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FAST MEAN REVERTING VOLATILITY IN FONG–VASICEK MODEL OF INTEREST RATES

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In Fong-Vasicek model of interest rates, the instantaneous interest rate (short rate) follows a mean reverting process and its volatility follows a Bessel square root process. We consider different time scales for the volatility and the short rate processes and we are interested in properties of the term structures, which correspond to fast mean reverting volatility. The main result of this paper is the proof of the monotone decrease of the difference between interest rates with the same value of short rate and volatility, as the speed of volatility evolution increases.

K e y w o r d s: interest rates, Fong-Vasicek model, fast mean reverting volatility

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1 INTRODUCTION

Term structure of interest rates is the dependence between the interest rates and the time to maturity. Short rate models are formulated in terms of a stochastic differential equation (or a system of them) for the instantaneous interest rate (short rate). Other interest rates are determined by bond prices, which are solutions of the partial differential equation. This equation is derived by constructing a non-stochastic portfolio and using the no-arbitrage principle (c.f. [3]). In [3], a review of short term models can be also found.

In this paper we deal with the Fong-Vasicek model of interest rates [2]. It is a two-factor model, in which the volatility itself is stochastic, too. We study the volatility, which evolves in a different time scale than the short rate. In particular, we are interested in fast mean reverting volatility. This is motivated by paper [1], where the series expansion of the bond price with respect to the parameter describing the time scale of volatility is derived. This expansion is used to study the asymptotics.

We are interested in fast mean reverting volatility too. However, we do not want to restrict ourselves only to the asymptotics, we are interested in the behaviour for the high (but finite) speed as well. We consider two interest rates with different times to maturities, corresponding to the same value of the short rate and volatility. We prove that under some conditions the difference between these interest rates is a decreasing function of the volatility speed.

2 FONG–VASICEK MODEL OF INTEREST RATES

In Fong-Vasicek model, the short rate follows the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}r &= \kappa_1(\theta_1 - r)\mathrm{d}t + \sqrt{y}\mathrm{d}w_1 \,, \\ \mathrm{d}y &= \tilde{\kappa_2}(\theta_2 - y)\mathrm{d}t + \tilde{v}\sqrt{y}\mathrm{d}w_2 \,, \end{aligned} \tag{1}$$

where $\rho \in (-1,1)$ is the correlation between the increments of the Wiener processes w_1 and w_2.

If the market prices of risk of the short rate and of the volatility are $\lambda_1\sqrt{y}$ and $\lambda_2\sqrt{y}$ respectively (for some constants λ_1 and λ_2), the price of bond with time to maturity τ has the form (see [2])

$$P(\tau, r, y) = A(\tau)e^{-B(\tau)r - C(\tau)y}. \tag{2}$$

Hence the interest rates are given by

$$R(\tau, r, y) = -\frac{\log P(\tau, r, y)}{\tau} = -\frac{\log A(\tau)}{\tau} + \frac{B(\tau)}{\tau}r + \frac{C(\tau)}{\tau}y \,. \tag{3}$$

There are several numerically efficient ways of computing the functions $A(\tau)$, $B(\tau)$ and $C(\tau)$ appearing in (2) and (3), see [2], [4], [5].

In what follows, we assume that the condition on λ_1 holds:

$$\lambda_1 \leq -\frac{1}{2\kappa_1}. \tag{4}$$

In [5] we derived some properties under this condition, which will be used in this paper. We also use the following

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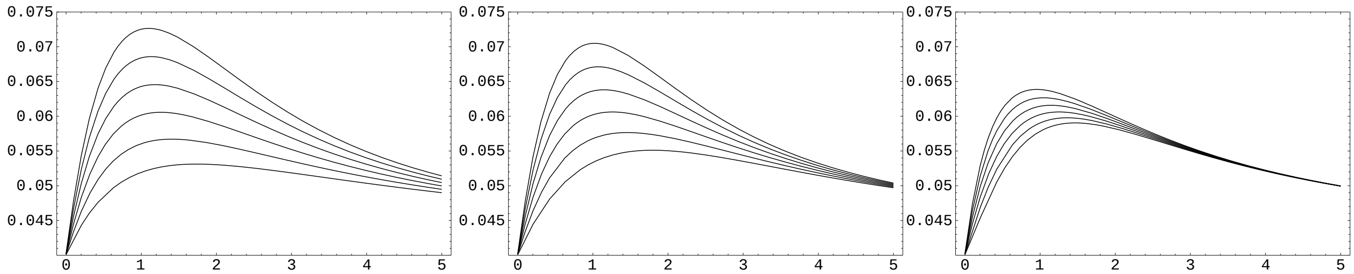

Fig. 1. The effect of increasing speed of the volatility evolution on term structures (time to maturity on the horizontal axis, interest rate on the vertical axis). The term structures in each figure correspond to different values of volatility.

characterization of the functions A, B, C: they satisfy the system of ordinary differential equations

$$\begin{aligned} \dot{A} &= -A(\kappa_1\theta_1 B + \tilde{\kappa}_2\theta_2 C)\,, \\ \dot{B} &= -\kappa_1 B + 1\,, \\ \dot{C} &= -\lambda_1 B - \tilde{\kappa}_2 C - \lambda_2\,, \end{aligned} \tag{5}$$

with initial conditions $A(0) = 1$, $B(0) = 0$, $C(0) = 0$ and they can be written in the following form:

$$\begin{aligned} B &= \big(1 - e^{-\kappa_1\tau}\big)/\kappa_1\,, \\ \dot{C} &= -\Big[\lambda_1 B + \frac{B^2}{2} + \big(\tilde{\kappa}_2 + \lambda_2\tilde{v} + \tilde{v}\rho B\big)C + \frac{\tilde{v}^2}{2}C^2\Big], \\ A &= \exp\Big(-\theta_1\tau + \theta_1 B - \tilde{\kappa}_2\theta_2\int_0^\tau C(s)\mathrm{d}s\Big)\,. \end{aligned} \tag{6}$$

3 FAST MEAN REVERTING VOLATILITY

If the volatility evolves in a time scale with the unit $\varepsilon > 0$, then the equation for y in (1) becomes

$$\mathrm{d}y = \frac{\tilde{\kappa_2}}{\varepsilon}(\theta_2 - y)\mathrm{d}t + \frac{\tilde{v}}{\sqrt{\varepsilon}}\sqrt{y}\mathrm{d}w_2.$$

The fast mean reverting volatility corresponds to small values of ε and the limit $\varepsilon \to 0$. If we define $\kappa_2 = \tilde{\kappa_2}/\varepsilon$, $v = \tilde{v}/\sqrt{\varepsilon}$, we obtain the stochastic differential equation

$$\mathrm{d}y = \kappa_2(\theta_2 - y)\mathrm{d}t + v\sqrt{y}\mathrm{d}w_2\,,$$

where the ratio κ_2/v^2 is constant. Hence we can fix the ratio

$$\kappa_2/v^2 = k \tag{7}$$

and consider the fast mean reversion as large values of v (and the corresponding values of κ_2) and limit $v \to \infty$.

4 THE MAIN RESULT

Firstly, we present a numerical example. In Fig. 1, there are terms structures for the same values of parameters κ_1, θ_1, θ_2 and k, the same values of short rate and volatility, but the increasing speed of volatility evolution. We notice that the differences between the interest rates decrease with increasing speed of volatility. In the rest of this paper, we prove this observation analytically.

Before the proof itself, we recall from [5] certain properties which hold if the condition (4) is satisfied: $C(\tau)$ is positive for $\tau > 0$ and

$$C(0) = 0\,,\ \ \dot{C}(0) = 0\,,\ \ \ddot{C}(0) = -\lambda_1\,. \tag{8}$$

Let us now define

$$D(\tau, v) = \frac{\partial C}{\partial v}(\tau, v)\,.$$

We show that there exists v_0 such that $D(\tau, v) < 0$ for all $\tau > 0$, whenever $v > v_0$. The proof consists of two steps. Firstly, we show that if $D = 0$ for some sufficiently large v and some $\tau > 0$, then $\dot{D} < 0$. Secondly, taking a large fixed v, we show that the function D is negative on some neighbourhood of $\tau = 0$. In the whole proof, we assume that the function C is sufficiently smooth.

Suppose that $D = 0$ for some $v > 0$ and $\tau > 0$. Differentiating the equation for C in (6), we obtain the equality for this point (τ, v):

$$\dot{D} = -[(2k\ v\ + \lambda_2 + \rho B)C + v\ C^2]\,. \tag{9}$$

The function B is bounded and k is a positive constant, hence there is v_0 such that for $v > v_0$ the following inequality holds for all $\tau > 0$:

$$2k\,v + \lambda_2 + \rho B > 0\,. \tag{10}$$

Since $C > 0$, if $D = 0$ for some $v > v_0$ and $\tau > 0$, from (8) and (9) we obtain $\dot{D} < 0$.

Differentiating the equation for C we obtain

$$\begin{aligned} &D(0, v) = 0\,,\ \ \dot{D}(0, v) = 0\,,\ \ \ddot{D}(0, v) = 0\,, \\ &\dddot{D}(0, v) = -(2k\ v\ + \lambda_2 + \rho B(0, v))\ddot{C}(0, v)\,. \end{aligned}$$

From (9) it follows that $2k\,v + \lambda_2 + \rho B(0, v) > 0$ for $v > v_0$; from (7) and (4) it follows that $\ddot{C}(0, v) < 0$. Hence for $v > v_0$

$$D(0, v) = 0\,,\ \ \dot{D}(0, v) = 0\,,\ \ \ddot{D}(0, v) = 0\,,\ \ \dddot{D}(0, v) < 0\,.$$

This implies that $D(\tau,v) < 0$ for τ from an interval of the form $(0,\tilde{\tau})$. This completes the proof.

Let us now consider $\tau > 0$. We have shown that $C(\tau,v)$ i s decreasing in v for $v > v_0$. Furthermore, the function C is positive for $\tau > 0$. Hence, there exists the limit of $C(\tau,v)$ as v approaches infinity. Define

$$L(\tau) = \lim_{v\to\infty} C(\tau,v)$$

for $\tau > 0$; it is clear that $L(\tau) \geq 0$. We show that the case when $L(\bar{\tau}) > 0$ for some $\bar{\tau} > 0$ leads to an unrealistic behaviour of the term structures (interest rates converge to infinity). Hence, we will further study only the case when $L(\tau) = 0$ for all τ.

Suppose that $L(\bar{\tau}) > 0$ for some $\bar{\tau} > 0$. Firstly, we show that $L(\tau) > 0$ on an interval $(\bar{\tau} - h, \bar{\tau})$ for some $h > 0$. Evaluating the derivative $\dot{C}$ in $\tau = \bar{\tau}$ gives

$$\dot{C} = -\big[(\lambda_1 B + B^2/2) + v\ (\lambda_2 + \rho B)C + v^2(k + C/2)C\big],$$

which approaches infinity as $v \to \infty$. Hence for $v > \tilde{v}$ the derivative $\dot{C}(\bar{\tau},v)$ is negative. It follows that for τ from an interval of the form $(\bar{\tau} - h, \bar{\tau})$ the inequality $C(\tau,v) > C(\bar{\tau},v)$ holds. Taking limit for $v \to \infty$ we obtain $L(\tau) > L(\bar{\tau})$, so L is positive also for $\tau \in (\bar{\tau} - h, \bar{\tau})$. We proceed to the computation of the limit of interest rates as $v \to \infty$. Using (3) and (6) , we write the interest rate $R(\tau,r,y,v)$ as

$$R = \theta_1(1 - \frac{B}{\tau}) + \frac{\theta_2 k}{\tau} v^2 \int_0^\tau C(s,v)\mathrm{d}s + \frac{B}{\tau} r + \frac{C}{\tau} y\,.$$

We show that the integral $\int_0^\tau C(s,v)\mathrm{d}s$ has a positive limit for $v \to \infty$, if τ is suitably chosen. It implies that for these maturities also the interest rates $R(\tau,r,y,v)$ converge to infinity. We obtain this assertion from the following computation:

$$\lim_{v\to\infty} \int_0^\tau C(s,v)\mathrm{d}s = \int_0^\tau \lim_{v\to\infty} C(s,v)\mathrm{d}s = \int_0^\tau L(s)\,. \quad (11)$$

Choosing τ such that the interval $(\bar{\tau} - h, \bar{\tau})$, in which L is positive, is contained in the interval $(0,\tau)$, yields to a positive value of the integral $\int_0^\tau L(s)\mathrm{d}s$. To justify the first equality in (10), it suffices to show that

$$\lim_{n\to\infty} \int_0^\tau C(s,v_n)\mathrm{d}s = \int_0^\tau \lim_{n\to\infty} C(s,v_n)\mathrm{d}s$$

for all sequences $v_1 < v_2 < \dots$. This equality holds because with the possible exception of some first terms, the sequence $\{C(s,v_n)\}_{n\geq 1}$ is decreasing and convergent for all $s \in (0,\tau)$, which allows us to apply Lebesgue's monotone convergence theorem.

We now analyze the second case left, when $L(\tau) = 0$ for all $\tau > 0$. Let us consider two values of the volatility y, without loss of generality we can assume that $y_1 < y_2$. Then $R(\tau,r,y_1,v) < R(\tau,r,y_2,v)$ and the difference between the two interest rates is

$$R(\tau,r,y_2,v) - R(\tau,r,y_1,v) = C(\tau,v)\frac{y_2 - y_1}{\tau}\,.$$

As a function of the variable v, the term $(y_2 - y_1)/\tau$ is a positive constant, and $C(\tau,v)$ is a decreasing function for large values of v. Hence also the difference between the interest rates decreases with increasing value of the parameter v. Recalling that the limit for $v \to \infty$ is $L(\tau)$, which equals zero, we conclude that the differences between the interest rates converge to zero and this convergence is monotone for large values of v.

5 CONCLUSIONS

We investigated fast mean reverting volatility in Fong-Vasicek model of interest rates. We proved that under specified conditions the difference between the interest rates corresponding to the same value of short rate and volatility is a decreasing function of the speed in which the volatility evolves.

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