum attained at some point $m_0 \in (0, \overline{m}]$, and in case of $m_0 < \overline{m}$, $\overline{m}$ may be a stationary point of L as well. In order to complete the verification of statement (ii), let us show that L cannot have two stationary points for $0 < p < 1$, $q > p$: From Lemma 1.11 we see that $K'(m)$ has the opposite sign to $\varrho(m)$ for any $m > 0$. Consequently K decreases on $(0, \overline{m}]$. However, if L had a relative minimum at some point $m_0 \in (0, \overline{m})$, and $\overline{m}$ were another stationary point of L, we would have $K(m_0) = L(m_0) < L(\overline{m}) = K(\overline{m})$ (see Lemma 1.11), a contradiction to $K(m_0) > K(\overline{m})$. $\square$

The properties of L ascertained in this section are summarised in Figure 3, which shows all the possible graphs of L with the corresponding sets of parameters in the (p, q)-plane, distinguished by colours. (Note that although we have not ruled out in Lemma 5.5 the possibility that $\overline{m}$ is a stationary point of L for $p \leq 0$, $q > -p$, it has no influence on the number of solutions of (1).) Using Lemma 1.8 , we can state the main result of this section. Recall that $L(u(0)) = l$ for any $u \in \mathcal{S}(l)$, and see also Lemmata 5.2, 5.3, 5.4 and 5.5 concerning $L(0)$, $\lim_{m\to\infty} L(m)$ and m_0.

5.6 Theorem
Assume (I.24) *and* $l > 0$. *Then* $\mathcal{N}(l) = \emptyset$, *and the following holds for the positive symmetric solutions of* (1)*:*

If $p > 0$ *and* $q > p$, *then*

$$|\mathcal{S}(l)| = \begin{cases} 2 & \text{if } l \in (L(m_0), L(0)), \\ 1 & \text{if } l \in \{L(m_0)\} \cup [L(0), \infty), \\ 0 & \text{otherwise}, \end{cases}$$

and L *decreases on* $(0, m_0]$ *and increases on* $[m_0, \infty)$, *see Figure 3.*

In all the other cases,

$$|\mathcal{S}(l)| = \begin{cases} 1 & \text{if } l \text{ is between } L(0) \text{ and } \lim_{m\to\infty} L(m), \\ 0 & \text{otherwise}, \end{cases}$$

and L *is strictly monotone, see Figure 3.*

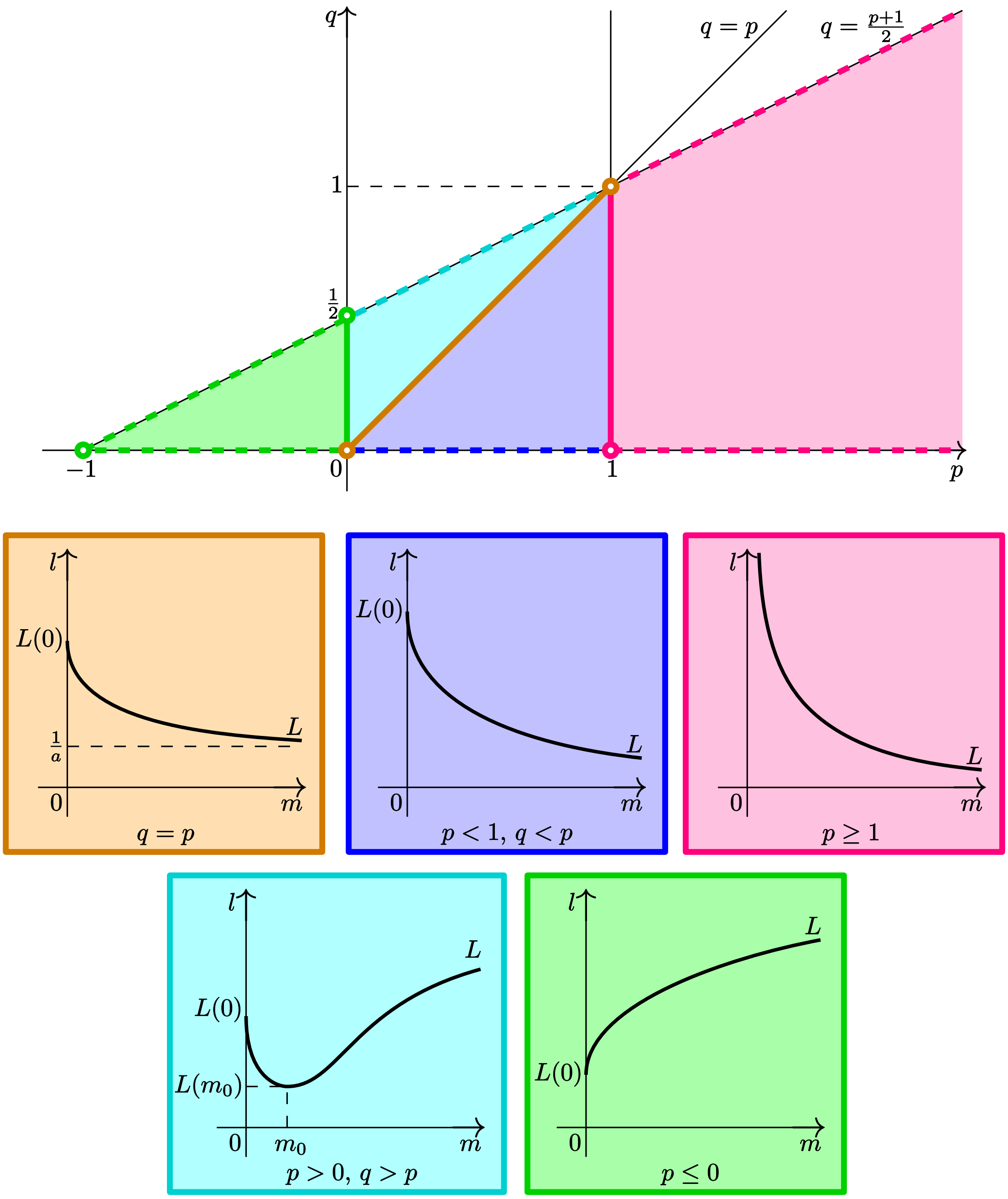

Figure 3: The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case III ($p > -1$, $0 < q < \frac{p+1}{2}$, $a > 0$) according to Lemmata 1.8, 5.2, 5.3, 5.4 and 5.5. See also Theorem 5.6.

6 Case IV ($p > -1$, $q = \frac{p+1}{2}$)

In this case we have from Lemma 1.5 that the time map is defined only for $q < a$, and is given by

$$L(m) = \frac{1}{\sqrt{2a}}\, I_p\Bigg(\underbrace{\left(\frac{a}{a-q}\right)^{\frac{1}{2q}}}_{=:r_{q,a}}\Bigg)\, m^{\frac{1-p}{2}}, \quad m > 0.$$

Thus, it is a bijection of $(0,\infty)$ onto $(0,\infty)$ for $p \neq 1$, and a constant function for $p = 1$. Namely, we can use (I.16) to derive that

$$L_{1,1,a}(m) = \frac{1}{\sqrt{a}} \ln \frac{\sqrt{a}+1}{\sqrt{a-1}} = \frac{1}{2\sqrt{a}} \ln\left(\frac{\sqrt{a}+1}{\sqrt{a-1}}\right)^2 = \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a}+1}{\sqrt{a}-1}.$$

Furthermore, solving (I.1) for $p = 1$, we obtain that $u_{m,1,a}(x) = m\,\mathrm{ch}(\sqrt{a}x)$. So according to Lemma 1.8, we can state the following:

6.1 Theorem
Let $p > -1$, $q = \frac{p+1}{2}$, $a > 0$. Then for arbitrary $l > 0$:

$$\mathcal{S}(l) = \begin{cases} \left\{ u_{m,p,a}\big|_{[-l,l]} \;:\; m = \left(\dfrac{\sqrt{2a}}{I_p(r_{q,a})}l\right)^{\frac{2}{1-p}} \right\} & \textit{if } p \neq 1,\ q < a, \\ \left\{x \mapsto m\,\mathrm{ch}(\sqrt{a}x), x \in [-l,l] \;:\; m > 0\right\} & \textit{if } p = 1,\ a > 1,\ l = \frac{1}{2\sqrt{a}}\ln\frac{\sqrt{a}+1}{\sqrt{a}-1}, \\ \emptyset & \textit{otherwise}, \end{cases}$$

$$\mathcal{N}(l) = \emptyset.$$

7 Case V ($p > -1$, $q > \frac{p+1}{2}$), symmetric solutions

Recall that due to Lemma 1.5 (iv), we have the following time maps in case V: $L_1 < L_2$ defined on $(0, M)$ and L defined on $\{M\}$. In this section we describe their behaviour for

$$p > -1,\ q > \frac{p+1}{2},\ a > 0. \tag{I.26}$$

7.1 Lemma (for $p > 1$ see [5, p. 57 and Lemma 3.3])
Assume (I.26). *Then* $R_1' > 0$, *while*

$$\lim_{m\to 0} \frac{R_1(m)}{m} = 1, \qquad \lim_{m\to M} R_1(m) = R(M) = \left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}$$

and $R_2' < 0$, *while*

$$\lim_{m\to 0} R_2(m) = \left(\frac{2a}{p+1}\right)^{\frac{1}{2q-p-1}} =: R_{2;p,q,a}(0) =: R_2(0), \qquad \lim_{m\to M} R_2(m) = R(M).$$

Moreover,

$$\frac{R_2(m)}{R_2(0)} = 1 - \frac{m^{p+1}}{(2q-p-1)R_2^{p+1}(0)} - \frac{2q+p}{2(2q-p-1)^2 R_2^{2(p+1)}(0)} m^{2(p+1)} + o\big(m^{2(p+1)}\big)$$

for $m \to 0$.

Proof: It is clear from Lemma 1.5 (iv) and (I.8) that $R_1' > 0$ and $R_2' < 0$. The limits of $R_1(m)$, $R_1(m)/m$ and $R_2(m)$ can be calculated in the same way as in [5] for $p > 1$, and the derivation of the asymptotic expansion of $R_2(m)$ for $m \to 0$ does not differ from that of $R(m)$ for $m \to 0$ and $m \to \infty$ in the proof of Lemma 5.1. $\square$

7.2 Definiton
For p, q, a satisfying (I.26) and $q < |p|$ set

$$\overline{m} := \overline{m}_{p,q,a} := \left(\frac{(p+q)(2q-p-1)}{2q(q-1)}\right)^{\frac{1}{p+1}} \left(\frac{a(p-q)}{q(q-1)}\right)^{\frac{1}{2q-p-1}}.$$

7.3 Lemma (for $p > 1$ see [5, Lemmata 3.1, 3.4, 3.3, 3.2 and 3.5])
If (I.26) *holds, then*

$$\lim_{m\to M} L_1(m) = L(M), \quad \lim_{m\to M} L_1'(m) = \infty,$$

$$\lim_{m\to 0} L_1(m) = \begin{cases} 0 & \textit{if } q > p, \\ \frac{1}{a} & \textit{if } q = p, \\ \infty & \textit{if } q < p, \end{cases} \tag{I.27}$$

and the following holds concerning the monotonicity of L_1*:*

(i) *If* $q \geq p$, *then* $L_1' > 0$.

(ii) *If* $q < p$, *then there exists such a point* $m_{0;p,q,a} =: m_0 \in [\overline{m}, M)$ *that*

$$L_1' < 0 \text{ on } (0, m_0), \quad L_1' > 0 \text{ on } (m_0, M).$$

Proof: It does not differ from the proof that can be found in [5] for $p, q > 1$. Let us mention that (I.27) is obtained as the consequence of

$$L_1(m) = \frac{1}{a} m^{q-p} + o\big(m^{q-p}\big), \qquad m \to 0. \tag{I.28}$$

$\square$

7.4 Lemma (for $p > 1$ see [5, Lemmata 3.1, 3.4 and 3.3])
If (I.26) *holds, then*

$$\lim_{m\to M} L_2(m) = L(M), \quad \lim_{m\to M} L_2'(m) = -\infty,$$

$$\lim_{m\to 0} L_2(m) = \begin{cases} \infty & \text{if } p \geq 1, \\ \dfrac{2}{1-p}\left(\dfrac{p+1}{2a}\right)^{\frac{q-1}{2q-p-1}} =: L_{2;p,q,a}(0) =: L_2(0) & \text{if } p \in (-1, 1). \end{cases}$$

Proof: The limits at M can be calculated in the same way as it was done in [5] for $p, q > 1$, while the proof of the second part of the lemma is essentially the same as that of Lemma 4.2. $\square$

7.5 Lemma
Assume (I.26) *with* $p < 1$. *Then*

(i) *if* $p \geq 0$ *or* $q < -p$ *or* $p \geq -\frac{1}{2}$, $q = -p$, *then* $L_2 < L_2(0)$ *in some neighbourhood of* 0,

(ii) *and if* $p < 0$, $q > -p$ *or* $p < -\frac{1}{2}$, $q = -p$, *then* $L_2 > L_2(0)$ *in some neighbourhood of* 0.

(See Figure 4 showing these two sets in the (p, q)*-plane.)*

Proof: We use the asymptotic expansions of $I_p(y)$ and $R_2(m)$ from Lemmata 2.4 and 7.1 respectively, and our goal is to find the second term of the asymptotic expansion of $L_2(m)$ for $m \to 0$, and to determine its sign. However, as we will see, it has eight different forms depending on the value of p and q.

All the asymptotic expansions in this proof will concern $y \to \infty$ and $m \to 0$.

1. For $-\frac{1}{3} < p < 1$ the expansion of $L_2(m)$ looks like that of $L(m)$, and is derived in the same way as in the proof of Lemma 5.3.

2. If $p = -\frac{1}{3}$, then writing $B_p + o(1)$ as $O(1)$ and $R_2(m)$ as $R_2(0)(1 + O(m^{2/3}))$, we obtain:

$$\begin{aligned} L_2(m) &= \frac{1}{2}\sqrt{\frac{3}{a}} R_2^{2/3}(m) + \frac{1}{2\sqrt{3a}} m^{2/3} \ln \frac{R_2(m)}{m} + O(m^{2/3}) \\ &= L_2(0) + \frac{1}{2\sqrt{3a}} m^{2/3} \ln \frac{1}{m} + O\big(m^{2/3}\big). \end{aligned}$$

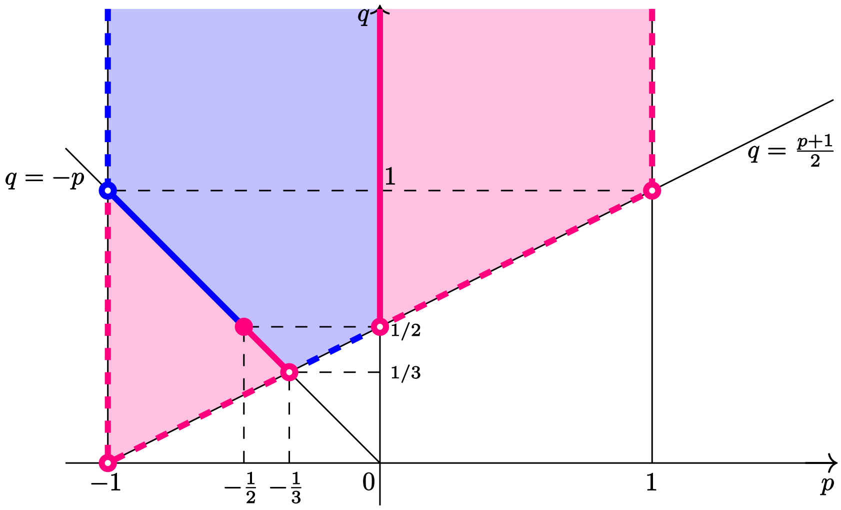

Figure 4: The behaviour of L_2 near 0 in case V for $p < 1$ according to Lema 7.5: if (p, q) belongs to the purple set, then $L_2 < L_2(0)$ near 0, and if (p, q) belongs to the blue set, then $L_2 > L_2(0)$ near 0. (Recall that $\lim_{m\to 0} L_2(m) = L_2(0)$, see Lemma 7.4.)

3. Now let $-1 < p < -\frac{1}{3}$. In general, we have the expansion

$$\frac{I_p(y)}{\sqrt{p+1}} = \frac{2}{1-p} y^{\frac{1-p}{2}} - \frac{1}{3p+1} y^{-\frac{3p+1}{2}} + \varrho_p(y)$$

for some function ϱ_p, which is given by different formulae depending on p, and will be specified later. It can be derived from Lemma 7.1 that

$$R_2^{\frac{1-p}{2}}(m) = R_2^{\frac{1-p}{2}}(0)\Bigg(1 - \frac{1-p}{2(2q-p-1)R_2^{p+1}(0)} m^{p+1} \\ - \frac{(1-p)(4q+3p+1)}{8(2q-p-1)^2 R_2^{2(p+1)}(0)} m^{2(p+1)} + o\big(m^{2(p+1)}\big)\Bigg)$$

and

$$R_2^{-\frac{3p+1}{2}}(m) = R_2^{-\frac{3p+1}{2}}(0)\left(1 + \frac{3p+1}{2(2q-p-1)R_2^{p+1}(0)} m^{p+1} + o\big(m^{p+1}\big)\right),$$

which yield:

$$L_2(m) = L_2(0) + C_{p,q,a} m^{p+1} + D_{p,q,a} m^{2(p+1)} \\ + \sqrt{\frac{p+1}{2a}}\, m^{\frac{1-p}{2}} \varrho_p\left(\frac{R_2(m)}{m}\right) + o\big(m^{2(p+1)}\big), \tag{I.29}$$

where

$$C_{p,q,a} = -\frac{2(p+q)}{(3p+1)(2q-p-1)R_{2;p,q,a}^{p+q}(0)} \begin{cases} > 0 & \text{if } q > -p, \\ = 0 & \text{if } q = -p, \\ < 0 & \text{if } q < -p, \end{cases}$$

$$D_{p,q,a} = -\frac{8q+p-1}{4(2q-p-1)^2 R_{2;p,q,a}^{q+2p+1}(0)}.$$

Using that $\varrho_p(y) = o(y^{-(3p+1)/2})$ and $R_2(m) = O(1)$, we can rewrite (I.29) in the form
$$L_2(m) = L_2(0) + C_{p,q,a} m^{p+1} + o\big(m^{p+1}\big), \tag{I.30}$$
thus further calculations are needed for $q = -p$.

(a) Let us consider $-q = p \in (-\frac{3}{5}, -\frac{1}{3})$. Since $\varrho_p(y) = B_p + o(1)$ and $O(m^{2(p+1)}) = o(m^{(1-p)/2})$, we have
$$L_2(m) = L_2(0) + \sqrt{\frac{p+1}{2a}} B_p m^{\frac{1-p}{2}} + o\Big(m^{\frac{1-p}{2}}\Big) \tag{I.31}$$
from (I.29). According to Lemma 2.4, $B_p < 0$ for $p \in (-\frac{1}{2}, -\frac{1}{3})$, and $B_p > 0$ for $p \in (-\frac{3}{5}, -\frac{1}{2})$. In the case $p = -\frac{1}{2}$ the expansion from Lemma 2.4 does not suffice for us, but we can use (I.18) together with
$$\sqrt{R_2(m)} = 4a - \sqrt{m} - \frac{m}{4a} + o(m)$$
to derive that
$$\begin{aligned} L_2(m) &= \frac{16a}{3}\sqrt{1 - \frac{\sqrt{m}}{2a} - \frac{m}{16a^2} + o(m)}\left(1 + \frac{\sqrt{m}}{4a} - \frac{m}{16a^2} + o(m)\right) \\ &= L_2(0) - \frac{m}{a} + o(m). \end{aligned}$$

(b) If $-q = p = -\frac{3}{5}$, then inserting $\varrho_p(y) = \frac{3}{8}\ln y + O(1)$ and $R_2(m) = O(1)$ in (I.29), we obtain that
$$L_2(m) = L_2(0) + \frac{3}{8\sqrt{5a}} m^{4/5} \ln\frac{1}{m} + O\big(m^{4/5}\big).$$

(c) Finally, for $-q = p \in (-1, -\frac{3}{5})$ we have
$$\varrho_p(y) = -\frac{3}{4(5p+3)} y^{-\frac{5p+3}{2}} + o\Big(y^{-\frac{5p+3}{2}}\Big),$$
which together with $R_2(m) = R_2(0) + o(1)$ and (I.29) yields
$$L_2(m) = L_2(0) + \underbrace{\frac{2p(p+1)}{(5p+3)(3p+1)^2 R_2^{p+1}(0)}}_{>0} m^{2(p+1)} + o\big(m^{2(p+1)}\big). \ \square$$

The next three lemmata deal with the monotonicity and the stationary points of L_2.

7.6 Lemma
Assume (I.26)*. The following holds:*

(i) *If $p \geq 0$ or $p \geq -\frac{1}{2}$, $q = -p$, then*
$$L_2' < 0 \text{ on } (0, M).$$

(ii) *If $p < 0$, $q > -p$ or $p < -\frac{1}{2}$, $q = -p$, then L_2 has a unique stationary point $m_{0;p,q,a} =: m_0 \in (0, M)$, while*

$$L_2' > 0 \text{ on } (0, m_0), \quad L_2' < 0 \text{ on } (m_0, M).$$

(iii) *If $q < -p$, then one of the following holds:*

A: $L_2' < 0$ on $(0, M)$,

B: $L_2' < 0$ on $(0, \overline{m})$, $L_2'(\overline{m}) = 0$ and $L_2' < 0$ on $(\overline{m}, M)$,

C: $L_2' < 0$ on $(0, m_1)$, $L_2' > 0$ on (m_1, m_2), and $L_2' < 0$ on (m_2, M) for some $m_1 = m_{1;p,q,a} \in (0, \overline{m})$, $m_2 = m_{2;p,q,a} \in [\overline{m}, M)$.

Proof: The case $p \geq 1$ is trivial, so let $p < 1$, and suppose that $m_0 \in (0, M)$ is a stationary point of L_2. Recall that $L_2' < 0$ near M due to Lemma 7.4.

Firstly, let us consider $q \geq 1$. Then $L_2''(m_0) < 0$, so there are only two possibilities: Either $L_2' < 0$ on $(0, M)$ or L_2 has a unique stationary point, which is a point of strict relative maximum. Lemma 7.5 guarantees that the first one holds for $p \geq 0$ and the second one for $p < 0$.

Now let $q < 1$. Consequently:

$$L_2''(m_0) < 0 \iff R_2(m_0) < \left(\frac{a(q-p)}{q(1-q)}\right)^{\frac{1}{2q-p-1}} =: \overline{R}_{2;p,q,a} =: \overline{R}_2. \tag{I.32}$$

Recall that $(R(M), R_2(0))$ is the range of R_2. The inequality $\overline{R}_2 > R(M)$ holds always, while $\overline{R}_2 < R_2(0)$ only for $q < -p$. (In the latter case, we have $R_2(\overline{m}) = \overline{R}_2$.) So if $q \geq -p$, then each stationary point of L_2 is a point of strict relative maximum, and by means of Lemma 7.5 we have again that $L_2' < 0$ for $p \geq 0$ and for $-q = p \in [-\frac{1}{2}, -\frac{1}{3})$, and L_2 has a unique stationary point for $p < 0$, $q > -p$ and for $p < -\frac{1}{2}$, $q = -p$.

From now on we will consider only $q < -p$ (thus, $-1 < p < -\frac{1}{3}$ and $q < 1$). So we have

$$L_2''(m_0) < 0 \iff m_0 > R_2^{-1}(\overline{R}_2) = \overline{m}.$$

It means that L_2 has at most one stationary point (a point of strict relative minimum) in $(0, \overline{m})$, at most one (a point of strict relative maximum) in $(\overline{m}, M)$, and $\overline{m}$ may be a stationary point as well. Suppose that $\overline{m}$ and some $m_2 > \overline{m}$ are both stationary points of L_2, thus L_2 increases on $[\overline{m}, m_2]$. Since K_2 decreases on $[\overline{m}, M)$, we have $L_2(\overline{m}) = K_2(\overline{m}) > K_2(m_2) = L_2(m_2)$ (see Lemma 1.11), a contradiction. Therefore, L_2 has at most one stationary point in $[\overline{m}, M)$. Furthermore, due to Lemma 7.5, only A, B or C can hold. □

7.7 Lemma

Assume (I.26) *and $q < -p$. There exists a continuous function $q^* : (-1, -\frac{1}{2}) \to \mathbb{R}$ such that $\frac{p+1}{2} < q^*(p) < -p$ for $p \in (-1, -\frac{1}{2})$, $\lim_{p\to -1/2} q^*(p) = \frac{1}{2}$, and the following holds:*

(i) *If $p \geq -\frac{1}{2}$, $q < -p$ or $p < -\frac{1}{2}$, $q < q^*(p)$, then*

$$L_2' < 0 \text{ on } (0, M).$$

(ii) *If $p < -\frac{1}{2}$ and $q = q^*(p)$, then $\overline{m}$ is a stationary point of L_2, while*

$$L_2' < 0 \text{ on } (0, \overline{m}), \quad L_2' < 0 \text{ on } (\overline{m}, M).$$

(iii) *If $p < -\frac{1}{2}$ and $q^*(p) < q < -p$, then L_2 has two stationary points $m_{1;p,q,a} =: m_1$, $m_{2;p,q,a} =: m_2$, while $m_1 < \overline{m} < m_2$, and*

$$L_2' < 0 \text{ on } (0, m_1), \quad L_2' > 0 \text{ on } (m_1, m_2), \quad L_2' < 0 \text{ on } (m_2, M).$$

In addition, for all $p \in (-1, -\frac{1}{2})$, $q = q^*(p)$ is the only solution of the equation

$$I_p(g(p,q)) - \underbrace{\frac{1}{1-p}\sqrt{\frac{2(q-p)(1-q)}{q}}\,\overset{*}{g}^{\frac{1-p}{2}}(p,q)}_{=:G^*(p,q)} =: f^*(p,q) = 0 \tag{I.33}$$

in $(\frac{p+1}{2}, -p)$, where

$$\overset{*}{g}(p,q) = \left(\frac{2q(q-1)}{(2q-p-1)(p+q)}\right)^{\frac{1}{p+1}}.$$

Proof: From Lemma 7.6 we already know that only A, B or C can hold for $q < -p$. Let us notice the crucial role of the sign of $L_2'(\overline{m})$: If it is $+$, then C holds, if 0, then B or C occurs, and if $-$, then A holds. So we derive the following condition:

$$L'_{2;p,q,a}(\overline{m}_{p,q,a}) > 0 \iff L_2(\overline{m}) - \frac{(1-q)\overline{R}_2^{\frac{2q-p-1}{2}}}{a(1-p)}\overline{R}_2^{\frac{1-p}{2}} > 0 \iff f^*(p,q) > 0$$

(see (I.32) for the definition of $\overline{R}_2$), and in the sequel we

1. find $\lim_{q\to(p+1)/2} f^*(p,q)$
2. and $\lim_{q\to -p} f^*(p,q)$,
3. and investigate the monotonicity of $f^*(p,\cdot)$.

Afterwards we will be able to describe the sets where f^* (or equivalently $L_2'(\overline{m})$) is positive, zero and negative, resp.

1. Since $\lim_{q\to(p+1)/2} \overset{*}{g}(p,q) = \infty$, using the first term of the asymptotic expansion of $I_p(y)$ for $y \to \infty$ (see Lemma 2.4), we obtain:

$$\lim_{q\to\frac{p+1}{2}} \frac{f^*(p,q)}{\overset{*}{g}^{\frac{1-p}{2}}(p,q)} = \frac{3p+1}{(1-p)\sqrt{p+1}} < 0,$$

thus

$$\lim_{q\to\frac{p+1}{2}} f^*(p,q) = -\infty.$$

2. We are going to find $\lim_{q\to -p} f^*(p,q)$, so we denote $-q-p =: r$ for the sake of simplicity. All the asymptotic expansions in this step will concern $r \to 0+$ or $y \to \infty$. We will see that the first two terms of the asymptotic expansions of $I_p(\overset{*}{g}(p,q))$ and $G^*(p,q)$ are identical, therefore we need to calculate the first three. We have:

$$\begin{aligned} G^*(p,q) &= \frac{2\sqrt{p+1}}{1-p}\sqrt{\frac{1+\frac{3p+1}{2p(p+1)}r+\frac{1}{2p(p+1)}r^2}{1+\frac{r}{p}}}\,\overset{*}{g}^{\frac{1-p}{2}}(p,q)\\ &= \frac{2\sqrt{p+1}}{1-p}\sqrt{1+\frac{p-1}{2p(p+1)}r+\frac{1}{2p^2(p+1)}r^2+O(r^3)}\;\overset{*}{g}^{\frac{1-p}{2}}(p,q)\\ &= \frac{2\sqrt{p+1}}{1-p}\left(1+\frac{p-1}{4p(p+1)}r-\frac{p^2-10p-7}{32p^2(p+1)^2}r^2+O\big(r^3\big)\right)\overset{*}{g}^{\frac{1-p}{2}}(p,q). \end{aligned}$$

It will be useful to write the asymptotic expansion of $I_p(y)$ in the form

$$\frac{I_p(y)}{\sqrt{p+1}} = \frac{2}{1-p}\left(1+\frac{p-1}{2(3p+1)}\frac{1}{y^{p+1}}\right)y^{\frac{1-p}{2}}+\varrho_p(y),$$

where function ϱ_p will be specified later. Joining the last formula with

$$\begin{aligned} \frac{1}{\overset{*}{g}^{p+1}(p,q)} &= \frac{3p+1}{2p(p+1)}r\frac{1+\frac{2}{3p+1}r}{1+\frac{2p+1}{p(p+1)}r+\frac{1}{p(p+1)}r^2}\\ &= \frac{3p+1}{2p(p+1)}r\left(1-\frac{4p^2+3p+1}{p(p+1)(3p+1)}r+O\big(r^2\big)\right), \end{aligned} \tag{I.34}$$

we obtain that

$$\begin{aligned} I_p(\overset{*}{g}(p,q)) = \frac{2\sqrt{p{+}1}}{1{-}p}&\left(1+\frac{p{-}1}{4p(p{+}1)}r+\frac{(1{-}p)(4p^2{+}3p{+}1)}{4p^2(p{+}1)^2(3p{+}1)}r^2+O\big(r^3\big)\right)\\ &\cdot\overset{*}{g}^{\frac{1-p}{2}}(p,q)+\sqrt{p{+}1}\,\varrho_p(\overset{*}{g}(p,q)), \end{aligned}$$

consequently

$$\begin{aligned} f^*(p,q) = &\left(\frac{\sqrt{p+1}(29p^3+21p^2+15p-1)}{16p^2(p+1)^2(3p+1)(p-1)}r^2+O\big(r^3\big)\right)\overset{*}{g}^{\frac{1-p}{2}}(p,q)\\ &+\sqrt{p+1}\,\varrho_p(\overset{*}{g}(p,q)). \end{aligned} \tag{I.35}$$

(a) Let $-\frac{3}{5} < p < -\frac{1}{3}$, thus $\varrho_p(y) = B_p + o(1)$. Since

$$\overset{*}{g}^{\frac{1-p}{2}}(p,q) = O\left(r^{\frac{p-1}{2(p+1)}}\right) = o\left(\frac{1}{r^2}\right),$$

we have

$$f^*(p,q) = \sqrt{p+1}\,B_p + o(1).$$

So $\lim_{q\to -p} f^*(p,q)$ is negative for $p \in (-\frac{1}{2}, -\frac{1}{3})$, zero for $p = -\frac{1}{2}$, and positive for $p \in (-\frac{3}{5}, -\frac{1}{2})$ due to Lemma 2.4.

(b) If $p = -\frac{3}{5}$, then inserting $\varrho_p(y) = \frac{3}{8}\ln y + O(1)$ and $\overset{*}{g}^{\frac{1-p}{2}}(p,q) = O(\frac{1}{r^2})$ in (I.35), we obtain that

$$f^*(p,q) = \frac{3\sqrt{5}}{8\sqrt{2}}\ln\frac{1}{r}+O(1) \;\longrightarrow\; \infty.$$

(c) For $p \in (-1, -\frac{3}{5})$ we have

$$\varrho_p(y) = \left(-\frac{3}{4(5p+3)}\frac{1}{y^{2(p+1)}} + o\left(\frac{1}{y^{2(p+1)}}\right)\right)y^{\frac{1-p}{2}}.$$

Thus, (I.35) yields

$$f^*(p,q) = \left(\frac{4(p+1)^{3/2}}{p(3p+1)(5p+3)(p-1)}r^2 + o(r^2)\right)\overset{*}{g}^{\frac{1-p}{2}}(p,q) \longrightarrow \infty.$$

(See (I.34).)

So we have derived that

$$\lim_{q\to -p} f^*(p,q) \begin{cases} < 0 & \text{if } -\frac{1}{2} < p < -\frac{1}{3}, \\ = 0 & \text{if } p = -\frac{1}{2}, \\ > 0 & \text{if } -1 < p < -\frac{1}{2}. \end{cases}$$

3. The increase of $f^*(p,\cdot)$ can be proved using

$$\begin{aligned} \frac{\partial f^*}{\partial q}(p,q) &= \sqrt{\frac{p+1}{\overset{*}{g}^{p+1}(p,q)-1}}\frac{\partial \overset{*}{g}}{\partial q}(p,q) - \frac{\sqrt{(2q-p-1)(p-q)(p+q)}}{2q}\frac{\partial \overset{*}{g}}{\partial q}(p,q) \\ &\quad + \frac{q^2-p}{(1-p)q\sqrt{2q(q-p)(1-q)}}\frac{1}{\sqrt{\overset{*}{g}^{p+1}(p,q)}}\overset{*}{g}(p,q) \\ &= \frac{1}{2q}\sqrt{\frac{(2q-p-1)(p+q)}{p-q}}\left(\frac{q^2-p}{q(1-q)(1-p)}\overset{*}{g}(p,q) + (p+q)\frac{\partial \overset{*}{g}}{\partial q}(p,q)\right) \end{aligned}$$

and

$$\frac{\partial \overset{*}{g}}{\partial q}(p,q) = -\frac{q^2-2pq+p}{q(1-q)(2q-p-1)(p+q)}\overset{*}{g}(p,q),$$

which yield

$$\frac{\partial f^*}{\partial q}(p,q) = \frac{p+q}{q^2(p-1)}\sqrt{\frac{(p+q)(p-q)}{2q-p-1}}\overset{*}{g}(p,q) > 0.$$

From *1.*, *2.* and *3.* we can see that if $p \in [-\frac{1}{2}, -\frac{1}{3})$, $q \in (\frac{p+1}{2}, -p)$, then $f^*(p,q) < 0$, i. e. $L_2' < 0$. Moreover, $f^*(p,\cdot)$ has a unique zero—denote it by $q^*(p)$—for all $p \in (-1, -\frac{1}{2})$, and

- if $\frac{p+1}{2} < q < q^*(p)$, then $L_2'(\overline{m}) < 0$, so A holds,
- if $q^*(p) < q < -p$, then $L_2'(\overline{m}) > 0$, so C holds with $m_2 > \overline{m}$,
- and if $q = q^*(p)$, then $L_2'(\overline{m}) = 0$, so either B holds or C with $m_2 = \overline{m}$. Nevertheless, we prove that only B can hold for $q = q^*(p)$: So suppose that C holds for some $p = p_0 \in (-1, -\frac{1}{2})$ and $q = q^*(p_0)$, consequently, $L'_{2;p_0,q^*(p_0),a}(\widetilde{m}) > 0$ for some $\widetilde{m} \in (0, M)$. From the definition of R_2 and the implicit function theorem it follows that $R_{2;p_0,\cdot,a}(\widetilde{m})$ is continuous, which together with (I.11), (I.8) and Theorem 2.6 guarantees the continuity of $L'_{2;p_0,\cdot,a}(\widetilde{m})$. Thus, $L'_{2;p_0,q^*(p_0)-\varepsilon,a}(\widetilde{m}) > 0$ if $\varepsilon > 0$ is small enough, giving a contradiction.

At this moment, assertions (i)–(iii) have been proved. Since f^* is continuous due to Theorem 2.6, from the implicit function theorem we have the continuity of q^* as well. So there only remains to find its limit at $-\frac{1}{2}$. Recall that $\lim_{q\to 1/2} f^*(-\frac{1}{2}, q) = 0$, and choose arbitrary $\varepsilon \in (0, \frac{1}{2})$. From the increase of $f^*(-\frac{1}{2}, \cdot)$ we have $f^*(-\frac{1}{2}, \frac{1}{2} - \varepsilon) < 0$, therefore $f^*(p, \frac{1}{2} - \varepsilon) < 0$ for all $p \in (-\frac{1}{2} - \delta, -\frac{1}{2})$ and some suitable $\delta \in (0, \frac{1}{2})$, and the increase of $f^*(p, \cdot)$ yields that $\frac{1}{2} - \varepsilon < q^*(p) < -p$ for $p \in (-\frac{1}{2} - \delta, -\frac{1}{2})$. So we conclude that $\lim_{p\to -1/2} q^*(p) = \frac{1}{2}$. $\square$

7.8 Lemma
There exists

$$\lim_{p\to -1} q^*(p) =: q^*(-1) \in (0, 1),$$

and it is the only solution of the equation

$$\varphi^*(q) := I_{-1}(\psi^*(q)) - \sqrt{\frac{1-q^2}{2q}}\,\psi^*(q) = 0$$

in $(0, 1)$, *where*

$$\psi^*(q) = \mathrm{e}^{\frac{q+1}{2q(1-q)}}.$$

Proof: Recall the definitions of f^* and $\overset{*}{g}$ from Lemma 7.7. An easy calculation and Theorem 2.6 yield that $\lim_{p\to -1} \overset{*}{g}(p, q) = \psi^*(q)$ and $\lim_{p\to -1} f^*(p, q) = \varphi^*(q)$ for all $q \in (0, 1)$. In the sequel we examine the behaviour of φ^*.

Since $\lim_{q\to 0} \psi^*(q) = \infty$ and $I_{-1}(y) = o(y)$ for $y \to \infty$ (see Lemma 2.5),

$$\varphi^*(q) = -\frac{1}{\sqrt{2q}}\big(1 + o(1)\big)\psi^*(q) \longrightarrow -\infty, \qquad q \longrightarrow 0.$$

Set $r := 1 - q$, and consider $r \to 0+$. Using Lemma 2.5 with $n = 4$ and the formulae

$$\begin{aligned}
\frac{1}{\sqrt{\ln \psi^*(q)}} &= \sqrt{r}\left(1 - \frac{r}{4} - \frac{5}{32}r^2 - \frac{13}{128}r^3 + O\big(r^4\big)\right),\\
\frac{1}{\ln \psi^*(q)} &= r\left(1 - \frac{r}{2} - \frac{r^2}{4} + O\big(r^3\big)\right),\\
\frac{1}{\ln^2 \psi^*(q)} &= r^2\big(1 - r + O\big(r^2\big)\big),\\
\frac{1}{\ln^3 \psi^*(q)} &= r^3\big(1 + O(r)\big),
\end{aligned}$$

we obtain that

$$I_{-1}(\psi^*(q)) = \sqrt{r}\left(1 + \frac{r}{4} + \frac{7}{32}r^2 + \frac{89}{128}r^3 + O\big(r^4\big)\right)\psi^*(q).$$

On the other hand,

$$\sqrt{\frac{1-q^2}{2q}} = \sqrt{r}\left(1 - \frac{r}{2}\right)^{1/2}(1-r)^{-1/2} = \sqrt{r}\left(1 + \frac{r}{4} + \frac{7}{32}r^2 + \frac{25}{128}r^3 + O\big(r^4\big)\right).$$

Thus,

$$\varphi^*(q) = \frac{r^{7/2}}{2}\psi^*(1-r)\bigl(1+O(r)\bigr) = \frac{r^{7/2}}{2}\mathrm{e}^{\frac{1}{r}+\frac{1}{2}}\bigl(1+O(r)\bigr) \longrightarrow \infty.$$

It is not hard to derive that

$$(\psi^*)'(q) = \frac{q^2+2q-1}{2q^2(1-q)^2}\psi^*(q)$$

and

$$\begin{aligned}(\varphi^*)'(q) &= \left(\frac{1}{\sqrt{\ln\psi^*(q)}} - \sqrt{\frac{1-q^2}{2q}}\right)(\psi^*)'(q) + \sqrt{\frac{2q}{1-q^2}}\frac{q^2+1}{4q^2}\psi^*(q)\\ &= \frac{1-q}{2q^2}\sqrt{\frac{1-q^2}{2q}}\psi^*(q) > 0.\end{aligned}$$

So we conclude that φ^* has a unique zero $q_0 \in (0,1)$. Since φ^* increases, and $\lim_{p\to-1} f^*(p,q) = \varphi^*(q)$, we have that for arbitrary $\varepsilon \in (0, \min\{q_0, 1-q_0\})$ there exists such $\delta > 0$ that

$$\forall p \in (-1, -1+\delta):\ \ f^*(p, q_0-\varepsilon) < 0 < f^*(p, q_0+\varepsilon).$$

Consequently,

$$\forall p \in (-1, -1+\delta):\ \ q_0 - \varepsilon < q^*(p) < q_0 + \varepsilon$$

due to the increase of $f^*(p,\cdot)$ (see step *3.* of the proof of Lemma 7.7) and therefore, $\lim_{p\to-1} q^*(p) = q_0$. □

Numerical calculations indicate that q^* is probably decreasing, concave, its graph touches the graph of $q = -p$ in $-\frac{1}{2}$, and $q^*(-1) \approx 0.730$.

We append Figure 5 with all the possible graphs of L_1 and L_2 and the corresponding sets of (p,q), based on the lemmata of this section. (Let us notice that the graph of q^* in it is the output of the numerical solution of (I.33).) These results are sufficient to determine the number of the symmetric solutions of (1) in case V depending on p, q, a, l (see Lemma 1.8) except for $p < -\frac{1}{2}$, $q^*(p) < q < -p$ because it is required to investigate, for which p, q is $L_2(0) > L_2(m_2)$. In view of Lemmata 7.6 (ii) and 7.7 (ii), it can be expected that this domain is divided by a continuous curve into three sets where $L_2(0) = L_2(m_2)$ for (p,q) lying on the curve, $L_2(0) < L_2(m_2)$ above it, and $L_2(0) > L_2(m_2)$ under it. This hypothesis is also consistent with numerical calculations and may be an object of further research.

So let us state the main result of this section.

7.9 Theorem
Suppose (I.26).

(a) *If* $q < p$, *then*

$$\bigl\{|\mathcal{S}(l)|\ :\ l > 0\bigr\} = \bigl\{0,1,2\bigr\}.$$

(b) *If* $q = p$*, then*
$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1\}.$$

(c) *If* $p \geq 1$ *and* $q > p$*, then*
$$|\mathcal{S}(l)| = 1 \quad \textit{for } l > 0.$$

(d) *If* $0 \leq p < 1$ *or* $p \geq -\frac{1}{2}$, $q \leq -p$ *or* $p < -\frac{1}{2}$, $q \leq q^*(p)$*, then*
$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1\}.$$

(e) *If* $p < 0$, $q > -p$ *or* $p < -\frac{1}{2}$, $q = -p$*, then*
$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1, 2\}.$$

(f) *If* $p < -\frac{1}{2}$ *and* $q^*(p) < q < -p$*, then*
$$\{|\mathcal{S}(l)| \ : \ l > 0\} = \{0, 1, 2, 3\}.$$

The exact dependence of $|\mathcal{S}(l)|$ *on* l *as well as the monotonicity properties of* L_1 *and* L_2 *are indicated in Figure 5. (Recall Lemma 1.8.)*

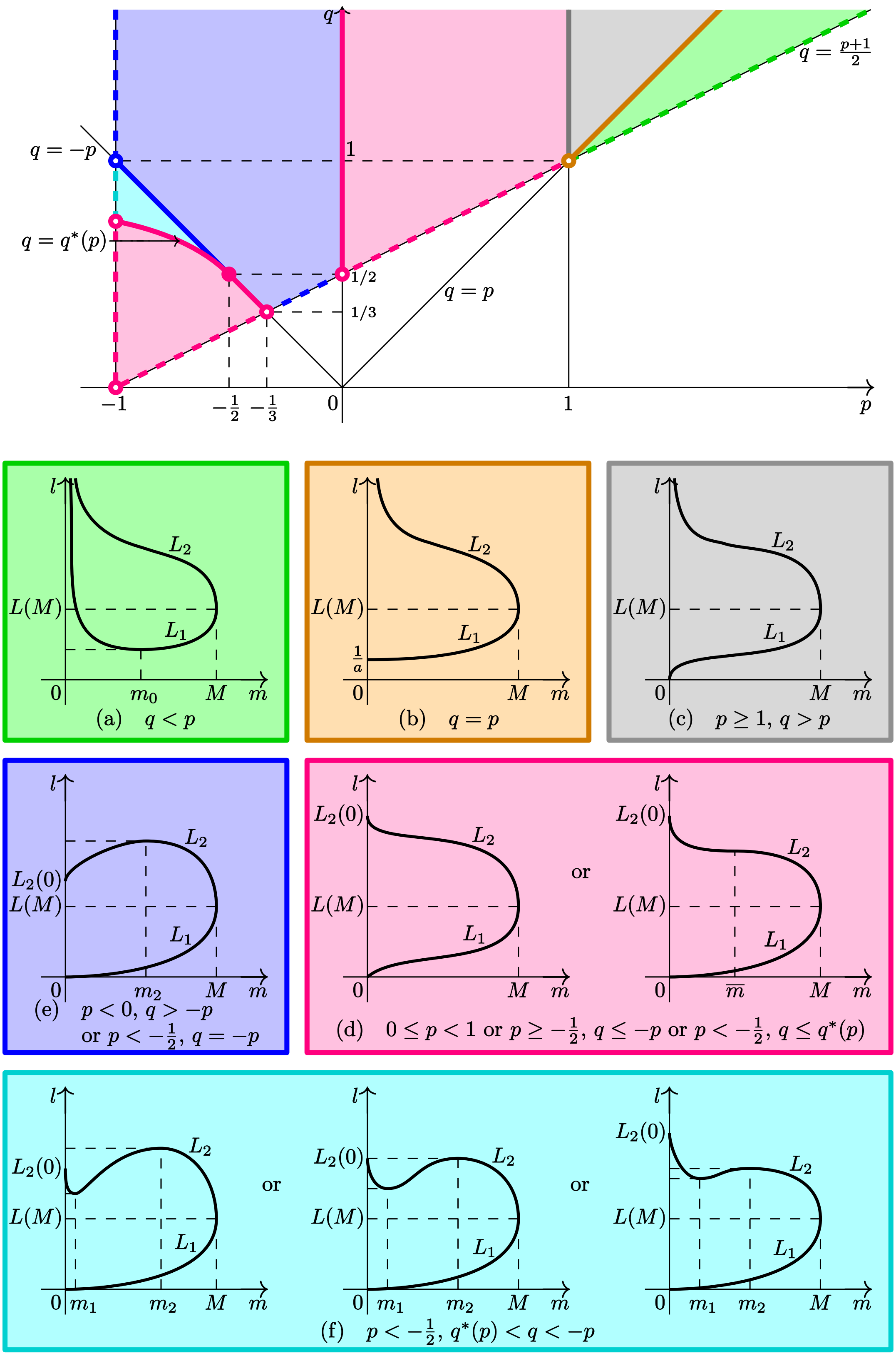

Figure 5: The relation between $m = u(0)$ and l for $u \in \mathcal{S}(l)$ in case V ($p > -1$, $q > \frac{p+1}{2}$, $a > 0$) according to Lemmata 1.8, 7.3, 7.4, 7.6, 7.7 and 7.8. See also Theorem 7.9.

8 Case V ($p > -1$, $q > \frac{p+1}{2}$), non-symmetric solutions

Assume (I.26) and $l > 0$. Then, following from Lemmata 1.5 (iv) and 1.8, (1) can possess positive non-symmetric solutions, and their number is determined by the properties of $L_1 + L_2$. We already know from Lemmata 7.3 and 7.4 that

$$\begin{aligned} &\lim_{m\to 0}(L_1+L_2)(m) = \begin{cases} \infty & \text{if } p \geq 1, \\ L_2(0) & \text{if } p \in (-1,1), \end{cases} \\ &\lim_{m\to M}(L_1+L_2)(m) = 2L(M). \end{aligned} \tag{I.36}$$

In this section the question of the monotonicity of $L_1 + L_2$ will be examined.

It was shown in [5, Theorems 34], that if (I.26) holds, then

$$1 < p \leq 4 \quad \text{or} \quad p > 4,\ q \geq p - 1 - \frac{1}{p-2} \tag{I.37}$$

is a sufficient condition for the decrease of L_1+L_2. However, we prove in Lemma 8.2 that $(L_1 + L_2)' < 0$ for all $p \geq 1$, without assuming (I.37). On the other hand, the case of $p < 1$ is much more complicated, and we have succeeded only in describing the behaviour of $L_1 + L_2$ near 0 and M (see Lemmata 8.6, 8.9, 8.10 and 8.11), except two special cases dealt with in Lemma 8.5.

The first lemma is essential for the proof of Lemma 8.2.

8.1 Lemma
If (I.26) *holds, then* $R_1R_2 < R^2(M)$.

Proof: Choose $p > -1$, $q > \frac{p+1}{2}$, $a > 0$, $m \in (0, M)$, and set $\alpha := R_2(m)/R(M)$. Evidently, $\alpha > 1$ (see (I.7)). Our aim is to prove that

$$R_1(m) < \frac{R(M)}{\alpha}. \tag{I.38}$$

Since $\mathcal{F}(m, \cdot)$ is decreasing on $(0, R(M)]$, (I.38) is equivalent to

$$\mathcal{F}\big(m, R_1(m)\big) > \mathcal{F}\left(m, \frac{R(M)}{\alpha}\right),$$

which can be rewritten in the form

$$\mathcal{F}\big(m, \alpha R(M)\big) - \mathcal{F}\left(m, \frac{R(M)}{\alpha}\right) > 0,$$

using the definition of $R_1(m)$ and $R_2(m)$. One can derive that

$$\mathcal{F}\big(m, \alpha R(M)\big) - \mathcal{F}\left(m, \frac{R(M)}{\alpha}\right) = \underbrace{R^{p+1}(M)}_{>0}\big(F_\alpha(2q) - F_\alpha(p+1)\big),$$

where

$$F_\alpha(x) := \frac{\alpha^x - \alpha^{-x}}{x}, \qquad x > 0, \tag{I.39}$$

therefore, the verification of the increase of F_α on $(0,\infty)$ will make the proof complete. Defining

$$G(z) := \big(z^2+1\big)\ln z - z^2 + 1, \qquad z > 1,$$

we have that

$$F'_\alpha(x) = \frac{G(\alpha^x)}{x^2\alpha^x}.$$

Thus, it suffices to prove that $G(z) > 0$ for $z > 1$. And it holds indeed because $G(1) = 0$, $G'(1) = 0$ and

$$G''(z) = 2\ln z + \frac{z^2-1}{z^2} > 0, \qquad z > 1. \qquad \square$$

8.2 Lemma
If (I.26) *holds with* $p \geq 1$, *then* $(L_1 + L_2)' < 0$.

Proof: Let $p \geq 1$, $q > \frac{p+1}{2}$, $a > 0$ and $m \in (0, M)$.

1. For arbitrary $y > 1$ we have

$$I_p(y) > \int_1^y \sqrt{\frac{p+1}{V^{p+1}-1}}\left(\frac{V}{y}\right)^p \mathrm{d}V = \frac{2}{y^p}\sqrt{\frac{y^{p+1}-1}{p+1}}.$$

Consequently,

$$L_i(m) > \frac{\sqrt{2}}{\sqrt{a}R_i^p(m)}\sqrt{\frac{R_i^{p+1}(m) - m^{p+1}}{p+1}} = \frac{R_i^{q-p}(m)}{a}, \qquad i = 1,2.$$

(Recall that $\mathcal{F}(m, R_i(m)) = 0$.) Using Lemma 1.9 and the last inequality, we obtain that

$$(L_1+L_2)'(m) \leq \frac{R^{q-p}(M)}{2am}\left(F\left(\frac{R_1(m)}{R(M)}\right) + F\left(\frac{R_2(m)}{R(M)}\right)\right),$$

where

$$F(x) := F_{p,q}(x) := (1-p)x^{q-p} + \frac{(2q-p-1)x^{q-p}}{1-x^{2q-p-1}}, \qquad x \in (0,1)\cup(1,\infty).$$

Thus,

$$F\left(\frac{R_1(m)}{R(M)}\right) + F\left(\frac{R_2(m)}{R(M)}\right) < 0 \tag{I.40}$$

is a sufficient condition for $(L_1+L_2)'(m) < 0$.

2. Let us prove that F is increasing on $(0,1)$ for all $p \geq 1$, $q > \frac{p+1}{2}$.
For this purpose, it is useful to introduce parameters

$$\alpha := p - 1, \qquad \beta := 2(p-q).$$

Thus, we consider $\alpha \geq 0$, $\beta < \alpha$. One can derive that

$$F(x) = -\alpha x^{-\beta/2} + \frac{(\alpha-\beta)x^{-\beta/2}}{1-x^{\alpha-\beta}},$$
$$F'(x) = \underbrace{\frac{x^{-\beta/2-1}}{2(1-x^{\alpha-\beta})^2}}_{>0} g\big(x^{\alpha-\beta}\big),$$

where

$$g(z) := g_{\alpha,\beta}(z) := \alpha\beta z^2 + \big(2\alpha^2 - 5\alpha\beta + \beta^2\big)z + \beta^2.$$

So it suffices to prove that $g > 0$ on $(0,1)$.

If $\beta \leq 0$, then the statement follows from the facts that $g(0) = \beta^2 \geq 0$, $g(1) = 2(\alpha-\beta)^2 > 0$, and g is concave. Therefore, assume $\beta > 0$. In that case, g is strictly convex, attaining its minimum at

$$\frac{-2\alpha^2 + 5\alpha\beta - \beta^2}{2\alpha\beta} =: z_{0;\alpha,\beta} =: z_0.$$

If $z_0 \leq 0$, then $g(z) > g(0) > 0$ for $z \in (0,1)$. If $z_0 > 0$, then

$$g(z_0) = \frac{\big(\alpha-\beta)^2(-4\alpha^2 + 12\alpha\beta - \beta^2\big)}{4\alpha\beta} = (\alpha-\beta)^2\left(z_0 + \frac{1}{2} + \frac{\beta}{4\alpha}\right) > 0,$$

yielding again that $g > 0$ on $(0,1)$.

So F is indeed increasing on $(0,1)$.

3. Lemmata 8.1 and 1.5 (iv) imply that

$$0 < \frac{R_1(m)}{R(M)} < \frac{R(M)}{R_2(m)} < 1.$$

Thus, due to *2.*,

$$F\left(\frac{R(M)}{R_2(m)}\right) + F\left(\frac{R_2(m)}{R(M)}\right) \leq 0$$

is a sufficient condition for (I.40). And since the range of $R_2/R(M)$ is a subset of $(1,\infty)$ (actually, it equals to $(1, R_2(0)/R(M))$, see Lemma 7.1), the verification of

$$\forall p \geq 1,\ q > \frac{p+1}{2},\ x > 1\colon \quad F\left(\frac{1}{x}\right) + F(x) \leq 0 \tag{I.41}$$

will finish the proof.

Let us reformulate (I.41) by means of α and β, and let us multiply the resulting inequality by $x^{\beta/2}(1-x^{\alpha-\beta})$, to obtain the equivalent assertion

$$\forall \alpha \geq 0,\ \beta < \alpha,\ x > 1\colon \quad u_{\alpha,\beta}(x) := \beta x^{\alpha} + \alpha x^{\alpha-\beta} - \alpha x^{\beta} - \beta \geq 0.$$

Trivially, $u_{0,\beta} \equiv 0$, so we will consider only $\alpha > 0$. Since $u_{\alpha,\beta}(1) = 0$, it suffices to prove that $u_{\alpha,\beta}$ is non-decreasing on $[1,\infty)$. However,

$$u'_{\alpha,\beta}(x) = \underbrace{\alpha x^{\beta-1}}_{>0}\underbrace{\big(\beta x^{\alpha-\beta} + (\alpha-\beta)x^{\alpha-2\beta} - \beta\big)}_{=:v_{\alpha,\beta}(x)}$$

with $v_{\alpha,\beta}(1) = \alpha - \beta > 0$, so it suffices to verity the non-decrease of $v_{\alpha,\beta}$ on $[1, \infty)$. And that is guaranteed by the equality

$$v'_{\alpha,\beta}(x) = \underbrace{(\alpha-\beta)x^{\alpha-2\beta-1}}_{>0}\underbrace{\left(\beta x^{\beta} + \alpha - 2\beta\right)}_{=:w_{\alpha,\beta}(x)},$$

$w_{\alpha,\beta}(1) = \alpha - \beta > 0$ and the non-decrease of $w_{\alpha,\beta}$. □

8.3 Remark
The proof of Lemma 8.2 was motivated by [6, Remark 5.3], where a sufficient condition for $(\overline{L}_1+\overline{L}_2)' < 0$ ($\overline{L}_1$ and $\overline{L}_2$ being the time maps associated with (II.1), see Definition 9.3), looking similar to (I.40), had been derived. That condition is based on a different integral estimate, and will be verified in the proof of Lemma 9.8.

Lemma 8.2—together with (I.36) and Lemma 1.8—leads to this result:

8.4 Theorem
If (I.26) *holds with $p \geq 1$, then*

$$|\mathcal{N}(l)| = \begin{cases} 2 & \textit{if } l > L(M), \\ 0 & \textit{if } l \leq L(M). \end{cases}$$

(See Lemma 1.5 and Definition 1.7 concerning $L(M)$.)

The rest of this section will be devoted to $p < 1$.

8.5 Lemma

(i) *If $p = 0$, $q = 1$, $a > 0$, then*

$$L_1 + L_2 \equiv 2 \ \textit{on}\ (0, M) = \left(0, \frac{a}{2}\right).$$

(ii) *If $p = -\frac{1}{2}$, $q = \frac{1}{2}$, $a > 0$, then*

$$L_1 + L_2 \equiv \frac{16a}{3} \ \textit{on}\ (0, M) = \left(0, a^2\right).$$

Proof: In the case of $q = p + 1 > 0$, (I.5) is quadratic in x^q, so one can solve it explicitely, obtaining

$$R_{1,2}(m) = \left(\frac{a}{q}\right)^{\frac{1}{q}}\left(1 \mp \sqrt{1 - \frac{2q}{a}m^q}\right)^{\frac{1}{q}}, \qquad m \in (0, M) = \left(0, \left(\frac{a}{2q}\right)^{\frac{1}{q}}\right). \quad \text{(I.42)}$$

(i) If $p = 0$ and $q = 1$, then by virtue of (I.17) and (I.42), we obtain

$$L_{1,2}(m) = \sqrt{2 - \frac{2a}{m} \mp 2\sqrt{1 - \frac{2m}{a}}} = 1 \mp \sqrt{1 - \frac{2m}{a}}, \qquad m \in (0, M).$$

(ii) Similarly,

$$L_{1,2}(m) = \frac{4a}{3}\left(1 \mp \sqrt{1 - \frac{\sqrt{m}}{a}}\right)\left(1 \mp \sqrt{1 - \frac{\sqrt{m}}{a}} + \frac{\sqrt{m}}{a}\right)$$
$$= \frac{8a}{3} \mp \frac{4a}{3}\sqrt{1 - \frac{\sqrt{m}}{a}}\left(2 + \frac{\sqrt{m}}{a}\right)$$

for $p = -\frac{1}{2}$, $q = \frac{1}{2}$ due to (I.18) and (I.42). □

Recall that

$$\lim_{m\to 0}(L_1 + L_2)(m) = L_2(0)$$

for $p < 1$ according to Lemmata 7.3 and 7.4.

8.6 Lemma
Assume (I.26) *with* $p < 1$. *Then*

(i) *if* $p > 0$ *or* $p = 0$, $q > 1$ *or* $q < -p$ *or* $p > -\frac{1}{2}$, $q = -p$, *then* $L_1 + L_2 < L_2(0)$ *in some neighbourhood of* 0,

(ii) *and if* $p = 0$, $q < 1$ *or* $p < 0$, $q > -p$ *or* $p < -\frac{1}{2}$, $q = -p$, *then* $L_1 + L_2 > L_2(0)$ *in some neighbourhood of* 0.

(See Figure 6 showing these two sets in the (p, q)*-plane.)*

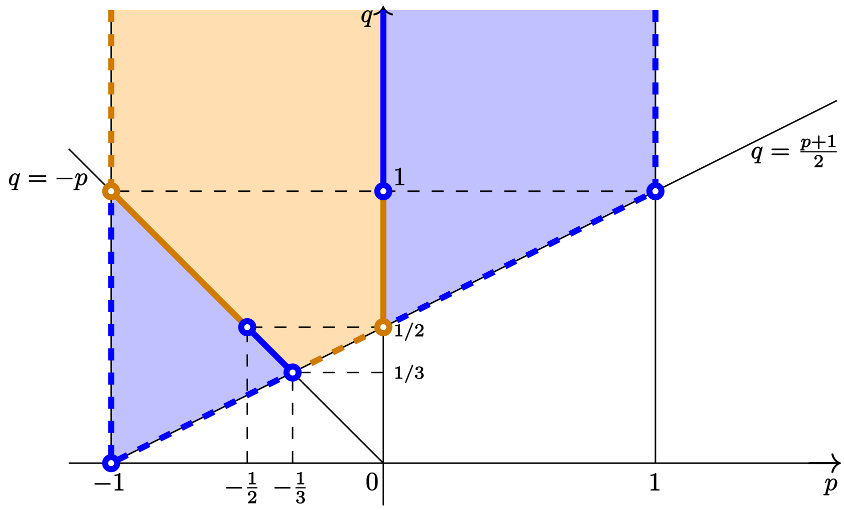

Figure 6: The behaviour of $L_1 + L_2$ near 0 in case V for $p < 1$ according to Lema 8.6: if (p, q) belongs to the blue set, then $L_1 + L_2 < L_2(0)$ near 0, and if (p, q) belongs to the brown set, then $L_1 + L_2 > L_2(0)$ near 0. (Recall that $\lim_{m\to 0}(L_1 + L_2)(m) = L_2(0)$, see (I.36).)

Proof: It is clear from Lemmata 7.3 and 7.5 that $L_1 + L_2 > L_2(0)$ near 0 if either $p < 0$, $q > -p$ or $p < -\frac{1}{2}$, $q = -p$. In order to verify the statement of the lemma for the remaining pairs (p, q), we will find the second term of the asymptotic expansion of $(L_1 + L_2)(m)$ for $m \to 0$, and determine its sign, using (I.28) and several equalities from the proof of Lemma 7.5. All the asymptotic expansions will concern $m \to 0$.

- If $p \in (0, 1)$, then $m^{q-p} = o(m^{(1-p)/2})$, so by means of step *1.* of the proof of Lemma 7.5 we have
$$(L_1 + L_2)(m) = L_2(0) + \sqrt{\frac{p+1}{2a}} B_p m^{\frac{1-p}{2}} + o\Big(m^{\frac{1-p}{2}}\Big), \tag{I.43}$$
while $B_p < 0$ (see Lemma 2.4).
- If $p = 0$, then according to step *1.* of the proof of Lemma 7.5,
$$(L_1 + L_2)(m) = L_2(0) - \frac{2q}{2q-1}\left(\frac{1}{2a}\right)^{\frac{q}{2q-1}} m + o(m)$$
for $q > 1$, and
$$(L_1 + L_2)(m) = L_2(0) + \frac{1}{a} m^q + o(m^q)$$
for $q < 1$.
- Now consider $q < -p$ (and consequently, $p < -\frac{1}{3}$). Using (I.30) and realising that $m^{q-p} = o(m^{p+1})$, we obtain
$$(L_1 + L_2)(m) = L_2(0) + \underbrace{C_{p,q,a}}_{<0} m^{p+1} + o\big(m^{p+1}\big).$$
- Finally, if $-q = p \in (-\frac{1}{2}, -\frac{1}{3})$, then the equality $m^{q-p} = o(m^{(1-p)/2})$ and (I.31) yield the asymptotic expansion of the form as in (I.43) with $B_p < 0$ due to Lemma 2.4. $\square$

To determine the behaviour of $L_1 + L_2$ near M is much more difficult. For this purpose, the second term of the corresponding asymptotic expansion will be investigated, the finding of which requires the following lemma:

8.7 Lemma
If (I.26) *holds, then*
$$\frac{R_{1,2}(m)}{R(M)} = 1 \mp \frac{\sqrt{M-m}}{\sqrt{qM}} - \frac{p+2q-2}{6qM}(M-m) + o(M-m), \qquad m \to M-.$$

Proof: Assume (I.26). From Lemma 7.1 we already know the first term of the asymptotic expansion of $R_{1,2}(m)/R(M)$ for $m \to M-$. The next two terms will be found by means of the method of undetermined coefficients from the proof of Lemma 5.1. However, let us first notice that (I.5), as an equation in m, has the explicit solution
$$m = x\left(1 - \frac{p+1}{2a} x^{2q-p-1}\right)^{\frac{1}{p+1}} =: r_{p,q,a}(x) =: r(x), \qquad x \in (0, R_2(0)),$$

which determines the inverse functions of R_1 and R_2, and will be an important tool of this proof.

All the asymptotic expansions appearing below will concern $m \to M-$ or $z \to 0$.

1. We search for such $d_1, d_2 > 0$ and $c_1 < 0$, $c_2 > 0$ that

$$\frac{R_i(m)}{R(M)} - 1 \sim c_i(M-m)^{d_i}$$

for $i = 1, 2$. (Recall that according to Lemma 7.1, $R_i/R(M)$ is increasing for $i = 1$ and decreasing for $i = 2$, which explains the choice of the sign of c_i.) Using the substitution

$$\frac{R_i(m)}{R(M)} - 1 =: z, \tag{I.44}$$

one obtains

$$A_i := \lim_{m \to M-} \frac{\frac{R_i(m)}{R(M)} - 1}{(M-m)^{d_i}} = \lim_{z \to 0\mp} \frac{z}{\big(M - r\big(R(M)(1+z)\big)\big)^{d_i}},$$

where $z \to 0\mp$ means $z \to 0-$ for $i = 1$ and $z \to 0+$ for $i = 2$. This limit (which should be finite and non-negative, determining the value of c_i) will be calculated using the asymptotic expansion of the denominator of the last fraction. Therefore, it is convenient to derive the equality

$$\begin{aligned} &M - r\big(R(M)(1+z)\big) \\ &= M - M(1+z)\left(\frac{2q}{2q-p-1} - \frac{p+1}{2q-p-1}(1+z)^{2q-p-1}\right)^{\frac{1}{p+1}} \\ &= M\underbrace{\left[1 - (1+z)\left(1 - \frac{p+1}{2q-p-1}\Big((1+z)^{2q-p-1} - 1\Big)\right)^{\frac{1}{p+1}}\right]}_{=:h(z)}. \end{aligned}$$

Approximating $(1+z)^{2q-p-1}$ with its 2nd order Maclaurin polynomial, one obtains

$$h(z) = qz^2 + o(z^2),$$

which results in

$$A_i = \lim_{z \to 0\mp} \frac{z}{(qM)^{d_i}|z|^{2d_i}}.$$

Consequently, $d_i = \frac{1}{2}$ and $c_i = A_i = \mp 1/\sqrt{qM}$.

2. Now we seek $c_i \neq 0$ and $d_i > \frac{1}{2}$ fulfilling

$$\frac{R_i(m)}{R(M)} - 1 \pm \frac{\sqrt{M-m}}{\sqrt{qM}} \sim c_i(M-m)^{d_i}$$

for $i = 1, 2$. So we have to calculate the corresponding limit

$$B_i := \lim_{m \to M-} \frac{\frac{R_i(m)}{R(M)} - 1 \pm \frac{\sqrt{M-m}}{\sqrt{qM}}}{(M-m)^{d_i}} = \lim_{z \to 0\mp} \frac{z \pm \sqrt{\frac{h(z)}{q}}}{(qM)^{d_i}|z|^{2d_i}}$$

((I.44) was used again), which requires the knowledge of one more term of the asymptotic expansion of $h(z)$. Therefore, we derive that

$$\begin{aligned} h(z) &= 1 - (1+z)\Big(1 - z + (1-q)z^2 - \frac{pq + 2q^2 - 5q + 3}{3} z^3 + o\big(z^3\big)\Big) \\ &= qz^2\Big(1 + \frac{p+2q-2}{3} z + o(z)\Big), \end{aligned}$$

which yields

$$B_i = \lim_{z \to 0\mp} \frac{-\frac{p+2q-2}{6} z^2 + o\big(z^2\big)}{(qM)^{d_i} |z|^{2d_i}},$$

meaning that $d_i = 1$ and $c_i = -\frac{p+2q-2}{6qM}$. □

The next step is to calculate the expansion of $L_1 + L_2$.

8.8 Lemma
If (I.26) *holds, then*

$$\begin{aligned} &(L_1 + L_2)(m) \\ &= 2L(M) + \left(\frac{\sqrt{2}(q-p+2)}{3\sqrt{q}} \left(\frac{R(M)}{M}\right)^{\frac{1-p}{2}} + (p-1) I_p\left(\frac{R(M)}{M}\right)\right) \frac{M-m}{\sqrt{2aM^{p+1}}} + o(M-m) \end{aligned}$$

for $m \to M-$. Recall that

$$\frac{R(M)}{M} = \left(\frac{2q}{2q-p-1}\right)^{\frac{1}{p+1}}.$$

Proof: Assume (I.26). Unless otherwise stated, all the asymptotic expansions within this proof will concern $x := \frac{M-m}{m} \to 0+$. So we have

$$L_i(m) = \frac{M^{\frac{1-p}{2}}}{\sqrt{2a}} \left(1 + \frac{p-1}{2} x + o(x)\right) I_p\left(\frac{R_i(m)}{M(1-x)}\right) \tag{I.45}$$

for $i = 1, 2$. By means of Lemma 8.7 and

$$I_p(y) = I_p(y_0) + \sqrt{2q-p-1}(y - y_0) - \frac{(2q-p-1)^{3/2}}{4} y_0^p (y-y_0)^2 + o\big((y-y_0)^2\big),$$

which holds for $y \to y_0 := \frac{R(M)}{M}$ (and follows from the definition of the Taylor polynomial), we obtain

$$\begin{aligned} I_p\left(\frac{R_i(m)}{M(1-x)}\right) &= I_p\left(y_0\left(1 \mp \frac{\sqrt{x}}{\sqrt{q}} + \frac{4q-p+2}{6q} x + o(x)\right)\right) \\ &= I_p(y_0) \mp \sqrt{\frac{2q-p-1}{q}} y_0 \sqrt{x} + \frac{\sqrt{2q-p-1}(q-p+2)}{6q} y_0 x + o(x). \end{aligned}$$

It can be inserted in (I.45), resulting in

$$\begin{aligned} L_i(m) = L(M) &\mp \frac{\sqrt{x}}{\sqrt{aR^{p-1}(M)}} \\ &+ \left(\frac{\sqrt{2}(q-p+2)}{3\sqrt{q}} y_0^{\frac{1-p}{2}} + (p-1) I_p(y_0)\right) \frac{x}{2\sqrt{2aM^{p-1}}} + o(x), \end{aligned}$$

which confirms the conclusion of the lemma. □

8.9 Lemma
Assume (I.26) *with* $p < 1$. *There exist continuously differentiable functions* $\widehat{q} : (-1, 1) \to \mathbb{R}$ *and* $\overline{q} : (-1, -\frac{1}{7}) \to \mathbb{R}$ *such that* $\widehat{q} > 1$ *on* $(-1, 0)$, $\widehat{q}(p) > \frac{p+1}{2}$ *for* $p \in [0, 1)$, $\frac{p+1}{2} < \overline{q}(p) < p + \sqrt{2p(p-1)}$ *for* $p \in (-1, -\frac{1}{7})$, *and the following holds:*

(i) *If* $q > \widehat{q}(p)$ *or* $p < -\frac{1}{7}$, $q < \overline{q}(p)$, *then* $L_1 + L_2 > 2L(M)$ *in some neighbourhood of* M.

(ii) *If* $p \geq -\frac{1}{7}$, $q < \widehat{q}(p)$ *or* $p < -\frac{1}{7}$, $\overline{q}(p) < q < \widehat{q}(p)$, *then* $L_1 + L_2 < 2L(M)$ *in some neighbourhood of* M.

In addition, for all $p \in [-\frac{1}{7}, 1)$, $q = \widehat{q}(p)$ *is given as the only solution of*

$$\frac{\sqrt{2}(q-p+2)}{3\sqrt{q}} g^{\frac{1-p}{2}}(p,q) + (p-1) I_p\big(g(p,q)\big) =: f(p,q) = 0 \tag{I.46}$$

in $(\frac{p+1}{2}, \infty)$, *where*

$$g(p,q) = \left(\frac{2q}{2q-p-1}\right)^{\frac{1}{p+1}}.$$

Similarly, for all $p \in (-1, -\frac{1}{7})$, $q = \overline{q}(p)$ *and* $q = \widehat{q}(p)$ *are the only solutions of* (I.46) *in* $[p + \sqrt{2p(p-1)}, \infty)$ *and* $(\frac{p+1}{2}, p + \sqrt{2p(p-1)}]$ *respectively.*

(See Figure 7 showing the graphs of $\widehat{q}$ *and* $\overline{q}$*, as obtained by numerical solution of* (I.46)*.)*

Proof: It is clear from Lemma 8.8 that $L_1 + L_2 > 2L(M)$ near M if $f(p,q) > 0$, while $L_1 + L_2 < 2L(M)$ near M if $f(p,q) < 0$. Obviously,

$$\lim_{q \to \infty} f(p,q) = \infty, \qquad p \in (-1, 1). \tag{I.47}$$

In the sequel we

1. find $\lim_{q \to \frac{p+1}{2}} f(p,q)$,
2. examine the monotonicity of $f(p, \cdot)$
3. and prove that $f(p, 1) < 0$ for all $p \in (-1, 0)$,

which will make us able to describe the sets of (p,q) where f is positive, zero or negative.

1. Let $p \in (-1, 1)$. Since $\lim_{q \to \frac{p+1}{2}} g(p,q) = \infty$, Lemma 2.4 can be used. We need only the first term of the asymptotic expansion of $I_p(y)$ for $y \to \infty$ to calculate

$$\lim_{q \to \frac{p+1}{2}} \frac{f(p,q)}{g^{\frac{1-p}{2}}(p,q)} = \frac{-7p-1}{3\sqrt{p+1}},$$

thus $\lim_{q \to \frac{p+1}{2}} f(p,q)$ is equal to ∞ for $p < -\frac{1}{7}$, and $-\infty$ for $p > -\frac{1}{7}$.

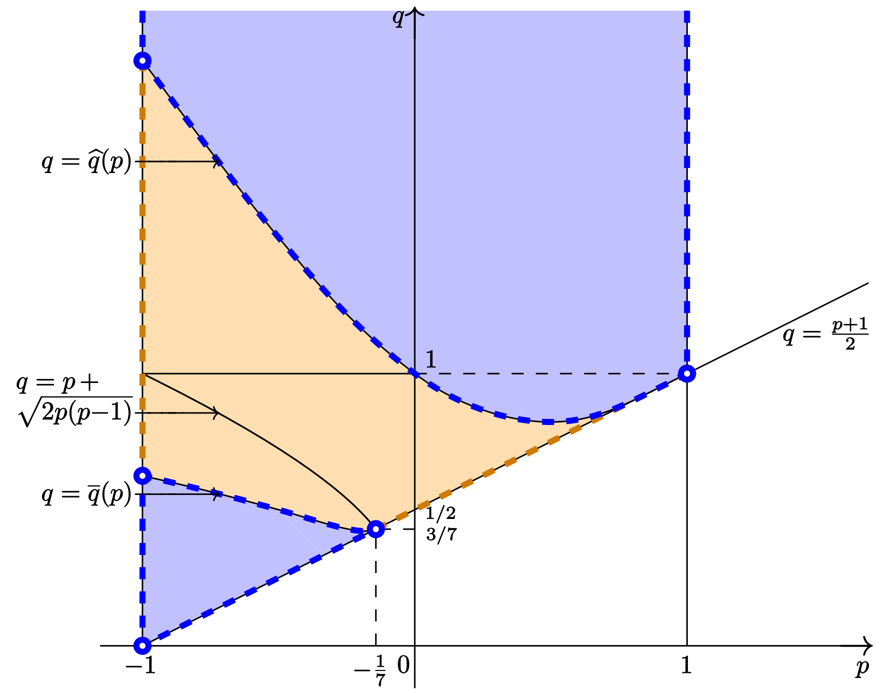

Figure 7: The behaviour of $L_1 + L_2$ near M in case V for $p < 1$ according to Lema 8.9: if (p, q) belongs to the blue set, then $L_1 + L_2 > 2L(M)$ near M, and if (p, q) belongs to the brown set, then $L_1 + L_2 < 2L(M)$ near M. (Recall that $\lim_{m\to M}(L_1 + L_2)(m) = 2L(M)$, see (I.36).)

Now assume that $p = -\frac{1}{7}$, and set $r := 2q - p - 1$. Approximating $I_p(y)$ with its two-term asymptotic expansion for $y \to \infty$, we obtain that

$$f\left(-\frac{1}{7}, q\right) = \underbrace{\left(\frac{7r + 36}{3\sqrt{7(7r + 6)}} - \frac{2\sqrt{6}}{\sqrt{7}}\right)}_{=O(r)} \underbrace{g^{4/7}\left(-\frac{1}{7}, q\right)}_{=O\left(\frac{1}{r^{2/3}}\right)} - \frac{8\sqrt{6}}{7\sqrt{7}} B_{-1/7} + o(1)$$

$$\longrightarrow -\frac{8\sqrt{6}}{7\sqrt{7}} B_{-1/7} < 0$$

for $r \to 0+$.

To sum up,

$$\lim_{q\to\frac{p+1}{2}} f(p, q) \begin{cases} > 0, & \text{if } p \in \left(-1, -\frac{1}{7}\right), \\ < 0, & \text{if } p \in \left[-\frac{1}{7}, 1\right). \end{cases}$$

2. Let $p \in (-1, 1)$ again. One can calculate that

$$\begin{aligned} \frac{\partial f}{\partial q}(p, q) &= \frac{q{+}p{-}2}{3q\sqrt{2q}} \frac{1}{\sqrt{g^{p+1}(p, q)}} g(p, q) + \frac{(1{-}p)(q{-}p{+}2)\sqrt{2q{-}p{-}1}}{6q} \frac{\partial g}{\partial q}(p, q) \\ &\quad + (p{-}1)\sqrt{\frac{p{+}1}{g^{p+1}(p, q){-}1}} \frac{\partial g}{\partial q}(p, q) \\ &= \frac{\sqrt{2q{-}p{-}1}}{6q^2}\left((p{+}q{-}2)g(p, q) + q(p{-}1)(p{+}5q{-}2)\frac{\partial g}{\partial q}(p, q)\right) \end{aligned}$$

and
$$\frac{\partial g}{\partial q}(p,q) = -\frac{g(p,q)}{q(2q-p-1)},$$
consequently,
$$\frac{\partial f}{\partial q}(p,q) = \underbrace{\left(q^2 - 2pq - p^2 + 2p\right)}_{=:\xi(p,q)} \underbrace{\frac{g(p,q)}{3q^2\sqrt{2q-p-1}}}_{>0}.$$

It is easy to see that
$$\xi(p,q) = 0 \iff p \le 0 \text{ and } q = p \pm \sqrt{2p(p-1)},$$
while $p - \sqrt{2p(p-1)} < \frac{p+1}{2}$ for all $p \le 1$, and $p + \sqrt{2p(p-1)} > \frac{p+1}{2}$ only if $p < -\frac{1}{7}$.

So we conclude that

- if $p \in [-\frac{1}{7}, 1)$, then $f(p,\cdot)$ increases on $(\frac{p+1}{2}, \infty)$,
- if $p \in (-1, -\frac{1}{7})$, then $f(p,\cdot)$ decreases on $(\frac{p+1}{2}, p + \sqrt{2p(p-1)}]$ and increases on $[p + \sqrt{2p(p-1)}, \infty)$.

3. In this step we prove that $f(p,1) < 0$ for all $p \in (-1,0)$, or equivalently,
$$I_p\left(\left(\frac{2}{1-p}\right)^{\frac{1}{p+1}}\right) > \frac{\sqrt{2}(3-p)}{3(1-p)}\left(\frac{2}{1-p}\right)^{\frac{1-p}{2(p+1)}}, \qquad p \in (-1,0). \qquad \text{(I.48)}$$

Our method is to gradually derive simpler and simpler sufficient conditions of (I.48), the last of which will be proved directly.

(a) Since $p \mapsto I_p(y)$ decreases on $\mathbb{R}$ for all $y > 1$ according to Theorem 2.6, a sufficient condition for (I.48) can be obtained replacing I_p on its left-hand side with I_0 (see also (I.17)). After squaring, this new inequality reads
$$\left(\frac{2}{1-p}\right)^{\frac{1}{p+1}} - 1 > \frac{1}{18}\left(\frac{2}{1-p} + 1\right)^2 \left(\frac{2}{1-p}\right)^{\frac{1-p}{p+1}}, \qquad p \in (-1,0).$$
Denoting $\frac{2}{1-p} =: x$, it simplifies to
$$\sqrt{x \cdot x^{\frac{1}{x-1}}} - 1 > \frac{(x+1)^2 x^{\frac{1}{x-1}}}{18}, \qquad x \in (1,2).$$
It is convenient to introduce the notation $\omega(x) := x^{1/(x-1)}$, by means of which the last inequality transforms to
$$-\frac{x^2 - 7x + 1}{9}\omega(x) - \frac{(x+1)^4}{324}\omega^2(x) > 1, \qquad x \in (1,2). \qquad \text{(I.49)}$$

(b) Let us prove that
$$\frac{x(4-x)}{8}\omega(x) > 1, \qquad x \in (1,2). \qquad \text{(I.50)}$$

Equivalently, it can be written as

$$\zeta(x) := x \ln x + (x-1)\big(\ln(4-x) - \ln 8\big), \qquad x \in (1,2).$$

We have

$$\zeta''(x) = \frac{2x^2 - 15x + 16}{x(x-4)^2},$$

and one can see that ζ'' is positive on $[1, x_0)$ and negative on $(x_0, 2]$, while $x_0 = (15 - \sqrt{97})/4$. Consequently, $\zeta' > \zeta'(1) = \ln \frac{3e}{8} > 0$ on $(1, x_0]$, and since $\zeta(1) = 0$, the positivity of ζ on $(1, x_0]$ follows. Therefore, the concavity of ζ on $[x_0, 2]$ with $\zeta(2) = 0$ ensures its positivity on $[x_0, 2)$, and (I.50) is verified.

Replacing the right-hand side of (I.49) with the left-hand side of (I.50), we obtain a sufficient condition for (I.49), which can be simplified to

$$\omega(x) < \frac{9\big(x^2 + 20x - 8\big)}{2(x+1)^4}, \qquad x \in (1,2). \tag{I.51}$$

(c) Our next auxiliary inequality is

$$\frac{6}{x+1} < \frac{9\big(x^2 + 20x - 8\big)}{2(x+1)^4}, \qquad x \in (1,2),$$

which is equivalent to

$$P(x) := 4x^3 + 9x^2 - 48x + 28 < 0, \qquad x \in (1,2),$$

and which can be proved realising that $P(1) = -7 < 0$, $P(2) = 0$ and $P'' > 0$ on $(1,2)$. It provides a sufficient condition for (I.51) in the form of

$$\omega(x) < \frac{6}{x+1}, \qquad x \in (1,2),$$

or equivalently,

$$\eta(x) := \ln x + (x-1)\big(\ln(x+1) - \ln 6\big) < 0, \qquad x \in (1,2),$$

which is a true inequality, since $\eta(1) = \eta(2) = 0$ and

$$\eta''(x) = \frac{(x-1)\big(x^2 + 3x + 1\big)}{x^2(x+1)^2} > 0, \qquad x \in (1,2).$$

Define

$$q_1(p) := \begin{cases} p + \sqrt{2p(p-1)} & \text{if } p \in \big(-1, -\frac{1}{7}\big), \\ \dfrac{p+1}{2} & \text{if } p \in \big[-\frac{1}{7}, 1\big). \end{cases}$$

As a consequence of *1.*, *2.* and *3.*, $\lim_{q \to q_1(p)} f(p,q) < 0$ for all $p \in (-1,1)$. Taking (I.47) and the increase of $f(p,\cdot)$ on $(q_1(p), \infty)$ into account as well, we obtain that

$$\forall p \in (-1,1)\colon \quad \exists! \widehat{q}(p) \in \big(q_1(p), \infty\big)\colon \quad f\big(p, \widehat{q}(p)\big) = 0.$$

Clearly, if $p \in (-1,1)$, $q > \widehat{q}(p)$, then $f(p,q) > 0$ and consequently, $L_1 + L_2 > 2L(M)$ near M. On the other hand, if $p \in (-1,1)$, $q \in [q_1(p), \widehat{q}(p))$, then $f(p,q) < 0$, and $L_1 + L_2 < 2L(M)$ near M. Furthermore, $\widehat{q} > 1$ on $(-1,0)$ due to *3.*, while the continuous differentiability of $\widehat{q}$ follows from the implicit function theorem and the continuous differentiability of f (see Theorem 2.7).

Similarly, since $f(p, q_1(p)) < 0$ for $p \in (-1, -\frac{1}{7})$, *1.* and *2.* imply that

$$\forall p \in \left(-1, -\tfrac{1}{7}\right): \quad \exists! \overline{q}(p) \in \left(\tfrac{p+1}{2}, q_1(p)\right): \quad f\left(p, \overline{q}(p)\right) = 0.$$

Again, $f(p,q)$ is positive for $p \in (-1, -\frac{1}{7})$, $q \in (\frac{p+1}{2}, \overline{q}(p))$, and negative for $p \in (-1, -\frac{1}{7})$, $q \in (\overline{q}(p), q_1(p)]$, making clear the behaviour of $L_1 + L_2$ near M for these values of p and q, and obviously, $\overline{q}$ is continuously differentiable. $\square$

The next lemma describes the basic properties of $\widehat{q}$.

8.10 Lemma
There exists

$$\lim_{p \to -1} \widehat{q}(p) =: \widehat{q}(-1) \in (1, \infty),$$

and it is the only solution of the equation

$$\varphi(q) := \frac{\sqrt{2}(q+3)}{3\sqrt{q}} \mathrm{e}^{\frac{1}{2q}} - 2I_{-1}\left(\mathrm{e}^{\frac{1}{2q}}\right) = 0 \tag{I.52}$$

in $[1, \infty)$. *Furthermore,* $\widehat{q} > 1$ *on* $(-1,0)$, $\widehat{q}(-\frac{1}{2}) = \frac{3}{2}$, $\widehat{q}(0) = 1$, $\widehat{q} < 1$ *on* $(0,1)$, *and* $\lim_{p \to 1} \widehat{q}(p) = 1$.

Proof: It is a part of Lemma 8.9 that $\widehat{q} > 1$ on $(-1,0)$. We also know from it that $L_1 + L_2 \neq 2L(M)$ near M for $p = 0$, $q \in (0, \infty) \setminus \{\widehat{q}(0)\}$, which, in view of Lemma 8.5 (i), yields $\widehat{q}(0) = 1$. It remains to

1. prove the existence and properties of $\lim_{p \to -1} \widehat{q}(p)$,
2. figure out $\widehat{q}(-\frac{1}{2})$
3. and prove that $\widehat{q} < 1$ on $(0,1)$.

We will obtain $\lim_{p \to 1} \widehat{q}(p)$ as a direct consequence of *3.* and $\widehat{q}(p) > \frac{p+1}{2}$.

1. Theorem 2.6 and some elementary calculations yield that $\lim_{p \to -1} f(p,q) = \varphi(q)$ for any $q > 0$ (see Lemma 8.9 for the definition of f).

Clearly, $\lim_{q \to \infty} \varphi(q) = \infty$. Since $I_{-1}(\mathrm{e}^{1/2q}) = O(\sqrt{q})\mathrm{e}^{1/2q}$ for $q \to 0$ due to Lemma 2.5,

$$\varphi(q) = \frac{\sqrt{2}}{\sqrt{q}} \mathrm{e}^{\frac{1}{2q}} \left(1 + O(q)\right) \longrightarrow \infty, \qquad q \longrightarrow 0.$$

It is not hard to derive that

$$\varphi'(q) = \frac{(q-1)(q+3)}{3q^2\sqrt{2q}} \mathrm{e}^{\frac{1}{2q}}, \qquad q > 0,$$

which implies that φ is decreasing on $(0,1]$ and increasing on $[1,\infty)$. Furthermore,

$$\varphi(1) = \frac{4}{3}\sqrt{2\mathrm{e}} - 2I_{-1}(\sqrt{\mathrm{e}}) < \frac{4}{3}\sqrt{2\mathrm{e}} - 2I_0(\sqrt{\mathrm{e}}) = 4\left(\frac{\sqrt{2\mathrm{e}}}{3} - \sqrt{\sqrt{\mathrm{e}}-1}\right) < 0$$

(see Theorem 2.6 and (I.17)).

So one can see that $\varphi|_{(1,\infty)}$ has a unique zero, which will be denoted by q_0. Since $\varphi = \lim_{p\to -1} f(p,\cdot)$, and it increases on $(1,\infty)$, we have that for arbitrary $\varepsilon \in (0, q_0 - 1)$ there exists $\delta > 0$ such that

$$\forall p \in (-1, -1+\delta): \quad f(p, q_0 - \varepsilon) < 0 < f(p, q_0 + \varepsilon)$$

and therefore,

$$\forall p \in (-1, -1+\delta): \quad q_0 - \varepsilon < \widehat{q}(p) < q_0 + \varepsilon,$$

following from the increase of $f(p,\cdot)$ on $(1,\infty)$ (see step *2.* of the proof of Lemma 8.9). Consequently, $\lim_{p\to -1} \widehat{q}(p) = q_0$.

2. One can calculate that

$$f\left(-\frac{1}{2}, q\right) = \frac{4\sqrt{2}q(2q+5)}{3(4q-1)^{3/2}} - \frac{3}{2}I_{-1/2}\left(\left(\frac{4q}{4q-1}\right)^2\right) = \frac{2\sqrt{2}\left(4q^2 - 8q + 3\right)}{3(4q-1)^{3/2}}$$

for $q > \frac{1}{4}$, which vanishes only for $q = \frac{1}{2}$ and $q = \frac{3}{2}$, meaning that $\widehat{q}(-\frac{1}{2}) = \frac{3}{2}$.

3. Now we prove that $f(p,1) > 0$ for all $p \in (0,1)$, guaranteeing that $\widehat{q} < 1$ on $(0,1)$. It is equivalent to

$$I_p\left(\left(\frac{2}{1-p}\right)^{\frac{1}{p+1}}\right) < \frac{\sqrt{2}(3-p)}{3(1-p)}\left(\frac{2}{1-p}\right)^{\frac{1-p}{2(p+1)}}, \qquad p \in (0,1), \tag{I.53}$$

which will be gradually simplified, similarly to step *3.* of the proof of Lemma 8.9.

(a) The first sufficient condition for (I.53) is

$$-\frac{x^2 - 7x + 1}{9}\omega(x) - \frac{(x+1)^4}{324}\omega^2(x) < 1, \qquad x > 2 \tag{I.54}$$

(again, $\omega(x) = x^{1/(x-1)}$), which can be derived in a way completely analogous to the corresponding part of the proof of Lemma 8.9.

(b) The opposite inequality of (I.50) does not hold for all $x > 2$. Instead,

$$\frac{\omega(x)}{2} < 1, \qquad x > 2 \tag{I.55}$$

will be used, which is equivalent to

$$\kappa(x) := (x-1)\ln 2 - \ln x > 0, \qquad x > 2,$$

and the validity of which follows from the facts that $\kappa(2) = 0$ and

$$\kappa'(x) = \ln 2 - \frac{1}{x} > \ln 2 - \frac{1}{2} > 0, \qquad x > 2.$$

Due to (I.55), 1 can be replaced with $\omega(x)/2$ on the right-hand side of (I.54), yielding a sufficient condition for (I.54), which can be rewritten as

$$\omega(x) > -\frac{18\big(2x^2 - 14x + 11\big)}{(x+1)^4}, \qquad x > 2.$$

(c) The final simplification will be done by virtue of the inequality

$$\frac{6}{x+1} > -\frac{18\big(2x^2 - 14x + 11\big)}{(x+1)^4}, \qquad x > 2,$$

equivalent to

$$Q(x) := x^3 + 9x^2 - 39x + 34 > 0, \qquad x > 2,$$

which holds since $Q(2) = 0$ and $Q'(x) > 9 > 0$ for $x > 2$. So now the only assertion to prove is

$$\omega(x) > \frac{6}{x+1}, \qquad x > 2.$$

And to do so, we just have to recall part (c) of step *3.* of the proof of Lemma 8.9, and to realise that $\eta(x) > 0$ for $x > 2$ because $\eta'(2) = \frac{5}{6} - \ln 2 > 0$ and $\eta'' > 0$ on $(2, \infty)$. $\square$

According to numerical calculations, $\widehat{q}(-1) \approx 2.151$, $\widehat{q}$ seems to be convex, having $\min \widehat{q} \approx 0.822 \approx \widehat{q}(0.495)$, and its graph seems to touch the graph of $q = \frac{p+1}{2}$ at 1.

Recall that the line $q = -p$ forms the border between those sets of (p, q) where $L_1 + L_2 < L_2(0)$ and $L_1 + L_2 > L_2(0)$ near 0 (see Lemma 8.6). According to Lemma 8.9, the graph of $\overline{q}$ plays a similar role in the behaviour of $L_1 + L_2$ near M. Therefore, if we are interested in the behaviour of $L_1 + L_2$ on $(0, M)$, we have to know the mutual position of these to curves.

8.11 Lemma
There exists

$$\lim_{p \to -1} \overline{q}(p) =: \overline{q}(-1) \in (0, 1),$$

and it is the only solution of the equation (I.52) *in* $(0, 1]$. *Furthermore,* $\overline{q}(p) < -p$ *for* $p \in (-1, -\frac{1}{2})$, $\overline{q}(-\frac{1}{2}) = \frac{1}{2}$, $\overline{q}(p) > -p$ *for* $p \in (-\frac{1}{2}, -\frac{1}{7})$ *and* $\lim_{p \to -1/7} \overline{q}(p) = \frac{3}{7}$.

Proof: The existence and properties of $\lim_{p \to -1} \overline{q}(p)$ can be validated the same way as it is done in step *1.* of the proof of Lemma 8.10 for $\lim_{p \to -1} \widehat{q}(p)$. And it is clear from step *2.* of the same proof and from the definition of $\overline{q}$ (or from

Lemma 8.5 (ii)) that $\overline{q}(-\frac{1}{2}) = \frac{1}{2}$. Further, since $\frac{p+1}{2} < \overline{q}(p) < p + \sqrt{2p(p-1)}$ for $p \in (-1, -\frac{1}{7})$ (see Lemma 8.9), the value of $\lim_{p\to -1/7} \overline{q}(p)$ is evident.

It remains to determine the sign of $\overline{q}(p)+p$ for $p \in (-1, -\frac{1}{3})$. (For $p \in [-\frac{1}{3}, -\frac{1}{7})$ we obviously have $-p \leq \frac{p+1}{2} < \overline{q}(p)$.) Let

$$\left.\begin{aligned} \mathit{\Gamma}(p) &:= g(p,-p) = \left(\frac{2p}{3p+1}\right)^{\frac{1}{p+1}}, \\ \mathit{\Phi}(p) &:= \frac{f(p,-p)}{(p-1)\sqrt{p+1}} = \frac{I_p(\mathit{\Gamma}(p))}{\sqrt{p+1}} - \frac{2\sqrt{2}}{3\sqrt{-p(p+1)}}\mathit{\Gamma}^{\frac{1-p}{2}}(p), \end{aligned}\right\} \qquad p \in \left(-1, -\tfrac{1}{3}\right).$$

We prove soon that

1. $\mathit{\Phi}$ decreases on $[-\frac{3}{7}, -\frac{1}{3})$,
2. $\mathit{\Phi} < 0$ on $(-\frac{1}{2}, -\frac{3}{7}]$
3. and $\mathit{\Phi} > 0$ on $(-1, -\frac{1}{2})$.

It will mean that $f(p,-p)$ is positive for $p \in (-\frac{1}{2}, -\frac{1}{3})$ and negative for $p \in (-1, -\frac{1}{2})$. Since for all $p \in (-1, -\frac{1}{3})$: $-p \in (\frac{p+1}{2}, p + \sqrt{2p(p-1)})$, $f(p,\cdot)$ decreases on $(\frac{p+1}{2}, p+\sqrt{2p(p-1)})$ (see step *2.* of the proof of Lemma 8.9) and $f(p, \overline{q}(p)) = 0$, the assertion of the lemma regarding the relationship between $\overline{q}(p)$ and $-p$ will follow.

1. Let $p \in (-1, -\frac{1}{3})$. We have

$$\mathit{\Gamma}'(p) = \left(\frac{1}{p(3p+1)} - \frac{1}{p+1}\ln\frac{2p}{3p+1}\right)\frac{\mathit{\Gamma}(p)}{p+1}$$

and

$$\begin{aligned} \left(\mathit{\Gamma}^{\frac{1-p}{2}}(p)\right)' &= \left(\sqrt{\frac{3p+1}{2p}}\mathit{\Gamma}(p)\right)' \\ &= \left(\frac{1-p}{2p(3p+1)} - \frac{1}{p+1}\ln\frac{2p}{3p+1}\right)\sqrt{\frac{3p+1}{2p}}\frac{\mathit{\Gamma}(p)}{p+1}. \end{aligned}$$

Thanks to Theorem 2.7, $\mathit{\Phi}$ is differentiable, and

$$\mathit{\Phi}'(p) = \underbrace{J_p(\mathit{\Gamma}(p))}_{<0} - \underbrace{\left(\frac{2p+1}{3p+1} + \frac{3p+2}{3(p+1)}\ln\frac{2p}{3p+1}\right)}_{=:H(p)}\underbrace{\sqrt{\frac{-3p-1}{p+1}}\frac{\mathit{\Gamma}(p)}{p(p+1)}}_{<0}.$$

Numerical calculations indicate that $\mathit{\Phi}$ is decreasing. If we could prove it, the proof would be complete (since we know that $\mathit{\Phi}(-\frac{1}{2}) = 0$). The non-positivity of H is a sufficient condition for it.

Instead of H, we will investigate h, defined as

$$\left.\begin{aligned} h(p) &:= \frac{3(p+1)}{3p+2}H(p) \\ &= \frac{3(p+1)(2p+1)}{(3p+1)(3p+2)} + \ln\frac{2p}{3p+1}, \end{aligned}\right\} \qquad p \in \left(-1, -\tfrac{1}{3}\right) \smallsetminus \left\{\tfrac{2}{3}\right\},$$

because it has a simpler derivative:

$$h'(p) = \frac{15p^2 + 15p + 4}{p(3p+1)^2(3p+2)^2} < 0.$$

Since $\lim_{p\to -1} h(p) = 0$, $h < 0$ on $(-1, -\frac{2}{3})$. One can also derive that $\lim_{p\to -2/3+} h(p) = \infty$ and $\lim_{p\to -1/3-} h(p) = -\infty$. Consequently, $h > 0$ on $(-\frac{2}{3}, p_0)$ and $h < 0$ on $(p_0, -\frac{1}{3})$ for some $p_0 \in (-\frac{2}{3}, -\frac{1}{3})$. It means that the sufficient condition for the decrease of Φ is met only for $p \in (p_0, -\frac{1}{3})$. Since $h(-\frac{3}{7}) = \ln 3 - \frac{6}{5} < 0$, we have $p_0 < -\frac{3}{7}$. (According to numerical calculations, $p_0 \approx -0.434$.)

2. The proof of $\Phi < 0$ on $(-\frac{1}{2}, -\frac{3}{7}]$ is based on the method of gradual simplification from step *3.* of the proof of Lemma 8.9.

(a) Let

$$\widetilde{\Phi}(p) := \frac{I_{-1/2}(\Gamma(p))}{\sqrt{p+1}} - \frac{2\sqrt{2}}{3\sqrt{-p(p+1)}}\Gamma^{\frac{1-p}{2}}(p), \qquad p \in \left(-1, -\tfrac{1}{3}\right).$$

Due to Theorem 2.6, $\widetilde{\Phi}(p) < 0$ is a sufficient condition for $\Phi(p) < 0$ for $p \in (-\frac{1}{2}, -\frac{3}{7}]$. (Naturally, the same holds even for $p \in (-\frac{1}{2}, -\frac{1}{3})$, but numerical calculations suggest that $\widetilde{\Phi} < 0$ on $(-\frac{1}{2}, p_1)$ and $\widetilde{\Phi} > 0$ on $(p_1, -\frac{1}{3})$ with $p_1 \approx -0.338$. This explains why we have executed step *1.*) Using (I.18), the condition we want to verify can be rewritten as

$$\left(\left(\frac{2p}{3p+1}\right)^{\frac{1}{2(p+1)}} - 1\right)\left(\left(\frac{2p}{3p+1}\right)^{\frac{1}{2(p+1)}} + 2\right)^2 < -\frac{1}{p}\left(\frac{2p}{3p+1}\right)^{\frac{1-p}{p+1}}, \qquad p \in \left(-\tfrac{1}{2}, -\tfrac{3}{7}\right],$$

or equivalently as

$$\left(x^{\frac{3x-2}{4(x-1)}} - 1\right)\left(x^{\frac{3x-2}{4(x-1)}} + 2\right)^2 < \frac{3x-2}{x^2}x^{\frac{3x-2}{x-1}}, \qquad x \in (2, 3],$$

where $x := \frac{2p}{3p+1}$. After introducing

$$\tau(x) := x^{\frac{3x-2}{4(x-1)}}, \qquad x > 1,$$

we can rearrange it into the form

$$3\tau^2(x) + \tau^3(x) + \frac{2-3x}{x^2}\tau^4(x) < 4, \qquad x \in (2, 3]. \tag{I.56}$$

(b) Now the inequality

$$\frac{2\tau^3(x)}{x^2} < 4, \qquad x \in (2, 3] \tag{I.57}$$

will be used. Its validity follows from its equivalent form

$$\zeta(x) := (x+2)\ln x - (x-1)4\ln 2 < 0, \qquad x \in (2, 3],$$

after realising that $\zeta(2) = 0$, $\zeta(3) = \ln \frac{243}{256} < 0$ and $\zeta''(x) = \frac{x-2}{x^2} > 0$ for $x \in (2,3)$. So the right-hand side of (I.56) can be replaced by the left-hand side of (I.57), yielding a sufficient condition for (I.56), which can be simplified to

$$\frac{x^2-2}{x^2}\tau(x) + \frac{2-3x}{x^2}\tau^2(x) < -3, \qquad x \in (2,3]. \tag{I.58}$$

(c) Let us now prove that

$$-\frac{2x+5}{3x}\tau(x) < -3, \qquad x \in (2,3]. \tag{I.59}$$

The given inequality can be rearranged into

$$\eta(x) := (2-x)\ln x + 4(x-1)\big(\ln(2x+5) - \ln 9\big) > 0, \qquad x \in (2,3].$$

One can derive that

$$\eta''(x) = \frac{P(x)}{x^2(2x+5)^2}$$

with

$$P(x) = 12x^3 + 68x^2 - 65x - 50.$$

Apparently, $P(x) > 75 > 0$ for $x \in (2,3)$ and consequently, η is strictly convex on $(2,3]$. And since $\eta(2) = 0$ and $\eta'(2) = \frac{8}{9} - \ln 2 > 0$, we have that $\eta > 0$ on $(2,3]$.

Thanks to (I.59), a sufficient condition for (I.58) follows, namely

$$\tau(x) > \frac{5x^2+5x-6}{3(3x-2)}, \qquad x \in (2,3].$$

(d) It is easy to see that

$$\frac{3x+4}{5} > \frac{5x^2+5x-6}{3(3x-2)}, \qquad x \in (2,3]$$

because it is equivalent to

$$Q(x) := 2x^2 - 7x + 6 > 0, \qquad x \in (2,3],$$

while $\frac{3}{2}$ and 2 are the roots of Q. So proving

$$\tau(x) > \frac{3x+4}{5}, \qquad x \in (2,3], \tag{I.60}$$

will finish step *2*. Let us express (I.60) in the form

$$\kappa(x) := (3x-2)\ln x + 4(1-x)\big(\ln(3x+4) - \ln 5\big) > 0, \qquad x \in (2,3].$$

We have

$$\kappa''(x) = -\frac{S(x)}{x^2(3x+4)^2},$$

where

$$S(x) = 9x^3 + 42x^2 - 96x - 32.$$

Since $S(2) = 16 > 0$ and $S'(x) > 180 > 0$ for $x > 2$, κ is strictly concave on $(2,3]$, which together with $\kappa(2) = 0$ and $\kappa(3) = \ln \frac{3^7 5^8}{13^8} > 0$ yields that $\kappa > 0$ indeed on $(2,3]$.

3. Parts (a), (b) and (c) of step *2.* are applicable for the proof of the positivity of Φ on $(-1, -\frac{1}{2})$ with minor changes.

(a) It suffices to prove that $\widetilde{\Phi} > 0$ on $(-1, -\frac{1}{2})$, which is equivalent to

$$3\tau^2(x) + \tau^3(x) + \frac{2-3x}{x^2}\tau^4(x) > 4, \qquad x \in (1,2). \tag{I.61}$$

(b) Since $\zeta(1) = \zeta(2) = 0$ and $\zeta''(x) < 0$ for $x \in (1,2)$, $\zeta > 0$ on $(1,2)$, yielding a sufficient condition for (I.61) in the form

$$\frac{x^2-2}{x^2}\tau(x) + \frac{2-3x}{x^2}\tau^2(x) > -3, \qquad x \in (1,2). \tag{I.62}$$

(c) We have $P(1) = -35 < 0$, $P(2) = 188 > 0$ and $P'(x) > 107 > 0$ for $x > 1$. Consequently, P has a unique root x_0 in $(1,2)$, and η is strictly concave on $(1, x_0]$ and strictly convex on $[x_0, 2)$. However, $\eta(1) = \eta(2) = 0$, and $\eta'(1) = 1 + 4\ln\frac{4}{9} < 0$, which ensure that $\eta < 0$ on $(1,2)$, and

$$\tau(x) < \frac{5x^2+5x-6}{3(3x-2)}, \qquad x \in (1,2)$$

is a sufficient condition for (I.62).

(d) As we have seen, $Q(\frac{3}{2}) = 0$ and therefore, we cannot proceed as in part (d) of step *2.* Instead, let us prove that

$$\frac{8x+4}{x+8} < \frac{5x^2+5x-6}{3(3x-2)}, \qquad x \in (1,2).$$

The desired inequality is equivalent to

$$T(x) := 5x^3 - 27x^2 + 46x - 24 > 0, \qquad x \in (1,2).$$

Let us notice that $T'' < 0$ on $(1, \frac{9}{5})$ and $T'' > 0$ on $(\frac{9}{5}, 2)$. And since $T(1) = T(2) = 0$ and $T'(2) = -2 < 0$, the positivity of T on $(1,2)$ follows.

Consequently, it suffices to prove that

$$\tau(x) < \frac{8x+4}{x+8}, \qquad x \in (1,2).$$

Let us reformulate it as

$$\mu(x) := 4(x-1)\big(\ln 4 + \ln(2x+1) - \ln(x+8)\big) - (3x-2)\ln x > 0, \qquad x \in (1,2).$$

After differentiating we obtain that

$$\mu''(x) = -\frac{U(x)}{x^2(x+8)^2(2x+1)^2},$$

where

$$U(x) = 12x^5 + 212x^4 - 161x^3 - 522x^2 + 736x + 128.$$

We have that $U(1) = 405 > 0$, $U'(1) = 117 > 0$, $U''(1) = 774 > 0$ and $U'''(x) > 4842 > 0$ for $x > 1$, meaning that μ is strictly concave on $(1,2)$. The last fact we have to realise is that $\mu(1) = \mu(2) = 0$. $\square$

Numerical calculations indicate that $\overline{q}(-1) \approx 0.624$, it has a unique stationary point (≈ -0.185, while $\overline{q}(-0.185) \approx 0.421$) as well as a unique inflection point (≈ -0.400), and its graph touches the graph of $q = \frac{p+1}{2}$ at $-\frac{1}{7}$.

For $p < 1$ we have succeeded in describing the behaviour of $L_1 + L_2$ only near 0 and M, except $p = 0$, $q = 1$ and $p = -\frac{1}{2}$, $q = \frac{1}{2}$, for which $L_1 + L_2$ is constant, and except $p \in (-1, 0) \cup (0, 1)$, $q = \widehat{q}(p)$ and $p \in (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, -\frac{1}{7})$, $q = \overline{q}(p)$, for which we have no information at all. However, using numerical calculations one can observe that $L_1 + L_2$ has probably at most one relative extremum for any $p \in (-1, 1)$, $q > \frac{p+1}{2}$, $(p, q) \notin \{(0, 1), (-\frac{1}{2}, \frac{1}{2})\}$. If it is true, the behaviour of $L_1 + L_2$ on $(0, M)$ is clear for all $p \in (-1, 1)$, $q \notin \{\widehat{q}(p), \overline{q}(p)\}$, and due to the continuous dependence of $L_{1;p,q,a}(m)$ and $L_{2;p,q,a}(m)$ on q, the proof of which is evident, even for $q = \widehat{q}(p)$ and $q = \overline{q}(p)$.

The results of this section concerning the properties of $L_1 + L_2$ are summarised in Figure 8, which shows the graphs of $L_1 + L_2$ and the corresponding sets of (p, q). Let us notice that the graphs of $\widehat{q}$ and $\overline{q}$ in it are the output of the numerical solution of (I.46).

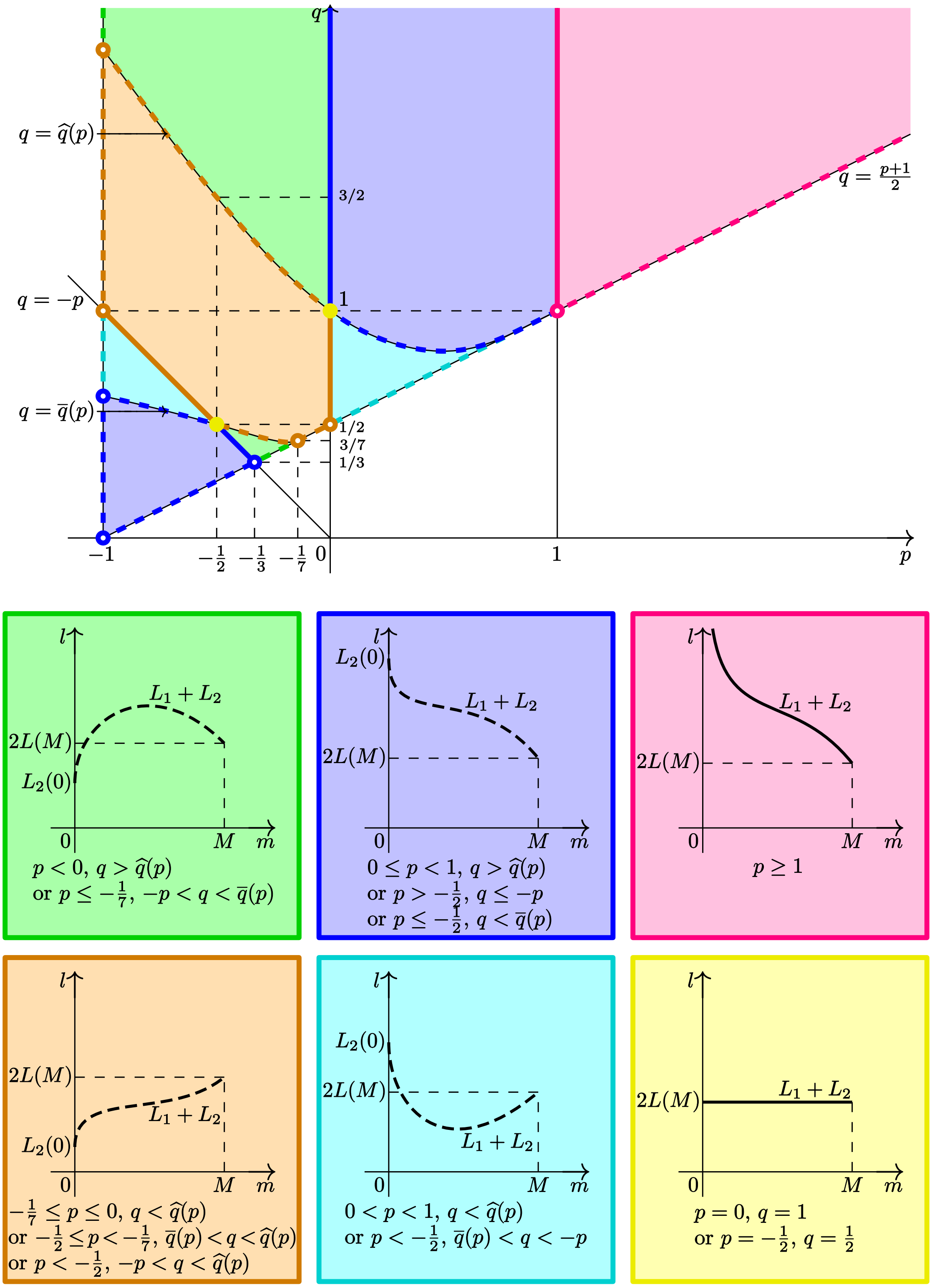

Figure 8: The behaviour of $L_1 + L_2$ in case V ($p > -1$, $q > \frac{p+1}{2}$, $a > 0$) according to (I.36) and Lemmata 8.2, 8.5, 8.6, 8.9, 8.10 and 8.11.
The dashed graphs mean that for those values of p and q the behaviour of $L_1 + L_2$ has been examined only near 0 and M, and the graph has been plotted assuming that $L_1 + L_2$ has at most one stationary point. (This assumption is consistent with numerical calculations.)

Chapter II

Some related results

9 Sign-changing non-antisymmetric solutions

This section will start by recalling the shooting method from [6]. Lemmata 9.1, 9.2, 9.5 and 9.6 will be stated under weaker assumptions on q than the corresponding assertions cited from [6], but we do not provide the proofs because they are unchanged.

Let $p \geq 1$, $q \in \mathbb{R}$, $a, l > 0$. If u is a sign-changing solution of (1) and x_0 is its zero, then $u(\cdot + x_0)$ solves

$$\begin{cases} u'' = a|u|^{p-1}u, \\ u(0) = 0, \\ u'(0) = \theta \end{cases} \tag{II.1}$$

for some $\theta \in \mathbb{R}$. Since $u \mapsto a|u|^{p-1}u$ is locally Lipschitz continuous on $\mathbb{R}$, (II.1) has a unique maximal solution, which is obviously odd. It will be denoted by $\overline{u}_{\theta,p,a}$ and its domain by $(-\overline{\Lambda}_{\theta,p,a}, \overline{\Lambda}_{\theta,p,a})$. Clearly, $\overline{u}_{0,p,a} \equiv 0$ on $\mathbb{R}$ and thus, $x_0 \in (-l, l)$ and $\theta \neq 0$. One can also see that u is strictly convex on the intervals where it has positive values, and strictly concave on the intervals where it has negative values. As a consequence, $\overline{u}'_{\theta,p,a} > 0$ if $\theta > 0$, and $\overline{u}'_{\theta,p,a} < 0$ if $\theta < 0$. In addition, $\overline{u}_{-\theta,p,a} = -\overline{u}_{\theta,p,a}$, therefore we will restrict our further considerations to $\theta > 0$.

Let us also introduce the notation $\mathcal{N}^{\pm}(l) = \mathcal{N}^{\pm}(l; p, q, a)$ for the set of all sign-changing non-antisymmetric (i. e. not odd) solutions of (1). Obviously, $\mathcal{N}^{\pm}(l)$ consists of all such functions $\pm\overline{u}_{\theta,p,a}(\cdot - (l_1 - l_2)/2)|_{[-l,l]}$ that $\theta > 0$, $l_1 + l_2 = 2l$, $l_1 \neq l_2$ and $0 < l_i < \overline{\Lambda}_{\theta,p,a}$, $\overline{u}'_{\theta,p,a}(l_i) = \overline{u}^q_{\theta,p,a}(l_i)$ for $i = 1, 2$.

9.1 Lemma (for $q > 1$ see [6, pp. 114–116])
Let $p \geq 1$, $q \in \mathbb{R}$, $a > 0$, and set $b := \frac{2a}{p+1}$. Then the following statements are equivalent for arbitrary $\theta, l > 0$:

(i) *$l < \overline{\Lambda}_{\theta,p,a}$ and $\overline{u}'_{\theta,p,a}(l) = \overline{u}^q_{\theta,p,a}(l)$,*

(ii) *the equation*

$$0 = \overline{\mathcal{F}}(\theta, x) := \overline{\mathcal{F}}_{p,q,a}(\theta, x) := x^{2q} - bx^{p+1} - \theta^2$$

with the unknown $x > 0$ has some solution $\overline{R}$, and

$$l = \theta^{-\frac{p-1}{p+1}} I_{p,b}\Big(\theta^{-\frac{2}{p+1}} \overline{R}\Big),$$

where

$$I_{p,b}(y) := \int_0^y \frac{\mathrm{d}s}{\sqrt{bs^{p+1}+1}}, \quad y \geq 0.$$

Clearly, $\overline{\mathcal{F}}(\theta,\cdot)$ has different behaviour for $q \in (-\infty,0)$, $\{0\}$, $(0,\frac{p+1}{2})$, $\{\frac{p+1}{2}\}$, $(\frac{p+1}{2},\infty)$. In the rest of this section, we will deal only with the third case.

9.2 Lemma (for $q>1$ see [6, p. 115])
Let $p \geq 1$, $0<q<\frac{p+1}{2}$, $a,\theta>0$, and let us introduce

$$\Theta := \Theta_{p,q,a} := \sqrt{\frac{p+1-2q}{p+1}} \left(\frac{q}{a}\right)^{\frac{q}{p+1-2q}}.$$

If $\theta > \Theta$, then $\overline{\mathcal{F}}(\theta,\cdot)$ has no zero. If $\theta=\Theta$, then the only zero of $\overline{\mathcal{F}}(\theta,\cdot)$ is

$$\left(\frac{q}{a}\right)^{\frac{1}{p+1-2q}} =: \overline{R}_{p,q,a}(\Theta) =: \overline{R}(\Theta).$$

If $\theta<\Theta$, then $\overline{\mathcal{F}}(\theta,\cdot)$ has two zeros, which will be denoted by $\overline{R}_{i;p,q,a}(\theta) =: \overline{R}_i(\theta)$, $i=1,2$, being

$$\overline{R}_1(\theta) < \overline{R}(\Theta) < \overline{R}_2(\theta).$$

9.3 Definiton
Let $p\geq 1$, $0<q<\frac{p+1}{2}$, $a>0$, and put

$$\overline{L}_i(\theta) := \overline{L}_{i;p,q,a}(\theta) := \theta^{-\frac{p-1}{p+1}} I_{p,b}\Big(\theta^{-\frac{2}{p+1}} \overline{R}_{p,q,a}(\theta)\Big)$$

for $i=1,2$ and $\theta \in (0,\Theta)$. We introduce $\overline{L}_{p,q,a}(\Theta) =: \overline{L}(\Theta)$ analogously. Functions $\overline{L}$, $\overline{L}_1$ and $\overline{L}_2$ will be called **time maps** (associated with (II.1)).

Using Lemmata 9.1 and 9.2, we can describe $\mathcal{N}^{\pm}(l)$ by means of the time maps:

9.4 Lemma
For all $p\geq 1$, $q \in (0,\frac{p+1}{2})$ and $a,l>0$:

$$\mathcal{N}^{\pm}(l) = \left\{ \pm \overline{u}_{\theta,p,a}\left(\cdot \pm \frac{\overline{L}_2(\theta)-\overline{L}_1(\theta)}{2}\right)\bigg|_{[-l,l]} \;:\; \overline{L}_1(\theta)+\overline{L}_2(\theta)=2l \right\},$$

where the two $\pm$ symbols on the right-hand side are independent (i. e. there are four sign-changing non-antisymmetric solutions corresponding to any $\theta>0$ satisfying $\overline{L}_1(\theta)+\overline{L}_2(\theta)=2l$).

We need to know the limits of $\overline{L}_1+\overline{L}_2$ at 0 and Θ, and whether $\overline{L}_1+\overline{L}_2$ is monotone. Therefore, we now cite the following two lemmata and afterwards state the new results.

9.5 Lemma (for $q>1$ see [6, Lemma 5.2])
If $p\geq 1$, $0<q<\frac{p+1}{2}$ and $a>0$, then

$$\lim_{\theta\to\Theta} \overline{L}_i(\theta) = \overline{L}(\Theta), \qquad i=1,2,$$

$$\lim_{\theta\to 0} \overline{L}_2(\theta) = \infty.$$

Let $p \geq 1$, $1 < q < \frac{p+1}{2}$ and $a > 0$. According to [6, Theorem 1.3 (iii)], if

$$(p-q)(2q+1-p)(p+1) \geq 2q(p-1)$$

or equivalently,

$$q > \frac{p(p-1)}{p+1}, \tag{II.2}$$

then $(\overline{L}_1 + \overline{L}_2)' < 0$. However, we prove this property in Lemma 9.8 without assuming (II.2), including also some $q \leq 1$.

9.6 Lemma (for $q > 1$ see [6, proof of Lemma 5.1])
If $p \geq 1$, $0 < q < \frac{p+1}{2}$, $a > 0$, $i \in \{1, 2\}$, then $\overline{L}_i$ is differentiable on $(0, \Theta)$, fulfilling

$$\overline{L}_i'(\theta) = -\frac{p-1}{(p+1)\theta}\overline{L}_i(\theta) + \frac{p+1-2q}{(p+1)q\theta}\frac{\overline{R}_i^{1-q}(\theta)}{1 - \frac{a}{q}\overline{R}_i^{p+1-2q}(\theta)}.$$

9.7 Lemma
If $p \geq 1$, $0 < q < \frac{p+1}{2}$ and $a > 0$, then $\overline{R}_1\overline{R}_2 < \overline{R}^2(\Theta)$.

Proof: It is much the same as the proof of Lemma 8.1. So let $p \geq 1$, $0 < q < \frac{p+1}{2}$, $a > 0$, $\theta \in (0, \Theta)$, and set $\alpha := \overline{R}_2(\theta)/\overline{R}(\Theta) > 1$. Using the increase of $\overline{\mathcal{F}}(\theta, \cdot)$ on $(0, \overline{R}(\Theta))$ and the definition of $\overline{R}_1(\theta)$ and $\overline{R}_2(\theta)$, one can see that it suffices to prove that

$$0 > \overline{\mathcal{F}}\big(\theta, \alpha\overline{R}(\Theta)\big) - \overline{\mathcal{F}}\left(\theta, \frac{\overline{R}(\Theta)}{\alpha}\right) = 2q\overline{R}^{2q}(\Theta)\big(F_\alpha(2q) - F_\alpha(p+1)\big)$$

(see (I.39) for the definition of F_α), which is a true inequality due to the increase of F_α. □

9.8 Lemma
If $a > 0$ and either $p = 1$, $q \in (0, 1)$ or $p > 1$, $q \in [\frac{1}{2}, \frac{p+1}{2})$, then $(\overline{L}_1 + \overline{L}_2)' < 0$.

Proof: Consider $p \geq 1$, $0 < q < \frac{p+1}{2}$, $a > 0$, $\theta \in (0, \Theta)$, and put $b := \frac{2a}{p+1}$. We will proceed similarly to the proof of Lemma 8.2.

1. We start with the estimate suggested in [6, Remark 5.3]:

$$I_{p,b}(y) > \frac{y}{\sqrt{by^{p+1} - 1}}, \qquad y > 0,$$

which results in

$$\overline{L}_i(\theta) > \overline{R}_i^{1-q}(\theta), \qquad i = 1, 2.$$

Applying this inequality to the formula included in Lemma 9.6, one can derive a sufficient condition for $(\overline{L}_1 + \overline{L}_2)'(\theta) < 0$ in the form of

$$\overline{F}\left(\frac{\overline{R}_1(\theta)}{\overline{R}(\Theta)}\right) + \overline{F}\left(\frac{\overline{R}_2(\theta)}{\overline{R}(\Theta)}\right) < 0, \tag{II.3}$$

where

$$\overline{F}(x) := \overline{F}_{p,q}(x) := (1-p)qx^{1-q} + \frac{(p+1-2q)x^{1-q}}{1-x^{p+1-2q}}, \qquad x \in (0,1)\cup(1,\infty).$$

2. Now we prove the increase of $\overline{F}$ on $(0,1)$. Setting

$$\alpha := p-1 \geq 0, \qquad \beta := 2(q-1) \in (-2,\alpha),$$

we obtain that

$$\overline{F}(x) = -\alpha\left(\frac{\beta}{2}+1\right)x^{-\beta/2} + \frac{(\alpha-\beta)x^{-\beta/2}}{1-x^{\alpha-\beta}} = F(x) - \frac{\alpha\beta}{2}x^{-\beta/2}.$$

Since F increases on $(0,1)$ due to step *2.* of the proof of Lemma 8.2, $\overline{F}$ increases on $(0,1)$ as well.

3. Using the same ideas as in step *3.* of the proof of Lemma 8.2, we can see that it suffices to verify the inequality

$$\overline{u}_{\alpha,\beta}(x) := \beta(\alpha+2)x^{\alpha} + \alpha(\beta+2)x^{\alpha-\beta} - \alpha(\beta+2)x^{\beta} - \beta(\alpha+2) \geq 0$$

for all $x > 1$ and α, β fulfilling either $\alpha = 0$, $\beta \in (-2,0)$ or $\alpha > 0$, $\beta \in [-1,\alpha)$. The first case is clear. In the second one we have that $\overline{u}_{\alpha,\beta}(1) = 0$ and

$$\overline{u}'_{\alpha,\beta}(x) = \underbrace{\alpha x^{\beta-1}}_{>0}\underbrace{\big(\beta(\alpha+2)x^{\alpha-\beta} + (\alpha-\beta)(\beta+2)x^{\alpha-2\beta} - \beta(\beta+2)\big)}_{=:\overline{v}_{\alpha,\beta}(x)},$$

so the verification ot the non-negativity of $\overline{v}_{\alpha,\beta}$ on $(1,\infty)$ will finish the proof. And since $\overline{v}_{\alpha,\beta}(1) = 2(\alpha-\beta)(\beta+1) \geq 0$ and

$$\overline{v}'_{\alpha,\beta}(x) = \underbrace{(\alpha-\beta)x^{\alpha-2\beta-1}}_{>0}\underbrace{\big(\beta(\alpha+2)x^{\beta} + (\beta+2)(\alpha-2\beta)\big)}_{=:\overline{w}_{\alpha,\beta}(x)},$$

we just need to observe that $\overline{w}_{\alpha,\beta} \geq 0$ on $(1,\infty)$ because $\overline{w}_{\alpha,\beta}(1) = 2(\alpha-\beta)(\beta+1) \geq 0$ and $\overline{w}_{\alpha,\beta}$ is non-decreasing. $\square$

9.9 Remark

The proof of Lemma 9.8 does not work for $p > 1$, $q \in (0,\frac{1}{2})$, $a > 0$, i. e. for $\alpha > 0$, $\beta \in (-2,-1)$, because in that case we have $\overline{u}'_{\alpha,\beta}(1) = \alpha\overline{v}_{\alpha,\beta}(1) < 0$, implying that $\overline{u}_{\alpha,\beta} < 0$ in the right neighbourhood of 1. In addition, numerical calculations suggest that if $p > 1$ is big enough and $q \in (0,\frac{1}{2})$ is small enough, then $\overline{L}_1 + \overline{L}_2$ has a stationary point where a minimum is attained.

Joining the results of Lemmata 9.4, 9.5 and 9.8, we immediately obtain the following assertion:

9.10 Theorem

If $a, l > 0$ and either $p = 1$, $q \in (0,1)$ or $p > 1$, $q \in [\frac{1}{2},\frac{p+1}{2})$, then

$$\left|\mathcal{N}^{\pm}(l)\right| = \begin{cases} 4 & \text{if } l > \overline{L}(\Theta), \\ 0 & \text{if } l \leq \overline{L}(\Theta). \end{cases}$$

(See Lemma 9.2 and Definition 9.3 concerning $\overline{L}(\Theta)$.)

10 Explicit solution of the Cauchy problem for $u'' = au^{-1/2}$ with $a > 0$

The subject of this section is the initial value problem

$$\begin{cases} u'' = au^{-1/2}, \\ u(0) = \alpha, \\ u'(0) = \beta, \end{cases} \tag{II.4}$$

which can be solved explicitely for any $a, \alpha > 0$ and $\beta \in \mathbb{R}$.

10.1 Theorem

Let $a, \alpha > 0$, $\beta \in \mathbb{R}$, and set $\gamma := \beta^2 - 4a\sqrt{\alpha}$. Problem (II.4) *possesses a unique maximal solution, which will be denoted by u, and which is given by the following formulae:*

(i) *If $\gamma < 0$, then*

$$u(x) = \Big(\frac{\gamma}{4a}\Big)^2 \check{I}\Big(\frac{6a^2}{|\gamma|^{3/2}}|x - x_0|\Big), \qquad x \in \mathbb{R},$$

where

$$\check{I}(w) = \Bigg(\Bigg(\sqrt[3]{w + \sqrt{w^2+1}} + \sqrt[3]{w - \sqrt{w^2+1}}\Bigg)^2 + 1\Bigg)^2, \qquad w \geq 0$$

and

$$x_0 = -\frac{\beta(6a\sqrt{\alpha} - \beta^2)}{6a^2}. \tag{II.5}$$

Furthermore, $\operatorname{sgn} x_0 = -\operatorname{sgn}\beta$*, the graph of u is symmetric with respect to $x = x_0$, and* $\min u = u(x_0) = (\frac{\gamma}{4a})^2$.

(ii) *If $\gamma = 0$, then setting*

$$d := \frac{2\alpha^{3/4}}{3\sqrt{a}} > 0,$$

we have

$$u(x) = \alpha\Big(\operatorname{sgn}\beta \cdot \frac{x}{d} + 1\Big)^{4/3}, \qquad x \in \mathcal{D}(u),$$

where

- *if $\beta > 0$, then $\mathcal{D}(u) = (-d, \infty)$, $u' > 0$, $\lim_{x\to -d} u(x) = 0$ and $\lim_{x\to -d} u'(x) = 0$,*
- *and if $\beta < 0$, then $\mathcal{D}(u) = (-\infty, d)$, $u' < 0$, $\lim_{x\to d} u(x) = 0$ and $\lim_{x\to d} u'(x) = 0$.*

(iii) *If $\gamma > 0$, then setting*

$$d := \frac{\gamma^{3/2} + |\beta|(6a\sqrt{\alpha} - \beta^2)}{6a^2} > 0,$$

we have

$$u(x) = \Big(\frac{\gamma}{4a}\Big)^2 \check{J}\Big(\frac{6a^2}{\gamma^{3/2}}\big(\operatorname{sgn}\beta \cdot x + d\big) - 1\Big), \qquad x \in \mathcal{D}(u),$$

where

$$\check{J}(w) = \begin{cases} \Bigg(\Big(\sqrt[3]{w+\sqrt{w^2-1}} + \sqrt[3]{w-\sqrt{w^2-1}}\Big)^2 + 1\Bigg)^2 & \textit{if } w \geq 1, \\ \Big(4\cos^2\dfrac{\arccos w}{3} - 1\Big)^2 & \textit{if } w \in [-1,1], \end{cases}$$

and

- *if* $\beta > 0$, *then* $\mathcal{D}(u) = (-d, \infty)$, $u' > 0$, $\lim_{x\to -d} u(x) = 0$ *and* $\lim_{x\to -d} u'(x) = \sqrt{\gamma}$,
- *and if* $\beta < 0$, *then* $\mathcal{D}(u) = (-\infty, d)$, $u' < 0$, $\lim_{x\to d} u(x) = 0$ *and* $\lim_{x\to d} u'(x) = -\sqrt{\gamma}$.

Proof: Choose $a, \alpha > 0$ and $\beta \in \mathbb{R}$. Since $u \mapsto au^{-1/2}$ is locally Lipschitz continuous on $(0,\infty)$, (II.4) has a unique maximal solution denoted by u, and its domain $\mathcal{D}(u)$ is an open interval. We will proceed as follows:

1. If u has a stationary point, we will express u by means of $u_{m,-1/2,a}$ for some $m > 0$ using the formulae from Theorem 1.1 and (I.18).
2. If u is strictly monotone, we will derive formulae for (II.4) analogous (and in analogous way) to that for (I.1).

Step *1.* will lead to assertion (i), while step *2.* to assertions (ii) and (iii).

1. Suppose that $x_0 \in \mathbb{R}$ is a stationary point of u. (We will see soon for which values of a, α, β this occurs.) Set $m := u(x_0) > 0$ and $v := u_{m,-1/2,a}$. Then clearly, $\operatorname{sgn} x_0 = -\operatorname{sgn}\beta$, $u(x) = v(x - x_0)$ for $x \in \mathbb{R}$, $\alpha = v(-x_0)$, $\beta = v'(-x_0)$, the graph of u is symmetric with respect to $x = x_0$, and $\min u = m$.

Our goal is to ascertain under which conditions on a, α and β, u possesses a stationary point, to express x_0 and m by means of a, α and β if that condition is met, and to derive an explicit formula for v.

Inserting $p = -\frac{1}{2}$ and $x = x_0$ in (I.4), one obtains

$$|\beta| = 2\sqrt{a(\sqrt{\alpha} - \sqrt{m})},$$

which is equivalent to

$$\sqrt{m} = -\frac{\gamma}{4a}.$$

So we have to require $\gamma < 0$, and afterwards we have

$$m = \Big(\frac{\gamma}{4a}\Big)^2. \tag{II.6}$$

Similarly, (I.2) yields

$$-\operatorname{sgn}\beta \cdot x_0 = |x_0| = \frac{m^{3/4}}{\sqrt{2a}} I_{-1/2}\Big(\frac{\alpha}{m}\Big), \tag{II.7}$$

from which, by means of (I.18), (II.5) follows.

As we have just seen, $\gamma < 0$ is a necessary condition for the existence of a stationary point of u. However, it is a sufficient condition as well: If we have $\gamma < 0$ and define m and x_0 as in (II.6) and (II.7), then comparing (II.7) with (I.2) for $p = -\frac{1}{2}$ and $x = -x_0$, and inserting $p = -\frac{1}{2}$ and $x = -x_0$ in (I.4), we can see that $u_{m,-1/2,a}(\cdot - x_0)$ solves (II.4) and consequently, it is identical with u, which therefore indeed has a stationary point.

Since $I_{-1/2}$ is a bijection of $[1, \infty)$ onto $[0, \infty)$ (see (I.3)), one can rewrite (I.2) as

$$u(x) = v(x - x_0) = \left(\frac{\gamma}{4a}\right)^2 I_{-1/2}^{-1}\left(\frac{8\sqrt{2}a^2}{|\gamma|^{3/2}}|x - x_0|\right), \qquad x \in \mathbb{R},$$

so it sufficies to prove that

$$I_{-1/2}^{-1}(z) = \check{I}\left(\frac{3z}{4\sqrt{2}}\right), \qquad z \geq 0. \tag{II.8}$$

By means of (I.18), $I_{-1/2}$ can be expressed as

$$\begin{aligned} I_{-1/2}(y) &= \frac{2\sqrt{2}}{3}\widetilde{I}\big(\sqrt{y} - 1\big), \qquad y \geq 1, \\ \widetilde{I}(Y) &= \sqrt{Y}(Y + 3), \qquad Y \geq 0. \end{aligned}$$

(Compare with (I.14).) Using Cardano's formula, we obtain that

$$\begin{aligned} I_{-1/2}^{-1}(z) &= \left(\widetilde{I}^{-1}\left(\frac{3z}{2\sqrt{2}}\right) + 1\right)^2, \qquad z \geq 0, \\ \widetilde{I}^{-1}(Z) &= \left(\sqrt[3]{\frac{Z}{2} + \sqrt{\left(\frac{Z}{2}\right)^2 + 1}} + \sqrt[3]{\frac{Z}{2} - \sqrt{\left(\frac{Z}{2}\right)^2 + 1}}\right)^2, \qquad Z \geq 0, \end{aligned}$$

so (II.8) follows, and the proof of (i) is complete.

2. Now let $\gamma \geq 0$. (Consequently, u is monotone.) Since u fulfils the equation in (II.4), we have

$$0 = u''(y)u'(y) - au^{-1/2}(y)u'(y) = \Bigg(\underbrace{\frac{(u'(y))^2}{2} - 2a\sqrt{u(y)}}_{=:\chi(y)}\Bigg)', \qquad y \in \mathcal{D}(u)$$

($\mathcal{D}(u)$ will be specified later). Apparently, χ is a constant function with the function value $\chi(0) = \frac{\gamma}{2}$. Thus, it is easy to see that

$$(u'(y))^2 = 4a\sqrt{u(y)} + \gamma, \qquad y \in \mathcal{D}(u). \tag{II.9}$$

The monotonicity of u ensures that $\operatorname{sgn} u'(y) = \operatorname{sgn} \beta$ for any $y \in \mathcal{D}(u)$. Therefore,

$$\frac{u'(y)}{\sqrt{4a\sqrt{u(y)} + \gamma}} = \operatorname{sgn} \beta, \qquad y \in \mathcal{D}(u).$$

Now choose $x \in \mathcal{D}(u)$, and integrate the last equality on $[0, x]$ with respect to y, using the substitution $u(y) =: v$:

$$\operatorname{sgn}\beta \cdot x = \int_\alpha^{u(x)} \frac{\mathrm{d}v}{\sqrt{4a\sqrt{v} + \gamma}}. \tag{II.10}$$

(a) If $\gamma = 0$ (case (ii)), then the integral in (II.10) can be easily calculated, yielding

$$\operatorname{sgn}\beta \cdot x = \frac{2}{3\sqrt{a}} u^{3/4}(x) - d, \qquad x \in \mathcal{D}(u).$$

Necessarily, $\operatorname{sgn}\beta \cdot x > -d$. One can also see that u is indeed given as in the lemma, and $\operatorname{sgn}\beta \cdot x > -d$ is a sufficient condition for $x \in \mathcal{D}(u)$. The limits of $u(x)$ and $u'(x)$ for $x \to -\operatorname{sgn}\beta \cdot d$ are clear as well (recall (II.9)).

(b) Now suppose $\gamma > 0$, which corresponds to case (iii). Using the substitution

$$\frac{4a\sqrt{v}}{\gamma} + 1 =: V$$

in (II.10), we obtain that

$$\operatorname{sgn}\beta \cdot x = \frac{\gamma^{3/2}}{12a^2}\left(\widetilde{J}\left(\frac{4a\sqrt{u(x)}}{\gamma} + 1\right) - \widetilde{J}\left(\frac{\beta^2}{\gamma}\right)\right), \qquad x \in \mathcal{D}(u),$$

where

$$\widetilde{J}(Y) = \frac{3}{2}\int_0^Y \frac{V-1}{\sqrt{V}}\,\mathrm{d}V = \sqrt{Y}(Y-3), \qquad Y \geq 1.$$

Consequently,

$$\operatorname{sgn}\beta \cdot x = \frac{\gamma^{3/2}}{12a^2}\left(\widetilde{J}\left(\frac{4a\sqrt{u(x)}}{\gamma} + 1\right) + 2\right) - d, \qquad x \in \mathcal{D}(u),$$

and since $\widetilde{J}$ is a bijection of $(1, \infty)$ onto $(-2, \infty)$, the necessity of $\operatorname{sgn}\beta \cdot x > -d$ and the validity of

$$u(x) = \left(\frac{\gamma}{4a}\right)^2 \left(\widetilde{J}^{-1}\left(\frac{12a^2}{\gamma^{3/2}}(\operatorname{sgn}\beta \cdot x + d) - 2\right) - 1\right)^2, \qquad x \in \mathcal{D}(u)$$

follow. On the other hand, $\operatorname{sgn}\beta \cdot x > -d$ is obviously also a sufficient condition for $x \in \mathcal{D}(u)$, and the limits of u and u' at $-\operatorname{sgn}\beta \cdot d$ are clear as well (see (II.9)).

It remains to find the inverse of $\widetilde{J}$. Apparently, $\widetilde{J}(Y) = Z$ is equivalent to

$$\left(\sqrt{Y}\right)^3 - 3\sqrt{Y} - Z = 0.$$

Treated as a cubic equation in $\sqrt{Y}$, it has a unique real root

$$\sqrt{Y} = \sqrt[3]{\frac{Z}{2} + \sqrt{\left(\frac{Z}{2}\right)^2 - 1}} + \sqrt[3]{\frac{Z}{2} - \sqrt{\left(\frac{Z}{2}\right)^2 - 1}} \quad \big(> 0\big)$$

for $Z > 2$, and a unique positive real root

$$\sqrt{Y} = 2\cos\frac{\arccos\frac{Z}{2}}{3}$$

for $Z \in [-2, 2]$. Consequently,

$$\widetilde{J}^{-1}(Z) = \sqrt{\check{J}\left(\frac{Z}{2}\right)} + 1, \qquad Z \geq -2,$$

which completes the proof of (iii). $\square$

Afterword

In this thesis we got familiar with the shooting method, which made it possible to simplify the question of the solvability of (1) to the question of the properties of the time maps, which are real functions of one real variable. Examining their properties, we were able to determine the number of positive symmetric solutions of (1) for $p > -1$, $q \geq 0$ and $p = -1$, $q = 0$ (see Theorems 3.1, 4.4, 5.6, 6.1 and 7.9), the number of its positive non-symmetric solutions for $p \geq 1$, $q > \frac{p+1}{2}$ (see Theorem 7.9) with some partial results for $p \in (-1, 1)$, $q > \frac{p+1}{2}$ (see Lemmata 8.6 and 8.9), and the number of its sign-changing non-antisymmetric solutions for $p = 1$, $q \in (0, 1)$ and $p > 1$, $q \in [\frac{1}{2}, \frac{p+1}{2})$ (see Theorem 9.10), while the number of its sign-changing antisymmetric solutions for $p \geq 1$, $q > 1$ is known from [6]. Let us also mention Theorem 10.1, which gives an explicit formula for the solution of the Cauchy problem for $u'' = au^{-1/2}$ assuming $a > 0$.

The predominant majority of the results mentioned above are new results achieved by the author. Theorems 8.4 and 9.10 provide the answers for two long-standing open questions arising in [5] and [6], while the other statements deal with values of parameters not considered before. The contents of Sections 1–7 with the exception of Theorem 2.7 were published in [14], while the results of Section 8 until Theorem 8.4 together with Section 9 have been submitted for publication.

The given topic has not been exhausted by this thesis at all. There remains to verify analytically the numerically predicted properties of q^* (see the paragraph below the proof of Lemma 7.8), $\widehat{q}$ (see the paragraph below the proof of Lemma 8.10) and $\overline{q}$ (see the paragraph below the proof of Lemma 8.11), as well as to determine the sign of $L_2(0) - L_2(m_2)$ in case V for $p < -\frac{1}{2}$, $q \in (q^*(p), -p)$ in dependence on p, q (see the second paragraph below the proof of Lemma 7.8), and to investigate the so far unknown properties of $L_1 + L_2$ in case V for $p < 1$ (see the second paragraph below the proof of Lemma 8.11). And naturally, a further goal can be to determine the number of positive solutions of (1) in cases VI–XIII, the number of its sign-changing antisymmetric solutions for $p \geq 1$, $q \geq 1$, and the number of its sign-changing non-antisymmetric solutions for $p > 0$, $q \in (0, \frac{1}{2})$. Moreover, one could also study the sign-changing solutions of (1) for $p \in (0, 1)$, $q \in \mathbb{R}$.

Throughout this whole thesis, we could get by only using the knowledge of real analysis (except for the use of Picard's existence theorem), but in spite of this, this topic cannot be called too simple or uninteresting. On the contrary, the author consideres it especially nice and hopes that the reader has acquired a similar impression.

Resumé

V predloženej dizertačnej práci sme skúmali riešiteľnosť okrajovej úlohy

$$\begin{cases} u''(x) = a|u(x)|^{p-1}u(x), & x \in (-l, l), \\ u'(\pm l) = \pm|u(\pm l)|^{q-1}u(\pm l), \end{cases}$$

kde $a, l > 0$.

Najväčšia časť práce bola venovaná štúdiu existencie a počtu jej kladných riešení, čiže kladných riešení úlohy

$$\begin{cases} u''(x) = au^p(x), & x \in (-l, l), \\ u'(\pm l) = \pm u^q(\pm l), \end{cases}$$

kde možno uvažovať ľubovoľné $p, q \in \mathbb{R}$. Daná úloha bola prvýkrát systematicky študovaná v článku [5], avšak iba pre $p, q > 1$. Naším cieľom bolo rozšíriť jeho výsledky pre čo najväčšiu množinu parametrov p, q.

Nástrojom k tomu – ako aj nástrojom citovaného článku – bola metóda streľby, ktorú možno zhrnúť nasledovne: Zrejme každé kladné riešenie spomenutej okrajovej úlohy pre zadané $l > 0$ sa dá získať z riešení tej istej diferenciálnej rovnice uvažovanej spolu so začiatočnými podmienkami $u(0) = m$ a $u'(0) = 0$ pre vhodné $m > 0$. Táto začiatočná úloha má jediné riešenie pre ľubovoľné $m > 0$. Vzťah medzi m a l je daný istými funkciami nazývanými zobrazenia dostrelu, pre ktoré možno odvodiť aj vzorec. Ten ukazuje nutnosť štúdia kladných symetrických (t. j. párnych) a kladných nesymetrických riešení osobitne. Navyše treba rozlíšiť trinásť prípadov – číslovaných I až XIII – ohľadom hodnôt parametrov p a q, pričom kladné nesymetrické riešenia existujú iba v prípapdoch V–VII.

V tejto práci sa nám podarilo určiť počet kladných symetrických riešení v prvých piatich prípadoch, spolu zahŕňajúcich $p > -1$, $q \geq 0$ a $p = -1$, $q = 0$, pričom sme si mohli všimnúť, že vlastnosti zobrazení dostrelu sú omnoho rozmanitejšie a ťažšie vyšetriteľné mimo množiny $p, q > 1$. Otázka počtu kladných nesymetrických riešení pre $p, q > 1$ bola v [5] zodpovedaná iba čiastočne, my sme však na ňu našli úplnú odpoveď a získali aj čiastočné výsledky pre tú časť prípadu V, ktorá nebola súčasťou [5].

Tiež sme sa zaoberali riešeniami uvažovanej okrajovej úlohy meniacimi znamienko, ktoré boli prvýkrát systematicky študované v [6] pre $p \geq 1$, $q > 1$. Pritom sme použili metódu streľby s tou zmenou, že začiatočné podmienky boli $u(0) = 0$ a $u'(0) = \theta$ pre vhodné $\theta \in \mathbb{R}$. Počet antisymetrických (t. j. nepárnych) riešení meniacich znamienko bol v [6] určený pre všetky $p \geq 1$, $q > 1$, avšak počet neantisymetrických riešení meniacich znamienko iba pre istú časť uvažovaných hodnôt parametrov. My sme vyšetrili aj zvyšnú časť spolu s niektorými doteraz

neuvažovanými hodnotami parametrov, konkrétne s $p = 1$, $q \in (0,1)$ a $p > 1$, $q \in [\frac{1}{2}, 1]$.

Na tomto mieste ešte spomenieme vedľajší výsledok práce v podobe explicitného vzorca pre riešenie Cauchyho úlohy pre $u'' = au^{-1/2}$, kde $a > 0$.

Vďaka metóde streľby sme mohli previesť otázku riešiteľnosti študovanej okrajovej úlohy na otázku priebehu zobrazení dostrelu, čiže reálnych funkcií jednej reálnej premennej, a teda sme ďalej už vystačili s prostriedkami reálnej analýzy. Vyšetriť ich priebeh však nebolo až také ľahké, lebo sú dané vzorcom obsahujúcim nevlastný parametrický integrál, ktorý sa dá vypočítať iba pre niektoré špeciálne hodnoty parametra p, kým jeho horná hranica je daná implicitne.

List of symbols

Here the reader can find all the symbols used at least twice in this thesis except their occurrences in proofs. References for their definitions are provided as well. Some symbols (namely, $L(0)$, $\overline{m}$ and m_0) have been introduced repeatedly for different values of their parameters, but all their definitions correspond with each other.

Positive solutions of (1)**:**

$\overline{a}$	Lemma 4.5
$b_k(p)$	Lemma 2.4
B_p	Lemma 2.4
$\mathcal{F}(m,x) = \mathcal{F}_{p,q,a}(m,x)$	Lemma 1.4
$I_p(y)$	Theorem 1.1
$J_p(y)$	Theorem 2.7
$K(m) = K_{p,q,a}(m)$	Definition 1.10
$K_1(m) = K_{1;p,q,a}(m)$	Definition 1.10
$K_2(m) = K_{2;p,q,a}(m)$	Definition 1.10
$L(m) = L_{p,q,a}(m)$	Definition 1.7
$L(0) = L_{p,q,a}(0)$	Lemma 4.2 (case II)
	Lemma 5.3 (case III)
$L_1(m) = L_{1;p,q,a}(m)$	Definition 1.7
$L_2(m) = L_{2;p,q,a}(m)$	Definition 1.7
$L_2(0) = L_{2;p,q,a}(0)$	Lemma 7.4 (case V)
$\Lambda_{m,p,a}$	Theorem 1.1
$\overline{m} = \overline{m}_{p,q,a}$	Lemma 5.5 (case III)
	Definition 7.2 (case V)
$m_0 = m_{0;p,q,a}$	Lemma 5.5 (case III)
	Lemma 7.3 (case V)
	Lemma 7.6 (case V)
$m_1 = m_{1;p,q,a}$	Lemma 7.7 (case V)
$m_2 = m_{2;p,q,a}$	Lemma 7.7 (case V)
$M = M_{p,q,a}$	Lemma 1.5
$\mathcal{N}(l) = \mathcal{N}(l;p,q,a)$	Definition 1.2
$\overline{p}$	Lemma 4.5 (case II)

p_s	Definition 2.2
$q^*(p)$	Lemma 7.7 (case V)
$q^*(-1)$	Lemma 7.8 (case V)
$\widehat{q}(p)$	Lemma 8.9 (case V)
$\widehat{q}(-1)$	Lemma 8.10 (case V)
$\overline{q}(p)$	Lemma 8.9 (case V)
$r_{q,a}$	Section 6 (case IV)
$R(m) = R_{p,q,a}(m)$	Lemma 1.5
$R_1(m) = R_{1;p,q,a}(m)$	Lemma 1.5
$R_2(m) = R_{2;p,q,a}(m)$	Lemma 1.5
$R_2(0) = R_{2;p,q,a}(0)$	Lemma 7.1 (case V)
$\mathcal{S}(l) = \mathcal{S}(l; p, q, a)$	Definition 1.2
$u_{m,p,a}$	Theorem 1.1

Sign-changing solutions of (1)**:**

$\overline{\mathcal{F}}(\theta, x) = \overline{\mathcal{F}}_{p,q,a}(\theta, x)$	Lemma 9.1
$I_{p,b}(y)$	Lemma 9.1
$\overline{L}(\Theta) = \overline{L}_{p,q,a}(\Theta)$	Definition 9.3
$\overline{L}_1(\theta) = \overline{L}_{1;p,q,a}(\theta)$	Definition 9.3
$\overline{L}_2(\theta) = \overline{L}_{2;p,q,a}(\theta)$	Definition 9.3
$\overline{\Lambda}_{\theta,p,a}$	beginning of Section 9
$\mathcal{N}^{\pm}(l) = \mathcal{N}^{\pm}(l; p, q, a)$	beginning of Section 9
$\overline{R}(\Theta) = \overline{R}_{p,q,a}(\Theta)$	Lemma 9.2
$\overline{R}_1(\theta) = \overline{R}_{1;p,q,a}(\theta)$	Lemma 9.2
$\overline{R}_2(\theta) = \overline{R}_{2;p,q,a}(\theta)$	Lemma 9.2
$\Theta = \Theta_{p,q,a}$	Lemma 9.2
$\overline{u}_{\theta,p,a}$	beginning of Section 9

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