Journal of ELECTRICAL ENGINEERING, VOL. 57, NO. 12/s, 2006, 1–5

WEIGHTED CENTRAL PATH IN SEMIDEFINITE PROGRAMMING ASSOCIATED WITH SYMMETRIZATION MAP $(XS + SX)/2$

Mária Trnovská *

Central path is an important tool in interior point methods. It is an analytic curve which "leads" to the optimal solution of the given problem. Inspired by the paper of M.Preiss, J.Stoer (see [7]) the existence of infeasible weighted central path in semidefinite programming associated with so called AHO-symmetrization map is shown.

Keywords: semidefinite programming, interior point methods, central path

2000 *Mathematics Subject Classification*: 90C22,90C25,90C51

1 INTRODUCTION

Semidefinite programming (SDP) is a special mathematical programming class which contains linear or convex quadratic programming as a special case. It became popular in the 1990's because of its applicability to various special problems like e.g. eigenvalue optimization. It is used in control theory, statistics and combinatorial optimization (see e.g. [4]) - many of the optimization problems from these fields can be formulated as SDP problems and then solved using interior point methods, or they can be relaxed using SDP to obtain better solution estimations than by linear programming relaxation.

Let S^n be the vector space of $n \times n$ real symmetric matrices, " $\bullet$ " the inner product defined on S^n as $\mathbf{X} \bullet \mathbf{Y} := trace(\mathbf{XY})$. We will denote S^n_+ (S^n_{++}) the set of all positive semidefinite (positive definite) matrices. We will write $\mathbf{X} \succeq 0$ $(\mathbf{X} \succ 0)$ if $\mathbf{X} \in S^n_+$ $(\mathbf{X} \in S^n_{++})$.

For given symmetric matrices $\mathbf{A}_1, \dots, \mathbf{A}_m, \mathbf{C} \in S^n$ and a vector $b \in R^m$ we can define primal SDP problem as

$$(P) \quad \min \left\{ \mathbf{C} \bullet \mathbf{X} \;\middle|\; \mathbf{A}_i \bullet \mathbf{X} = b_i;\ i = 1, \dots, m;\ \mathbf{X} \succeq 0 \right\}$$

and dual SDP problem as

$$(D) \quad \max \left\{ b^T y \;\middle|\; \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C};\ \mathbf{S} \succeq 0 \right\}$$

We will denote

$$\mathcal{P} = \{\mathbf{X} \in S^n | \mathbf{A}_i \bullet \mathbf{X} = b_i,\ i = 1, \dots, m;\ \mathbf{X} \succeq 0\},$$

$$\mathcal{D} = \{y \in R^m | \sum_{i=1}^m \mathbf{A}_i y_i \preceq \mathbf{C}\},$$

the primal, resp. dual feasible set and

$$\mathcal{P}^0 = \{\mathbf{X} \in S^n | \mathbf{A}_i \bullet \mathbf{X} = b_i,\ i = 1, \dots, m;\ \mathbf{X} \succ 0\}$$

$$\mathcal{D}^0 = \{y \in R^m | \sum_{i=1}^m \mathbf{A}_i y_i \prec \mathbf{C}\}$$

the primal, resp. dual strictly feasible set.

2 CENTRAL PATH IN SDP

In SDP the following two assumptions are usually made:

1. The matrices $\mathbf{A}_1, \dots, \mathbf{A}_m$ are linearly independent;
2. $\mathcal{P}^0 \neq \emptyset, \mathcal{D}^0 \neq \emptyset$.

The assumption 1 ensures the one-to-one corespondence between the dual variables y and $\mathbf{S}$. The assumption 2 (also referred to as the interior point assumption) follows from the strong duality theorem and is equivalent to the fact that the (primal and dual) optimal solution sets are nonempty and bounded (see [8]). Under these assumptions we obtain the neccesary and sufficient condition of optimality:

$(\mathbf{X}, y, \mathbf{S})$ is optimal $\Leftrightarrow$

$$\begin{array}{ll} \mathbf{A}_i \bullet \mathbf{X} = b_i, \quad i = 1, \dots, m, & \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}, & \mathbf{X} \succeq 0, \\ \mathbf{XS} = 0, & \mathbf{S} \succeq 0. \end{array} \tag{1}$$

Interior point methods for SDP usually work with the system

$$\begin{array}{ll} \mathbf{A}_i \bullet \mathbf{X} = b_i, \quad i = 1, \dots, m, & \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}, & \mathbf{X} \succ 0, \\ \mathbf{XS} = \mu \mathbf{I}, & \mathbf{S} \succ 0, \end{array} \tag{2}$$

* Department of Applied Mathematics and Statistics, Faculty of Mathematics, Physics and Informatics, Comenius University Mlynská dolina, 842 48 Bratislava, Slovakia E-mail: trnovska@pc2.iam.fmph.uniba.sk

Research supported by Comenius University grant UK/332/2006 and VEGA-grant 1/1000/04.

ISSN 1335-3632

where the last optimality condition in (1) is perturbed. It is well known that $\forall \mu > 0$ there exists unique solution $(\mathbf{X}_\mu, y_\mu, \mathbf{S}_\mu)$ of the system (2) (see [4]). The map

$$\mu \mapsto (\mathbf{X}_\mu, y_\mu, \mathbf{S}_\mu)$$

implicitly defined by system (2) is called the *central path*. It is an analytic curve in the interior of the primal-dual feasible set. Most interior point methods follow the central path to reach the optimal solution. Since the behavior of the central path is important for the interior point algorithms, its properies are often studied (see e.g. [1],[2],[3]).

In order to preserve the symmetricity in the algorithms the condition $\mathbf{XS} = \mu\mathbf{I}$ is usually replaced with an equivalent condition $(\mathbf{XS} + \mathbf{SX})/2 = \mu\mathbf{I}$. The map $(\mathbf{X}, \mathbf{S}) \mapsto (\mathbf{XS} + \mathbf{SX})/2$ is often referred to as the AHO-symmetrization map.

3 MOTIVATION AND GOAL

If strictly feasible solutions don't exist (the assumption 2 is not satisfied) or are unknown we have to deal with so called infeasible interior point methods. They are connected with the system

$$\begin{array}{ll} \mathbf{A}_i \bullet \mathbf{X} = b_i + \mu\triangle b_i, \ i = 1, \dots, m, & \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C} + \triangle\mathbf{C}, & \mathbf{X} \succ 0, \\ \frac{1}{2}(\mathbf{XS} + \mathbf{SX}) = \mu\mathbf{W}, & \mathbf{S} \succ 0, \end{array} \tag{3}$$

where $\triangle b \in R^m, \triangle\mathbf{C} \in S^n, \mu > 0, \mathbf{W} \succ 0$. Our goal is to analyse whether it is possible to define the $\mathbf{W}$-weighted central path as the unique solution of the system (3) for fixed $\triangle b \in R^m, \triangle\mathbf{C} \in S^n$, i.e. to show that under some assumptions the following theorem holds.

Theorem 1. Let $\triangle b, \triangle\mathbf{C}$ be fixed. If the system (3) has a solution for some $\mu = \mu_0 > 0, \mathbf{W} = \mathbf{W}_0 \succ 0$ then it has a unique solution for all $\mu \in (0, \mu_0\rangle$ and all $\mathbf{W} \succ 0$.

This result was already shown for semidefinite nonlinear complementarity problems ([5], [6]) and semidefinite linear complementarity problems ([7]). The authors Monteiro, Pang (see [5]) and Monteiro, Zanjacomo (see [6]) used deep results of nonlinear analysis. However we were motivated by the approach of Preiss, Stoer (see [7]) which was more elementary - it was essentialy based on implicit function theorem.

4 EXISTENCE OF WEIGHTED SDP CENTRAL PATH

In the next we willl consider the following assumptions:
1. The matrices $\mathbf{A}_1, \dots, \mathbf{A}_m$ are linearly independent;
2. The system (1) is solvable.

Define

$$\begin{array}{c} \mathcal{A} : S^n \to R^m, \quad \mathcal{A}(\mathbf{X}) = [\mathbf{A}_1 \bullet \mathbf{X}, \dots, \mathbf{A}_m \bullet \mathbf{X}] \\ \tilde{\mathcal{A}} : R^m \to S^n, \quad \tilde{\mathcal{A}}(y) = \sum_{i=1}^m \mathbf{A}_i y_i. \end{array}$$

Choose $\mathbf{X}^0 \in S^n_{++}, y^0 \in R^m, \mathbf{S}^0 \in S^n_{++}, \mu^0 > 0$ so that $\mathbf{X}^0\mathbf{S}^0 + \mathbf{S}^0\mathbf{X}^0 \succ 0$ and let

$$\triangle b = \frac{\mathcal{A}(\mathbf{X}^0) - b}{\mu^0}, \qquad \triangle\mathbf{C} = \frac{\tilde{\mathcal{A}}(y^0) + \mathbf{S}^0 - \mathbf{C}}{\mu^0}$$

$$\mathbf{W}^0 = \frac{\mathbf{X}^0\mathbf{S}^0 + \mathbf{S}^0\mathbf{X}^0}{2\mu^0}.$$

Then $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ is a solution of system (3) for $\mu = \mu^0$ and $\mathbf{W} = \mathbf{W}^0$. Let $\triangle b, \triangle\mathbf{C}$ as well as $\mathbf{X}^0, y^0, \mathbf{S}^0, \mu_0, \mathbf{W}_0$ be fixed throughout the paper.

Consider the map $F : S^n \times R^m \times S^n \to R^m \times S^n \times S^n$ with parameters $\mu, \mathbf{W}$:

$$F_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S}) = \begin{pmatrix} \mathcal{A}(\mathbf{X}) - b - \mu\triangle b \\ \tilde{\mathcal{A}}(y) + \mathbf{S} - \mathbf{C} - \mu\triangle\mathbf{C} \\ \frac{1}{2}(\mathbf{XS} + \mathbf{SX}) - \mu\mathbf{W} \end{pmatrix}$$

The Fréchet derivative of $F_{\mu,\mathbf{W}}$ is the linear map $DF_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S})[\triangle\mathbf{X}, \triangle y, \triangle\mathbf{S}] =$

$$\begin{pmatrix} \mathcal{A}(\triangle\mathbf{X}) \\ \tilde{\mathcal{A}}(\triangle y) + \triangle\mathbf{S} \\ \frac{1}{2}(\triangle\mathbf{X}\mathbf{S} + \mathbf{S}\triangle\mathbf{X}) + \frac{1}{2}(\mathbf{X}\triangle\mathbf{S} + \triangle\mathbf{S}\mathbf{X}) \end{pmatrix}$$

The proof of the next lemma can be found in [4].

Lemma 1. If $\mathbf{X} \succ 0, \mathbf{S} \succ 0, \mathbf{XS} + \mathbf{SX} \succeq 0$ then $DF_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S})$ is regular linear map.

For $\mu > 0, \mathbf{W} \succ 0$ we will denote

$$(\mathbf{X}_{(\mu,\mathbf{W})}, y_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})})$$

- the solution of (3), if it exists.

Lemma 2. Let $\mathcal{O}(\mathbf{W}_0) \subseteq S^n_{++}$ be a bounded neighbourhood $\mathbf{W}_0$. Then the set

$$\{(\mathbf{X}_{(\mu,\mathbf{W})}, y_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})}) \mid \mu \in (0, \mu_0\rangle, \mathbf{W} \in \mathcal{O}(\mathbf{W}_0)\}$$

is bounded.

Proof. Assumption 2 states that there exists

$$(\mathbf{X}^*, y^*, \mathbf{S}^*)$$

- the solution of (1). Recall $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ is the solution of (3) for $\mu = \mu_0, \mathbf{W} = \mathbf{W}_0$ and let

$$(\mathbf{X}_{(\mu,\mathbf{W})}, y_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})})$$

be the solution of (3) for $\mu \in (0, \mu_0\rangle, \mathbf{W} = \mathcal{O}(\mathbf{W}_0)$. Define

$$\hat{\mathbf{X}} = \tfrac{\mu}{\mu_0}\mathbf{X}^0 + \left(1 - \tfrac{\mu}{\mu_0}\right)\mathbf{X}^*$$
$$\hat{y} = \tfrac{\mu}{\mu_0}y^0 + \left(1 - \tfrac{\mu}{\mu_0}\right)y^*$$
$$\hat{\mathbf{S}} = \tfrac{\mu}{\mu_0}\mathbf{S}^0 + \left(1 - \tfrac{\mu}{\mu_0}\right)\mathbf{S}^*$$

Clearly

$$\mathcal{A}(\hat{\mathbf{X}}) = b + \mu\triangle b,$$
$$\tilde{\mathcal{A}}(\hat{y}) + \hat{\mathbf{S}} = \mathbf{C} + \mu\triangle\mathbf{C}$$

and $\hat{\mathbf{X}} \succ 0, \hat{\mathbf{S}} \succ 0$. From this it follows that

$$\mathcal{A}(\hat{\mathbf{X}} - \mathbf{X}_{(\mu,\mathbf{W})}) = 0$$
$$\tilde{\mathcal{A}}(\hat{y} - y_{(\mu,\mathbf{W})}) + (\hat{\mathbf{S}} - \mathbf{S}_{(\mu,\mathbf{W})}) = 0$$

and hence $(\hat{\mathbf{X}} - \mathbf{X}_{(\mu,\mathbf{W})}) \bullet (\hat{\mathbf{S}} - \mathbf{S}_{(\mu,\mathbf{W})}) = 0$. Moreover it can be seen that

$$\hat{\mathbf{X}} \bullet \hat{\mathbf{S}} \le \mu_0 tr(\mathbf{W}_0) + \mathbf{X}^0 \bullet \mathbf{S}^* + \mathbf{S}^0 \bullet \mathbf{X}^*$$

From this relation and from boundedness of $\mathcal{O}(\mathbf{W}_0)$ we obtain that

$$\hat{\mathbf{X}} \bullet \mathbf{S}_{(\mu,\mathbf{W})} + \mathbf{X}_{(\mu,\mathbf{W})} \bullet \hat{\mathbf{S}} = \hat{\mathbf{X}} \bullet \hat{\mathbf{S}} + \mathbf{X}_{(\mu,\mathbf{W})} \bullet \mathbf{S}_{(\mu,\mathbf{W})} \le \gamma.$$

for some $\gamma > 0$. So the set

$$\{(\mathbf{X}_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})}) \mid \mu \in (0,\mu_0\rangle, \mathbf{W} \in \mathcal{O}(\mathbf{W}_0)\}$$

is included in the simplex

$$\{(\mathbf{X},\mathbf{S}) \mid \mathbf{X} \succeq 0, \mathbf{S} \succeq 0, \hat{\mathbf{X}} \bullet \mathbf{S} + \mathbf{X} \bullet \hat{\mathbf{S}} \le \gamma\}$$

which is bounded if $\hat{\mathbf{X}} \succ 0, \hat{\mathbf{S}} \succ 0$. The boundedness of the set

$$\{(\mathbf{X}_{(\mu,\mathbf{W})}, y_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})}) \mid \mu \in (0,\mu_0\rangle, \mathbf{W} \in \mathcal{O}(\mathbf{W}_0)\}$$

follows from Assumption 1.

■

Consider the map
$G : S^n \times R^m \times S^n \times R \times S^n \to R^m \times S^n \times S^n$
such that

$$G(\mathbf{X}, y, \mathbf{S}, \mu, \mathbf{W}) = F_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S}).$$

Obviously $G(\mathbf{X}^0, y^0, \mathbf{S}^0, \mu_0, \mathbf{W}_0) = 0$. In the proof of the next lemma we will use the technique of analytic continuation and the implicit function theorem.

Lemma 3:. Let $\mu_1 \in (0,\mu_0\rangle, \mathbf{W}_1 \succ 0$.

$$\phi : \langle 0,1\rangle \to (0,\mu_0\rangle \times S^n_{++}, \qquad \phi(t) = (\mu_t, \mathbf{W}_t)$$

be a continuous path from $\phi(0) = (\mu_0, \mathbf{W}_0)$ to $\phi(1) = (\mu_1, \mathbf{W}_1)$. Then for all $t \in \langle 0,1\rangle$ the system

$$G(\mathbf{X}, y, \mathbf{S}, \mu_t, \mathbf{W}_t) = 0$$

has a locally unique solution $(\mathbf{X}^t, y^t, \mathbf{S}^t)$ where $\mathbf{X}^t \succ 0, \mathbf{S}^t \succ 0$. Moreover there exists a unique function g which is defined and analytic on some neighbourhood of $\phi(t)$, $g(\phi(t)) = (\mathbf{X}^t, y^t, \mathbf{S}^t)$ and $G(g(\phi(t)), \phi(t)) = 0$.

Proof. For $t \in \langle 0,1\rangle$ consider the system

$$G(\mathbf{X}, y, \mathbf{S}, \phi(t)) = 0, \quad \mathbf{X} \succ 0, \quad \mathbf{S} \succ 0. \qquad (4)$$

$(\mathbf{X}^0, y^0, \mathbf{S}^0)$ is solution of this system for $t = 0$. From Lemma 1 it follows that the partial Fréchet derivative $DG(\mathbf{X}, y, \mathbf{S}, \phi(t))$ concerning the variables $\mathbf{X}, y, \mathbf{S}$ is regular in $(\mathbf{X}^0, y^0, \mathbf{S}^0, \phi(0))$. From the implicit function theorem we obtain that there is an analytic function g defined on some neighbourhood of $(\mu_0, \mathbf{W}_0)$ so that

$$g(\phi(0)) = g(\mu_0, \mathbf{W}_0) = (\mathbf{X}^0, y^0, \mathbf{S}^0)$$
$$G(g(\phi(t)), \phi(t)) = G(g(\mu_t, \mathbf{W}_t), \mu_t, \mathbf{W}_t) = 0$$

on some neighbourhood of $t = 0$. Actually there is a maximal $\bar{t} \in (0,1\rangle$ so that

$$g(\mu_t, \mathbf{W}_t) = (\mathbf{X}^t, y^t, \mathbf{S}^t) \quad \forall t \in \langle 0, \bar{t})$$

is locally unique solution of (4). Lemma 2 implies that the set $\{(\mathbf{X}^t, y^t, \mathbf{S}^t) \,|t \in \langle 0, \bar{t})\}$ is bounded. So there exists $\{t_k\}_{i=1}^{\infty} \in \langle 0, \bar{t})$ such that $\lim_{k\to\infty} t_k = \bar{t}$ and

$$\lim_{k\to\infty} g(\phi(t_k)) = \lim_{k\to\infty} (\mathbf{X}^{t_k}, y^{t_k}, \mathbf{S}^{t_k}) = (\bar{\mathbf{X}}, \bar{y}, \bar{\mathbf{S}}).$$

Then we have

$$\mathcal{A}(\mathbf{X}^{t_k}) = b + \mu_{t_k}\triangle b,$$
$$\tilde{\mathcal{A}}(y^{t_k}) + \mathbf{S}^{t_k} = \mathbf{C} + \mu_{t_k}\triangle\mathbf{C},$$
$$\mathbf{X}^{t_k}\mathbf{S}^{t_k} + \mathbf{S}^{t_k}\mathbf{X}^{t_k} = 2\mu_{t_k}\mathbf{W}_{t_k},$$
$$\mathbf{X}^{t_k} \succ 0, \ \mathbf{S}^{t_k} \succ 0.$$

By taking limit with $k \to \infty$ we obtain

$$\mathcal{A}(\bar{\mathbf{X}}) = b + \mu_{\bar{t}}\triangle b,$$
$$\tilde{\mathcal{A}}(\bar{y}) + \bar{\mathbf{S}} = \mathbf{C} + \mu_{\bar{t}}\triangle\mathbf{C},$$
$$\bar{\mathbf{X}}\bar{\mathbf{S}} + \bar{\mathbf{S}}\bar{\mathbf{X}} = 2\mu_{\bar{t}}\mathbf{W}_{\bar{t}},$$
$$\bar{\mathbf{X}} \succeq 0, \ \bar{\mathbf{S}} \succeq 0.$$

Hence $\bar{\mathbf{X}}\bar{\mathbf{S}} + \bar{\mathbf{S}}\bar{\mathbf{X}} \succ 0$ we have that $\bar{\mathbf{X}} \succ 0$, $\bar{\mathbf{S}} \succ 0$ and therefore partial Fréchet derivative $DG(\mathbf{X}, y, \mathbf{S}, \phi(t))$ concerning the variables $\mathbf{X}, y, \mathbf{S}$ is regular in the point $(\bar{\mathbf{X}}, \bar{y}, \bar{\mathbf{S}}, \phi(\bar{t}))$. By using Implicit function theorem again and the maximality $\bar{t}$ we obtain $\bar{t} = 1$.

■

It remains to show the global uniquenes of the solution of the system (3). It follows from the next lemma and the uniquess of the analytic continuation stated in Corollary 1.

Lemma 4:. If the system

$$\begin{array}{ll} \mathcal{A}(\mathbf{X}) = b + \nu\triangle b & \mathbf{X} \succ 0 \\ \tilde{\mathcal{A}}(y) + \mathbf{S} = \mathbf{C} + \nu\triangle\mathbf{C} & \mathbf{S} \succ 0 \\ (\mathbf{X}\mathbf{S} + \mathbf{S}\mathbf{X})/2 = \nu\mathbf{I} & \end{array} \qquad (5)$$

has a solution for some $\nu > 0$ then the solution is unique.

Proof. Suppose there are two solutions $(\mathbf{X}_1, y_1, \mathbf{S}_1)$, $(\mathbf{X}_2, y_2, \mathbf{S}_2)$. Let

$$(\triangle\mathbf{X}, \triangle y, \triangle\mathbf{S}) = (\mathbf{X}_1, y_1, \mathbf{S}_1) - (\mathbf{X}_2, y_2, \mathbf{S}_2).$$

Hence $\mathbf{X}_i \succ 0, \mathbf{S}_i \succ 0,\ i = 1,2$, from $(\mathbf{X}_i\mathbf{S}_i + \mathbf{S}_i\mathbf{X}_i)/2 = \nu\mathbf{I}$ we obtain $\mathbf{X}_i\mathbf{S}_i = \nu\mathbf{I}$. From that follows

$$(\mathbf{X}_1 + \mathbf{X}_2)\triangle\mathbf{S} + \triangle\mathbf{X}(\mathbf{S}_1 + \mathbf{S}_2) = 0.$$

Therefore

$$0 = \triangle\mathbf{X} \bullet \triangle\mathbf{S} =$$
$$= -tr((\mathbf{S}_1 + \mathbf{S}_2)^{\frac{1}{2}}\triangle\mathbf{X}(\mathbf{X}_1 + \mathbf{X}_2)^{-1}\triangle\mathbf{X}(\mathbf{S}_1 + \mathbf{S}_2)^{\frac{1}{2}})$$

From the fact that $(\mathbf{S}_1 + \mathbf{S}_2)^{\frac{1}{2}} \succ 0$ and $(\mathbf{X}_1 + \mathbf{X}_2)^{-1} \succ 0$ and properties of the trace it follows that $\triangle\mathbf{X} = 0$. The equality $\triangle\mathbf{S} = 0$ can be proved similarly.

■

Lemma 5:. If the system

$$\begin{array}{ll} \mathcal{A}(\mathbf{X}) = b + \nu\triangle b & \mathbf{X} \succ 0 \\ \tilde{\mathcal{A}}(y) + \mathbf{S} = \mathbf{C} + \nu\triangle\mathbf{C} & \mathbf{S} \succ 0 \\ (\mathbf{XS} + \mathbf{SX})/2 = \nu\mathbf{W} & \end{array} \quad (6)$$

has a solution for some $\nu > 0$ then the solution is unique.

Proof. Let $(\mathbf{X}_W^1, y_W^1, \mathbf{S}_W^1)$ and $(\mathbf{X}_W^2, y_W^2, \mathbf{S}_W^2)$ be two solution of system (6) and let $(\mathbf{X}_I, y_I, \mathbf{S}_I)$ be the unique solution of (5). From Lemma 3 and the convexity of S_{++}^n it follows that we can choose the path

$$\phi : \langle 0, 1\rangle \to R_{++} \times S_{++}^n \qquad \phi(t) = (\nu, t\mathbf{I} + (1-t)\mathbf{W}).$$

and make the analytic continuation along this path from $(\mathbf{X}_W^1, y_W^1, \mathbf{S}_W^1)$, resp. $(\mathbf{X}_W^2, y_W^2, \mathbf{S}_W^2)$ to $(\mathbf{X}_I, y_I, \mathbf{S}_I)$, which is the unique solution of the system (5). We can make the analytic continuation along the inverse path $\phi^{-1}(t) = (\nu, (1-t)\mathbf{I} + t\mathbf{W})$ from $(\mathbf{X}_I, y_I, \mathbf{S}_I)$ to both $(\mathbf{X}_W^1, y_W^1, \mathbf{S}_W^1)$ and $(\mathbf{X}_W^2, y_W^2, \mathbf{S}_W^2)$. The uniqueness of the analytic continuation (stated in Lemma 3) implies thet $(\mathbf{X}_W^1, y_W^1, \mathbf{S}_W^1) = (\mathbf{X}_W^2, y_W^2, \mathbf{S}_W^2)$.
■

At the beginning of this paragraph we showed that we can find $\triangle b, \triangle\mathbf{C}$ and $\mu_0 > 0, \mathbf{W}_0 \succ 0$ so that there exists a solution of system (3) for $\mu = \mu_0$ and $\mathbf{W} = \mathbf{W}_0$. From the convexity of the set $(0, \mu_0\rangle \times S_{++}^n$ it follows that any pair $(\mu, \mathbf{W}) \in (0, \mu_0\rangle \times S_{++}^n$ can be joined with $(\mu_0, \mathbf{W}_0)$ by a straight line. If we let $\triangle b, \triangle\mathbf{C}$ be fixed, then Lemma 3 implies that there exists a solution of system (3) for any parameters $(\mu, \mathbf{W}) \in (0, \mu_0\rangle \times S_{++}^n$. From Lemma 5 we obtain that this solution is unique. We actually proved Theorem 1.

5 CONCLUSION

If $\triangle b, \triangle\mathbf{C}$ and $\bar{\mu} > 0$ are appropriately chosen and the weight $\mathbf{W} \succ 0$ is fixed, then Theorem 1 states that there exists unique solution of the system (3) for $\mu \in (0, \bar{\mu}\rangle$. Then the map

$$\mu \mapsto (\mathbf{X}_\mu, y_\mu, \mathbf{S}_\mu),$$

where $(\mathbf{X}_\mu, y_\mu, \mathbf{S}_\mu)$ is the solution of (3) is called the weighted central path in SDP. Contrary to the central path it does not lie in the primal-dual feasible set. The existence of the weighted central path was shown for semidefinite linear and nonlinear complementarity problems. As it was mentioned above we used the techniques of Preiss and Stoer ([7]) to prove that the weighted central path in semidefinite programming can be well defined.

References

[1] GOLDFARB, D.—SCHEINBERG, K.: Interior point trajectories in semidefinite programming, SIAM J. on Optimization **8** (1998), 347-886.

[2] HALICKÁ, M.: Analyticity of the central path at the boundary point in semidefinite programming, European Journal of Operational Research **143** (2002), 311-324.

[3] HALICKÁ, M.—de KLERK, E.—ROOS, C.: On the convergence of the central path in semidefinite optimization, SIAM J. on Optimization **12** (2002), 1090-1099.

[4] De KLERK, E.: Aspects of Semidefinite Programming - Interior Point Algorithms and Selected Applications, Kluwer Academic Publishers, Netherlands, 2002.

[5] MONTEIRO, R.D.C.— PANG, J-S.: On two interior point mappings for nonlinear semidefinite complementarity problems, Math Oper. Res. **21,** (1998), 39-60,.

[6] MONTEIRO, R.D.C.— ZANJACOMO, P.: General interior-point maps and existence of weighted paths for nonlinear semidefinite complementarity problems, Math Oper. Res. **25,** (2000), 381-399,.

[7] PREISS, M.— STOER, J.: Analysis of infeasible-interior-point paths arising with semidefinite linear complementarity problems, Optimization Online (2003), working paper.

[8] TRNOVSKÁ, M.: Strong Duality Conditions in Semidefinite Programming, Journal of Electrical Engineering, **56,** (2005).

Received 31 May 2006

Mária Trnovská (RNDr) is a PhD student of applied mathematics at the Department of Applied Mathematics and Statistics, Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava. Her supervisor Margaréta Halická is an associate professor at the Faculty of Mathematics, Physics and Informatics, Comenius University.