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Dynamical systems with discrete Lyapunov functionals

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1. Introduction.

We study global properties of dynamical systems generated by the scalar parabolic equation

$$u_t = u_{\xi\xi} + f(t, \xi, u, u_\xi), \ \xi \in S^1 \tag{1.1}$$

and their C^1 small perturbations. Here f is C^2 in all variables and 1-periodic in t.

Long time behavior of the solutions of these equations as well as the equations in the same form on an interval with the separated boundary conditions have been investigated in many papers (see [An1], [AF], [BF1], [BF2], [BPS], [Br], [CCH], [CM], [CP], [FM], [FR], [FS], [He2], [HR], [Kw], [M1], [M3], [Na], [SY]). Let us mention now three works having direct impact on the questions solved in Theorems 1.1 and 1.2 below.

Fiedler and Mallet-Paret proved a Poincaré-Bendixson theorem in [FM-P] for the semiflow given by autonomous equation (1.1), i.e. when f does not depend on t. A description of ω-limit sets is also available for nonautonomous equations (1.1) in case f is independent of ξ (see [FS]). On the other hand, Fiedler and Sandstede [FS] showed that, in the general case, $\omega-$limit sets of solutions can be as complicated as those of solutions of smooth 1-periodic vector fields in $\mathbb{R}^2$. More specifically, to any C^2 $1-$periodic vector field in the plane one can associate a nonlinearity f such that (1.1) yields the same dynamics in some invariant plane as the given vector field. One of our theorems complements these results; we show that each $\omega-$limit set of (1.1) can be imbedded in the plane. Chen and Poláčik studied chain reccurent set for the period map of the nonautonomous equation in the form (1.1) under the Dirichlet boundary conditions in [CP]. They described a way how to prove that for a broad class of small C^1 perturbations of this period map the ω-limit sets are just single points. Similar perturbation results for separated boundary conditions as well as for equations on S^1 are proved in this work, using a completely different, unified method.

Before stating our results we review some properties of equation (1.1).

Let X be any fractional power space associated with the operator $u \mapsto -u_{\xi\xi}$: $H^2(S^1) \to L^2(S^1)$, denoted by A, such that the embedding relation $X \hookrightarrow C^1(S^1)$ is satisfied. Then by the standard theory ([He1]) equation (1.1) induces a local semiflow $(t, u(t, \cdot)) \in [0, \infty) \times X$, where $u(t, u_0)$ is the solution of (1.1) in the sense of ([He1]). If the solution $u(t, \cdot)$ of (1.1) is bounded in X norm then it is global and its orbit is precompact in X. We are interested in asymptotic behavior of these solutions. In fact we restrict ourselves to the solutions starting in a bounded open set $B \subset X$. For the next two theorems we assume the following dissipativity condition fullfiled for a large subclass of equations (1.1) (see [Ha])

$$(D) \qquad \begin{array}{l} \text{there is a } T_0 > 0 \text{ such that for any } u_0 \in cl(B) \text{ the solution of (1.1)} \\ \text{with } u(0, \cdot) = u_0 \text{ is global and } u(t, \cdot) \in B \text{ for all } t > T_0. \end{array}$$

This condition also follows from pointwise dissipativity and compactness of the flow (see [Ha]). Having (D), in the nonautonomous case we can define the Poincaré map F by

$$Fu_0 = u(1, u_0), \quad u_0 \in B.$$

By [He1] the restriction of the map F to the set B belongs to the following Banach space

$$C^1(B, X) := \{G : B \to X \ : \ G \text{ is continuously Fréchet differentiable with bounded derivative on } B\},$$

with the norm

$$\mid G \mid_{C^1(B,X)} = \sup_{x \in B} \mid G(x) \mid_X + \sup_{x \in B} \mid DG(x) \mid_{L(X,X)} .$$

Because of (D) all $F^n u_0, \ u_0 \in B, n \in \mathbb{N}$ are defined. Thus we can define ω-limit set of $u_0 \in B$ for F as the set of all limits of convergent subsequences of the sequence $F^n u_0, \ n = 1, 2, \ldots$.

Theorem 1.1. *Assume (D) for equation (1.1). Let F be the Poincaré map for (1.1) and denote by B_1 the set $\bigcup_{n=1}^{N_0} F^n(B)$ for an $N_0 > T_0$. Then there is an $\epsilon > 0$ such that for any map $G : X \to X$ with*

$$\sup_{x \in B_1} \mid Gx - Fx \mid_X, \quad \sup_{x \in B} \mid DG(x) - DF(x) \mid_{L(X,X)} < \epsilon$$

the ω-limit set of any $u_0 \in B$ for G is homeomorphic to a subset in the plane.

Now we turn to the autonomous case when (D) is satisfied for (1.1). Then equation (1.1) defines maps $S_t : X \to X, \ t > 0$ by the formula

$$S_t u_0 := u(t, u_0).$$

Obviously $S_{t_1+t_2} = S_{t_1} \circ S_{t_2}, \ t_1, t_2 \geq 0$ and $\lim_{t \to 0} S_t u_0 = u_0, \ u_0 \in X$. Such a family of maps is called a semiflow on X. By [He1] we know that each map $S_t, \ t > 0$ restricted to B is from $C^1(B, X)$. and the map

$$(t, u) \mapsto \frac{d}{dt} S_t u : \mathbb{R}^+ \times X \to X$$

is defined and continuous.

We say that family of maps $S_t : X \to X, \ t > 0$ with the above properties is a semiflow on X C^1 on B.

By the ω-limit set of a point $u_0 \in B$, denoted by $\omega(u_0)$, we mean the set of all limits of all convergent sequences $S_{t_n} u_0, \ n = 1, 2, \ldots$ where $t_n, n = 1, 2, \ldots$ are unbounded increasing sequences of positive numbers. The ω-limit set of any point $u_0 \in B$ is invariant under the semiflow S_t, i.e $S_t(\omega(u_0)) = \omega(u_0)$. Therefore for any point x in $\omega(u_0)$ we can associate a map

$$t \mapsto S_t x : \ \mathbb{R} \to \omega(u_0),$$

where for $t > 0$ $S_{-t} x$ is a point whose S_t image is x. If this map has limits for both $t \to +\infty$ and $t \to -\infty$ then we call its image a connecting orbit of equilibria.

Theorem 1.2. *Assume (D) for an autonomous equation (1.1). Let $S_t, t > 0$ be the semiflow given by (1.1) and denote by B_1 the set $\bigcup_{0<t\leq T_0} S_t(B)$. Then there is an $\epsilon > 0$ such that for any semiflow S'_t, $t > 0$ on X C^1 on B such that*

$$\sup_{x\in B_1} \mid S'_t x - S_t \mid_X, \quad \sup_{x\in B} \mid DS'_t(x) - DS_t(x) \mid_{L(X,X)} < \epsilon, \quad \textit{for all} \ \ t \in [1,2]$$

the ω-limit set of any $u_0 \in B$ for S'_t consists of either a single periodic orbit or equilibria and their connecting orbits.

Note that in the above theorems we do not assume small C^1 perturbations to be injective or compact. The possible applications of these theorems are parabolic equations on thin annulus (see [HR]) or perturbations of the nonlinearity in (1.1) by a small delay.

Now we state immediate application for abstract parabolic equations allowing for instance small C^1 dependence on nonlocal terms containing in nonlinearity of (1.1).

Consider the abstract parabolic equation

$$u_t = Au + f_0(t,u) + \epsilon g(t,u), \tag{1.2}$$

where $\epsilon > 0$, A is the sectorial operator on $L^2(S^1)$ given above, $f_0(\cdot,\cdot) : [0,+\infty) \times X \to L^2(S^1)$ is a C^1 1-periodic function representing for fixed $t \in [0,+\infty)$ the Nemitskii operator

$$u(\cdot) \mapsto f(t, u(\cdot), u_\xi(\cdot)) : \ X \to L^2(S^1),$$

$g(\cdot,\cdot) : [0,+\infty) \times X \to L^2(S^1)$ is any C^1 1-periodic globally Lipschitz function.

Corollary 1.1. *Let the operator f_0 be such that for equation (1.1) (D) is satisfied. Then there is an $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0]$ the ω-limit set of any $u_0 \in B$ for the Poincaré map associated with equation (1.2) is embedded into the plane.*

If moreover the functions f_0 and g do not depend on t, the ω-limit set of any $u_0 \in B$ for the semiflow associated with (1.2) consists of either a single periodic orbit or some equilibria and their connecting orbits.

Note that in the case when a Poincaré-Bendixson theorem is valid for C^1 small perturbations of a finite dimensional systems (cf. Theorem 8.2) arbitrarily small C^0 perturbations could have a chaos as it is proved in [Ge]. It is obvious that the C^1 structural stability of ω-limit sets stated in the above results does not take place for general dynamical systems even in finite dimensions. The needed structure here is provided by the properties of the linear equation (1.1), i.e. when f is linear in the variables u, u_ξ and not necessary depending periodically on t. That structure is used in most works devoted to the study of dynamical properties of the solutions of the same kind of equations as (1.1) (see e.g. [An1], [AF], [BF2], [BPS], [Br], [CCH], [CM], [CP], [CLM-P1], [CLM-P2], [FM], [FR], [FS], [He2], [M1], [M2], [M3], [Na], [SY]). The crucial properties are compactness of the solution operator and monotonicity of number of zeros (see [Ni], [M2], [He2], [An2]) for solutions of these linear equations. They are useful for dynamical study because the difference of any two solutions of nonlinear equation (1.1) $u(t,\cdot)-v(t,\cdot)$ satisfies such a linear equation. The compactness is standard for even more general parabolic equations (see [He1]). In [An1] the full strenght of the monotonicity is proved, namely, any nonzero solution of the linear equation (1.1) has immediately finite number of zeros nonincreasing with time and strictly dropping at each time when the solution has a double zero. These properties led to the definition of an abstract discrete Ljapunov functional (cf. Definition 2.1).

In [FM-P] it is showed for autonomous equations that ω-limit sets are injectively projected by the following map

$$u(\cdot) \mapsto (u(\xi_0), u_\xi(\xi_0)) : \ X \to \mathbb{R}^2, \quad \xi_0 \in S^1$$

into the plane. This follows from the fact that the zero number is constant along the difference of two trajectories in the ω-limit sets. To prove this, the Poincaré-Bendixson Theorem in the plane was used in [FM-P]. We show that the stability of dimensions of ω-limit sets is a consequence of the mentioned constancy of a discrete Ljapunov functional constructed for each C^1 small perturbation from a cone structure of the unperturbed problem (cf. Theorem C and Corollary 2.3 in Section 2). To accomplish that we need the existence of an exponential separation (cf. Definition 2.3) for vector bundle maps "strongly" preserving a cone structure. This is formulated in the next section in Theorem B. This theorem is also the main abstract result obtained here with possible other applications to the dynamics of equation (1.1) (cf. Remark 8.1).

The paper is organized as follows. In the next section we formulate all abstract results. In Section 3 the mentioned constancy of an abstract Ljapunov functional for a discrete dynamical system is proved. Theorem B and C indicated above are proved in the following two Sections 4 and 5. Their consequences are shown in Section 6. Section 7 is devoted to the proof of Theorems 1.1 and 1.2 from abstract results. In the last section we apply abstract results to the equation in the form (1.1) on the segment under the Dirichlet boundary conditions (cf. Theorem 8.1) and to monotone cyclic feedback systems.

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2. Abstract results.

Let $\mathcal{X}$ be a metric space and $G : \mathcal{X} \to \mathcal{X}$ a map. If $z \in \mathcal{X}$ we say that the following set

$\omega(z) = \{u \in \mathcal{X}$: there is a sequence of natural numbers $n_k \to +\infty$ as $k \to +\infty$ such that $\lim_{k\to+\infty} G^{n_k} z = u\}$.

is the ω−limit set of z for G. By $O^+(z)$ for $z \in \mathcal{X}$ we denote the positive orbit of z, i.e. the set $\{F^i z : \; i \in \mathbb{N} \cup \{0\}\}$. The omega limit set of any point z is closed. If moreover the positive orbit of z is precompact then $\omega(z)$ is compact.

A subset K of $\mathcal{X}$ is said to be invariant under G if $G(K) = K$. Note that if G is continuous, injective and K is compact, invariant under G then G restricted to K is a homeomorphism on K. In this case for any $x \in K$ the set $\{G^i x : \; i \in \mathbb{Z}\}$ is defined and called the orbit of x.

Denote by D the set $\{(x, x) : x \in \mathcal{X}\}$, by $\mathcal{X}^{(2)}$ the induced metric space of the metric space $\mathcal{X} \times \mathcal{X}$ on the set $\mathcal{X} \times \mathcal{X} \setminus D$ and by $\Omega(z)$ the set $\omega(z) \times \omega(z) \setminus D$.

Definition 2.1. Let $\mathcal{X}$ be a compact metric space and let $G : \mathcal{X} \to \mathcal{X}$ be a continuous, injective map. By a *discrete Ljapunov functional for G on $\mathcal{X}$* we mean a function $\theta : \mathcal{X}^{(2)} \to \mathbb{N} \cup \{0\}$ satisfying the following axioms (A1-A3) with a natural number μ

(A1) *Monotonicity*: $\theta(x, y) \geq \theta(G^m x, G^m y)$ for any point $(x, y) \in \mathcal{X}^{(2)}$ and $m \geq \mu$.

(A2) *Dropping property at the points of discontinuity*: Let θ be discontinuous at a point $(G^\mu x, G^\mu y)$ for some $(x, y) \in \mathcal{X}^{(2)}$. Then $\theta(G^{2\mu} x, G^{2\mu} y) < \theta(x, y)$.

(A3) *Shifted lower semicontinuity*: Let $(x_n, y_n), \; n = 1, 2, ...$ be a convergent sequence of points from $\mathcal{X}^{(2)}$ with the limit $(x, y) \in \mathcal{X}^{(2)}$. Then

$$\theta(G^\mu x, G^\mu y) \leq \liminf_{n\to+\infty} \theta(x_n, y_n).$$

We remark that the zero number for the image of positively invariant subset under the period map for equation (1.1) satisfy the axioms above with $\mu = 1$. We actually prove stronger properties of functionals we use (cf. Remark 5.1).

Theorem A. *Let X be a compact metric space and G be an injective continuous map of X. If there is a discrete Ljapunov functional θ for G on $\mathcal{X}$, then for each $z \in X$ and $(x, y) \in \omega(z) \times \omega(z) \setminus D$ the functional θ is constant on the sequence $\{(G^n x, G^n y) : \; n \in \mathbb{Z}\}$.*

Now we are going to introduce several definitions needed for the formulation as well as for the proof of Theorem B stated below.

Definition 2.2. Let F be a homeomorphism of a compact metric space K, and Y a metric space. Further let $\{Y_x, \; x \in K\}$ a family of subsets of Y and $\{R_x, \; x \in K\}$ be a family of continuous maps $R_x : Y_x \to Y, \; x \in K$. Then we say that:
i) the set

$$\bigcup_{x\in K}\{x\}\times Y_x \text{ or shortly } K\times(Y_x)$$

is a bundle of sets over K (or shortly a bundle). If there is a subset Y_0 such that $Y_0 = Y_x$ for all $x \in K$ we write the bundle $K \times (Y_x)$ as $K \times Y_0$.
ii) the family of sets $\{Y_x,\ x \in K\}$, or the corresponding bundle is continuous iff for any convergent sequences $x_n \in K,\ y_n \in Y_{x_n},\ n = 1.2.\dots$ their limits, $x \in K$ and $y \in Y$, respectively, satisfy $y \in Y_x$.
iii) a pair $(F, \{R_x,\ x \in K\})$, or shorter (F, R) is a bundle map on the bundle $K \times (Y_x)$ iff

$$R_x(Y_x) \subset Y_{Fx}.$$

iv) the family $\{Y_x,\ x \in K\}$ or the corresponding bundle $K \times (Y_x)$ is invariant (positively invariant) under a pair (F, R) iff for all $x \in K$

$$R_x(Y_x) = Y_{Fx} \quad (R_x(Y_x) \subset Y_{Fx}).$$

v) a bundle $K \times \{Z_x,\ x \in K\}$ is a subbundle of the bundle $K \times (Y_x)$ iff $Z_x \subset Y_x$ for all $x \in K$.
vi) a bundle $K \times (Y_x)$ is a $k-$dimensional vector bundle iff Y is a Banach space and each $Y_x,\ x \in K$ is a $k-$ dimensional linear subspace of Y.
vii) a bundle map (F, R) on a vector bundle $K \times (Y_x)$ is a vector bundle map iff each $R_x,\ x \in K$ is a linear bounded map from Y_x to Y_{Fx}.
viii) R^n_x for $x \in K$ and $n \in \mathbb{N}$ is the map

$$R_{F^{n-1}x} \circ \cdots \circ R_{Fx} \circ R_x : Y_x \mapsto Y_{F^n x}.$$

Let Y be a Banach space and k a natural number. Then by $G(k, Y)$ we denote the metric space of $k-$dimensional linear subspaces of a Banach space Y. The distance of any two elements in $G(k, Y)$ is given by the Hausdorff distance of the unit spheres in the corresponding linear subspaces. Having $v_1, \dots, v_k$ any k vectors in Y by $[v_1, \dots, v_k]$ we denote the linear subspace generated by these vectors. For any subset A in Y we denote by $G_k(A)$ or simply G_kA the set of all $k-$dimensional linear subspaces lying in A. We will also write GA if it is clear which k is used.

Now consider any $k-$dimensional vector subbundle $K \times (Y_x)$ of a vector bundle $K\times Y$ where K is a compact metric space. Then $K\times(Y_x)$ is a continuous subbundle of $K \times Y$ iff the map $x \mapsto Y_x :\ K \to G(k, Y)$ is continuous. This is a consequence of Definition 2.2ii) and the definition of the metric in $G(k, Y)$.

For two Banach spaces $Y_1,\ Y_2,\ L(Y_1, Y_2)$ denotes the Banach space of linear bounded operators from Y_1 to Y_2. Let J be from $L(Y, Y)$,where Y is a Banach space. Then by G_kJ or GJ we denote the map defined for elements $E \in G(k, Y)$, on which J is injective, by $GJ(E) = J(E)$. In Sections 3 and 4 by X is denoted a Banach space with norm $|\cdot|$. By S we denote its unit sphere, and by X^* the dual Banach space. If a map J is in $L(X, X)$ then by J^* we denote its adjoint map in $L(X^*, X^*)$. For any linear subspace L in X^* we define the anihilator of L in X as follows

$$Anih(L) := \{v \in X :\ l(v) = 0 \ \text{ for all } l \in L\}.$$

Also if L is a $k-$tuple of functionals from X^* then $Anih(L)$ means the anihilator of the linear space generated by these vectors. Analogously for a linear subspace E in X we define the anihilator of E in X^*.

A crucial concept is a $k-cone$ in X defined as follows.

A closed subset C in X is a $k-$cone iff $\lambda C = C$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and there are $k-$dimensional space V_0 and $k-$ codimensional space L_0 such that $V_0 \subset C$ and $L_0 \cap C = \{0\}$.

Sometimes, when it is clear which k we have in mind, we say simply that C is a cone.

For the next definition we need one additional notation. Let X be a Banach space, $\{T_x, x \in K\}$ a family of operators in $L(X, X)$, and F a homeomorphism of a compact metric space K. Then by (F^{-1}, T^*) is denoted the vector bundle map on $K \times X$ defined by the homeomorphism F^{-1} on the compact set K and the family of compact maps $\{T^*_{F^{-1}x},\ x \in K\}$.

Definition 2.3. Let X be a Banach space, K a compact metric space, F a homeomorphism of K and $\{T_x,\ x \in K\}$ a family of maps in $L(X, X)$ continuously dependent on $x \in K$. We say that the vector bundle (F, T) *admits a $k-$dimensional continuous separation along* K or that there is a $k-$dimensional continuous separation for (F, T) on K iff there are $k-$dimensional subbundles $K \times \{V_x, x \in K\}$ and $K \times \{L_x, x \in K\}$ of the bundles $K \times X$ and $K \times X^*$, respectively, such that
i) $K \times (V_x)$ and $K \times (L_x)$ are continuous bundles, invariant under the bundle maps (F, T) and (F^{-1}, T^*), respectively.
ii) (*exponential separation*) There are constants $M > 0$ and $0 < \gamma < 1$ such that

$$\mid T_x^n w \mid \leq M\gamma^n \mid T_x^n v \mid$$

for all $x \in K,\ w \in Anih(L_x) \cap S,\ v \in V_x \cap S$ and $n \in \mathbb{N}$.
If C is a $k-$cone and $V_x \subset C$ and $Anih(L_x) \cap C = \{0\}$ for all $x \in K$ then we say that (F, T) *admits a $k-$dimensional continuous separation associated to the cone* C.

The following theorem gives sufficient conditions for the existence of the just introduced object.

Theorem B. *Let X be a Banach space, C a $k-$cone in X, K a compact metric space, F a homeomorphism of K and $\{T_x,\ x \in K\}$ a family of maps in $L(X, X)$ continuously dependent on $x \in K$. Assume that each $T_x,\ x \in K$, is compact and for any $x \in K$ and $v \in C \setminus \{0\}$ there is an open neighborhood of v in X mapped into $C \setminus \{0\}$ by the map T_x. Then the vector bundle map (F, T) admits a $k-$dimensional continuous separation along K associated with the $k-$cone C.*

Here we give a review of results generalized by Theorem B and its semiflow analog Corollary 2.2 below. The next of them concerns the case when C is a $1-$cone given as a union of two convex positive cones with nonempty interior. If K is a point, Theorem B is proved in [Pe] for finite-dimensional X and if X is any Banach space it is the Krein-Rutman Theorem. With described $1-$cones the theorem is proved in finite dimensions in [Ru] and for any Banach space in [PT]. There is also the result in [Mi] with a family of operators coming from linear reaction diffusion equations. The first result when C is not a $1-$cone seems to be the following one in [FO1]. If K is a point, $dim(X) < +\infty$, C a $k-$cone with nonempty interior the theorem is proved. Actually for this result in [FO1] is used little weaker definition of a $k-$cone, namely, C is a closed set invariant under the multiplication of nonzero real numbers and containing at least one k-dimensional subspace of X and not any $k+1$ dimensional subspace. But if there is any linear map satisfying the assumptions of Theorem B in this case, i.e. $T(C \setminus \{0\}) \in \int(C)$, this set has the same properties as our $k-$cone. Further, there are several results of transversal instersections of invariant manifolds for some monotone systems of equations ([FO2]) and equations of type (1.1) (see [An1], [He2], [CCH]) which include the existence of invariant bundles with K consisting of a connecting orbit (mostly for flows) of two equilibria or periodic orbits and the family of operators given by the linearizations of these equations. In [CLM-P1] the conclusion of Theorem B is proved for K a point and linear operators are time-one maps of linear equations of type (1.1) with Dirichlet boundary conditions. Results in [CLM-P2] concern linear nonautonomous equations of type (1.1) with various boundary conditions. We describe the main results in [CLM-P2] for equations $u_t = u_{\xi\xi} + b(t,\xi)u$ with Dirichlet boundary conditions and $b(t,\cdot)$ are from any but fixed ball in $L_\infty(\mathbb{R} \times [0,1])$ endowed with the weak star topology. The solution operators for these equations together with the usual skew-product flow on these balls give the vector bundle semiflow for which $k-$dimensional continuous separations are proved with any $k \in \mathbb{N}$.

The method of the proof of Theorem B has some common features with that in [PT] proving the theorem with $1-$cone consisting of two positive convex cones with nonempty interiors. The essential difference between them manifests in showing the existence of a dual invariant bundle. In [PT] this bundle was given by Tikhonov fixed point theorem due to involved convexity, without using "strong" monotonicity. Here, for general $k(\geq 2)$-cones and bundle maps (F,T) no fixed point theorem could be applied without utilizing "strong" preserving of cones by bundle maps.

The next theorem gives sufficient conditions under which the ω-limit sets for small C^1 perturbations of a map F_0 are embedded into the d-dimensional linear space.

Theorem C. *Let X be a Banach space, $\mathcal{U}_0$ an open subset of X, $F_0 \in C^1(\mathcal{U}_0, X)$, K_0 a compact subset of $\mathcal{U}_0$ invariant under F_0, $N \in \mathbb{N}$ or $N = +\infty$, C_i, $1 \leq i < N$ a k_i-cone , $C_N = X$ and Π a $d-$codimensional linear subspace of X. Further let $C_i \supset C_{i-1}$, $k_i > k_{i-1}$ for any $1 < i < N$ and $C_0 = \{0\}$. Suppose that there is a continuous family of compact operators $\{T_{(x,y)} \in L(X,X),\ (x,y) \in \mathcal{U}_0 \times \mathcal{U}_0\}$ such that for any $x, y \in \mathcal{K}_0$ the following holds*
i) $T_{(x,y)}(x-y) = F_0x - F_0y$.
ii) for any $1 \leq i \leq N$ and $v \in C_i \setminus \{0\}$ there is a neighborhood of v mapped into $C_i \setminus \{0\}$ by $T_{(x,y)}$.
iii) $v \in X \setminus \{0\}$ implies $T_{(x,y)}v \in C_j \setminus \{0\}$ for some $j \in \mathbb{N}$.
iv) $T_{(x,y)}(C_i) \cap (C_i \setminus C_{i-1}) \cap \Pi = \emptyset$ for all $1 \leq i \leq N$.
Then there exist an open neighborhood of K_0 denoted by $\mathcal{V}$ and an $\epsilon > 0$ such that for any $G \in C^1(\mathcal{V}, X)$ with

$$| F_0 |_{\mathcal{V}} - G |_{C^1(\mathcal{V},X)} < \epsilon$$

the $\omega-$limit set for G of any point $x_0 \in \mathcal{V}$, with all $G^n x_0$, $n \in \mathbb{N}$ defined and

$cl(\{G^n x_0 : \ n \in \mathbb{N}\} \subset \mathcal{V}$, is homeomorphic to a subset in $\mathbb{R}^d$.

If in the above theorem $d = 1$ then we obtain that ω-limit sets from the statement of the theorem are embedded into the line. If F is the time one map for the equation of the form (1.1) on the segment under the separated boundary conditions the assumptions of the following corollary are satisfied (see Theorem 8.1).

Corollary 2.1. *Let the assumptions of Theorem C be satisfied. Moreover suppose that Π is $1-$codimensional linear subspace of X and the following assumption is satisfied*

(A) if $v \in C_i \setminus C_{i-1}$ and $T_{(x,y)}v \in C_i \setminus C_{i-1}$ for some $v \in X$, $(x, y) \in K_0 \times K_0$, $1 \leq i \leq N$ then the vectors v and $T_{(x,y)}v$ lie in the same component of $X \setminus \Pi$.

Then there is an $\epsilon > 0$ and open neighborhood $\mathcal{V}$ of K_0 such that for any $G \in C^1(\mathcal{V}, X)$ with

$$| F_0 |_{\mathcal{V}} - G |_{C^1(\mathcal{V},X)} < \epsilon$$

and any distinct $x, y \in \mathcal{V}$, with all $G^n x, G^n$, $n \in \mathbb{N}$ defined, there is an $n_0 \in \mathbb{N}$ such that all $G^n x - G^n y$ lie in the same halfspace of $X \setminus \Pi$ or $| G^n x - G^n y |$ exponentially tends to zero as $n \to +\infty$. Moreover, the ω-limit set for G of any $x_0 \in \mathcal{V}$, with all $G^n x_0$, $n \in \mathbb{N}$ defined and $cl(\{G^n x_0 : \ n \in \mathbb{N}\}) \subset \mathcal{V}$, is just a single point.

For the formulation of analogous assertions for flows we need several definitions.

Let $\mathcal{X}$ be a metric space and $\mathcal{U}_0$ a subset in $\mathcal{X}$. We say that a family of maps $S_t \in C(\mathcal{U}_0, \mathcal{X})$, $t > 0$ is a *semiflow continuous on* $\mathcal{U}_0$ iff
a) the function $t \mapsto S_t x : (0, +\infty) \to \mathcal{X}$ is uniformly continuous and $\lim_{t \to 0} S_t x = x$ for all $x \in \mathcal{U}_0$.
b) if $x \in \mathcal{U}_0$ and t_1, $t_2 > 0$ such that $S_{t_1+t_2} x \in \mathcal{U}_0$ then

$$S_{t_2} \circ S_{t_1} x = S_{t_1+t_2} x.$$

If moreover $\mathcal{X}$ is a Banach space and the function $(t, x) \mapsto \frac{d}{dt} S_t x : (0, +\infty) \times \mathcal{U}_0 \to \mathcal{X}$ is continuous then we call S_t *regularizing semiflow continuous on* $\mathcal{U}_0$.

The image of the map $t \mapsto S_t x : [0, +\infty) \to \mathcal{X}$ is called a positive orbit of $x \in \mathcal{U}_0$. By the $\omega-$limit set of the point $x \in \mathcal{U}_0$ for a semiflow S_t on $\mathcal{U}_0$ we mean the set of limits of all convergent sequences $S_{t_n} x$ where $t_n \to +\infty$.

Let K be a compact metric space. We say that a semiflow S_t on K is a continuous flow on K if S_t is a homeomorphism of K for all $t > 0$. Let moreover $\{T_x^t : \ t > 0, \ x \in K\}$ be a family of bounded linear maps of a Banach space X with the following properties:

j) the function $(x, t) \mapsto T_x^t : \ K \times (0, +\infty) \to L(X, X)$ is continuous .
jj) $T_x^{t_1+t_2} = T_{S_{t_2} x}^{t_1} \circ T_x^{t_2}$ for all $t_1, t_2 > 0$ and $x \in K$.

Then the pairs of families of maps (S_t, T^t) is called a *vector bundle (or skew product) semiflow on* $K \times X$. We denote it by (S_t, T^t).

If C is a $k-$cone in X we give an analogue of the Definition 2.3 as follows:
We say that a vector bundle semiflow (S_t, T^t) on $K \times X$ admits a $k-$dimensional continuous separation along K associated with the $k-$cone C if there are $k-$dimensional

subbundles $K \times \{V_x : x \in K\}$ and $K \times \{L_x : x \in K\}$ of the bundles $K \times X$ and $K \times X^*$, respectively, such that the following properties are satisfied:
i) $K \times (V_x)$ and $K \times (L_x)$ are continuous bundles .
ii) $T_x^t V_x = V_{S_t x}$ and $T_x^{t*} L_{S_t x} = L_x$ for all $t > 0,\ x \in K$.
iii) *(exponential separation)* There are constants $M > 0$ and $0 < \gamma < 1$ such that one has

$$| T_x^t w | \leq M\gamma^t | T_x^t v |$$

for all $x \in K,\ w \in Anih(L_x) \cap S,\ v \in V_x \cap S$ and $t \geq 1$.

The following result is a continuous analog of Theorem B, and it actually follows from that theorem.

Corollary 2.2. *Let X be a Banach space, C a $k-$cone in X, K a compact metric space, (S_t, T^t) a vector bundle semiflow on $K \times X$. Assume that each $T_x^t,\ x \in K,\ t \geq 1$ is compact and maps an open neighborhood of any $v \in C \setminus \{0\}$ into $C \setminus \{0\}$. Then the vector bundle semiflow (S_t, T^t) admits a $k-$dimensional continuous separation along K associated with the $k-$cone C.*

Theorem C has a semiflow analog in the next corollary.

Corollary 2.3. *Let X be a Banach space, $\mathcal{U}_0$ an open subset of X, $N \in \mathbb{N}$ or $N = +\infty$, $C_i,\ 1 \leq i < N$ a sequence of k_i-cones, $C_0 = \{0\}, C_N = X$ and $C_i \supset C_{i-1},\ k_i > k_{i-1}$ for all $1 < i < N$. Further let S_t be a semiflow continuous on $\mathcal{U}_0$, Π a $d-$codimensional linear subspace of X and $K_0 \subset \mathcal{U}_0$ a compact and invariant set under each map $S_t,\ t > 0$.*

Suppose that for any $(x, y) \in \mathcal{U}_0 \times \mathcal{U}_0$ and any $\frac{1}{2} \leq t \leq 1$ there is a compact operator $T_{(x,y)}^t \in L(X, X)$ with the following properties :
i) $T_{(x,y)}^t (x - y) = S_t x - S_t y$.
ii) if $v \in C_i \setminus \{0\}$ then there is an open neighborhood of v mapped into $C_i \setminus \{0\}$ by $T_{(x,y)}^t$.
iii) $v \in X \setminus \{0\}$ implies $T_{(x,y)}^t v \in C_j \setminus \{0\}$ for some $j \in \mathbb{N}$.
iv) if $0 \neq T_{(x,y)}^t v \in ((C_i \setminus C_{i-1}) \cap \Pi)$ for an $1 \leq i \leq N$ then $v \notin C_i$.
v) the function $(t, x, y) \mapsto T_{(x,y)}^t : [\frac{1}{2}, 1] \times \mathcal{U}_0 \times \mathcal{U}_0 \to L(X, X)$ is continuous.

Then there exist an open neighborhood of K_0 denoted by $\mathcal{V}$ and an $\epsilon > 0$ such that for any semiflow S'_t continuous on $\mathcal{V}$ with $S'_t \in C^1(\mathcal{V}, X),\ t \in [\frac{1}{2}, 1]$ and

$$| S_t |_{\mathcal{V}} - S'_t |_{C^1(\mathcal{V},X)} < \epsilon \quad \textit{for all } \frac{1}{2} \leq t \leq 1$$

the $\omega-$limit set of any $x_0 \in \mathcal{V}$ for S'_t, for which the closure of its positive orbit lies entirely in $\mathcal{V}$, is homeomorphic to a subset in $\mathbb{R}^d$.

A Poincaré Bendixsom Theorem is stated in the following corollary.

Corollary 2.4. *In addition to the assumptions of Corollary 2.3 suppose that the semiflow S_t is regularizing and $d = 2$. Then there exist an open neighborhood of K_0 denoted by $\mathcal{V}$ and an $\epsilon > 0$ such that for any regularizing semiflow S'_t continuous on $\mathcal{V}$ with $S'_t \in C^1(\mathcal{V}, X),\ t \in [\frac{1}{2}, 1]$ and*

$$| S_t |_{\mathcal{V}} - S'_t |_{C^1(\mathcal{V},X)} < \epsilon \quad \textit{for all } \frac{1}{2} \leq t \leq 1$$

the $\omega-$ limit set of any $x_0 \in \mathcal{V}$, for which the closure of its positive orbit lies entirely in $\mathcal{V}$, consists of either a single periodic orbit or equilibria and their connecting orbits.

We rephrase the result of the last corollary by saying that $x_0 \in \mathcal{U}$ has the PB property for the semiflow S'_t.

3. Discrete Ljapunov functionals on omega-limit sets.

We begin with a definition of a function. Let u, v be two different points in X. Then sequence $\theta(G^i u, G^i v)$, $i = 1, 2, \ldots$ is eventually constant. Indeed, it is eventually nonincreasing because of Axiom A1 and has only natural or zero values. Therefore there is a minimal number $m \in \mathbb{N} \cup \{0\}$ from which the above sequence is constant. Hence θ is continuous at the point $(G^{m+1}u, G^{m+1}v)$ because in the other case by Axiom A2 we obtain $\theta(G^m u, G^m v) > \theta(G^{m+\mu}u, G^{m+\mu}v)$, contradicting the definition of the number m. Denote the number $m+1$ by $\eta(u, v)$. We shall use it in the following lemma.

Lemma 3.1. *If $z \notin \omega(z)$ then θ is bounded on $\Omega(z)$.*

Proof: We show that θ is bounded on $\{G^m z\} \times \{G^{i+m}z : i \in \mathbb{N}\}$ for some $m \in \mathbb{N}$ and then the shifted lower semicontinuity of θ gives boundedness of θ on $\Omega(z)$.

Denote the compact set $cl(O^+(z) \setminus \{z\})$ by C. Then $z \notin C$ because $z \notin \omega(z)$. By the definition of η the function θ is continuous on the set

$$\{(G^{\eta(z,u)}z, G^{\eta(z,u)}u) : u \in C\}.$$

Hence, using the continuity of G, we find for any $u \in C$ an open neighborhood U such that the function θ is constant on sets $\{G^{\eta(z,u)}z\} \times G^{\eta(z,u)}(U)$. Since C is compact there are points u_1, u_2, ..., u_k in C such that their corresponding neighborhoods U_i, $i = 1$, ..., k cover C. Denote

$$m = max\{\eta(z, u_i),\, 1 \le i \le k\} \quad \text{and} \quad \tau = max\{\theta(G^{\eta(z,u_i)}z, G^{\eta(z,u_i)}u_i),\, 1 \le i \le k\}.$$

The monotonicity of θ implies that τ is the upper bound of θ on the set $\{G^{m+\mu}z\} \times G^{m+\mu}(C)$.

Now take any $(u, v) \in \Omega(z)$. Since $\omega(z)$ is invariant under G^μ there are points $\bar{u}$ and $\bar{v}$ lying in $\omega(z)$ such that $u = G^\mu \bar{u}$, $v = G^\mu \bar{v}$. Therefore we can choose two increasing sequences of natural numbers such that

$$\bar{u} = \lim_{k \to +\infty} G^{m_k} z \quad \text{and} \quad \bar{v} = \lim_{k \to +\infty} G^{n_k} z.$$

In addition we may suppose that $n_k > m_k$, $k = 1, 2, \ldots$. Since $G^{n_k - m_k} z \in C$, we have

$$\theta(G^m z,\, G^{n_k - m_k + m} z) \le \tau.$$

Hence, by the monotonicity property of θ and Axiom A3, we finally obtain

$$\tau \ge \liminf_{k \to +\infty} \theta(G^{m_k} z, G^{n_k} z) \ge \theta(G^\mu \bar{u},\, G^\mu \bar{v}).$$

Thus $\theta(u, v) \le \tau$. Since (u, v) was an arbitrary point in $\Omega(z)$, τ is the upper bound of θ on $\Omega(z)$. ♢

The next lemma follows from Lemma 3.1.

Lemma 3.2. *If $z \notin \omega(z)$ then for any $(u,v) \in \Omega(z)$ such that*

$$\theta(u,v) > \theta(Gu,Gv)$$

there is a natural number N such that

$$\theta(G^{-N}u, G^{-N}v) > \theta(G^N u, G^N v)$$

and, moreover, θ is continuous at the points $(G^{-N}u, G^{-N}v)$ and $(G^N u, G^N v)$.

Proof: Take any (u,v) from the assumptions of the lemma. Then a sequence of natural numbers $\theta(G^{-i\mu}u, G^{-i\mu}v)$, $i = 1, 2 \ldots$, is nondecreasing by Axiom A2 and by Lemma 3.1 it is also bounded from above. Therefore this sequence is eventually constant. Hence by Axiom A1 the sequence $\theta(G^{-i}u, G^{-i}v)$ is eventually constant as well as the sequence $\theta(G^i u, G^i v)$, $i = 1, 2 \ldots$. Hence there exists $N > 2$ such that

$$\theta(G^{N-\mu}u, G^{N-\mu}v) = \theta(G^N u, G^N v) = \theta(G^{N+\mu}u, G^{N+\mu}v)$$

$$\theta(G^{-N-\mu}u, G^{-N-\mu}v) = \theta(G^{-N}u, G^{-N}v) = \theta(G^{-N+\mu}u, G^{-N+\mu}v)$$

Because of Axioms A1, A2 the number N has the property needed in the lemma. $\diamondsuit$

Now we state three propositions the proofs of which will be given later. The first one is a discrete version of Lemma 2.1 in [FM-P]. We give here a modification of that proof to our setting. In the sequel we will use a notion of the α-limit set for a point in a ω-limit set. It is defined as the ω-limit set of the same point for the inverse of the injective function G. By $O(x)$ we denote an orbit of a point x lying in some ω-limit set, i.e. the set $O(x) = \{G^i x, i \in \mathbb{N}\}$. The semiorbit of z, i.e. the set $\{G^i z : i \in \mathbb{N}\}$, is denoted by $O^+(z)$.

Proposition 3.1. *Let $w \in \omega(z)$. Then*

$$\theta(u,v) = \theta(Gu,Gv) \tag{3.1}$$

for all $(u,v) \in cl(O(w)) \times cl(O(w)) \setminus D$.

Proposition 3.2. *Let $z \notin \omega(z)$, $(u,v) \in \Omega(z)$. If $u \notin \alpha(u)$ or $v \notin \alpha(v)$, then*

$$\theta(u,v) = \theta(Gu,Gv)$$

Proposition 3.3. *Let $z \notin \omega(z)$, $(u,\, v) \in \Omega(z)$. If $u \in \alpha(u)$ and $v \in \alpha(v)$ then*

$$\theta(u,v) = \theta(Gu,Gv).$$

It is easy to see that Theorem A follows from these propositions. Indeed, if $z \in \omega(z)$ then the theorem follows from Proposition 3.1 with $w = z$. If $z \notin \omega(z)$, Propositions 3.2 and 3.3 give $\theta(u,v) = \theta(Gu,Gv)$ for all (u,v) in $\Omega(z)$ and thus establish Theorem A in this case.

Proof of Proposition 3.1: First we show (3.1) for (u, v) coming from iterates of w.

Take any two natural numbers $m,\ n$ such that $G^m w \neq G^n w$. Since $w \in \omega(z)$ there is an increasing sequence of natural numbers $n_k,\ k = 1,\ 2 \ldots$ such that $G^{n_k} z \to w$ as $k \to +\infty$. Using Axioms A1, A3 and the continuity of G we obtain

$$\theta(G^m w, G^n w) \ \geq\ \theta(G^{m+\mu} w, G^{n+\mu} w) \ \geq\ \liminf_{k\to+\infty} \theta(G^{m+n_k-\mu} z, G^{n+n_k-\mu} z) \ \geq\ \theta(G^m w, G^n w).$$

Hence (3.1) follows for $(G^m w, G^n w)$.

Now take any $(u, v) \in cl(O(w)) \times cl(O(w)) \setminus D$. Suppose on the contrary that (3.1) is false for this (u, v). We have $G^{n_k} w \to u$ and $G^{m_k} w \to v$ as $k \to +\infty$ for some sequences of integer numbers $n_k, m_k, k = 1,\ 2\ , \ldots$. Moreover we may suppose that $n_k \neq m_k$ for all $k \in \mathbb{N}$. Note that θ is constant on some neighborhood of $(G^{\eta(u,v)} u, G^{\eta(u,v)} v)$. Denote this constant θ_0. Hence using subsequently Axioms A1, validity (3.1) for iterates of w and Axiom A3 we obtain

$$\theta(G^{-mu} u, G^{-\mu} v) \geq \theta(u, v) \ >\ \theta(Gu, Gv) \ \geq\ \theta_0 \ \geq\ \lim_{k\to+\infty} \theta(G^{n_k+\eta(u,v)} w, G^{m_k+\eta(u,v)} w)$$

$$= \lim_{k\to+\infty} \theta(G^{n_k-2\mu} w, G^{m_k-2\mu} w) \ \geq\ \theta(G^{-\mu} u, G^{-\mu} v)$$

.

Thus (3.1) is established in all needed cases.$\diamondsuit$

Proof of Proposition 3.2: Take any $(u', v') \in \Omega(z)$ satisfying the assumptions of the proposition. We shall suppose that $u' \notin \alpha(u')$, the other case, $v' \notin \alpha(v')$, is analogous. We proceed by contradiction. Thus, let $(u', v') \in \Omega(z)$, $u' \notin \alpha(u')$ and $\theta(u', v') > \theta(Gu', Gv')$. By Lemma 3.1 there are negative iterates of $u',\ v'$, denoted by $u,\ v$, respectively, and $m \in \mathbb{N}$ such that

$$\theta(u, v) > \theta(G^m u, G^m v) \tag{3.2}$$

and θ is continuous at the points $(u, v),\ (G^m u, G^m v)$. Since u lies on the same orbit as u' we have $u \notin \alpha(u)$.

Because $\alpha(u) \subset cl(O(u))$ we can apply Proposition 3.1 for $w = u$ which, in conjunction with (A2) yields the continuity of θ at each point of the compact set $\alpha(u) \times \{u\}$. Moreover, θ is continuous at (u, v). Therefore there exist open sets U, V, U_1, ..., U_k such that

$$u \in U,\ v \in V,\ \alpha(u) \subset \textstyle\bigcup_{i=1}^{k} U_i \text{ and } \theta \text{ is constant on each } U_i \times U,\ i = 1, \ldots, k$$

and also on $U \times V$. Since the function $(\overline{u}, \overline{v}) \to \theta(G^m \overline{u}, G^m \overline{v})$ is continuous at (u, v) we take U and V above smaller, if necessary, such that

$$\{\theta(G^m u, G^m v)\} \ =\ \theta(G^m(U) \times G^m(V)) \tag{3.3}$$

.

Note that $U \cap O^+(z)$ is nonempty because $u \in \omega(z)$. So choose any $x \in U \cap O^+(z)$. Since all large negative iterates of u lie in the open neighborhood $\bigcup_{i=1}^{k} U_i$ of $\alpha(u)$ and $v \in \omega(x) = \omega(z)$ there is an $n \geq \mu$ such that $G^n x \in V$ and $G^{-n} u \in U_j$ for a $1 \leq j \leq k$.

At this point we are ready to derive a contradiction. Since θ is constant on the sets $U_j \times U$, $U \times V$ and $G^{-n}u \in U_j, x \in U,\ v, G^n x \in V$ we have

$$\theta(G^{-n}u, u) = \theta(G^{-n}u, x) \quad \text{and} \quad \theta(u, G^n x) = \theta(u, v).$$

Hence, by the monotonocity property of θ we obtain the following inequality

$$\theta(G^{-n}u, u) \geq \theta(u, v). \tag{3.4}$$

Further, since $(G^m x, G^{m+n} x) \in (G^m(U) \times G^m(V))$ because $(x, G^n x) \in U \times V$, the fact that θ is constant on $U \times V$ and (3.3) give

$$\theta(G^m u, G^m v) = \theta(G^m x, G^{m+n} x)$$

Hence using (3.2) we obtain

$$\theta(u, v) > \theta(G^m x, G^{m+n} x). \tag{3.5}$$

Now, the fact that θ is locally constant near each point of $O(u) \times O(u) \setminus \mathcal{X}^{(2)}$ together with $u \in \omega(x)$ implies the existence of an arbitrarily large $m_1 \in \mathbb{N}$ such that the following holds

$$\theta(G^{m_1} x, G^{m_1+n} x) = \theta(u, G^n u).$$

Therefore, choosing such m_1 greater than $m + \mu$, Axiom A1 together with (3.5) implies that

$$\theta(u, v) > \theta(u, G^n u).$$

From this inequality and (3.4) we obtain

$$\theta(G^{-n}u, u) > \theta(u, G^n u).$$

This is a contradiction with the conclusion of Proposition 3.1 for $w = u$.

Proof of Proposition 3.3: Suppose on the contrary that there is a $(u_0, v_0) \in \Omega(z)$ such that $u_0 \in \alpha(u_0),\ v_0 \in \alpha(v_0)$ and

$$\theta(u_0, v_0) > \theta(Gu_0, Gv_0).$$

If $u_0 \in \alpha(v_0)$ or $v_0 \in \alpha(u_0)$ then the points lie in the closure of the orbit of one of them. But in this case the last inequality contradicts the conclusion of Proposition 3.1. Thus we may assume that $u_0,\ v_0$ are such that $u_0 \notin \alpha(v_0),\ v_0 \notin \alpha(u_0)$. By

Lemma 3.2 we find a $(u_1, v_1) \in \Omega(z)$ and natural numbers m, r with the following properties

$$G^m u_1 = u_0, \quad G^m v_1 = v_0 \tag{3.6}$$

$$\theta(u_1, v_1) \; > \; \theta(G^r u_1, G^r v_1) \tag{3.7}$$

and the function θ is continuous at its arguments in the last inequality. Since u_0, v_0 lie in their α-limit sets not containing the other point of these two, the same holds for u_1, v_1 because of (3.6).

Now we are going to use the fact that for any point with compact α-limit sets, including this point, there is a point w_1 arbitrarily close to w such that $\omega(w_1) = \alpha(w)$ (see Corollary 11.5 in [Ma]). Thus, we can change the points u_1, v_1 by the other points u, v such that all what we have said about the relation of the former points and their α-limit sets is satisfied for the later points and their ω-limit sets. Moreover u, v can be chosen arbitrarily close to the respective points u_1 and v_1, hence by the continuity of θ in the arguments in (3.7) the inequality (3.7) holds for (u, v) as well. Therefore it is sufficient to show the impossibility of the following assertion (S):

There exists $(u, v) \in \Omega(z)$ and $r \in \mathbb{N}$ such that $u \in \omega(u)$, $v \in \omega(v)$, $u \notin \omega(v)$, $v \notin \omega(u)$, the function θ is continuous at the point (u, v) and

$$\theta(u, v) \; > \; \theta(G^r u, G^r v). \tag{3.8}$$

The contradiction is reached as follows. First we construct two sequences $\{u_1, u_2, ..., u_k\}$ in $\omega(u)$ and $\{v_1, v_2, ..., v_k\}$ in $\omega(v)$ such that

$$\theta(u_i, v_i) \; \leq \; \theta(G^r u, G^r v) \tag{3.9}$$

for $1 \leq i \leq k$ and

$$\theta(u, v) \; = \; \theta(u_k, v_k) \tag{3.10}$$

The number k above will be equal to a number $k_1 + k_2 + l_1 + l_2 + 1$ defined later (using Lemma 3.4 below). Hence combining (3.10) with the inequality (3.9) for $i = k$ we obtain that $\theta(u, v) \leq \theta(G^r u, G^r v)$ contradicting (3.8).

For the construction of the above sequence we need two observations formulated in two lemmas below with the proofs postponed to the end of this section.

Lemma 3.3. *There exist open neighborhoods U and V of the points u and v, respectively, and a natural number N such that for any $(x, y) \in U \times \omega(v)$ and $(x', y') \in \omega(u) \times V$ it holds*

$$\theta(x, y) \; \geq \; \theta(G^m u, G^m y) \tag{3.11}$$

$$\theta(x', y') \; \geq \; \theta(G^m x', G^m v) \tag{3.12}$$

for all $m \geq N$.
Moreover the function θ is constant on $U \times V$.

Lemma 3.4. *Let $\mathcal{M}$, $\mathcal{N}$ be infinite subsets of $\mathbb{N}$. Then there are natural numbers k_1, k_2, l_1, l_2 and $m_1, m_2, m_3 \in \mathcal{M}, n_1, n_2, n_3 \in \mathcal{N}$ such that*

$$n_3 \;>\; k_1 m_1 + k_2 m_2 + m_1$$

and

$$m_3 \;=\; l_1 n_1 + l_2 n_2 + n_3 - k_1 m_1 + k_2 m_2.$$

Denote

$$\mathcal{M} \;=\; \{m \in \mathbb{N} : m \geq max(N, r+\mu) \;\text{ and } G^m u \,\in\, U\}$$

$$\mathcal{N} \;=\; \{n \in \mathbb{N} : n \geq max(N, r+\mu) \;\text{ and } G^n u \,\in\, V\}$$

Note that $\mathcal{M}$ and $\mathcal{N}$ are infinite sets because u and v are contained in their respective ω-limit sets. Thus $\mathcal{M}$ and $\mathcal{N}$ satisfy the assumptions of Lemma 3.4 . Thus we have natural numbers k_1, k_2, l_1, l_2 and $m_1, m_2, m_3,\; n_1, n_2, n_3$ as in the lemma. Define the announced sequences as follows

$u_i \;=\; G^{m_1} u,\; v_i \;=\; G^{i m_1} v$ for $1 \;\leq\; i \;\leq\; k_1$

$u_{k_1+j} \;=\; G^{m_2} u,\; v_{k_1+j} \;=\; G^{j m_2 + k_1 m_1} v$ for $1 \;\leq\; j \;\leq\; k_2$

$u_{k_1+k_2+1} \;=\; G^{n_3 - k_1 m_1 - k_2 m_2} u,\; v_{k_1+k_2+1} \;=\; G^{n_3} v$

$u_{k_1+k_2+i+1} \;=\; G^{i n_1 + n_3 - k_1 m_1 - k_2 m_2} u,\; v_{k_1+k_2+i+1} \;=\; G^{n_1} v$ for $1 \;\leq\; i \;\leq\; l_1$

$u_{k_1+k_2+l_1+j+1} \;=\; G^{j n_2 + l_1 n_1 + n_3 - k_1 m_1 - k_2 m_2} u,\; v_{k_1+k_2+l_1+j+1} \;=\; G^{n_2} v$ for $1 \;\leq\; j \;\leq\; l_2$

Now it is easy to verify (3.9) using induction and Lemma 3.3, as follows.

Since $m_1 > r + \mu$ the monotonicity property of θ implies (3.9) for $i = 1$. Let (3.9) holds for some $i \in \{1, 2, ..., k_1 + k_2\}$. Note that putting

$$s = 1 \quad \text{if} \quad 1 \leq i < k_1$$

$$s = 2 \quad \text{if} \quad k_1 \leq i < k_1 + k_2$$

we have

$$u_{i+1} \;= G^{m_s} u \quad \text{and} \quad v_{i+1} \;=\; G^{m_s} v_i.$$

Then taking $x = u_i,\; y = v_i$ and $m = m_s$ in (3.11) we obtain the following inequality

$$\theta(u_i, v_i) \;\geq\; \theta(G^{m_s} u, G^{m_s} v_i) \;=\; \theta(u_{i+1}, v_{i+1}).$$

This together with the induction hypothesis gives (3.9) for $i + 1$.

If $i = k_1 + k_2 + 1$ then (3.9) follows similarly as in the preceding induction step but putting in (3.11)

$$x = u_{k_1+k_2},\ y = v_{k_1+k_2} \quad \text{and} \quad m = n_3 - k_1 m_1 - k_2 m_2 \ (> m_1 > N)$$

In order to prove (3.9) for the remaining i we can proceed by the same way as for i less than $k_1 + k_2$. It is sufficient to change the role of u by v as well as the role of (3.11) by (3.12), k_1 and k_2 by l_1 and l_2, m_s by analogously defined n_s.

Since θ is constant on $U \times V$ and

$$u_k = G^{m_3} u \in U \text{ and } v_k = G^{n_2} v \in V,$$

the equality (3.10) also holds. Thus, it remains to prove lemmas 3.3 and 3.4.

Proof of Lemma 3.3: We shall find U, an open neighborhood of u, and $N \in \mathbb{N}$. Analogously we find V and a possibly different $N \in \mathbb{N}$ satisfying (3.12) on $\omega(u) \times V$. But the monotonicity of θ gives that the maximum of these N's satisfies both (3.11) and (3.12). By the continuity of θ at (u, v), we can choose U and V smaller, if necessary, such that, in addition, θ will be constant on $U \times V$.

Take any point $y \in \omega(v)$ and recall that $u \notin \omega(v)$. Denote $n(y) = \eta(u, y)$. Then by the definition of η the function θ is locally constant on some neighborhood of the point $(G^{n(y)}u, G^{n(y)}y)$. Therefore, by the continuity of G, there is an open neighborhood $U_y \times V_y$ of the point (u, y) in $\mathcal{X}^{(2)}$ such that θ is constant on $G^{n(y)}U_y \times G^{n(y)}V_y$. Because of compactness of $\omega(v)$ we can choose a finite subset $\{y_1, ..., y_l\}$ in $\omega(v)$ such that the sets V_{y_i}, $i = 1, ..., l$ cover $\omega(v)$. Denote

$$U := \bigcap_{i=1}^{l} U_{y_i}, \quad N := \max_{1 \le i \le l} n(y_i) + \mu$$

Then, for any $(x, y) \in U \times \omega(v)$ we have $y \in V_{y_i}$, for some $i \in \{1, ..., l\}$. Henceforth Axiom A1 gives (3.11) for all $m \geq N$ by the following sequence of inequalities

$$\theta(x, y) \geq \theta(G^{n(y_i)}x, G^{n(y_i)}y) = \theta(G^{n(y_i)}u, G^{n(y_i)}y) \geq \theta(G^m u, G^m y)$$

The proof of Lemma 3.3 is complete.$\diamondsuit$

Proof of Lemma 3.4: This lemma follows from the Euclides algorithm for integers and infiniteness of the sets $\mathcal{M}$ and $\mathcal{N}$. The details follows.

Denote by $d(a, b)$ the largest common divisor of two natural numbers a and b. Choose m_1 and n_1 to be arbitrary numbers in $\mathcal{M}$ and $\mathcal{N}$, respectively. Then we find m_2 and n_2 such that there are infinite subsets $\mathcal{M}'$ of $\mathcal{M}$ and $\mathcal{N}'$ of $\mathcal{N}$ such that all their elements are divisible by $d(m_1, m_2)$ and by $d(n_1, n_2)$, respectively. Denote

$$n := d(n_1, n_2)$$

$$d := d(n, d(m_1, m_2))$$

Further choose a natural number $\tilde{r}$ which multiplied by d appears infinitely often as a remainder at dividing numbers of $\mathcal{M}'$ by n. By the Euclides algorithm, n can be written as $n = pn_1 + qn_2$ where $p,\ q$ are integers. Using similar expressions for $d(m_1, m_2)$ and d we find that

$$\tag{3.13} \tilde{r} \cdot d \;=\; l'_2 n_2 + l'_1 n_1 - k'_1 m_1 - k'_2 m_2$$

for some quadruplet of integers (k'_1, k'_2, l'_1, l'_2). Denote by K a natural number greater than absolute values of all numbers in this quadruple. Then we choose k_1 and k_2 to be

$$k_1 = Kn_1 + k'_1,\ k_2 = Kn_2 + k'_2$$

which together with

$$\bar{l}_1 = Km_1 + l'_1,\ \bar{l}_2 = Km_2 + l'_2$$

gives a quadruple $(k_1, k_2, \bar{l}_1, \bar{l}_2)$ of the natural numbers satisfying (3.13) with obvious replacements.

Let L be a natural number greater than $\mid p \mid$ and $\mid q \mid$. Then

(3.14) all multiples of n greater than $4Ln_1^2 n_2^2$ can be written as $i_1 n_1 + i_2 n_2$ for some natural numbers i_1, i_2

This can be seen from the following formula involving integers $t > 2Ln_1 n_2$ and $0 \leq s < 2n_1 n_2$

$$(t \cdot 2n_1 n_2 + s)n \;=\; (tn_2 n + s \cdot p)n_1 + (tn_1 n + s \cdot q)n_2.$$

Now we take any number n_3 from $\mathcal{N}'$ greater than $(k_1 + 1)m_1 + k_2 m_2$. By the choice of $\tilde{r}$ and n_3 we have an arbitrarily large $m \in \mathcal{M}$ such that $m - (\bar{l}_1 n_1 + \bar{l}_2 n_2 + n_3 - k_1 m_1 - k_2 m_2)$ is divisible by n. Therefore, in view of (3.14), the remaining numbers m_3, l_1 and l_2 are easily found. $\diamondsuit$

4. Existence of exponential separation.

This section is divided into three parts . In part *a)* we show that Theorem B follows from Theorem 4.1 below. In part *c*) the steps of the proof of Theorem 4.1 are formulated in five claims. The proofs of this claims use observations included in part *b*). There, we outline the proof of Theorem 4.1 for the finite dimensional X. Then we indicate a problem arising in infinite dimensions for overcoming of which Proposition 4.2 and 4.3 will be useful.

Recall that by S we denote the boundary of the closed unit ball $\mathbb{B}$ in the Banach space X.

a) Reducing the problem. The first purpose of this part is to create a $k-$cone C_1 such that $T_x^2 = T_{Fx} \circ T_x(C_1 \setminus \{0\}) \subset \ C \setminus \{0\} \subset int(C_1)$ for each $x \in K$.
We shall use the continuity of the vector bundle map (F, T), the compactness of K and the following property rewritten from the assumption of preserving C by (F, T):
for every $v \in C \setminus \{0\}$ and $x \in K$ there is a real number $\delta(x, v) > 0$ such that

$$T_x(v + 2\delta(x, v)\mathbb{B}) \subset C \setminus \{0\},$$

Let L_0 be the $k-$codimensional linear subspace of X from the definition of the $k-$cone C. We show in a moment that for any $v \in C \setminus \{0\}$ there is a $\delta(v) > 0$ such that $T_x^2(v + \delta(v)\mathbb{B}) \subset C \setminus \{0\}$ for any $x \in K$. Then we find a desired $k-$cone C_1 as follows.

Consider the set

$$\bigcup_{v \in S} ((v + \delta(v)\mathbb{B}) \cap S) \setminus L_0.$$

This is an open neighborhood of $C \cap S$ in the metric space S with the induced topology from X. Therefore there is a closed set $\mathcal{C}_1 \subset S$ such that $C \cap S \subset int_S(\mathcal{C}_1)$ $\subset \mathcal{C}_1 \subset S \setminus L_0$. Hence it is easy to see that the set

$$C_1 := \bigcup_{\lambda \in \mathbb{R}} \lambda \mathcal{C}_1$$

is the desired $k-$cone.

In order to find $\delta(v)$ for a fixed $v \in C \setminus \{0\}$ and any $x \in K$ consider the set

$$T_x v + \delta(Fx, T_x v)\mathbb{B}$$

which is mapped by T_{Fx} to $C \setminus \{0\}$. The union of these sets over $x \in K$ is an open neighborhood of the compact set $\{T_x v : \ x \in K\}$. Thus the continuous dependence of $T_x \in L(X, X)$ on $x \in K$ gives the existence of a $\delta(v) > 0$ independent on $x \in K$ such that

$$\bigcup_{x \in K} T_x(v + \delta(v)\mathbb{B}) \subset \bigcup_{x \in K} (T_x v + \delta(Fx, T_x v)\mathbb{B})$$

Now it is easy to see that $\delta(v)$ has the properties required above.

Since $C \setminus \{0\} \subset int(C_1)$, $T_x^2(C_1 \setminus \{0\}) \subset int(C_1)$ and the $k-$cone C_1 contains a $k-$dimensional subspace without zero in its interior. So for the vector bundle map (F^2, T^2) we can apply the next theorem the proof of which is postponed to part c) of this section.

Theorem 4.1. *Let (F,T) be a vector bundle map on $K \times X$ and C_1 a $k-$cone in X containing a $k-$dimensional linear subspace in $int(C_1) \cup \{0\}$. Suppose that T_x is compact and $T_x(C_1 \setminus \{0\}) \subset int(C_1)$ for all $x \in K$. Then (F,T) admits a $k-$dimensional continuous separation on K associated with the $k-$cone C_1.*

So we have a $k-$dimensional continuous separation for (F^2, T^2) along K associated to the $k-$cone C_1 above. Using Lemma 4.1 below we next obtain a $k-$dimensional continuous separation for (F,T) along K associated to the same cone C_1. But since $T_x(C_1) \subset C$ for all $x \in K$ the invariance of the vector bundles in this separation gives that the same separation is associated also to the $k-$cone C, i.e. a separation required in the statement of Theorem B. Therefore in order to finish our reduction of the proof of Theorem B to the proof of Theorem 4.1 we just need to show the following lemma.

Lemma 4.1. *Let (F,T) be a vector bundle on $K \times X$ and C, C_1 k-cones such that $T_x^m(C_1 \setminus \{0\}) \subset C \setminus \{0\}$ for an $m \in \mathbb{N}$ and $T_x(C) \subset C \subset int(C_1) \cup \{0\}$ for every $x \in K$. Assume that the bundle map (F^m, T^m) admits a $k-$dimensional continuous separation along K associated with the k-cone C_1. Then there is a $k-$ dimensional continuous separation for (F,T) along K with the same invariant vector bundles as for the $k-$dimensional continuous separation for (F^m, T^m) along K.*

Proof. Denote by $K \times (V_x)$ and $K \times (L_x)$ invariant $k-$dimensional subbundles of $K \times X$ and $K \times X^*$, respectively, in the separation for (F^m, T^m). First we show the property i) from the definition of the separation for (F,T) by using the exponential separation for (F^m, T^m) with the corresponding constants denoted by M' and γ'.

The invariance of the bundle $K \times (V_x)$ under the bundle map (F^m, T^m) and the finite dimensionality of each V_x, $x \in K$ imply that every $T_x(V_x)$ is an isomorphic image of V_x under the map T_x. The continuous dependence of V_x, L_x and T_x on $x \in K$ yields that the vector spaces $T_x(V_x)$ and $T^*_{Fx}(L_{Fx})$ continuously depend on $x \in K$. By the assumptions of the lemma we have that $T_x^n u \in C_1 \setminus \{0\}$ for all $u \in C_1 \setminus \{0\}$, $n \geq m$ and $x \in K$. Hence, using positive invariance of the bundles $K \times (V_x)$ and $K \times (Anih(L_x))$ under the bundle map (F^m, T^m) on $K \times X$ we obtain that $T_x(V_x) \cap Anih(L_{Fx}) = \{0\}$ and $Anih(T^*_{Fx}(L_{Fx})) \cap C_1 = \{0\}$. Therefore the continuous dependence of the vector spaces $T_x(V_x)$ and $T_x^*(L_{Fx})$ on $x \in K$ together with the compactness of K give a constant $c > 0$ such that the following property holds:
whenever we have $x \in K$, $0 \neq v \in V_x$ and $y \in K$, $0 \neq w \in Anih(T_y^*(L_{Fy}))$ together with uniquely associated $v_1 \in V_{Fx}$, $w_1 \in Anih(L_{Fx})$, $v_2 \in V_y$, $w_2 \in Anih(L_y)$ such that

$$T_x v = v_1 + w_1 \quad \text{and} \quad w = v_2 + w_2$$

then

$$\frac{| w_1 |}{| v_1 |} < c \quad \text{and} \quad \frac{| w_2 |}{| v_2 |} > c.$$

Note that in the last inequality, on the left hand side, assumes infinite value when $w \in Anih(L_y)$. We now show that in this property one actually has

$$\frac{| w_1 |}{| v_1 |} = 0 \quad \text{and} \quad \frac{| w_2 |}{| v_2 |} = +\infty, \tag{4.1}$$

which means $T_x v \in V_{Fx}$ and $w \in L_y$. Hence $T_x(V_x) = V_{Fx}$ and $T^*_{Fy}(L_{Fy}) = L_y$ because of the arbitrariness of $x,\ y \in K, v \in V_x \setminus \{0\}, w \in Anih(T^*_y(L_{Fy}))$.

For the proof of (4.1) we use arbitrary but fixed $p \in \mathbb{N}$, $x, y \in K$, $0 \neq v \in V_x$, $0 \neq w \in Anih(T^*_y(L_{Fy}))$. Let v_1, w_1, v_2 and w_2 be as above. Then, by the invariance of the bundles $K \times (V_x)$, $K \times (L_x)$, there is an element $v' \in V_{F^{-mp}x}$ with the property

$$T^{mp}_{F^{-mp}x} v' = v$$

and an element $w' \in Anih(T^*_{F^{mp+1}y}(L_{F^{mp+1}y}))$ with the property

$$T^{mp}_y w = w'.$$

Moreover, similarly as for $T_x v$ and w, we have the expressions

$$T_{F^{-mp}x} v' = v_3 + w_3 \quad \text{and} \quad w' = v_4 + w_4$$

with $v_3 \in V_{F^{-mp+1}x}$, $w_3 \in Anih(L_{F^{-mp+1}x})$, $v_4 \in V_{F^{mp}y}$ and $w_4 \in Anih(L_{F^{mp}y})$. Clearly,

$$T^{mp}_{F^{-mp+1}x} v_3 = v_1, \quad T^{mp}_{F^{-mp+1}x} w_3 = w_1, \quad T^{mp}_y v_2 = v_4, \quad T^{mp}_y w_2 = w_4$$

and

$$\frac{| w_3 |}{| v_3 |} < \ c \ < \frac{| w_4 |}{| v_4 |}.$$

Using the exponential separation for (F^m, T^m) we obtain

$$\frac{| w_1 |}{| v_1 |} = \frac{| T^{mp}_{F^{-mp+1}x}(w_3) |}{| T^{mp}_{F^{-mp+1}x}(v_3) |} \leq M'\gamma'^p \frac{| w_3 |}{| v_3 |} \ < \ cM'\gamma'^p$$

and

$$c \ < \ \frac{| w_4 |}{| v_4 |} = \frac{| T^{mp}_y w_2 |}{| T^{mp}_y v_2 |} \ \leq \ M'\gamma'^p \frac{| w_2 |}{| v_2 |}.$$

Thus for any $p \in \mathbb{N}$ it is proven that

$$\frac{| w_1 |}{| v_1 |} \leq cM'\gamma'^p \quad \text{and} \quad c < M'\gamma'^p \frac{| w_2 |}{| v_2 |}$$

which can be satisfied only in the case that (4.1) holds.

It remains to show that (F, T) satisfies the property ii) in Definition 2.2 with the bundles $K \times (V_x)$, $K \times (L_x)$ and some constants $M > 0$ and $0 < \gamma < 1$.

By the continuous dependence of $T_x \in L(X, X)$ on $x \in K$ we can find a constant $c_1 > 0$ such that

$$| T_x^q |_{L(X,X)} < c_1 \ \text{ for all } \ x \in K,\ 0 \le q < m.$$

The equality $T_x^m(V_x) = V_{F^m x}$ for any $x \in K$ implies injectivity of every $T_x^q,\ 0 < q < m$, on the finite dimensional space V_x. Thus for each $x \in K$ we can find a constant $c_2 > 0$ such that

$$| v | \le c_2 \mid T_x^q v \mid \quad \text{for all} \ \ v \in V_x \ 0 \le q < m.$$

Moreover, the continuous dependence of T_x and V_x on $x \in K$ implies that the constant c_2 above can be chosen independent of $x \in K$.

Now using the above estimates, the exponential separation for (F^m, T^m) and the invariance of the bundle $K \times (V_x)$ under (F^m, T^m), we obtain for any $x \in K,\ v \in S \cap V_x,\ w \in S \cap Anih(L_x), p \in \mathbb{N}$ and $0 \le q < m$

$$| T_x^{pm+q} w | \le c_1 \mid T_x^{pm} w | \le c_1 M' \gamma'^p \mid T_x^{mp} v | \le c_1 M' \gamma^{mp} \mid T_x^{mp} v \mid$$

$$\le c_1 M' \gamma^{mp} c_2 \mid T_x^{mp+q} v | \le c_1 c_2 M' \gamma^{-m} \gamma^{mp+q} \mid T_x^{mp+q} v |,$$

where $\gamma = \gamma'^{\frac{1}{m}}$. Thus, since each $n \in \mathbb{N}$ can be written in the form $n = mp + q,\ p \in \mathbb{N}$ and $0 \le q < m$, putting $M = c_1 c_2 M' \gamma^{-m}$, we have

$$| T_x^n w | \le M \gamma^n \mid T_x^n v |,$$

for all x, v and w as above. The needed exponential estimate for (F, T) on K with the bundles $K \times (V_x)$ and $K \times (L_x)$ is thus established. The proof of the lemma is complete.$\diamondsuit$

b) Useful observations. Consider a $k-$cone C_1 from the assumptions of Theorem 4.1. First we will introduce an angle of two vectors in $int(C_1)$. Later we will use the notation $\mathcal{S}(C_1)$ for the set of all compact maps $J \in L(X, X)$ such that $J(C_1 \setminus \{0\}) \subset int(C_1)$ and the notation GC_1^* for the following set

$$\{L \in G(k, X^*) : \ Anih(L) \cap C_1 = \{0\}\}.$$

The set $\mathcal{S}(C_1)$ is also considered as the topological subspace of $L(X, X)$.

Now for any $u,\ v \in int(C_1)$ define

$$\alpha_0(u, v) = inf\{\alpha \ge 0 : \ \beta v - u \in int(C_1) \ \text{ for all } \ \beta \ge \alpha\} \tag{4.2}$$

The correctness of this definition can be seen as follows. The set on the right hand side of (4.2) is bounded below by zero. This set is also nonempty, since $v \in int(C_1)$ implies that for all sufficiently large β the vector $v - \frac{u}{\beta}$ lies in a small neighborhood of v.

The crucial property of this number is that for any map J from $\mathcal{S}(C_1)$ and any $u,\ v \in int(C_1)$ with $\alpha_0(u, v) > 0$, images of $u,\ v$ by J satisfy

$$\alpha_0(J(u), J(v)) < \alpha_0(u, v). \tag{4.3}$$

Indeed, if $\beta v - u \in C_1$ then $J(\beta v - u) \in int(C_1)$ and for $\beta = \alpha_0(u,v)$ it implies that for all β' less than $\alpha_0(u,v)$ and sufficiently close to it we have

$$\beta' J(v) - J(u) \in int(C_1),$$

and thus (4.3) holds.

Further consider fixed u and v in $int(C_1)$. Since for any positive real numbers r, s

$$\beta rv - su \ \in \ C_1 \text{ iff } \beta \frac{r}{s} v - u \in C_1,$$

we have

$$\alpha_0(su, rv) = \frac{r}{s} \alpha_0(u,v).$$

Therefore the number

$$\alpha(u,v) := \alpha_0(u,v) \cdot \alpha_0(v,u)$$

has the following properties of the angle of two nonzero vectors

$$\alpha(u,v) = \alpha(v,u) = \alpha(\frac{u}{\mid u \mid}, \frac{v}{\mid v \mid}).$$

Note that if the vectors u and v generate a plane contained in C_1 then $\alpha(u,v) = 0$. We say that $\alpha(u,v)$ is the angle of the vectors u and v. It is an analog of the Hilbert projective metric on unit vectors in the interior of a positive convex cone. However, while the Hilbert projective metric is a continuous function of those vectors, $\alpha(u,v)$ is only upper semicontinuously dependent on u, v. The part "only" of the last assertion is caused by the nonconvexity of the $k-$cone C_1. The upper semicontinuity of $\alpha(u,v)$ follows from the fact that if vectors $u_i, \ v_i \in int(C_1)$ converge to u and v, respectively, when $i \to +\infty$, and if $\beta v - u \in int(C_1)$ then for all sufficiently large i the vectors $\beta u_i - v_i$ also lie in $int(C_1)$.

Actually we want to use an angle of any two elements in $G(int(C_1))$. Thus take two such elements $E_1, \ E_2$. As their angle we define the number

$$\alpha(E_1, E_2) = sup\{\alpha(u,v) : u \in E_1 \setminus \{0\}, \ v \in E_2 \setminus \{0\}\}$$

Useful properties of this angle are collected in Proposition 4.1.

Proposition 4.1. *Let* $E_1, \ E_2 \in G(int(C_1))$. *Then one has*
i) $\alpha(E_1, E_2) = 0$ *iff* $E_1 = E_2$
ii) If $E_{1i}, \ E_{2i}$ *converge to* $E_1, \ E_2$ *, respectively, when* $i \to +\infty$ *and* $\alpha(E_{1i}, \ E_{2i})$ *is defined for all* i *then*

$$\limsup_{i \to \infty} \alpha(E_{1i}, \ E_{2i}) \leq \alpha(E_1, \ E_2)$$

iii) If $J \in \mathcal{S}(C_1)$ and $\alpha(E_1, E_2) \neq 0$ then

$$\alpha(J(E_1), J(E_2)) < \alpha(E_1, E_2).$$

Proof. Here is an outline of the proof. Statement i) follows from the fact that $k+1-$ dimensional subspaces of X cannot lie in C_1. The parts ii) and iii) are consequences of their mentioned analogues of angles of vectors and one additional observation. Namely, for any E_1, E_2 from the assumptions of the proposition, the sets $E_1 \cap S$ and $E_2 \cap S$ are compact and the function $\alpha(\cdot,\cdot)$ is upper semicontinuous, so $\alpha(\cdot,\cdot)$ reaches its supremum on $(E_1 \cap S) \times (E_2 \cap S)$. It means that there are vectors $u \in E_1$ and $v \in E_2$ such that $\alpha(E_1, E_2) = \alpha(u, v)$. Detailed proofs of i)-iii) are next.

i) Suppose on the contrary that $\alpha(E_1, E_2) = 0$ and $E_1 \neq E_2$. Then there is a vector $e \in E_2 \setminus E_1$ such that $\alpha(e, u) = 0$ for all $u \in E_1$. Since $u \in E_1$ implies $(-u) \in E_1$ we have that $\alpha(e, u) = \alpha(e, -u) = 0$ for all $u \in E_1$. Hence by definition of $\alpha(\cdot,\cdot)$ it holds $\beta_1 e + \beta_2 u \in C_1$ for all $u \in E_1$ and $\beta_1, \beta_2 \in \mathbb{R}$. So a $k+1-$dimensional subspace lies in C_1 and it contradicts the definition of a $k-$cone.

ii) Let $u_i \in E_{1i}$ and $v_i \in E_{2i}$ be such that $\alpha(E_{1i}, E_{2i}) = \alpha(u_i, v_i)$. Then since E_{1i} and E_{2i} converge to E_1 and E_2, respectively , we can choose a subsequence $i_m,\ m = 1, 2, \ldots$ from any subsequence of natural numbers such that u_{i_m}, v_{i_m} converge to some vectors, $u \in E_1$ and $v \in E_2$, respectively. Choosing an appropriate subsequence of $i_m,\ m = 1, 2, \ldots$, by upper semicontinuity of $\alpha(\cdot,\cdot)$ we obtain that

$$\limsup_{m\to\infty} \alpha(E_{1i}, E_{2i}) = \lim_{m\to\infty} \alpha(E_{1i_m}, E_{2i_m}) = \lim_{m\to\infty} \alpha(u_{i_m}, v_{i_m}) \leq \alpha(u, v) \leq \alpha(E_1, E_2).$$

Hence ii) is established.

iii) Since J is injective on E_1 and E_2, it is an isomorphism on its images of E_1 and E_2, respectively . Therefore if $u \in J(E_1)$ and $v \in J(E_2)$ are such that $\alpha(J(E_1), J(E_2)) = \alpha(u, v)$ then by (4.3) we obtain

$$\alpha(J(E_1), J(E_2)) = \alpha(u, v) < \alpha(J^{-1}(u), J^{-1}(v)) = \alpha(E_1, E_2).$$

So the proof of the proposition is finished.$\diamondsuit$

Now we give an outline of the proof of Theorem 4.1 for finite dimensional X. First, we take for each $x \in K$ the set GC_1 at the point $F^{-n}x$ and then we squeeze it by taking $GT^n_{F^{-n}x}(GC_1)$. Denote this set by $\mathcal{V}^n_x$. Since the closure of the union of all $GT_x(GC_1),\ x \in K$ is a compact set in $int(GC_1)$, we have the limit $\mathcal{V}^0_x$ of $\mathcal{V}^n_x$ as $n \to +\infty$ in the sense of ω-limit sets. This has to be V_x because of the strong monotonicity of $\alpha(\cdot,\cdot)$. To prove it , not knowing at this time that the bundle $K \times (\mathcal{V}^0_x)$ is continuous, we take its continuous hull. This is the bundle $K \times (\mathcal{V}_x)$ where $\mathcal{V}_x$ is the limit of $\mathcal{V}_y$ as $y \to x$ in the sense of ω-limit sets. This bundle is invariant under (F, GT). Further we consider $\alpha(x)$-the maximal angle of the points in $\mathcal{V}_x$. The function $\alpha : K \to [0, +\infty)$ is uppersemicontinuous by the continuity of

$K \times (\mathcal{V}_x)$ and the uppersemicontinuity of $\alpha(\cdot,\cdot)$ (cf. Prop. 4.1ii)). Since K is also compact, the function α is bounded and takes its maximum on K. By the invariance of $K \times (\mathcal{V}_x)$ under (F, GT) and the strong monotonicity of $\alpha(\cdot,\cdot)$ (cf. Prop.4.1iii)) this maximum is 0. Thus $\alpha(x) = 0$ for all $x \in K$ implying by Proposition 4.1i) that each $\mathcal{V}_x$ is a k-dimensional subspace of X. Since $K \times (\mathcal{V}_x)$ is also continuous and invariant under (F, GT) we have obtained the desired $k-$dimensional invariant bundle $K \times (V_x)$.

Here, in the finite dimensional case, GC_1^* and (F^{-1}, GT^*) have the same properties as GC_1 and (F, GT), respectively. Therefore, analogously as above we obtain a dual bundle $K \times (L_x)$.

In order to prove the exponential separation consider for each $x \in K$ the set

$$C_x(\delta) = \{u \in X : \ u = v + w \text{ where } v \in V_x, \ w \in Anih(L_x) \text{ and } \mid w \mid \leq \delta \mid v \mid\}.$$

If $\delta > 0$ is sufficiently small then $K \times (C_x(\delta))$ is a subbundle of the bundle $K \times (int(GC_1)$. Hence, due to the construction of $K \times (\mathcal{V}_x)$, which is in fact $K \times (V_x)$, we obtain that all normalized sequences $T_x u, \ x \in K, \ u \in C_x(\delta)$ has the limit in the union of all $V_x, \ x \in K$. This implies that there is a $0 < \gamma < 1$ and $n \in \mathbb{N}$ such that

$$\frac{\mid T_x^n w \mid}{\mid T_x^n v \mid} \leq \gamma \frac{\mid w \mid}{\mid v \mid}, \quad x \in K, \ u = v + w \in C_x(\delta).$$

From this using Lemma 4.1 we obtain the exponential separation completing the outline of the proof of Theorem 4.1 with $dim(X) < \infty$.

If X has infinite dimensions it may happen that the sets above depending on $n \in \mathbb{N}$ are always noncompact even in the limit. So the crucial part of the proof of Theorem 4.1 (see Claims 1 and 2) creates at any point $x \in K$ a cone $C_x \subset int(C_1)$ such that the closure of the set $GT_x(GC_x)$ is already compact in $G(k, X)$.

In order to obtain such cones C_x, we need a property, stated in Proposition 4.2, of a compact subset of the set of compact operators in $\mathcal{S}(C_1)$. For example, for just one operator $J \in \mathcal{S}(C_1)$, this property says, in a different way, that the set $GJ^*(GC_1^*)$ is a precompact subset of GC_1^*.

Before we give a general statement we introduce some notation. First consider for any $(l_1, \ldots, l_k) \in (X^*)^k, \ (v_1, \ldots, v_k) \in X^k$ the $k \times k$ matrix with elements

$$\iota(l_1, \ldots, l_k, v_1, \ldots, v_k)_{i,j} = l_i(v_j)$$

for $i, j = 1, \ldots, k$.

The matrix $\iota(l_1, \ldots, l_k, v_1, \ldots, v_k)$ is regular iff functionals $l_1, \ldots, l_k$ as well as vectors $v_1, \ldots, v_k$ are linearly independent and $Anih([l_1, \ldots, l_k]) \cap [v_1, \ldots, v_k] = \{0\}$
.

By the assumption on the cone C_1, we can find k linearly independent unit vectors $e_1, \ldots, e_k$ generating a $k-$dimensional linear subspace V_0 contained in $int(C_1) \cup \{0\}$. Fix such an $k-$tuple $e_1, \ldots, e_k$. Take for any given $L \in GC_1^*$ an element of $(X^*)^k$ denoted by $(l_1', \ldots, l_k')$ such that $[l_1', \ldots, l_k'] = L$. Since $V_0 \subset int(C_1) \cup \{0\}$ the matrix $\iota(l_1', \ldots, l_k', e_1, \ldots, e_k)$ is regular. The representant of any given $L \in GC_1^*$ is chosen to be $(l_1, \ldots, l_k) \in (X^*)^k$ such that

$$(l_1,\dots,l_k) = (\iota(l'_1,\dots,l'_k,e_1,\dots,e_k))^{-1}(l'_1,\dots,l'_k);$$

thus the matrix

$$\iota(l_1,\dots,l_k,e_1,\dots,e_k) = Id,$$

where Id is the $k \times k$ identity matrix. We denote $(l_1,\dots,l_k)$ also by the letter L.

In this representation GC_1^* corresponds to the following set

$$\mathcal{L} := \{(l_1,\dots,l_k) \in (X^*)^k : \ Anih([l_1,\dots,l_k]) \cap int(C_1) = \emptyset \text{ and } \iota(l_1,\dots,l_k,e_1,\dots,e_k) = Id\}$$

Next we define a representation of GJ^* for $J \in \mathcal{S}(C_1)$. Note that these maps J are injective on $V_0 = [e_1,\dots,e_k] \subset C_1$ and hence $Je_1,\dots,Je_k$ are linearly independent vectors in $int(C_1)$. So for any $(l_1,\dots,l_k) \in \mathcal{L}$ the matrix

$$A_J(L) := \iota(l_1,\dots,l_k,Je_1,\dots,Je_k)$$

is regular. Thus GJ^* for $J \in \mathcal{S}(C_1)$ corresponds to the map $\mathcal{J}^* : \mathcal{L} \to \mathcal{L}$ defined by

$$\mathcal{J}^*(L) = (A_J(L))^{-1}(J^*l_1,\dots,J^*l_k)$$

for any $L \in \mathcal{L}$ and $J \in \mathcal{S}(C_1)$.

Proposition 4.2. *Let $\mathbb{J}$ be any compact subset of maps in $\mathcal{S}(C_1)$. Then the following set*

$$\{\mathcal{J}^*(L) : \ L \in \mathcal{L} \text{ and } J \in \mathcal{S}(C_1)\}$$

is precompact in $(X^*)^k$.

For the proof of Proposition 4.2 we need the following lemma.

Lemma 4.2. *The set $\mathcal{L}$ is bounded and closed in $(X^*)^k$. Moreover, each sequence in $\mathcal{L}$ has a subsequence $(l_1^p,\dots,l_k^p) \in \mathcal{L}$, $p = 1,2,\dots$ such that for each $i = 1,\dots,k$ the sequence l_i^p, $p = 1,2,\dots$ has a weak star limit l_i and $(l_1,\dots,l_k)$ lies in $\mathcal{L}$.*

Proof of Lemma 4.2. In order to prove the boundedness of $\mathcal{L}$, realize that there is a $\delta > 0$, such that for any

$$(e'_1,\dots,e'_k) \ \in \ (e_1+\delta\mathbb{B}) \times \dots \times (e_k+\delta\mathbb{B})$$

$e'_1,\dots,e'_k$ are linearly independent vectors generating a subspace in $int(C_1) \cup \{0\}$. We claim that

$$l_i(e_i - v) > 0 \text{ for all } (l_1,\dots,l_k) \in \mathcal{L},\ v \in \delta\mathbb{B} \text{ and } \ i = 1,\dots,k. \tag{4.4}$$

Indeed, in the opposite case there are $i \in \{1,\ldots,k\}, (l_1,\ldots l_k) \in \mathcal{L}$ and a $v \in \delta\mathbb{B}$ such that $l_i(e_i - v) = 0$. Moreover, we already know that $l_i(e_j) = 0$ for all $j \in \{1,\ldots k\} \setminus \{i\}$. Therefore for linearly independent vectors $e_1,\ldots,e_{i-1}, e_i - v, e_{i+1},\ldots,e_k$ generating a subspace E' contained in $int(C_1) \cup \{0\}$ the matrix

$$\iota(l_1,\ldots,l_k,e_1,\ldots,e_{i-1},e_i - v,e_{i+1},\ldots,e_k)$$

is nonregular. But this contradicts the fact that $Anih([l_1,\ldots l_k]) \cap E' = \{0\}$ and $l_1,\ldots,l_k$ are linearly independent functionals.

Now using (4.4) we obtain that $1 = l_i(e_i) > l_i(v)$ for all $v \in \delta B,\ (l_1,\ldots,l_k) \in \mathcal{L}$ and $i \in \{1,\ldots,k\}$. This implies that

$$\text{if } (l_1,\ldots,l_k) \in \mathcal{L} \text{ then } \mid l_i \mid \leq \frac{1}{\delta} \text{ for all } i \in \{1,\ldots,k\}.$$

Thus the boundedness of $\mathcal{L}$ is established.

The closedness of $\mathcal{L}$ in $(X^*)^k$ follows from the sequential weak star compactness of $\mathcal{L}$ which we are going to show next.

Take any sequence $(l_1^{p'},\ldots,l_k^{p'}) \in \mathcal{L},\ p' = 1,2,\ldots$. Since $\mathcal{L}$ is bounded in $(X^*)^k$ the Banach Alaouglu Theorem gives a subsequence of the previous sequence, denoted by $(l_1^p,\ldots,l_k^p),\ p = 1,2,\ldots$, such that for each $i \in \{1,\ldots,k\}$ the sequence l_i^p has a weak star limit $l_i \in X^*$. Hence, since $\iota(l_1^p,\ldots,l_k^p,e_1,\ldots,e_k) = Id$ for all p, we have

$$\iota(l_1,\ldots,l_k,e_1,\ldots,e_k) = Id \tag{4.5}$$

Now we claim that

$$Anih([l_1,\ldots,l_k]) \cap int(C_1) = \emptyset \tag{4.6}$$

.

Suppose on the contrary that there is a $v \in Anih([l_1,\ldots,l_k]) \cap int(C_1)$. Thus if we denote $c_i^p := l_i^p(v)$ for any $p \in \mathbb{N},\ i \in \{1,\ldots,k\}$ then $c_i^p \to l_i'(v) = 0$ as $p \to \infty$ for all $i = 1,\ldots,k$. Since $v \in int(C_1)$, for all sufficiently large p we also have

$$v - c_1^p e_1 - \ldots c_k^p e_k \in int(C_1).$$

But this vector lies in $Anih([l_1^p,\ldots,l_k^p])$, too, contradicting $(l_1^p,\ldots,l_k^p) \in \mathcal{L},\ p = 1,2,\ldots$.

By (4.5) and (4.6) we obtain $(l_1,\ldots,l_k) \in \mathcal{L}$. This completes the proof of the lemma.$\diamondsuit$

Proof of Proposition 4.2.: In order to prove this proposition it is sufficient to show that for any convergent subsequence of the sequence $J_m,\ m = 1,2,\ldots$ from the compact set $\mathbb{J} \in L(X,X)$ with a limit J and any $L_m \in \mathcal{L},\ m = 1,2,\ldots$ there is a subsequence of the sequence $\mathcal{J}_m^*(L_m)$ converging to an element in $\mathcal{L}$.

We have that $J_m^*,\ m = 1,2,\ldots$ is a sequence of compact operators with the limit J^*, also a compact operator, and $\mathcal{L}$ is bounded in $(X^*)^k$ by Lemma 4.2. Therefore we may suppose, without loss of generality, that if $L_m = (l_1^m, \dots, l_k^m)$ then for each $i \in \{1, \dots, k\}$ the sequence $\mathcal{J}_m^*(l_i^m)$ converges to an $l_i' \in X^*$. The above lemma allows us to assume that the sequence $(l_1^m, \dots, l_k^m)$ has for all its component $i = 1, \dots, k$ a weak star limit $(l_1, \dots, l_k)$ creating $L := (l_1, \dots, l_k) \in \mathcal{L}$. Hence for any $v \in X$

$$l_i^m(Jv) \to l_i(Jv) \text{ as } m \to \infty, \ 1 \le i \le k.$$

Moreover, the convergence of J_m to J with the boundedness of $\mathcal{L}$ implies that

$$l_i^m(J_m v - Jv) \to 0 \text{ as } m \to \infty,$$

so $l_i^m(J_m v) \to l_i(Jv)$ for any $v \in X, \ i = 1, \dots, k$. This means that

$$(J_m^* l_1^m, \dots, J_m^* l_k^m) \to (l_1', \dots, l_k') = (J^* l_1, \dots, J^* l_k)$$

as well as

$$A_{J_m}(L_m) \to A_J(L)$$

as $m \to \infty$. So $\mathcal{J}_m^*(L_m)$ converges to $\mathcal{J}^*(L)$ in $(X^*)^k$. Since $\mathcal{L}$ is closed in $(X^*)^k$ the proposition is proved. $\diamondsuit$

The next proposition will be used twice in the proof of Theorem 1 (see Claims 1 and 3). For its formulation we need several definitions of the properties of a bundle $K \times (Y_x)$ and the bundle map $(\tilde{F}, \{R_x, \ x \in K\})$ on it. Here $\tilde{F}$ is a homeomorphism of a compact metric space K, each $Y_x, \ x \in K$, is a subset of a metric space Y and $R_x : Y_x \mapsto Y_{\tilde{F}x}$ is a continuous map for any $x \in K$. We say that

i) the bundle $K \times (Y_x)$ is nontrivial iff $Y_x \neq \{0\}$ for all $x \in K$.
ii) the bundle $K \times (Y_x)$ is compact iff from any sequence $y_n \in Y_{x_n}, \ n = 1, 2, \dots$, where $x_n \to x$ as $n \to +\infty$, we can choose convergent subsequence $y_{n_i}, \ i = 1, 2, \dots$ with a limit y in Y_x. Obviously any compact bundle $K \times (Y_x)$ is also continuous.
iii) the bundle map $(\tilde{F}, R)$ on $K \times (Y_x)$ is compact iff

$$cl(\bigcup_{x \in K} R_x(Y_x))$$

is a compact subset of Y.
iv) the bundle map $(\tilde{F}, R)$ on $K \times (Y_x)$ is continuous iff for any sequences $x_i \to x_0$ and $y_i \to y_0$, where $x_i \in K$ and $y_i \in Y_{x_i}$ for all $i \ge 0$, the sequence $R_{x_i} y_i$ has the limit $R_{x_0} y_0$ when $i \to +\infty$.
v) the maximal invariant subbundle $K \times (Z_x)$ for the bundle map $(\tilde{F}, R)$ on $K \times (Y_x)$ is defined by

$$\begin{aligned} Z_x = \{z \in Y_x :\ & \text{there is a sequence } z_i, \ i \in \mathbb{Z}, \\ & \text{such that } R_{\tilde{F}^i x}(z_i) = z_{i+1} \text{ for all } i \in \mathbb{Z}, \text{ and } z_0 = z\} \end{aligned}$$

Proposition 4.3. *Let $\tilde F$ be a homeomorphism of a compact metric space K, Y a metric space, $K\times(Y_x)$ a subbundle of the bundle $K\times Y$ and $(\tilde F,R)=(\tilde F,\{R_x,\ x\in K\})$ a bundle map on $K\times(Y_x)$. Suppose that the bundle $K\times(Y_x)$ is continuous and the bundle map $(\tilde F,R)$ is continuous and compact. Then the following bundle*

$$Z_x := \{y:\ y=\lim_{p\to\infty} R^{m_p-1}_{\tilde F^{-m_p}x_p} y_p$$
$$\text{for some sequences } m_p\to+\infty \text{ and } x_p\to x \text{ as } p\to\infty, \text{ and } y_p\in Y_{\tilde F^{-m_p}x_p}\}$$

is nontrivial, compact and it is equal to the maximal invariant subbundle for $(\tilde F,R)$.

Proof: It is easy to see that the maximal invariant subbundle is contained in the bundle $K\times(Z_x)$. Since any $R^{m_p-1}_{\tilde F^{-m_p}x_p}y_p$ from the definition of Z_x is contained in Y_{x_p} the continuity of the bundle $K\times(Y_x)$ implies that $K\times(Z_x)$ is a subbundle of $K\times(Y_x)$. The nontriviality of $K\times(Z_x)$ is assured by the compactness of the bundle map $(\tilde F,R)$.

Now we show that for any $x\in K$ and $z\in Z_x$ one has $R_x(z)\in Z_x$ and that there exists a $z'\in Z_{\tilde F^{-1}x}$ such that $z=R_{\tilde F^{-1}x}z'$. Thereby we prove the invariance of the bundle $K\times(Z_x)$.

By the definition of Z_x there is a sequence $m_p\in\mathbb N,\ x_p\in K$ and $y_p\in Y_{\tilde F^{-m_p}x_p}$ such that

$$x_p\to x \ \text{ and } R^{m_p-1}_{\tilde F^{-m_p}x_p}y_p\to z \text{ as } p\to+\infty.$$

Therefore, using the continuity of the bundle map $(\tilde F,R)$, we obtain

$$R_xz=R_x(\lim_{p\to\infty}R^{m_p-1}_{\tilde F^{-m_p}x_p}y_p)=\lim_{p\to\infty}R_{x_p}\circ R^{m_p-1}_{\tilde F^{-m_p}x_p}y_p=\lim_{p\to\infty}R^{m_p}_{\tilde F^{-m_p}x_p}y_p.$$

Hence, since $\tilde Fx_p\to\tilde Fx$ as $p\to\infty$, we have $R_xz\in Z_{\tilde Fx}$ by the definition of $Z_{\tilde Fx}$.

Further consider the sequence

$$y'_p=R^{m_p-2}_{\tilde F^{-m_p}x_p}y_p\in Y_{\tilde F^{-1}x_p},\ p=1,2,\ldots.$$

Obviously

$$R_{x_p}y'_p\to z \text{ as } p\to\infty. \tag{4.7}$$

Moreover, each convergent subsequence of y'_p has its limit in $Z_{\tilde F^{-1}x}$. Since y'_p is contained in the precompact set $\bigcup_{x\in K}R_x(Y_x)$ for all sufficiently large p (such that $m_p>2$), it has a convergent subsequence with a limit z'. We already know that $z'\in Z_{\tilde F^{-1}x}$. Combining (4.7) with the continuity of the bundle map $(\tilde F,R)$, we also obtain $R_xz'=z$. Thus z' has properties required above.

In order to prove the compactness of the bundle $K\times(Z_x)$, take any sequence $z_p\in Z_{x_p},\ p=1,2,\ldots$ where $x_p\in K,\ p=1,2,\ldots$ is a sequence with limit $x\in K$. Then we can choose for any p a natural number m_p and points $x'_p\in K$, $y_p\in Y_{\tilde F^{-m_p}x'_p}$ in Y such that

$$\text{the distance of } z_p \text{ from } R^{m_p-1}_{\tilde{F}^{-m_p}x'_p} y_p \text{ in } Y$$

$$\text{and the distance of } x'_p \text{ from } x_p \text{ in } K$$

go both to zero as $p \to \infty$ and moreover $m_p \to +\infty$.

So $x'_p \to x$ as $p \to \infty$ and the compactness of $(\tilde{F}, R)$ gives at least one convergent subsequence of the sequence

$$R^{m_p-1}_{\tilde{F}^{-m_p}x'_p} y_p, \ p = 1, 2, \dots.$$

Then $z_p, \ p = 1, 2, \dots$ has clearly a convergent subsequence as well. The definition of Z_x implies that each such convergent subsequence has its limit in Z_x. Therefore the compactness of the bundle $K \times (Z_x)$ is proved and the proof of the proposition is complete.

c) Proof of Theorem 4.1. In the first claim below we state the existence of a "generalized" invariant bundle $K \times (\mathcal{L}_x)$. We will use the notation $\mathcal{T}^*_x$ for the representation, indicated above, of the map GT^*_x restricted to GC^*_1 where $x \in K$ and $\{T_x, \ x \in K\}$ is the family of operators from the assumptions of Theorem 4.1. The set $\mathcal{L} \subset {X^*}^k$ is considered with the induced topology from ${X^*}^k$.

Claim 1. *Denote by $K \times \{\mathcal{L}_x, \ x \in K\}$ the maximal invariant subbundle for the bundle map $(F^{-1}, \mathcal{T}^*)$ on the bundle $K \times \mathcal{L}$. Then the bundle $K \times (\mathcal{L}_x)$ is nontrivial and compact.*

This claim is an immediate consequence of Proposition 4.3 applied to the bundle map $(F^{-1}, \mathcal{T}^*)$ on the bundle $K \times \mathcal{L}$. Here the assumptions of Proposition 4.3 are trivially satisfied with two exceptions. The first one is the compactness of the bundle map $(F^{-1}, \mathcal{T}^*)$. It follows from Proposition 4.2 applied to the compact family of maps $\{T_x, \ x \in K\}$. The second one is the continuity of the bundle map $(F^{-1}, \mathcal{T}^*)$. It is an easy consequence of the continuity of the functions $(L, J) \in (\mathcal{L}, \mathcal{S}(C_1)) \mapsto A_J(L)$ and $(l, J) \in (X^*, L(X, X)) \mapsto J^*l \in X^*$.

Our next step is to show the existence of a bundle of cones $K \times (C_x)$ invariant under (F, T) and such that the bundle map (F, GT) restricted to $K \times (GC_x)$ is compact. Recall that $V_0 = [e_1, \dots, e_k]$ denotes a $k-$dimensional linear subspace in $int(C_1) \cup \{0\}$. The main ingredients in this step are the obtained bundle $K \times (\mathcal{L}_x)$ above , Proposition 4.2 and a number β, similar to "secans" of an angle between v and V_0 with respect to some subspace of X whose anihilator in X^* is a $L \in \mathcal{L}$. In order to define this number consider any $L \in \mathcal{L}$. For any $v \in X$ let $P_L(v)$ be the projection of the vector v into the $k-$ dimensional space V_0 along the $k-$codimensional space $Anih(L)$. Since for any $(l_1, \dots, l_k) \in \mathcal{L}$ we have the following equality of matrices

$$\iota(l_1, \cdots, l_k, e_1, \cdots, e_k) = Id,$$

we can write for any $v \in X$

$$P_L(v) = \sum_{i=1}^{k} l_i(v) e_i$$

It is easy to see that the map

$$X \times \mathcal{L} \ni (v, L) \mapsto P_L(v) \in X$$

is continuous if we use product topology on $X \times \mathcal{L}$.

Now we define a function, possibly taking the infinite value, on $(C_1 \setminus \{0\}) \times \mathcal{L}$ by

$$\beta(v, L) = \frac{| v |}{| P_L(v) |}.$$

The restriction of β to $int(C_1) \times \mathcal{L}$ obviously takes only finite values and is continuous.

Claim 2. *Let $K \times (\mathcal{L}_x)$ be the bundle map from Claim 1 and $V_0 \in G(int(C_1))$. Then for all sufficiently large $\tau > 0$ the cones*

$$C_x := \{v \in C_1 : \ \beta(v, L) \leq \tau \ for\ all \ \ L \in \mathcal{L}_x\} \cup \{0\}, \ x \in K,$$

create a continuous subbundle $K \times (C_x)$ of the bundle $K \times C_1$ positively invariant under (F, T). Moreover, the bundle map (F, GT) restricted to the bundle $K \times \{GC_x, \ x \in K\}$ is compact.

In order to prove this claim consider the following subset of $S \cap C_1$

$$\mathcal{A} := \{\frac{T_x v}{| T_x v |} : \ x \in K, \ v \in (C_1 \backslash \{0\}) \text{ and } \beta(T_x v, L) \geq \beta(v, \mathcal{T}^*_{Fx}(L)), \text{ for a } L \in \mathcal{L}_{Fx}\}$$

We are going to show that $\mathcal{A}$ is precompact in X.

Fix any $x \in K$, $L \in \mathcal{L}_{Fx}$ and $v \in C_1 \setminus \{0\}$ such that the inequality in the definition of $\mathcal{A}$ is satisfied. Then

$$v = e + w = P_{\mathcal{T}^*_{Fx}(L)}(v) + w,$$

where $e \in V_0$ and $w \in Anih(\mathcal{T}^*_{Fx}(L))$. Hence

$$T_x v = T_x e + T_x w,$$

where $T_x w \in Anih(L)$. Therefore

$$P_L(T_x v) = P_L(T_x e).$$

Thus we can write the inequality in the definition of $\mathcal{A}$ as

$$(4.8) \qquad | T_x v | \geq \frac{| P_L(T_x v) |}{| P_{\mathcal{T}^*_{Fx}(L)}(v) |} | v | = | P_L(T_x \frac{e}{| e |}) || v | .$$

This inequality will be used later on. Now we show that the closure of the set

$$\mathcal{B} = \{P_L(T_x(e)) : \ x \in K, \ e \in V_0 \cap S, \ L \in \bigcup_{x \in K} \mathcal{L}_x\}$$

is bounded away from zero in X. Since $V_0 \cap S$ is compact and the compact operators $T_x,\ x \in K$, create a compact subset in $\mathcal{S}(C_1)$, the set

$$\bigcup_{x \in K} T_x(V_0 \cap S)$$

is a compact subset in $int(C_1)$. Therefore, using the compactness of $\bigcup_{x \in K} \mathcal{L}_x \subset \mathcal{L}$, the continuity of the function $(v, L) \mapsto P_L(v)$, we obtain that $\mathcal{B}$ is a compact subset of $int(C_1) \setminus \{0\}$. So, there is a positive minimum of the norms of vectors in $\mathcal{B}$ denoted by $c > 0$. Then in view of (4.8) we have

$$| T_x v | \geq c | v |$$

for any $x \in K$ and v in the definition of the set $\mathcal{A}$. Hence $\mathcal{A}$ is a subset of the following set

$$\{\frac{T_x v}{| T_x v |} :\ v \in S,\ x \in K \ \text{ and } \quad | T_x v | \geq c | v |\}. \tag{4.9}$$

We claim that this set is precompact.

Indeed, consider any sequence of the unit vectors $v_n,\ n = 1, 2, \dots$ such that $| T_{x_n} v_n | \geq c > 0$ for some sequence of points $x_n, n = 1, 2, \dots$ in K. By compactness of K and continuous dependence of $T_x \in L(X, X)$ on $x \in K$ we may also assume, without loss of generality, that

$$x_n \to x \ \text{ and } \quad | T_x v_n - T_{x_n} v_n | \to 0 \ \text{ as } \ n \to +\infty$$

for some $x \in K$. Therefore, since T_x is a compact operator, there is a subsequence of natural numbers $n_i,\ i = 1, 2, \dots$ such that the following sequence of vectors (with the norms not less than c)

$$T_{x_{n_i}} v_{n_i},\ i = 1, 2, \dots \tag{4.10}$$

has a limit $v \in X$ with the norm greater or equal to $c > 0$. Thus the normalized sequence of the vectors in (4.10) converges to $\frac{v}{|v|}$.

Since $\mathcal{A}$ is a subset of the precompact set in (4.9), $\mathcal{A}$ is also precompact. Next we use this fact to find a number τ_0 such that for all $\tau > \tau_0$ the corresponding sets $C_x,\ x \in K$ have the required properties in the claim.

By definition of $\mathcal{L}_x$ for any $x \in K$ and $L \in \mathcal{L}_x$ there is a L' in $\mathcal{L}_{Fx}$ such that $\mathcal{T}^*_{Fx} L' = L$. Hence if $v \in Anih(L) \cap C_1$ then $T_x v \in Anih(L') \cap (int(C_1) \cup \{0\})$. Thus $v = 0$, implying $Anih(L) \cap S \cap C_1 = \emptyset$. Therefore for any $v \in S \cap C_1$ and $L \in \bigcup_{x \in K} \mathcal{L}_x$ we have $P_L(v) \neq 0$. Thus the function $\beta(\cdot, \cdot)$ is finite and continuous on $(C_1 \setminus \{0\}) \times (\bigcup_{x \in K} \mathcal{L}_x)$. Hence the compactness of $cl(\mathcal{A})(\subset S \cap C_1)$ gives the existence of an upper bound τ_0 of the set

$$\{\beta(v, L) :\ v \in \mathcal{A},\ x \in K,\ L \in \bigcup_{x \in K} \mathcal{L}_x\}.$$

Now take any $\tau > \tau_0$ and let $C_x,\ x \in K$ be the corresponding family of sets from the statement of the claim. Each $C_x,\ x \in K$ is obviously closed because of the continuity of the function $\beta(\cdot,\cdot)$. These sets are $k-$cones since they also contain $V_0 \setminus \{0\}$ in their interior and they are subsets of the $k-$cone C_1 invariant under multiplication by all real numbers. We show the positive invariance of $K \times (C_x)$ under (F,T).

If $v \in C_x \setminus \{0\}$ for a $x \in K$ and $L \in \mathcal{L}_{Fx}$ then we have the alternative

1) $\beta(T_x v, L) < \beta(v, \mathcal{T}^*_{Fx}(L))$
or
2) $\beta(T_x v, L) \geq \beta(v, \mathcal{T}^*_{Fx}(L))$

In the first case, $v \in C_x \setminus \{0\}$ implies $\beta(v, \mathcal{T}^*_{Fx}(L)) \leq \tau$, so $\beta(T_x v, L) < \tau$. The second one tells that

$$\frac{T_x(v)}{\mid T_x v \mid} \in \mathcal{A},$$

thus $\beta(T_x v, L) \leq \tau$. Thus for any $v \in C_x \setminus \{0\}$ and $L \in \mathcal{L}_{Fx}$ we have obtained $\beta(T_x v, L) \leq \tau$, i.e. $T_x v \in C_{Fx}$.

The continuity of the bundle $K \times (C_x)$ follows immediately from the continuity of the extended function $\beta(\cdot,\cdot)$ and of the bundle $K \times (\mathcal{L}_x)$. It remains to be proven that the bundle map (F, GT) restricted to the bundle $K \times (GC_x)$ is compact. For this it is sufficient to show that the closure of the set

$$\mathcal{C} := \{\frac{T_x v}{\mid T_x v \mid} : \ x \in K, \ v \in C_x \setminus \{0\}\}$$

is compact.

If we consider $v \in C_x$ with $\mid P_L(v) \mid = 1$ for some $x \in K$ and $L \in \mathcal{L}_x$ then $\mid v \mid \leq \tau$. Hence, as $\{T_x, \ x \in K\}$ is a compact family of compact operators in $L(X,X)$ and $K \times (C_x)$ is a continuous bundle, the compactness of $\mathcal{C}$ can fail only in the case when there is a sequence $v_n \in C_{x_n}$ with $\mid P_{L_n}(v_n) \mid = 1$, $x_n \in K$ and $L_n \in \mathcal{L}_{x_n}$ such that

$$\mid T_{x_n} v_n \mid \to 0 \ \text{ as } \ n \to +\infty.$$

In this case, since the set $\bigcup_{x \in K} \mathcal{L}_x$ is compact and the function $(v, L) \mapsto P_L(v)$ is continuous, we may suppose that the following sequence

$$P_{L'_n}(T_{x_n} v_n) \quad \text{where } \mathcal{T}^*_{Fx_n}(L'_n) = L_n, \ n = 1, 2, \cdots,$$

approaches 0 as $n \to +\infty$.
All vectors in this sequence belong to the set $\mathcal{B}$. Indeed as it can be easily seen from the considerations just before (4.8)

$$P_{L'_n}(T_{x_n} v_n) = P_{L'_n}(T_{x_n}(P_{L_n} v_n))$$

implying $P_{L'_n}(T_{x_n} v_n) \in \mathcal{B}$. We thus have a contradiction since the closure of $\mathcal{B}$ is separated from zero by the ball of radius $c > 0$. The proof of Claim 2 is finished.

Now take the bundle of cones $K \times (C_x)$ for fixed and sufficiently large τ provided by Claim 2. Hence the bundle $K \times (GC_x)$ is invariant under the bundle map (F, GT) and $\bigcup_{x \in K} GT_x(GC_x)$ is a compact subset of GC_1. Moreover, the continuity of the function $(v, J) \in C_1 \setminus \{0\} \times \mathcal{S}(C_1) \mapsto \frac{Jv}{|Jv|} \in S$ gives that the bundle map (F, GT) is continuous. Therefore applying Proposition 4.3 to the bundle map (F, GT) restricted to the bundle $K \times (GC_x)$ we obtain the first part of the following claim.

Claim 3. *Denote the maximal invariant subbundle for the bundle map (F, GT) restricted to the bundle $K \times (GC_x)$ by $K \times \{\mathcal{V}_x,\ x \in K\}$. Then this bundle is nontrivial and compact. Moreover each $\mathcal{V}_x,\ x \in K$ contains only one element associated with a $k-$dimensional subspace. The element in each $\mathcal{V}_x$ is denoted by V_x.*

The second part of this claim is a consequence of Proposition 4.1. To explain it, define a real function $\alpha : K \to \mathbb{R}$ as follows

$$\alpha(x) = sup\{\alpha(E_1, E_2) : E_1, E_2 \in \mathcal{V}_x\},\ x \in K.$$

The compactness of the bundle $K \times (\mathcal{V}_x)$ together with the uppersemicontinuity of $\alpha(\cdot, \cdot)$ imply the following:
the function α is bounded from above, in the definition of the value of $\alpha(x)$ we can take the maximum and finally the function α is upper semicontinuous on K.
Hence the function α reaches its supremum on the compact set K, say at a point $x_0 \in K$. Then $\alpha(x_0) = 0$.

Indeed, otherwise we have

$$0 < \alpha(x_0) = \alpha(E_1, E_2) \ \text{ for some } \quad E_1 \neq E_2 \in \mathcal{V}_{x_0}.$$

By the invariance of $K \times (\mathcal{V}_x)$ under (F, GT), there are E_1' and E_2' such that

$$GT_{F^{-1}x_0}(E_i') = E_i,\ i = 1, 2.$$

Hence by Proposition 4.1iii)

$$\alpha(F^{-1}x_0) \geq \alpha(E_1', E_2') > \alpha(E_1, E_2) = \alpha(x_0) = \max_{x \in K} \alpha(x)$$

which is impossible.

Thus $\alpha(K) = 0$. Therefore Proposition 4.1i) gives a unique V_x for each $x \in K$ such that $\mathcal{V}_x = \{V_x\}$.

Now use Claim 3 for another τ, say τ', which is greater than already fixed τ. Then we have corresponding bundle of cones $K \times (C_x')$ and the maximal invariant subbundle for (F, GT) restricted to the bundle $K \times (GC_x')$. We denote this maximal invariant subbundle by $K \times (\mathcal{V}_x')$. By Claim 3 we know that each $\mathcal{V}_x',\ x \in K$ contains just one element. Since $K \times (GC_x)$ is a subbundle of $K \times (GC_x')$ the corresponding maximal invariant subbundles have the same property, i.e. $K \times (\mathcal{V}_x)$ is a subbundle of $K \times (\mathcal{V}_x')$. Consequently these bundles coincide because each $\mathcal{V}_x$ and $\mathcal{V}_x',\ x \in K$ contains only one element. This is going to be used in the proof of "generalized" exponential separation stated in the following claim.

Claim 4. *Let $V_x,\ x \in K$ be the $k-$dimensional subspaces from Claim 3. Then there is a $n \in \mathbb{N}$ and $0 < \gamma' < 1$ such that*

$$| T_x^n w | \leq \gamma' \, | T_x^n v || w | \tag{4.11}$$

for any $x \in K,\ v \in V_x \cap S$ and $w \in Anih(L)$ with $L \in \mathcal{L}_x$.

It is sufficient to prove this claim for the vectors w with a fixed norm.

Let τ' be a number introduced before the claim. Since $\tau' > \tau$, the set $GC_x \subset int(GC'_x)$ for any $x \in K$. Hence the compactness of the subbundle $K \times (\{V_x\})$ of the bundle $K \times (GC_x)$ implies that there exists a $\delta > 0$ such that for any $x \in K$

$$\{v + w : v \in V_x \cap S,\ L \in \mathcal{L}_x,\ w \in Anih(L) \cap (\delta \mathbb{B})\} \subset C'_x.$$

Moreover, the number $\delta > 0$ can be chosen so small that for any $x \in K$ and any $v,\ w$ from the corresponding sets there is an $E \in GC'_x$ such that $v + w \in E$. Now suppose that the claim is false, where in (4.11) $\mid w \mid = \delta$. Then we have infinite sequences $n_i \in \mathbb{N},\ x_i \in K,\ v_i \in V_{x_i} \cap S,\ L_i \in \mathcal{L}_{x_i},\ w_i \in Anih(L_i) \cap (\delta S),\ E_i \in GC'_{x_i},\ i = 1, 2, \dots$ such that

$$\frac{\mid T^{n_i}_{x_i} w_i \mid}{\mid T^{n_i}_{x_i} v_i \mid} \geq \frac{\delta}{2}, \tag{4.12}$$

$v_i + w_i \in E_i$ for all $i \in \mathbb{N}$ and $n_i \to +\infty$ as $i \to \infty$.

First note that the denominators in the fractions above are nonzero because of the injectivity of T_x on V_x for any $x \in K$. The invariance of $K \times (\mathcal{L}_x)$ under the bundle map $(F^{-1}, \mathcal{T}^*)$ implies for any L_i the existence of $L'_i \in \mathcal{L}$ such that $\mathcal{T}^{*n_i}_{F^{n_i}x_i}(L'_i) = L_i$. Hence $T^{n_i}_{x_i}(w_i) \in Anih(L'_i)$, and thus $T^{n_i}_{x_i} w_i \notin int(C_1)$ for all $i \in \mathbb{N}$. This will be used later.

Since K is compact we may suppose that

$$F^{n_i} x_i \to x \ \text{ as } i \to \infty \text{ for some } x \in K.$$

Then applying Claim 3 to the bundle $K \times (GC'_x)$ in view of Proposition 4.3 we have

$$GT^{n_i}_{F^{-n_i}(F^{n_i}x_i)}(E_i) = GT^{n_i}_{x_i}(E_i) \to GV_x \ \text{ as } i \to +\infty. \tag{4.13}$$

(We also use compactness of the bundle map (F, GT) restricted to the bundle $K \times (GC_x)$.)

Hence, passing to a subsequence, we may assume that there is $u \in V_x \cap S$ with

$$\frac{T^{n_i}_{x_i}(v_i + w_i)}{\mid T^{n_i}_{x_i}(v_i + w_i) \mid} \to u \tag{4.14}$$

as $i \to +\infty$.

Since (4.12) holds, $T^{n_i}_{x_i} w_i \neq 0$ and we can denote for any $i \in \mathbb{N}$

$$\bar{w}_i = \frac{T^{n_i}_{x_i} w_i}{\mid T^{n_i}_{x_i} w_i \mid} \quad \text{and} \quad \bar{v}_i = \frac{T^{n_i}_{x_i} v_i}{\mid T^{n_i}_{x_i} w_i \mid},$$

where obviously $\mid \bar{w}_i \mid = 1$.

Then from (4.14) we obtain

$$(4.15) \qquad \frac{\bar{v}_i + \bar{w}_i}{\mid \bar{v}_i + \bar{w}_i \mid} \to u \in V_x.$$

By (4.12), $\mid \bar{v}_i \mid \leq \frac{2}{\delta}$ for all $i \in \mathbb{N}$, hence we have all $\mid \bar{v}_i + \bar{w}_i \mid$ bounded by $1 + \frac{2}{\delta}$. Therefore we can pass to a subsequence, if necessary, such that the sequence $\bar{v}_i,\ i = 1, 2, \ldots$ has a limit $\bar{v} \in V_x$ by the continuity, and the sequence $\mid \bar{v}_i + \bar{w}_i \mid$ has a limit $\delta' \geq 0$. Combining this with (4.15) we obtain

$$\bar{w}_i \to \delta' u - \bar{v} \in V_x \quad \text{as} \quad i \to +\infty.$$

But $\bar{w}_i$ are unit vectors in the complement of $int(C_1)$. So $\bar{w}_i$ cannot converge to a vector in $V_x \subset int(C_1) \cup \{0\}$. This contradiction proves the claim.

Now it is easy to prove the following claim.

Claim 5. *Let $K \times (\mathcal{L}_x)$ be the bundle from Claim 1. Then each $\mathcal{L}_x,\ x \in K$ contains just one element. This element is denoted by L_x.*

Suppose by contradiction that there is a $x \in K$ and $L \neq L'$ from $\mathcal{L}_x$. Choose $u \in Anih(L') \setminus Anih(L)$ and let be δ be as in the proof of Claim 4. Then $u = v + w$ where $0 \neq v \in V_x$ and $0 \neq w \in Anih(L)$. Using (4.11) we have for all sufficiently large $p \in \mathbb{N}$

$$(4.16) \qquad \frac{\mid T_x^{np} w \mid}{\mid T_x^{np} v \mid} < \gamma'^p \frac{\mid w \mid}{\mid v \mid} < \delta.$$

(We used here also the fact that $T_x(Anih(L_x)) \subset Anih(L_{Fx})$ for any $x \in K,\ L_x \in \mathcal{L}_x$ following from the invariance of the bundle $K \times (\mathcal{L}_x)$ under the bundle map (F^{-1}, T^*) on $K \times \mathcal{L}$.)
Therefore $T_x^{np}(v + w) \in C'_x \subset C_1$, and consequently

$$(4.17) \qquad T_x^{np+1}(v + w) \in int(C_1)$$

for all sufficiently large p. But the invariance of $K \times (\mathcal{L}_x)$ under $(F^{-1}, \mathcal{T}^*)$ implies $T_x^j(v+w) \notin int(C_1)$ for all $j \in \mathbb{N}$, as we saw in the proof of Claim 4. This contradicts (4.17). Thus we proved Claim 5.

Continuity of the bundle $K \times (\mathcal{L}_x)$ implies immediately the continuous dependence of L_x on $x \in K$ in $\mathcal{L}$ as well as the continuity of the bundle of the corresponding $k-$dimensional vector spaces in X^* denoted by the same letters. So far we have obtained continuous bundles $K \times (V_x)$ (see Claim 3) and $K \times (L_x)$ invariant under the bundle maps (F, T) and (F, T^*), respectively, such that $V_x \subset int(C_1) \cup \{0\}$ and $Anih(L_x) \cap int(C_1) = \emptyset$. The exponential separation for these invariant bundles remains to be proved. But this can be done using the estimate (4.16) similarly as in Lemma 4.1. The proof of Theorem 4.1 is finished.$\diamondsuit$

5. Discrete Ljapunov functionals for perturbations.

In this section we prove Theorem C using Theorems B and A, and Corollary 2.1. First, in Proposition 5.1 below we state the existence of cones suitable for the construction of a discrete Ljapunov functional for a small C^1 perturbations. This construction is given in Corollary 5.1. Since we do not assume injectivity or compactness of perturbations we use two more consequences of Proposition 5.1 formulated in Corollaries 5.2-5.3. Then we show how mentioned statements together with Theorem A imply Theorem C. Before the proof of Proposition 5.1 we derive two lemmas of their own interest. After that proof we improve the conclusion of Theorem C and prove Corollary 2.1.

Proposition 5.1. *Suppose that the assumptions of Theorem C are satisfied. Then there are numbers $N_0,\ m_0 \in \mathbb{N},\ 0 \leq \lambda_0 < 1,\ \epsilon_0 > 0$, an open bounded neighborhood $\mathcal{V}_0$ of K_0 and a sequence of sets $\mathcal{C}_1 \subset int(\mathcal{C}_2) \cup \{0\} \subset \mathcal{C}_2 \subset \ldots \mathcal{C}_{N_0-1} \subset int(\mathcal{C}_{N_0}) \cup \{0\} \subset \mathcal{C}_{N_0}$ with each $\mathcal{C}_i$ a k_i-cone such that for any $G \in C^1(\mathcal{V}_0, X)$ with*

$$| G - F |_{\mathcal{V}_0} |_{C^1(\mathcal{V}_0,X)} < \epsilon_0$$

and $x, y \in \mathcal{V}_0$ the following holds

i) if $x - y \in \mathcal{C}_i \setminus \{0\}$ for an $1 \leq i \leq N_0$ and $G^m x, G^m y$ are defined for a $m \geq m_0$ then $G^m x - G^m y \in int(\mathcal{C}_i) \setminus \{0\}$.

ii) if $G^{2m_0}x,\ G^{2m_0}y$ are defined, and $x - y$ as well as $G^{2m_0}x - G^{2m_0}y$ lie in $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ for an $i \in \{1, 2, \ldots, N_0\}$, then $G^{m_0}x - G^{m_0}y \notin \Pi$. (Here $\mathcal{C}_0 = \{0\}$.)

iii) if $G^m x, G^m y$ are defined with $G^m x - G^m y \notin \mathcal{C}_{N_0}$ and $m \geq m_0$ then

$$| G^m x - G^m y | \leq \lambda_0^m \mid x - y \mid .$$

Moreover we have that

iv) there is a projection P_0 of X on k_{N_0}-dimensional space and a constant $\tilde{c} > 0$ such that

$$\mathcal{C}_{N_0} \subset \{v \in X : | (I - P_0)v | \leq \tilde{c} \mid P_0 v \mid\}.$$

For the next corollary we put $\mathcal{C}_{N_0+1} := X$ and $\mathcal{C}_0 = \{0\}$.

Corollary 5.1. *Suppose that the assumptions of Theorem C are satisfied and $N_0, m_0, \lambda_0,\ \epsilon_0, \mathcal{V}_0$ and G are as in Proposition 5.1. Assume that $\mathcal{X} \subset \mathcal{V}_0$ is a compact set positively invariant under G such that G is injective on $\mathcal{X}$. Then the function θ defined at each $(x, y) \in \mathcal{X} \times \mathcal{X} \setminus \{(x, x) :\ x \in \mathcal{X}\}$ by*

$$\theta(x, y) := i \quad \textit{if} \quad x - y \in int(\mathcal{C}_i) \setminus int(\mathcal{C}_{i-1})$$

is a discrete Ljapunov functional for G on $\mathcal{X}$.

Proof. First, let us note that θ is defined for all (x, y) with distinct $x, y \in \mathcal{X}$ since $int(\mathcal{C}_i) \setminus int(\mathcal{C}_{i-1})$, $1 \leq i \leq N_0 + 1$ is a partition of the set X.

Axiom A1 with $\mu = m_0$ follows immediately from the definition of θ and Proposition 5.1i). If the function θ is continuous at the point (x, y) then the statement of Axiom A3 for (x, y) follows from Axiom A1.

Now let θ be discontinuous at the point $(x, y) \in \mathcal{X} \times \mathcal{X}$ with distinct x, y. We are going to prove that

$$\theta(x', y') \geq \theta(x, y) - 1 \geq \theta(G^{m_0}x, G^{m_0}y) \tag{5.1}$$

for all x', y' sufficiently close to x, y, respectively.

First, let us explain how (5.1) helps to show Axiom A2 and the remaining part of Axiom A3. Axiom A3 with $\mu = m_0$ for the point (x, y) of discontinuity of θ follows from (5.1) and Axiom A1 immediately. To prove Axiom A2 suppose that θ is discontinuous at the point $(G^{m_0}x, G^{m_0}y)$. For this point (5.1) gives

$$\theta(G^{m_0}x, G^{m_0}y) > \theta(G^{2m_0}x, G^{2m_0}y).$$

This together with the already known inequality

$$\theta(x, y) \geq \theta(G^{m_0}x, G^{m_0}y)$$

proves Axiom A2.

To show (5.1) denote $\theta(x, y) = i$. If $x - y$ lies in the open set $int(\mathcal{C}_i) \setminus \mathcal{C}_{i-1}$ then θ is continuous at (x, y). Since (x, y) is the point of discontinuity of θ, $x - y \in \partial\mathcal{C}_{i-1} \subset \mathcal{C}_{i-1}$. Hence, $i > 1$ and by Proposition 5.1i) we have

$$G^{m_0}x - G^{m_0}y \in int(\mathcal{C}_{i-1}).$$

This yields the inequality on the right hand side of (5.1).

If $i > 2$ then $\mathcal{C}_{i-2} \subset int(\mathcal{C}_{i-1})$ implies that

$$\partial\mathcal{C}_{i-1} \cap int(\mathcal{C}_{i-2}) = \emptyset.$$

This takes place also for $i = 2$. Hence $x - y$ lies in the open set $X \setminus int(\mathcal{C}_{i-2})$ giving the inequality on the left hand side of (5.1). The proof of the corollary is complete. $\diamondsuit$

Remark 5.1 Actually we have proved that the function θ satisfies the inequality

$$\theta(x', y') \geq \theta(G^{m_0}x, G^{m_0}y)$$

for all x', y' sufficiently close to $x, y \in \mathcal{X}$, respectively, and if θ is discontinuous this inequality is strong.

Corollary 5.2. *Suppose that the assumptions of Theorem C are satisfied and $N_0, m_0, \lambda_0, \epsilon_0, \mathcal{V}_0$ and G are as in Proposition 5.1.*

Let x_0 be a point in $\mathcal{V}_0$ such that the set $\{G^n x_0 : n \geq 1\}$ is defined. Then this set is precompact in X.

Proof. Denote $x_j := G^j x_0, \; c_1 := diam(\mathcal{V}_0)$. The corollary follows from the fact that for any $\eta > 0$ there is a $p_0 \in \mathbb{N}$ and a finite subset $\mathcal{P}$ of $\mathbb{N}$ such that

$$\{x_n : n \geq p_0\} \subset \bigcup_{p \in \mathcal{P}} B(x_p, \eta),$$

where $B(x, r)$ denotes a ball in X with center x and radius r. To prove this fact fix any $\eta > 0$. Take p_0 such that

$$p_0 > m_0 \quad \text{and} \quad \lambda_0^p c_1 < \eta.$$

Then if $p_0 < p < q$ and

$$x_p - x_q \notin \mathcal{C}_{N_0}$$

we obtain by Proposition 5.1iii) that

$$(5.2) \qquad \mid x_p - x_q \mid \leq \lambda_0^p \mid x_0 - x_{q-p} \mid \leq \lambda_0^p c_1 < \eta.$$

If for $p_0 < p < q$ we have

$$x_p - x_q \in \mathcal{C}_{N_0}.$$

then Proposition 5.1iv) gives

$$(5.3) \quad \mid x_p - x_q \mid \leq \mid (I - P_0)(x_p - x_q) \mid + \mid P_0(x_p - x_q) \mid \leq (c_1 + 1) \mid P_0(x_p - x_q) \mid .$$

Since $O^+(x)$ is bounded and P_0 is a compact operator the set $P_0(O^+(x))$ is pre-compact in $P_0(X)$. Hence there is a finite subset $\mathcal{P}$ of the set $\{p \in \mathbb{N} : p \geq p_0\}$ such that

$$P_0(O^+(x)) \subset \bigcup_{p \in \mathcal{P}} B(P_0 x_p, \frac{\eta}{c_1 + 1}).$$

Therefore , if $q \geq p_0$, by (5.2) we obtain that

$$x_q - x_p \in \mathcal{C}_{N_0} \quad \text{for all } p \in \mathcal{P} \text{ implies} \quad \mid x_p - x_q \mid < \eta, \ p \in \mathcal{P}$$

and by (5.3) we have that

$$x_q - x_p \notin \mathcal{C}_{N_0} \quad \text{for a } p \in \mathcal{P} \text{ implies} \quad \mid x_p - x_q \mid < \eta \quad \text{for the same } p.$$

Thus, the set $\mathcal{P}$ and the number p_0 satisfy the statement of the fact with the number η fixed above. The corollary is proved. $\diamondsuit$

Corollary 5.3. *Suppose that the assumptions of Theorem C are satisfied and $N_0, m_0, \lambda_0, \epsilon_0, \mathcal{V}_0$ and G are as in Proposition 5.1.*

Let $\mathcal{I}$ be a subset of $\mathcal{V}_0$ invariant under G. Then the restriction of G to $\mathcal{I}$ is injective and $x - y \in \mathcal{C}_{N_0}$ for all distinct $x, y \in \mathcal{I}$.

Proof. We prove that

$$(5.4) \qquad x - y \in \mathcal{C}_{N_0} \setminus \{0\} \quad \text{for all distinct } x, y \in \mathcal{I}.$$

This using Proposition 5.1i) gives

$$G^{m_0} x - G^{m_0} y \in int(\mathcal{C}_{N_0}) \setminus \{0\}$$

yielding injectivity of G on $\mathcal{I}$.

In order to show (5.4) suppose on the contrary that $x - y \notin \mathcal{C}_{N_0} \setminus \{0\}$. Since $\mathcal{I}$ is invariant under G there are points $x_m, y_m \in \mathcal{I}$ for each $m \in \mathbb{N}$ such that

$$G^m x_m = x, \; G^m y_m = y.$$

Denote $c_2 := diam(\mathcal{I})$. Then by Proposition 5.1iii) we obtain that

$$\mid x - y \mid \leq \lambda_0^m \mid x_m - y_m \mid \leq \lambda_0^m c_2, \quad \text{for all } m \geq m_0.$$

Hence, since $0 \leq \lambda_0 < 1$ the distance of x from y is 0 contradicting $x \neq y$. The corollary is proved.$\diamondsuit$

Proof of Theorem C. Let i_0, m_0, ϵ_0 and $\mathcal{V}_0$ be as in Proposition 5.1. Take $\mathcal{V} := \mathcal{V}_0$ and $\epsilon := \epsilon_0$. Consider any $G \in C^1(\mathcal{V}, X)$ such that

$$\mid G - F \mid_{\mathcal{V}} \mid < \epsilon$$

and $x_0 \in \mathcal{V}$ with all $G^n x_0, \; n \geq 1$ defined and $cl\{G^n x_0, \; n \geq 1\} \subset \mathcal{V}$.

If $\omega(x_0)$ is a periodic orbit then the conclusion of Theorem C follows immediately. Thus assume $\omega(x_0)$ is not a periodic orbit. Then $G^n x_0 \neq G^m x_0$ for all $n \neq m$. Moreover, applying Corollary 5.3 to $\omega(x_0)$, which is obviously invariant, we obtain that G is injective on $\omega(x_0)$. Hence G is injective on the set $cl\{G^n x_0 : \; n \geq 0\}$. Denote this set by $\mathcal{X}$. By Corollary 5.2 $\mathcal{X}$ is also compact. Therefore by Corollary 5.1 we obtain the discrete Ljapunov functional θ for G on $\mathcal{X}$. Theorem A applied to this functional and G gives

$$\theta(x, y) = \theta(G^n x, G^n y) \quad \text{for all distinct } x, y \in \omega(x_0) \text{ and } n \in \mathbb{Z}.$$

Hence, by Corollary 5.3 applied to $\mathcal{I} = \omega(x_0)$ and Proposition 5.1ii) we have

$$x - y \notin \Pi \quad \text{for all distinct } x, y \in \omega(x_0). \tag{5.5}$$

Now denote by Π' a d-dimensional subspace of X such that $\Pi \oplus \Pi' = X$. Further denote by π the projection of X on Π' along Π. Then (5.5) implies that π is injective map on $\omega(x_0)$. Hence, since $\omega(x_0)$ is compact and π is continuous the ω-limit set of x_0 is homeomorphic to the subset $\pi(\omega(x_0))$ in Π'. The theorem is proved. $\diamondsuit$

For the next two lemmas we assume the following assumption (H):
The map F is a homeomorphism of a compact metric space K, X is a Banach space, $\{T_x, \; x \in K\}$ is a family of compact operators in $L(X, X)$ continuously dependent on $x \in K$. Further $C_1 \subset C_2 \subset \ldots \subset C_i \subset \ldots$ is a sequence of cones with length N less or equal ∞ such that each C_i is a k_i-cone and $k_i < k_{i+1}$ for $i = 1, 2, \ldots$. Moreover for any $x \in K$, $i = 1, 2, \ldots$ and $0 \neq v \in C_i$ there is an open neighborhood of v in X mapped by T_x into C_i and

$$T_x(X \setminus \{0\}) \in \bigcup_{j=1}^{N} C_j \setminus \{0\}.$$

For any fixed i this assumption allow us Theorem B applied to the bundle map (F, T) on the bundle $K \times X$. We obtain that there are continuous k_i-dimensional subbundles $K \times (V_x^i)$ and $K \times (L_x^i)$ of the bundles $K \times X$ and $K \times X^*$, respectively, invariant under the bundle map (F, T) on $K \times X$.

Lemma 5.1. *Let the assumption (H) be satisfied and $N = +\infty$. Then for any $0 < \lambda < 1$ there is an $i_0 \in \mathbb{N}$ such that*

$$(5.6) \qquad | T_x u | \leq \lambda | u |,$$

for any $x \in K$ and $u \in Anih(L_x^{i_0}) \setminus \{0\}$.

Proof: Take any $0 < \lambda < 1$ and suppose that (5.6) is false for each $i_0 \in \mathbb{N}$. This means that there is a sequence of points $x_i \in K$ and $u_i \in Anih(L^i_{x_i}) \cap S$ such that

$$(5.7) \qquad | T_{x_i} u_i | \geq \lambda$$

for all $i \in \mathbb{N}$.

Since K is compact we can find a convergent subsequence of the sequence x_i. Let $x \in K$ be its limit. Then, since the corresponding sequence of operator T_{x_i} converges to the compact operator T_x and $u_i \in S$ for all $i \in \mathbb{N}$, passing to further subsequences we achieve that $T_{x_i} u_i$ has a limit $u \in X$. Thus, in addition to (5.7), we may suppose without loss of generality that $x_i \to x \in K$ and $T_{x_i} u_i \to u \in X$ as $i \to +\infty$. By (5.7), obviously $| u | \geq \lambda > 0$, and, by the assumptions of Theorem C, we have $T_x u \in C_{j_0} \setminus \{0\}$ for a $j_0 \in \mathbb{N}$.

Using the same procedure as in the part $a)$ of Section 4 we can find other k_i-cones C'_i such that $T^2_y(C'_i \setminus \{0\}) \subset C_i \setminus \{0\} \subset int(C'_i) \setminus \{0\}$ for any $y \in K,\ i \in \mathbb{N}$ and $C'_i \subset C'_{i-1}$ for all $i > 1$. Since we also have $T_y(C_i) \subset C_i$ for all $y \in K$ and $i \in \mathbb{N}$ we can use Lemma 4.1 for the vector bundle (F, T) and the cones C_i, C'_i and any fixed $1 \in \mathbb{N}$. Therefore the dual invariant bundles for the k_i-dimensional continuous separation for (F, T) along K associated to the k_i-cone C_i are the same as corresponding bundles for k_i-dimensional continuous separation for (F^2, T^2) along K associated to the k_i-cone C'_i. This follows using Lemma 4.1 as in the proof of Theorem B . So $L^i_y \cap C'_i = \{0\}$ for any $y \in K,\ i \in \mathbb{N}$, and hence $u_i \notin C'_i$ for all $i \in \mathbb{N}$. Thus for any $i \in \mathbb{N}$ we have $T^2_{x_i} u_i \notin C'_i$ by the positive invariance of each bundle $K \times (L^i_x)$ under the bundle map (F^2, T^2). Since $C'_{j_0} \subset C'_i$ for all $i \geq j_0$, we have

$$T^2_{x_i} u_i \notin C'_{j_0}.$$

But since $T^2_{x_i} u_i$ approaches $T_x u \in C_{j_0} \setminus \{0\} \subset int(C'_{j_0})$ as $i \to +\infty$ we obtain a contradiction. The lemma is proved.$\diamondsuit$

Up to the end of this section we say that a vector subbundle of a vector bundle $K \times Y$ is compact iff the Grassmanian analog of the vector subbundle is a compact bundle of sets (cf. the definition ii) before the Proposition 4.3).

Lemma 5.2. *Let the assumption (H) be satisfied. Then $V^i_x \subset V^{i+1}_x,\ L^i_x \subset L^{i+1}_x$ for all $x \in K,\ 1 \leq i < N$. Moreover*

$$(V^{i+1}_x \cap Anih(L^i_x)) \oplus V^i_x = V^{i+1}_x \quad \textit{for all } x \in K,\ 1 \leq i < N$$

and each bundle $K \times (V^{i+1}_x \cap Anih(L^i_x)),\ 1 \leq i < N$ is a compact subbundle of $K \times X$ invariant under the bundle map (F, T) on $K \times X$.

Proof. Fix $1 \leq i < N$. Let us note that the bundles $K \times (V_x^i),\ K \times (V_x^{i+1})$ are continuous bundles with fixed dimension for each of them. Hence, they are also compact. The same property of the dual bundles $K \times (L_x^i),\ K \times (L_x^{i+1})$ implies that the bundles $K \times (Anih(L_x^i)),\ K \times (Anih(L_x^{i+1}))$ are continuous.

First we show that $V_x^i \subset V_x^{i+1}$ for all $x \in K$. Suppose on the contrary that there is a $x \in K$ and $u \in V_x^i$ such that $u \notin V_x^{i+1}$. Write $u := v + w$ with $v \in V_x^{i+1},\ 0 \neq w \in Anih(L_x^{i+1})$. Since the bundle $K \times (V_x^{i+1})$ is invariant under the bundle map (F,T) on $K \times X$ there are $u_n \in V_{F^{-n}x}^i,\ n \in \mathbb{N}$ such that

$$u = T_{F^{-n}x}^n u_n.$$

Write

$$u_n := v_n + w_n \quad \text{with } v_n \in V_{F^{-n}x}^{i+1},\ w_n \in Anih(L_{F^{-n}x}^{i+1}).$$

The positive invariance of the bundles $K \times (V_x^i)$ and $K \times (Anih(L_x^i))$ under the bundle map (F,T) on $K \times X$ together with the uniqueness of the decomposition $u = v + w$ gives

$$T_{F^{-n}x}^n w_n = w, \quad T_{F^{-n}x}^n v_n = v.$$

We claim that there is a $c > 0$ such that

$$|\, w_n \,| \leq c \,|\, v_n \,| \,. \tag{5.8}$$

Otherwise there is an infinite sequence $n_j,\ j = 1, 2, \dots$ such that

$$\frac{|v_{n_j}|}{|w_{n_j}|} \to 0 \quad \text{as} \quad i \to \infty. \tag{5.9}$$

Since $K \times (V_x^{i+1})$ is a compact bundle, passing to a subsequence if necessary, we obtain that

$$F^{-n_j}x \to x_0 \quad \text{and} \quad \frac{u_{n_j}}{|u_{n_j}|} \to u_0 \in V_{x_0}^i \quad \text{as } j \to \infty.$$

Write the last sequence of unit vectors in the following way

$$\frac{u_{n_j}}{|\, u_{n_j} \,|} = \frac{v_{n_j} + w_{n_j}}{|\, v_{n_j} + w_{n_j} \,|} = \frac{\frac{v_{n_j}}{|w_{n_j}|} + \frac{w_{n_j}}{|w_{n_j}|}}{|\, \frac{v_{n_j}}{|w_{n_j}|} + \frac{w_{n_j}}{|w_{n_j}|} \,|} \to u_0 \in V_{x_0}^i.$$

Thus, using (5.9) we obtain that the sequence

$$\frac{w_{n_j}}{|w_{n_j}|} \in Anih(L_{F^{-n_j}x}^{i+1}) \text{ has the limit } u_0.$$

Hence, by continuity of the bundle $K \times Anih(L_x^{i+1})$ we have

$$u_0 \in Anih(L_{x_0}^{i+1}) \setminus \{0\},$$

consequently $u_0 \notin C_{i+1}$. Since $C_i \subset C_{i+1}$, $u_0 \notin C_i$. But $u_0 \in V_{x_0}^i \subset C_i$. This contradiction proves (5.8).

Now let $M > 0,\ 0 < \gamma < 1$ be constants for the exponential separation for (F, T) along K associated with the k_{i+1}-cone C_{i+1}. Then using (5.8) we obtain that

$$\frac{|w|}{|v|} \le M\gamma^n \frac{|w_{n_j}|}{|v_{n_j}|} \le M\gamma^{n_j} c \to 0 \quad \text{as } j \to +\infty.$$

This contradicts $w \neq 0$ and proves $V_x^{i+1} \subset V_x^i,\ x \in K$.

In order to prove $L_x^i \subset L_x^{i+1}$ for fixed $x \in K$, assume the opposite. So, there is an $u \in Anih(L_x^{i+1})$ such that $u = v + w$ with $v \in V_x^i,\ w \in Anih(L_x^i)$ and $v \neq 0$. Consider

$$u_n = T_x^n u = T_x^n v + T_x^n w := v_n + w_n, \quad n \in \mathbb{N}.$$

By the invariance of $K \times (V_x^i)$ under (F, T) each $v_n,\ n \ge 1$ is nonzero. Compactness of $K \times (V_x^i)$ gives a sequence $n_j,\ j = 1, 2, \dots$ and $x_0 \in K,\ v_0 \neq 0$ such that

$$\frac{v_{n_j}}{|v_{n_j}|} \to v_0 \in V_{x_0}^i \quad \text{as } j \to +\infty. \tag{5.10}$$

By the exponential separation for (F, T) associated with C_i we have

$$\frac{|w_{n_j}|}{|v_{n_j}|} \to 0 \ \text{ as } j \to \infty.$$

Hence, using (5.10), positive invariance and continuity of the bundle $K \times (Anih(L_x^i))$ we obtain that

$$Anih(L^i_{F^{n_j}x}) \ni \frac{u_{n_j}}{|\, u_{n_j}\, |} \to v_0 \in V_{x_0}^i \cap Anih(L_{x_0}^{i+1}).$$

But, $Anih(L_{x_0}^{i+1}) \cap C_{i+1} = \{0\}$ and $V_{x_0}^i \subset C_i$, which together with $C_i \subset C_{i+1}$ imply $v_0 = 0$. This contradicts $|\, v_0 \,| = 1$ and proves $L_x^{i+1} \subset L_x^i,\ x \in K$.

Now denote $K \times (V_x^{i+1} \cap Anih(L_x^i)) := K \times (W_x^i)$. Since the bundle $K \times (V_x^{i+1})$ is compact and the bundle $K \times (Anih(L_x^i))$ is continuous the bundle $K \times (W_x^i)$ is compact. For each $x \in K$ we also have $Anih(L_x^{i+1}) \oplus V_x^{i+1} = Anih(L_x^i) \oplus V_x^i = X$ and the inclusions $Anih(L_x^{i+1}) \subset Anih(L_x^i),\ V_x^i \subset V_x^{i+1}$. Hence $W_x^i \oplus V_x^i = V_x^{i+1}$ for all $x \in K$. Therefore each W_x^i has the dimension $k_{i+1} - k_i$. By the above inclusions and the positive invariance of the bundles $K \times (V_x^{i+1})$ and $K \times (Anih(L_x^i))$ under (F, T) we also have $T_x(W_x^i) \subset W_{Fx}^i$ for all $x \in K$. Since each T_x is injective on V_x^{i+1} and $dim(W_x^i) = dim(W_{Fx}^i)$ we obtain $T_x(W_x^i) = W_{Fx}^i$. Thus the bundle $K \times (W_x^i)$ is invariant under (F, T) on $K \times X$, completing the proof of the lemma. $\diamondsuit$

Proof of Proposition 5.1. Suppose that the assumptions of Theorem C are satisfied for a compact set K_0 in a Banach space X, a map F_0, a $d-$codimensional space Π, a nested family of k_i cones $C_i,\ i = 1, 2, \dots$ and a family of operators $\{T_{(x,y)} : x, y \in K_0\}$.

First we are going to use Theorem B and Lemmas 5.1, 5.2 for $K := K_0 \times K_0$, the map F given by the formula

$$F(x, y) = (F_0 x, F_0 y), \quad (x, y) \in K;$$

$\{T_{(x,y)} : (x,y) \in K\}$-the family of compact operators in $L(X,X)$; $C_1 \subset C_2 \subset \ldots \subset C_i \subset \ldots$ the nested family of k_i-cones with $k_{i+1} > k_i$ for all $1 \leq i < N$, $C_N = X$ and Π- $d-$codimensional plane in X.

By the assumptions of the theorem we have

$$T_{(x,y)}(C_i) \cap (C_i \setminus C_{i-1}) \cap \Pi = \emptyset \text{ for any } (x,y) \in K,\ i = 1, 2, \ldots, \tag{5.11}$$

$$F_0 x - F_0 y = T_{(x,y)}(x-y) \text{ for all } (x,y) \in K \tag{5.12}$$

and for any $0 \neq v \in X$ there is an $j \in \mathbb{N}$ such that

$$T_{(x,y)} v \in C_j \setminus \{0\}. \tag{5.13}$$

Since K_0 is an invariant subset for the continuous map F_0, the function F is continuous and surjective. Applying (5.13) to x, y, v such that $0 \neq v = x - y$ we obtain that F_0 is injective on K_0 due to the formula (5.12), Hence F is a homeomorphism of K. We have shown that all the assumptions of Theorem B for the bundle map (F,T) and each k_i-cone C_i, $i = 1, 2, \ldots$ are satisfied as well as all the assumptions of Lemma 5.2 and Lemma 5.1 if $N = \infty$.

By Theorem B we have for each $i = 1, 2, \ldots$ continuous bundles $K \times (V^i_{(x,y)})$, $K \times (L^i_{(x,y)})$ and constants $M_i > 0$, $0 < \gamma_i < 1$ for the corresponding exponential separations. Denote by $P^i_{(x,y)}$ the projection of X on $V^i_{(x,y)}$ along $Anih(L^i_{(x,y)})$ for each $(x,y) \in K$, $i = 1, 2, \ldots$. By $Q^i_{(x,y)}$ we denote the projection $Q^i_{(x,y)} := I - P^i_{(x,y)}$. Denote also $V^0_{(x,y)} = \{0\}$, $L^0_{(x,y)} = \{0\}$, $P^0_{(x,y)} = 0$, $Q^0_{(x,y)} = I$ for all $(x,y) \in K$. If $N = \infty$ apply Lemma 5.1 for fixed $0 < \lambda_1 < 1$. Then there is an $i_0 \in \mathbb{N}$ such that

$$\text{if} \quad P^{i_0}_{(x,y)} u = 0 \quad \text{then} \quad | T_{(x,y)} u | \leq \lambda_1 \mid u \mid . \tag{5.14}$$

Denote $N_0 := N$ if $N < \infty$ and $N_0 := i_0$ if $N = \infty$. Further define for $1 \leq i \leq N_0$, $s \geq 0$ the sets

$$\mathcal{C}_i(s) := \bigcup_{(x,y) \in K} \{v \in X : \mid Q^i_{(x,y)} v \mid \leq s \mid P^i_{(x,y)} v \mid\}$$

$$\mathcal{D}_i(s) := \bigcup_{(x,y) \in K} \{v \in X : \mid P^i_{(x,y)} v \mid \leq s \mid Q^i_{(x,y)} v \mid\}.$$

Note that if $0 \leq s_1 < s_2$ and $1 \leq i \leq N_0$ then $\mathcal{C}_i(s_1) \subset \mathcal{C}_i(s_2)$ as well as $\mathcal{D}_i(s_1) \subset \mathcal{D}_i(s_2)$. We are going to show that for a small interval of values s the above sets are k_i-cones and vectors from $C_i(s)$ satisfy some estimates.

Let us notice that for each $1 \leq i \leq N_0$ the bundles $K \times (V^i_{(x,y)})$, $K \times (L^i_{(x,y)})$ are compact and the following relations are satisfied

(5.15)

$$\mathcal{C}_i(0) = \bigcup_{(x,y)\in K} V^i_{(x,y)} \subset C_i,\ \ \mathcal{D}_i(0)\cap C_i = \bigcup_{(x,y)\in K} Anih(L^i_{(x,y)})\cap C_i = \{0\},\ \ 1\le i\le N_0.$$

Therefore there is a $1 \ge \delta_0 > 0$ such that

$$\mathcal{C}_i(s)\cap\mathcal{D}_i(s) = \{0\} \quad \text{for all} \quad 0\le s\le\delta_0 \text{ and all } 1\le i\le N_0. \tag{5.16}$$

Each $\mathcal{C}_i(s),\ 0 < s \le \delta_0,\ 1\le i\le N_0$ is a k_i-cone. Let us show it.

Because of (5.15) it is sufficient to prove that $\lambda\mathcal{C}_i(s) = \mathcal{C}_i(s)$ for all $\lambda\in\mathbb{R}\setminus\{0\}$, which is obvious, and $\mathcal{C}_i(s)$ is a closed subset of X. This closedness follows from the fact that $\{P^i_{(x,y)}, Q^i_{(x,y)} :\ (x,y)\in K\}$ is a compact set of continuous projections of X.

Another consequence of (5.16) is the following inclusion

$$\mathcal{C}_i(s)\ \subset\ \mathcal{C}_i(\delta_0)\ \subset\ \{v\in X : \mid Q^i_{(x,y)}v \mid\le \tfrac{1}{\delta_0}\mid P^i_{(x,y)}v\mid\} \tag{5.17}$$

valid for all $0 < s\le\delta_0,\ 1\le i\le N_0$ and $(x,y)\in K$.

Now define the sets

$$W^i_z(s) := \{u\in X :\mid Q^i_z u + P^{i-1}_z u\mid\le s\mid Q^{i-1}_z P^i_z u\mid\},\ z\in K,\ 1\le i\le N_0,\ 0\le s,$$

$$\mathcal{W}_i(s) := \bigcup_{z\in K} W^i_z(s),\ 0\le s,\ 1\le i\le N_0.$$

Note that for all $1\le i\le N_0$ and $z\in K$ we have

$$W^i_z(0) = V^i_z\cap Anih(L^{i-1}_z)\subset (C_i\setminus C_{i-1})\cup\{0\}$$

and if $0\le s_1 < s_2$ then $W^i_z(s_1)\subset W^i_z(s_2)$ as well as $\mathcal{W}_i(s_1)\subset\mathcal{W}^i_z(s_2)$. For the proof of statement ii) we need the existence of δ_1 such that

$$\Pi\cap W_i(\delta_1) = \{0\} \quad \text{for all } 1\le i\le N_0, \tag{5.18}$$

and

(5.19)

$$\{u\in X : \mid P^{i-1}_{(x,y)}u\mid\le s\mid Q^{i-1}_{(x,y)}u\mid,\ \mid Q^i_{(x,y)}u\mid\le s\mid P^i_{(x,y)}u\mid,\ (x,y)\in K\}\subset\mathcal{W}_i(\tfrac{2s}{1-s})$$

for all $0\le s<1,\ 1\le i\le N_0$.

The equality (5.18) follows from the compactness each of the bundle $K\times(V^i_z\cap Anih(L^{i-1}_z))$, $1\le i\le N_0$ and the fact that $K\times(V^i_z\cap Anih(L^{i-1}_z)\cap\Pi) = K\times(\{0\})$ for all $1\le i\le N_0$. This fact is an immediate consequence of (5.11) and the invariance of all bundles $K\times(V^i_z\cap Anih(L^{i-1}_z))$ under (F,T) on $K\times X$ (cf. Lemma 5.2). To show (5.19) take any u from the set on the left hand side in (5.19). Write

u in the form $u = w_1 + w_2 + v$ where $w_1 \in Anih(L_z^i)$, $w_2 \in V_z^i \cap Anih(L_z^{i-1})$ and $v \in V_z^{i-1}$. Then we have

$$\mid v \mid \leq s \mid w_1 + w_2 \mid \leq s \mid w_1 \mid + s \mid w_2 \mid \quad \text{and} \quad \mid w_1 \mid \leq s \mid w_2 + v \mid \leq s \mid w_2 \mid + s \mid v \mid .$$

Hence, adding the resulting inequalities we obtain that

$$\mid w_1 + v \mid \leq \mid w_1 \mid + \mid v \mid \leq \frac{2s}{1-s} \mid w_2 \mid .$$

This proves (5.19).

Now denote

$$\tau = max(sup\{\mid T_z \mid_{L(X,X)}: \; z \in K\}, 1)$$

$$\varrho = max(sup\{\mid Q_z^{i+1} \mid_{L(X,X)} \mid P_z^i \mid_{L(X,X)}: \; 1 \leq i \leq N_0, \; z \in K\}, 1).$$

Take $0 < \delta$ such that $\delta < \delta_0$ and $\frac{2\delta}{1-\delta} < \delta_1$, and if $N = \infty$ we also require that

$$\lambda_0 := 2\delta + \frac{\delta\tau + \lambda_1}{1-\delta} \; < \; 1 \tag{5.20}$$

Our choice of the sequence of cones $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{N_0}$ is given by

$$\mathcal{C}_i := \mathcal{C}_i((2\varrho)^{i-N_0}\delta), \quad 1 \leq i \leq N_0.$$

Let us show that Lemma 5.2 and the definition of this sequence implies that

$$\mathcal{C}_1 \subset int(\mathcal{C}_2) \cup \{0\} \subset \mathcal{C}_2 \subset \ldots \subset \mathcal{C}_{N_0-1} \subset int(\mathcal{C}_{N_0}) \cup \{0\} \subset \mathcal{C}_{N_0}$$

and we already know that each $\mathcal{C}_i, \; 1 \leq i \leq N_0$ is a k_i-cone. Let $1 \leq i < N_0$ and $v \in \mathcal{C}_i((2\varrho)^{i-N_0}\delta)$. Then there is a $z \in K$ such that

$$\mid P_z^i v \mid (2\varrho)^{i-N_0}\delta \geq \mid Q_z^i v \mid .$$

Due to Lemma 5.2 we have

$$Q_z^{i+1} v = Q_z^{i+1} Q_z^i v \;\; \text{and} \;\; P_z^i v = P_z^i P_z^{i+1} v \;\; \text{for all } z \in K, \, v \in X \text{ and } 1 \geq i < N_0.$$

Therefore

$$\mid Q_z^{i+1} v \mid = \mid Q_z^{i+1} Q_z^i v \mid \leq \mid Q_z^{i+1} \mid_{L(X,X)} \mid Q_z^i v \mid \leq \mid Q_z^{i+1} \mid_{L(X,X)} (2\varrho)^{i-N_0}\delta \mid P_z^i v \mid$$

$$= \mid Q_z^{i+1} \mid_{L(X,X)} (2\varrho)^{i-N_0}\delta \mid P_z^i P_z^{i+1} v \mid \leq \mid Q_z^{i+1} \mid_{L(X,X)} \mid P_z^i \mid_{L(X,X)} (2\varrho)^{i-N_0}\delta \mid P_z^{i+1} v \mid$$

$$\leq \varrho (2\varrho)^{i-N_0}\delta \mid P_z^{i+1} v \mid < (2\varrho)^{i+1-N_0}\delta \mid P_z^{i+1} v \mid .$$

Hence $v \in \mathcal{C}^{i+1}((2\varrho)^{i+1-N_0}\delta)$.

Now we are going to obtain estimates needed to establish statements i)-iii).

Denote

$$M = max\{M_i :\ 1 \le i \le N_0\},\ \gamma = max\{\gamma_i :\ 1 \le i \le N_0\} \quad \text{and} \quad c = \tfrac{1}{\delta}.$$

Obviously $0 < \gamma < 1$. Then there is a $m_1 \in \mathbb{N}$ such that

$$\text{(5.21)} \qquad cM\gamma^m \ \le\ (8\varrho)^{-N_0}\delta \quad \text{for all } m \ge m_1.$$

This using the exponential separations means that we have

(5.22)
$$\text{if } \mid Q^i_{(x,y)}v \mid \le c \mid P^i_{(x,y)}v \mid \quad \text{then} \quad \mid Q^i_{(F_0^m x, F_0^m y)}T^m_{(x,y)}v \mid \le (8\varrho)^{-N_0}\delta \mid P^i_{(F_0^m x, F_0^m y)}T^m_{(x,y)}v \mid$$

for all $m \ge m_1,\ (x,y) \in K,\ v \in X,\ 1 \le i \le N_0$. Note that this estimate together with (5.17) and (5.12) shows statement i) for $F_0,\ x,y \in K_0$.

The key estimate to show statement iii) is the following

$$\text{(5.23)} \qquad \text{if} \quad \mid P^{N_0}_{(x,y)}u \mid \le \delta \mid Q^{N_0}_{(x,y)}u \mid \quad \text{then} \quad \mid T_{(x,y)}y \mid \le (\lambda_0 - 2\delta) \mid u \mid,$$

where $(x,y) \in K,\ u \in X$. Let us show it. Write $u = P^{N_0}_{(x,y)}u + Q^{N_0}_{(x,y)}u := v + w$. If $\mid v \mid \le \delta \mid w \mid$ then by (5.14) and (5.20) we have

$$\mid T_{(x,y)}(v+w) \mid \le \mid T_{(x,y)} \mid \mid v \mid + \lambda_1 \mid w \mid \le \mid T_{(x,y)} \mid \delta \mid w \mid + \lambda_1 \mid w \mid \le$$

$$\frac{\mid T_{(x,y)} \mid \delta + \lambda_1}{1-\delta}((1-\delta) \mid w \mid) \le (\lambda_0 - 2\delta)((1-\delta) \mid w \mid) \le (\lambda_0 - 2\delta) \mid v + w \mid .$$

Thus (5.23) is established.

Choose m_0 such that

$$\text{(5.24)} \qquad m_0 > m_1 \quad \text{and} \quad (\lambda_0 - \delta)^{m-m_1}(\delta+\tau)^{m_1} \le \lambda_0^m \quad \text{for all } m \ge m_0.$$

Our next aim is to prove the following statement:

There is an $\epsilon_1 > 0$ such that if $z \in K$ and a family of operators $T'_n \in L(X,X),\ 1 \le n \le 2m_0$ satisfy

$$\mid T'_n - T_{F^n z} \mid_{L(X,X)} < \epsilon_1 \quad \text{for all} \quad 1 \le n \le 2m_0$$

then

j) $T'_m \circ T'_{m-1} \circ \cdots \circ T'_1(\mathcal{C}_i \setminus \{0\}) \subset int(\mathcal{C}_i) \setminus \{0\}$ for all $m_0 \le m \le 2m_0$, $1 \le i \le N_0$.

jj) if u and $T'_{2m_0} \circ T'_{2m_0-1} \circ \cdots \circ T'_1 u$ lie in $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ for an $1 \leq i \leq N_0$ then $T_{m_0} \circ T_{m_0-1} \circ \cdots \circ T'_1 u \notin \Pi$.

jjj) if $u \notin \mathcal{C}_{N_0}$ and $T'_m \circ T'_{m-1} \circ \cdots \circ T'_1 u \notin \mathcal{C}_{N_0}$ for an $m_0 \leq m \leq 2m_0$ then

$$\mid T'_m \circ T'_{m-1} \circ \cdots \circ T'_1 u \mid \leq \lambda_0^m \mid u \mid .$$

In order to prove this statement denote for each $z \in K$ and $1 \leq i \leq N_0$ the sets

$$C^i_z(s) = \{u \in X : \mid Q^i_z u \mid \leq s \mid P^i_z u \mid\}$$

$$D^i_z(s) = \{u \in X : \mid P^i_z u \mid \leq s \mid Q^i_z u \mid\}$$

From (5.22), in this notation, we obtain that

$$T^m_z(C^i_z(c)) \subset C^i_{F^m z}((8\varrho)^{-N_0}\delta) \quad \text{for all} \quad z \in K,\ 1 \leq i \leq N_0,\ m_1 \leq m \leq 2m_0$$

Hence, we can find $\epsilon_1 > 0$ such that $0 < \epsilon_1 < \delta$ and for all $z \in K$ and families of operators $T'_1, T_2, \ldots, T'_{2m_0}$ in $L(X, X)$ with

$$\text{(5.25)} \qquad \mid T'_n - T_{F^n z} \mid_{L(X,X)} < \epsilon_1 \quad \text{for all} \quad 1 \leq n \leq 2m_0$$

we have

$$\text{(5.26)} \qquad T'_m \circ T'_{m-1} \circ \cdots \circ T'_1(C^i_z(c) \setminus \{0\}) \subset C^i_{F^m z}((4\varrho)^{-N_0}\delta) \setminus \{0\},$$

for all $m_1 \leq m \leq 2m_0,\ 1 \leq i \leq N_0$.

Now, let

$$T'_m \circ T'_{m-1} \circ \cdots \circ T'_1 u \notin C^{N_0}_{F^m z}((4\varrho)^{-N_0}\delta) \quad \text{for some } z \in K,\ m_0 \leq m \leq 2m_0,\ 1 \leq i \leq N_0$$

and $T'_1,\ T_2, \ldots, T'_{2m_0}$ be as in (5.25). Hence, by (5.26) we have

$$T'_n \circ T'_{n-1} \circ \cdots \circ T'_1 u \notin C^i_{F^n z}(c) \quad \text{for all} \quad 1 \leq n \leq m - m_1.$$

Since $c \geq \frac{1}{\delta}$, due to the definitions of the sets $C^i_z(s),\ D^i_z(s)$ we also have that

$$X \setminus C^i_{F^n z}(c) \subset D^i_{F^n z}(\delta) \setminus \{0\}, \quad 1 \leq n \leq m - m_1.$$

Thus,

(5.27)
if $T'_m \circ T'_{m-1} \circ \cdots \circ T'_1 u \notin C^i_{F^m z}((4\varrho)^{-N_0}\delta)$
for some $z \in K$, $m_0 \leq m \leq 2m_0,\ 1 \leq i \leq N_0$ then
$T'_n \circ T'_{n-1} \circ \cdots \circ T'_1 u \in X \setminus C^i_{F^n z}(c) \subset D^i_{F^n z}(\delta)\ \{0\}$ for all $\quad 1 \leq n \leq m - m_1$.

Hence, using (5.26) and the fact that $m_0 \leq 2m_0 - m_1$ we obtain that

$$\text{if } u \in C_z^i(c) \text{ and } T'_{2m_0} \circ T'_{2m_0-1} \circ \cdots \circ T'_1 u \in C^i_{F^{2m_0}z}(c) \setminus C^{i-1}_{F^{2m_0}z}((4\varrho)^{-N_0}\delta)$$

for some $z \in K,\ 1 \leq i \leq N_0$ then

$$T_{m_0} \circ T_{m_0-1} \circ \cdots \circ T'_1 u \in C^i_{F^{m_0}z}((4\varrho)^{-N_0}\delta) \cap D^{i-1}_{F^{m_0}z}(\delta).$$

Therefore using (5.19) for $s = \delta$ and the estimate $\frac{2\delta}{1-\delta} \leq \delta_1$ we also have by (5.18) that

$$C^i_{F^{m_0}z}((4\varrho)^{-N_0}\delta) \cap D^{i-1}_{F^{m_0}z}(\delta) \setminus \{0\} \subset C^i_{F^{m_0}z}(\delta) \cap D^{i-1}_{F^{m_0}z}(\delta) \setminus \{0\} \subset W_i(\frac{2\delta}{1-\delta}) \setminus \{0\} \subset \mathcal{W}_i(\delta_1) \setminus \{0\}.$$

Hence the equality (5.18) gives $T_{m_0} \circ T_{m_0-1} \circ \cdots \circ T'_1 u$ does not lie in Π. Thus we have obtained that

(5.28)
$$\begin{aligned}&\text{if } u \in C_z^i(c) \text{ and } T'_{2m_0} \circ T'_{2m_0-1} \circ \cdots \circ T'_1 u \in C^i_{F^{2m_0}z}(c) \setminus C^{i-1}_{F^{2m_0}z}((4\varrho)^{-N_0}\delta)\\ &\text{for some } z \in K,\ 1 \leq i \leq N_0 \text{ then}\\ &T_{m_0} \circ T_{m_0-1} \circ \cdots \circ T'_1 u \notin \Pi.\end{aligned}$$

Now, using (5.27) for $i = N_0$, (5.23) and the fact that $\epsilon_1 < \delta$ we obtain that

$$\text{if} \quad T'_m \circ T'_{m-1} \circ \cdots \circ T'_1 u \notin C^{N_0}_{F^m z}((4\varrho)^{-N_0}\delta) \quad \text{for some } z \in K, u \in X \text{ and a } m_0 \leq m \leq 2m_0$$

then

$$| T'_n \circ T'_{n-1} \circ \cdots \circ T'_1 u | \leq (\lambda_0 - \delta)^n | u | \quad \text{for all} \quad 1 \leq n \leq m - m_1.$$

From this by (5.24) and $m \geq m_0$ we obtain

$$| T'_m \circ T'_{m-1} \circ \cdots \circ T'_1 u | \leq (\lambda_0 - \delta)^{m-m_1} (\tau + \epsilon_1)^{m_1} | u |$$

$$\leq (\lambda_0 - \delta)^{m-m_1} (\tau + \delta)^{m_0} | u | \leq \lambda_0^m | u | .$$

Thus we have

$$\begin{aligned}&\text{if } T'_m \circ T'_{m-1} \circ \cdots \circ T'_1 u \notin C^{N_0}_{F^m z}((4\varrho)^{-N_0}\delta)\\ &\text{for some } z \in K, u \in X \text{ and a } m_0 \leq m \leq 2m_0 \quad \text{then}\\ &| T'_m \circ T'_{m-1} \circ \cdots \circ T'_1 u | \leq \lambda_0^m | u |.\end{aligned} \tag{5.29}$$

By (5.17) we have

(5.30)
$$\mathcal{C}_i \subset C_z^i(c) \quad \text{and} \quad C_z^i((4\varrho)^{-N_0}\delta) \setminus \{0\} \subset int(\mathcal{C}_i) \quad \text{for all } z \in K \text{ and } 1 \leq i \leq N_0.$$

Hence, (5.26), (5.28) and (5.29) imply the statements j),jj) and jjj), respectively.

Now we find $\mathcal{V}_0$ and $\epsilon_0 > 0$ needed for the statement of the proposition.

Define for any $\mathcal{U} \subset \mathcal{U}_0$ and $G \in C^1(\mathcal{U}, X)$ the family of the operators $\{T'_{(x,y)} : (x,y) \in \mathcal{U} \times \mathcal{U}\}$ as follows

$$T'_{(x,y)} = T_{(x,y)} + (\int_0^1 (DG - DF_0)(sx + (1-s)y)\, ds) \in L(X,X),\ (x,y) \in K.$$

Obviously

$$T'_{(x,y)}(x-y) = Gx - Gy,\ x, y \in \mathcal{U}, \tag{5.31}$$

and if $\mid G - F_0 \mid_{\mathcal{U}} \mid_{C^1(\mathcal{U},X)} < \epsilon$ then

$$\mid T'_{(x,y)} - T_{(x,y)} \mid_{L(X,X)} < \epsilon.$$

Now it is easy to find an open neighborhood $\mathcal{V}_0$ of K_0 and $\epsilon > 0$ such that for any $G \in C^1(\mathcal{V}_0, X)$ with

$$\mid G - F_0 \mid_{\mathcal{V}_0} \mid_{C^1(\mathcal{V}_0,X)} < \epsilon_0$$

we obtain that if $(G^m x, G^m y) \in \mathcal{V}_0 \times \mathcal{V}_0$ is defined for a $(x,y) \in \mathcal{V}_0 \times \mathcal{V}_0$ and a $m_0 \le m \le 2m_0$ then there is a $z \in K$ such that

$$\mid T'_{(G^n x, G^n y)} - T_{F^n z} \mid \le \epsilon_1 \quad \text{for all} \quad 1 \le n \le m.$$

Hence, using (5.31) from statements j),jj) and jjj) we obtain statements i),ii) and iii), respectively, for $m_0 \le m \le 2m_0$. Using repeatedly statements i),iii) for $m = m_0, \ldots, 2m_0$ we obtain these statements for all $m \ge m_0$.

We showed all properties of the sequence of cones $\mathcal{C}_1, \ldots, \mathcal{C}_{N_0}$ except of Proposition 5.1iv). Therefore take $z_0 \in K$. By (5.17) we have

$$\mathcal{C}_{N_0} \subset C_{z_0}^{N_0}(\frac{1}{\delta}).$$

Hence, as P_0 we can take $P_{z_0}^{N_0}$ and $\tilde{c} := \frac{1}{\delta}$. The proof of the proposition is complete.◇

In the proofs of the next corollary and Corollary 2.1 we use the notation from the previous proof. Next corollary states that homeomorphism of ω-limit sets into $\mathbb{R}^d$ from Theorem C is even Lipschitz imbedding of these sets to $\mathbb{R}^d$. Here we say that a map from a subset in a metric space into another metric space is a Lipschitz imbedding iff this map is Lipschitz, injective and its inverse from its range is also Lipschitz map.

Corollary 5.4. *Let the assumption of Theorem C be satisfied. Then there exist an open neighborhood of K_0 denoted by $\mathcal{V}$ and an $\epsilon > 0$ such that for any $G \in C^1(\mathcal{V}, X)$ with*

$$\mid F \mid_{\mathcal{V}} - G \mid_{C^1(\mathcal{V},X)} < \epsilon$$

any projection of X along Π on a transversal d-dimensional space to Π is a Lipschitz imbedding of the $\omega-$limit set for G of any point $x_0 \in \mathcal{V}$, with all $G^n x_0$, $n \in \mathbb{N}$ defined and $cl(\{G^n x_0 : \ n \in \mathbb{N}\} \subset \mathcal{V}$.

Proof. Let Π' be a d-dimensional subspace of X such that $\Pi \oplus \Pi' = X$. Denote by π the projection of X to Π' along Π. We also define the sets

$$\pi_s = \{u \in X : \mid (I - \pi)u \mid \leq s \mid \pi u \mid\}.$$

Note that $\pi_0 = \Pi$. In the proof of Proposition 5.1 we can find δ_1 in (5.18) so small that the following holds

$$\pi_{\delta_1} \cap \mathcal{W}_i(\delta_1) = \{0\} \quad \text{for all } 1 \leq i \leq N_0.$$

Then modifying in the above proof, if necessary, chosen constants δ, m_1, m_0, ϵ_0 and neighborhood $\mathcal{V}_0$, the next stronger version of statement ii) of Proposition 5.1 holds

ii)' if $G^{m_0}x$, $G^{m_0}y$ are defined and $x - y$ as well as $G^{2m_0}x - G^{2m_0}y$ lie in $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ for an $i \in \{1, 2, \ldots, N_0\}$ then $G^{m_0}x - G^{m_0}y \notin \pi_{\delta_1}$.

From the proof of Theorem C we know that either $\omega(x_0)$ is a periodic orbit or each distinct $x, y \in \omega(x_0)$ admits an $1 \leq i \leq N_0$ such that

$$G^n x - G^n y \in \mathcal{C}_i \setminus \mathcal{C}_{i-1} \quad \text{for all } n \in \mathbb{N}. \tag{5.32}$$

Note that it is easy to see using Proposition 5.1i) that if $\omega(x_0)$ is a periodic orbit then (5.32) is satisfied. Thus, from (5.32) using ii)' we obtain that

$$x - y \in \pi_{\delta_1} \quad \text{for all distinct } x, y \in \omega(x_0).$$

This together with $\pi \in L(X, X)$ means that π is a Lipschitz imbedding of $\omega(x_0)$ to d-dimensional space Π'. $\diamondsuit$

Proof of Corollary 2.1. Denote by X^+ and X^- the two open halfspaces of X separated by Π. Recall that for all $1 \leq i \leq N_0$ the bundle $K \times (V_z^i \cap Anih(L_z^{i-1}))$ is invariant under the bundle map (F, T) on $K \times X$ and $(K \times (V_z^i \cap Anih(L_z^{i-1})) \cap \Pi) = K \times (\{0\})$. Hence, by assumption (A) of the corollary we obtain that

$$\text{if } \ u \in V_z^i \cap Anih(L_z^{i-1})) \cap X^{\pm} \quad \text{then} \quad T^n u \in X^{\pm} \text{ for all } 1 \leq n, \ 1 \leq i \leq N_0$$

Hence, choosing in (5.25) $\epsilon_1 > 0$ smaller if necessary, the following stronger version of (5.28) combined with (5.30) takes place

$$\begin{aligned}&\text{if } u \in \mathcal{C}_i \setminus \mathcal{C}_{i-1} \cap X^{\pm} \text{ and } T'_{2m_0} \circ T'_{2m_0-1} \circ \cdots \circ T'_1 u \in \mathcal{C}_i \setminus \mathcal{C}_{i-1} \text{ for an } 1 \leq i \leq N_0 \\ &\text{then } T_{m_0} \circ T_{m_0-1} \circ \cdots \circ T'_1 u \in X^{\pm}.\end{aligned}$$

Hence we obtain that

$$\text{if } u \text{ and } T'_{3m_0} \circ T'_{3m_0-1} \circ \cdots \circ T'_1 u \text{ lie in } \mathcal{C}_i \setminus \mathcal{C}_{i-1} \text{ for an } 1 \leq i \leq N_0$$

then

$$T'_n \circ T'_{n-1} \circ \cdots \circ T'_1 u \in X^+ (\text{or } X^-) \text{ for all } m_0 \leq n \leq 2m_0.$$

Then proceeding as in the proof of Proposition 5.1 we find $\mathcal{V}_0$ and $\epsilon_0 > 0$ such that in addition to statements i)-iv) of Proposition 5.1 we obtain the following statement

v) if $G^{3m_0}x, G^{3m_0}y$ are defined and both $x - y$, $G^{3m_0}x - G^{3m_0}y$ lie in $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ for an $1 \leq i \leq N_0$ then $G^n x - G^n y \in X^+$(or X^-) for all $m_0 \leq n \leq 2m_0$.

Now, take any map G from an ϵ_0-neighborhood in $C^1(\mathcal{V}_0, X)$ of $F_0 \mid_{\mathcal{V}_0}$. By Proposition 5.1i) we have that for any distinct $x, y \in \mathcal{V}_0$ with all $G^n x, G^n y,\ n \geq 1$ defined there is an $n_0 \in \mathbb{N}$ such that
either

$$G^n x - G^n y \notin \mathcal{C}_{N_0} \text{ for all } n \geq n_0 \tag{5.33}$$

or

$$G^n x - G^n y \in \mathcal{C}_i \setminus \mathcal{C}_{i-1} \text{ for all } n \geq n_0 \text{ and an } i \in \{1, 2, \ldots N_0\}. \tag{5.34}$$

In case (5.33) Proposition 5.1iii) implies that $G^n x - G^n y$ converges exponentially to $\{0\}$. In case (5.34) using repeated applying of v) we obtain that the sequence $G^n x - G^n y,\ n = 1, 2, \ldots$ eventually lie in one of the halfspaces X^+ or X^-.

In order to show the second part of the corollary take any $x_0 \in \mathcal{V}$, with all $G^n x_0,\ n \in \mathbb{N}$ defined and $cl(\{G^n x_0 :\ n \in \mathbb{N}\}) \subset \mathcal{V}$. Then by Corollary 5.2 we know that the set $\{G^n x_0 :\ n \in \mathbb{N}\}$ is precompact. Thus $\omega(x_0)$ is nonempty. If $x_0 = Gx_0$ then we are done. Thus let $Gx_0 \neq x_0$. Then apply the first part of the corollary to $x = x_0$ and $y = Gx_0$. Hence, we obtain that $\mid G^n x_0 - G^{n+1} x_0 \mid$ exponentially tends to 0 or the sequence $G^n x_0 - G^{n+1} x_0$ eventually lie in one of the halfspaces X^+ (or X^-). The former possibility clearly imply that $G^n x_0$ converges to a point as $n \to \infty$. The latter one yields for all sufficiently large $n < m$ that $G^n x_0 - G^m x_0$ lie in X^+ (or X^-). Hence, $\omega(x_0)$ is a single point. The proof is complete. $\diamondsuit$

6. Proofs of Corollaries 2.2-2.4.

Proof of Corollary 2.2: By Theorem B we have that the vector bundle (S_1, T^1) admits a $k-$dimensional continuous separation along K associated to the $k-$cone C. For $x \in X$ let $V_x \subset X$, $L_x \subset X^*$ be the subspaces of this separation. Let M and γ be constants from the exponential separation for (S_1, T^1). Using Lemma 4.1 we obtain that the bundles $K \times (V_x)$ and $K \times (L_x)$ are invariant under the bundle maps (S_t, T^t) for all rational $t > 0$. Hence by the continuity of the function $t \longrightarrow T_x^t : (0, +\infty) \to L(X, X)$ for all $x \in K$ we obtain that the equalities

$$T_x^t V_x = V_{S_t x}, \quad T_x^{*t} L_{S_t x} = L_x, \quad x \in K$$

hold also for all irrational $t > 0$.

It remains to prove the exponential separation. Denote $c_1 = sup\{| T_x^t |_{L(X,X)}: x \in K, t \in [1, 2]\}$. Since each V_x is a $k-$dimensional space and $K \times (V_x)$ is an invariant bundle for all bundle maps (S_t, T^t), for each $x \in K$ and $t \geq 1$ there is a constant c_2 such that

$$| v | \leq c_2 \mid T_x^t v \mid \quad \text{for all } v \in V_x.$$

Since $K \times (V_x)$ is a continuous bundle and the map $(x, t) \mapsto T_x^t : K \times (0, +\infty) \to L(X, X)$ is continuous we can make the constant c_2 independent on $x \in K$ and $t \in [1, 2]$.

Denote $[t]$ the integer part of t and $\{t\} := t - [t]$. Using the exponential estimate for (S_1, T^1) we obtain that for all $x \in K$, $w \in Anih(L_x)$, $v \in V_x$, $t \geq 1$ one has

$$| T_x^t w |=| T_x^{[t-1]+1+\{t\}} w | \leq c_1 M \gamma^{[t]-1} \mid T_x^{[t]-1} v \mid$$

$$\leq c_1 M \gamma^{[t]-1} c_2 \mid T_x^{[t]+\{t\}} | \leq (c_1 c_2 M \gamma^{-2}) \gamma^t \mid T_x^t v \mid .$$

This verifies the needed exponential estimate and completes the proof of Corollary 2.2.$\diamondsuit$

For the proof of the next two corollaries we need the following semiflow analog of Proposition 5.1. Recall also the definition of sets π_s. Let Π' be a linear subspace of X transversal to Π and let π be the projection of X to this space along Π. Then we have sets

$$\pi_s = \{u \in X : | (I - \pi) u | \leq s \mid \pi u |\}, \quad s \geq 0.$$

Proposition 6.1. *Suppose that the assumptions of Corollary 2.3 are satisfied. Then there are numbers $T_0 \in \mathbb{R}$, $N_0 \in \mathbb{N}$, $0 \leq \lambda < 1$, $\delta_1 > 0$, $\epsilon_0 > 0$, an bounded open neighborhood $\mathcal{V}_0$ of K_0 and a sequence of sets $\mathcal{C}_1 \subset int(\mathcal{C}_2) \cup \{0\} \subset \mathcal{C}_2 \subset \ldots \mathcal{C}_{N_0-1} \subset int(\mathcal{C}_{N_0}) \cup \{0\} \subset \mathcal{C}_{N_0}$ with each $\mathcal{C}_i$ a k_i-cone such that for any continuous semiflow S'_t on $\mathcal{V}_0$ with $S'_t \in C^1(\mathcal{V}_0, X)$, $t \in [\frac{1}{2}, 1]$ and*

$$| S_t |_{\mathcal{V}_0} - S'_t |_{C^1(\mathcal{V}_0, X)} < \epsilon \quad \textit{for all } \frac{1}{2} \leq t \leq 1$$

and $x, y \in \mathcal{V}_0$ the following holds *i) if $x-y \in \mathcal{C}_i \setminus \{0\}$ for an $1 \leq i \leq N_0$ and $S'_t x, S'_t y$ are defined for a $t \geq T_0$ then $S'_t x - S'_t y \in int(\mathcal{C}_i)$.*

ii) if $S'_{2T_0}x,\ S'_{2T_0}y$ are defined and $x-y$ as well as $S'_{2T_0}x - S'_{2T_0}y$ lie in $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ for an $i \in \{1,2,\ldots,N_0\}$ then $S'_{T_0}x - S'_{T_0}y \notin \pi_{\delta_1}$. (Here $\mathcal{C}_0 = \{0\}$.)

iii) if $S'_t x, S'_t y$ are defined with $S'_t x - S'_t y \notin \mathcal{C}_{N_0}$ and $t \geq T_0$ then

$$| S'_t x - S'_t y | \leq \lambda^t \mid x-y \mid . \tag{6.1}$$

Moreover we have that

iv) there is a projection P_0 of X on k_{N_0}-dimensional space and a constant $\tilde{c} > 0$ such that

$$\mathcal{C}_{N_0} \subset \{v \in X : \mid (I-P_0)v \mid \leq \tilde{c} \mid P_0 v \mid\}.$$

Proof. The proposition follows from Corollary 2.2 by the same way as Proposition 5.1 was proved using Theorem B with just one exception. Here statement ii) is stronger then that of Proposition 5.1. Its proof is similar to the proof of ii)' in the proof of Corollary 5.4. ◇

Having the above statement, Corollaries 2.3 and 2.4 do not require special proofs. Let us briefly discuss it. Corollary 2.3 follows from this proposition and a semiflow analog of Theorem A similarly as Theorem C was proved using Theorem A. The semiflow analog of Theorem A could be proved by replacing discrete maps G by semiflows. Proposition 6.1 provide sufficient information to produce more general abstract zero number, as it is introduced in [FM-P], which does not change arguing of the proof of a Poincaré-Bendixson Theorem in [FM-P]. We give here proofs of these corollaries based on Proposition 6.1 and Theorem C. In the proof of Corollary 2.4 we use also the conclusion of Corollary 2.3. Note that from the next two proofs and a semiflow analog of Theorem A, one can reconstruct a proof of Poincaré - Bendixson Theorem in the unperturbed case of Corollary 2.4, not using Theorems B and C.

Proof of Corollary 2.3 Take any continuous semiflow S'_t satisfying (6.1) and any $x_0 \in \mathcal{V}_0$ for which the closure of its positive orbit lies entirely in $\mathcal{V}_0$. The semiorbit of x_0 with respect to the semiflow S'_t is precompact. Indeed, this is a continous time analog of the precompactness of semiorbits for maps proved in Corollary 5.2. Namely, it follows from two facts contained in Proposition 6.1. The first one is the exponential contraction of the difference of two points on a trajectory when time increases until their difference does not lie in $\mathcal{C}_{N_0}$ (see Proposition 6.1iii)). Due to Proposition 6.1iv) the second fact says that P_0 is a Lipschitz imbedding to finite dimensional space $P_0(X)$ of any sets for which the difference of any two points lie in C_{N_0} . The detailed proof follows the lines of the proof of Corollary 5.2 proved by Proposition 5.1iii)-iv).

Note that Proposition 6.1 implies that the map S'_t for any $t \in [\frac{1}{2}, 1]$ satisfies the conclusions of Proposition 5.1 for a $m_0 > \frac{T_0}{t}$. Since Theorem C is the consequence of Proposition 5.1 and Theorem A, we conclude that the ω-limit set of x_0 with respect to any $S'_t,\ t \in [\frac{1}{2}, 1]$ is homeomorphic to a subset in $\mathbb{R}^d$. Hence, the corollary follows from the next lemma. This lemma is proved in [Ak] (Ch.6, Proposition 3a)). Nevertheless we give here its proof for the convenience of the reader.

Lemma 6.1. *Let $\mathcal{X}$ be a compact metric space and S_t a continuous semiflow on $\mathcal{X}$. Then for any $x_0 \in \mathcal{X}$ and $0 < \tau < 1$ there is a $t_0 \in [\tau, 1]$ such that the $\omega-$limit set of x_0 for the semiflow S_t coincides with the $\omega-$limit set of x_0 for the map S_{t_0}.*

Proof of Lemma 6.1. Denote the $\omega-$limit set of x_0 for the semiflow S_t by $\omega(x_0)$ and and for the map $S_\tau, \ \tau > 0$ by $\omega(S_\tau, x_0)$. Since $\mathcal{X}$ is compact, so is the set $\omega(x_0)$. Hence, there is a countable dense subset $\{x_n, \ n \in \mathbb{N}\}$ in $\omega(x_0)$. For each $x_n, \ n = 1, 2, \ldots$ there is a sequence of positive numbers $t_i^n, \ i = 1, 2, \ldots$ such that

$$\lim_{i\to\infty} t_i^n = 0 \quad \text{and} \quad \lim_{i\to\infty} S_{t_i^n} x_0 = x_n. \tag{6.2}$$

We show that the following set

$$\mathcal{G} = \{\tau \in \mathbb{R}^+ : \ \omega(x_0) = \omega(S_\tau, x_0)\}$$

is residual in $\mathbb{R}^+$. Define the sets

$$\mathcal{G}_{nN\epsilon} := \{\tau : \text{ there are } i > N \text{ and } l > N \text{ such that } \mid l\tau - t_i^n \mid < \epsilon\}.$$

We claim that the following set

$$\mathcal{G}' := \bigcap_{n\in\mathbb{N}} \bigcap_{N\in\mathbb{N}} \bigcap_{\epsilon=\frac{1}{2},\frac{1}{3},\ldots} G_{nN\epsilon}$$

is a subset of $\mathcal{G}$. Let us show it. Take any $\tau \in \mathcal{G}'$. Then for all n there are increasing sequences $i_k, \ l_k, \ k = 1, 2, \ldots$ such that $\mid l_k\tau - t_{i_k}^n \mid \to 0$ as $k \to \infty$. Then by the uniform continuity of the map $t \mapsto S_t x_0 : [0, +\infty) \to X$ we have

$$\mid S_{l_k\tau} x_0 - S_{t_{i_k}^n} x_0 \mid = \mid S_{|l_k\tau - t_{i_k}^n|} y_k - y_k \mid \to 0 \quad \text{ as } k \to \infty,$$

where $y_k = S_{t_{i_k}^n} x_0$. Hence by (6.2) we have $S_{l_k\tau} x_0 \to x_n$ as $k \to +\infty$. Thus, we conclude that

$$\omega(x_0) = cl\{x_n : \ n \in \mathbb{N}\} \subset cl(\omega(S_\tau, x_0)) = \omega(S_\tau, x_0) \subset \omega(x_0),$$

i.e. $\omega(x_0) = \omega(S_\tau, x_0)$.

We show that each $\mathcal{G}_{nN\epsilon}$ is open and dense in $\mathbb{R}^+$ implying that $\mathcal{G}$ is a residual subset of $\mathbb{R}^+$. The openness of the set $\mathcal{G}_{nN\epsilon}$ is clear. To prove its density consider any interval $[\tau_1, \tau_2]$ with $0 < \tau_1 < \tau_2$. Then there is a $T > 0$ such that

$$[T, \infty) \subset \bigcup_{l>N} l[\tau_1, \tau_2] \tag{6.3}$$

implying that $[\tau_1, \tau_2]$ contains a point from $\mathcal{G}_{nN\epsilon}$. The inclusion (6.3) follows from the fact that

$$l\tau_1 < (l+1)\tau_1 < l\tau_2 < (l+1)\tau_2 \quad \text{for all sufficiently large } l$$

yielding in particular that for those l we have that the set $l[\tau_1,\tau_2]\cup(l+1)[\tau_1,\tau_2]$ is an interval. The lemma is proved. $\diamondsuit$

Proof of Corollary 2.4. We use here Proposition 6.1 for d=2. Take $\epsilon=\epsilon_0$, $\mathcal{V}=\mathcal{V}_0$. Further take any continuous semiflow S'_t satisfying (6.1) and any $x_0\in\mathcal{V}$ for which the closure of its positive orbit lies entirely in $\mathcal{V}$. Recall from the proof of Corollary 2.4 that the semiorbit of x_0 is precompact and the ω-limit set of x_0 coincide with the ω-limit set of x_0 with respect to a map S'_{t_0} for a $t_0\in[\frac{1}{2},1]$. By Proposition 5.1 we can use Corollary 5.4 for S'_{t_0} and x_0. Hence, π is a Lipschitz imbedding of $\omega(x_0)$ to $\mathbb{R}^2$. Moreover, recall that the discrete Ljapunov functional for S'_{t_0} on $cl(\{{S'_{t_0}}^n x_0:\ n\in\mathbb{N}\})$ given by Corollary 5.1 is constant and not greater than N_0 on all sequences ${S'_{t_0}}^n x-{S'_{t_0}}^n y$, $n\in\mathbb{Z}$ where x,y are any distinct points from $\omega(x_0)$. Since $\omega(x_0)$ is connected this implies by Proposition 5.1i) applied to the map S'_{t_0} that there is an $i\in\{1,2,\dots N_0\}$ such that

$$x-y \ \text{ lies in the open set } int(\mathcal{C}_i)\setminus\mathcal{C}_{i-1} \quad \text{for all distinct } x,y\in\omega(x_0). \tag{6.4}$$

Hence, using statement j) from the proof of Proposition 5.1 applied to $T'_n={DS'_{t_0}}^n(x)$, $n=1,2,\dots,m_0$, $x\in\omega(x_0)$ we obtain also that

(6.5)
$$\text{if } \tfrac{d}{dt}S'_t x\,|_{t=0}\neq 0 \quad\text{then}\quad \tfrac{d}{dt}S'_t x\,|_{t=0}={DS'_{t_0}}^m\tfrac{d}{dt}S'_t\,|_{t=-m_0t_0}\in int(\mathcal{C}_i)\setminus\mathcal{C}_{i-1}\quad\text{for all } x\in\omega(x_0).$$

Note also that the map S'_{t_0} is injective on $\omega(x_0)$ (cf. Corollary 5.3) implying that if $y\in\omega(x_0)$ and its semiorbit is not an equilibrium or periodic orbit then the map $t\mapsto S'_t y:(0,+\infty)\to X$ is injective.

Lemma 6.2. *The ω-limit set of x_0 with respect to the semiflow S'_t is a single periodic orbit or a subset of equilibria or there is a $T_1>0$ such that the map π is Lipschitz imbedding of the set $cl(\{S'_t x_0:\ t\geq T_1\}$ to $\mathbb{R}^2$.*

Before we give the proof of the above lemma we complete the proof of the corollary. If $\omega(x_0)$ is a single periodic orbit or a subset of equilibria we are done. If $\omega(x_0)$ is not a set of equilibria or a single periodic orbit we can apply Lemma 6.2. Thus we have $T_1>0$ such that the map $\pi:\ cl(\{S'_t x_0:\ t\geq T_1\}\to\mathbb{R}^2$ is a Lipschitz imbedding. Since the flow S'_t is regularizing, we obtain a continuous vector field $g(y)$ on the set $\pi(cl(\{S'_t x_0:\ t\geq T_1\})\subset\mathbb{R}^2$ by taking

$$g(y):=\pi(\tfrac{d}{dt}S'_t y)=\tfrac{d}{dt}\pi(S'_t y)\quad \text{for all } y\in cl(\{S'_t x_0:\ t\geq T_1\}.$$

Therefore we can apply to the semiflow $\pi\circ S'_t$ on $\pi(cl(\{S'_t x_0:\ t\geq T_1\})$ the standard Poincaré-Bendixson Theorem in the plane. Indeed, the proof of this theorem uses only the fact that given semiflow on the closure of an orbit in the plane has the regularizing property (see also the proof of Proposition 1 in [FM-P]). Hence, since π is a homeomorphism from $\omega(x_0)$ onto $\pi(\omega(x_0))$ the PB property of x_0 for the semiflow S'_t is proved.$\diamondsuit$

Proof of Lemma 6.2. First we prove that if $\omega(x_0)$ is not a set of periodic orbits then there is a $T_2>0$ such that

(6.6) $\qquad x-y\ \in int(\mathcal{C}_i)\setminus\mathcal{C}_{i-1}\quad$ for all distinct $x,y\in cl(\{S_t'x_0:\ t\geq T_2\}$.

We consider any equilibrium also as periodic orbit.

In order to prove this statement, suppose on the contradiction that $\omega(x_0)$ is not a of periodic orbits and (6.6) is not satisfied for any T_2. Then by (6.4) we may assume that there are sequences $t_n,\ \delta_n>0,\ n\in\mathbb{N}$ such that $t_n\to+\infty$ as $n\to\infty$ and

either

a) $S_{t_n}'x_0-S_{t_n+\delta_n}'x_0\notin int(\mathcal{C}_i)$ for all $n\in\mathbb{N}$,

or

b) $S_{t_n}'x_0-S_{t_n+\delta_n}'x_0\in\mathcal{C}_{i-1}$ for all $n\in\mathbb{N}$.

In case b) Proposition 6.1 and $S_{t_1}'x_0-S_{t_1+\delta_1}x_0\in\mathcal{C}_{i-1}$ imply that $S_t'x_0-S_{t+\delta_1}x_0\in\mathcal{C}_{i-1}$ for all $t>t_1+T_0$. Due to (6.4) and the closedness of C_{i-1} we obtain that $y-S_{\delta_1}y==$ for all $y\in\omega(x_0)$. This contradicts the asumption on $\omega(x_0)$ not to be a set of periodic orbits.

In case a) we may suppose that the sequence δ_n has a limit s in $\mathbb{R}\cup\{+\infty\}$. To prove that a) leads to the contradiction we distinguish three cases $s=0$, $0<s<\infty$ and $s=+\infty$

1. $s=0$. Since $\omega(x_0)$ is not a subset of equilibria there is a $y\in\omega(x_0)$ such that the derivative of the trajectory through y is nonzero, i.e. $\frac{d}{dt}S_t'y\mid_{t=0}\neq 0$. Since $y\in\omega(x_0)$, we can pass to the subsequence, if necessary, such that there is a sequence of positive numbers $s_n<t_n-T_0,\ n=1,2,\ldots$ with the property $S_{s_n}x_0\to y$ as $n\to\infty$. By Proposition 6.1i) we also have that $S_{s_n+\delta_n}'x_0-S_{s_n}'x_0\notin int(\mathcal{C}_i)$ for all $n\in\mathbb{N}$. Hence, by the regularizing property of S_t' we have

$$\lim_{n\to\infty}\frac{S_{s_n+\delta_n}'x_0-S_{s_n}'x_0}{\delta_n}=\frac{d}{dt}S_t'y\mid_{t=0}\notin int(\mathcal{C}_i).$$

This contradicts (6.5) and the existence of the nonzero derivation $\frac{d}{dt}S_t'y\mid_{t=0}$.

2. $0<s<+\infty$. Since $\omega(x_0)$ is not a set of periodic orbits there is an $y\in\omega(x_0)$ such that the map $t\mapsto S_t'y:[0,+\infty)\to X$ is injective. Then we can pass to a subsequence $s_n,\ n=1,2,\ldots$, if necessary, such that $s_n<t_n-T_0$ for all $n\in\mathbb{N}$ and $S_{s_n}'x_0$ converges to y as $n\to\infty$. Hence, by Proposition 6.1i) we have that

$$\lim_{n\to\infty}(S_{s_n}'x_0-S_{s_n+\delta_n}'x_0)=y-S_sy\notin int(\mathcal{C}_i).$$

This contradicts (6.4).

3. $s=+\infty$. By Proposition 6.1i) we may suppose that all $x_0-S_{\delta_n}'x_0,\ n=1,2,\ldots$ does not lie in $int(\mathcal{C}_i)$ and moreover, passing to a subsequence if necessary, $S_{\delta_n}'x_0$ converge to a $y\in\omega(x_0)$ as $n\to\infty$. Since $t_n\to\infty$ Proposition 6.1i) implies also that $S_t'x_0-S_{\delta_n+t}'x_0$ do not lie in $int(\mathcal{C}_i)$ for all sufficiently large n. Hence $S_t'x_0-S_t'y\notin int(\mathcal{C}_i)$ for all $t\geq 0$. Thus using (6.4) and precompactnes of the semiorbits of x_0 and of y we obtain that

$$\lim_{t\to+\infty}(S_t'x_0-S_t'y)=0.$$

This means that $\omega(x_0) = \omega(y)$. Since $\omega(x_0)$ is not a single periodic orbit the map $t \mapsto S'_t y : [0, +\infty) \to X$ is injective. We know that π is a Lipschitz imbedding of $\omega(x_0)$ as well as $cl(\{S'_t y :\ t \geq 0\}$ to R^2. Hence, the map $t \mapsto \pi(S'_t y) : [0, +\infty) \to R^2$ is also injective. Therefore πy does not lie in the ω-limit set of πy with respect to the semiflow $\pi \circ S'_t$ given by the vector field $\pi(\frac{d}{dt} S'_t y)$ on the set $cl(\{S'_t y :\ t \geq 0\}$. Since $\pi \mid_{\omega(x_0)}$ is a homeomorphism from $\omega(x_0)$ to its image we have $y \notin \omega(y)$ contradicting $y \in \omega(x_0) = \omega(y)$.

Thus the lemma is proved if $\omega(x_0)$ containes at least one nonperiodic orbit. To finish the proof of the lemma it is sufficient to show that $\omega(x_0)$ consisting of periodic orbits not all being equilibria is a single periodic orbit. Therefore suppose on the contrary that $\omega(x_0)$ containes at least one nontrivial periodic orbit and no nonperiodic orbits. Then due to connectivity of $\omega(x_0)$ and continuity of the function $x \mapsto S'_t x : \omega(x_0) \to \omega(x_0)$ we obtain infinitely many different periodic orbits not being eqilibria in $\omega(x_0)$. Hence, the semiflow $\pi \circ S'_t$ defined on $\pi(\omega(x_0)) \subset \mathbb{R}^2$ containes theree nontrivial periodic orbits p_1, p_2, p_3 such that the orbits p_2 and p_3 lie in different components of $\mathbb{R}^2 \setminus p_1$. This contradicts the fact that $\pi(\omega(x_0)) \setminus p_1$ is contained only in one of the components of $\mathbb{R}^2 \setminus p_1$. To show this fact consider a point $y \in \omega(x_0)$ not lying on the periodic orbit $p \in \omega(x_0)$ with $\pi(p) = p_1$. Since $y \in \omega(x_0)$ there ie an sequence $t_n,\ n = 1, 2,$ with $t_{n+1} - t_n > T_0,\ n \in BbbN$ and $S_{t_n} x_0 \to y$ as $n \to \infty$. Moreover, because of (6.4) we may suppose that

$$S'_{t_n} x_0 - z \in int(\mathcal{C}_i) \setminus \mathcal{C}_{i-1} \quad \text{for all } z \in p \text{ and } n \in BbbN.$$

Hence

$$S'_{t_m} x_0 - S'_{t_m - t_n} z \in int(\mathcal{C}_i) \setminus \mathcal{C}_{i-1} \quad \text{for all } M > n,$$

Since $t_m \to \infty$ as $m \to \infty$, Proposition 6.1i) and ii) gives

$$S'_t x_0 - z \notin \Pi \quad \text{for all } z \in p \text{ and } t > T_0.$$

Therefore connected sets $\pi(p)$ and $\pi(\{S'_t x_0 : t > T_0\})$ do not intersect. This proves the fact above completing the proof of the lemma. $\diamondsuit$

7. Proofs of Theorems 1.1 and 1.2.

We prove Theorem 1.1 and 1.2 using Theorem C for $d = 2$ and Corollary 2.4, respectively.

Throughout this section the dot in $u(t,\cdot)$ represents the space variable ξ. Let $u(t,\cdot),\ v(t,\cdot),\ t \geq 0$ be two global solutions in the sense of [He1] of the equation (1.1). Then their difference $w(t,\cdot) := u(t,\cdot) - v(t,\cdot)$ satisfies the following equation

$$w_t = w_{\xi\xi} + a(t,\xi)w + b(t,\xi)w_\xi, \quad t > 0,\ \xi \in S^1 \tag{7.1}$$

with

$$a(t,\xi) = \int_0^1 \frac{\partial}{\partial u} f(t, su(t,\xi) + (1-s)v(t,\xi), su_\xi(t,\xi) + (1-s)v_\xi(t,\xi))\, ds$$

$$b(t,\xi) = \int_0^1 \frac{\partial}{\partial p} f(t, su(t,\xi) + (1-s)v(t,\xi), su_\xi(t,\xi) + (1-s)v_\xi(t,\xi))\, ds$$

The derivation $u_\xi(t,\cdot)$ satisfy (7.1) with $u(t,\cdot) = v(t,\cdot)$.

Now we review two consequences of results in [He1] and [An2] for the equation (7.1).

By [He1] we obtain that equation (7.1) generates the family of compact operators $T(t,s) \in L(X,X),\ t > s \geq 0$ with the following properties:

a) if $w(t,\cdot)$ is any solution of (7.1) then

$$w(t,\cdot) = T(t,s)w(s,\cdot),\ t > s \geq 0$$

b) $T(t,s)T(s,r) = T(t,r)$ for all $t > s > r \geq 0$.
c) $\lim_{t\to s} T(t,0) = T(s,0)$ in $L(X,X)$ for all $s > 0$.
d) for any fixed t and s the operators $T(t,s)$ are change continuously in $L(X,X)$ when $u(0,\cdot)$ and $v(0,\cdot)$ are varied in X.

Let $z(u)$ denote the number, possibly infinite, of zeros of a function $u \in X$. Fix an arbitrary point $\xi_0 \in S^1$. Denote by π the following map

$$u(\cdot) \longrightarrow (u(\xi_0), u_\xi(\xi_0)) : X \to \mathbb{R}^2.$$

This map is linear and continuous since $X \hookrightarrow C^1(S^1)$.

Now let $w(t,\cdot),\ t > 0$ be a global solution of (7.1) with $a(\cdot,\cdot),\ b(\cdot,\cdot)$ being C^1 on $(0,\infty) \times S^1$. Then using [An2] we obtain that

α) if $w(0,\cdot) \neq 0$ then $z(w(t,\cdot))$ is finite for each $t > 0$.
β) $z(w(t,\cdot)) \leq z(w(s,\cdot)$ for any $t > s \geq 0$.
γ) if $w(t_0,\cdot)$ has a multiple zero for a $t_0 > 0$ then

$$z(w(t_0 - \epsilon,\cdot)) \geq z(w(t_0 + \epsilon,\cdot)) + 2$$

for any $0 < \epsilon < t_0$.

The last statement γ), in other words says that at a multiple zero of a solution the zero number of the solution decrease at least by two, it is the consequence of Theorem B in [An2] and the fact that each $\xi \in S^1$ is an interior point of S^1.

We also use the following consequence of α) and γ)

δ) if $w(0, \cdot) \neq 0$ then for any $s > 0$ there are only finitely many time instances $t > s$ such that $w(t, \cdot)$ has a multiple zero on S^1.

Let B be the ball from the assumptions of Theorems 1.1 and 1.2. It is well known (see [He1]) that the semiflow S_t defined by (1.1), where the nonlinearity is independent of t, is a continuous regularizing semiflow on B. If the nonlinearity of (1.1) is 1-periodic in t then the time-one map F defined by (1.1) lies in $C^1(B, X)$.

Now let x and y be from B. Then by the assumptions of Theorems 1.1 and 1.2 there are global solutions of (1.1) with $u(0, \cdot) = x$ and $v(0, \cdot) = y$. Denote by $T^t_{(x,y)}$ the operator $T(t, 0)$ corresponding to the solutions $u(t, \cdot),\ v(t, \cdot)$. Then using a)-d) we obtain that

1) if f does not depend on t then

$$S_t x - S_t y = T^t_{(x,y)}(x - y),\ x, y \in B,\ t > 0$$

and if f is 1-periodic in t then

$$Fx - Fy = T^1_{(x,y)}(x - y), \quad x, y \in B$$

2) the function $(x, y, t) \longrightarrow T^t_{(x,y)} : B \times B \times (0, \infty) \to L(X, X)$ is continuous with the range in the set of compact operators; and if f does not depend on t, for any $x, y \in B,\ t, s > 0$ we have

$$T^{t+s}_{(x,y)} = T^t_{(S_s x, S_s y)} T^s_{(x,y)}$$

Now we define a family of k_i-cones as follows

$$C_i := cl\{u(\cdot) \in X :\ u(\cdot) \ \text{ has only simple zeros and } \ z(u(\cdot)) \leq 2i\}$$

We also put $C_0 = \{0\}$.

Take any $i \in \mathbb{N}$. Let us show that C_i is $(2i + 1)$-cone.

First identify any function $w(\cdot) \in X$ with a 2π-periodic function $w(\cdot) \in C^1(\mathbb{R})$. Then functions $u_n(t, \xi) = e^{-n^2 t} cos\, n\xi,\ v_n(t, \xi) = e^{-n^2 t} sin\, n\xi$ solve the equation

$$w_t = w_{\xi\xi}, \quad \xi \in [0, 2\pi],\ t \in \mathbb{R} \tag{7.2}$$

$$w(t, 0) = w(t, 2\pi)$$

$$w_\xi(t, 0) = w_\xi(t, 2\pi)$$

Now define the $2i + 1$-dimensional subspace

$$V_0 = span\{u_l(0, \cdot), v_l(0, \cdot) :\ 0 \leq l \leq i\}$$

and the $2i+1$-codimensional subspace of X

$$L_0 := cl(span\{u_l(0,\cdot), v_l(0,\cdot) :\ l > i\}).$$

We have $V_0 \subset C_i$ and $L_0 \cap C_i = \{0\}$. Indeed, for any $w(\cdot) \in X$ we can write its Fourier series as follows

$$w(\cdot) = a_0 + \sum_{n=1}^{\infty} a_n u_n(0,\cdot) + b_n v_n(0,\cdot).$$

If $w(\cdot) \in V_0$ is nonconstant there is the last nonzero pair of numbers among (a_n, b_n), $n = 1, 2, \ldots$, say for $n = m$. Obviously $m \le i$. Then the function $w(t,\xi) = a_0 + \sum_1^m a_l e^{-l^2 t} cos\, l\xi + b_l e^{-l^2 t} sin\, l\xi$ is the solution of (7.2) with the properties

$$\text{(7.3)} \qquad w(0,\cdot) = w(\cdot) \ \text{ and } \lim_{t\to-\infty} e^{m^2 t} w(t,\cdot) = a_m cos\, m\xi + b_m sin\, m\xi$$

Since $a_m cos\, m\xi + b_m sin\, m\xi = c \cdot sin\,(m\xi + d)$ for some $c \neq 0,\ d \in \mathbb{R}$, this function has exactly $2m$ zeros and all of them simple. Hence by (7.3) we have $z(w(t,\cdot)) = 2m$ for all $(-t)$ sufficiently large. Thus by property β) above, the zero number of $w(\cdot)$ is not greater than $2m$. Since $w(t,\cdot) \to w(0,\cdot)$ when $t \to 0$, by β) and δ) we have $w(\cdot) \in C_i$.

If $0 \neq w(\cdot) \in L_0$ then there is the first nonzero pair among (a_n, b_n), $n = 1, 2, \ldots$, denoted again by (a_m, b_m), with $m > i$. Let $w(t,\cdot)$, $t \ge 0$ be the solution of (7.3) with $w(0,\cdot) = w(\cdot)$ and suppose, on the contrary, that $w(\cdot) \in C_i$. Then by δ) and β) we have

$$\text{(7.4)} \qquad z(w(t,\cdot)) \le 2i \ \text{ for all } \ t > 0$$

(see also statement 4) below). But

$$\lim_{t\to\infty} e^{m^2 t} w(t,\cdot) = a_m u_m(0,\cdot) + b_m v_m(0,\cdot)$$

implying that

$$z(w(t,\cdot)) = 2m > 2i$$

for all t sufficiently large. This contradicts (7.4).

Since also C_i is closed and $\lambda C_i = C_i$ for any $\lambda \in \mathbb{R} \setminus \{0\}$ we conclude that C_i is $2i+1$-cone.

Now, by α) we have that

3) for any $w \in X \setminus \{0\}$, $t > 0$, $(x,y) \in B \times B$

$$T^t_{(x,y)} w \in C_j$$

for a $j \in \mathbb{N}$.

Next we prove the following statement

4) for any $(x,y) \in B \times B,\ t > 0,\ i \in \mathbb{N}$ and $w \in C_i \setminus \{0\}$ there is a neighborhood $\mathcal{W}$ of w such that

$$T^t_{(x,y)} w' \in C_i \setminus \{0\} \ \text{ for all } \ w' \in \mathcal{W}.$$

From δ) we obtain a $t_0 \in (0,t)$ such that $T^{t_0}_{(x,y)} w$ has only simple zeros. Since $w \in C_i$ there are functions $w_j \in X,\ j = 1,2,\dots$ each with at most $2i$ zeros such that $w_j \to w$ as $j \to \infty$. By β) each function $T^{t_0}_{(x,y)} w_j$ has at most $2i$ zeros. Hence, since $T^{t_0}_{(x,y)} w_j$ converges to the function $T^{t_0}_{(x,y)} w$ with only simple zeros, this latter function has at most $2i$ zeros. Since these zeros are all simple, there is an open neighborhood $\mathcal{W}$ of w such that all functions from the set $T^{t_0}_{(x,y)}(\mathcal{W})$ has the same number of zeros. So by β) we obtain that $T^{t_0}_{(x,y)}(\mathcal{W})$ is contained in C_i.

Denote $\Pi = \pi^{-1}(0)$. Note that Π consists of functions that have a multiple zero at ξ_0. Since $\pi \in L(X, \mathbb{R}^2)$ is surjective Π is 2-codimensional subspace in X. We prove that γ) and δ) imply

5) if $T^{t_0}_{(x,y)} w \in (C_i \setminus C_{i-1}) \cap \Pi$ for $t_0 > 0,\ (x,y) \in B \times B,\ w \in X,\ i \in \mathbb{N}$ then $w \notin C_i$.

Indeed, suppose on the contrary that $w \in C_i$. Hence, since $T^{t_0}_{(x,y)} w$ lies in an open set $X \setminus C_{i-1}$, we obtain using δ), that there is an arbitrarily small $\eta > 0$ such that $T^{t_0+\eta}_{(x,y)} w$ has exactly $2i$ zeros, all simple. Therefore by γ) we have that $T^{t_0-\eta}_{(x,y)} w$ has at least $2i+2$ zeros. This together with δ) implies that there is an $0 < \eta_1 < t_0$ such that the function $T^{t_0-\eta_1}_{(x,y)} w$ has at least $2i+2$ zeros and all simple, and consequently does not lie in C_i. This contradicts β) and $w \in C_i$.

As K_0 in Theorem C and Corrolary 2.4 we take

$$\mathcal{A}_1 = \bigcap_{N \in \mathbb{N}} cl(\bigcup_{n \geq N} F^n(B)) \quad \text{and} \quad \mathcal{A}_2 = \bigcap_{T > 0} cl(\bigcup_{t \geq T} S_t(B)),$$

respectively. These sets are attractors for dynamical systems F and S_t, respectively (see [Ha]). Obviously

$$\mathcal{A}_1 \subset cl(F^N(B)) \subset cl(F^{N_0}(B)),\ N \geq N_0,$$

$$\mathcal{A}_2 \subset cl(S_T(B)) \subset cl(S_{T_0}(B)),\ T \geq T_0,$$

where N_0 and T_0 are as in the dissipativity assumption (D) of Theorems 1.1 and 1.2. Hence, since the maps F and $S_t,\ t > 0$ are compact by [He1], the sets $\mathcal{A}_1$ and $\mathcal{A}_2$ are compact invariant subsets in B for F and S_t, respectively.

Now it is easy to check that statements 1)-5) above together with the above choice of $X,\ S_t,\ F,\ \mathcal{U}_0 := B,\ \Pi,\ K_0$ and $C_i, i \in \mathbb{N}$ satisfy the assumptions of Theorem C with $d = 2$ and Corollary 2.4. Therefore there are open neighborhoods $\mathcal{V}_i$ of $\mathcal{A}_i$ and $\epsilon_i > 0,\ i = 1,2$ such that for any map $G \in C^1(\mathcal{V}_1, X)$ and any regularizing continuous semiflow $S'_t \in C^1(\mathcal{V}_2, X),\ t > 0$ with

$$(7.5) \qquad |\, G - F \,|_{C^1(B,X)} < \epsilon_1 \quad \text{and} \quad \sup_{t \in [\frac{1}{2},1]} |\, S'_t - S_t \,|_{C^1(B,X)} < \epsilon_2$$

the ω-limit set of any point $x_0 \in B$ for the map G is embedded into the plane and the ω-limit set of any point $x_0 \in B$ for the semiflow S'_t has the PB property.

Now take any open neighborhoods $\mathcal{V}'_i$, $\mathcal{W}_i$ of $\mathcal{A}_i$ such that $\mathcal{W}_i \subset cl(\mathcal{W}_i) \subset \mathcal{V}'_i \subset cl(\mathcal{V}'_i) \subset \mathcal{V}_i,\ i = 1, 2$. Recall that $\mathcal{A}_1$ and $\mathcal{A}_2$ are intersections of nested compact sets of $cl(F^n(B)),\ n > N_0$ and $cl(S_T(B)),\ T > T_0$, respectively. Therefore there are $N_1 > N_0$ and $T_1 > T_0$ such that $cl(F^{N_1}(B)) \subset \mathcal{W}_1$ and $cl(S_{T_1}(B)) \subset \mathcal{W}_2$. Hence we can make $\epsilon_i,\ i = 1, 2$ smaller, if necessary, such that whenever a map G and a continuous semiflow S'_t has the property

$$| G - F |_{C^0(B_1,X)} < \epsilon_1, \quad \sup_{t\in[\frac{1}{2},1]} | S'_t - S_t |_{C^0(B_1,X)} < \epsilon_2 \tag{7.6}$$

then

$$cl(G^N(B)) \subset \mathcal{V}'_1 \ \text{ for all } N_1 \le N \le 2N_1 \text{ and}$$

$$cl(S'_T(B)) \subset \mathcal{V}'_2 \ \text{ for all } T_1 \le T \le 2T_1.$$

Hence we also have

$$cl(\bigcup_{n\ge N_1} G^n(B)) \subset \mathcal{V}_1, \tag{7.7}$$

$$cl(\bigcup_{t\ge T_1} S'_t(B)) \subset \mathcal{V}_2 \tag{7.8}$$

for G and S'_t as above.

Now take any map $G : X \to X$ with its restriction to B being in $C^1(B, X)$ and any C^1 semiflow $S'_t,\ t > 0$ such that

$$\sup_{x\in B_1} | Gx - Fx |, \ \sup_{x\in B} | DG(x) - DF(x) | < \epsilon_1$$

and

$$\sup_{x\in B_2} | S'_t x - S_t |_X, \ \sup_{x\in B} | DS'_t(x) - DS_t(x) |_{L(X,X)} < \epsilon_2,$$

where B_1 and B_2 are defined in Theorem 1.1 and Theorem 1.2 , respectively. We also take any $x_0 \in B$. Note that positive trajectories of S'_t with precompact orbits are uniformly continuous due to the regularizing property of the semiflow S'_t. Obviously (7.6) is satisfied for the restrictions of such G and S'_t to the respective sets B_i. Therefore by (7.7) and (7.8) we have that

$$G |_B^n (G^{N_1} x_0) \in \mathcal{V}_1, \ n \ge 1 \text{ and } \ S'_t |_B (S'_{T_1} x_0) \in \mathcal{V}_2, \ t > 0.$$

Moreover, since the restrictions of G and S'_t to the respective $\mathcal{V}_i$ satisfy (7.5) the conclusions of Theorems 1.1 and 1.2 hold for the map G and the semiflow S'_t. This completes the proofs of Theorems 1.1 and 1.2.

8. Other applications.

In this section we prove convergence to equilibria for C^1 small perturbations of dissipative period maps given by equation (8.1) below, and a Poincaré-Bendixson Theorem for C^1 small perturbations of monotone cyclic feedback equations.

Consider the equation

$$u_t = u_{\xi\xi} + f(t, \xi, u, u_\xi), \quad \xi \in [0,1], \tag{8.1}$$

$$u(t,0) = u(t,1) = 0, \quad t > 0,$$

where $f \in C^2$ is 1-periodic in t. Let X be a fractional power space associated with the operator $u \mapsto -u_{\xi\xi}: \ H^2(0,1) \cap H_0^1(0,1) \to L^2(0,1)$ that satisfy the following embedding relation

$$X \hookrightarrow C^1([0,1]).$$

We are interested in solutions of (8.1) starting in an open set B in X. We impose the following dissipativity condition on (8.1)

(D) there is a $T_0 > 0$ such that for any $u_0 \in cl(B)$ the solution of (8.1) with $u(0,\cdot) = u_0$ is global and $u(t,\cdot) \in B$ for all $t > T_0$.

Let $F : X \to X$ be the time-one map for equation (8.1). Then $F\mid_B \in C^1(B,X)$, as follows from [He1].

Theorem 8.1. *Assume (D) . Let F be the Poincaré map for (1.1) and denote by B_1 the set $\bigcup_{n=1}^{N_0} F^n(B)$ for a $N_0 > T_0$. Then there is an $\epsilon > 0$ such that for any map $G : X \to X$ with*

$$\sup_{x\in B_1} \mid Gx - Fx \mid, \quad \sup_{x \in B} \mid DG(x) - DF(x) \mid < \epsilon$$

the ω-limit set of any point $x_0 \in B$ for G is a single point.

Proof. Let $u(t,\cdot), \ v(t,\cdot), t \geq 0$ be solutions of (8.1) starting from $x \in B$ and $y \in B$, respectively. Their difference $w(t,\cdot)$ satisfies equation (7.1) with Dirichlet boundary conditions. For any $(x,y) \in B \times B$ we associate, as in the proof of Theorems 1.1 and 1.2, the compact operator $T^1_{(x,y)}$.

Let $\phi \in X$ and $z(\phi)$ denote number of zeros of the function ϕ in the open interval $(0,1)$. For $w(t,\cdot)$ from [He1] and [An] we obtain the analogous statements of a)-d) and $\alpha) - \delta)$ from the previous section with just two exceptions. Denote by π the map

$$u(\cdot) \mapsto u_\xi(0): \ X \to \mathbb{R},$$

and let $\Pi = \pi^{-1}(0)$. The formulation of $\gamma)$ has to be replaced here as follows if $w(t_0,\cdot)$ has a mutiple zero on $[0,1]$ for a $t_0>0$ then for $0<\eta<t_0$ we have $z(w(t_0-\eta,\cdot))\geq z(w(t_0+\eta,\cdot))+1$.

The immediate consequence of this is the following statement

(A') if $z(w(t_1,\cdot))=z(w(t_2,\cdot))$ for $0<t_1<t_2$ then $\pi(w(t_1,\cdot))\,\pi(w(t_2,\cdot))\geq 0$.

Define the sets

$C_i = cl\{\phi\in X:\ \phi$ has no multiple zero and the number of zeros in (0,1) is less than $i\}$.

Similarly as in the proof above, using the analogs of $\alpha)-\delta)$ for the equation

$$u_t=u_{\xi\xi},\quad \xi\in[0,1]$$

with Dirichlet boundary conditions we obtain that each $C_i,\ i\in\mathbb{N}$ is a $i-$cone.

Using the analogs of a)-d) and $\alpha)-\delta)$ we obtain here the analogs of statements 1)- 5) with just one exception. Here we do not consider f not depending on t (cf. statement 2) in Section 7).

Statement (A') imply assumption (A) in Corollary 2.1 for the family of operators $\{T^1_{(x,y)}:\ (x,y)\in B\times B\}$. Now the theorem follows from Corollary 2.1 using the same arguments by which Theorem 1.1 was proved.$\diamondsuit$

The conclusion of the above theorem is proved in the unperturbed case in [BPS]. In [CP] a different approach is proposed to show the same kind of the result.

Another possible application of our abstract results are monotone feedback delay equations considered in [M-P]. It seems feasible to obtain for C^1 small perturbation of nonlinearities of these equations in the periodically forced case imbedding of ω-limit sets for the period map into the plane or a Poincaré-Bendixson Theorem in the autonomous case.

Now consider systems of the form

$$\frac{d}{dt}u_i=f^i(u_i,u_{i-1}),\quad i=1,2,\dots,n\ (mod\ n). \tag{8.2}$$

We assume the nonlinearity $f=(f^1,f^2,\dots,f^n)$ is defined on a nonempty open set $O\in\mathbb{R}^n$ with the property that each coordinate projection $O^i\subset\mathbb{R}^2$ of O onto the (x^i,x^{i-1}) plane is convex and that $f^i\in C^1(O^i)$. The following assumption makes from (8.2) a monotone cyclic feedback system (see [M-PSm])

(8.3)
$\delta^i\frac{\partial f^i(u_i,u_{i-1})}{\partial u_{i-1}}>0$ for some $\delta^i\in\{-1,+1\}$ and all $(u_i,u_{i-1})\in O^i,\quad 1\leq i\leq n.$

The product

$$\sigma = \delta^1\delta^2 \dots \delta^n$$

characterizes the entire system as one with negative feedback ($\sigma = -1$) or positive feedback ($\sigma = +1$). For such systems the Poincaré-Bendixson Theorem is proved in [M-PSm].

Define the function N (see [M-PSm]), taking values in $\{0, 1, 2, \dots, n\}$, by

$$N(v) = card\{i : \ \delta^i v_i v_{i-1} < 0\}$$

on the set $\mathcal{N} = \{v \in \mathbb{R} : \ v_i \neq 0, \ 1 \leq i \leq n\}$. If $\sigma = -1$ then N takes only odd values and only even values if $\sigma = +1$. Note that Proposition 1.1 in [M-PSm] implies analogs of statements $\alpha) - \delta)$ from the previous section.

Further we consider only the case $\sigma = -1$. The other one is analogous. Then the sets

$$C_k := cl(\{v \in \mathcal{N} : \ N(v) \leq k\})$$

are $(2k-1)$-cones for all k with $2k - 1 \leq n$. This follows similarly as in the proof of Theorems 1.1 and 1.2. considering the system $\frac{d}{dt}u_1 = -u_n$, $\frac{d}{dt}u_i = u_{i-1}$ for all $1 < i \leq n$ and using monotonicity of the function N on solutions of this system (see Proposition 1.1 in [M-PSm]). Obviously $\mathbb{R}^n$ is a $n-$cone. We are interested in solutions emanating in a bounded open set $B \subset cl(B) \subset O$. Impose the following dissipativity condition on solutions of (8.2)

(D) there is a $T_0 > 0$ such that for any $u_0 \in cl(B)$ the solution of (8.2) with $u(0, \cdot) = u_0$ is global and $u(t, \cdot) \in B$ for all $t > T_0$.

Using Proposition 1.1 in [MP-Sm] and standard properties of differential equations generating by smooth vector fields in $\mathbb{R}^n$ we can verify, analogously as in the proof of Theorem 1.2, all the assumptions of Corollary 2.4 for K_0-the attractor of (8.2) in B. Hence by the same way as we proved Theorem 1.2 we obtain the following theorem.

Theorem 8.2. *Let $S_t, t > 0$ be the semiflow defined by equation (8.2) on the set $B_1 := \{u(t, u_0) : \ t \geq 0, \ u_0 \in B\}$. Assume (8.3) and (D). Then there is an $\epsilon > 0$ such that for any regularizing semiflow S'_t continuous on B_1 with $S'_t \in C^1(B, X), \ t \in [\frac{1}{2}, 1]$ and*

$$| \, S_t \, |_{B_1} - S'_t \, |_{C^0(B_1, X)}, \ | \, S_t \, |_B - S'_t \, |_{C^1(B, X)} < \epsilon \quad \textit{for all } \frac{1}{2} \leq t \leq 1$$

the $\omega-$ limit set of any $x_0 \in B$, for which the closure of its positive orbit lies entirely in B, consists of either a single periodic orbit or equilibria and their connecting orbits.

Note that Gedeon in [Ge] proved chaos for equations (8.2) with f in arbitrarily small C^0 neighborhood of a subclass of nonlinearities in (8.2) giving monotone cyclic negative feedback systems.

Remark 8.1 a) It is possible to improve Theorem 1.1 in the following way. Chain reccurent sets of maps G are imbedded into $\mathbb{R}^2$. This enables to say that $\omega-$ limit

sets for asymptotically 1-periodic in t nonlinearities f in equation (1.1) are subsets of the product of homoeomorphic image of a compact set in the plane and the interval (see also [CP] for the corresponding result under the Dirichlet boundary conditions). We will prove it in a forthcoming paper.
b) If dissipative smooth semiflows or maps satisfy a cone condition then they admit an inertial manifold (see [M-PSe]). Strong cone conditions even imply C^1 smoothness of these manifolds. As it is seen from the proofs of Theorems 1.1 and 1.2 a cone condition is provided for semiflows as well as period maps given by (1.1) which is strengthen by Theorem B. In [Te] we prove the existence of these (C^1) manifolds for general dissipative equations of type (1.1) with various boundary conditions (for previous results see [Br], [CL], [Kw], [Mk]).

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