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ON A VOLATILITY AVERAGING IN A TWO-FACTOR INTEREST RATE MODEL *

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Abstract. In this paper we deal with the Fong-Vašíček two-factor interest rate model for valuing term structures. The volatility of the short rate process is assumed to be stochastic and it satisfies a stochastic differential equation of the mean reversion type. The equation for the zero coupon bond price is a linear parabolic equation in two space dimensions. These spatial dimensions correspond to the short rate and volatility. It is shown that this equation possesses an explicit solution giving rise to study further properties of the two-factor model analytically. Knowing the density distribution of the stochastic volatility we are yet able to perform averaging of the bond price and the term structure with respect to stochastic volatility. Unlike the short rate known from the market date on daily basis the volatility of the short rate process is unknown and can be hardly estimated from historical data. Therefore such a volatility averaging is of special importance when applying two-factor interest rate models to market data analysis.

Key words. two-factor interest rate model, Fong-Vašíček model, Fokker-Planck equation, volatility averaging

AMS subject classifications. 91B28, 35K05

1. Introduction. Term structure models describe a functional dependence between the time to maturity of a discount bond and its present price. The time structure of bond prices (or yields) is a function of time to maturity, state variables like e.g. instantaneous interest rate as well as several model parameters. Continuous interest rate models are formulated in terms of stochastic differential equations for the instantaneous interest rate (or short rate) as well as equations for other relevant quantities like e.g. volatility of the short rate process. In one-factor models there is a single stochastic differential equation for the short rate. The volatility of the short rate process is given in a deterministic way. It is assumed to be constant (the Vašíček model) or it is a function of the short rate itself (the Cox, Ingersoll, and Ross model). Beside these two simple models there is a wide range of other models including, in particular, Brennan–Schwartz model, Hull–White model, Ho–Lee model, Merton model and many other models. Based on this assumption made on the form of the short rate process one-can derive a linear scalar parabolic equation for the bond price as function of the current short rate and time to maturity. The reader is reffered to the book by Kwok [7] for detailed discussion on applications and properties of one-factor interest rate models.

Unlike one-factor interest rate model in which the volatility of the process is assumed to be deterministic it is reasonable to conjecture that the market changes the volatility of the underlying process for the short rate. In two-factor continuous interest rate models we allow other quantities including, in particular, volatility to have a stochastic behavior driven by another stochastic differential equation. In this paper we focus our attention to the so-called Fong-Vašíček two-factor model [5] in which the volatility of the short rate process satisfies a stochastic differential equation

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of the mean reversion type. As a consequence of the multidimensional Itô's lemma (see [7]) the corresponding equation for the bond price is now a linear parabolic equation in two space dimensions. These spatial dimensions correspond to the short rate and volatility.

The main goal of this paper is to investigate a special case of two-factor interest rate models referred to as the Fong-Vašíček model [5]. In this model we will prove that the governing PDE possesses a separable solution (see Section 4) giving rise to study further properties of the two-factor model analytically. We also analyze the stochastic process driving the volatility. It is well known that its density distribution is a solution to the Fokker-Planck partial differential equation and can be expressed analytically in the case the volatility undergoes the Bessel square root process of a mean reversion type. Knowing the density distribution of the stochastic volatility we are yet able to construct an algorithm for volatility averaging of the bond price and the term structure. Unlike the short rate which is known from the market data on daily basis the volatility of the short rate process is unknown and can be hardly estimated from historical data. Therefore such a volatility averaging is of special importance for practitioners. The algorithm of volatility averaging and its numerical implementation is the main result of this paper.

The paper is organized as follows. In the next section we recall a general form of a two-factor interest rate model with stochastic volatity. Both stochastic equations for the short rate as well as for the volatility are Orstein-Uhlenbeck processes often referred to as mean reverting processes in the financial literature. A parabolic partial differential equation for the bond price is also recalled in this section. In section 3 we analyze stochastic differential equation driving the volatility of the short rate. We investigate the density distribution of stochastic volatility. Section 4 is devoted to description and numerical realization of the algorithm for averaging of the bond price and the term structures with respect to the stochastic volatility.

2. The two factor model with a stochastic volatility. Any continuous interest rate model is derived from a basic assumption made on the form of a stochastic process driving the instantaneous interest rate $r_t, t \in [0,T]$. We will assume that the instantaneous interest rate (short rate) r and its volatility y satisfy the following system of stochastic differential equations of the Orstein-Uhlenbeck type:

$$dr = \kappa_1(\theta_1 - r)dt + \sqrt{y} r^\gamma dw_1 \tag{2.1}$$

$$dy = \kappa_2(\theta_2 - y)dt + v y^\delta dw_2 \tag{2.2}$$

where $\theta_1 > 0$ is a given constant characterizing the long term average interest rate, $\theta_2 > 0$ is the long term average volatility, $\gamma, \delta \geq 0$ are model parameters, $v > 0$ is the constant volatility of the volatility, i.e. the volatility of the stochastic process for the short rate volatility, $\kappa_1, \kappa_2 > 0$ are rates of reversion for the short rate and volatility, resp., w_1, w_2 are two Wiener processes (c.f. [7]) with correlation $\rho \in [-1,1]$, i.e. $\rho dt = E(dw_1(t)dw_2(t))$. In this model the short rate r_t mean reverts toward an unconditional mean θ_1, whereas the volatility mean reverts toward θ_2 value. In Fig. 2.1 we present a result of a numerical simulation of a process $\{r(t), y(t)), t \in [0,T]\}$ satisfying (2.1)-(2.2) with parameters $\kappa_1 = \kappa_2 = 1, \theta_1 = 0.06, \theta_2 = 0.3, v = 1, \gamma = 0, \delta = 0.5, T = 10$. We chose the correlation $\rho = 0.5$.

It is worthwhile noting that $y(t) \to \sigma^2 := \theta_2$ as $t \to \infty$ in the case $v = 0$. The equation for the short rate then reduces to $dr = \kappa_1(\theta_1 - r)dt + \sigma r^\gamma dw_1$ which a

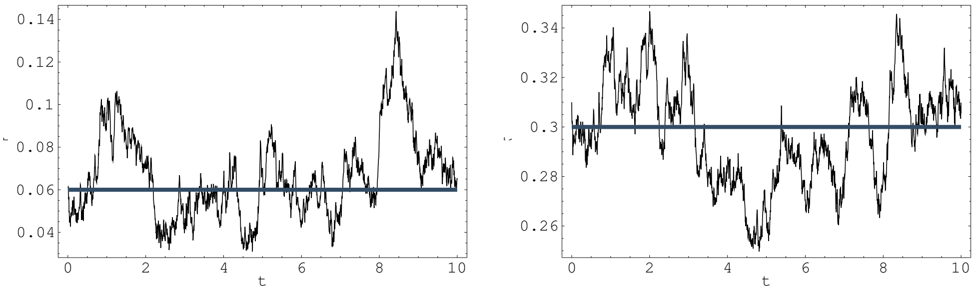

FIG. 2.1. *Simulation of a solution* (r, y) *to the system of stochastic differential equations (2.1)-(2.2). The long term means* θ_1 *and* θ_2 *are depicted by a vertical blue line.*

stochastic equation driving the short rate in the Vašíček model ($\gamma = 0$) or CIR model ($\gamma = 1/2$) or generalized CIR model ($\gamma = 3/2$).

Let us denote by t and T the present time and maturity, resp. If we denote by τ the time to maturity T, i.e. $\tau = T - t$, then it is well known (see [7]) that the bond price $P = P(\tau, r, y)$ is a classical solution of the partial differential equation

$$-\frac{\partial P}{\partial \tau} + (\kappa_1(\theta_1 - r) - \tilde{\lambda}_1 \sqrt{y} r^\gamma)\frac{\partial P}{\partial r} + (\kappa_2(\theta_2 - y) - \tilde{\lambda}_2 v y^\delta)\frac{\partial P}{\partial y} +$$
$$\frac{1}{2}(\sqrt{y} r^\gamma)^2 \frac{\partial^2 P}{\partial r^2} + \frac{1}{2}(v y^\delta)^2 \frac{\partial^2 P}{\partial y^2} + (\sqrt{y} r^\gamma)(v y^\delta)\rho \frac{\partial^2 P}{\partial r \partial y} - rP = 0 \tag{2.3}$$

with the initial condition $P(0, r, y) = 1$ and such that the above equation is satisfied up to the boundary of the 2D spatial domain $\{(r, y), 0 \le r, 0 \le y\}$. Here $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ stand for the so-called market price of risk and are, in general, assumed to be functions depending on r and y only (c.f. [5, 7]).

The term structure R (or yield) is a functional dependence of time to maturity τ and the current value of the short rate r. In a two-factor model it also depends on the stochastic volatility y, i.e. $R = R(\tau, r, y)$. It is related to the bond price P as: $P = e^{-R\tau}$. It means

$$R(\tau, r, y) = -\frac{\log P(\tau, r, y)}{\tau}\,. \tag{2.4}$$

3. Volatility averaging of bond prices and term structures. A solution $P(\tau, r, y)$ of the partial differential equation (2.3) gives us the bond prices for given values of the short rate r and the variable y. Unlike the short rate r the volatility y is not an observable variable in the real market. It suggests investigation of $P(\tau, r, y)$ for the given τ and r as function of the random variable y.

In what follows we will assume that the value of the short rate r at the time τ to maturity is known from the market data. The hidden parameter in the model is the volatility y which is supposed to be driven by the stochastic differential equation (2.2). In order to find the distribution of $P(\tau, r, y)$, one has to know distribution of y

We remind ourselves (c.f. Goodman *et al.* [2]) that if the process $y(t)$ satisfies a stochastic differential equation

$$dy(t) = a(t, y(t))dt + b(t, y(t))dw,$$

then, with regard to the Feynman-Katz formula, the conditional density $f(t, y|y(0) = y_0)$ of the random variable $y(t)$ satisfying an initial condition $y(0) = y_0$ is a solution

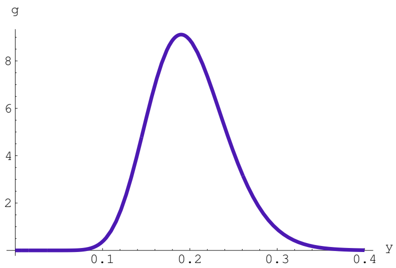

FIG. 3.1. *A limiting density distribution g of the y variable.*

to the Fokker-Planck partial differential equation

$$-\frac{\partial f}{\partial t} - \frac{\partial (af)}{\partial y} + \frac{\partial^2}{\partial y^2}\left(\frac{b^2 f}{2}\right) = 0, \ \ \tau > 0\,, \tag{3.1}$$

with the initial condition $f(0, y) = \delta_0(y - y_0)$ where δ_0 is the Dirac function.

In general, the density function $f(\tau, y)$ is not known in a closed form for arbitrary $\delta > 0$. However, if we consider a special case in which the variable y follows the so-called Bessel square root process $dy = \kappa_2(\theta_2 - y)dt + vy^{\frac{1}{2}}dw_2$ i.e. the parameter $\delta = \frac{1}{2}$, then the density of $y(t)$ subject to the initial condition $y(0) = y_0$ can be expressed by an explicit formula (see [7])

$$f(t, y|y(0) = y_0) = \begin{cases} 0 & \text{for } y \le 0, \\ c_0 e^{-p-q}\left(\frac{q}{p}\right)^{\frac{\alpha-1}{2}} I_{\alpha-1}(2\sqrt{pq}) & \text{for } y > 0, \end{cases} \tag{3.2}$$

where

$$c_0 = \frac{2\kappa_2}{v^2\left(1 - e^{-\kappa_2 t}\right)}, \ p = c_0 y_0 e^{-\kappa_2 \tau}, \ q = c_0 y, \ \alpha = \frac{2\kappa_2\theta_2}{v^2},$$

and I_m is the modified Bessel function of the first kind of the order m (see [1]). Taking into account properties of the modified Bessel function we are yet able to compute the limiting distribution g, i.e. the limit $g(y)$ of the density function $f(t, y|y(0) = y_0)$ for $t \to \infty$. In the case of $\delta = \frac{1}{2}$ we can compute the limit

$$g(y) := \lim_{t\to\infty} f\left(t, y|y(0) = y_0\right) = \begin{cases} 0 & \text{for } y \le 0, \\ \frac{1}{\Gamma(\alpha)}\lambda^\alpha e^{-\lambda y} y^{\alpha-1} & \text{for } y > 0, \end{cases} \tag{3.3}$$

which is a density of the Gamma distribution $\Gamma(\lambda, \alpha)$ (see Fig. 3.1) with parameters

$$\lambda = \frac{2\kappa_2}{v^2}, \quad \alpha = \frac{2\kappa_2\theta_2}{v^2}. \tag{3.4}$$

Assuming that the process (2.1)-(2.2) runs for a long enough time, we may approximate the distribution of $y(t)$ for large t with a limiting distribution g found above. The averaged price of a bond and the averaged term structure with respect to the volatility are then given by

$$\langle P(\tau, r, y)\rangle_y = \int_0^\infty P(\tau, r, y)g(y)dy, \quad \langle R(\tau, r, y)\rangle_y = \int_0^\infty R(\tau, r, y)g(y)dy\,.$$

4. Algorithm for volatility averaging in the Fong-Vašíček model. In this section we discuss the main result of this paper - the method of volatility averaging for a special class of two-factor models which is referred to as the Fong-Vašíček model [5]. Our algorithm consists in finding a solution P to the governing PDE (2.3). Knowing the density distribution of the stochastic volatility we are then able to compute the bond price averaged with respect to stochastic volatility. In a general case of model parameters we have to solve the governing equation (2.3) numerically. However, in the special case in which the short rate r and its volatility y evolves according to the following system of stochastic differential equations:

$$\begin{aligned} dr &= \kappa_1(\theta_1 - r)dt + \sqrt{y}dw_1\,, \\ dy &= \kappa_2(\theta_2 - y)dt + v\sqrt{y}dw_2 \end{aligned} \tag{4.1}$$

we are able to simplify the computation significantly as equation (2.3) can be separated and subsequently it can be reduced to a system of ODEs which can be easily solved by the Runge-Kutta method. Notice that the above system of stochastic differential equations corresponds to the choice of $\gamma = 0$ and $\delta = \frac{1}{2}$ in (2.1)-(2.2) and it is called the Fong-Vašíček model in the literature. It can be viewed as a natural extension of the Vašíček model ([10]), a one-factor model given by the first of the above equations with a constant y. In the Fong-Vašíček model the market prices $\tilde{\lambda}_i, i = 1, 2$ of risk appearing in (2.3) are defined as

$$\tilde{\lambda}_1 = \lambda_1\sqrt{y},\ \tilde{\lambda}_2 = \lambda_2\sqrt{y}\,.$$

For a fixed y, the term $\tilde{\lambda}_1$ corresponds to the market price of risk in the Vašíček model.

In what follows we will look for a solution of (2.3) in the form

$$P(\tau, r, y) = A(\tau)e^{-B(\tau)r - C(\tau)y}\,. \tag{4.2}$$

Inserting the above ansatz into (2.3) and by collecting the terms multiplying r and y variables one sees that the bond price P is a solution to the parabolic PDE (2.3) iff the functions $A = A(\tau),\ B = B(\tau), C = C(\tau), \tau \in (0, T]$, satisfy the following system of ordinary differential equations:

$$\begin{aligned} \dot{A} &= -A\left(\kappa_1\theta_1 B + \kappa_2\theta_2 C\right)\,, \\ \dot{B} &= -\kappa_1 B + 1\,, \\ \dot{C} &= -\lambda_1 B - \kappa_2 C - \lambda_2 v C - \frac{B^2}{2} - \frac{v^2C^2}{2} - v\rho BC \end{aligned} \tag{4.3}$$

with initial conditions $A(0) = 1,\ B(0) = 0,\ C(0) = 0$. Integrating the equation for B yields

$$B(\tau) = \left(1 - e^{-\kappa_1\tau}\right)/\kappa_1$$

and hence C satisfies the differential equation

$$\dot{C}(\tau) + \lambda_1 B(\tau) + \frac{B(\tau)^2}{2} + \left(\kappa_2 + \lambda_2 v + v\rho B(\tau)\right) C(\tau) + \frac{v^2}{2}C(\tau)^2 = 0\,.$$

The above ODE can be solved numerically by means of the Runge-Kutta method. In our numerical simulations we used the Runge-Kutta method of the 4th order with adaptive step size. Notice that the coefficient B has exactly the same form as the one in the one-factor Vašíček model (see [7]). Finally, by integrating the equation for A, we obtain

$$A(\tau) = \exp\left(-\theta_1\tau + \theta_1 B(\tau) - \kappa_2\theta_2 \int_0^\tau C(s)ds\right).$$

In what follows we will make a structural assumption on the model parameters guaranteeing feasibility of the bond price P given by (4.2). By a feasible solution we mean a function P such that $0 < P(\tau, r, y) < 1$ for $\tau \in (0, T]$ (the bond price is less than its value in the maturity), the function $r \mapsto P(\tau, r, y)$ is decreasing (larger short rates mean lower present bond prices), the function $y \mapsto P(\tau, r, y)$ is decreasing (higher volatility of the short rate means lower bond prices). In order to ensure these properties it suffices to prove $0 < A(\tau) < 1$, $B(\tau) > 0$ and $C(\tau) > 0$ for $\tau \in (0, T]$. In the following we will show that under the assumption

$$\lambda_1 \le -\frac{1}{2\kappa_1} \tag{4.4}$$

we have $C(\tau) > 0$ for $\tau \in (0, T]$. Indeed, since $C(0) = 0$ and $B(0) = 0$ it follows from the differential equation for C that $\dot{C}(0) = 0$ and so $\ddot{C}(0) = -\lambda_1 > 0$. Hence $C(\tau) > 0$ in some neighborhood of $\tau = 0$. Now it suffices to show that $\dot{C}(\tau) > 0$ whenever $C(\tau) = 0$. This is true because for $C(\tau) = 0$ the derivative $\dot{C}(\tau)$ is equal to

$$\dot{C}(\tau) = -\frac{1 - e^{-\kappa_2\tau}}{2\kappa_1^2}\left(2\lambda_1\kappa_1 + 1 - e^{-\kappa_1\tau}\right) > -\frac{1 - e^{-\kappa_2\tau}}{2\kappa_1^2}\left(2\lambda_1\kappa_1 + 1\right) \ge 0\,.$$

Because C is continuous it must be bounded from above on any compact interval $[0, T]$. Furthermore $A(\tau)$ never attains zero value as the integral $\int_0^\tau C(s)ds$ is always finite for $\tau \in [0, T]$. As $B(\tau) < \tau$ for any $\tau > 0$ we have

$$0 < A(\tau) < \exp\left(-\theta_1(\tau - B(\tau))\right) < 1\,,$$

and, as $B(\tau) \to 1/\kappa_1$ as $t \to \infty$ we have $A(\tau) \to 0$ for $\tau \to \infty$. In Fig. 4.1 we present behavior of a solution A, B, C to (4.3). In the left column we show their behavior for parameter values satisfying the structural condition (4.4). The corresponding solution P is feasible. On the other hand, the right column contains plots of a solution A, B, C to (4.3) in the case the condition (4.4) is violated. Negative values of C imply unfeasibility of the solution P. Fig. 4.2 depicts a 3D plot of a feasible solution P for one instant of time to maturity τ.

In the remaining part of this section we will find averaged values of the bond price $\langle P(\tau, r, y)\rangle_y$ and the term structure $\langle R(\tau, r, y)\rangle_y$ with respect to the stochastic volatility y by assuming the limiting distribution g of y. These averaged values will be given in terms of $A(\tau), B(\tau), C(\tau)$. Moreover, we will prove that their variance of $P(\tau, r, y)$ and $R(\tau, r, y)$ for fixed r converges to zero for $\tau \to \infty$.

We already know that the asymptotic distribution of y is the Gamma distribution $\Gamma(\lambda, \alpha)$ with parameters $\lambda = \frac{2\kappa_2}{v^2}, \alpha = \frac{2\kappa_2\theta_2}{v^2}$. Then the average value of $P(\tau, r, y)$ with respect to the asymptotic distribution g of y is given by the integral $\langle P(\tau, r, y)\rangle_y = \int_0^\infty P(\tau, r, y)g(y)dy$. Substituting (4.2), (3.3) and evaluating the integral gives

$$\langle P(\tau, r, y)\rangle_y = A(\tau)e^{-B(\tau)r}\left(1 + \frac{C(\tau)}{\lambda}\right)^{-\alpha}.$$

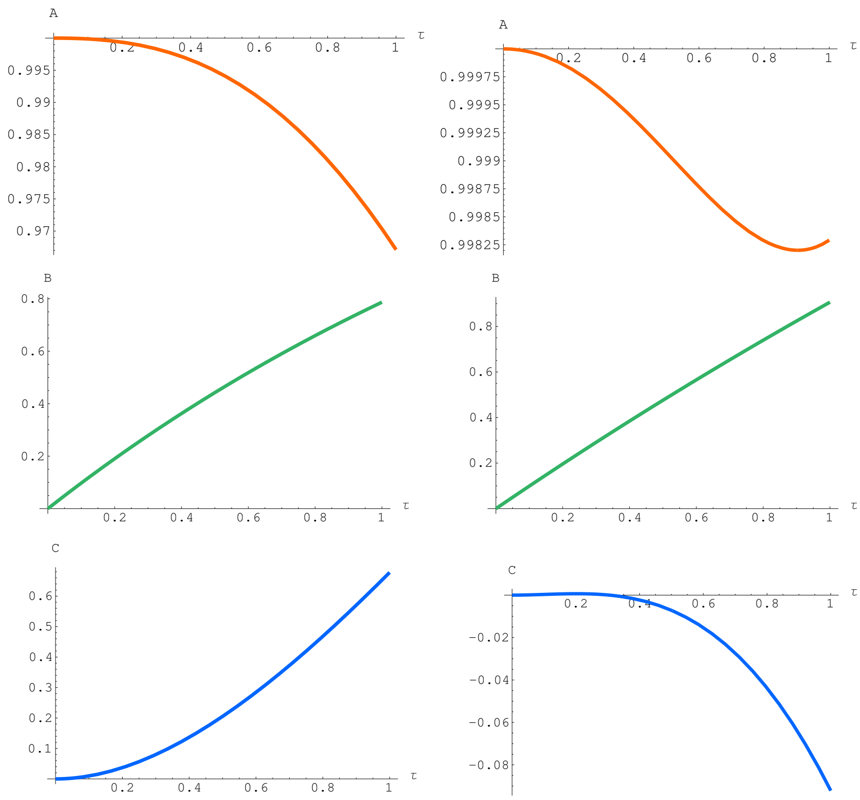

FIG. 4.1. *Graphs of the functions A, B, C for two different sets of parameters; $\lambda_1 = -2, \kappa_1 = 0.5$ (left column) and $\lambda_1 = -0.1, \kappa_1 = 0.2$ (right column). In both cases we chose $\lambda_2 = -3, \kappa_2 = 0.2, \theta_1 = 0.04, \theta_2 = 0.2, v = 0.1, \rho = 0.5$.*

The average of P^2 can be computed as $\left\langle P^2(\tau,r,y)\right\rangle_y = A^2(\tau)e^{-2B(\tau)r}\left(1+\frac{2C(\tau)}{\lambda}\right)^{-\alpha}$ and hence the variance of $P(\tau,r,y)$ can be expressed as:

$$Var_y(P(\tau,r,y)) = A^2(\tau)e^{-2B(\tau)r}\left(\left(1+\frac{2C(\tau)}{\lambda}\right)^{-\alpha} - \left(1+\frac{C(\tau)}{\lambda}\right)^{-2\alpha}\right)$$

By the mean value theorem the difference of negative α powers in the above expression is equal to $-\frac{\alpha}{\lambda^2}C(\tau)^2\xi^{-\alpha-1}$ for some ξ from the interval $\left(1+\frac{2C(\tau)}{\lambda}, \left(1+\frac{C(\tau)}{\lambda}\right)^2\right)$. Hence $\xi > 1$ and therefore

$$Var_y(P(\tau,r,y)) < A^2(\tau)e^{-2B(\tau)r}\frac{\theta_2 v^2}{2\kappa_2}C^2(\tau)\,.$$

Since $C(\tau)$ is bounded and $A(\tau)\to 0$ and $B(\tau)\to\frac{1}{\kappa_1}$ for $\tau\to\infty$ so we conclude that

$$Var_y(P(\tau,r,y)) \to 0 \text{ for } \tau\to\infty\,.$$

The function $y\mapsto P(\tau,r,y)$ is strictly convex because $\partial_y^2 P(\tau,r,y) = C(\tau)^2 P(\tau,r,y) > 0$. Hence, by Jensen's inequality, we have

$$\left\langle P(\tau,r,y)\right\rangle_y = \int_0^\infty P(\tau,r,y)g(y)dy > P(\tau,r,\int_0^\infty yg(y)dy) = P(\tau,r,\theta_2)\,,$$

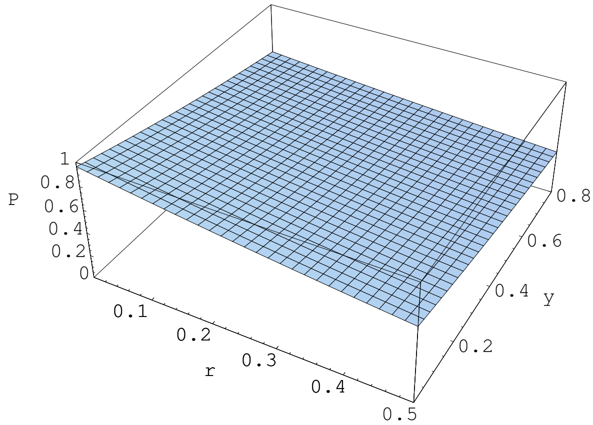

FIG. 4.2. *A solution $P = P(\tau, r, y)$ to the PDE for the bond price.*

i.e. the averaged bond price $\langle P(\tau, r, y)\rangle_y$ is always greater than the bond price $P(\tau, r, \theta_2)$ corresponding to the mean value $\theta_2 = \langle y \rangle$ of the stochastic volatility y. Notice that $P = e^{-R\tau}$ and $P = Ae^{-Br-Cy}$. Thus

$$R(\tau, r, y) = -\frac{\log P(\tau, r, y)}{\tau} = -\frac{\log A(\tau)}{\tau} + \frac{B(\tau)}{\tau} r + \frac{C(\tau)}{\tau} y\,.$$

It means that R is linear in the y variable. Since the expected value of y is θ_2 and its variance $Var(y) = \frac{\alpha}{\lambda^2}$ we obtain

$$\langle R(\tau, r, y)\rangle_y = R(\tau, r, \langle y\rangle) = R(\tau, r, \theta_2)$$

and

$$Var_y(R(\tau, r, y)) = \left(\frac{C(\tau)}{\tau}\right)^2 Var(y) = \frac{v^2\theta_2}{2\kappa_2}\frac{C^2(\tau)}{\tau^2}\,.$$

As $C(\tau)$ is bounded and $\frac{1}{\tau^2} \to 0$ for $\tau \to \infty$ we obtain

$$Var_y(R(\tau, r, y)) \to 0 \text{ for } \tau \to \infty\,.$$

In Fig. 4.4 we show the dependence of variances $Var_y(P(\tau, r, y))$ and $Var_y(R(\tau, r, y))$ as a functions of time τ to maturity for a fixed short rate value $r = 0.04$. Both of them have unique local maximum. Interestingly enough, the position of the maximum of the variance for P is large than the one for variance of R.

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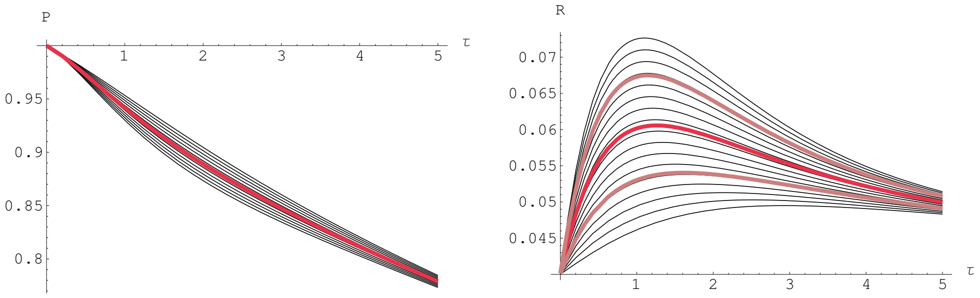

FIG. 4.3. *The price $P = P(\tau, r, y)$ for a fixed short rate $r = 0.04$ (left) and the structure $R = R(\tau, r, y), \tau \in [0, T]$, (right) is plotted for several values of the y variable. Curves P and R corresponding to the mean value of y are plotted in the middle in red. The 95% confidence interval of the term structure R is bounded by upper and lower pink curves.*

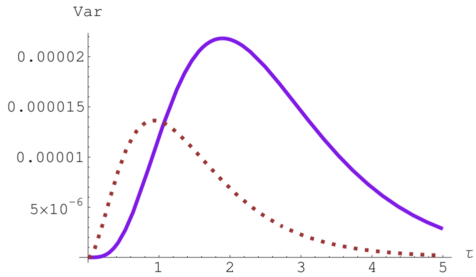

FIG. 4.4. *The variance $Var_y(R(\tau, r, y))$ (dashed curve) and $Var_y(P(\tau, r, y))$ (solid line) of the term structure R and the bond price P, resp., as a function of time τ to maturity for fixed short rate $r = 0.04$.*

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