COMENIUS UNIVERSITY BRATISLAVA

FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Mathematical Analysis and Calibration of a Multifactor Panel Model for Credit Spreads and Risk-free Interest Rate

Dissertation Thesis

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Dissertation Thesis in Applied Mathematics

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Mathematical Analysis and Calibration of a Multifactor Panel Model for Credit Spreads and Risk-free Interest Rate

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Supervisor: **prof. RNDr. Daniel Ševčovič, CSc.**

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Univerzita Komenského v Bratislave
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ZADANIE ZÁVEREČNEJ PRÁCE

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Študijný program: aplikovaná matematika (Jednoodborové štúdium, doktorandské III. st., externá forma)
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Typ záverečnej práce: dizertačná
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Názov: Mathematical Analysis and Calibration of a Multifactor Panel Model for Credit Spreads and Risk-free Interest Rate

Cieľ: In this thesis we develop a new multifactor panel model for the credit spreads and the risk-free interest rate. The interest rate for every country is modeled by a two-factor Cox-Ingersoll-Ross process. One of the factors, the risk-free rate, beeing common for all countries. The secon factor, credit spread, is individual for every country. The model will be calibrated using the sovereign yield curves of the euro area countries from for the whole existence of the euro area and the results will be discussed.

Školiteľ: prof. RNDr. Daniel Ševčovič, CSc.
Katedra: FMFI.KAMŠ - Katedra aplikovanej matematiky a štatistiky

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Cieľ: V dizertačnej práci navrhujeme nový viacfaktorový panelový model pre kreditné spredy a bezrizikovú úrokovú mieru. Úroková miera pre každú krajinu je modelovaná dvojfaktorovým Cox, Ingersoll, Ross procesom. Jeden faktor, bezriziková úroková miera je spoločná pre všetky krajiny, kým druhý faktor, kreditný spred, je individuálny pre každú krajinu. Model bude kalibrovaný na výnosových krivkách krajín eurozóny počas celého obdobia jej vzniku a diskutujeme výsledky

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I would like to thank my supervisor Daniel Ševčovič for his inspiring ideas, valuable guidance in my research and overall support. I would also like to thank to my wife Anna for her patience, support and all the efforts made. Without her, I would never be able to finish my research. My thanks belong also to my children Ľuboš and Katarína for their understanding of the time I was not able to devote to them. Finally I would like to thank to my fellow colleagues at the National Bank of Slovakia for the inspiring and supporting environment they provide.

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Abstract

In this thesis a new panel model is developed for the sovereign bond yields in the euro area. We assume that the short-rate of a particular euro area country is a sum of two unobserved underlying processes: the risk-free rate and a credit spread. The risk-free rate is common for all euro area countries, while the credit spread is idiosyncratic for every country. Both the risk-free rate and the credit spreads are modeled by the well-known one-factor Cox, Ingersoll, Ross or Chan, Karolyi, Longstaff, Sanders process. The model is calibrated using daily data on yield curves for the whole existence of the euro area until February 3, 2012 and the results are discussed.

Key words: panel model, multifactor model, credit spread, risk-free rate

Abstrakt

V dizertačnej práci navrhujeme nový panelový model pre výnosové krivky štátov eurozóny. Predpokladáme, že okamžitá úroková miera je súčtom dvoch nepozorovateľných procesov: bezrizikovej sadzby a kreditného spredu. Bezriziková sadzba je spoločná pre všetky krajiny eurozóny, kým kreditný spred je samostatný pre každú krajinu. Aj bezriziková úroková miera aj kreditné spredy sú modelované všeobecne známym Cox, Ingersoll, Ross, alebo Chan, Karolyi, Longstaff, Sanders procesom. Model je kalibrovaný na denných dátach výnosových kriviek jednotlivých krajín eurozóny od začiatku jej existencie až do 3. februára 2012 a analyzujeme a zhrnieme výsledky.

Kľúčové slová: panelový model, viacfaktorový model, kreditný spred, bezriziková sadzba

Preface

My first encounter with the concept of sovereign credit risk was during my master studies at the Comenius University in Bratislava. I had the opportunity to work on my master's thesis at the National Bank of Slovakia on the topic of managing credit risk of the investment portfolio of the National Bank of Slovakia. Although I joined another department at the National Bank in 2006, the sovereign credit risk remained in my interest.

Usually the sovereign debt of the developed countries is considered to be a credit-risk-free investment. This is supported also by the global banking capital standard The Basel Capital Accord, developed by the Basel Committee on Banking Supervision. In all its versions, Basel I, Basel II and even in Basel III developed in 2010-2011, the global standard does not require banks to hold any capital for their investment in sovereign bonds with a credit rating of AA- or higher. However I learned that the yield of a sovereign debt portfolio might vary significantly in time based on the interest rate developments as well as the perceived riskiness of the issuing countries or market sentiment. Also history and even the recent events remind us that even sovereigns tend to default from time to time. This thesis is written in the middle of a severe financial and sovereign debt crisis. Greece defaulted in March this year despite coordinated support from the European Union and who knows, what is yet to happen.

It is impossible to find a real risk-free investment in the market. This makes the problem of separation of the interest rate risk from the sovereign credit risk a delicate matter. However the creation of the euro area provides a great opportunity for this task. Usually one of the low risk investments is considered to be the risk-free investment and only the differences in riskiness compared to this base investment are studied. In my work I would like to answer the fundamental question, how much of the yield is actually the risk-free part and how much can be attributed to the credit risk.

The euro area forms an environment where the difference between individual country yields should be mainly driven by different levels of credit risk, as circumstances should be equal. This was the major motivation to propose the model presented in this thesis. In our model we try to estimate the common risk-free rate and the individual credit spreads from the panel of euro area yield curves and we seek the most suitable formulation of the processes which govern the risk-free rate and credit spreads.

First we tried to formulate the model using the Cox, Ingersoll, Ross processes with no correlation between the risk-free rate and credit spreads in order to obtain analytical solutions to the yield curves. However this assumption proved to be nonrealistic for the data we used. Therefore we chose the general Chan, Karolyi, Longstaff, Sanders process with correlated risk-free rate and credit spreads in the final model estimation. To my best knowledge no such model has been studied so far.

Contents

Introduction

All investors in the financial market have their own risk profiles and try to find investment opportunities that best suit their risk appetite. Many investors are traditionally conservative, like insurance companies or pension funds, and seek investments with minimum credit risk. The reason is that they lack the expertise to assess credit risk properly. Also many other economic or financial models use the risk-free rate as a benchmark to discount future cash-flows. But what really can be considered as a risk-free investment?

Usually government bonds issued by the domestic government or by other high rated governments are considered to be credit-risk-free investments. Recently also bonds issued by high rated globally active financial or non-financial corporations are gaining that status. However events like the collapse of Lehman Brothers, an AAA rated investment bank, on September 15, 2008, the downgrade of the United States by Standard & Poor's on August 5, 2011 [111] , or the current sovereign crisis of the euro area with the downgrade of nine euro area sovereigns by Standard & Poor's on January 13, 2012 [110] including the AAA rated France and Austria and with the default of Greece on March 8, 2012 [79] remind us once again that credit risk is related with every investment involving another counterparty, no matter how negligible it might be viewed in advance.

This applies, of course, also to sovereign investments although investors tend to forget the lessons learned in the past as history is full of sovereign defaults or debt restructuring where investors lost a significant part of their investment. According to Reinhart, Rogof and Savastano [101] there were 33 external sovereign defaults[1] in Europe in nineteenth century and 35 defaults among the Middle Income Countries between 1970

[1] External default – default or restructuring on obligations to foreign creditors

and 2001. This number rises dramatically if internal sovereign defaults[2] are considered as well [100].

One can find swap rates or even interbank interest rates to be a better proxy for the risk-free rate. However this does not fully solve the credit risk problem either. The collapse of Lehman Brothers in September 2008 marked the significant increase in risk aversion and financial turmoil. The lack of trust lead to a freeze in the unsecured interbank markets and interbank rates rose significantly [44]. Many prime international financial institutions including Lehman Brothers, AIG, Dexia and many others were nationalized, sold, liquidated or acquired state aid. So the sovereign acts as the backstop for the financial markets and troubles in the financial system can lead to the sovereign crisis as demonstrated by Iceland or Ireland (for further details see Chapter 1.2). According to Bell and Pain [6] a number of banks including big ones defaulted in recent years.

But still not enough effort is put into measuring sovereign credit risk. This is partially justified by the difficulties in splitting the sovereign yield into the risk-free part and the credit spread in an environment when the sovereign bond is the least risky asset in the market. However the creation of the euro area in 1999 provides a unique opportunity to this kind of studies. At the time this thesis is written seventeen countries in the euro area form a currency union using the single European currency, the Euro. This forms an environment where government bonds with different levels of credit risk are issued, while having all other properties virtually the same. Using this broad range of data it should be possible to define the common risk-free rate for the euro area and the unique credit spreads for every country.

This opportunity was explored by Geyer, Kossmeier and Pichler [63] and Puig [99], who studied the credit spreads of the euro area countries over the German yield curve. However the German yield was considered to be the risk-free rate in both approaches. Kaminska, Meldrum and Smith [83] studied a model involving interest rates of the United States, United Kingdom and German yields and exchange rates between the US dollar, British pound and the Euro. Corzo and Schwartz [28], Lacko and Stehlíková [88] and Zíková and Stehlíková [129] studied the convergence of domestic

[2] Internal default – default on obligations to domestic creditors or more often the debt amortization through hyperinlfation

interest rates to the European interest rate in countries adopting the Euro during the period before the Euro adoption. Monfort and Renne [96] studied the default and liquidity risks of ten euro area countries in an arbitrage-free framework again with Germany as the risk-free rate.

The purpose of this thesis is to develop a new model for the risk-free rate and credit spreads of the euro area sovereigns. The model assumes that the short-rate for every sovereign is a sum of the risk-free rate common to the whole euro area and a unique credit spread for every country. Both the risk-free rate and credit spreads are assumed to be unobserved variables in the market as well as the short-rate for every country. The only observable variables are yields of the sovereign bonds. Both the risk-free rate and credit spreads are modeled using the Cox, Ingersoll, Ross (CIR) and the Chan, Karolyi, Longstaff, Sanders (CKLS) processes. The advantage of the model is that it allows us to describe the true risk-free rate for the euro area and asses how much of the sovereign yield can be attributed to the risk-free part and how much is the compensation for the credit risk of the issuing country. The model is calibrated using daily data on euro area yield curves for the whole period of existence of the euro area until February 3, 2012.

The thesis is organized as follows. The historical and recent examples of sovereign defaults are described in Chapter 1. In Chapter 2 the major interest rate models are described and in Chapter 3 the recent literature is discussed. The proposed model and its properties are described in Chapter 4 and the calibration method based on Ševčovič and Urbánová Csajková [117] is described in Chapter 5. Calibration results are discussed in Chapter 6.

A shortened description of the final specification of the model in Section 4.4, with a shortened description of the calibration method and data from Chapter 5 and the results for the final model specification in Section 6.5 have been submitted for publication [116].

1. Sovereign credit risk

Government bonds, especially of countries with high rating, are usually considered to be risk-free investments. Other investments considered to be risk-free are swap contracts or bonds of large internationally operating corporates or financial institutions. However recent events like the collapse of Lehman Brothers in 2008, recent sovereign crisis of the euro area leading to default of Greece on March 8, 2012 [79] or the downgrade of the United States by S&P on August 5, 2011 [111] and the nine euro area countries on January 12, 2012 [110] warns us that the credit risk is related with every investment, even if it might be considered negligible at first glance.

But not only recent events, but also history shows that sovereigns or their governments defaults on their obligations from time to time, or as is the practice in modern times restructure their debts under unfavorable conditions to their bondholders, like haircuts on the principal of the bond (the term default will be used to denote both default and debt restructuring from this point onwards in this thesis). This situation keeps repeating although the default brings major negative consequences to the defaulted economy.

De Paoli, Hoggart and Saporta [37] summarize papers studying impact of sovereign defaults on the local economy. They consider the major negative consequences to be decreased access to market funding for an extended period of time, higher interest rate costs in the future, need to issue new debt in a global, not local currency and when combined with banking or currency crises also a substantial negative shock to the output of the economy, despite a one-off positive demand shock caused by the default.

1.1. Historical default developments

Despite all these negative consequences countries tend to default time to time. As an example we can point towards France, which according to Reinhart, Rogoff a Savastano [101] defaulted on their obligations eight times between years 1500 and 1800 and Spain, which defaulted thirteen times between 1500 and 1900. Portugal, Austria and Germany defaulted five times in the nineteen's century and Greece four times. The total number of defaults in Europe was 33 in the nineteenth century including also Bulgaria, Holland and Russia.

Similar examples can be found among emerging markets as well. According to Reinhart, Rogoff and Savastano [101, Table 1] Venezuela leads the Emerging markets ranking with nine defaults since 1824, followed by Mexico (8), Brazil and Colombia (7) and Turkey (6). Other examples include Philippines, Egypt, Chile and Argentina with the latest default episode in 2001. All these countries except Colombia experienced a default in the last 30 years.

The stunning picture is that there are countries, which never defaulted like Australia, Canada, New Zealand, Norway, United Kingdom (UK), United States (US), India, Korea, Singapore or Thailand. On the other hand "*many of the Latin American countries that have been experiencing severe debt problems today also experienced debt problems in the 1980s. And in the 1930s. And in the 1870s. And in the 1820s. And generally, other times as well." (*Reinhart, Rogoff a Savastano [101], p. 6*)* The authors call these countries serial defaulters and conclude that defaults have become their way of life.

More recent default examples can be also found, including Greece (2012), Argentina (2001), Ecuador (2000, 1984) Russia (1991, 1998) Iran (1992), Iraq (1990) and many others. According to Reinhart, Rogoff and Savastano [101, Table 3] there were thirty-five sovereign defaults in total between 1970 and 2001. According to Reinhart and Rogof [100] the situation is even worse if internal defaults, restructurings and hyperinflation periods are considered as well.

1.2. Major developments in the current financial crisis

After the collapse of Lehman Brothers in 2008 Iceland, Hungary and Ukraine were among the first sovereign victims. However they managed to survive with emergency loans from the International Monetary Fund (IMF) and from other countries. According to Andresen [2] the IMF approved loans to Hungary, Ukraine, Iceland, Belarus and Latvia amounting USD 39 billion within months. According to the IMF Factsheet as of January 12, 2012 [73] the IMF support was received by Bosnia, Greece, Kosovo, Romania, Serbia, Ukraine, Armenia, Ireland, Moldova, Portugal, Poland and Macedonia. A year later the situation worsened also in the euro area starting with Greece, spreading later to Ireland and Portugal, Cyprus and finally to core euro area countries like Spain, Italy and Belgium. In the following paragraphs the situation in major troubled individual countries, which is different from country to country, is described in more detail.

Iceland

Iceland was the first sovereign victim of the financial crisis. The reason can be clearly attributed to the extremely huge operations of the banking sector, which in times of trouble was too big to save for the Icelandic government. According to Burgess, Braithwaite and O'Connor [17] the external indebtedness of the Icelandic banking sector in the second quarter of 2008 reached 600 % of the GDP of Iceland. During the first week of October 2008 three major Icelandic banks, Glitnir, Landsbanki and Kaupthing, were nationalized, Icelandic kronor fell sharply until the trading was suspended and CDS spreads soared, while investor demanded advance payments, what happened for the first time since Brazil in 2002 (Burgess, Braithwaite a O'Connor [17]). On October 8, 2008 the Central Bank of Iceland abandoned its currency peg with the Euro [19] and introduced capital controls on October 10, 2008 [20], which are in place until present [21], although they have been amended several times.

According to The Economist [122] the country with around 300 thousands inhabitants after long and tough negotiations reached an agreement on a loan from the UK, German Dutch governments in the amount of USD 6.3 billion at the interest rate of 5.5 % p.a. to pay the guaranteed depositors of Landsbanki foreign branches. At the same time, this agreement was a condition for Iceland to be able to draw the stabilization aid funds from the IMF amounting USD 2.1 billion and a further loan from the Nordic governments amounting USD 2.5 billion [15]. After the government resignation and rejection in June 2009 the deal was finally approved in the Icelandic Parliament on August 28, 2011 [126].

At that time the default of Iceland was seen very possible, e.g. by Danielson [36] *„… it is hard to see how the Icelandic state could service the debt created by the Icesave*[3] *obligations to the UK and the Netherlands, making government default likely*." Even the Icelandic Prime Minister Geer Harde saw the default as a real option (quoted in Burgess, Braithwaite and O'Connor [17]): *„The danger is real that the Icelandic economy would be sucked, along with banks, under the waves and the nation would become bankrupt."* However the crisis was managed and the economic situation of Iceland improves according to the OECD [97].

Hungary

Troubles occurred in 2008 also in Hungary. The Hungarian government asked for the IMF Stand-by-Agreement in amount of SDR[4] 10.5 billion (equal of EUR 12.5 billion) on November 4, 2008 for seventeen months. Together with the support of the European Union (EU) the amount reached EUR 25 billion. According to the IMF Country report [70] the high level of indebtedness and imbalances, mainly driven by the foreign exchange lending to private households, resulted in decreased investor interest for Hungarian assets. The situation worsened dramatically in October 2008. The public deficit reached 8 % of GDP in 2002 – 2006 and foreign indebtedness reached 97 % of GDP. It was concluded that the IMF support is necessary along with strong fiscal restrictions and policies to reduce macroeconomic imbalances. The Stand-by-Agreement was granted on November 6, 2008 [71].

The situation in Hungary stabilized and Hungary managed to return to financial markets with a successful issue on August 3, 2009 [18]. However elections in April 2010 and the establishment of the new government lead to shift in economic policies and to a stop of the IMF program and concerns regarding the long-term sustainability of public finances have gradually increased since then. Contagion from the euro area sovereign debt

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[3] Icesave was a Landsbanki branch in Great Britain and Netherlands with deposits amounting to 4.5 bil. GBP in Great Britain and 1 bil. EUR in Netherlands.

[4] SDR – Special Drawing Right – IMF currency unit

crises and recent policies adopted in Hungary lead to an economic slowdown and renewed investors concerns about the creditworthiness of Hungary. A further EU/IMF support to Hungary cannot be ruled out, however firm fiscal commitments and changes in laws most criticized by the EU must be resolved [74].

Greece

Greece was at the onset of the euro area sovereign crisis in 2010. At the end of 2009 fears spread among investors about the sustainability of public finances in certain euro area countries including Greece. The Greek public deficit jumped in 2009 to 12.7 %[5] of the GDP [61] from the estimate of 2 % of GDP in November 2008. The main reasons for this development include the slowdown in the GDP, unrealistic assumptions used in the forecasts, lax fiscal developments in the past and statistical miscalculations of data. According to the European Statistical Office (Eurostat) Greece have been among the top three indebted countries and their debt-to-GDP ratio exceeded 94 % since 1995. Debt-to-GDP increased every year since 2003 from 97.4 % to 129.3 % in 2009 and 144.9 % in 2010 and since 2006 the ratio is highest in the EU[6].

This significant indebtedness was coupled with another Greek specialty – misreporting of data to the Eurostat. In its report to the European Council on January 8, 2010 [48] the Eurostat summarizes the history of reliability of Greek data. In the 2004 report [57] Eurostat showed how Greek statistical authorities had misreported figures on deficit and debt in the years between 1997 and 2003 in order to fulfill the criteria for joining the euro area with a assistance of some major world banks. According to the 2010 report [48] since 2003 Eurostat expressed reservations on the Greek data five times and when the reservation was not expressed this was due to the corrections made due to Eurostat interventions during the submission period. Eurostat at the same time expressed its inability to validate 2009 data. The data were finally validated only on November 15, 2010 [58].

The new government elected in October 2009 announced a package including austerity measures, independence of the statistical authorities and structural reforms for the

[5] According to the up-to-date figure from Eurostat the actual deficit was 15.8 % of the GDP
[6] Source: Eurostat

economy. However the package was not enough to restore investor confidence and yields on Greek bonds moved up by two percentage points in second half of January 2010 (the developments of the yields are described in more detail in Section 6.1). The increased costs of financing had to be accepted by Greece also on the primary market, which further increased the problems with the public deficit.

Another major blow came on April 27, 2010 when Standard and Poor's (S&P) downgraded Greece to BB+ [91], which is a sub-investment grade. Greek yields soared and markets froze. On May 2, 2010 the euro area and the IMF [75] announced the joint rescue loan for Greece in amount of EUR 110 billion at the interest rate of 5.5 % and conditional on more austerity measures adopted by the Greek government. Slovakia later denies its part of the loan to Greece [123]. On May 3, 2010 the European Central Bank (ECB) also supported Greece by suspending the rating limit for the eligibility of Greek bonds in the ECB operations [45]. The eligibility was applied to all outstanding and new issued Greek bonds. On May 9, 2010 the European Council agreed to establish the temporary European Financial Stability Facility (EFSF) in the amount of EUR 500 billion to help along with the IMF countries in trouble.

The austerity measures were opposed by Greek public and riots were held in major Greek cities. Since then the situation followed the same pattern. Greek yields gradually increased, new austerity measures were announced and new rescue plans arranged. More riots were held as the measures hit harder and harder the Greek general public and the GDP forecasts turned more and more negative, which worsened the situation even further. Greece entered the vicious circle which ended in default in March 2012.

It was originally hoped that with the rescue agreement Greece would be able to return to market financing in 2012. However it was soon clear that due to deeper recession and revised deficit and debt this hope will not come true. In June S&P downgraded Greece again to CCC, the lowest rating in the world [103]. After political turbulence in Greece another austerity package was adopted at the end of June 2010. The situation calmed and long-term Greek yields declined below 10 % p.a. however they started to rise again in the last quarter of 2010 and a new deal was inevitable.

At the same time European leaders felt that the temporary EFSF cannot secure stability in the euro area and a permanent mechanism is needed. This was approved by the European Council on October 28 and 29, 2010 and amendments to the European treaties were approved by the European Council on December 16 and 17, 2010. Slovak Parliament denied the approval on October 11, 2011 which led to the loss of confidence of the government and new elections in 2012 [124]. The mechanism was subsequently approved by the Slovak Government on February 1, 2012 [125] and an amended Treaty was approved by the European Council on February 2, 2012 [55]. President of the European Council stated: *"The objective is to have the mechanism in force in July 2012 a year earlier then originally planned. The ESM[7] will be an international financial institution, based in Luxembourg and founded by euro area member states, with an initial maximum lending volume of €500 billon, to be achieved by €700 billion of subscribed capital (€80 billion in paid-in shares and €620 billion in callable shares).*" [55]

During 2011 the Greek need for further assistance increased and a preliminary second rescue package was approved by the European Council on July 21, 2011 [53]. The new package amounted EUR 109 billion loan from the IMF/ESFS to Greece. Maturity of the loans to Greece was extended to 15 years up to 30 years and interest rate lowered to 3.5 %. Voluntary support by the private sector was envisaged as well. A new austerity package was envisaged too. The European Council decided in October 26 and 27, 2011 to leverage the EFSF to EUR 1 trillion, recapitalize EU banks and a voluntary 50 % haircut on Greek sovereign bonds with final details to be agreed [54]. The second rescue loan was increased to EUR 130 billion. The package was finalized by the European Council on February 21, 2012 [56]. The details contained the public sector acceptance of a nominal haircut of 53.5 %, the retroactive lowering of interest rate to Euribor + 150 basis points on Greek loans and the IMF/EFSF loan of up to EUR 130 billion.

According to Clearstream [27] and the Greek Ministry of Finance [62] the details concerning the Private Sector Involvement (debt restructuring) are following. On February 24, 2012 Greek Parliament amended Greek law in order to retroactively introduce collective action clauses to bonds issued under Greek law. The collective action clauses

[7] ESM – European Stability Mechanism

were already part of the issue conditions for bonds issued under British law. Under this amendment if bondholders representing at least 90 % of the overall Greek debt voluntarily agree to exchange them, the exchange will be eligible on all bonds under the Greek law. If at least 75 % of the overall debt bondholders agree to the exchange, Greek government has the right to unilaterally force the exchange too all bondholders under the Greek law. If less than 75 % bondholders participate, no exchange will proceed.

The Greek government offered the exchange for the period of February 24, 2012 until March 8, 2012 covering total of 135 ISIN codes. During this period bondholders representing 85.8 % of the debt tendered for offer and Greece invoked the collective clauses to force the complete exchange of bonds issued under Greek law and to those foreign bonds where the offer was agreed by bondholders. This represents total of EUR 197 billion in issued bonds.

For every EUR 1 000 of face value of the old bonds the bondholders will receive EUR 315 face value of a mixture of 20 bonds issued on March 12, 2012 with maturities from 11 to 30 years bearing 2 % p.a. coupon rate. They will further receive face value of EUR 315 of a GDP-linked security paying an interest of 1% if the Greek GDP exceeds certain threshold. Finally they will receive face value of EUR 75 of both a one-year and a two-year note issued by the EFSF and a one-year EFSF note with the face value equal to the accrued interest on the bonds. The settlement was completed on March 12, 2012 and the overall loss to bondholders is around 53.5 % of the current market value of Greek bonds, which is around 73.5 % of total principal amount. On March 9, 2012 the International Swaps and Derivatives Association issued a public statement that the Greek restructuring is considered as a credit event for the CDS[8] contracts following the exercise of the collective action clauses by Greece [79].

It is not entirely clear, why Greece was not allowed to default in 2010. Some economists like Roubini [102] argued that Greece should leave the euro area, introduce drachma and devaluate it to gain competitiveness. However this solution was unacceptable from the political perspective and maybe also due to fears that other European countries might follow, especially major southern economies like Italy or Spain. Other reason for the

[8] CDS – Credit Default Swap

postponed debt restructuring might be that the exposure to Greek debt and CDS written on Greece was not transparent. On the other hand after the European Union decided to rescue Greece, its response often lacked decisiveness and offered only short-term solutions in order to buy more time. This happened despite the fact that the situation was often calmed only by ECB interventions like the securities purchase program or Long-Term Refinancing Operations [46]. Only market situation, contagion to Ireland and Portugal and later to Cyprus, Italy, Spain and Belgium forced the Union to propose decisive actions in late 2011/early 2012.

Ireland

The Irish case is quite similar to the case of Iceland because sovereign troubles have not been caused by government over expenditure, but due to financial aid to the banking sector, which proved to be too big to save for the Irish government. However in the case of Ireland, the banks did not expand that much internationally, but domestically, funding the economic growth of the past decade and the real estate bubble. Ireland as a small and open economy was hit very hard by the financial and economic crisis in 2008. Due to its close ties with the UK Ireland was among the first European countries in recession with the real GDP contracting by 3 % in 2008 and by 7 % in 2009[9] and the unemployment rate rising from 4.9 % at the end of 2007 to 8.5 % in 2008, 12.9 % in 2009[9] and property prices declined.

In order to help the troubled banks Irish government guaranteed all Irish bank deposits and debt on September 29, 2008 for the period of two years [47]. This was not enough and the government announced recapitalization plan for the banks in amount of EUR 10 billion [105]. The situation worsened in December 2010 when it was revealed that Anglo Irish Bank's director was hiding his loans from the bank in amount of EUR 87 million. He and two other directors resigned [16]. The Anglo Irish Bank was finally nationalized on January 21, 2009, after the government decision dated January 15, 2009 and the legislation approval on January 20, 2009 [77]. Government continued to support the banking system with EUR 3.5 billion recapitalization of the Allied Irish Bank (AIB) and Bank of Ireland on February 11, 2009 [78]. This resulted in Irish deficit rising to

[9] Source: Eurostat

second highest among the European countries recording 14.2 % of the GDP and increasing the debt level to 65.2 % of GDP[10]. However the situation appeared still to be manageable and Irish yields remained contained throughout 2009.

The situation worsened in 2010 and the support for Irish banks lead to the astonishing public deficit of 32 % of the GDP in 2010[10], which lead to the debt increase to 92.5 % of GDP in 2010. This situation was considered unsustainable by markets and yields of Irish bonds rose significantly in third quarter 2010. Ireland finally negotiated support from the EFSF and IMF [76] and was the first country to obtain funds from it. According to the description of the Irish support on the European Commission's website [49] the total support reached EUR 85 billion provided by the IMF, EFSF, EFSM, UK, Denmark and Sweden. Total of EUR 35 billion was used to support the ailing Irish banking system.

Despite all the effort made Moody's downgraded Ireland to BB+ in July 2011 [85]. The reaction of European Union was to ease the loan conditions. In July 2011 the interest rate was lowered and maturity extended [53]. After that the yields of Irish bonds plummeted and now are in the range from 5 to 7.5 % p.a. We can therefore conclude that Ireland is on a good way to solve its problems.

Portugal

The causes of the Portugal crisis are rooted in the last decade of economic development. After the period of strong growth at the end of the second millennium Portugal experienced a period of subdued growth in the first decade of the third millennium with annual real GDP growth rates varying between -0.9 % and 2.4 %[10]. This level was significantly below par in the EU. However in terms of public deficit Portugal was among the leaders with the deficit exceeding 2.9 % of GDP every year since 2000. This period was a missed opportunity to consolidate public finance in Portugal. The public debt increased from 48.2 % of GDP in 2000 to 68.3 at the verge of the crisis in 2007, while at the same time the overall debt level in the EU decreased from 61.9 % to 59 %[10.] Portugal thus did not build the buffers which were needed since 2008.

In December 2009 S&P decreased the outlook on Portuguese rating to negative voicing pessimism on the country's capacity to strengthen its public finances and reduce

[10] Source: Eurostat

debt. The reasons for this move were the expected high public deficit and debt in 2009 and the fact, that decreasing the deficit would be complicated by structural weaknesses in the economy and weak competitiveness that would hamper growth [1].

These fears materialized with the public deficit reaching 10.1 % of GDP in 2009 and 9.8 % in 2010. Together with the economic contraction in 2009 this drove the public debt level to 83 % in 2009 and 93.3 % in 2010, which is fourth highest in the EU behind Greece, Italy and Belgium and higher than Ireland and Iceland[10]. Portugal was downgraded in April 2010 [92] and yields started to rise. They have steadily increased during 2010 and breached 7 % p. a. in early 2011 and reached 9 – 10 % when Portugal finally got the EU/IMF loan on May 17, 2011 in the amount of EUR 78 billion [52]. However, the situation is far from over and Portuguese yields have risen dramatically over the last year reaching 20 % p. a. occasionally. Portugal might need to follow the Greek path in the future.

Possible contagion to other countries

In third quarter 2011 yields of other euro area countries increased. Countries most affected list Cyprus, Italy, Spain and Belgium. In case of Cyprus the reason were mostly economic ties with troubled Greece. In case of Italy and Belgium high indebtedness is mostly of concern and austerity packages have been agreed by the new governments in late 2011. In Spain, economic non-competitiveness results in slow economic growth and high unemployment, while its debt levels remain low in comparison with other EU crisis. However contagion fears prompted Spain to expand austerity measures to contain the high public deficit.

It was mainly these contagion fears that prompted the European Union to finally agree on key reforms including the inevitable Greek default. The positive effect is the agreement on the fiscal rules enforcement in order to protect the euro area that was agreed on the December European Council meeting. But still much progress in this area needs to be made and the actions taken contain themselves elements of possible sovereign risk contagion, which is documented by the S&P downgrade of nine euro area countries on January 13, 2012 [110].

2. Interest rate models

The most widely used interest rate models are described in this chapter. Only a brief overview of a wide range of models used is presented in this chapter, with a concentration on the short-rate models, which will be used later in the thesis. Other models will not be used. Comprehensive overviews of interest rate models can be found e.g. in Chan, Karolyi, Longstaff and Sanders [22], Duan and Simonato [40] and Lacko and Stehlíková [88].

Definition 1. *The standard Wiener process* $\{W_t, t \geq 0\}$ *is a continuous stochastic process with the following properties:*

- *for every* $s > 0$*,* $t \geq s \geq 0$ *the difference* $W_t - W_s$ *is a normally distributed random variable with zero mean and variance* $t - s$*,*
- *for every* $t_1 \geq s_1 \geq 0$*,* $t_2 \geq s_2 \geq 0$ *the differences* $W_{t_1} - W_{s_1}$ *and* $W_{t_2} - W_{s_2}$ *are independent random variables,*
- $W_0 = 0$.

Let us denote the special difference of the Wiener process $W(t+dt) - W(t)$ as dW.

Definition 2. *The* n*-dimensional standard Wiener process is the process* $\{\mathbf{W}_t, t \geq 0\} = \{(W_1(t), \ldots, W_n(t)), t \geq 0\}$ *where* $W_i(t)$ *are one-dimensional standard Wiener processes and the correlation between their increments* $Cor\left[dW_i^t, dW_j^t\right] = \rho_{i,j}(t)dt$ *for* $i \neq j$.

Let us denote $P(t,T)$ the price at time t of a zero-coupon bond with maturity at time T.

The value of the bond at maturity must be equal to its par value, which is assumed to be one for reason of simplicity, thus $P(T,T) = 1$. The yield to maturity $R(t,T)$ at time t of this particular bond is defined by:

$$R(t,T) = -\frac{1}{T-t}\ln(P(t,T)) \,. \tag{2.1}$$

The term structure of interest rates or the yield curve is the functional dependence of the yield to maturity on the residual time to maturity $\tau = T - t$. The instantaneous rate or the short rate is defined as the limit of (2.1) when time nears the maturity of the bond and is defined by:

$$r_t = \lim_{T \to t^+} R(t,T) = -\left.\frac{\partial \ln P(t,T)}{\partial T}\right|_{T=t} . \tag{2.2}$$

The forward rate $f(t,T_1,T_2), T_1 < T_2$ is the yield agreed at time t with which a zero-coupon bond is bought at time T_1 with maturity at time T_2. The forward rate is defined by:

$$f(t,T_1,T_2) = -\frac{1}{T_2 - T_1}\ln\left(\frac{P(t,T_2)}{P(t,T_1)}\right). \tag{2.3}$$

The instantaneous forward rate is defined as the limit of (2.3) when T_1 nears T_2:

$$F(t,T) = \lim_{T_1 \to T_2} f(t,T_1,T_2) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T}. \tag{2.4}$$

Note that $F(t,t) = r_t$. The models of interest rates differ by modeling either the short rate or the instantaneous forward rate and by the number and type of factors used in the model. In the following sections the most common interest rate models are described.

Proposition 1. (Itô lemma) *(Kwok [86], p. 29). Let $f(t, X_t)$ be a C^2 smooth, non-random function and $\{X_t, t \geq 0\}$ is a stochastic process defined by:*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \,, \tag{2.5}$$

where W_t is a standard Wiener process. Then the stochastic process $Y_t = f(t, X_t)$ satisfies the following stochastic differential:

$$dY_t = \left(\frac{\partial f}{\partial t} + \mu(t, X_t)\frac{\partial f}{\partial X_t} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial X_t^2}\right)dt + \sigma(t, X_t)\frac{\partial f}{\partial X_t}dW_t . \tag{2.6}$$

Proposition 2. (Multidimensional Itô lemma) *(Itô [80], Theorem 6). Let* $f(t,X_t):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ *be a* C^2 *smooth, non-random, function* $\mu(t,X_t):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ *,* $\sigma(t,X_t):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n$ *and* $\{X_t, t\geq 0\}$ *is a* $n-$*dimensional stochastic process defined by:*

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)d\mathbf{W}_t, \qquad (2.7)$$

where $\mathbf{W}_t$ *is a* n*-dimensional standard Wiener process. Then the stochastic process* $Y_t = f(t,X_t)$ *satisfies the following stochastic differential:*

$$dY_t = \frac{\partial f}{\partial t}dt + \left(\nabla_X f\right)dX + \frac{1}{2}(dX)'(\nabla_X^2 f)(dX). \qquad (2.8)$$

2.1. One-factor short-rate models

A general equilibrium one-factor short-rate model is, according to Kwok [86], described by the following process:

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t, \qquad (2.9)$$

where W_t is a standard Wiener process, $\mu(t,r_t)$ and $\sigma(t,r_t)$ are functions defining the trend and volatility of the short-rate process.

Proposition 3. *(Kwok [86], p. 321). Let* r_t *be a short-rate process defined by (2.9). The price of a discount*[11] *bond* $P(t,\tau,r_t)$ *at time* t *with residual time to maturity* τ *is a solution to this partial differential equation:*

$$-\frac{\partial P}{\partial \tau} + \left[\mu(t,r_t) - \lambda(t,r_t)\sigma(t,r_t)\right]\frac{\partial P}{\partial r_t} + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 P}{\partial r_t^2} - r_t P = 0, \qquad (2.10)$$

with a terminal condition $P(t,0,r_t)=0$ *for all* r_t *and* t*, where* $\lambda(t,r_t)$ *is the market price of risk function.*

[11] The terms zero-coupon bond and discount bond are equivalents in this thesis.

For a thorough insight into this topic the reader is referred to [86]. The actual derivation of the price of a zero-coupon bond is performed for the specific model proposed in Chapter 4. A thorough comparison of one-factor models was performed by Chan, Karolyi, Longstaff and Schwartz [22].

Vasicek model

The first equilibrium short-rate model was developed by Vasicek [127] and is of the form:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \qquad (2.11)$$

where W_t is a standard Wiener process and κ, θ and σ are constants. Note that Vasicek model is a special case of (2.9). One major benefit of this model is the explicit solution to the price of a zero-coupon bond.

Proposition 4. *(Special case of Proposition 3) Suppose that the short-rate r_t is defined by (2.11) and $\lambda(t, r_t) = \lambda$ is constant. There exists an explicit solution to the partial differential equation (2.10) of the form:*

$$\begin{aligned} P(t,\tau,r_t) &= A(\tau)e^{-B(\tau)r_t}, \\ B(\tau) &= \frac{1-e^{-\kappa\tau}}{\kappa}, \qquad \tau > 0, \qquad (2.12) \\ A(\tau) &= \exp\left([B(\tau)-\tau]\left[\theta - \frac{\sigma^2}{2\kappa^2} - \frac{\sigma\lambda}{\kappa}\right] - \frac{\sigma^2 B(\tau)^2}{4\kappa} \right). \end{aligned}$$

However, the major drawback of the model is the unrealistic assumption of constant volatility and the resulting fact that the model admits negative values for the short-rate with nonzero probability. Both these drawback are addressed in the CIR model.

CIR model[12]

In the CIR model, developed by Cox, Ingersoll and Ross [31], the volatility of the short-rate process is assumed to depend also on the level of the short-rate:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\,dW_t\,, \tag{2.13}$$

where W_t is a standard Wiener process and κ, θ and σ are constants, which is also a special case of (2.9). As in the case of the Vasicek model, there exists an explicit solution for the price of a zero-coupon bond also for the CIR model.

Proposition 5. *(Special case of Proposition 3) Suppose that the short-rate* r_t *is defined by (2.13) and* $\lambda(t, r_t) = \lambda\sqrt{r_t}$ *. There exists an explicit solution to the partial differential equation (2.10) of the form:*

$$\begin{aligned} P(t,\tau,r_t) &= A(\tau)e^{-B(\tau)r_t}, \\ B(\tau) &= \frac{2[e^{\eta\tau}-1]}{(\kappa+\lambda\sigma+\eta)[e^{\eta\tau}-1]+2\eta},\ \tau > 0, \\ A(\tau) &= \left[\frac{\eta e^{(\kappa+\lambda\sigma+\eta)\tau/2}}{e^{\eta\tau}-1}B(\tau)\right]^{2\kappa\theta/\sigma^2}, \end{aligned} \tag{2.14}$$

where $\eta = \sqrt{(\kappa+\lambda\sigma)^2 + 2\sigma^2}$.

If the condition $2\theta\kappa > \sigma^2$ is fulfilled the CIR process is zero with zero probability.

Proposition 6. *(Kwok [35], Chapter 7). Suppose that the short-rate* r_t *is defined by (2.13),* $2\theta\kappa > \sigma^2$ *and* $r_0 = r(t_0)$*. Then for every* $t > t_0$*, the probability density function of the CIR process is:*

$$f(x) = \begin{cases} 0 & x \le 0 \\ ce^{-a-b}\left(\dfrac{b}{a}\right)^{q/2} I_q(2\sqrt{ab}) & x > 0 \end{cases}, \tag{2.15}$$

where $c = \dfrac{2\kappa}{\sigma^2(1-e^{-\kappa(t-t_0)})}$, $a = cr_0e^{-\kappa(t-t_0)}$, $b = cr$, $q = \dfrac{2\kappa\theta}{\sigma^2} - 1$ *and* I_q *is the modified Bessel function of the first kind and order* q.

[12] CIR model – Cox, Ingersoll, Ross model as published in [30].

Let us denote $\alpha = 2\kappa\theta / \sigma^2$ and $\beta = 2\kappa / \sigma^2$.

Proposition 7. *(Kwok [35], Chapter 7). The limit of (2.15) for* $t \to \infty$ *is the probability density function of a random variable with a Gamma distribution* $\Gamma(\alpha, \beta)$ *and which has the following form:*

$$f(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x > 0 \end{cases}. \qquad (2.16)$$

Proposition 8. *(Kwok [35], Chapter 7). The cumulative distribution function of this random variable with the density function (2.16) is defined by:*

$$G(x | \kappa, \sigma, \theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-\beta x} dt. \qquad (2.17)$$

CKLS model[13]

Chan, Karolyi, Longstaff, Sanders [22] studied the generalized version of the Vasicek and CIR models, which is defined by:

$$dr_t = (a + br_t)dt + \sigma r_t^\gamma dW_t, \qquad (2.18)$$

where W_t is a standard Wiener process and a, b, γ and σ are constants. The Vasicek and CIR models are special cases of (2.18) for $\gamma = 0$ and $\gamma = 1/2$ respectively. They studied the constraints imposed on the parameter γ and found out that the optimal value of this parameter is approximately $3/2$. However the only cases where there exists an explicit solution to the price of a zero-coupon bond are Vasicek and CIR models.

[13] CKLS model – Chan, Karolyi, Longstaff, Sanders model as published in [22].

Other special cases of the CKLS model include models by:

- Merton [93]:

$$dr_t = \mu dt + \sigma dW_t\,, \tag{2.19}$$

- Dothan [39]:

$$dr_t = \sigma r_t dW_t\,, \tag{2.20}$$

- Brennan and Schwartz [13]:

$$dr_t = \kappa(\theta - r_t)dt + \sigma r_t dW_t\,, \tag{2.21}$$

- Cox, Ingersoll and Ross [30]:

$$dr_t = \sigma r_t^{3/2} dW_t\,, \tag{2.22}$$

CEV (Constant elasticity of volatility) proposed by Cox [29] and Cox and Ross [32]:

$$dr_t = \beta r_t dt + \sigma r_t^{\gamma} dW_t\,. \tag{2.23}$$

2.2. No-arbitrage short-rate models

In one-factor short-rate models the current value of the short-rate together with the value of the market price of risk determines the whole term structure of interest rates. Naturally this model yield curve does not perfectly matches the real yield curve. In the no-arbitrage class of models the desired outcome is to perfectly match the current market yield curve. This is achieved by introducing time-varying coefficients to the short-rate models, which enables us to perfectly match the market yield curves. The examples of no-arbitrage models include the following models (W_t denotes the standard Wiener process):

- Ho and Lee model [67] based on the Merton model where the parameter μ in (2.19) is time-varying:

$$dr_t = \mu(t)dt + \sigma dW_t\,, \tag{2.24}$$

- Hull and White model [69] based on the CKLS model where all the parameters except the parameter γ in (2.18) are time-varying:

$$dr_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)r_t^{\gamma} dW_t, \tag{2.25}$$

- Black, Derman and Toy [10] based on the Brennan-Schwartz model where, $\theta = 0$ and the remaining two parameters in (2.21) are time-varying:

$$dr_t = -\beta(t)r_t dt + \sigma(t)r_t dW_t. \tag{2.26}$$

2.3. No-arbitrage forward rate models

The forward rate models fit into the well known Heath, Jarrow, Morton framework, published in [66]. The framework defines the forward rate process in the form:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \tag{2.27}$$

where W_t is the standard Wiener process. The individual models are obtained by specifying the form of the volatility function $\sigma(t,T)$. The models are usually able to model the whole term-structure of interest rates, however, they often lose the Markovian property.

2.4. Two-factor short-rate models

The major disadvantage of one-factor models is the dependence of the whole term structure of interest rates on a single factor – the short rate. This allows only for one shape of the term structure over time, which in reality is not true. By addition of another factor to the formulation of the model it is possible to achieve different shapes of the term structure of interest rates. The general two-factor short rate model is defined by:

$$\begin{aligned} dr_t &= \mu_r(r_t, y_t, t)dt + \sigma_r(r_t, y_t, t)dW_t^1, \\ dy_t &= \mu_y(r_t, y_t, t)dt + \sigma_y(r_t, y_t, t)dW_t^2, \end{aligned} \tag{2.28}$$

where W_t^1 and W_t^2 are standard Wiener processes and their increments are constantly correlated with a correlation coefficient ρ.

Proposition 9. *Suppose that the short-rate process is defined by (2.28). The price of a zero-coupon bond* $P(t,T,r_t,y_t)$ *is defined by:*

$$\frac{\partial P}{\partial t}+(\mu_r-\lambda_1\sigma_r)\frac{\partial P}{\partial r}+(\mu_y-\lambda_2\sigma_y)\frac{\partial P}{\partial y}+\frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2}+\frac{1}{2}\sigma_y^2\frac{\partial^2 P}{\partial y^2}+\sigma_r\sigma_y\rho\frac{\partial^2 P}{\partial r\partial y}-rP=0 \quad (2.29)$$

For the derivation of (2.29) and thorough analysis of the model the reader is referred e.g. to Stehlíková [112]. We perform this exercise for the model proposed in Chapter 4.

There exists a range of possibilities what to take as the second factor in the model. Based on this choice the models can be divided into several broad categories:

- Models where the second factor is one of the constant parameters from the one-factor models. For the analysis of these type of models see e.g. Stehlíková [112]:
    - models where the second factor is the long-term equilibrium rate θ, e.g. the model developed by Balduzzi, Das and Foresi [5]:

$$\begin{aligned} dr_t &= \kappa(\theta_t - r_t)dt + \sigma_r(r_t)dW_t^1, \\ d\theta_t &= \mu(\theta_t)dt + \sigma_\theta(\theta_t)dW_t^2, \end{aligned} \quad (2.30)$$

    - models where the second factor is the volatility parameter σ, e.g. the model developed by Fong and Vasicek [60]:

$$\begin{aligned} dr_t &= \kappa_1(\theta_1 - r_t)dt + \sqrt{\sigma_t}dW_t^1, \\ d\sigma_t &= \kappa_2(\theta_2 - \sigma_t)dt + \nu\sqrt{\sigma_t}dW_t^2, \end{aligned} \quad (2.31)$$

      or model by Anderson and Lund [3] where the short rate follows a one factor CKLS model with stochastic volatility described by a logarithmic Vasicek model:

$$\begin{aligned} dr_t &= \kappa_1(\theta_1 - r_t)dt + \sigma_t r_t^\gamma dW_t^1, \\ d\ln(\sigma_t^2) &= \kappa_2(\theta_2 - \ln(\sigma_t^2))dt + \xi dW_t^2. \end{aligned} \quad (2.32)$$

- Models, where the second factor is another, often unobservable, macroeconomic variable such as:
    - Inflation, see e.g. Cox, Ingersoll and Ross [31] for the model with the following form, where the real value of the nominal payoff of the zero-coupon bond at maturity is $1/p(T)$ where:

$$\begin{aligned} dr_t &= \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\,dW_t, \\ dp_t &= \mu(p_t)dt + \sigma(p_t)dW_t^2, \end{aligned} \tag{2.33}$$

    - Consol Rate[14], see e.g. Brennan and Schwartz [14] for the study of the model with the following form:

$$\begin{aligned} dr_t &= (\theta_r + \kappa_r(l_t - r_t))dt + \sigma_r r_t dW_t^r, \\ dl_t &= (\theta_l + \kappa_1 r_t + \kappa_2 l_t))dt + \sigma_l l_t dW_t^l, \end{aligned} \tag{2.34}$$

    or Schaefer and Schwartz [106], for the study of a slightly modified version of the model where $s_t = l_t - r_t$, which is driven by the orthogonality of the random variables s_t and l_t, which is confirmed by [4]:

$$\begin{aligned} ds_t &= \kappa_s(\theta - s_t)dt + \sigma_s dW_t^s, \\ dl_t &= \theta(t, l_t, s_t)dt + \sigma_l\sqrt{l_t}\,dW_t^l, \end{aligned} \tag{2.35}$$

    or Christiansen [26] for the study of a slightly modified version of this model in the GARCH form:

$$\begin{aligned} dl_t &= (\theta_l + \kappa_l l_t))dt + \sigma_l l_t^{\gamma_l} dW_t^l, \\ ds_t &= (\theta_s + \kappa_s s_t))dt + \sigma_s s_t^{\gamma_s} dW_t^s. \end{aligned} \tag{2.36}$$

- Models where the second factor is an variable exogenous to the studied economy, such as:
    - Models with European interest rate r_t^e and domestic interest rate r_t^d, where the trend in the domestic interest rate is to converge to the European rate with a spread θ_d see e.g. Corzo and Schwartz [28]. The Wiener processes

[14] Consol-rate – is the yield of a bond which never pays back the principal, but is paying coupons for eternity.

increments dW_t^d and dW_t^e are constantly correlated with the correlation coefficient ρ. Corzo and Schwart [28] studied the model inspired by the Vasicek model:

$$\begin{aligned} dr_t^e &= \kappa_e(\theta_e - r_t^e)dt + \sigma_e dW_t^e, \\ dr_t^d &= (\theta_d + \kappa_d(r_t^e - r_t^d))dt + \sigma_d dW_t^d, \end{aligned} \tag{2.37}$$

- CIR version of the previous model developed by Lacko and Stehlíková [88]:

$$\begin{aligned} dr_t^u &= \kappa_u(\theta_u - r_t^u)dt + \sigma_u \sqrt{r_t^u} dW_t^u, \\ dr_t^d &= (\theta_d + \kappa_d(r_t^u - r_t^d))dt + \sigma_d \sqrt{r_t^d} dW_t^d, \end{aligned} \tag{2.38}$$

- CKLS version of the previous model developed by Zíková and Stehlíková [129]:

$$\begin{aligned} dr_t^e &= \kappa_e(\theta_e - r_t^e)dt + \sigma_e (r_t^e)^{\gamma_e} dW_t^e, \\ dr_t^d &= (\theta_d + \kappa_d(r_t^e - r_t^d))dt + \sigma_d (r_t^d)^{\gamma_d} dW_t^d. \end{aligned} \tag{2.39}$$

- Models where the short rate is composed of several processes.
  - Two-factor Vasicek model:

$$\begin{aligned} r_t &= r_t^1 + r_t^2, \\ dr_t^1 &= \kappa_1(\theta_1 - r_t^1)dt + \sigma_1 dW_t^1, \\ dr_t^2 &= \kappa_2(\theta_2 - r_t^2)dt + \sigma_2 dW_t^2, \end{aligned} \tag{2.40}$$

  - Two-factor CIR model, see e.g. Cox, Ingersoll and Ross [31] or Chen, Scott [24]:

$$\begin{aligned} r_t &= r_t^1 + r_t^2, \\ dr_t^1 &= \kappa_1(\theta_1 - r_t^1)dt + \sigma_1 \sqrt{r_t^1} dW_t^1, \\ dr_t^2 &= \kappa_2(\theta_2 - r_t^2)dt + \sigma_2 \sqrt{r_t^2} dW_t^2. \end{aligned} \tag{2.41}$$

2.5. *Multi-factor interest rate models*

Of course there is no need to stop with two factors and more factors can be added. Examples of such models are given below, but they are not further studied in this thesis in more detail due to their complexity.

Three factor model with stochastic volatility and long-term equilibrium rate

The model was developed by Chen [23] and has the following form:

$$\begin{aligned} dr_t &= \kappa(\theta_t - r_t)dt + \sqrt{\sigma_t}\,dW_t^1, \\ d\theta_t &= \mu(\theta_t)dt + \sigma_\theta(\theta_t)dW_t^2, \\ d\sigma_t &= \kappa_2(\theta_2 - \sigma_t)dt + \nu\sqrt{\sigma_t}\,dW_t^2. \end{aligned} \tag{2.42}$$

Multi-factor CIR model

The multi-factor CIR model is a model of the form:

$$\begin{aligned} r &= r_1 + r_2 + \cdots + r_m, \\ dr_i &= \kappa_i(\theta_i - r_i)dt + \sigma_i\sqrt{r_i}\,dW_i, \end{aligned} \tag{2.43}$$

where all factors r_i are defined by a one-factor CIR model. For more information see e.g. Schlögl and Sommers [107]. Similar models can be derived using Vasicek or CKLS specification of (2.43).

Exponentially affine models

Duan and Simonato [40] offer even a broader definition of interest rate models, the so called exponential affine or completely affine models described first by Duffie and Kan [42]. In this model the yield to maturity is an affine function of an abstract state vector X_t, defined by:

$$d\boldsymbol{X}_t = U(\boldsymbol{X}_t, \psi)dt + \Sigma(\boldsymbol{X}_t, \psi)d\boldsymbol{W}_t, \tag{2.44}$$

where $\boldsymbol{W}_t$ is a $n \times 1$ vector or independent Wiener processes, ψ is a $p \times 1$ parameter vector and $U(\cdot)$ a $\Sigma(\cdot)$ are $n \times 1$ and $n \times n$ function sufficiently regular in order to provide a single solution to (2.44). For more details the reader is referred to Duan and Simonato [40]. No restriction on the market price of risk function is placed in this class of models. Note that all models described in this chapter above are special cases of this model.

Duffee [41] formulated the so called essentially affine term-structure models where both the yield to maturity and the market price of risk is an affine function of the abstract state-space variable X_t defined by (2.44).

2.6. LIBOR market model

The LIBOR market model is usually a multifactor no-arbitrage model of the so called LIBOR rate process (see Definition 5 below). In principle it is possible to construct also a one-factor LIBOR market model, but this is considered too restrictive to describe the dynamics of the yield curve properly [108]. The reader is referred to Brace, Gatarek, Musiela [12] and Jamshidan [81] for a thorough insight into the theory behind LIBOR market models. The following description of the model follows Jamshidan [81].

Definition 3. *Suppose that* τ *is sufficiently large,* $\{F_t, 0 \le t \le \tau\}$ *is a filtration and* (Ω, F_t, P) *is a probability space. Let* ε *be the collection of continuous semi-martingales on* $[0,\tau]$ *with respect to* (Ω, F_t, P). *The vector* $B = (B_1, \ldots, B_n) | B_i \in \varepsilon$ *is called a price system or a market.*

Definition 4. *The price system* B *is called arbitrage-free if there exist a* $\xi, \xi \in \varepsilon, \xi > 0$ *with* $\xi_0 = 1$ *such that* ξB_i *are martingales with respect to* (Ω, F_t, P) *for all* $1 \le i \le n$. *The process* ξ *is called the state price deflator [109].*

Definition 5. *Suppose that the set of zero-coupon bond prices with maturities* $0 \le T_1 \le T_i \le T_n \le \tau$ *is a strictly positive arbitrage-free price system given by the following system of stochastic differential equations:*

$$\frac{dB_i}{B_i} = \mu_i dt + \sigma_i dW = \mu_i dt + \sum_{k=1}^{d} \sigma_{ik} dW_k,$$
$$\frac{d\xi}{\xi} = -rdt - \phi dW = -rdt - \sum_{k=1}^{d} \phi_k dW_k, \qquad (2.45)$$

where $\boldsymbol{W} = \{W_1, \cdots W_d\}$ *is a* $d-$*dimensional Wiener process. The* $(n-1)-$*dimensional process given by:*

$$L_i = \delta_i^{-1}\left(\frac{B_i}{B_{i+1}} - 1\right), \qquad (2.46)$$

where B_i *is given by (2.45) and* $\delta_i > 0$ *for* $i = 1 \ldots n-1$ *is called the* $(n-1)-$*dimensional LIBOR process.*

Proposition 10. *The dynamics of the LIBOR process is given by:*

$$dL_i = L_i \gamma_i dW^{i+1}, \qquad (2.47)$$

where

$$\gamma_i = \delta_i^{-1}(1 + \delta_i L_i)(\sigma_i - \sigma_{i+1}) / L_i,$$
$$dW^j = dW + (\phi - \sigma_j)dt. \qquad (2.48)$$

Definition 6. *The LIBOR process given by (2.47) and (2.48) is called a LIBOR market model if the relative volatilities* γ_i *are deterministic:*

$$\gamma_i(t, \omega) = \gamma_i(t), \text{ for } i = 1 \ldots n-1. \qquad (2.49)$$

3. Overview of calibration methods

A number of recent studies focused on the calibration methods for interest rates models. This fact is driven by the growing volume and range of interest rate derivatives available on the market. These calibration methods can be divided into several broad categories according to the approach used: statistical analysis of the short-rate time-series, comparison of the real yield curves and modeled yield curves, comparison of real yield curves with approximate formulae for the bond price. The benefit of using less complicated models with analytical solutions allows us to directly compare and minimize the differences of the model yields with the real yields. A comprehensive survey of the calibration methods is for example in Urbánová Csajková [121].

The use of more advanced models which better capture the dynamics of the interest rates however come at the cost, that only approximate formulas for the bond prices can be obtained. The fall of the socialist block in Europe in 1989 opened a number of small new financial markets, which are yet not fully integrated into the European market and often driven by local currency developments. Therefore a number of studies focused on calibration of the known models on these markets, e.g. for the central European countries.

Another type of studies investigated the so called convergence models where the domestic short-rate is driven by the short-rate exogenous to the studied economy, e.g. the European short-rate.

3.1. Recent interest rate modeling research

Monfort and Renne [96] developed a no-arbitrage regime switching affine term-structure model (ATSM) for ten euro area sovereign yield curves to estimate the default risk and liquidity risk premium embedded in these yields. The model is driven by five macroeconomic factors $Y_t = (Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}, Y_{5,t})$ and the regime variable z_t where $z_t = (1,0)'$ and $z_t = (0,1)'$ indicates tranquil and crisis periods. They model the innovations of the factors Y_t in the following way:

$$Y_t = \mu z_t + \Phi Y_{t-1} + \Omega(z_t)\varepsilon_t, \tag{3.1}$$

where ε_t are independently and identically normally $N(0,I)$ distributed.

The risk-free short-rate is defined by:

$$r_{t+1} = a_1 z_t + b_1 Y_t. \tag{3.2}$$

Price of a risk-free zero-coupon bond with residual maturity τ is given by:

$$B_{0,t,\tau} = E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \ldots - r_{t+\tau}\right)\right], \tag{3.3}$$

where $E_t^{\mathbb{Q}}$ is the expectation operator at time t under the risk-neutral probability measure $\mathbb{Q}$. The price of a defaultable and illiquid zero-coupon bond of country n at time t if the country has not defaulted before time t is given by:

$$B_{n,t,\tau} = E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \ldots - r_{t+\tau} - \lambda_{n,t+1} - \ldots - \lambda_{n,t+\tau}\right)\right], \tag{3.4}$$

where $\lambda_{n,t+i} = \lambda_{n,t+i}^c + \lambda_{n,t+i}^l$ is the hazard rate for country n consisting of the credit-risk related part $\lambda_{n,t+i}^c$ and the liquidity related part $\lambda_{n,t+i}^l$. They showed that $\lambda_{n,t+i}$ is also an affine function of (z_t, Y_t) and thus $B_{n,t,\tau}$ is an exponentially affine function of (z_t, Y_t) and the resulting yield of a sovereign bond is an affine function of (z_t, Y_t) define by:

$$R_{n,t,\tau} = \frac{1}{\tau}\left(-c_{n,\tau} z_t - f_{n,\tau} y_t\right). \tag{3.5}$$

Monfort and Renne [96] used the German yield curves as the risk-free rate and thus used the level, slope and curvature of the German yield curve as first three factors in Y_t. The other two factors are first two principal components of the 10 year spreads vs. Germany of the four countries France, Italy, Netherlands and Spain. The liquidity driving factor was chosen to be the spread between German sovereign bonds and KfW[15] agency bonds. The model was estimated using monthly data since the start of the euro area excluding Greece using a two step approach. In the first step the dynamics of Y_t and z_t are estimated by maximizing the log-likelihood function. In the second step the risk-neutral dynamics of $\left(z_t, Y_t\right)$ and the hazard rates $\lambda_{n,t}$ are estimated using non-linear least squares. The crisis period was estimated to start in September 2008 until the end of the sample with two short breaks in late 2009 and early 2010. The resulting fit of the credit spreads over the German yield curve is satisfactory with the standard deviation being 18 basis points.

Kaminska, Meldrum and Smith [83] develop a joint three country model for the interest rates of the US, UK and the euro area. The vector of state variables z_t is defined by the first-order VAR process:

$$z_{t+1} = \Phi z_t + \Omega^{1/2}\varepsilon_{t+1}, \tag{3.6}$$

where ε_t are independently and identically normally $N(0, I)$ distributed and Φ and Ω are square matrices of the corresponding dimension. The logarithm of the bond price and the yield to maturity is an affine function of the state vector z_t:

$$\begin{aligned} \ln P_t^n &= A_n + B_n z_t, \\ y_t^n &= \frac{A_n}{n} + \frac{B_n}{n} z_t, \end{aligned} \tag{3.7}$$

where n denotes the number of periods to maturity. They estimated the model using zero-coupon forward rates derived from government yield curves of the US, UK and Germany using two common factors, one individual factor derived from the principal component analysis and the exchange rates.

A similar model just for single country environment has been estimated by Kim and Orphanides [84] for the US or by Joyce, Lindholdt and Sorensen [82] for the UK and for two countries e.g. by Benati [7] or Diez de los Rios [38]. Medvedev [94] analyzed an affine model where the state space variable follows a mean-reverting CIR type process.

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[15] KfW is a German agency which bonds are guaranteed by the Federal Republic of Germany. Thus the difference between the yields of these two bonds should be only liquidity driven.

Geyer, Kossmeier and Pichler [63] studied the yield spreads of the euro area countries over the German bond yields. The reasons for the yield spreads still to be present after the creation of the European Monetary Union are country-specific default risk, due to different fiscal policies or vulnerabilities to macroeconomic shocks, different liquidity in the bond market for different countries and a small probability of breaking up of the euro area and introduction of national currencies. At that time this possibility was considered very small, which is quite in contrast with the developments since 2010.

Their model (originated by Duffie and Singleton [43]) assumes the risk-free short-rate r depends on a set of state-space variables x_t driven by Itô processes. The German yield curve is considered to be the default free term structure. The default of another country C is assumed to be the first jump of the Cox process[16] with intensity $h_C(y_t)$ driven by set of state-space variables y_t driven by Itô processes. In case of default a fraction L_C of the pre-default market value is lost. Duffie and Singleton [43] showed that the short-rate process of such a country is given by $r + S_C = r + h_C L_C$, where S_C is called short-spread. Geyer, Kossmeier and Pichler [63] assumed that both r and S_C are affine function of latent factors x_t and y_t which are driven either by Vasicek or CIR models.

They estimated the model on weekly data for the panel of Austria, Belgium, Italy and Spain for maturities from two to nine years from January 1999 to May 2002. The surprising result of their calibration was that the model is well fit using only two global factors, thus neglecting any individual risk factors of the particular countries.

The opposite result was achieved by Puig [99] where idiosyncratic factors have been found dominant in explaining the spreads. She estimated a linear regression model for the spreads of the 10-year government bond yields of the euro area countries over the German 10-year government yield except Greece and Luxembourg using daily data from January, 1 1999 to December 31, 2005.

[16] See Lando, D.: On Cox processes and credit risky securities. Review of Derivatives Research 2 (1998), 99-120

3.2. *Direct calibration methods*

Direct calibration methods can be used to calibrate models where analytical formulas for the price of the bonds exist. If not, approximate formulas need to be used to derive the price of bonds as described in 3.3 to be able to use the direct methods. The well known methods are briefly described below as well as the min-max method developed by Ševčovič and Urbánová Csajková [117] [118], which is an appealing alternative to the standard methods.

Maximum likelihood estimator

The aim of this approach is to find the set of model parameters which maximizes the logarithm of the likelihood function given the observed state variables. If the state variable is not observable proxy variables could be used like e.g. in Chan, Karolyi, Longstaff and Sanders [22]. Another obstacle to use this approach is the need for an analytical formula for the log likelihood function, which is known only in some cases. Furthermore, even if the log likelihood function is known the maximum does not necessarily exist as pointed out by Stehlíková [112] in the case of a discretized CKLS model. If the log likelihood function is not known some approximate likelihood functions have been proposed like in Chen and Scott [24].

Generalized method of moments

The Generalized method of moments was developed by Hansen [65]. The parameter estimates are obtained so that the moments of the discretized model for the short rate are fit as good as possible. The moments are approximated by their sample counterparts. The major advantage of this method is that the distribution of the residuals does not have to be normal. This method has been used e.g. by Chan, Karolyi, Longstaff and Sanders [22].

Markov Chain Monte Carlo

The Markov Chain Monte Carlo method assumes a distribution of the parameter vector θ and the state-space variables X conditionally on the observed prices Y. The method is based on generating random samples from this distribution $p\left(\theta, X \middle| Y\right)$. However this distribution is usually not known and only its marginal distributions

$p(\theta|X,Y)$ and $p(X|\theta,Y)$ are known. The method assumes that these two marginal distributions can fully describe the joint distribution. This method was studied e. g. by Feng and Xie [59].

Kalman filter

In a multifactor model with unobserved variables the aim is to estimate both the parameters of the model and the time-dynamics of the unobserved variables. This problem is solved by the Kalman filter approach. The approach is based on a state-space representation of the term-structure model with the state-space variables being unobservable and following Markov process. Examples of usage of this approach are Benati [7], Chen and Scott [24], Duan and Simonato [40], Geyer, Kossmeier and Pichler [63] and Kaminska, Meldrum and Smith [83].

Two-phase minmax method

Ševčovič and Urbánová Csajková [117] [118] and Urbánová Csajková [121] proposed a new two-phase min-max method to calibrate a one factor CIR model. In the first phase the four initial CIR model parameter space is reduced to three essential parameters by introducing the following transformation:

$$\beta = e^{-\eta}, \ \xi = \frac{\kappa + \lambda + \eta}{2\eta}, \ \rho = \frac{2\kappa\theta}{\sigma^2}. \tag{3.8}$$

Proposition 11. *(Urbánová Csajková [121], p. 33) The transformation (3.8) is a smooth mapping and the preimage of the domain of the new parameters β, ξ and ρ is a smooth λ-parameterized curve in R^4.*

In the space of these three new parameters the weighted least squares sum in the following form is minimized:

$$U(\beta,\xi,\rho) = \frac{1}{m}\sum_{j=1}^{m}\frac{1}{n}\sum_{i=1}^{n}(R_j^i - \tilde{R}_j^i)^2\tau_j^2, \tag{3.9}$$

where R_j^i is the market yield curve at time i with m maturity buckets, $\tilde{R}_j^i$ is the theoretical yield curve modeled by the CIR model with parameters β, ξ and ρ. For the sake of better numerical properties they transformed the cost function (3.9) to an equivalent form:

$$U(\beta,\xi,\rho)=\frac{1}{m}\sum_{j=1}^{m}\Big(\big(\tau_j E(R_j)-B(\tau_j)E(R_0)+\ln A(\tau_j)\big)^2+D(\tau_j R_j+B(\tau_j)R_0)\Big), \quad (3.10)$$

where $E(X)$ and $D(X)$ are mean and standard deviation of a random variable X and R_0 is the observed short-rate. For generic input data it is assumed that a unique global minimum of (3.10) exists. Let us denote this global minimum $\hat{\beta}$, $\hat{\xi}$ and $\hat{\rho}$. The global minimum is searched for using a variant of evolution algorithm with 300 generations.

In the second phase the logarithm of the likelihood function of the CIR process is maximized.

Proposition 12. *(Bergstrom [8]) The logarithm of the likelihood function of the CIR process has the following form:*

$$\ln L(\kappa,\sigma,\theta)=-\frac{1}{2}\sum_{t=2}^{n}\left(\ln v_t^2+\frac{\varepsilon_t^2}{v_t^2}\right), \quad \text{(3.11)}$$

where $v_t^2=\dfrac{\sigma^2}{2\kappa}(1-e^{-2\kappa\Delta t})r_{t-\Delta t}$, $\varepsilon_t=r_t-e^{-\kappa\Delta t}r_{t-\Delta t}-\theta(1-e^{-\kappa\Delta t})$.

The maximum of (3.11) is searched for among all the triplets $(\kappa_\lambda,\sigma_\lambda,\theta_\lambda)$, for which the function U is equal to the global minimum $U(\hat{\beta},\hat{\xi},\hat{\rho})$:

$$\ln L^{r}=\ln L(\kappa_{\bar{\lambda}},\sigma_{\bar{\lambda}},\theta_{\bar{\lambda}})=\max_{\lambda}\ln L(\kappa_\lambda,\sigma_\lambda,\theta_\lambda). \quad (3.12)$$

The resulting $\bar{\kappa}=\kappa_{\bar{\lambda}}$, $\bar{\sigma}=\sigma_{\bar{\lambda}}$, $\bar{\theta}=\theta_{\bar{\lambda}}$ and $\bar{\lambda}$ are considered to be the estimates of the parameters of the CIR model and the corresponding market price of risk.

The proposed algorithm was tested by calibrating the EURO-LIBOR[17], USD-LIBOR[17], EURIBOR[18] yield curves, where it achieved good results and on interbank rates for Czech Republic, Hungary, Poland and Slovakia where the results were sufficient only for PRIBOR[19].

Extended minmax method for a multi-factor model

Sutóris [115] in his master's thesis studied the extended version of the minmax method of Ševčovič and Urbánová Csajková (see above) for a multi-factor CIR model. It is not possible to use the original method directly as the decomposition of the short rate to the individual factors in (2.43) is not observable. It is necessary to estimate the value of the individual factors for every time observation along the estimation of the parameters. However, the number of parameters equals $N \times n + 4 \times n$, where N is the number of time observations and n is the number of CIR factors, which is very high. Sutóris proposed to effectively solve this problem by solving the problem of factor estimation as an inner problem in the estimation if the CIR parameters. Let us introduce the transformation of parameters as in (3.8):

$$\beta_i = e^{-\eta_i}, \ \xi_i = \frac{\kappa_i + \lambda_i + \eta_i}{2\eta_i}, \ \rho_i = \frac{2\kappa_i\theta_i}{\sigma_i^{\ 2}}. \tag{3.13}$$

Let us define the parameter vectors $\bar{\beta} = \{\beta_i, i = 1\ldots n\}$, $\bar{\xi} = \{\xi_i, i = 1\ldots n\}$ and $\bar{\sigma} = \{\sigma_i, i = 1\ldots n\}$ and the vector of the factors $\bar{r} = \{r_i, i = 1\ldots n\}$.

Proposition 13. *The modeled yield to maturity of the multifactor CIR model (2.43) is given by:*

$$\tilde{R}(\bar{\beta},\bar{\xi},\bar{\rho},\bar{r},\tau_j) = \frac{1}{\tau}\sum_{i=1}^{m}\left(-\ln A(\beta_i,\xi_i,\rho_i,\tau) + B(\beta_i,\xi_i,\rho_i,\tau)r_i\right). \tag{3.14}$$

[17] LIBOR – London interbank offered rate are interest rates published by the British Bankers' Association and calculated by Thompson Reuters. The rates are published for 10 currencies and maturities varying from overnight up to 1 year. The rates are computed as trimmed average of unsecured interbank deposits of contributor banks in the London interbank market.
[18] EURIBOR – Euro interbank offered rate are interest rates in the euro area interbank market computed as trimmed average of contributor banks for maturities from 1 week up to 1 year.
[19] PRIBOR – Prague interbank offered rate is the interbank rate published by the Czech National Bank.

The cost function is defined by:

$$U(\overline{\beta},\overline{\xi},\overline{\rho},\overline{r}(\cdot)) = \sum_{j=1}^{k}\sum_{i=1}^{n}(R_j^i - \tilde{R}^i(\overline{\beta},\overline{\xi},\overline{\rho},\overline{r}(i),\tau_j))^2\tau_j^2, \tag{3.15}$$

where R_j^i is the real market yield curve at time i with k maturities and $\tilde{R}^i$ is the modeled yield to maturity defined by (3.14) and $r(\cdot) = \{r(t), t \in 1 \ldots n\}$. As $r(\cdot)$ is not known the loss function (3.15) is minimized over the parameters $\overline{\beta}$, $\overline{\xi}$, $\overline{\rho}$ and the short-rate factor decomposition $r(\cdot)$:

$$\begin{aligned} &\min_{\overline{\beta},\overline{\xi},\overline{\rho},\overline{r}(\cdot)} U(\overline{\beta},\overline{\xi},\overline{\rho},\overline{r}(\cdot)), \\ &r_i(t) \geq 0, \sum_{i=1}^{m} r_i(t) = r(t). \end{aligned} \tag{3.16}$$

The problem (3.16) is solved in a two step approach:

$$\min_{\overline{\beta},\overline{\xi},\overline{\rho}} \min_{\overline{r}(\cdot)} U(\overline{\beta},\overline{\xi},\overline{\rho},\overline{r}(\cdot)). \tag{3.17}$$

3.3. Approximate solutions to bond price in interest rate models

Since Chan, Karolyi, Longstaff and Sanders [22] published their comparison of one-factor models, models where no analytical solution exists gained prominence in the research. As in these models no analytical formula for the price of the zero-coupon bond exists approximate solutions had to be found in order to price interest rate derivatives using these classes of models. One avenue was explored by Choi and Wirjanto [25], who studied a slightly modified version of the CKLS model defined by:

$$dr = \left(a + br + \lambda(t,r)\sigma r^{\gamma}\right)dt + \sigma r^{\gamma} dW. \tag{3.18}$$

where W is a standard Wiener process and a, b, σ and γ are constants and $\lambda(t,r)$ is the market price of risk.

Proposition 14. *Suppose that the short rate processes is defined by (3.18). The price of a zero-coupon bond is defined by:*

$$-\frac{\partial P}{\partial \tau}+\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 P}{\partial r^2}+(a+br)\frac{\partial P}{\partial r}-rP=0\,, \qquad (3.19)$$

with the initial condition $P(t,0,r)=1$ *for all* t *and* $r>0$.

Such an equation does have an analytical solution only for $\gamma=0$ and $\gamma=0.5$, which correspond to Vasicek and CIR models.

Proposition 15. *(Choi and Wirjanto [25], Theorem 2). The approximate price of a zero-coupon bond* P^{ap} *satisfying (3.19) is given by:*

$$\begin{aligned} \ln P^{ap}(\tau,r) &= -rB+\frac{a}{b}(\tau-B)+\left(r^{2\gamma}+q\tau\right)\frac{\sigma^2}{4b}\left[B^2+\frac{2}{b}(\tau-B)\right]- \\ &\quad -q\frac{\sigma^2}{8b^2}\left[B^2(2b\tau-1)-2B\left(2\tau-\frac{3}{b}\right)+2\tau^2-\frac{6\tau}{b}\right], \qquad . \qquad (3.20) \\ q(r) &= \gamma(2\gamma-1)\sigma^2 r^{2(2\gamma-1)}+2\gamma r^{2\gamma-1}(a+br), \\ B(\tau) &= \left(e^{b\tau}-1\right)/b. \end{aligned}$$

Stehlíková [112] studied the uniqueness of the solution to (3.19), the accuracy of the approximate formula (3.20) and proposed a better approximate solution with a higher accuracy.

Definition 7 *(Stehlíková [112], Definition 1). A complete solution to (3.19) is a function* $P(\tau,r)$ *with continuous partial derivatives satisfying* $P(0,r)=0$, $\left|P(\tau,r)\right|\le Me^{-mr^\delta}$ *and* $\left|\frac{\partial P}{\partial r}\right|\le M$ *for all* $r>0$ *and* $t\in(0,T)$ *where* M, m *and* δ *are constants.*

Proposition 16. *(Stehlíková [112], Theorem 2). For* $\frac{1}{2}\le\gamma<\frac{3}{2}$ *and* $2a>\sigma^2$ *there exist a unique complete solution to (3.19).*

The better approximate solution suggested by Stehlíková is defined by:

$$\ln P^{ap2}(\tau,r) = \ln P^{ap}(\tau,r) - c_5(r)\tau^5 - c_6(r)\tau^6, \qquad (3.21)$$

where c_5 and c_6 are lengthy functions of a, b, γ, σ and r.

Proposition 17. *(Stehlíková [112], Theorem 4). The difference between* $\ln P^{ap2}$ *and the exact solution to* (3.19) is $o(\tau^6)$ for $\tau \to 0^+$.

Stehlíková [113] suggested another approximate formula for the solution to (3.19). The idea is to take the explicit solution to the bond price for the Vasicek model and substitute the constant volatility in the by the CKLS type volatility. The formula has the following form:

$$\ln P^{ap3}(\tau,r) = \left(\frac{a}{b} + \frac{\sigma^2 r^{2\gamma}}{2b^2}\right)\left(\frac{1-e^{b\tau}}{b} + \tau\right) + \frac{\sigma^2 r^{2\gamma}}{4b^3}\left(1-e^{b\tau}\right)^2 + \frac{1-e^{b\tau}}{b} r. \qquad (3.22)$$

Proposition 18. *(Stehlíková [113]). The difference between* $\ln P^{ap3}$ *and the exact solution to* (3.19) is $o(\tau^3)$ for $\tau \to 0^+$.

The accuracy is not better than the accuracy of the previously mention approximations; however the idea was picked up by Halgašová [64] to develop an approximation formula for the two-factor CKLS model. Halgašová [64] derived the explicit solution to the two-factor Vasicek model with correlated factors in the form of:

$$P(t,\tau,r_1,r_2) = A(\tau)e^{-B_1(\tau)r_1 - B_2(\tau)r_2}$$
$$\ln A(\tau) = \ln A_1(\tau) + \ln A_2(\tau) + \rho\frac{\sigma_1\sigma_2}{\kappa_1\kappa_2}\left[\tau - B_1(\tau) - B_2(\tau) + \frac{1-e^{(\kappa_1+\kappa_2)\tau}}{\kappa_1+\kappa_2}\right], \qquad (3.23)$$

where $A_i(\tau)$ and $B_i(\tau)$ are defined by (2.12).

She then proposed the approximate solution to the bond price for a two-factor CKLS model in the form of:

$$
\begin{aligned}
&P^{ap4}(t,\tau,r_1,r_2) = A^{ap}(\tau)e^{-B_1(\tau)r_1 - B_2(\tau)r_2},\\
&\qquad \ln A^{ap}(\tau) = \ln A_1^{ap}(\tau) + \ln A_2^{ap}(\tau) + \rho\frac{\sigma_1\sigma_2 r_1^{\gamma} r_2^{\gamma}}{b_1 b_2}\left[\tau - B_1(\tau) - B_2(\tau) + \frac{e^{(b_1+b_2)\tau} - 1}{b_1 + b_2}\right],\\
&\qquad \ln A_i^{ap}(\tau) = \left(\tau - B_i(\tau)\right)\left[\frac{a_i}{b_i} + \frac{\sigma_1^2 r_1^{2\gamma}}{2b_i^2}\right] + \frac{\sigma_1^2 r_1^{2\gamma}}{4b_i^3}B_i^2(\tau),\\
&\qquad\quad B_i(\tau) = \frac{e^{b_i\tau} - 1}{b_i}.
\end{aligned}
\tag{3.24}
$$

Proposition 19. ***(****Halgašová [64]****).*** *The difference between* $\ln P^{ap4}$ *and the exact solution* is of the order $o(\tau^3)$ for $\tau \to 0^+$.

The main results of her work should be published soon. One of the observations in her work was that the cost function's sensitivity to the correlation parameters was very low.

The same idea was used by Zíková and Stehlíková [129] to derive the approximate solution to the zero-coupon bond price for a CKLS convergence model based on a Vasicek type convergence model.

3.4. Analysis of the convergence models

Corzo and Schwartz [28] proposed and described a model for a country entering the euro area based on a Vasicek model. The trend in the domestic interest rate is to converge to the European rate with a spread θ_d while the European rate is modeled by the Vasicek process. The model is described by (2.37). The Wiener processes increments for the domestic and European interest rate dW_d and dW_u are constantly correlated with the correlation coefficient ρ.

The reason for such a model is that after the irrevocably fixed exchange rate is set, the risk-free rate of the two countries should be the same, otherwise arbitrage would exist. They also observed the convergence of interest rates in the countries before forming the euro area. However, the government bond yields can vary due to different credit risk in individual countries.

Proposition 20. *Suppose that the short rate processes for the domestic and European interest rates are given by (2.37),* $\lambda_d(t, r_t^d) = \lambda_d$ *and* $\lambda_e(t, r_t^e) = \lambda_e$*. The price of the domestic bond is given by:*

$$P(r_e, r_d, \tau) = e^{A(\tau) - D(\tau) r_d - U(\tau) r_e}, \tag{3.25}$$

where the functions $A(\tau)$, $D(\tau)$ *and* $U(\tau)$ *are given by:*

$$\begin{aligned} D(\tau) &= \frac{1 - \mathrm{e}^{-\kappa_d \tau}}{\kappa_d}, \\ U(\tau) &= \frac{\kappa_d D(\tau)(1 - e^{-\kappa_e \tau})}{\kappa_e}, \\ A(\tau) &= \frac{1}{2}\tau \begin{bmatrix} -2\theta_d D(\tau) + U(\tau)(-2\kappa_e \theta_e + U(\tau)\sigma_e^2 + 2\sigma_e \lambda_e) + \\ + D(\tau)(2U(\tau)\rho\sigma_d\sigma_e + \sigma_d(2\lambda_d + D(\tau)\sigma_d)) \end{bmatrix}. \end{aligned} \tag{3.26}$$

Using the generalized method of moments they estimated the parameters for 1-month interbank rates in Spain. The result was quite good both in-sample[20] and out-of-sample[21]. When used to price zero-coupon bonds this model provided lower errors then the standard Vasicek model, especially for longer maturities.

Already in their paper Corzo and Schwartz mention that it is possible to do a similar analysis also for the CIR model. This is the topic of the paper by Lacko and Stehlíková [88] and Lacko's master thesis [87]. They investigate the model defined by (2.38). The Wiener processes increments dW_d and dW_u are also constantly correlated with the correlation coefficient ρ. The price of a zero-coupon domestic bond is also searched for in the form of (3.25).

[20] in-sample – the model fit is compared using data which have been used in the parameter estimation.
[21] out-of-sample – the model fit is compared on a data set, which has not been used in the parameter estimation.

Proposition 21. *Suppose that the short rate processes for the domestic and European interest rates is given by (2.38) and the price of a zero-coupon domestic bond is of the form (3.25). The partial differential equation for the price of the domestic zero-coupon bond is defined by:*

$$-\frac{\partial P}{\partial t}+(\theta_d+\kappa_d(r_u-r_d)-\lambda_d(r_d,r_u)\sigma_d\sqrt{r_d})\frac{\partial P}{\partial r_d}+(\kappa_u(\theta_u-r_u)-\bar{\lambda}_u r_u\sigma_u)\frac{\partial P}{\partial r_u}$$
$$+\frac{\sigma_d^2 r_d}{2}\frac{\partial^2 P}{\partial r_d^2}+\frac{\sigma_u^2 r_u}{2}\frac{\partial^2 P}{\partial r_u^2}+\rho\sigma_d\sigma_u\sqrt{r_u r_d}\,\frac{\partial^2 P}{\partial r_u\partial r_d}-r_d P=0 \quad , \qquad (3.27)$$

where $\bar{\lambda}_u\sqrt{r_u}$ *is the European market price of risk, where* $\bar{\lambda}_u>0$ *is a constant and* $\bar{\lambda}_d(r_d,r_u)$ *is the domestic market price of risk.*

Proposition 22. *Suppose that* $\rho=0$ *and* $\lambda_d(r_d,r_u)=\bar{\lambda}_d\sqrt{r_d}$. *The partial differential equation governing the price of the zero-coupon domestic bond (3.27) is transformed into the following system of ordinary differential equations for the functions* $A(\tau)$, $D(\tau)$ *a* $U(\tau)$ *in (3.25):*

$$\begin{aligned}\dot{D}&=1-(\kappa_d+\bar{\lambda}_d\sigma_d)-\frac{\sigma_d^2}{2}D^2,\\ \dot{U}&=\kappa_d D=(\kappa_u+\bar{\lambda}_u\sigma_u)U-\frac{\sigma_u^2}{2}U^2,\\ \dot{A}&=-\theta_d D-\theta_u\kappa_u U,\end{aligned} \qquad (3.28)$$

with the initial condition $A(0)=D(0)=U(0)=0$.

Proposition 23. *The solution to the ordinary differential equation for* $D(\tau)$ *is given by:*

$$D(\tau)=\frac{D_+(1-\mathrm{e}^{k\tau})}{1-(D_+/D_-)\mathrm{e}^{k\tau}}, \qquad (3.29)$$

where $k=\sqrt{(\kappa_d+\bar{\lambda}_d\sigma_d)^2+2\sigma_d^2}$, $D_+=-\dfrac{\kappa_d+\bar{\lambda}_d\sigma_d+k}{\sigma_d^2}<0$, $D_-=-\dfrac{\kappa_d+\bar{\lambda}_d\sigma_d-k}{\sigma_d^2}>0$.

The equations for $A(\tau)$ and $U(\tau)$ could be solved only numerically. They also showed that the difference between the price of a zero-coupon bond in case of zero and non-zero correlation is of the order τ^3 where τ is the remaining time to maturity. Lacko calibrated the model on O/N BRIBOR[22] and EONIA[23] data before Slovak republic entered the euro area.

[22] O/N BRIBOR is the overnight interbank rate on the Slovak market, which was published by the National Bank of Slovakia before January 1, 2009.
[23] EONIA – Euro Overnight Index Average is calculated as a weighted average of all unsecured overnight deposits on the interbank market among the corresponding banks.

4. Multifactor interest rate model

In this chapter a new panel model for sovereign interest rates of the euro area countries is developed. The idea, inspired by Leško [90], is based on the multifactor CIR or CKLS model which we broadened to a multiple equation model for multiple countries.

4.1. *Description of the model*

Let us assume that the unobserved short-rate interest rate r_t^i of a given euro area country i at every point in time t is defined by a two-factor CIR model .The short-rate is a sum of a risk-free short-rate rf_t and a credit spread cs_t^i of country i over the risk-free rate and both these factors are unobserved in the market. As all the euro area countries have the same currency, the Euro, in a no-arbitrage environment the risk-free rate has to be common for all countries. On the other hand the credit spreads mirror the idiosyncratic credit risk inherent in the investments in sovereign debt of a particular country. Therefore the credit spreads are unique for each country. In principle, it is possible for two countries to have the same credit spread if the risk profile of these two countries is the same. However, this cannot be assumed in advance and for all countries. Let us also assume that both the risk-free rate and the credit spreads are described by the one factor CIR model (2.13). The model can be written in the following form:

$$r_t^i = rf_t + cs_t^i, \tag{4.1}$$

$$drf_t = \kappa(\theta - rf_t)dt + \sigma\sqrt{rf_t}dW_t \tag{4.2}$$

$$dcs_t^i = \kappa_i(\theta_i - cs_t^i)dt + \sigma_i\sqrt{cs_t^i}dW_t^i, \text{ for } i = 1\ldots n, \tag{4.3}$$

where W_t and W_t^i are standard Wiener processes, $i = 1\ldots n$ represents individual Member States of the euro area and t is time. Here coefficients κ, θ and σ are the mean-reversion spread, long-term equilibrium rate and volatility, respectively, for the risk-free rate. The coefficients κ_i, θ_i and σ_i have the same interpretation for the credit spread of country i. This notation will be used throughout the whole thesis.

4.2. The price of a zero-coupon sovereign bond

Theorem 1. *Suppose that the short rate processes r_t^i for euro area countries are defined by (4.1), (4.2) and (4.3) and the increments of the Wiener processes dW_t and dW_t^i are constantly correlated with correlation coefficient ρ_i. The governing equation for the price $P^i = P^i(t,T,r_t^i)$ of a zero-coupon sovereign bond of country i with remaining time to maturity τ at time point t is defined by:*

$$-\frac{\partial P^i}{\partial \tau}+\left[\kappa(\theta - rf_t)+\lambda\sqrt{rf_t}\sigma\right]\frac{\partial P^i}{\partial rf_t}+\left[\kappa_i(\theta_i - cs_t^i)+\lambda_i\sqrt{cs_t^i}\sigma_i\right]\frac{\partial P^i}{\partial cs_t^i}$$
$$+\frac{1}{2}\sigma^2 rf_t\frac{\partial^2 P^i}{\partial (rf_t)^2}+\sigma\sigma_i\rho_i\sqrt{rf_t}\sqrt{cs_t^i}\frac{\partial^2 P^i}{\partial rf_t\partial cs_t^i}+\frac{1}{2}\sigma_i^2 cs_t^i\frac{\partial^2 P^i}{\partial (cs_t^i)^2}-(rf_t+cs_t^i)P^i=0, \quad (4.4)$$

with the initial condition $P^i(t,0,rf_t,cs_t^i)=1$ for all t, $rf_t>0$ and cs_t^i.

Proof: To derive the price of a zero-coupon sovereign bond the methodology of Kwok [86] for the general two-factor short-rate model is followed. Using Proposition 2 (Multidimensional Itô's lemma) the stochastic differential of the price P^i is defined by:

$$dP^i=\begin{pmatrix}\frac{\partial P^i}{\partial t}+\kappa(\theta-rf_t)\frac{\partial P^i}{\partial rf_t}+\kappa_i(\theta_i-cs_t^i)\frac{\partial P^i}{\partial cs_t^i}+\frac{1}{2}\sigma^2 rf_t\frac{\partial^2 P^i}{\partial (rf_t)^2}+ \\ +\sigma\sigma_i\rho_i\sqrt{rf_t}\sqrt{cs_t^i}\frac{\partial^2 P^i}{\partial rf_t\partial cs_t^i}+\frac{1}{2}\sigma_i^2 cs_t^i\frac{\partial^2 P^i}{\partial (cs_t^i)^2}\end{pmatrix}dt$$
$$+\sigma\sqrt{rf_t}\frac{\partial P^i}{\partial rf_t}dW_t+\sigma_i\sqrt{cs_t^i}\frac{\partial P^i}{\partial cs_t^i}dW_t^i. \quad (4.5)$$

Let us denote the terms corresponding to dt, dW_t and dW_t^i in (4.5) as $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{\sigma}^i$. Then (4.5) can be written in the form of:

$$dP^i=\tilde{\mu}dt+\tilde{\sigma}dW_t+\tilde{\sigma}^i dW_t^i. \quad (4.6)$$

Let us construct a risk-less portfolio Π composed of three bonds with different maturities T_1, T_2 and T_3 where Δ_1, Δ_2 and respectively Δ_3 are number of bonds in the portfolio:

$$\Pi = \Delta_1 P^i(T_1) + \Delta_2 P^i(T_2) + \Delta_3 P^i(T_3)\,. \tag{4.7}$$

The change in the value of the portfolio is:

$$d\Pi = \Delta_1 dP^i(T_1) + \Delta_2 dP^i(T_2) + \Delta_3 dP^i(T_3)\,, \tag{4.8}$$

what can be rewritten in the following way by substituting (4.6) into (4.8):

$$\begin{aligned} d\Pi = \left[\Delta_1\tilde{\mu}(T_1) + \Delta_2\tilde{\mu}(T_2) + \Delta_3\tilde{\mu}(T_3)\right]\; dt + \left[\Delta_1\tilde{\sigma}(T_1) + \Delta_2\tilde{\sigma}(T_2) + \Delta_3\tilde{\sigma}(T_3)\right]\; dW_t \\ + \left[\Delta_1\tilde{\sigma}^i(T_1) + \Delta_2\tilde{\sigma}^i(T_2) + \Delta_3\tilde{\sigma}^i(T_3)\right]\; dW_t^i \end{aligned}\,. \tag{4.9}$$

To obtain a risk less portfolio the stochastic part of the equation (4.9) must be eliminated. This is achieved when the terms corresponding to dW_t and dW_t^i in (4.9) are zero, which can be obtained by a suitable choice of Δ_1, Δ_2 and Δ_3 such that:

$$\begin{aligned} \Delta_1\tilde{\sigma}(T_1) + \Delta_2\tilde{\sigma}(T_2) + \Delta_3\tilde{\sigma}(T_3) = 0, \\ \Delta_1\tilde{\sigma}^i(T_1) + \Delta_2\tilde{\sigma}^i(T_2) + \Delta_3\tilde{\sigma}^i(T_3) = 0. \end{aligned} \tag{4.10}$$

In such a case the change in the value of the portfolio is deterministic:

$$d\Pi = \left[\Delta_1\tilde{\mu}(T_1) + \Delta_2\tilde{\mu}(T_2) + \Delta_3\tilde{\mu}(T_3)\right] dt\,. \tag{4.11}$$

In a no arbitrage world the yield of such a portfolio must be equal to the instantaneous rate resulting in the following equation:

$$d\Pi = \left[\Delta_1\tilde{\mu}(T_1) + \Delta_2\tilde{\mu}(T_2) + \Delta_3\tilde{\mu}(T_3)\right] dt = r\Pi = r\left[\Delta_1 P^i(T_1) + \Delta_2 P^i(T_2) + \Delta_3 P^i(T_3)\right]. \tag{4.12}$$

Combining (4.10) and (4.12) we get the following linear system for the mix of bonds in the portfolio:

$$\begin{pmatrix} \tilde{\sigma}(T_1) & \tilde{\sigma}(T_2) & \tilde{\sigma}(T_3) \\ \tilde{\sigma}^i(T_1) & \tilde{\sigma}^i(T_2) & \tilde{\sigma}^i(T_3) \\ \tilde{\mu}(T_1) - rP^i(T_1) & \tilde{\mu}(T_2) - rP^i(T_2) & \tilde{\mu}(T_3) - rP^i(T_3) \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.13}$$

The linear system (4.13) has a nontrivial solution only when the matrix of the system has not full rank, which means that there must exist functions $\lambda(t, rf_t, cs_t^i)$ and $\lambda_i(t, rf_t, cs_t^i)$ such that:

$$\tilde{\mu}(T_j) - rP^i(T_j) = \lambda(t, rf_t, cs_t^i)\tilde{\sigma}(T_j) + \lambda_i(t, rf_t, cs_t^i)\tilde{\sigma}^i(T_j). \tag{4.14}$$

The functions $\lambda(t, rf_t, cs_t^i)$ and $\lambda_i(t, rf_t, cs_t^i)$ are market prices of risk and do not depend on the maturity T because the maturities in (4.7) the above calculations have been chosen arbitrarily.

Let us denote the remaining time to maturity $\tau = T - t$. By substituting τ and (4.5) into (4.14) we get the partial differential equation for the price of a zero coupon bond given by (4.4). □

Theorem 2. *Suppose that the correlation coefficient $\rho_i = 0$ for $i = 1 \ldots n$ in (4.4) and the market prices of risk are the same as in one-factor CIR models $\lambda(t, r) = \bar{\lambda}\sqrt{rf_t}$ and $\lambda_i(t, r) = \bar{\lambda}_i\sqrt{cs_t^i}$. The solution to (4.4) is obtained in a separated form:*

$$\begin{aligned} P^i(t, \tau, r_t^i) &= P_{rf}^i(t, \tau, rf_t) P_{cs}^i(t, \tau, cs_t^i) \\ &= \boldsymbol{A}(\tau)\exp\left(-B(\tau)rf_t - B_i(\tau)cs_t^i\right) = A(\tau)A_i(\tau)\exp\left(-B(\tau)rf_t - B_i(\tau)cs_t^i\right) \end{aligned}, \tag{4.15}$$

where $P_{rf}^i(t, \tau, rf_t)$ and $P_{cs}^i(t, \tau, cs_t^i)$ are solutions to the one-factor CIR models for rf_t and cs_t^i in the form of (2.14):

$$\begin{aligned} &P_{rf}^i(t, \tau, rf_t) = A(\tau)e^{-B(\tau)rf_t}, P_{cs}^i(t, \tau, cs_t^i) = A_i(\tau)e^{-B_i(\tau)cs_t^i}, \\ &A(\tau) = \left[\frac{\eta e^{(\kappa+\bar{\lambda}\sigma+\eta)\tau/2}}{e^{\eta\tau}-1}B(\tau)\right]^{2\kappa\tau/\sigma^2}, \ A_i(\tau) = \left[\frac{\eta_i e^{(\kappa_i+\bar{\lambda}_i\sigma_i+\eta_i)\tau/2}}{e^{\eta_i\tau}-1}B_i(\tau)\right]^{2\kappa\,\tau/\sigma_i^2}, \\ &B(\tau) = \frac{2[e^{\eta\tau}-1]}{(\kappa+\bar{\lambda}\sigma+\eta)[e^{\eta\tau}-1]+2\eta}, \ B_i(\tau) = \frac{2[e^{\eta_i\tau}-1]}{(\kappa_i+\bar{\lambda}_i\sigma_i+\eta_i)[e^{\eta_i\tau}-1]+2\eta_i}, \\ &\eta = \sqrt{(\kappa+\bar{\lambda}\sigma)^2+2\sigma^2}, \ \eta_i = \sqrt{(\kappa_i+\bar{\lambda}_i\sigma_i)^2+2\sigma_i^2}. \end{aligned} \tag{4.16}$$

Proof. The partial derivatives of (4.15) are:

$$\frac{\partial P^i}{\partial \tau} = \frac{\dot{\boldsymbol{A}}}{\boldsymbol{A}} P^i + (-\dot{B} rf - \dot{B}_i cs_t^i) P^i = \frac{\dot{A}}{A} P^i + \frac{\dot{A}_i}{A_i} P^i + (-\dot{B} rf - \dot{B}_i cs_t^i) P^i,$$
$$\frac{\partial P^i}{\partial rf_t} = -B(\tau) P^i, \frac{\partial P^i}{\partial cs_t^i} = -B_i(\tau) P^i, \tag{4.17}$$
$$\frac{\partial^2 P^i}{\partial (rf_t)^2} = B^2(\tau) P^i, \frac{\partial^2 P^i}{\partial (cs_t^i)^2} = B_i^2(\tau) P^i.$$

By substituting (4.17) into (4.4) we obtain:

$$-\frac{\dot{\boldsymbol{A}}}{\boldsymbol{A}} P^i - (-\dot{B} rf_t - \dot{B}_i rfcs_t^i) P^i - \left[\kappa(\theta - rf_t) + \lambda \sqrt{rf_t}\,\sigma\right] B(\tau) P^i$$
$$-\left[\kappa_i(\theta_i - cs_t^i) + \lambda_i \sqrt{cs_t^i}\,\sigma_i\right] B_i(\tau) P^i + \frac{1}{2}\sigma^2 rf_t B^2(\tau) P^i + \frac{1}{2}\sigma_i^2 cs_t^i B_i^2(\tau) P^i - \left(rf_t + cs_t^i\right) P^i = 0 \quad .(4.18)$$

By substituting the market price functions into (4.18) and by eliminating the nonzero bond price P^i we obtain:

$$-\frac{\dot{\boldsymbol{A}}}{\boldsymbol{A}} - (-\dot{B} rf_t - \dot{B}_i cs_t^i) - \left[\kappa(\theta - rf_t) + \bar{\lambda}\sigma rf_t\right] B(\tau)$$
$$-\left[\kappa_i(\theta_i - cs_t^i) + \bar{\lambda}_i \sigma_i cs_t^i\right] B_i(\tau) + \frac{1}{2}\sigma^2 rf_t B^2(\tau) + \frac{1}{2}\sigma_i^2 cs_t^i B_i^2(\tau) - (rf_t + cs_t^i) = 0 \quad . \tag{4.19}$$

As (4.19) must hold true for all values of rf_t and cs_t^i the terms corresponding to rf_t and cs_t^i must sum up to zero which gives the following system of ordinary differential equations:

$$\frac{\dot{\boldsymbol{A}}}{\boldsymbol{A}} + \kappa\theta B + \kappa_i \theta_i B_i = 0,$$
$$\dot{B} + (\kappa - \bar{\lambda}\sigma) B + \frac{1}{2}\sigma^2 B^2 - 1 = 0, \tag{4.20}$$
$$\dot{B}_i + (\kappa_i - \bar{\lambda}_i \sigma_i) B_i + \frac{1}{2}\sigma_i^2 B_i^2 - 1 = 0,$$

with the initial conditions $\boldsymbol{A}(0) = 1$, $B(0) = 0$ and $B_i(0) = 0 \; for \; i = 1 \ldots n$.

The ordinary differential equations for $B(\tau)$ and $B_i(\tau)$ are the same as in the case of one-factor CIR model. It is well known (see e.g. Kwok [86]) that the solution to the ordinary differential equations for $B(\tau)$ and $B_i(\tau)$ admits the solution in the form of (4.16). The ordinary differential equation for $\boldsymbol{A}(\tau)$ can be rewritten in the following way by introducing the substitution $\boldsymbol{a} = \ln(\boldsymbol{A})$:

$$\dot{\boldsymbol{a}} = -\kappa\theta B - \kappa_i\theta_i B_i. \tag{4.21}$$

The solution to (4.21) can be obtained by integrating:

$$\boldsymbol{a}(\tau) = \int_0^\tau \left(-\kappa\theta B(s) - \kappa_i\theta_i B_i(s)\right)ds = -\int_0^\tau \kappa\theta B(s)ds - \int_0^\tau \kappa_i\theta_i B_i(s)ds. \tag{4.22}$$

Both integrals in (4.22) are exactly the same as in the case of the one-factor CIR model therefore (4.22) can be rewritten in the following way:

$$\ln(\boldsymbol{A}(\tau)) = \ln(A(\tau)) + \ln(A_i(\tau)) = \ln(A(\tau)A_i(\tau)), \tag{4.23}$$

Which yields the desired outcome that $\boldsymbol{A}(\tau) = A(\tau)A_i(\tau)$, where $A(\tau)$ and $A_i(\tau)$ are defend by (4.16). □

The assumption of zero correlation between the risk-free rate rf_t and credit spreads cs_t^i in Theorem 2 might be viewed as non-realistic. However, theoretically, there is no reason why the risk-free rate process should depend on the credit spread processes. Although the level of the risk-free interest rate increases the costs of financing of the country, the credit spread of this country depends more on its structural position and views of the investors than on the cost of financing. Therefore the zero correlation assumption can be justified from the theoretical point of view. Note that no assumption on the correlation between the credit spreads of the individual countries is required to calculate the prices of their zero-coupon bonds.

4.3. Yield of a zero-coupon sovereign bond

Proposition 24. *Suppose that the price of the zero-coupon bond of country i is given by (4.15) and (4.16). The yield of a zero coupon bond of country i is given by the formula:*

$$R^i(\tau) = -\frac{\ln(A(\tau)) + \ln(A_i(\tau)) - B(\tau)rf_t - B_i(\tau)cs_t^i}{\tau}. \tag{4.24}$$

Proof. Proposition 24 is a direct result of substituting (4.15) into (2.1) where A, A_i, B, B_i are defined by (4.16).

4.4. *Final model specification*

During the preliminary and full calibration described in Chapter 6 it was revealed that the assumption of zero correlation between the risk-free rate and credit spreads is unrealistic for the observed time period. The dependence is strongly negative. Therefore we are forced to adjust the model specification in order to accommodate the correlation between the credit spread and the risk-free rate. There are basically two possibilities. One is to use the two-factor Vasicek model which offers analytical solution for the bond price, or to use the two factor CKLS model, however at the cost of the need to use approximate solutions for the bond price. In principle we could use also the two-factor CIR model with correlation, however as we would also need to use the approximate solutions for the bond-price it is reasonable also to enhance the model specification to the CKLS type, which is a broader model than the CIR model. We decided to use the latter approach. The final model specification is following:

$$r_t^i = rf_t + cs_t^i, \tag{4.25}$$

$$drf_t = \kappa\left(\theta - rf_t\right)dt + \sigma rf_t^{\gamma} dW_t, \tag{4.26}$$

$$dcs_t^i = \kappa_i\left(\theta_i - cs_t^i\right)dt + \sigma_i\left(cs_t^i\right)^{\gamma_2} dW_t^i, \text{ for } i = 1\ldots n, \tag{4.27}$$

$$Cov\left(dW_t, dW_t^i\right) = \rho_i dt\ . \tag{4.28}$$

We decided not to go as far as to specify a separate power for every credit spread in (4.27) in order to limit the number of parameters to be estimated.

Of course, there is no analytical formula for the price of a zero-coupon bond for the model defined by (4.25) - (4.28). Therefore we will use the approximate formula developed by Halgašová [64] for the two factor CKLS model for every country i. The approximate formula is defined by:

$$P_i^{ap}(t,\tau,r_1,r_2) = a_i^{ap}(\tau)e^{-B(\tau)rf_t - B_i(\tau)cs_t^i},$$

$$\ln a_i^{ap}(\tau) = \ln A^{ap}(\tau) + \ln A_i^{ap}(\tau) + \rho_i \frac{\sigma\sigma_i rf_t^{\gamma}\left(cs_t^i\right)^{\gamma_2}}{\kappa\kappa_i}\left[\tau - B(\tau) - B_i(\tau) + \frac{1-e^{(\kappa+\kappa_i)\tau}}{\kappa+\kappa_i}\right],$$

$$\ln A^{ap}(\tau) = \left(B(\tau)-\tau\right)\left[\theta - \frac{\sigma^2 rf_t^{2\gamma}}{2\kappa^2} - \frac{\sigma\lambda rf_t^{\gamma}}{2\kappa}\right] + \frac{\sigma^2 rf_t^{2\gamma}}{4\kappa}B^2(\tau), \tag{4.29}$$

$$\ln A_i^{ap}(\tau) = \left(B_i(\tau)-\tau\right)\left[\theta_i - \frac{\sigma_i^2\left(cs_t^i\right)^{2\gamma_2}}{2\kappa_i^2} - \frac{\sigma_i\lambda_i\left(cs_t^i\right)^{\gamma_2}}{2\kappa_i}\right] + \frac{\sigma_i^2\left(cs_t^i\right)^{2\gamma_2}}{4\kappa_i}B_i^2(\tau),$$

$$B(\tau) = \frac{1-e^{\kappa\tau}}{\kappa}, B_i(\tau) = \frac{1-e^{\kappa_i\tau}}{\kappa_i}.$$

The yield of a zero-coupon bond price is then given by:

$$R^i(\tau) = -\frac{\ln(a_i^{ap}(\tau)) - B(\tau)rf_t - B_i(\tau)cs_t^i}{\tau}, \tag{4.30}$$

where a_i^{ap}, B and B_i are given by (4.29).

4.5. *Possible extensions of the model*

One possible extension of the model would be to include countries which entered the euro area at a later stage into the model. After the entry date the model specification would be the same, only the number of countries would increase. However before the entry it makes sense to model the development of the interest rate of the entering country using the convergence model as developed by Lacko and Stehlíková [88].

Let us denote country a the country, which is entering the euro area at time E. The proposed model structure for the entering country could be defined in the following way:

$$r_t^a = \begin{cases} rf_t^a + cs_t^a, t \le E \\ rf_t + cs_t^a, t > E \end{cases}, \tag{4.31}$$

where rf_t and cs_t^a are defined by (4.2) and (4.3).

The process for the domestic risk-free rate before the entry date E would have the form of the convergence model:

$$drf_t^a = \left(\theta_{rf}^a + \kappa_{rf}^a (rf_t - rf_t^a)\right) dt + \sigma_{rf}^a \sqrt{rf_t^a}\, dW_t^{arf} \, . \qquad (4.32)$$

It would be interesting to investigate, whether there are structural changes to the credit spread after the entry into the euro area as well.

Another possible extension is to formulate a regime switching processes for the risk-free rate and credit spreads in (4.2) and (4.3) having different parameters in the calm and stress periods during the business cycle.

Third possible extension is to use three factors in formulation of the model (4.1) where the third factor could be interpreted as the liquidity premium individual for each country as the liquidity of the bond markets differ from country to country.

5. Calibration methodology

In this chapter the methods used for the calibration of the model are described. In the first section the methodology for the CIR model without correlation is described and in Section 5.2 the calibration methodology for the full model specification is described.

5.1. *Calibration methodology for the model with zero correlation*

The goal of the calibration is to minimize fit the real market yield curves as best as possible with the model yield curves. For the measurement of the goodness of fit the unweighted least squares sum of differences between the observed yield curves of the respective euro area countries and the theoretical yields from our model. Let us denote $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_n)$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{r} = \left\{ \left(rf_t, cs_t^1, \ldots, cs_t^n \right), t > 0 \right\}$. The theoretical yield of country i with residual maturity τ_j observed at time t $R_t^i(\tau_j) = R_t^i(\tau_j, \kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \lambda, \lambda_i, rf_t, cs_t^i) = R_t^i(\tau_j, \boldsymbol{\kappa}, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\lambda}, \boldsymbol{r})$ is defined by (4.24). The cost function is then defined by:

$$U\left(\boldsymbol{\kappa}, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\lambda}, \boldsymbol{r}\right) = \sum_{t=1}^{N} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\tilde{R}_t^i(\tau_j) - R_t^i(\tau_j) \right)^2, \tag{5.1}$$

where $\tilde{R}_t^i(\tau_j)$ is the real yield of a zero-coupon government bond of country i with residual maturity τ_j observed at time t. That is the sum of residuals through every country i, every maturity τ_j and every time observation t. Note that the value of the cost function depends not only on the parameters of the CIR processes for the risk-free rate and credit spreads, but also on the actual development of these processes, because they are unobserved variables.

Also note that the parameters κ and λ in the modeled yield to maturity given by (4.24) and (4.16) appear only in the term $\kappa+\lambda\sigma$. The cost function is therefore constant for $\kappa+\lambda\sigma$ being constant. One of them is a free parameter. It is therefore not possible to calibrate the original model parameters directly as we would obtain a continuum of optimal solutions.

To avoid this, we follow the extended version of the min max optimization developed by Ševčovič and Urbánová Csajková [117] to calibrate the model. Let us introduce the following new essential parameters:

$$\beta = e^{-\eta},\ \xi = \frac{\kappa+\lambda\sigma+\eta}{2\eta},\ \tilde{\rho} = \frac{2\kappa\theta}{\sigma^2},\ \beta_i = e^{-\eta_i},\ \xi_i = \frac{\kappa_i+\lambda_i\sigma_i+\eta_i}{2\eta_i},\ \tilde{\rho}_i = \frac{2\kappa_i\theta_i}{\sigma_i^2}\text{[24]}. \quad (5.2)$$

Let us denote $D, D_i = (0,\infty)^3 \times \mathbb{R} \subset \mathbb{R}^4$ and $\Omega, \Omega_i = (0,1)^2 \times (0,\infty) \subset \mathbb{R}^3$. Using the result of Ševčovič and Urbánová Csajková [117], which they showed for a one factor CIR model, for every single CIR process in (4.2) and (4.3) the transformation $T: D \to \Omega$ defined in (5.2) is a smooth mapping and for every $(\hat{\beta}, \hat{\xi}, \hat{\rho})$ the preimage $T^{-1}(\hat{\beta}, \hat{\xi}, \hat{\rho}) = \left\{(\kappa_\lambda, \theta_\lambda, \sigma_\lambda, \lambda) \in \mathbb{R}^4, \lambda \in \hat{J}\right\}$, $\hat{J} = (-\infty, -(2\hat{\xi}-1)\ln\hat{\beta})$, is a smooth λ-parameterized curve in $D \subset \mathbb{R}^4$ or $D_i \subset \mathbb{R}^4$.

After some straightforward calculations the functions A, A_i, B, B_i in (4.24) can be expressed in terms of the new parameters:

$$\begin{aligned}
A(\tau) &= \left(\frac{\beta^{(1-\xi)\tau}}{\xi(1-\beta^\tau)+\beta^\tau}\right)^{\tilde{\rho}}, \\
A_i(\tau) &= \left(\frac{\beta_i^{(1-\xi_i)\tau}}{\xi_i(1-\beta_i^\tau)+\beta_i^\tau}\right)^{\tilde{\rho}_i}, \\
B(\tau) &= -\frac{1}{\ln\beta}\frac{1-\beta^\tau}{\xi(1-\beta^\tau)+\beta^\tau}, \\
B_i(\tau) &= -\frac{1}{\ln\beta_i}\frac{1-\beta_i^\tau}{\xi_i(1-\beta_i^\tau)+\beta_i^\tau}.
\end{aligned} \quad (5.3)$$

[24] In [117] the third parameter was ρ. However in order to avoid confusion with the correlation parameter in the final model specification we denote the transformed CIR parameter $\tilde{\rho}$ and $\tilde{\rho}_i$.

Let us denote the vector of the new parameters as:

$$\psi = (\phi, \phi_1, \ldots, \phi_N) = (\beta, \xi, \tilde{\rho}, \ldots \beta_n, \xi_n, \tilde{\rho}_n) \,. \tag{5.4}$$

The yield of the zero-coupon bond of country i time to maturity τ_j at the time t in terms of the new parameters (5.4) is given by:

$$R_t^i(\tau_j, \psi, \boldsymbol{r}) = R_t^i(\tau_j, \phi, \phi_i, rf_t, cs_t^i) = -\frac{\ln A(\phi, \tau_j) A_i(\phi_i, \tau_j) - B(\phi, \tau_j) rf_t - B_i(\phi_i, \tau_j) cs_t^i}{\tau_j}, \tag{5.5}$$

where A, A_i, B, B_i are given by (5.3).

The cost function (5.1) represented in terms of the new parameters is given by:

$$U(\psi, \boldsymbol{r}) = U(\phi, \phi_1, \ldots, \phi_n, \boldsymbol{r}) = \sum_{t=1}^{N} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\tilde{R}_t^i(\tau_j) - R_t^i(\tau_j, \psi, \boldsymbol{r}) \right)^2, \tag{5.6}$$

where the modeled yield of the zero-coupon bond is given by (5.5).

Contrary to [117] the unweighted least squares sum is used. This choice is due to a broad range of maturities used from 3 months up to ten years. Maturity weighting would result in excessive weight to be put on the longer part of the yield curve compared to the short-term part of the yield curve.

The transformed parameters are defined on an open space $\Omega \times \Omega_i^n = (0,1)^{2n+2} \times (0,\infty)^{n+1} \subset \mathbb{R}^{3n+3}$. However $\beta \to 0$ only if κ, λ or $\sigma \to \infty$ and $\beta \to 1$ only if κ, λ and $\sigma \to 0$. $\xi \to 0$ or $\xi \to 1$ only if $\sigma \to 0$ or κ, λ or $\sigma \to \infty$. $\tilde{\rho} \to 0$ only if κ or $\theta \to 0$ or $\sigma \to \infty$ and $\tilde{\rho} \to \infty$ only if κ or $\theta \to \infty$ or $\sigma \to 0$. The same reasoning applies to β_i, ξ_i and $\tilde{\rho}_i$. These are all degenerate solutions. Therefore we will search for the estimate of the parameters ψ on the following compact space:

$$B = \left\{ \psi \in \mathbb{R}^{3n+3} \middle| \begin{array}{l} \delta \le \beta \le 1-\delta, \delta \le \beta_i \le 1-\delta \\ \delta \le \xi \le 1-\delta, \delta \le \xi_i \le 1-\delta \\ \delta \le \tilde{\rho} \le \tilde{\rho}_{\max}, \delta \le \tilde{\rho}_i \le \tilde{\rho}_{\max} \end{array} \right\}, \tag{5.7}$$

where δ is small enough and $\tilde{\rho}_{\max}$ is large enough.

The estimate of the parameters ψ is obtained by minimizing the cost function (5.6) over the parameters ψ and the time-series of the risk-free rate $rf(\cdot)$ and credit spreads $cs^i(\cdot)$ subject to the following conditions:

$$\begin{gathered}\min_{\psi,\boldsymbol{r}} U(\psi,\boldsymbol{r}), \\ \boldsymbol{r} \geq 0, \psi \in B.\end{gathered} \tag{5.8}$$

The number of parameters to be estimated is very large, $3(n+1)+(n+1)N$. It is therefore suitable to minimize the objective function in (5.8) using a two step approach, clearly we have:

$$\min_{\psi,\boldsymbol{r}} U(\psi,\boldsymbol{r}) = \min_{\psi} \min_{\boldsymbol{r}} U(\psi,\boldsymbol{r}), \tag{5.9}$$

where the inner problem in (5.9) has the following form:

$$\begin{gathered}\min_{\boldsymbol{r}} U(\psi,\boldsymbol{r}) \\ \boldsymbol{r} \geq 0\end{gathered} \tag{5.10}$$

For a fixed value of the parameter vector ψ the yield to maturity function (5.5) is an affine function in the variables $rf(\cdot)$ and $cs^i(\cdot)$. Thus the inner problem (5.10) is a ordinary least squares estimate with constraints $\boldsymbol{r}_t \geq 0$ for all t.

Let us denote

$$S_C = \left\{ \boldsymbol{r} \geq 0, \sum_{i=0}^{n} \boldsymbol{r}_i \leq C \right\} \subset \mathbb{R}^{n+1}, \tag{5.11}$$

where C is a sufficiently large constant.

Proposition 25. *For every fixed value ψ the inner minimization problem* (5.10) *has a unique global minimum.*

Proof. Let us investigate the function $\boldsymbol{r} \to U(\psi,\boldsymbol{r})$ for a fixed parameter vector ψ, such that $\phi \in \Omega$ and $\phi_i \in \Omega$ for all i. It is easy to show that $B > 0$, $A > 0$ for all $\phi \in \Omega$ and $B_i > 0$ and $A_i > 0$ for all $\phi_i \in \Omega_i$ for all i. Therefore $R_t^i(\tau_j,\psi,\boldsymbol{r}) \to \infty$ if $rf_t \to \infty$ or $cs_t^i \to \infty$ and thus also $U(\psi,\boldsymbol{r}) \to \infty$ if $rf_t \to \infty$ or $cs_t^i \to \infty$. This cannot be a minimum of the function $\boldsymbol{r} \to U(\psi, \boldsymbol{r})$. Therefore the minimum is attained on a compact space S_C defined by (5.11), where C is a sufficiently large constant.

It is simple calculus to show that for a fixed value of the parameters ψ the Hessian of the function $\boldsymbol{r} \to U(\psi, \boldsymbol{r})$ does not depend on $\boldsymbol{r}$ and has the following form:

$$\nabla_{rr} U(\psi, \boldsymbol{r}) = 2N\left(\sum_{j=1}^{m} \frac{1}{\tau_j^2}\right)\begin{pmatrix} nB^2 & BB_1 & BB_2 & \cdots BB_n \\ BB_1 & B_1^2 & 0 & 0 \quad 0 \\ BB_2 & 0 & B_2^2 & \ddots \quad \vdots \\ \vdots & \vdots & \ddots & \ddots \quad 0 \\ BB_n & 0 & \cdots & 0 \quad B_n^2 \end{pmatrix}. \tag{5.12}$$

For all $\boldsymbol{x} \in \mathbb{R}^{n+1}$:

$$\begin{aligned} \boldsymbol{x}'\nabla_{rr} U(\psi, \boldsymbol{r})\boldsymbol{x} &= 2N\left(\sum_{j=1}^{m} \frac{1}{\tau_j^2}\right)\left[nB^2 x_0^2 + \sum_{i=1}^{n} B_i^2 x_i^2 + 2\sum_{i=1}^{n} BB_i x_0 x_i\right] \\ &= 2N\left(\sum_{j=1}^{m} \frac{1}{\tau_j^2}\right)\left[\sum_{i=1}^{n} \left(Bx_0 + B_i x_i\right)^2\right] \geq 0 \end{aligned}. \tag{5.13}$$

Therefore $\nabla_{rr} U(\psi, \boldsymbol{r})$ is a positive definite matrix for all $\boldsymbol{r} \geq \boldsymbol{0}$ and thus $\boldsymbol{r} \to U(\psi, \boldsymbol{r})$ is a strictly convex function. It is well known that a strictly convex function on a compact has a unique global minimum. □

From a numerical point of view it is a quadratic programming problem, which can be efficiently and reliably solved. In practice we used the Matlab function *lsqnonneg* to solve the inner problem, which uses the algorithm described in Lawson and Hanson [89, Chapter 23].

Let us denote

$$\boldsymbol{r}(\psi) = \arg\min_{\boldsymbol{r} \in S} U\left(\psi, \boldsymbol{r}\right), \tag{5.14}$$

the obtained unique minimum of the inner problem (5.10) for a fixed parameter value ψ.

Proposition 26. *The functions* $\psi \to \boldsymbol{r}(\psi)$ *as well as* $\psi \to U\left(\psi, \boldsymbol{r}(\psi)\right)$ *are continuous.*

Proof. To prove the proposition we need to show that for every sequence $\{\psi_n\} \underset{n\to\infty}{\to} \psi$ also $\boldsymbol{r}(\psi_n) \to \boldsymbol{r}(\psi)$. We know that $\boldsymbol{r}\left(\psi_n\right) \in S_C$ for C sufficiently large. As S_C is a compact there exist a subsequence $\left\{\psi_{n_k}\right\}_{k=1}^{\infty}$ such that $\left\{\boldsymbol{r}\left(\psi_{n_k}\right)\right\} \underset{k\to\infty}{\to} \tilde{\boldsymbol{r}} \in S_C$. From the definition of $\boldsymbol{r}(\psi_{n_k})$ it follows that $U\left(\psi_{n_k}, \boldsymbol{r}\right) \geq U\left(\psi_{n_k}, \boldsymbol{r}(\psi_{n_k})\right)$ for every $\boldsymbol{r} \in S_C$. Passing to the limit for $k \to \infty$ we obtain $U\left(\psi, \boldsymbol{r}\right) \geq U\left(\psi, \tilde{\boldsymbol{r}}\right)$ for every $\boldsymbol{r} \in S_C$. According to Proposition 25 the function $\boldsymbol{r} \to U(\psi, \boldsymbol{r})$ is strictly convex on S_C and the inner problem (5.10) has an unique minimum $\boldsymbol{r}\left(\psi\right)$. Thus $\tilde{\boldsymbol{r}} \equiv \boldsymbol{r}\left(\psi\right), \boldsymbol{r}(\psi_n) \to \boldsymbol{r}(\psi)$ *and consequently* $U\left(\psi_n, \boldsymbol{r}(\psi_n)\right) \to U\left(\psi, \boldsymbol{r}(\psi)\right)$. □

It is well known that a continuous function attains a minimum on a compact space, although it may not be unique. Therefore the outer problem:

$$\min_{\psi} U\left(\psi, \boldsymbol{r}(\psi)\right). \tag{5.15}$$

has a solution on B although it may not be unique. As the outer problem need not be a good function to optimize the simulated annealing method is then used, in contrast to the evolution strategies used in [117]. We used $\delta = 10^{-12}$ and $\tilde{\rho}_{\max} = 250$. The outer problem was solved using the Matlab function *simulannealbnd* from the Genetic algorithm and Direct Search Toolbox.

As the last step the likelihood function of the CIR models over the parameters $\lambda, \lambda_1, \ldots, \lambda_N$ is maximized in order to find the original parameters of the CIR models. Under the assumption of independence between the risk-free rate and credit spreads of individual countries and also among the credit spreads this can be done in isolation for every single CIR process. In the following section the arguments hold true both for the risk-free rate rf_t and for the credit spreads cs_t^i and their corresponding parameters.

Proposition 27. *(Bergstrom [8]) The discretized model corresponding to (2.13) has the following form:*

$$r_{t+\Delta t} - r_t = (\theta - r_t)\left(1 - e^{-\kappa\Delta t}\right) + \varepsilon_{t+\Delta t}, \tag{5.16}$$

where $\varepsilon_{t+\Delta t}$ *is a normally distributed random variable with zero mean and dispersion* $\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\Delta t}\right) r_t$.

Proposition 28. *(Bergstrom [8]) The logarithm of the likelihood function for model (5.16) has the form:*

$$\ln L(\kappa, \theta, \sigma) = -\frac{1}{2}\sum_{t=2}^{N}\left(\ln v_t^2 + \frac{\varepsilon_t^2}{v_t^2}\right), \tag{5.17}$$

where $v_{t+\Delta t}^2 = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t}) r_t$, $\varepsilon_{t+\Delta t} = r_{t+\Delta t} - e^{-\kappa\Delta t} r_t - \theta(1 - e^{-\kappa\Delta t})$.

Let us find the global minimum of the likelihood function (5.17) over the λ-parameterized curve $\left\{(\kappa_\lambda, \theta_\lambda, \sigma_\lambda, \lambda) \in \mathbb{R}^4, \lambda \in \hat{J}\right\}$, $\hat{J} = (-\infty, -(2\hat{\xi} - 1)\ln\hat{\beta})$. The resulting optimal values $(\bar{\kappa}, \bar{\theta}, \bar{\sigma}, \bar{\lambda}) = (\kappa_{\bar{\lambda}}, \theta_{\bar{\lambda}}, \sigma_{\bar{\lambda}}, \bar{\lambda}) = \max_{\lambda \in \hat{J}} \ln L(\kappa_\lambda, \theta_\lambda, \sigma_\lambda)$ are the estimates of the original parameters of the CIR model. This process is done for the risk-free rate rf_t and for all credit spreads cs_t^i.

To assess the goodness of fit the average distance from the real data defined as the square root of the Mean Square Error:

$$AvgErr^2 = \frac{1}{Nnm}\left(\sum\left(\tilde{R}_t^i(\tau_j) - R_t^i(\tau_j)\right)^2\right) = \frac{1}{Nnm} U(\psi, \boldsymbol{r}), \tag{5.18}$$

where $\tilde{R}_t^i(\tau_j)$ is the real yield of a zero-coupon government bond of country i with residual maturity τ_j observed at time t and $R_t^i(\tau_j)$ is the modeled yield defined by (5.5).

5.2. *Calibration methodology for the final model specification*

Although in Ševčovič and Urbánová Csajková [117] a parameter transformation was suggested also for the Vasicek model, it cannot be applied to the approximate bond price given by (4.29) because the terms A^{ap}, A_i^{ap} in (4.29) involve rf_t and cs_t^i. It is also not necessary to introduce the transformation as no parameter is free in the approximate bond price formula. Therefore we will calibrate the model directly using the original parameter vector including correlations between the risk-free rate and credit spreads and the power parameters $\tilde{\psi} = (\tilde{\phi}, \tilde{\phi}_1, \ldots, \tilde{\phi}_N, \boldsymbol{\rho}, \gamma, \gamma_2) = (\kappa, \theta, \sigma, \lambda, \ldots, \kappa_n, \theta_n, \sigma_n, \lambda_n, \rho_1, \ldots \rho_n, \gamma, \gamma_2)$.

The cost function is then defined by:

$$U\left(\tilde{\psi}, \boldsymbol{r}\right) = \sum_{t=1}^{N} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\tilde{R}_t^i(\tau_j) - R_t^i(\tau_j, \tilde{\psi}, \boldsymbol{r}) \right)^2, \tag{5.19}$$

where the modeled yield of the zero-coupon bond is given by (4.30).

The estimate of the parameters $\tilde{\psi}$ is obtained by minimizing the cost function (5.19) over the parameters $\tilde{\psi}$ and the time-series of the risk-free rate $rf(\cdot)$ and credit spreads $cs^i(\cdot)$ subject to the following conditions:

$$\begin{gathered} \min_{\tilde{\psi}, \boldsymbol{r}} U\left(\tilde{\psi}, \boldsymbol{r}\right), \\ \boldsymbol{r} \geq 0, \\ \kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \gamma, \gamma_i \geq \delta, \\ -1 \leq \rho_i \leq 1. \end{gathered} \tag{5.20}$$

The number of parameters to be estimated is again very large, $4(n+1) + (n+1)N + 2 + n$. We could proceed as in the previous case and to try a two-step approach to (5.20) as we did in (5.9). However the inner problem in this case would not be a quadratic programming problem as the yield of the zero-coupon bond is no longer an affine function of $\boldsymbol{r}$. The inner problem thus becomes computationally cumbersome. Therefore proceed in the following way. As the estimate of the factors $\boldsymbol{r}$ in the CIR model is quite robust, we consider this estimate to be sufficient and we solve only the outer problem:

$$\begin{gathered}\min_{\tilde{\psi}} U\left(\tilde{\psi}, \boldsymbol{r}(\hat{\psi})\right), \\ \kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \gamma, \gamma_i \geq \delta, \\ -1 \leq \rho_i \leq 1.\end{gathered} \tag{5.21}$$

where $\hat{\psi}$ is the optimal solution to (5.15).

Again the outer problem was solved using the Matlab function *simulannealbnd* from the Genetic algorithm and Direct Search Toolbox with $\delta = 10^{-12}$. We estimated the optimal $\boldsymbol{r}$ for the problem (5.20) at the end of the algorithm to prove that it does not significantly differ from $\boldsymbol{r}(\hat{\psi})$.

6. Results of real market data calibration

In this chapter calibration results of the model proposed in Chapter 4 are presented. In the first two sections the data used and identified potential structural breakpoints are described. First a test calibration was performed using 250 time observations starting from August 1, 2000 (Greek data were first available) to test the method and results it yields. However this preliminary calibration already revealed some properties of the used data and model, which had an impact on the future calibration of the whole model. Therefore the preliminary results are presented in Section 6.3 to illustrate the behavior of the model. The full results are then presented in Section 6.4 and the results for the final model specification in Section 6.5.

6.1. Real market data used

At the time when this thesis is written, there are 17 members of the euro area including overseas territories. The euro area was initially formed by eleven Member States of the European Union, namely Austria, Belgium, Finland, France, Germany, Ireland, Italy, Luxembourg, Netherlands, Portugal and Spain. In the first stage the euro was introduced into non-cash operations on January 1, 1999 and into cash circulation on January 1, 2002. Greece haven't fulfilled the initial criteria and joined the euro area later on January 1, 2001. Further Member States joined the euro area in the following years: Slovenia on January 1, 2007, Cyprus and Malta on January 1, 2008, Slovakia on January 1, 2009 and Estonia on January 1, 2011. For detailed information the reader is referred to the website of the European Central Bank (ECB).

According to the ECB some other countries, Monaco, San Marino and Vatican City, are voluntarily using the Euro as the legal tender on a contractual basis with the

European Community. Other are using the Euro without a formal agreement, examples being Andorra, Montenegro and Kosovo. However, these countries were not taken into account when calibrating the model because they are not members of the euro area and moreover they are quite small, so market prices of their bonds either do not exist or are affected but low liquidity.

Euro denominated zero-coupon curves from Bloomberg since January 1, 1999 until February 3, 2012, provided under the license agreement [11], have been used to calibrate the model. Examples of the data are shown in Figure 1. Data before January 1, 1999 could not be used because the debt was denominated in different currencies and therefore the yields were subject to currency risk. However the curves were not available for all seventeen euro area countries for the whole period for various reasons.

Table 1 General government gross debt

	EUR billions				% of GDP			
Country	**1995**	**2000**	**2005**	**2010**	**1995**	**2000**	**2005**	**2010**
Euro area (17 countries)	4 015	4 694	5 719	7 819	72%	69%	70%	85%
Germany	1 067	1 232	1 526	2 062	56%	60%	69%	83%
Italy	1 071	1 300	1 514	1 843	121%	109%	105%	118%
France	675	826	1 145	1 591	56%	57%	66%	82%
Spain	295	374	392	642	63%	59%	43%	61%
Netherlands	243	225	266	370	76%	54%	52%	63%
Belgium	282	272	279	341	130%	108%	92%	96%
Greece	95	141	195	329	97%	103%	100%	145%
Austria	124	138	157	206	68%	66%	64%	72%
Portugal	53	62	96	161	59%	49%	63%	93%
Ireland	42	40	44	144	81%	38%	27%	93%
Finland	57	58	66	87	57%	44%	42%	48%
Slovakia	3,3	10,8	13,4	27,0	22%	50%	34%	41%
Slovenia	2,8	5,5	7,7	13,7	19%	26%	27%	39%
Cyprus	3,6	5,9	9,5	10,7	52%	60%	69%	62%
Luxembourg	1,2	1,4	1,8	7,7	7%	6%	6%	19%
Malta	1,0	2,3	3,4	4,3	35%	55%	70%	69%
Estonia	0,2	0,3	0,5	1,0	8%	5%	5%	7%

General government gross debt in millions of EUR and in percentage of GDP in the euro area countries
Source: Eurostat, Series code tsieb090

Data for Greece were only available since August 1, 2000 as the exchange rate between Greek Drachma and the Euro was fixed on June 19, 2000. Data for Cyprus, Estonia, Luxembourg and Malta were not available due to the small outstanding public debt of these countries (see Table 1) and the resulting low liquidity of their government bond markets. Data for Slovenia and Slovakia were available since January 1, 2008 and January 1, 2009 respectively. Again data before this date could not be used due to the currency risk. Due to these restrictions the model is calibrated for the initial eleven members of the euro area without Luxembourg and with Greece. Daily data since August 1, 2000 until February 3, 2012 have been used, totaling 3,002 time points.

The data for all countries were available for the maturities of 3 months, 6 months, one year and up to ten years. Total of $m = 12$ maturity buckets have been used. Twelve maturities for $n = 11$ countries and $N = 3002$ time points total 396 264 observations.

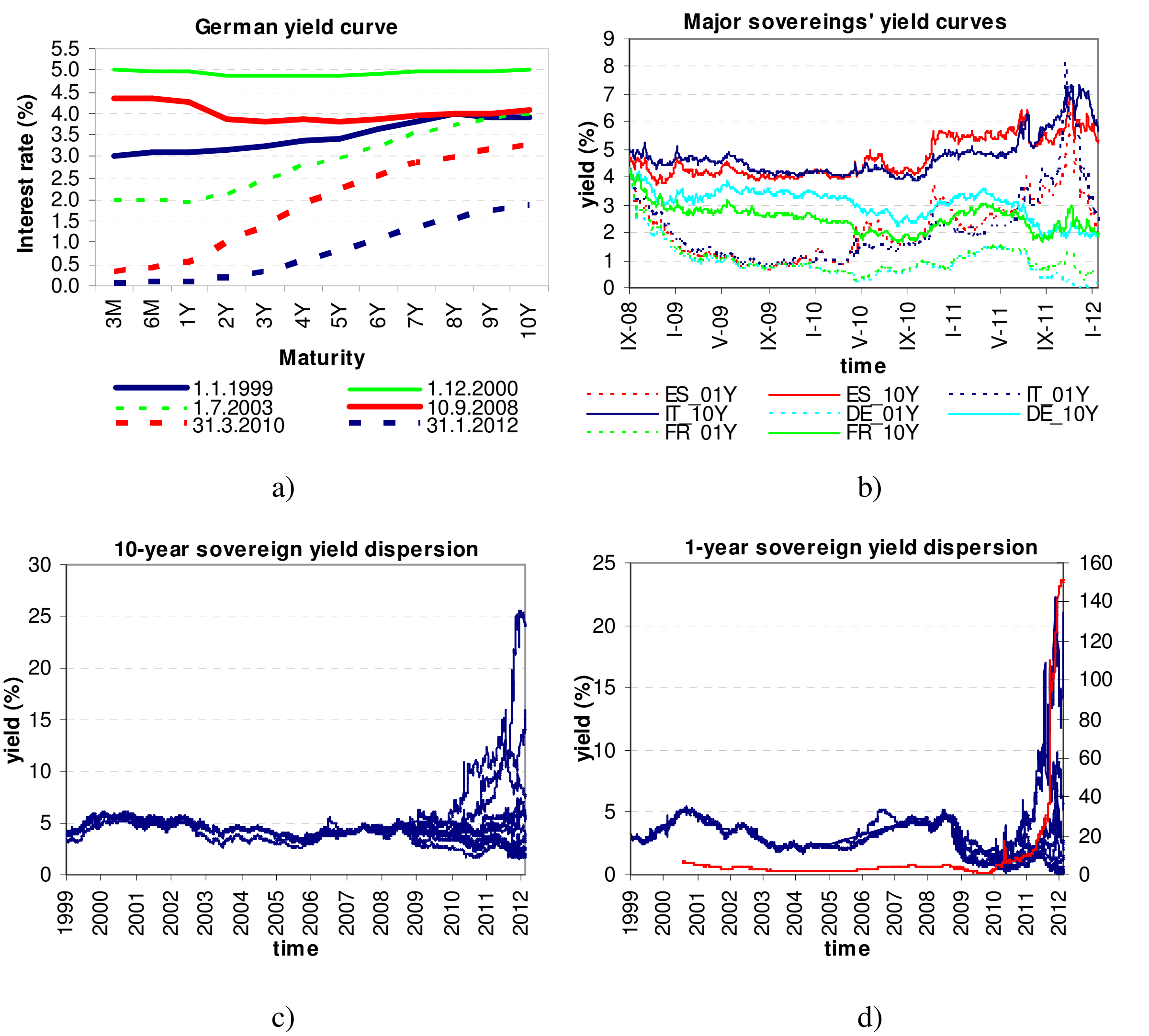


Figure 1 Examples of development in government bond yields over the whole existence of the euro area. The shape of the German yield curve in different points in time a), the evolution of 1-year and 10-year yields of the major euro area sovereigns b). The distribution of individual country yields for maturities of 10 years c) and 1 year d). In d) Greece is in red on the right-hand side axis, all other countries on the left-hand side axis. Source: Bloomberg.

Some special developments which need further attention are illustrated in Figure 2. In Figure 2a and 2b the convergence of Greek yields to the euro area yield cluster is clearly visible. Despite the fact that the exchange rate between the Drachma and the Euro was fixed on June 19, 2000, the yields converged only in January 2001 when Greece officially entered into the euro area. The difference and speed of convergence is more pronounced at the short-term end of the yield curve, while the ten year yields converged less and remained highest among euro area members.

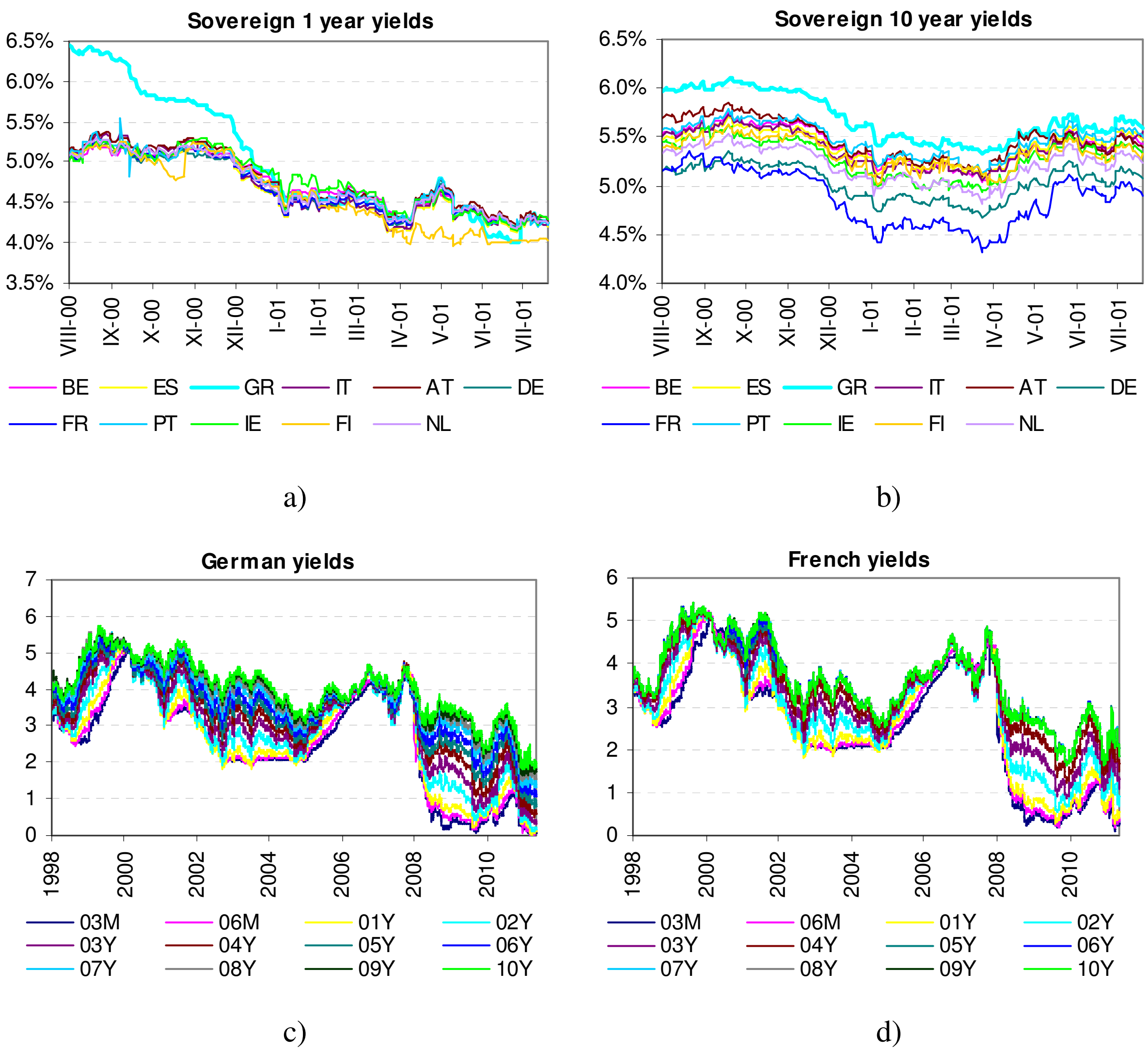
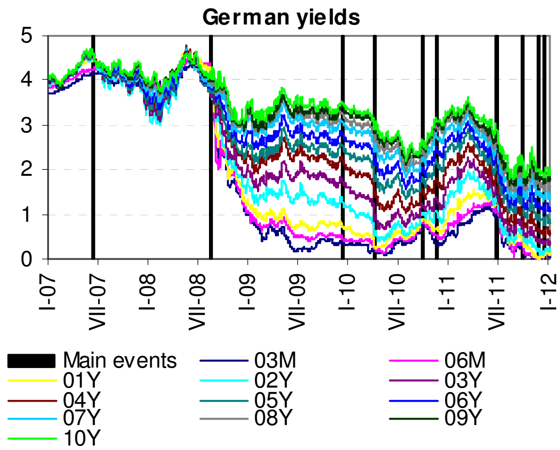
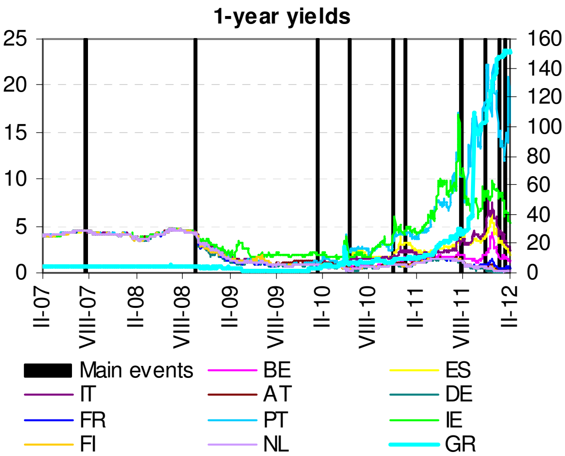


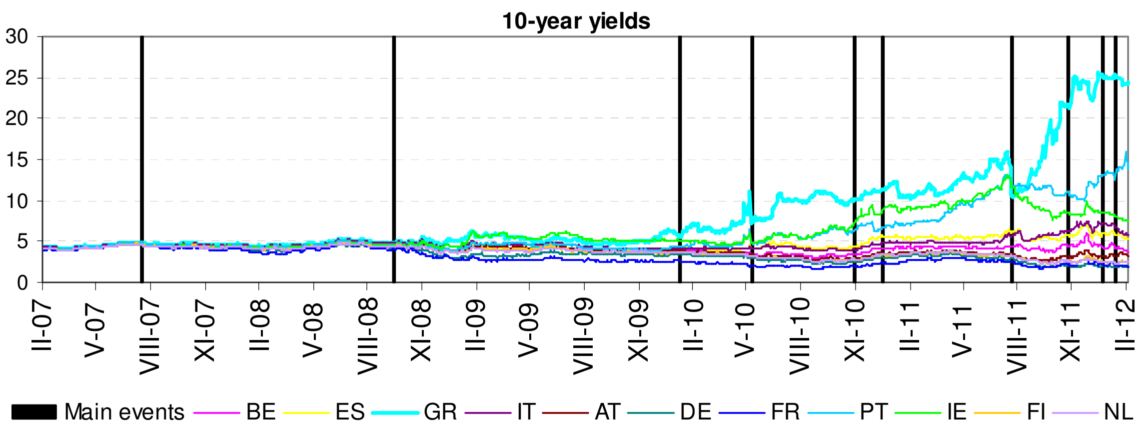
Figure 2 Examples of special developments in government bond yields. The 1-year a) and 10-year b) yields of the euro area countries at the entry of Greece. Yields of Germany c) and France d).
Source: Bloomberg.



a)



b)



c)

Figure 3 Development during the financial crises (yields in percentage points). German yield curves a), yield curves of the euro area countries for 1-year b) and 10-year c). Main events in the chronological order: July 16, 2007 (start of the subprime crisis), collapse of Lehman Brothers, Eurostat report on Greek data, establishment of the EFSF, European Council agrees ESM, Treaty amendments for ESM, second loan to Greece, European Council announces Greek PSI package (October 26, 2011), ECB 3-year Long-term refinancing operations, S&P downgrades 9 euro area countries. In b) Greece on the right-hand side axis and all other countries on left-hand side axis. Source: Bloomberg.

Note the low value of the 10-year French yields in Figure 2b compared to German yield. This situation is repeated during 2003 – 2005 and again during 2009 – 2011. France is the low outlier in Figure 1c in both periods. This is quite surprising as Germany is traditionally considered to be the least risky country. The reason is explained in Figures 2c and 2d, where the developments in yield curves of Germany and France is shown. Both countries share the same periods of flat and steep yield curves with 2001 – 2005 and

2009 – 2012 being the latter. However the difference between the two countries is that during the periods of the steep yield curve, the French curve is steep until 5-year maturity and flat above 5-year maturity, while the German is steep all the way up to 10-year maturity. Therefore German yields are at par or lower then French up to 5-year maturity, but above that during the periods of steep yield curve French yields are lower than the German ones. We are not able to explain this development, but it might be the reason for different calibration results for France compared to other euro area countries as described in Chapter 6.

In Figure 3 the development in yield curves is described in more detail. For reasoning of the main events refer to Chapter 1.2 and to the identification of potential break points below. Not much in the yield developments can be observed since the sub-prime crisis until the collapse of Lehman Brothers. This period is characterized by a flat yield curve and a very mild differentiation in sovereign long-term yields.

After the collapse of Lehman Brothers the yield curves fell down and steepened sharply. While the 1-year yields fell for all countries, 10 year yields remained broadly the same or fell slightly for the bulk of euro area countries, dropped significantly for Germany and France and even increased for Ireland and Greece. So the first differentiation among the euro area began at the short end as well as on the long end of the yield curve. At that time Ireland was considered more risky than Greece due to the banking crisis. But still the differentiation was only mild.

The true differentiation started at the end of 2009 when fears about the sustainability of Greek public debt increased. This was fueled by the Eurostat revelation about the quality of Greek data. Portugal yields joined the Irish yields and Spanish and Italian yields started to divert from the other countries as well as they started to rise slowly, while the other yields decreased. The establishment of the ESFS managed to decrease the yields of the three most risky countries, but only for a short period of time. At the same time the other yields continued to decrease except Italy and Spain, where long-term yields were stable and short-term yields increased.

After the announcement of the permanent ESM both short-term as well as long-term yields of all euro area countries increased and this situation remained until the planned Greek debt restructuring was announced at the European Council in July 2011. During that period also Belgian yields increased comparatively to the other less risky sovereigns. The decrease of yields after the European Council in July 2011 was once again only short-lived. Greek yields started to rise again about a month after the summit and Portuguese yields later during the year as well as Italian, Spanish and Belgian. Only Irish yields benefited and started to converge to the less risky countries as well as the yields of the less risky sovereigns. The yields of the more risky countries peaked in November 2011 and are gradually decreasing since then except Portugal.

6.2. Identification of potential structural break points

The calibration will be tested for potential structural breaks in the model parameters. Four potential break points have been identified from the economic development during the existence of the euro area. The macroeconomic development is shown in Figure 4. The euro area was formed before the peak of the dot-com bubble and the data series used for the estimation begins in the middle of the bubble peak. The first potential break point correspond to the stabilization of the economy in 2003 after the slowdown caused by the burst of the dot-com bubble. The first break point is March, 12 2003, which is marked as the lowest point of the Euro Stoxx 50 stock index after the burst of the dot-com bubble.

The economy experienced then a steady recovery until 2007, when the sub-prime mortgage crises started in the USA. The major turbulences started during the summer of 2007. According to the Federal Reserve's Chairmen Bernanke [9] the US subprime mortgage delinquencies reached 16 % in August 2007, which has taken toll on the financial markets and the strains intensified. The possible second break point was set to July 16, 2007, which was the highest value of the Euro Stoxx 50 index. Since this date also the interest rates started to fall.

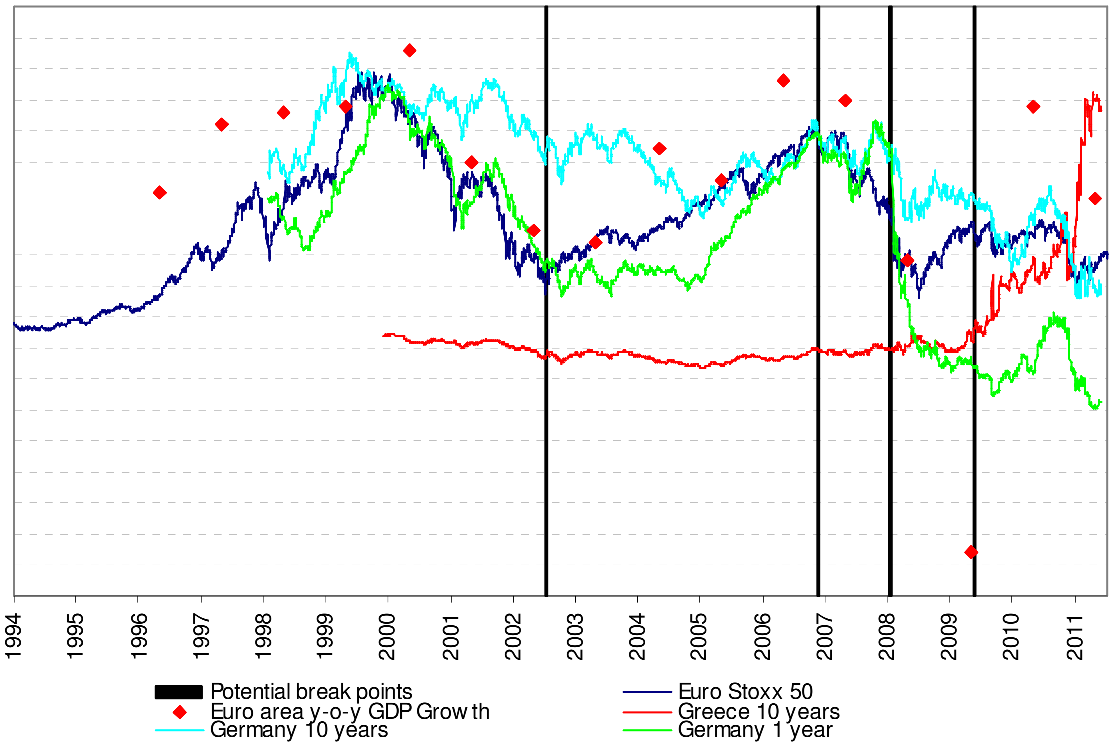


Figure 4 Determination of the potential break points. The values of the individual series were scaled in order to fit onto the same scale.
Source: Bloomberg, Eurostat

The natural choice of the third possible break point is September 15, 2008 when Lehman Brothers filed for bankruptcy under the US bankruptcy law. This was a major event, which caused significant turbulence in the financial markets including heavy losses in the stock market and freezing of the interbank market.

The fourth possible break point is also a natural choice, as on January 8, 2010 the European statistical office revealed that Greece was intentionally and significantly underestimating the level of their public debt and deficit. On this day and on the subsequent day the yield on Greek 10 year government bond jumped up by 25 basis points each day and within 2 weeks breached 7 % p. a. the situation worsened thereafter and resulted finally in the restructuring of the Greek debt on March 12, 2012 with significant losses for private investors as described in Chapter 1.

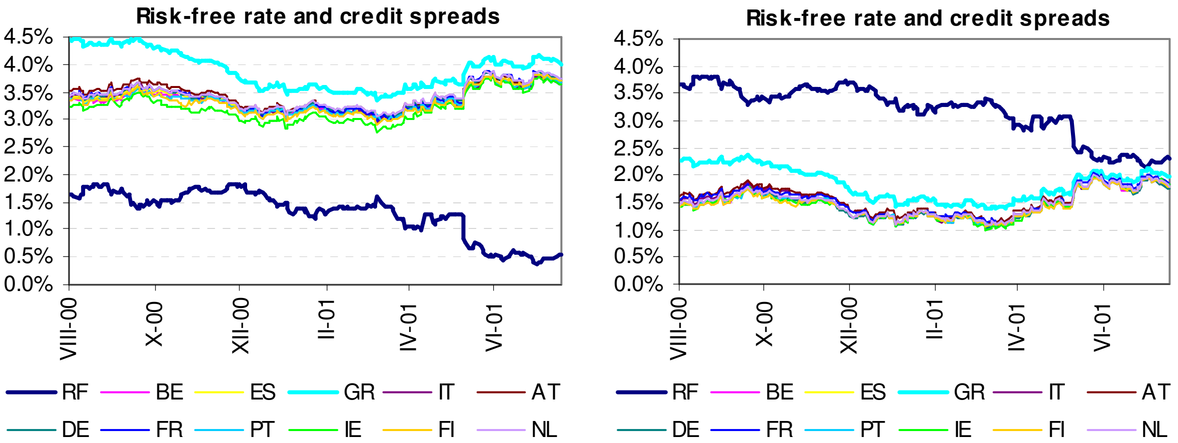
6.3. *Preliminary results*

First thing to remark is that due to the summation formulation of the model (4.1) the values of the risk-free rate and credit spreads are almost interchangeable. This is illustrated in Figures 5a and 5b. The estimated absolute daily changes in the values of the risk-free rate and credit spreads are very similar in both cases. However the levels are substantially different.

Note that if the model equations (4.2) and (4.3) would be formulated in the form of the Vasicek model (2.11) the result would be a continuum of pairs (rf_t, cs_t^i) where $rf_t + cs_t^i = c$ and c is a constant with the corresponding continuum of pairs of the (θ, θ_i) where $\theta + \theta_i = c_2$ and c_2 being a constant as well. The credit spreads of individual countries together with their θ_i s would keep the differences from each other, but the level of credit spreads and the risk-free rate would be impossible to estimate. This empirical result is confirmed also by Dai and Singleton [35].

However, this is not true for the CIR model studied here. The level of the risk-free rate and credit spread influences also the variance of the stochastic process for rf_t and cs_t^i and thus also the price of the zero-coupon bond. However this effect and also the effect on the cost function (5.1) is quite small. The parameters in both cases are substantially different, but the average error difference is only 0.55 basis points between the two cases.

The average error in Figure 5a is slightly better then the average error in Figure 5b. This is contradictory to our expectations. Our economically based expectations are that in the recessionary period the real interest rates declines and the credit spreads would rise. Therefore we would expect that in the good times of 2000 the risk-free rate would be high in order to be able to decline in the stress periods ahead. The second expectation is that in the late period considered the credit spreads would be raised significantly due to the euro area sovereign debt crises, so the fact that it starts in low ranges is welcomed. However these expectations are fulfilled in the best estimate presented in Figure 6.



a) b)

Figure 5 Examples of the preliminary calibration. Estimates of the risk-free rate and credit spreads for 250 days for the whole yield curve from 3 months to 10 years for different parameters a) and b). RF denotes the risk-free rate, country abbreviations the individual credit spread of the respective countries.

The second remark is that the model yields significantly better results for maturities from one to ten years then for the short-term maturities of three and six months. Especially the three month maturity estimates were quite poor. The estimates for the full yield curve and the truncated yield curve are presented in Figure 6.

The results for the truncated yield curve from 1 year to 10 years resembles the Figure 5a. The overall average error decreases from 13.56 basis points to 9.56 basis points when the three month maturity is left out and to 6.1 basis points when also the six month maturity is left out. The average error for the three month maturity exceeded 16 basis points for every single country, while for the other maturities the average error exceeded 10 basis points only occasionally. This can be explained by the fact that due to transaction costs and usually lower yield on the short end of the yield curve the investors usually hold such investments to maturity and do not trade with them. The resulting low liquidity on the short end of the yield curve results in imperfect pricing. If the three month maturity is left out from calibration of the model the estimates average error for the six month maturity increases, but for all other maturity it decreases. If also the six month maturity is left out the results are further improved but not that significantly to justify to left out also the six month maturity. It was therefore decided to leave the three month maturity bucket out of the full calibration thus eleven maturity buckets remain for the full calibration.

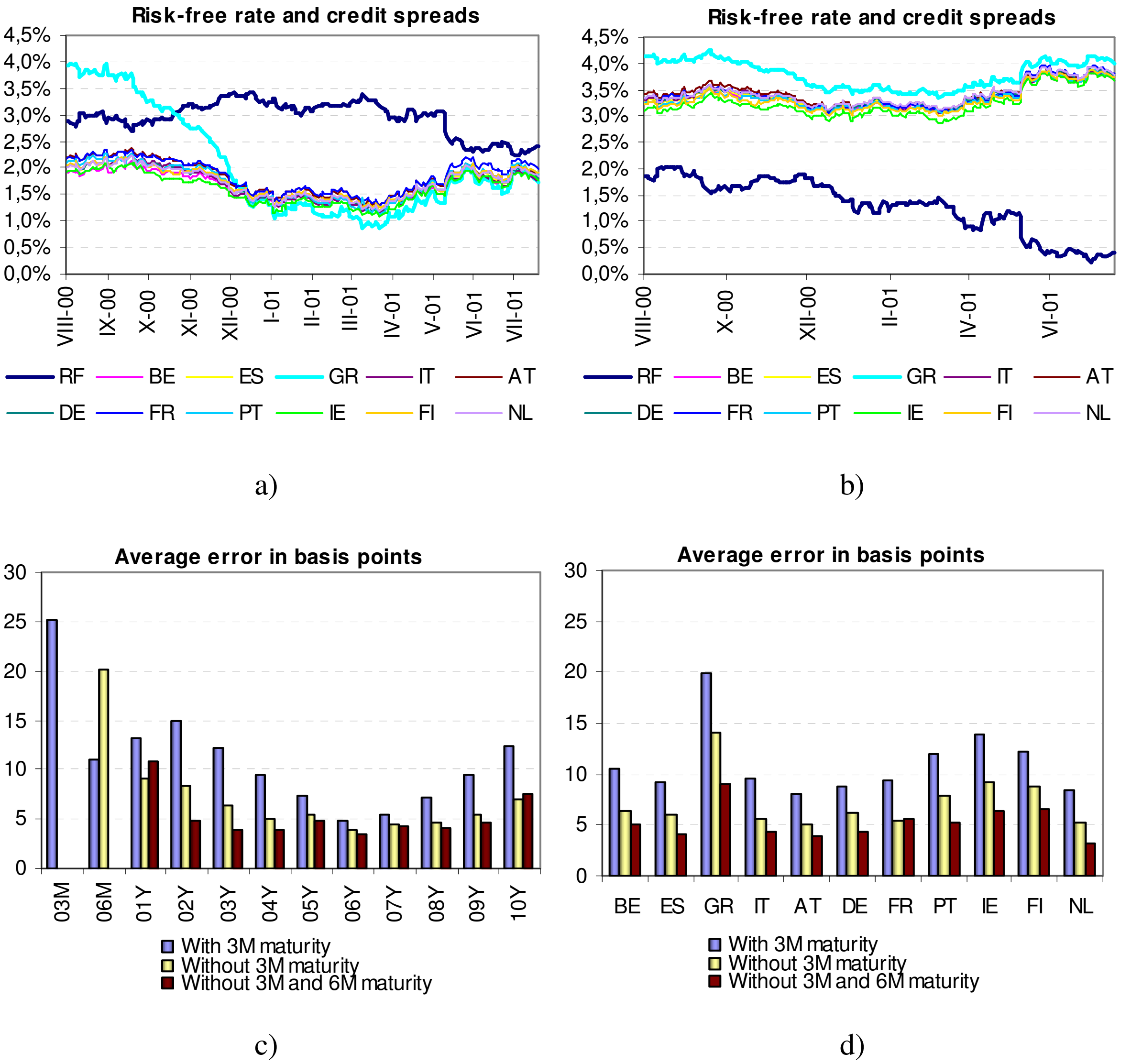


Figure 6 Summary results of the preliminary calibration. Estimates of the risk-free rate and credit spreads for 250 days for the whole yield curve from 3 months to 10 years a) and for the truncated yield curve from 6 months to 10 years b). Average error in basis points calculated for all countries for the respective maturities c) and for all maturities for the respective countries d). RF denotes the risk-free rate, country abbreviations the individual credit spread of the respective countries.

The calibration results are quite poor for Greece compared to other countries, especially for the three month maturity bucket where the average error for Greece exceeded 50 basis points. The average error was below 10 basis points only for the seven year maturity, while for the other maturities the average error ranged from 12 to 25 basis points. This is clearly due to the convergence of the Greek yields to the group of the other euro area yields until January 1, 2001. The overall average error excluding Greece would be 10.2 basis points for the whole curve and decreases to 6.6 basis points when the three month maturity is excluded and to 4.4 basis points when also the six month maturity is excluded. It is a coincidence that the countries where the calibrations was most imprecise are Greece, Ireland and Portugal, the countries which were at the onset of the euro area debt crises, however including Finland, which did not experience any difficulties in the recent crisis.

Table 2 Parameter estimates for the risk-free rate and credit spreads of euro area sovereigns for the initial 250 time observations and the full yield curve.

Country	κ, κ_i	θ, θ_i	σ, σ_i	λ, λ_i	Average error	Adjustment speed
Risk-free rate	0.4618	0.0204	0.0299	-3.41	NA	0.0018
Austria	0.2302	0.0080	0.0116	-20.07	7.99	0.0009
Belgium	0.1403	0.0142	0.0074	-21.46	10.44	0.0005
Finland	0.1042	0.0163	0.0062	-17.51	12.15	0.0004
France	0.0402	0.0114	0.0045	-10.52	9.37	0.0002
Germany	0.1168	0.0110	0.0044	-27.17	8.75	0.0004
Greece	1.6023	0.0080	0.0155	-77.21	19.81	0.0061
Ireland	0.1303	0.0180	0.0049	-25.69	13.84	0.0005
Italy	0.1830	0.0127	0.0066	-27.36	9.53	0.0007
Netherlands	0.1200	0.0138	0.0062	-19.96	8.42	0.0005
Portugal	0.1935	0.0203	0.0101	-11.06	12.01	0.0007
Spain	0.1472	0.0144	0.0061	-23.67	9.15	0.0006

Estimates of the parameters $\kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \lambda, \lambda_i$ for the CIR processes of the risk-free rate and credit spreads of the euro area countries for 250 time observations. The average error is in basis points. The adjustment speed is the magnitude of the mean reversion in the discretized CIR model (5.16).

The estimated parameters are broadly in range with the expectations. The long-term equilibrium rates are 2 % for the risk-free rate and 0.8 – 2.03 % for the credit spreads. The lowest value is attained by Austria and Greece. The lowest value for Austria is somewhat justified as it belonged to countries with the AAA rating until January 2012, although the expectations would be clearly to have Germany and France with the lowest spreads. However the Greek value is influenced by the convergence during the first half of the observation period. In order for the mean-reversion process to be significant enough to pull the rate to the other credit spreads the estimated value of the Greek long-term equilibrium spread is too low. Interestingly Ireland and Portugal have the highest equilibrium spreads and these countries received aid during the current crisis.

The estimated volatilities are quite significant and range from 24 % of the average credit spread for France up to 75 % for Greece and 101 % for the risk-free rate. Most countries are in the range below 45 %, the exceptions are besides Greece also Austria (64 %) and Portugal (58 %). Together with the estimated parameters κ, κ_i, this results in the risk-free rate and credit spreads being mainly driven by the stochastic part of the process. The stochastic term is of an order higher than the mean-reverting speed. The only exception is Greece during the convergence period. However after the credit spread converted to the other euro area spreads the process started to be driven by the stochastic part as well. This means that the even in the case of a significant distance from the long-term equilibrium rate, the processes are driven mainly by further shocks and not by the mean-reversion. The risk-free rate has higher mean-reversion speed than the credit spreads and also a higher volatility.

According to Pearson and Tong-Sheng Sun [98] the adjusted short-rate $r^* = (1 - \lambda\sigma B)r$ can be interpreted as the expected return on the bond and the market price of risk λ as a risk premium factor. For both the risk-free rate as well as credit spreads the estimates of the market prices of risk are negative, resulting in a higher expected return of the bond than the rate. This suggests that there are other risks than credit risk not captured by the model, perhaps liquidity risk being the most prominent. The risk premium increases with time to maturity, which is expected because in the long-term investment horizon the bondholder is exposed to more risks. The risk premium is significantly lower for the risk-free rate than for the credit spreads, roughly 25 % of the credit spreads. The risk premium for the risk-free rate ranges from 2.5 % to 27.5 % based on maturity. This suggests that there is not that much extra risk connected to the risk-free rate. The range for the credit spreads is from a couple percent for the three month maturity up to 100% or even 200 % for the ten year maturity. The extreme case is Greece, where the convergence caused the risk premium range from 28.5 % up to 292 %.

6.4. Full calibration results with zero correlation

The results of the full calibration using the data from August 1, 2000 until February 3, 2012 for maturities from six months up to ten years are presented in Figure 7. The model estimates are in line with the expectations until the collapse of the Lehman Brothers. We can observe the risk-free rate to drop during the dot-com bubble economic slowdown in 2001 – 2003 when the euro area GDP growth dropped from 3.8 % in 2000 to

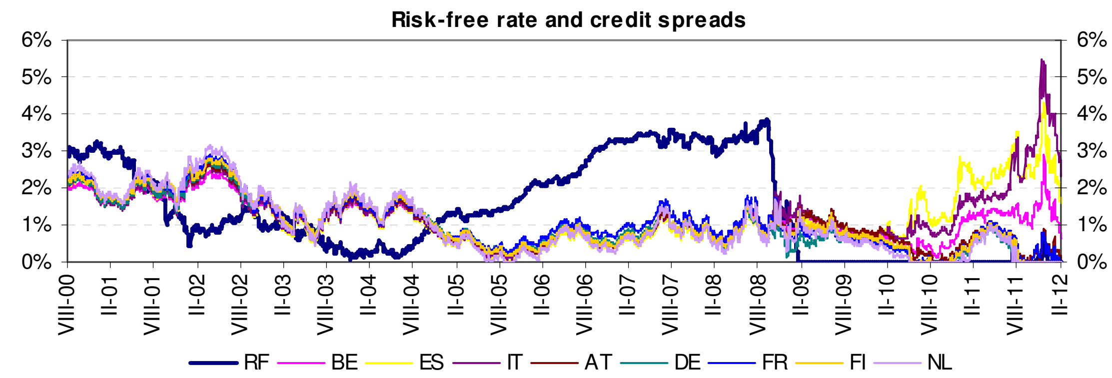
0.7 % in 2003[25]. The credit spreads remained elevated during that period. During the boom period afterwards that ended in 2008 the risk-free rate increased, while the credit spreads decreased and remained subdued around or below 1 %.

Shortly after the collapse of Lehman Brothers the risk-free rate sharply drops to around 1.5 % in October 2008, while in January 2009 it jumps further down and eventually reaches zero in mid January 2009. The risk-free rate remains zero until the end of the observation period with only a handful of points above zero, mostly between July and December 2011. This suggests that in such deep financial crisis the usual requirement of non-negative interest rates might be in contradiction with the fundamental economic situation. The situation does not change even if the model is calibrated only on the period after the collapse of Lehman Brothers. We tried to estimate also the development of the risk-free rate and credit spreads relaxing the nonnegativity constraint for the risk-free rate and credit spreads. We can see that the estimates differ only in the area where the nonnegativity constraints are active. See Figure 9. The risk-free rate drops as low as -1.66 % and recovers to positive values only in the second half of 2011, but return to negative territory in the end of 2011 during the intensified discussions about the Greek default and the details of the debt restructuring.

The development in credit spreads after the bankruptcy of Lehman Brother is also in line with expectations. During the volatile period immediately after collapse of Lehman Brother we can observe increased dispersion in credit spreads with the German spreads being significantly lowest due to the flight-to-safety market behavior. After easing of the situation the dispersion among spreads narrowed again. The spreads started to widen again after the publication of the Eurostat report on Greek data [57] on January 11, 2010.

The first spreads to increase were Greek, Portuguese and Irish spreads, later followed by Italian and Spanish and finally also by the Belgian spreads. The other spreads remained subdued and even declined further, except the period November 2010 to July 2011 when the uncertainty about the future steps and ability of the euro area to solve the crisis were heightened.

[25] Source: Eurostat



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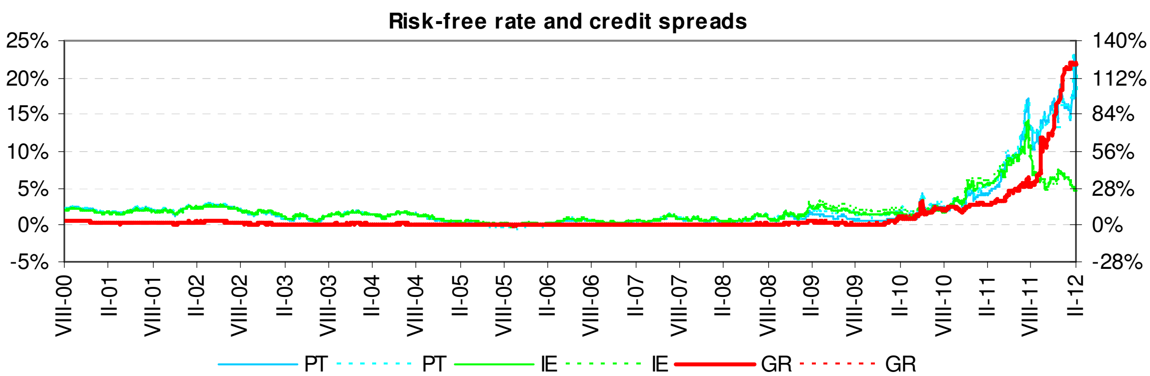

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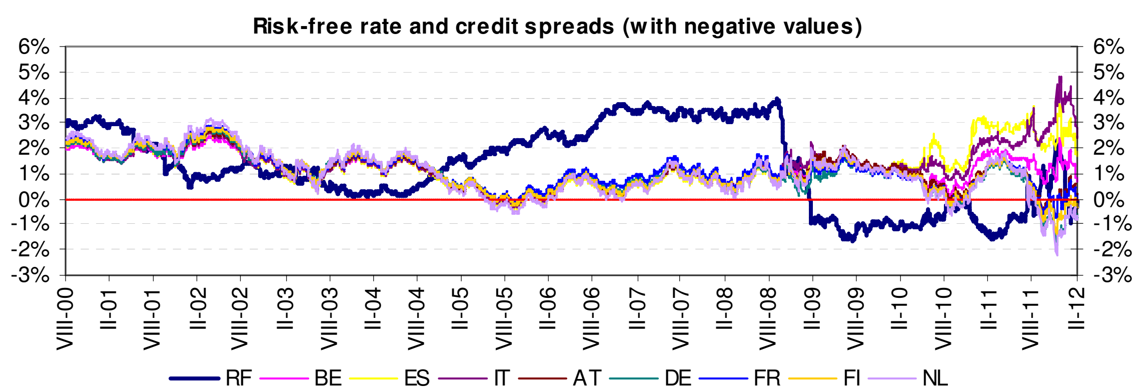
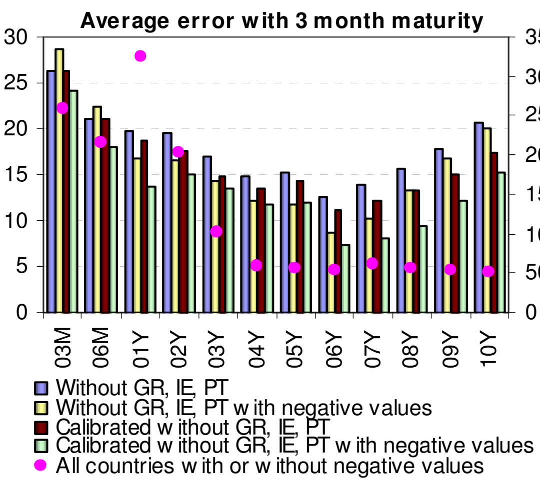


c)

Figure 7 Summary results of the full period calibration. Estimates of the risk-free rate and credit spreads for the full yield curve for the less risky countries with nonnegativity constraint a), without nonnegativity constraint c) and for Greece, Portugal and Ireland b) both with (solid line) and without (dashed line) nonnegativity constraints. In b) GR on right hand side axis, PT and IE on left hand side axis. RF denotes the risk-free rate, country abbreviations the individual credit spread of the respective countries.

During the calmer part of the crisis in the third quarter of 2010 and since August 2011 the spreads of the sovereign with the lowest spreads, Germany, Netherlands, Finland

and partially France, were also zero. The model was not able to capture the unusually low yields of these countries as the observed yields were significantly lower than the constant term[26] in (4.24). This also supports the idea that in such extraordinary circumstances it is plausible that interest rates might become negative.



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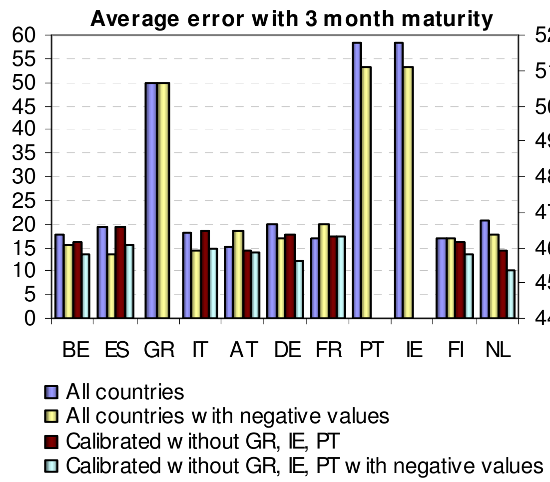

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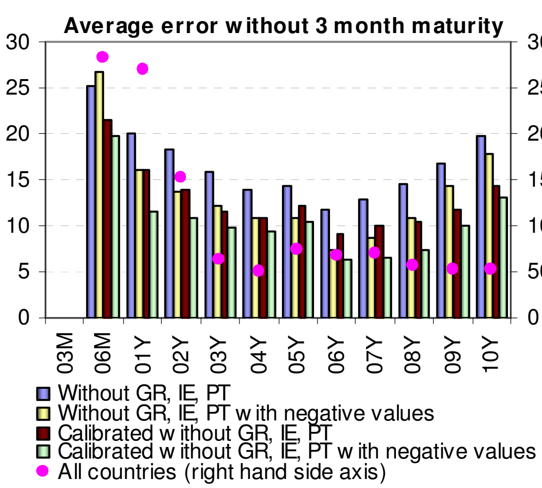

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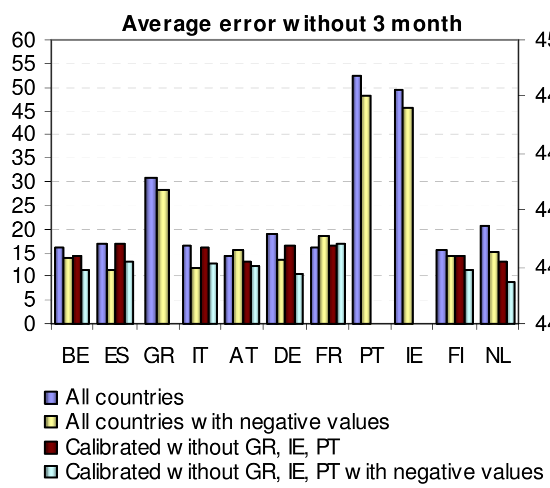


d)

Figure 8 Average error in basis points for the full calibration of the full yield curve a) and b) and of the truncated yield curve with maturities from 6 months to ten years c) and d). In b) and d) Greece on the right-hand side axis. RF denotes the risk-free rate, country abbreviations the individual credit spread of the respective countries.

The overall model fit is much worse than the fit using only the first 250 observations. The overall average error with and without the three month maturity is 156

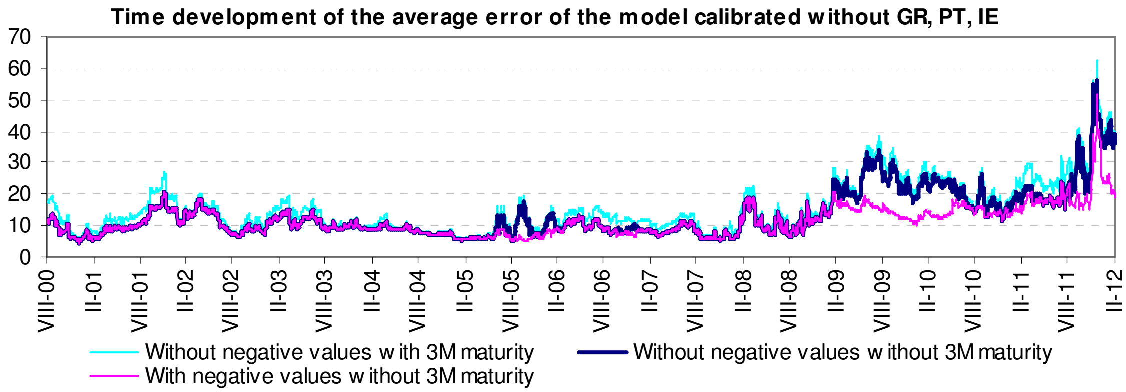
[26] By constant we mean the term which do not depend on the risk-free rate and credit spread.

and 131 basis points respectively. However this is mostly due to Greece and partially due to Portugal and Ireland. The model was not able to capture the unusually high yields of countries which would default without coordinated international support or which defaulted as Greece. The overall average error is 31 and 26 basis points respectively without Greece and 18 and 16 basis points respectively without Greece, Portugal and Ireland. The average error improves by 1.5 and 3 basis points respectively if negative values for the risk-free rate and credit spreads are allowed. If Greece, Portugal and Ireland are excluded from calibration the overall average error is 17 basis points or 14 without the three month maturity. The estimate is further improved by 1.5 and 3 basis points respectively if negative values for the risk-free rate and credit spreads are allowed. The average error is displayed in Figure 8.

The fit is significantly worse since the collapse of Lehman Brothers, as the credit spread could not capture all the dynamics in the yield curve in the period when the risk-free rate was zero. Also the inability to capture the very low yields of the least risky sovereigns during the crisis contributed significantly to the worse overall fit. The fit improves significantly if negative interest rates are allowed. For the results see Figure 9. If Greece, Portugal and Ireland are excluded from the model, the overall shape of the rates does not change significantly. The risk-free rate is somewhat higher while the credit spreads are somewhat lower. In both cases the risk-free rate is zero in the crisis period.

However, here the troubles start. In order to find the original parameters we would like to maximize the likelihood function. However if the risk-free rate or the credit spread is zero, the likelihood function is not defined. We tried to circumvent the problem by defining the likelihood function as zero for the points where the risk-free rate or credit spreads are zero. This allowed us to calculate the likelihood function and estimate the original parameters of the CIR processes.

The estimates of the volatilities ranged from 0.003 to 0.025 for the risk-free rate. Also the estimate for the risk-free rate were good with $\kappa = 0.61$ and $\theta = 0.018$. We can observe and increased mean-reversion speed and a decreased long-term equilibrium rate compared to the preliminary calibration. However the estimates for the mean-reversion speed and the long run rate for the credit spreads were negative, with e.g. $\theta_{BE} = -0.495$ and $\kappa_{BE} = -0.003$, which cannot be considered satisfactory.



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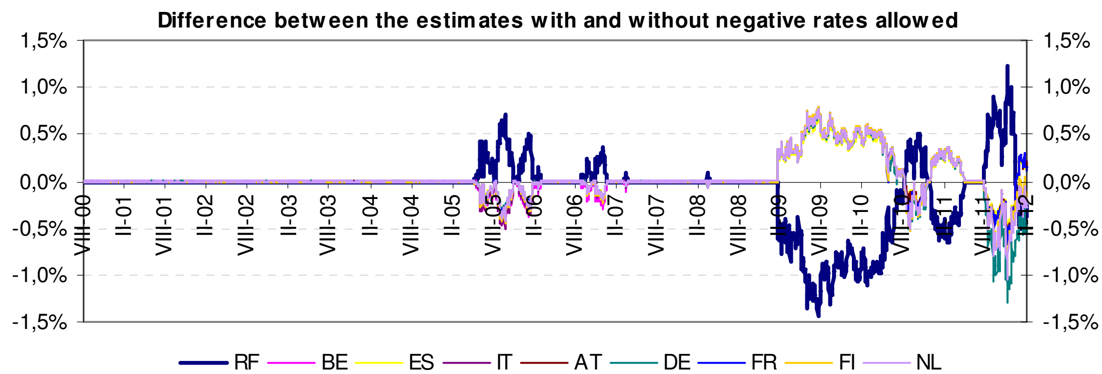

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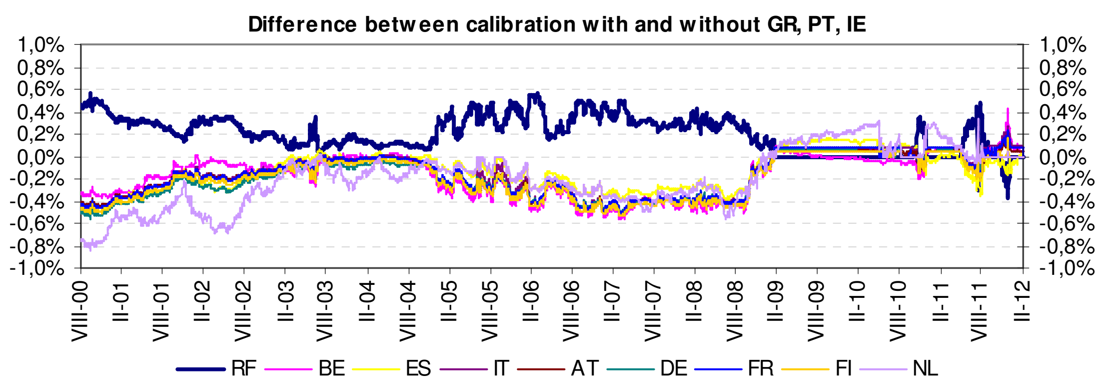


c)

Figure 9 Comparison of the calibration with and without negative rates and with and without GR, PT, IE. Average error for every day of the observed period a). Estimates of the risk-free rate and credit spreads in the model excluding GR, PT, IE with allowed negative rates minus estimates of the risk-free rate and credit spreads in the model excluding GR, PT, IE without allowed negative rates b). Estimates of the risk-free rate and credit spreads using all data including GR, PT, IE minus estimates when GR, PT and IE were excluded from the model with negative values not allowed c).

What is even worse is that the assumption of zero correlation between the risk-free rate and credit spreads for the studied data is violated. For the whole period the correlation ranges between -0.13 for Portugal to -0.44 for Finland. The correlation is present also if Greece, Portugal and are excluded from the model. The correlation increases if the three month maturity is retained in the model to -0.22 to -0.5 and to -0.29 to -0.56 if the period is limited to prior of the collapse of Lehman Brothers. For the preliminary calibration the correlation even reached -0.7. There is also a strong positive correlation between credit spreads. The correlation is lower for Greece, Portugal and Ireland and stronger for the other credit spreads ranging from 0.2 to 0.9.

We tried to solve this issue by introducing the multivariate likelihood function for the vector $\boldsymbol{r}$, where each item in $\boldsymbol{r}$ is described by the discretized model (5.16). The error vector $\boldsymbol{\varepsilon}_{t+\Delta t}$ has a multinormal distribution with zero mean and a covariance matrix $\Sigma = \left(\sigma_{i,j}\right)_{i,j=0\ldots n}$ defined by:

$$\sigma_{i,j} = \sigma_{j,i} = \rho_{i,j} \frac{\sigma_i \sigma_j}{2\sqrt{\kappa_i \kappa_j}} \sqrt{1-e^{-2\kappa_i \Delta t}} \sqrt{1-e^{-2\kappa_j \Delta t}} \sqrt{\boldsymbol{r}_i \boldsymbol{r}_j} \ , \tag{6.1}$$

where $\boldsymbol{r}_0 = rf_t$, $\boldsymbol{r}_i = cs_t^i$, $\kappa_0 = \kappa$, $\sigma_0 = \sigma$, $\rho_{i,i} = 1$ and $\rho_{0,i} = \rho_{i,0} = \rho_i$ for all $i > 0$.

The multivariate log likelihood function for this model is then defined by:

$$\ln L(\psi) = \sum_{t=2}^{N} \left(-\frac{n+1}{2} \ln(2\pi) - \frac{1}{2} \det(\Sigma) - \frac{1}{2} \boldsymbol{\varepsilon}_t' \Sigma^{-1} \boldsymbol{\varepsilon}_t \right). \tag{6.2}$$

We tried to find the maximum of (6.2) with a simple correlation structure $\rho_i = a, i \in \{GR, IE, PT\}$, $\rho_i = b, i \notin \{GR, IE, PT\}$, $\rho_{i,j} = c$, $i, j \in \{GR, IE, PT\}$, $\rho_{i,j} = d$, $i, j \notin \{GR, IE, PT\}$ and $\rho_{i,j} = \rho_{j,i} = e$, $i \in \{GR, IE, PT\}, j \notin \{GR, IE, PT\}$.

The estimated coefficients maximizing the multivariate log-likelihood function are presented in Table 3 for all countries and in Table 4 for the model excluding Greece, Ireland and Portugal. Generally the estimates of volatilities are good and comparable to those estimated in the preliminary calibration. Most of them slightly decreased except Finland, France and Netherlands. The processes are still volatility driven. The estimates of the mean reversion speed and long term equilibrium rate can be considered satisfactory only for the risk-free rate. This applies also to the estimation excluding Greece, Portugal and Ireland.

Table 3 Parameter estimates for the risk-free rate and credit spreads of euro area sovereigns for the full period without three month maturity using the multivariate likelihood function.

Country	κ, κ_i	θ, θ_i	σ, σ_i	λ, λ_i	Average error	Adjustment speed
Risk-free rate	0.4561	0.0242	0.0253	0.297	NA	0.00175
Austria	**-0.0114**	**-0.1660**	0.0049	0.106	14.50	-0.00004
Belgium	**-0.0473**	**-0.0317**	0.0029	-0.222	16.16	-0.00018
Finland	0.0074	0.2558	0.0103	-0.137	15.74	0.00003
France	0.0021	0.3932	0.0035	0.216	16.10	0.00001
Germany	0.0065	0.2493	0.0048	0.106	19.18	0.00003
Greece	0.4117	0.0313	0.0155	-0.541	**445.15**	0.00158
Ireland	0.0073	0.3200	0.0049	-0.365	**49.62**	0.00003
Italy	**-0.0008**	**-3.0554**	0.0085	-0.033	16.48	-0.00000
Netherlands	0.1114	0.0318	0.0158	-0.074	20.69	0.00043
Portugal	0.0843	0.0466	0.0101	-0.241	**52.46**	0.00032
Spain	0.0104	0.2456	0.0073	-0.247	16.97	0.00004

Estimates of the parameters $\kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \lambda, \lambda_i$ for the CIR processes of the risk-free rate and credit spreads of the euro area countries for full time period without three month maturity. The average error is in basis points. The adjustment speed is the magnitude of the mean reversion in the discretized CIR model (5.16).

Interestingly estimates of the market prices of risk decreased significantly and the risk premium are in absolute value lower than 0.03 for almost every country and maturity. Exceptions are the risk-free rate and Greece, where for the 10 year maturity the risk-premium reaches to 0.1 for the risk-free rate and -0.1 for Greece. The market prices of risk are positive for the risk-free rate and the lowest risky countries like Germany, France and Austria and negative for all the others. In the model without Greece, Portugal and Ireland the risk premium is lower than 0.015 for all countries and maturities suggesting that the model captures correctly the risks embedded in the investments in sovereign bonds.

The estimated correlations between the risk-free rate and credit spreads are -0.03 for Greece, Portugal and Ireland and -0.09 for the other countries. The correlations among credit spreads are 0.3 – 0.35. However the correlation in the residuals even increased and ranges from -0.36 to -0.65 between the risk-free rate and credit spreads and 0.1 – 0.95 between credit spreads. If Greece, Portugal and Ireland are excluded from the model, the correlation between the risk-free rate and credit spreads is estimated to be -0.03 and the correlation among credit spreads 0.3. However the remaining correlation in the residuals is -0.5 between risk-free rate and credit spreads and 0.4 – 0.9 for credit spreads. Therefore the correlation problem was not fixed and the estimate of the original parameters of the CIR processes is not good as well. Therefore we decided to try the final model specification and its results are described in the next section.

Table 4 Parameter estimates for the risk-free rate and credit spreads of euro area sovereigns for the full period without three month maturity and without Greece, Portugal and Ireland using the multivariate likelihood function.

Country	κ, κ_i	θ, θ_i	σ, σ_i	λ, λ_i	Average error	Adjustment speed
Risk-free rate	0.3208	0.0294	0.0305	0.028	NA	0.00123
Austria	**-0.0051**	**-0.3070**	0.0108	-0.110	13.18	-0.00002
Belgium	**-0.0060**	**-0.3007**	0.0070	-0.144	14.37	-0.00002
Finland	0.0047	0.3331	0.0060	0.173	14.36	0.00002
France	0.0034	0.1201	0.0020	0.002	16.57	0.00001
Germany	**-0.0122**	**-0.0873**	0.0040	0.048	16.68	-0.00005
Italy	0.0028	0.7553	0.0064	-0.081	16.16	0.00001
Netherlands	**-0.0067**	**-0.2195**	0.0059	0.055	13.10	-0.00003
Spain	**-0.0299**	**-0.0466**	0.0051	-0.080	16.73	-0.00011

Estimates of the parameters $\kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \lambda, \lambda_i$ for the CIR processes of the risk-free rate and credit spreads of the euro area countries for full time period without three month maturity and without Greece, Portugal and Ireland. The average error is in basis points. The adjustment speed is the magnitude of the mean reversion in the discretized CIR model (5.16).

6.5. Full calibration results with non-zero correlation

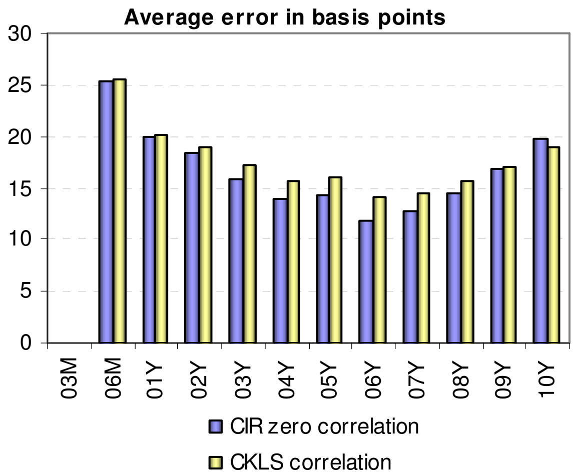
In this section results for the full period calibration using the approximate formula for the CKLS specification of the model with correlations are presented. The estimates of the risk-free interest rate and the credit spreads were taken from the estimation of the model with zero correlation. Summary results are presented in Table 5. The estimated powers in the CKLS model are 0.57 for the risk-free rate and 0.46 for the credit spreads, so the CIR model specification is optimal. The estimated correlations are in line with the calculated ones with the exception of Netherlands, where the estimated correlation is higher than the calculated. The correlation for Greece, Ireland, Portugal and France is lower than for the other countries.

Table 5 Parameter estimates for the final model specification for the risk-free rate and credit spreads of euro area sovereigns for the full period without three month maturity.

Country	κ, κ_i	θ, θ_i	σ, σ_i	λ, λ_i	1 ρ_i	2 ρ_i	Average error	Adjust-ment speed
Risk-free rate	0.420	0.023	0.063	0.178	NA	NA	NA	0.0016
Austria	0.030	0.075	0.072	-0.126	-0.35	-0.38	16.35	0.0001
Belgium	0.036	0.091	0.108	-0.066	-0.42	-0.43	18.53	0.0001
Finland	0.045	0.078	0.012	0.204	-0.48	-0.44	18.31	0.0002
France	0.047	0.009	0.171	-0.134	-0.31	-0.25	18.08	0.0002
Germany	0.025	0.043	0.091	-0.166	-0.43	-0.42	18.67	0.0001
Greece	0.413	0.003	0.665	-0.249	-0.28	-0.32	442.29	0.0016
Ireland	0.019	0.100	0.120	-0.140	-0.29	-0.29	47.72	0.0001
Italy	0.075	0.058	0.002	-0.748	-0.43	-0.39	18.11	0.0003
Netherlands	0.105	0.030	0.021	-0.490	-0.61	-0.43	16.12	0.0004
Portugal	0.084	0.058	0.080	-0.014	-0.12	-0.13	51.98	0.0003
Spain	0.064	0.066	3.2E-06	0.522	-0.42	-0.39	18.87	0.0002

Estimates of the parameters $\kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \lambda, \lambda_i, \rho_i$ of the final model specification for the processes of the risk-free rate and credit spreads of the euro area countries for full time period without three month maturity. The average error is in basis points. The adjustment speed is the magnitude of the mean reversion in the discretized model (5.16). Estimated 1) and calculated 2) correlation coefficients.

The estimated mean-reversion speed and long-term equilibrium rate for the risk-free rate are almost the same as for the model with zero correlation. However the volatility estimate is significantly higher. But still the process is mainly volatility driven. This can be confirmed also for the credit spreads. The estimated volatility is higher except for Italy and Spain and for these two countries the process is driven by the mean-reversion for low spreads around 1 %. Greece has the highest volatility, which is expected, but the second highest volatility for France is surprising. The estimated mean-reversions are higher, but not enough to drive the process. Quite surprising are the high levels of the long-term equilibrium rates for the credit spreads. The estimate for Greece is misleading. Due to the high levels of credit spreads during the crisis, the estimated spread during the benign period is close to zero for much of the period, which drives the equilibrium rate to be very low. However the low estimate for France cannot be explained. Otherwise the ranking of the long-term equilibrium rate is quite in line with expectations. The estimated market-prices of risk are low and negative except the risk-free rate and Finland. The risk-premium is significant mostly for Greece followed by France, Ireland and Germany. This suggests that for France and Germany there is some other aspect the model cannot capture. This is quite obvious for Greece and Ireland.



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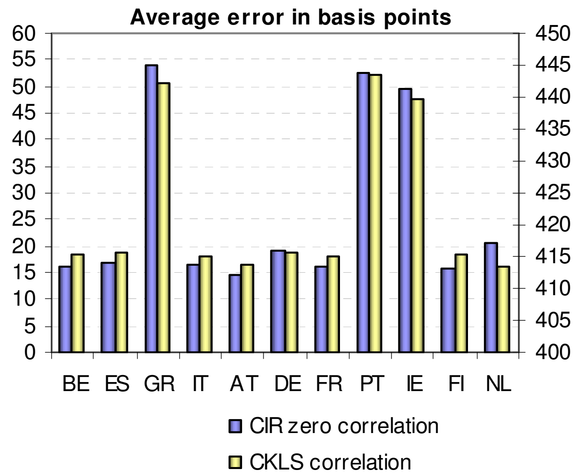

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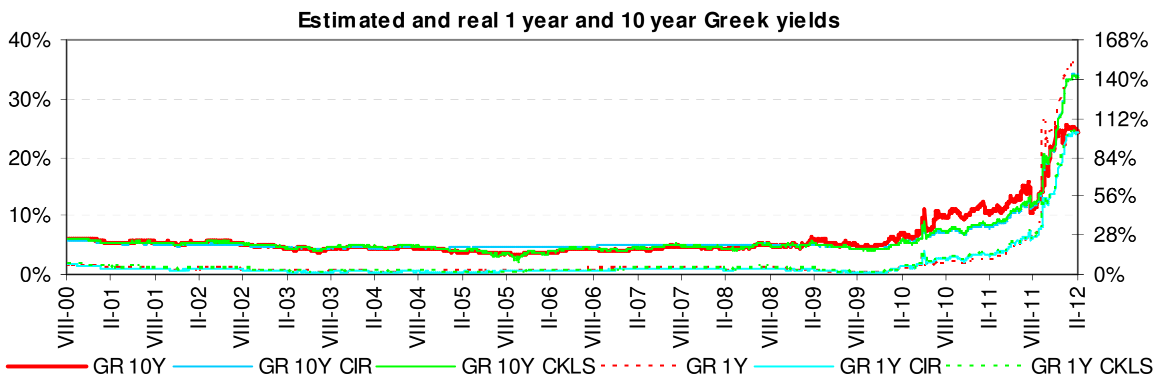

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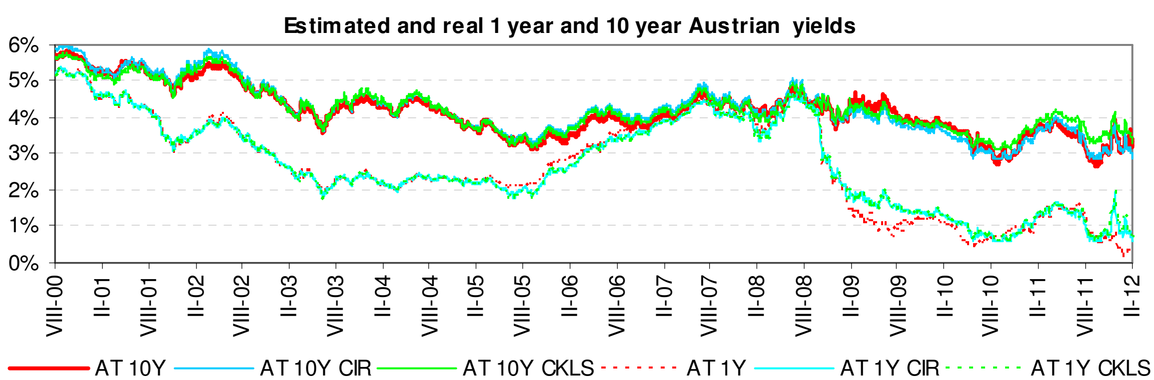


d)

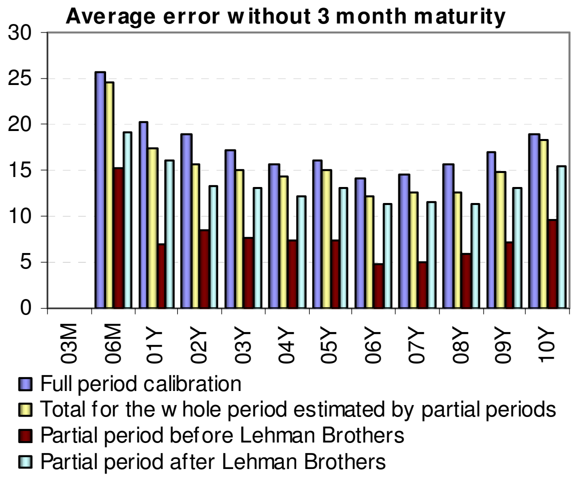
Figure 10 Comparison between the estimates of the CIR models without correlation and the CKLS model with correlation. Average error in basis points for the full calibration of the truncated yield curve with maturities from 6 months to ten years without GR, PT, IE a) and b). Comparison of the fit of the Greek c) and Austrian d) yield curve. In b) Greece on the right-hand side axis. In c) the one year rates are on the right-hand side axis. In c) and d) the CIR denotes the CIR model with zero correlation, CKLS denotes the CKLS model with correlation and the rate without CIR and CKLS denotes the real yield. RF denotes the risk-free rate, country abbreviations the individual credit spread of the respective countries.

The overall model fit is as bad as in the case of zero correlation with the average error being 130 basis points. This is the same as in the case of zero correlation, while the cost function is somewhat better. However this is again mostly due to Greece, Portugal and Ireland, without whom the average error decreases to 17 basis points, one basis point higher than in the case of the model with zero correlation. Thus this model can better capture the yield curves of Greece, Portugal and Ireland and the expense of poorer fit for the other countries.

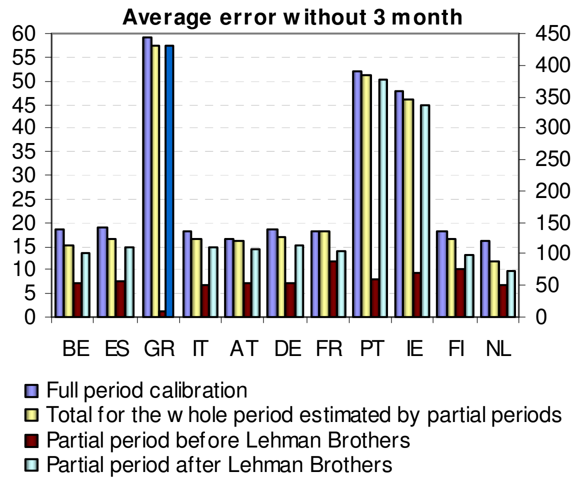
Comparison between the model with zero-correlation and the model final specification is shown in Figure 10. Two countries are displayed, Greece with the worst fit and Austria with the best fit. We can observe that both models give almost the same results for the short-term maturities and differ more for the long-term maturities and that the fit is better for short-term maturities and worse for the long-term maturities. The highest differences both between the models and between the models and the observed yields are during the financial crises, where the rates should be zero.

We estimated the model also for two periods – before the collapse of Lehman Brothers and after the collapse of Lehman Brothers. The results are shown in Figure 11. The estimated shape of the risk-free rate and credit spreads is similar to the one estimated for the full period, however it is more pronounced. That means that generally in "good" times the model estimated on the shorter period of time tend to estimate the risk-free rate higher and the credit spread lower and in "bad" time vice versa. Generally the fit is very good for the period before the collapse of Lehman Brother and worse afterwards, which is expected.

The overall fit is improved. However the model was not able to estimate Greek, Portuguese and Irish yields during the crisis. Before the collapse of Lehman Brother the average error was 10 basis points including Greece, Portugal and Ireland. France had the worst fit with average error of 12 basis points followed by Finland with 10. Netherlands had the best fit with the average error of 6.7 basis points. The six month maturity had the worst fit with average error of 15 basis points while the other maturities were captured relatively good with the average error ranging from 4.8 to 9.6 basis points. During the financial crisis after the collapse of Lehman Brothers the model fit worsens, but is still better than the fit of the model calibrated for the full period.



a)



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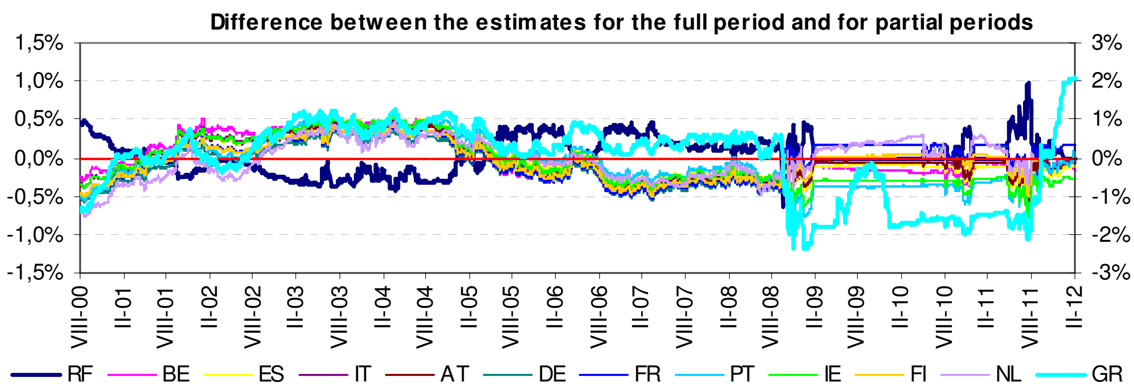


c)

Figure 11 Comparison between the estimate of the CKLS model with correlation for the full period of time and for the separate periods before and after the collapse of Lehman Brothers. Average error in basis points for the truncated yield curve with maturities from 6 months to ten years without GR, PT, IE a) and b). Estimated risk-free rate and credit spreads for the full period minus estimated risk-free rate and credit spreads for the partial periods c). In b) Greece on the right-hand side axis. RF denotes the risk-free rate, country abbreviations the individual credit spread of the respective countries.

The periods also differ with respect to the model properties. For the full period the CIR specification was justified. However, for the period before the collapse of Lehman Brothers the CIR specification is confirmed only for the risk-free rate, where the estimated coefficient is 0.55, but for the credit spreads it increases to 1. For the period after the collapse of Lehman Brothers the situation is opposite. The estimated power for the risk-free rate rises to 1.64 while for the credit spreads it is 0.74. The estimated correlations are in the first period in line with the real ones except Belgium, where the correlation is overestimated and Germany, where it is underestimated. However in the second period the correlations are poorly estimated and there is little correlation observed in the data. However this could be caused by the fact that there are only 316 days out of 855 since the collapse of Lehman Brothers where the risk-free rate is not zero.

Table 6 Parameter estimates for the risk-free rate and credit spreads of euro area sovereigns for the period before and after the collapse of Lehman Brothers without three month maturity for the final model specification.

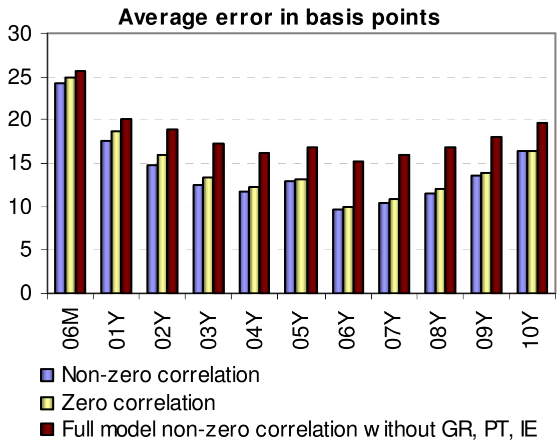
	Before collapse of Lehman Brothers						After collapse of Lehman Brothers					
	κ,κ_i	θ,θ_i	σ,σ_i	λ,λ_i	1 ρ_i	2 ρ_i	κ,κ_i	θ,θ_i	σ,σ_i	λ,λ_i	1 ρ_i	2 ρ_i
RF	0,240	0,033	0,021	0,198	NA	NA	0,136	2E-05	0,281	5,132	NA	NA
AT	0,006	0,177	0,271	-0,072	-0,44	-0,49	0,025	0,293	0,219	-0,035	-0,19	-0,11
BE	0,009	0,176	0,283	-0,014	-0,98	-0,44	0,176	0,070	0,368	-0,034	-0,91	-0,14
FI	0,008	0,153	0,046	0,026	-0,51	-0,51	0,139	0,071	0,192	-0,015	-0,22	0,18
FR	0,002	0,220	0,087	0,211	-0,54	-0,55	0,411	0,027	2,104	-0,115	-0,91	-0,12
DE	0,025	0,045	0,208	-0,129	-0,18	-0,51	0,226	0,036	0,407	-0,282	-0,95	-0,16
GR	0,009	0,208	0,051	0,063	-0,45	-0,56	0,452	4E-04	4,882	-0,389	-0,42	0,04
IE	0,187	0,010	0,124	-1,430	-0,46	-0,47	0,051	0,170	0,288	-0,186	-0,84	0,09
IT	0,084	0,024	0,128	-0,515	-0,42	-0,52	0,171	0,075	0,005	-0,641	0,16	-0,14
NL	0,010	0,131	0,000	0,840	-0,56	-0,52	0,173	0,051	0,412	-0,232	0,18	-0,14
PT	0,030	0,056	0,245	-0,071	-0,17	-0,29	0,098	0,114	1,381	-0,096	-0,23	0,03
ES	0,004	0,339	0,078	0,010	-0,57	-0,45	0,043	0,189	0,862	-0,096	-0,70	-0,13

Estimates of the parameters $\kappa,\kappa_i,\theta,\theta_i,\sigma,\sigma_i,\lambda,\lambda_i,\rho_i$ of the final model specification for the processes of the risk-free rate and credit spreads of the euro area countries for the two periods separated by the collapse of Lehman Brothers without three month maturity. Estimated 1) and calculated 2) correlation coefficients.

However the parameter estimates are not that encouraging at all. Parameter estimates are provided in Table 6. Estimates for the risk-free rate are good, with the long-term equilibrium rate dropping to almost zero after the collapse of Lehman Brothers and a dramatical increase in volatility. The mean –reversion is slowing down. For Portugal and Ireland we observe significant increase in the equilibrium rate and volatilities. For Greece we observe a significant increase in volatility as well. However the equilibrium rate drops from 20 % to zero, which cannot be justified. Also the estimates of the equilibrium rate for other countries are unrealistically high in both periods. All volatilities are significantly higher during the crisis period, but their values are extremely higher than then for the whole period of time. Also most of the volatilities in the first period are significantly higher than for the full period of time. On the other side we see higher mean-reversion speed, but not enough in order to significantly influence the development of the credit spread processes.

Finally the model performed quite well if estimated without Greece, Portugal and Ireland. The results for the whole period are presented in Figure 12 and Table 7. The estimated powers are 0.62 for the risk-free rate and 0.6 for the credit spread, so again close to the CIR specification. This is consistent with the model with zero correlation giving similar, but slightly worse results as the model with correlation. The estimates are much better than in the model with all countries. The average error is 14.7 basis points, which is half a basis point better than the model with zero correlation. The estimated correlations range from -0.25 to -0.4. The calculated correlations are practically zero. However this is caused by the fact that the risk-free rate and credit spreads are zero for almost a third of the whole observations period and of course in different time periods. If the correlations are calculated without the zero observations the estimated correlations are quite good.

The difference between the risk-free rate and credit spreads do not differ too much between the models with and without zero-correlation for most of the period. The risk-free rate is lower in the beginning of the observation period until mid 2002, then a little bit higher and vise-versa for the credit spreads. The estimated credit spreads during the good times in 2006 are even zero, which suggest that the investors were disregarding this risk very much during that times. During the financial crisis the estimated credit spreads in the model with correlation are higher for the risky countries and lower for the less risky countries. Again the fact that the processes are mainly volatility driven is confirmed. The stochastic term is on average of an order higher than the deterministic term. The long-term-equilibrium rates are well sorted with Italy and Spain having the highest, followed by Belgium, while on the lowest side we have France, Germany and Netherlands. The market price of the risk-free rate is positive and small, which results in no risk-premium. Belgium has also a positive market price of risk, which is unexpected, but the resulting negative risk-premium is below 1 %. For the other countries the market prices of risk are negative which results in positive risk-premium. However these are also small and below 2 % for the longest maturities and even smaller for the shorter ones.



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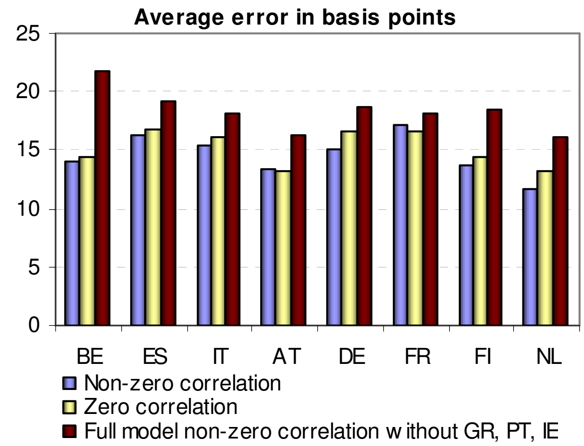

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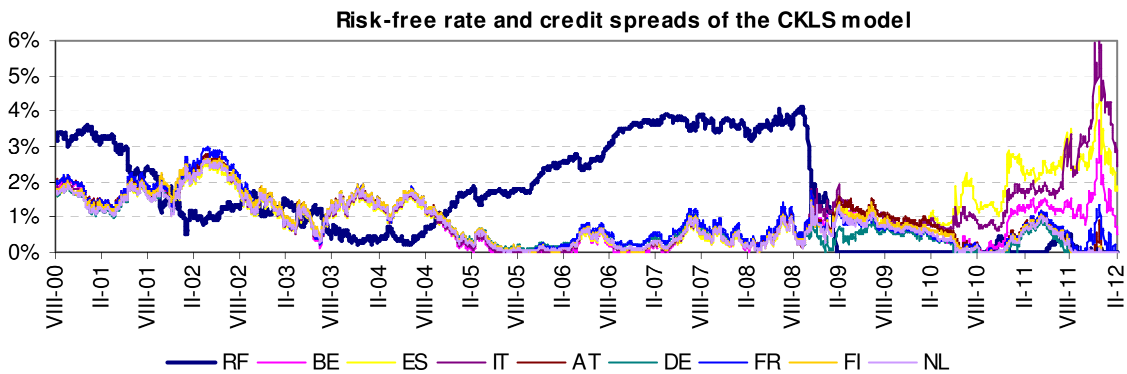

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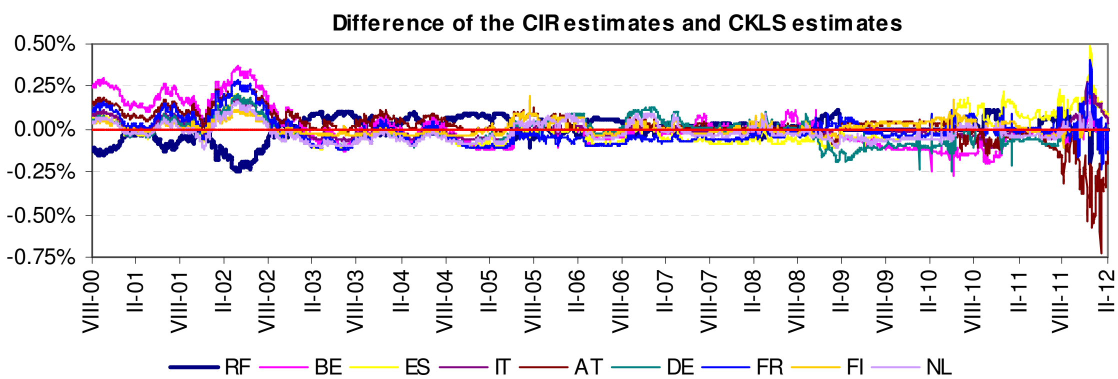


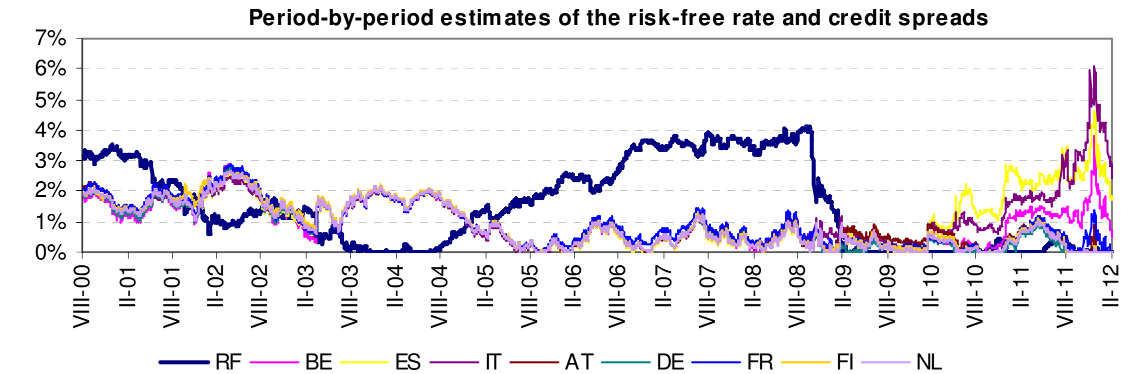
d)

Figure 12 Comparison between the estimate of the model with and without correlation for the full period of time for the truncated yield curve with maturities from 6 months to ten years estimated without Greece, Portugal and Ireland. Average error in basis points a) and b). Estimated risk-free rate and credit spreads c) and the difference between the estimated risk-free rate and credit spreads with and without correlation d). RF denotes the risk-free rate, country abbreviations the individual credit spread of the respective countries.

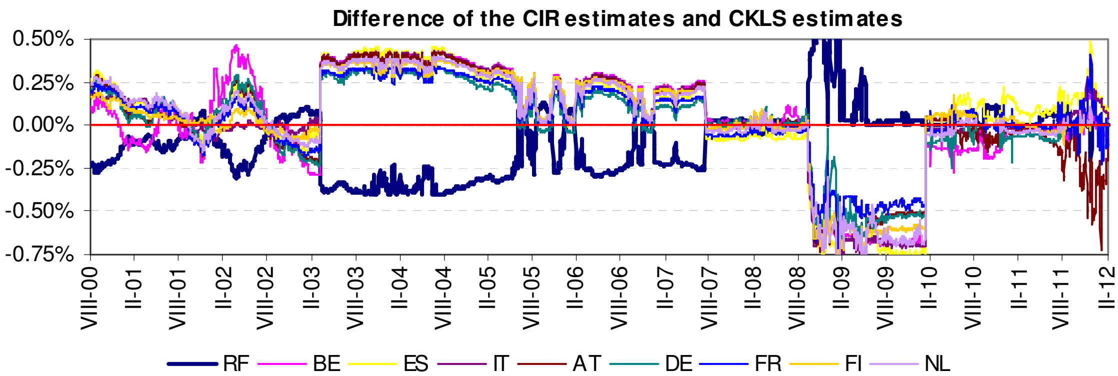
Table 7 Parameter estimates for the final model specification for the risk-free rate and credit spreads of euro area sovereigns for the full period without three month maturity calibrated without Greece, Portugal and Ireland.

Country	κ, κ_i	θ, θ_i	σ, σ_i	λ, λ_i	1 ρ, ρ_i	2 ρ, ρ_i	3 ρ, ρ_i	Average error
Risk-free rate	0.3060	0.0266	0.1084	0.0109	NA	NA	NA	NA
Austria	0.0421	0.0547	0.0163	-0.4688	-0.335	0.000	-0.15	13.30
Belgium	0.0500	0.0724	0.0100	0.5064	-0.397	-0.007	-0.35	14.09
Finland	0.0312	0.0599	0.0997	-0.1240	-0.248	0.001	-0.20	13.63
France	0.1153	0.0105	0.0530	-0.4374	-0.309	0.000	-0.45	17.14
Germany	0.0628	0.0238	0.1089	-0.2027	-0.383	0.010	-0.37	15.14
Italy	0.0299	0.0928	0.0986	-0.0820	-0.384	0.007	-0.43	15.38
Netherlands	0.0564	0.0362	0.0794	-0.2322	-0.344	0.000	-0.20	11.73
Spain	0.0233	0.0991	0.1312	-0.1019	-0.385	-0.002	-0.36	16.34

Estimates of the parameters $\kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \lambda, \lambda_i, \rho_i$ of the final model specification for the processes of the risk-free rate and credit spreads of the euro area countries without GR, PT, and IE for full time period without three month maturity. The average error is in basis points. Correlation coefficients - estimated 1), calculated 2) and calculated without zero observations 3).



a)



b)

Figure 13 Comparison between the period-by-period estimates and the full period estimate of the model with correlation for the truncated yield curve with maturities from 6 months to ten years estimated without Greece, Portugal and Ireland. Period-by-period estimated risk-free rate and credit spreads a) and the difference between the period-by-period and full period estimated risk-free rate and credit spreads b). RF denotes the risk-free rate, country abbreviations the individual credit spread of the respective countries.

The model was finally calibrated on the individual time periods between the possible break points identified in Chapter 6.2 using the final model specification. We will denote them as P1 – P5. The comparison to the full period estimates are presented in Figure 13 and Figure 14. The estimated risk-free rate and credit spreads are the same in P3 and P5, while significant differences are present in P2 and mainly in P4 after the collapse of Lehman Brothers. The movements in the estimates are symmetric in the sense that if the risk-free rate is higher than credit spreads are lower and vice-versa. We can observe another period of zero risk-free interest rate during the economic slowdown in 2003 – 2004 and also a period of zero spreads during the following boom in 2005 – 2006.

The overall average error decreased from 14.4 to 13.3 basis points. The value of the cost function (5.19) improves by 18 %. Out of this improvement 58 % can be attributed to P4 and 32 % to P2. The improvement is for every maturity except the two year and three year maturity and for every country. In P2 all countries gained significant improvements except Netherlands and in P4 except Austria and Germany. Periods P1, P3 and P5 combine only to 10.6 % of the total improvement. In these periods the development was mixed with some countries gaining and some loosing resulting in the overall improvement. Regarding the time aspect generally the long-term part of the yield curve was estimated more precisely at the expense of the short-term part in P5 and for the middle part of the curve in P3. The estimate in P5 is far worse that in other periods due to the need of zero values for the risk-free rate and credit spreads of the least risky countries in order to capture the very low yields of these countries.

Regarding the powers in the CKLS model specification these differ significantly only in P4, where the risk-free rate's power is 0.12 and the credit spreads' power is 1.07. This strongly increases the volatility present in the risk-free rate caused by the sharp drop after the collapse of Lehman Brothers. Otherwise the parameters are quite stable around 0.6 and around 0.5 during P1 suggesting the CIR model specification for most of the time. This explains also only minor improvement in the model estimates compared to the model with zero correlation.

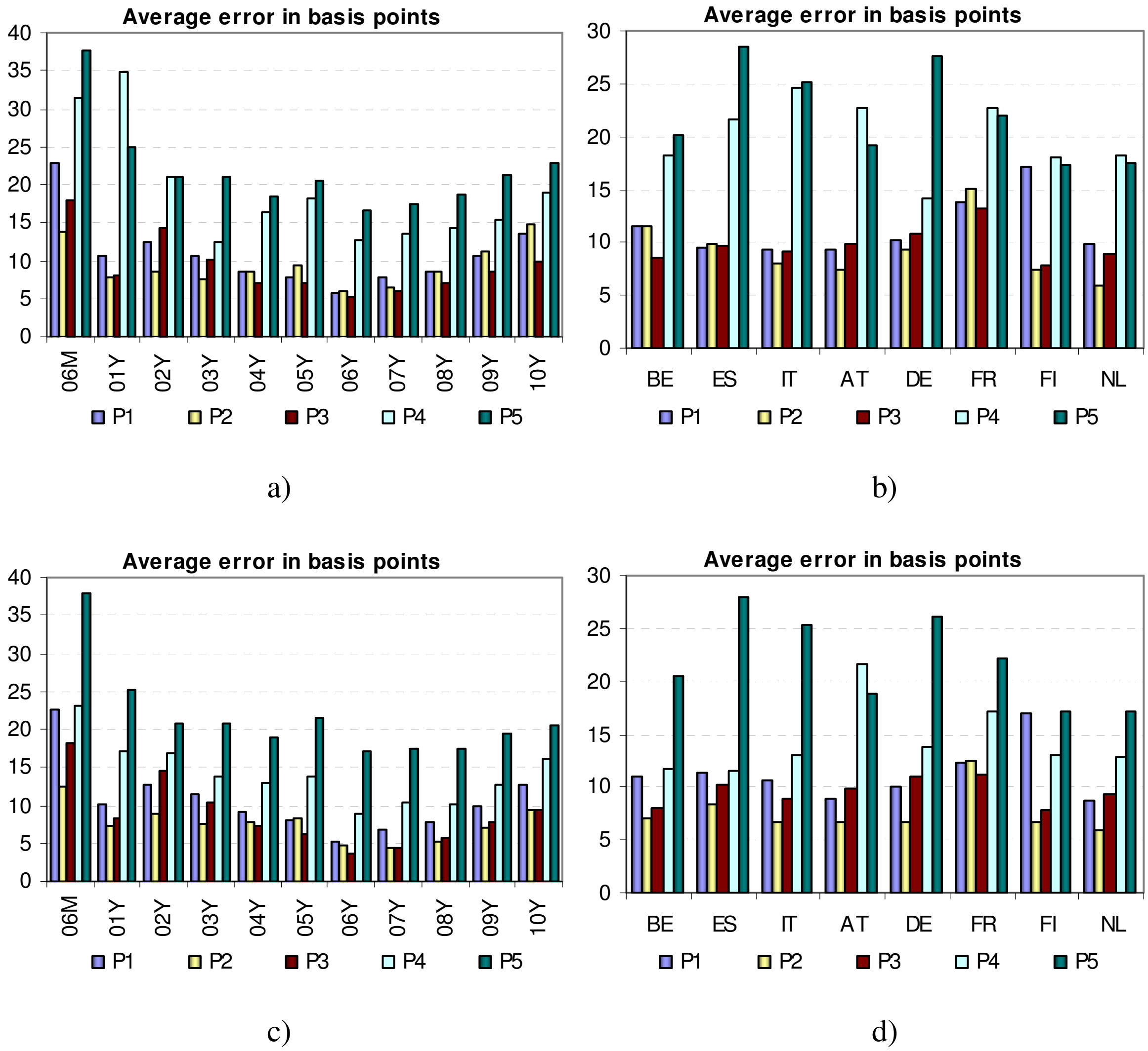


Figure 14 Comparison between the period-by-period estimates and the full period estimate of the model with correlation for the truncated yield curve with maturities from 6 months to ten years estimated without Greece, Portugal and Ireland. Average error in basis points for the full period calibration a) and b) and for the period-by-period calibration. P1 – August 1, 2000 – March 11, 2003, P2 – March 12, 2003 – July 15, 2007, P3 – July 16, 2007 – September 14, 2009, P4 – September 15, 2009 – January 8, 2010, P5 – January 9, 2010 – February 3, 2012

The parameter estimates are presented in Table 9. The estimated correlations are mostly in line with reality in P1 – P3. In P4 and P5 the calculated correlations are zero, however this is due to the zero estimates for the risk-free rate and some credit spreads. The estimated correlations are in line with P2 and therefore can be considered satisfactory.

Table 8 Average error and estimates of the powers for the period-by-period estimates for the final model specification for the risk-free rate and credit spreads of euro area sovereigns for the full period without three month maturity calibrated without Greece, Portugal and Ireland.

	Period-by-period			Full period		
Period	γ	γ_2	**Average error**	γ	γ_2	**Average error**
P1	0.4840	0.5111	11.49	0.6242	0.6000	11.68
P2	0.5773	0.4740	7.84	0.6242	0.6000	9.72
P3	0.6145	0.5983	9.63	0.6242	0.6000	9.89
P4	0.1237	1.0690	14.69	0.6242	0.6000	20.31
P5	0.6244	0.5738	22.26	0.6242	0.6000	22.59
Total	NA	NA	13.29	0.6242	0.6000	14.69

Average error and estimates of the parameters γ, γ_2 of the final model specification for the processes of the risk-free rate and credit spreads of the euro area countries without GR, PT, and IE for the period-by-period and full period calibration without three month maturity. The average error is in basis points. P1 – August 1, 2000 – March 11, 2003, P2 – March 12, 2003 – July 15, 2007, P3 – July 16, 2007 – September 14, 2009, P4 – September 15, 2009 – January 8, 2010, P5 – January 9, 2010 – February 3, 2012

The estimated parameters for the risk-free rate are quite robust. The long-term equilibrium rate ranges between 2.4 % and 3.2 % and the mean-reversion speed is around 0.3. The volatility ranges between 0.07 and 0.1. Although in P4 the estimated volatility decreases to 0.007 due to the very low power (0.12) the volatility is increased significantly, especially for low values of the risk-free rate. The actual volatility is thus increased despite the lower estimate of σ. The estimated market price of risk is positive and low, resulting with a negative risk-premium of less than 1% for all maturities. This is a property that would be expected from the risk-free rate.

The estimated volatilities and mean reverting speeds of the credit spreads are quite stable as well. The credit spread processes are also volatility driven in all periods except P4 where the changed power results in the mean reversion to gain importance for spreads near zero. The estimated volatilities are comparable to the risk-free rate, while the mean reverting speeds are significantly lower than for the risk-free rate. The estimates of the long-term equilibrium rates are quite volatile. We can observe similar parameter estimates in P3, P5 as for the full period estimates. In these periods the ordering of the estimates is the same as the experience during the financial crisis with France and Germany having the lowest spreads, followed by Netherlands, Austria and Finland while Italy and Spain having the highest spreads with Belgium a little behind them. The estimates vary from around 1% for France and 2% for Germany to almost 10 % for Italy and Spain. However for all the countries except France and Germany the equilibrium spread levels are way above the actual development of the spreads during the whole observed period.

Table 9 Parameter estimates for the period-by-period estimates for the final model specification for the risk-free rate and credit spreads of euro area sovereigns for the truncated yield curve without three month maturity calibrated without Greece, Portugal and Ireland.

		RF	AT	BE	FI	FR	DE	IT	NL	ES
κ, κ_i	P1	0.384	0.105	0.108	0.128	0.104	0.164	0.044	0.101	0.223
	P2	0.286	0.011	0.011	0.010	0.008	0.018	0.049	0.017	0.027
	P3	0.294	0.045	0.053	0.035	0.115	0.086	0.031	0.059	0.033
	P4	0.518	0.026	0.030	0.017	0.019	0.008	0.059	0.067	0.041
	P5	0.303	0.041	0.058	0.027	0.114	0.054	0.030	0.062	0.029
	T	0.306	0.042	0.050	0.031	0.115	0.063	0.030	0.056	0.023
θ, θ_i	P1	0.024	0.029	0.031	0.016	0.017	0.018	0.026	0.019	0.012
	P2	0.029	0.135	0.175	0.127	0.065	0.074	0.037	0.083	0.050
	P3	0.026	0.053	0.075	0.059	0.014	0.022	0.096	0.034	0.070
	P4	0.032	0.077	0.098	0.128	0.014	0.144	0.058	0.043	0.078
	P5	0.026	0.048	0.066	0.070	0.008	0.024	0.092	0.034	0.099
	T	0.027	0.055	0.072	0.060	0.010	0.024	0.093	0.036	0.099
σ, σ_i	P1	0.078	0.089	0.154	0.058	0.055	0.052	0.123	0.046	0.206
	P2	0.073	0.022	0.008	0.089	0.117	0.020	0.015	0.003	0.028
	P3	0.093	0.020	0.012	0.102	0.065	0.095	0.085	0.108	0.126
	P4	0.007	0.031	0.001	0.019	0.110	0.125	0.320	0.007	0.057
	P5	0.099	0.022	0.005	0.095	0.064	0.112	0.105	0.080	0.123
	T	0.108	0.016	0.010	0.100	0.053	0.109	0.099	0.079	0.131
λ, λ_i	P1	0.036	-0.229	-0.159	-0.500	-0.320	-0.502	-0.242	-0.540	-0.266
	P2	0.028	-0.000	0.180	-0.046	-0.029	-0.141	-0.409	-0.390	-0.183
	P3	0.021	-0.469	0.504	-0.126	-0.418	-0.221	-0.082	-0.228	-0.164
	P4	0.085	-2.998	-1.218	-0.667	-1.826	-0.585	-0.143	18.569	1.617
	P5	0.181	-0.492	0.510	-0.119	-0.431	-0.189	-0.093	-0.234	-0.098
	T	0.011	-0.469	0.506	-0.124	-0.437	-0.203	-0.082	-0.232	-0.102
1 ρ_i	P1	NA	-0.694	-0.647	-0.788	-0.638	-0.626	-0.631	-0.699	-0.711
	P2	NA	-0.345	-0.140	-0.358	-0.391	-0.352	-0.317	-0.341	-0.267
	P3	NA	-0.763	-0.588	-0.246	-0.559	0.102	-0.529	-0.627	-0.603
	P4	NA	-0.186	-0.309	-0.274	-0.227	-0.009	-0.504	-0.111	-0.380
	P5	NA	-0.335	-0.398	-0.249	-0.307	-0.383	-0.386	-0.344	-0.387
	T	NA	-0.335	-0.397	-0.248	-0.309	-0.383	-0.384	-0.344	-0.385
2 ρ_i	P1	NA	-0.747	-0.775	-0.712	-0.705	-0.724	-0.739	-0.726	-0.730
	P2	NA	-0.339	-0.156	-0.353	-0.401	-0.359	-0.320	-0.337	-0.295
	P3	NA	-0.564	0.545	-0.254	-0.538	-0.541	-0.617	-0.581	-0.609
	P4	NA	0.003	0.023	-0.010	0.011	0.010	0.009	0.013	0.028
	P5	NA	0.002	-0.022	0.003	-0.002	0.012	0.018	0.002	-0.003
	T	NA	0.000	-0.007	0.001	0.000	0.010	0.007	0.000	-0.002

Estimates of the parameters $\kappa, \kappa_i, \theta, \theta_i, \sigma, \sigma_i, \lambda, \lambda_i, \rho_i$ of the final model specification for the processes of the risk-free rate and credit spreads of the euro area countries without GR, PT, and IE for the period-by-period calibration without three month maturity. The average error is in basis points. Correlation coefficients - estimated 1), calculated 2). P1 – P5 denotes individual periods. T denotes the full period calibration estimates.

In P1 we observe plausible equilibrium levels for credit spreads ranging from 1.2 % to 3.1 %, which correspond to the actual development in credit spreads. However the value of the cost function for the full period estimate is in P1 higher than the P1 estimate only by

1 % and the improvement in the average error is only 0.07 basis points. The P1 estimate has a very high cost function value for other periods compared to their best estimates. The estimates in P2 and P4 are strange with the equilibrium rates ranging from 3.7 % for Italy to 17.4 % for Belgium and from 1.4 % for France to 14.4 % for Germany for the two periods respectively. These values, except the French estimates, are completely unrelated to the actual credit spread developments during these two periods and also do not correspond to the perceived riskiness of the countries.

The estimated values of the market prices of risk are negative, with the only exception being Belgium in P1, P3, P4 and the full period. We are not able to explain the positive values for Belgium, however the negative risk premium in this case is very small. For all other countries the risk premium is small in P2 for all maturities, but quite high in the other periods, especially for larger maturities.

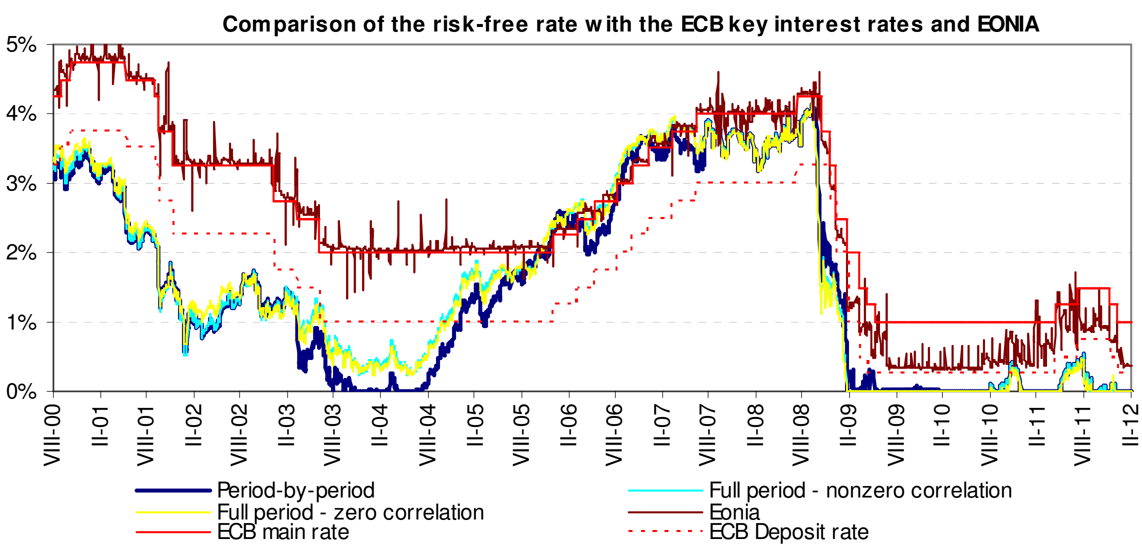


Figure 15 Comparison of the estimated risk-free rate with the ECB Key Interest Rates and the EONIA. The risk-free rate estimated period-by-period and for the full period with and without zero correlation. The ECB main rate is the minimum bid rate for the variable rate tenders up to October 15, 2009 and the fixed rate for tenders since then. Source: ECB[27], Thomson-Reuters, Own calculations.

Finally in Figure 15 we compare our estimates of the risk-free rate with the ECB Key interest rates and the EONIA overnight rate, which is a proxy for the short-rate. We can clearly see that the EONIA is highly volatile and copies the ECB main rate until the

[27] www.ecb.int

collapse of Lehman Brothers and the ECB deposit rate afterwards. The development of the estimated risk-free rate is in line with the ECB main rate and EONIA during the boom in 2006 – 2008, while it is in the period before and during the financial crisis.

7. Conclusions

In this thesis we developed a new multifactor panel model for the euro area risk-free rate and credit spreads of the euro area countries. The model assumes that the short-rate for every sovereign is a sum of the risk-free rate common to the whole euro area and a unique credit spread for every country. Both the risk-free rate and credit spreads are assumed to be unobserved variables in the market as well as the short-rate for every country. The only observable variables are yields of the sovereign bonds. Both the risk-free rate and credit spreads are modeled using the Cox, Ingersoll, Ross (CIR) and the Chan, Karolyi, Longstaff, Sanders (CKLS) processes using the recently developed approximate bond price formula for this kind of model.

The advantage of the model is that it allowed us to describe the true risk-free rate for the euro area and asses how much of the sovereign yield can be attributed to the risk-free part and how much is the compensation for the credit risk of the issuing country. We did not need to specify any existing asset as the risk-free asset. We also managed to estimate the specification of the model (the power parameter in the CKLS model) and correlations between the risk-free rate and credit spreads and to observe their changes in time.

We calibrated the model using euro area zero-coupon sovereign yield curves from August 1, 2000 to February 3, 2012 with maturities between three months and ten years. We managed to split the yield between the common risk-free rate of the euro area and the individual credit spreads of the euro area countries. The resulting general development and shape of the risk-free rate and credit spreads are robust and do not change significantly if estimated on sub-periods of data both with regard to time or group of countries used. Their shapes correspond to the economic expectations and to the ECB Key Interest rates, when in periods of economic slowdown the risk-free rate decreases, while in booming periods it increases. The credit spread behavior is opposite to the risk-free rate with quite a strong negative correlation between them. Our result suggest that in severe economic recession as the one experienced after the collapse of Lehman Brothers in September 2008 the usual assumption of nonnegative interest rates is unrealistic. Our estimates suggest that the risk-free rate should drop as low as -1.6 %. Even the credit spreads of the countries perceived to be least risky would need to drop into negative territory in order to capture the extremely low yields of these countries during the recent financial crisis. These conclusions hold also if the model is estimated only on the crisis period data.

The levels of the risk-free rate and credit spreads are not that robust as the overall shape. If the model is estimated on sub-periods of data, the general trend is that the risk-free rate and credit spread values increase for their high values compared to the full period estimate and decrease for the low values of their estimates. The model was not able to capture the development in yields of the countries which defaulted like Greece, or which needed coordinated international support like Ireland and Portugal. For the other countries the model performed quite well. The estimates were more precise for the middle part of the yield curve, less precise for the long-term part of the curve and worse for the very-short term part of the curve due to low liquidity of the short-term part of the sovereign yield curves.

Our estimates showed that the assumption of zero correlation between the risk-free rate and credit spreads is non-realistic and a strong negative correlation ranging between -0.3 and -0.7 is present in the data. However the CIR specification of the model can be confirmed except the immediate period after the collapse of Lehman Brothers where the risk-free rate process is closer to the Vasicek model and the credit spread volatility depends on the credit spreads with the power of 1.

Generally all processes are mainly driven by the stochastic part of the process rather than the mean reversion trend. The estimated parameters for the risk-free rate are quite robust both with regard to time periods and country groups used for the estimation. The estimated parameters for the credit spreads were quite volatile and showed in some cases unrealistically high estimates for the long-term equilibrium spreads.

List of symbols

AIB – Allied Irish Bank

AIG – American International Group

AT – Austria

ATSM – Affine term-structure model

BE – Belgium

BRIBOR – Bratislava inter-bank offered rate

CDS – Credit Default Swap

CIR – Cox-Ingersoll-Ross

CKLS – Chan, Karolyi, Longstaff, Sanders

DE – Germany

ECB – European Central Bank

EFSF – European Financial Stability Facility

EFSM – European Financial Stabilisation Mechanism

EONIA – Euro overnight index average

ES – Spain

ESM – European Stability Mechanism

EU – European Union

EUR – the Euro

EURIBOR – Euro inter-bank offered rate

Eurostat – European statistical office

FI – Finland

FR – France

GBP – British pound

GDP – Gross domestic product

GR – Greece

IE – Ireland

IMF – International Monetary Fund

ISIN – International Securities Identification Number

IT – Italy

LIBOR – London inter-bank offered rate

NL – Netherlands

OECD – Organisation for Economic Co-operation and Development

P, P(t, T), P(t, T, r), P(τ, r) – zero-coupon bond price

P1 – August 1, 2000 – March 11, 2003

P2 – March 12, 2003 – July 15, 2007,

P3 – July 16, 2007 – September 14, 2009

P4 – September 15, 2009 – January 8, 2010,

P5 – January 9, 2010 – February 3, 2012

PRIBOR – Prague inter-bank offered rate

PT – Portugal

RF – The risk-free rate

R(t, T) - yield to maturity

S&P – Standard and Poor's rating agency

U – Cost functional

UK – United Kingdom

US – United States of America

USD – US dollar

VAR – Vector auto-regressive

κ – Speed of the mean reversion of the short rate process

σ – Volatility of the short rate process

θ – Expected long-term interest rate of the short rate process

γ – Power in the volatility specification in the CKLS model

λ – Market price of risk

ρ - Correlation coefficient between the risk-free rate and credit spreads

τ – Time to maturity, $\tau = T - t$

T – Maturity of zero-coupon bond

t – Actual time

CD-ROM content

- Thesis
  - Thesis_Sestak_final.pdf
  - ŠESTÁK, Ľuboš: Mathematical Analysis and Calibration of a Multifactor Panel Model for Credit Spreads and Risk-free Interest Rate [Dissertation Thesis].
- Autoreferát
  - Autoreferat_Sestak_final.pdf
- Matlab Source-Codes
  - For the CIR model in transformed variables
    - kts – function $(\beta,\xi,\tilde{\rho},\lambda) \to (\kappa,\theta,\sigma)$
    - llik – the loglikelihood function (5.17)
    - llikMulti – the multivariate loglikelihood function (6.2)
    - odhadN – function to find $\min_{\psi} U(\psi,\boldsymbol{r}(\psi))$
    - rfcs – function $\psi \to \boldsymbol{r}(\psi)$
    - UN – function $U(\psi,\boldsymbol{r}(\psi))$
    - VelkeAT - – function for the A, A_i in (5.3)
    - VelkeBT - – function for the B, B_i in (5.3)
    - YieldModel - function for the yield of a zero-coupon bond
  - For the CKLS model in original variables
    - odhadNApVasCKLS – function to find $\min_{\tilde{\psi}} U(\tilde{\psi},\boldsymbol{r}(\hat{\psi}))$
    - rfcsApVasCKLS – function $\tilde{\psi} \to \arg\min_{\boldsymbol{r}} U(\tilde{\psi},\boldsymbol{r})$
    - UNApVasCKLS – function $U(\tilde{\psi},\boldsymbol{r})$
    - VelkeAApVasCKLS - – function for the $\ln(a)$ in (4.29)
    - VelkeBVas - – function for the B, B_i in (4.29)
    - YieldModelApVasCKLS - function for the approximate yield of a zero-coupon bond

The raw data cannot be provided according to the license agreement [11].

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