# Averaged bond prices for Fong-Vasicek and the generalized Vasicek interest rates models *

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**Abstract**

In short rate interest rate models, the behaviour of the short rate is given by a stochastic differential equation (1-factor models) or a system of stochastic differential equations (multifactor models). Interest rates with different maturities are determined by bond prices, which are solutions of the parabolic partial differential equation. We consider the generalized 2-factor Vasicek model and Fong-Vasicek model with stochastic volatility. In the 2-factor Vasicek model, the short rate is a sum of two independent Ornstein-Uhlenbeck processes. The bond price is a function of maturity and level of each of the components of the short rate. In Fong-Vasicek model, the volatility of the short rate is stochastic. The bond price is a function of maturity, short rate and volatility. In both cases, we do not observe all values necessary to obtain a bond price. Therefore, we propose the averaging of the bond prices. We consider the limiting probability distribution of unobservable variables. In this way, we obtain the averaged bond prices depending only on the maturity and short rate. We prove that there is no 1-factor model yielding the same bond prices as are the averaged values described above.

**Keywords:** interest rate models, 2-factor models, term structure averaging

## 1 *INTRODUCTION*

Term structure models describe the dependance between the time to maturity of a discount bond and its present price which implies the interest rate. Continuous short rate models are formulated in terms of a stochastic differential equation, or a system of them, for the instanteneous interest rate $r$ (short rate). The bond prices, and hence the term structures of the interest rates, are then obtained by solving the partial differential equation. We consider 2-factor generalized Vasicek and Fong-Vasicek two factor models. In the generalized Vasicek model, the bond price is a function of maturity, and the level of the short rate components, which are unobservable in the market separately. In Fong-Vasicek model, the bond price is a function of maturity, the level of the short rate and the level of volatility. The volatility is an unobservable parameter.

Since each of these model contains unobservable quantities, the interesting questions are the properties of the averaging of the bond prices with respect to the distribution of these unobservale variables. This is motived by papers [10] about averaging in stochastic volatility models of stock prices and [7] about averaging in stochastic volatility models of bond prices which are used in the series expansion of the prices. The asymptotic distribution of the hidden process is used. It can be justified if the processes have been evolving for a sufficiently long time.

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The averaged bond prices are functions of maturity and short rate, as the bond prices do in 1-factor models. We study a question whether there is a 1-factor model, yielding the same bond prices as are the averaged prices from the 2-factor models. Firstly, we consider Fong-Vasicek model and the 1-factor models which do not allow negative short rate. Although this assumption may be quite reasonable, it may seem quite restrictive in this setting. In both 2-factor Vasicek and Fong-Vasicek models, the short rate may become negative. For 2-factor Vasicek model, we study the question above without this nonnegativity assumption.

The paper is organized as follows. In section 2, we describe 1-factor and 2-factor models a present their properties which will be needed later. In section 3, we compute the averaged values in 2-factor Vasicek and Fong-Vasicek model. In section 4, we state the theorem and give the main idea of the proof. The proofs for the two models considered are in the subsections 4.1 and 4.2. The last section contains some concluding remarks.

## 2 DESCRIPTION OF THE MODELS

### 2.1 One factor models

In 1-factor models, the process describing the short rate (overnight rate), is assumed to be given by

$$dr = \mu(t,r)dt + \sigma(t,r)dw, \tag{1}$$

where $\mu(t,r)$ and $\sigma(t,r)$ are non-stochastic smooth functions. If $\mu(t,r) = \kappa(\theta - r)$, $\kappa > 0$, the process has the property of mean-reversion to the level $\theta$. A popular class of models is obtained by taking $\sigma(t,r) = \sigma r^\gamma$. It includes Vasicek model with $\gamma = 0$ ([16]), Cox-Ingersoll-Ross model with $\gamma = \frac{1}{2}$ ([8]), an important article on comparison of the models with different $\gamma$ is [4]. It started a discussion on the suitable choice of the volatility function in the interest rate models.

If the short rate evolves according to (1), then the discount bond with maturity $T$ has the price $P(t,r)$ depending on the time $t$ and the current level of the short rate $r$. It is given by the following partial differential equation:

$$\begin{aligned} -\frac{\partial P}{\partial t} + (\mu - \lambda_{mpr}\sigma)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP &= 0, \ \ t \in (0,T), \\ P(T,r) &= 1, \end{aligned} \tag{2}$$

where $\lambda_{mpr} = \lambda_{mpr}(t,r)$ is the market price of risk. The interest rates are then obtained from the bond prices by $R(t,r) = -\frac{\log P(t,r)}{T-t}$ (see [11]).

For the specific choices of the market price of risk in Vasicek and CIR models, it is known that the bond price can be written in the closed form.

If $\lambda_{mpr}(t,r) = \lambda$ in Vasicek model, then the price of bond with time to maturity $\tau = T-t$ has the form

$$P(\tau,r) = A(\tau)e^{-B(\tau)r}.$$

The functions $A(\tau)$ and $B(\tau)$ satisfy the following system of ordinary differential equations

$$\begin{aligned} \dot{A}(\tau) &= (\lambda\sigma - \kappa\theta)A(\tau)B(\tau) + \frac{1}{2}\sigma^2 A(\tau)B(\tau)^2, \\ \dot{B}(\tau) &= -\kappa B(\tau) + 1, \end{aligned} \tag{3}$$

which can be solved in the closed form:

$$\begin{aligned} A(\tau) &= \exp\left(\left(\frac{1-e^{-\kappa\tau}}{\kappa}-\tau\right)\left(\theta-\frac{\sigma\lambda}{\kappa}-\frac{\sigma^2}{2\kappa^2}\right)+\frac{\sigma^2}{4\kappa^2}(1-e^{-\kappa\tau})^2\right), \\ B(\tau) &= \frac{1-e^{-\kappa\tau}}{\kappa}. \end{aligned}$$

Similarly, if $\lambda_{mpr}(t,r)=\lambda\sqrt{r}$ in CIR model then the price of bond with time to maturity $\tau=T-t$ has again the form

$$P(\tau,r)=A(\tau)e^{-B(\tau)r}.$$

The functions $A(\tau)$ and $B(\tau)$ satisfy the following system of ordinary differential equations

$$\begin{aligned} \dot{A}(\tau) &= -\kappa\theta A(\tau)B(\tau) \\ \dot{B}(\tau) &= -(\kappa+\lambda\sigma)B-\frac{1}{2}\sigma^2B(\tau)^2+1 \end{aligned}$$

with initial conditions $A(0)=1$, $B(0)=0$. Its solution is given by

$$A(\tau)=\left(\frac{2\phi e^{(\phi+\psi)\tau/2}}{(\phi+\psi)(e^{\phi\tau}-1)+2\phi}\right)^{\frac{2\kappa\theta}{\sigma^2}},\ B(\tau)=\frac{2(e^{\phi\tau}-1)}{(\phi+\psi)(e^{\phi\tau}-1)+2\phi},$$

where

$$\psi=\kappa+\lambda\sigma,\ \phi=\sqrt{\psi^2+2\sigma^2}.$$

There are several possibilities of generalizing 1-factor models, leading to multifactor models. They include replacing a constant parameter of 1-factor model by a random one, evolving according to the stochastic differential equation (e.g. stochastic volatility models [1], [9]), adding another relevant quantity (consol rate in [3], European interest rate in [6], [12]), composition of short rate from more components (generalized Vasicek and CIR models in [8], [2], consol rate and the spread between the short rate and consol rate in [13], [5]). In 2-factor models, the bond prices are the solutions of the partial dfferential equation similarly as in the 1-factor case (see for example [11]). They are functions of the maturity and the factors of the short rate process.

We describe two of them, which we study in the rest of the paper.

### *2.2 Two-factor generalized Vasicek model*

In generalized 2-factor Vasicek model [2], the short rate is a sum of two independent processes of the form

$$\begin{aligned} r &= r_1+r_2, \\ dr_1 &= \kappa_1(\theta_1-r_1)dt+v_1dw_1, \\ dr_2 &= \kappa_2(\theta_2-r_2)dt+v_2dw_2. \end{aligned}$$

If the market prices of risk for both components of the short rate are constant and given by $\lambda_1$ and $\lambda_2$ respectively, the bond price can be written in the form

$$\Pi(\tau,r_1,r_2)=P_1(\tau,r_1)P_2(\tau,r_2),$$

where $P_1$ and $P_2$ are solutions of the bond price equations from 1-factor Vasicek models, corresponding to each of the short rate components.

### 2.3 Fong-Vasicek model with stochastic volatility

In Fong-Vasicek model [9], the volatility of short rate is assumed to be stochastic. The evolution of the short rate and volatility is given by the following system of stochastic differential equations:

$$dr = \kappa_1(\theta_1 - r)dt + \sqrt{y}dw_1,$$
$$dy = \kappa_2(\theta_2 - y)dt + v\sqrt{y}dw_2,$$

the correlation between $dw_1$ and $dw_2$ is denoted by $\rho$. If the market prices of risk of short rate and volatility are defined to be $\tilde{\lambda}_1 = \lambda_1\sqrt{y}$, $\tilde{\lambda}_2 = \lambda_2\sqrt{y}$, then the bond price has the form $P(\tau, r, y) = A(\tau)e^{-B(\tau)r - C(\tau)y}$. It splits the partial differential equation into a system of ordinary differential equations for $A = A(\tau),\ B = B(\tau)\ C = C(\tau)$:

$$\begin{aligned}
\dot{A} &= -A\left(\kappa_1\theta_1 B + \kappa_2\theta_2 C\right),\\
\dot{B} &= -\kappa_1 B + 1,\\
\dot{C} &= -\lambda_1 B - \kappa_2 C - \lambda_2 v C - \frac{B^2}{2} - \frac{v^2C^2}{2} - v\rho BC,
\end{aligned}$$

with initial conditions

$$A(0) = 1,\ B(0) = 0,\ C(0) = 0.$$

Integration of the equation for $B$ yields

$$B(\tau) = \frac{1}{\kappa_1}\left(1 - e^{-\kappa_1\tau}\right),$$

and $C$ satisfies the differential equation

$$\dot{C}(\tau) + \lambda_1 B(\tau) + \frac{B(\tau)^2}{2} + \left(\kappa_2 + \lambda_2 v + v\rho B(\tau)\right)C(\tau) + \frac{v^2}{2}C(\tau)^2 = 0,\ C(0) = 0.$$

Integration of the equation for $A$ yields

$$A(\tau) = \exp\left(-\theta_1\tau + \theta_1 B(\tau) - \kappa_2\theta_2\int_0^\tau C(s)ds\right).$$

In the following we assume that $\lambda_1 \leq -\frac{1}{2\kappa_1}$. Under this condition we have shown in [15] that the solution of bond pricing is feasible, i.e. it satisfies the natural condition that $0 < P(\tau, r, y) < 1$. We also make use of the following values of the derivatives of the function $C(\tau)$ for $\tau = 0$:

$$\dot{C}(0) = 0,\ \ddot{C}(0) = -\lambda_1. \tag{4}$$

## 3 BOND PRICES AVERAGING

### 3.1 Two-factor generalized Vasicek model

Since the limit distributions of $r_1$ and $r_2$ are normal distributions $N\left(\theta_1, \sigma_1^2\right)$ and $N\left(\theta_2, \sigma_2^2\right)$ respectively, where $\sigma_i^2 = \frac{v_i^2}{2\kappa_i}$, it follows from their independence that the limit distribution of $r_1 + r_2$ is $N\left(\theta_1 + \theta_2, \sigma_1^2 + \sigma_2^2\right)$. If the correlation of $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ is $\rho$, it is known that the conditional distribution of $Y$ conditioned by $X = x$ is $N\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2)\right)$. Using this property for $X = r_1 + r_2$, $Y = r_1$, we obtain

$$r|r_1 + r_2 \sim N(\mu_c, \sigma_c^2),$$

where

$$\mu_c = \theta_2 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}(r-\theta_1-\theta_2),\ \sigma_c^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}.$$

Let us denote the bond price in the two factor model by $\Pi(\tau, r_1, r_2)$ and the averaged value by $P(\tau, r)$ where $r = r_1 + r_2$. We consider the averaging with respect to the limit distribution under the condition $r_1 + r_2 = r$. Then

$$\begin{aligned} P(\tau, r) &= \int_{-\infty}^{\infty} A_1(\tau)A_2(\tau)e^{-B_1(\tau)r_1 - B_2(\tau)(r-r_1)}\frac{1}{\sqrt{2\pi\sigma_c^2}}e^{-\frac{(r_1-\mu_c)^2}{2\sigma_c^2}}dr_1 = \\ &= \tilde{A}(\tau)e^{-\tilde{B}(\tau)r}, \end{aligned} \quad (5)$$

where, using the notation $\tilde{A} = \tilde{A}(\tau)$, $\tilde{B} = \tilde{B}(\tau)$, $A_1 = A_1(\tau)$, $A_2 = A_2(\tau)$, $B_1 = B_1(\tau)$, $B_2 = B_2(\tau)$,

$$\begin{aligned} \tilde{A} &= A_1A_2\exp\left(-(B_1-B_2)\left(\theta_1 - (\theta_1+\theta_2)\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right) + \frac{1}{2}\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}(B_1-B_2)^2\right), \\ \tilde{B} &= \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}B_1 + \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}B_2. \end{aligned}$$

### 3.2 *Fong-Vasicek model with stochastic volatility*

The limit distribution of the volatility process is known to be the gamma distribution with the density

$$f(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda y}y^{\alpha-1},$$

where $\lambda = \frac{2\kappa_2}{v^2}$, $\alpha = \frac{2\kappa_2\theta_2}{v^2}$ for $y > 0$ and zero otherwise. Let us denote the bond price in Fong-Vasicek model by $\Pi(\tau, r, y)$. Then its averaged value $P(\tau, r)$ with respect to the limit distribution of $y$ is given by

$$\begin{aligned} P(\tau, r) &= \int_0^{\infty} A(\tau)e^{-B(\tau)r - C(\tau)y}\frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda y}y^{\alpha-1}dy = \\ &= A(\tau)e^{-B(\tau)r}\left(1 + \frac{C(\tau)}{\lambda}\right)^{-\alpha}. \end{aligned}$$

## 4 *AVERAGED BOND PRICES AND 1-FACTOR MODELS*

We study a problem whether there are functions $\mu$ and $\sigma$ such that the bond prices in the short rate model $dr = \mu dt + \sigma dw$ are the same as the averaged prices from 2-factor model. We restrict ourselves to certain processes. Drift and volatility of the process, as well as the market price of risk $\lambda_{mpr}$ are assumed to be time-independent. In the case of Fong-Vasicek model, we require the volatility to be zero for zero level of short rate. This condition is needed for the nonnegativity of short rate. In the case of generalized 2-factor Vasicek model, we consider a wider class of processes and we omit this condition.

**Theorem 1** *Consider the averaged bond prices $P(\tau, r)$ in 2-factor model, for which the fuctions $P$, $\frac{\partial P}{\partial r}$, $\frac{\partial^2 P}{\partial r^2}$ are bounded and the following conditions on the 1-factor model:*

1. *functions $\mu$, $\sigma$, $\lambda_{mpr}$ depend only on $r$ (and not on $\tau$),*

*2. functions $\mu$, $\sigma$, $\lambda_{mpr}$ are continuous in $r = 0$,*

*3. $\sigma(0) = 0$,*

*Then there is no such 1-factor interest rate model, for which the averaged bond prices satisfy the bond pricing PDE up to the boundary $r = 0$.*

### 4.1 Proof for Fong-Vasicek model with stochastic volatility

Firstly, we give the main idea of the proof. This idea works also for the 2-factor generalized Cox-Ingersoll-Ross model, for which we used it in [14].

Suppose that there is such 1-factor model. By limit $r \to 0$ in the PDE (2) we obtain that

$$\mu(r=0) = \left.\frac{\partial_\tau P}{\partial_r P}\right|_{r=0}.$$

In particular, the right hand side of the equality above is independent on $\tau$. This necessary condition leads to a contradiction.

To perform this proof, we start with computing the partial derivatives of $P$:

$$\begin{aligned}\frac{\partial P}{\partial \tau} &= \left(\frac{\dot{A}}{A} - \dot{B}r - \frac{\alpha \dot{C}}{\lambda + C}\right) P,\\ \frac{\partial P}{\partial r} &= -BP,\\ \frac{\partial^2 P}{\partial r^2} &= B^2 P.\end{aligned}$$

Since all the functions are bounded, the necessary condition for $P$ to be a bond price in a 1-factor model is:

$$\left.\frac{\frac{\dot{A}}{A} - \dot{B}r - \frac{\alpha}{\lambda+C}\dot{C}}{-B}\right|_{r=0} = \frac{\frac{\dot{A}}{A} - \frac{\alpha}{\lambda+C}\dot{C}}{-B} = \frac{-\kappa_1\theta_1 B - \kappa_2\theta_2 C - \dot{C}\frac{\alpha}{\lambda+C}}{-B} = k$$

for all $\tau > 0$ and some constant $0 < k < \infty$. Then

$$-\kappa_1\theta_1 B - \kappa_2\theta_2 C - \alpha\frac{\dot{C}}{\lambda + C} + kB = 0 \tag{6}$$

for all $\tau > 0$. Hence also the derivative of the left hand side is identically zero, so

$$-\kappa_1\theta_1\dot{B} - \kappa_2\theta_2\dot{C} - \alpha\frac{\ddot{C}(\lambda + C) - \dot{C}^2}{(\lambda + C)^2} + k\dot{B} = 0.$$

Since this equality holds for all $\tau > 0$, also the limit for $\tau \to 0+$ equals zero. Using the initial condition for $C$ and the values of its derivatives (4), it yields

$$\kappa_1\theta_1 + \lambda_1\theta_2 = k.$$

Substituting this expression for $k$ into (6) gives

$$(\kappa_1\theta_1 B + \kappa_2\theta_2 C)(\lambda + C) + \alpha\dot{C} - (\lambda + C)(\kappa_1\theta_1 + \lambda_1\theta_2)B = 0.$$

After substitution of the equation for $\dot{C}$ and the rearrangement, we obtain

$$\kappa_2\theta_2 C^2 - \alpha\lambda_2 v C - \frac{1}{2}\alpha B^2 - \frac{1}{2}\alpha v^2 C^2 - \alpha v \rho BC - 2\lambda_1 \alpha B - \lambda_1 \theta_2 BC = 0.$$

Now, we use a similar idea as before. Since the function on the left hand side equals zero for all $\tau > 0$, also its derivative is identically zero and hence the limit of the derivative for $\tau \to 0+$ is zero, too. This yields

$$2\lambda_1\alpha = 0,$$

which is a contradiction, since neither $\lambda_1$ nor $\alpha$ is equal to zero.

### 4.2 Two-factor Vasicek model without the requirement of nonnegativity of the short rate

In this section, we prove that for 2-factor Vasicek model, Theorem 1 holds also without the condition on zero volatility when the short rate is zero. As explained in the introduction, omitting this condition allows the short rate to become negative which is also a property of the original 2-factor model. However, even without this condition, a 1-factor model yielding the same bond prices as are averaged values from 2-factor Vasicek model does not exist.

Suppose that the averaged bond price is the solution of a 1-factor model bond valuation PDE. Substituting it to this PDE yields

$$-\frac{\dot{\tilde{A}}(\tau)}{\tilde{A}(\tau)} + \dot{\tilde{B}}(\tau) r - (\mu - \lambda_{mpr}\sigma)(r)\tilde{B}(\tau) + \frac{1}{2}\sigma^2(r) B^2(\tau) - r = 0. \tag{7}$$

It follows that $(\mu - \lambda_{mpr}\sigma)(r)\tilde{B}(\tau) - \frac{1}{2}\sigma^2(r)\tilde{B}(\tau)^2$ is a linear function of $r$ of the form

$$(\mu - \lambda_{mpr}\sigma)(r)\tilde{B}(\tau) - \frac{1}{2}\sigma^2(r)\tilde{B}(\tau)^2 = k_1(\tau) + k_2(\tau) r. \tag{8}$$

Morover, we show that the following stronger condition has to be satisfied:

$$\begin{aligned} \sigma^2(r) &= l_1 + l_2 r, \quad \text{where } l_2 \neq 0, & (9)\\ (\mu - \lambda_{mpr}\sigma)(r) &= l_3 + l_4 r, \quad \text{where } l_4 \neq 0. & (10) \end{aligned}$$

It means that the terms $(\mu - \lambda_{mpr}\sigma)(r)$ and $\sigma^2(r)$ do not contain nonlinear terms that could eventually vanish in $(\mu - \lambda_{mpr}\sigma)(r)\tilde{B}(\tau) - \frac{1}{2}\sigma^2(r)\tilde{B}(\tau)^2$.

Then we get

$$\left(-\frac{\dot{\tilde{A}}(\tau)}{\tilde{A}(\tau)} - l_3\tilde{B}(\tau) + \frac{1}{2}l_1\tilde{B}^2(\tau)\right) + r\left(\dot{\tilde{B}}(\tau) - l_4\tilde{B}(\tau) + \frac{1}{2}l_2\tilde{B}^2(\tau) - 1\right) = 0.$$

Thus, the equation for $\tilde{B}$ reads as

$$\begin{aligned} \dot{\tilde{B}}(\tau) - l_4\tilde{B}(\tau) + \frac{1}{2}l_2\tilde{B}^2(\tau) - 1 &= 0 \quad \text{with } l_2, l_4 \neq 0\\ B(0) &= 0. \end{aligned}$$

This is an equation of the same form as the one appearing in Cox-Ingersoll-Ross model and its solution is known in the closed form and it is given in the second section. However, the function

$$\tilde{B}(\tau) = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} B_1(\tau) + \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} B_2(\tau) = c_0 + c_1 e^{-\kappa_1\tau} + c_2 e^{-\kappa_2\tau}$$

for some constants $c_0$, $c_1$ and $c_2$, is not a function of this type.

To finish the proof, we prove (9) and (10). Firstly, we write the PDE (7) in terms of $B_1(\tau)$ and $B_2(\tau)$ only, using the expression (5) for the averaged bond price and the system (3) which is satisfied by its components:

$$
\begin{aligned}
-B_1(\tau)\left(\lambda_1 v_1 - \kappa_1(\theta_1+\theta_2)\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right) - B_1(\tau)^2\left(\frac{1}{2}v_1^2 - \kappa_1\sigma_c^2\right) &\quad - \\
-B_2(\tau)\left(\lambda_2 v_2 - \kappa_2\theta_2 - \kappa_2\left(\theta_1 - (\theta_1+\theta_2)\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\right) - B_2^2(\tau)\left(\frac{1}{2}v_2^2 - \kappa_2\sigma_c^2\right) &\quad - \\
-\left(\sigma_c^2(\kappa_1+\kappa_2)B_1(\tau)B_2(\tau)\right) + \left(-\kappa_1\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}B_1(\tau) - \kappa_2\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}B_2(\tau)\right)r &\quad + \\
+\frac{1}{2}\sigma^2(r)\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}B_1(\tau) + \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}B_2(\tau)\right)^2 &\quad - \\
-(\mu - \lambda_{mpr}\sigma)(r)\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}B_1(\tau) + \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}B_2(\tau)\right) &\quad = \quad 0.
\end{aligned}
$$

The equality holds for all $r$ and $\tau > 0$. Hence also the derivative of the left hand side with respect to $\tau$ is identically zero and its limit as $\tau \to 0+$ is zero, too. This yields

$$
\begin{aligned}
-\left[\left(\lambda_1 v_1 - \kappa_1(\theta_1+\theta_2)\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right) + \left(\lambda_2 v_2 - \kappa_2\theta_2 - \kappa_2\left(\theta_1 - (\theta_1+\theta_2)\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\right)\right] &\quad + \\
+\left[-\kappa_1\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} - \kappa_2\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right]r - (\mu - \lambda_{mpr}\sigma)(r) &\quad = \quad 0.
\end{aligned}
$$

The proposition (9) follows, with

$$
l_4 = -\kappa_1\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} - \kappa_2\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \tag{11}
$$

Hence also $\sigma^2(r)$ is a linear function of the form $l_1 + l_2 r$, as claimed in (10). What remains, is showing that $l_2 \neq 0$.

From (7) we see that the linear coefficient $k_2(\tau)$ of $(\mu - \lambda_{mpr}\sigma)(r)\tilde{B}(\tau) - \frac{1}{2}\sigma^2(r)\tilde{B}(\tau)^2$. in (8) is

$$
k_2(\tau) = \dot{\tilde{B}}(\tau) - 1 = -\kappa_1\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} - \kappa_2\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}. \tag{12}
$$

From (11) we see that the linear coefficient in $(\mu - \lambda_{mpr}\sigma)(r)\tilde{B}(\tau)$ is

$$
\left(-\kappa_1\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} - \kappa_2\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}B_1(\tau) + \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}B_2(\tau)\right),
$$

which is not equal to (12). Hence, the linear coefficient in $\sigma^2(r)$ is not zero, which finishes the proof.

## 5 CONCLUSION

We considered the 2-factor Vasicek and Fong-Vasicek models of interest rates and the averaged bond prices with respect to the asymptotic distribution of the unobservable processes. Such averaged values are functions of the maturity and the short rate. This property is shared with solutions to 1-factor interest rate models. Therefore, we studied a natural question whether there is a one factor model yielding the same bond prices as those obtained by averaging. We proved that the answer is negative for both models, when we assume that the 1-factor model does not allow negative short rate. Even without this condition, the answer remains negative for the 2-factor Vasicek model. The answer for this particular question for Fong-Vasicek model is still open.

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