Univerzita Komenského v Bratislave

Fakulta matematiky, fyziky a informatiky

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Autoreferát dizertačnej práce

KVALITATÍVNA A KVANTITATÍVNA ANALÝZA MODELOV OCEŇOVANIA DERIVÁTOV AKTÍV BLACK-SCHOLESOVHO TYPU SO VŠEOBECNOU FUNKCIOU VOLATILITY

na získanie akademického titulu *philosophiae doctor*

v odbore doktorandského štúdia:
9.1.9 Aplikovaná matematika

Bratislava, 2014

Dizertačná práca bola vypracovaná v dennej forme doktorandského štúdia na katedre aplikovanej matematiky a štatistiky Fakulty matematiky, fyziky a informatiky Univerzity Komenského v Bratislave

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Introduction

Pricing financial derivatives belongs to actual topics on financial markets. As markets have become more sophisticated, more complex contracts than simple buy or sell trades have been introduced. They are known as financial derivatives, derivative securities or just derivatives. There exist many kinds of financial markets, e.g. stock markets, bonds markets, currency markets or foreign exchange markets, commodity markets or futures and options markets. On option markets derivative products are traded.

A derivative is defined as a financial instrument whose value depends on (or derives from) the values of other, more basic underlying variables. Very often the variables underlying derivatives are the prices of traded assets. As an example, an asset option is a derivative whose value is dependent on the price of a asset. However, derivatives can be dependent on almost any variable.

A *European call* option is a contract with the following conditions: At a prescribed time in the future, known as the expiration date, the holder of the option may purchase a prescribed asset, known as the underlying asset for a prescribed amount, the exercise price or strike price. For the holder of the option this contract is a right, not an obligation. The other party of the contract, the writer, must sell the asset if the holder chooses to buy it. Since the option is the right with no obligation for the holder, it has some value, paid for at the time of opening the contract. The right to sell the option is called a *put option*. A put option allows its holder to sell the asset on a certain date for a prescribed amount. The writer is then obligated to buy the asset.

Options are used for hedging but also for speculations. Hedgers use derivatives to reduce the risk that they face from potential future movements in a market variable. Speculators use them to bet on the future direction of a market variable. Arbitrageurs take offsetting positions in two or more instruments to lock in a profit.

One of the most common methods of valuing stock options is the Black–Scholes method introduced in 1973. Economists Myron Scholes and Robert Merton and theoretical physicist Fischer Black derived and analysed a pricing model by means of a solution to a certain partial differential equation.

This thesis deals with the nonlinear models of Black–Scholes type, which are becoming more and more important since they take into account many effects that are not included in the linear model.

The main goals of the thesis can be summarized as follows:

- **Review of existing nonlinear models.** We review option pricing models of the Black–Scholes type with a general function of volatility. They provide more accurate values than the classical linear model by taking into account more realistic assumptions, such as transaction costs, the risk from an unprotected portfolio, large investor's preferences or illiquid markets.

- **Novel nonlinear models.** The main goals of the thesis is to derive models with variable transaction costs. We extend the models by two more new examples of realistic variable transaction costs that are decreasing with the amount of transactions. Using the Risk adjusted pricing methodology we derive a novel option pricing model under transaction costs and risk of the unprotected portfolio.

- **Solving the model by Gamma equation.** We show that the generalizations of the classical Black–Scholes model, including the novel model, can be solved by transformation of the fully nonlinear parabolic equation into a quasi-linear parabolic equation for which one can construct an effective numerical scheme for approximation of the solution.

- **Numerical scheme and experiments.** The aim of this part is to propose an efficient numerical discretization of the Gamma equation, including, in particular, the model with variable transaction costs. The numerical scheme is based on the finite volume approximation of the partial derivatives entering the equation to be solved.

The structure of the thesis is as follows:

In Section 1 we recall and summarize the nonlinear Black–Scholes option pricing models and the form of models with variable transaction costs. We review for example the Jumping volatility model due to Avellaneda, Levy and Paras [4], Leland model [26], the model with investor's preferences from Barles & Soner [6], the model with linear decreasing transaction costs depending on volume of trading stocks proposed by Avellaneda, Levy and Paras [4], non–arbitrage liquidity model developed by Bakstein and Howison [5] and Risk Adjusted Pricing Methodology model proposed by Kratka [22] and its generalization by Jandačka and Ševčovič in the work [21].

The main Subsection 1.7 develops a general theory of models with variable transaction costs. The main idea is in defining the modified transaction cost function $\tilde{C}$ when using the transaction costs measure, defined as the expected value of a change of the transaction cost per unit time interval Δt and price S. We also give the properties of this function to confirm its generality. Special cases of transaction costs function and their modification $\tilde{C}$ are also included. We mention the constant transaction costs function used in the Leland model [26] and also the linearly decreasing one from the model studied by Amster et al. [1]. We present and analyse two more new examples of realistic variable transaction costs that are decreasing with the amount of transactions, particularly, the piecewise linear non–increasing function and the exponentially decreasing function. By considering these functions, we solved the difficulty with possibly negative transaction costs that arises in the model proposed by Amster et al. [1].

Section 2 brings the main contribution in the form of a novel option pricing model under the transaction costs and the risk of an unprotected portfolio. It is a model with variable transaction costs with a general modified function of transaction costs $\tilde{C}$ and at the same time there is a possibility to control the risk of an unprotected portfolio. We show that this novel model is a generalization of the Leland model

[26], the model with linear decreasing transaction cost depending on the volume of transaction [1] and also of the Risk adjusted pricing methodology model [22], [21]. We give also detailed analysis behind the optimization of hedging time.

Section 3 we introduce the Gamma equation proposed in [21] by Jandačka and Ševčovič as the main instrument to solve the nonlinear models including the novel one. The method includes the derivation of the Gamma equation, transformation of the initial and boundary conditions and also backward transformation of the solution.

The advantage of using the transformation to the Gamma equation lies in the fact that we can use an efficient numerical scheme, introduced in Section 4. The construction of numerical approximation of a solution to Gamma equation is based on the derivation of a system of difference equations to be solved at every discrete time step. We give also the Mathematica code for the model with variable transaction cost. Finally we consider the modelling of a bid–ask spread and perform extensive comparisons.

1 Motivation for Studying Nonlinear Models

Analysing real market data we can see there is a need of nonlinear models, where $\sigma > 0$ is now not constant, but is a function of the option price V itself. We focus on case, where volatility σ depends of second derivative $\partial_S^2 V$ of the option price (hereafter referred to a Γ), the price of an underlying asset S and the time to expiration $\tau = T - t$, as Ševčovič, Stehlíková and Mikula state in [30], i.e.

$$\hat{\sigma} = \hat{\sigma}(S\partial_S^2 V, S, \tau). \tag{1}$$

On the one hand, in case of the constant $\sigma > 0$ in (37) represents the classical Black–Scholes equation derived by Black and Scholes in [7]. On the other hand, if $\sigma > 0$ is a function of a solution, equation (37) represents the nonlinear generalization of the Black–Scholes equation.

The motivation for studying the nonlinear Black–Scholes equation (37) with volatility having a general form of (1) arises from traditional option pricing models taking into account non–trivial transaction costs due to buying and selling assets, market feedbacks and illiquid market effects due to large traders choosing given stock–trading strategies, risk from a volatile (unprotected) portfolio or investors preferences, etc. There is an increase of interest in studying nonlinear Black–Scholes model, because it takes into account more realistic assumptions, that can impact volatility, drift and price of an asset.

One of the basic nonlinear models is the Leland model [26] which including transaction costs arising by hedging the portfolio with call or put options. This model was later extended by Hoggard, Whalley and Wilmott [19] for more general option types. Another nonlinear model is a model adjusted with jumping volatility known from Avellaneda and Paras [3]. Models including feedback and illiquid market effects due to large traders choosing given stock–trading strategies was developed by

Frey and Patie [16], Frey and Stremme [17], During and et al. [13], Schönbruchen and Wilmott [31]. There is also a nonlinear generalization proposed by Barles and Sonner[6] for the description of imperfect replication and investor's preferences. Another model that takes into account risk from unprotected portfolio is proposed by Kratka [22] and Jandačka and Ševčovič in [21], [30].

One of the models dealing with transaction costs is model proposed by Grossinho and Morais [18]. The model proposed by Avellaneda, Levy and Paras [4] is aligned with the Barles and Soner model [6] where it is assumed that investor's preferences are characterized by an exponential utility function. The next is the Risk adjusted pricing methodology (RAPM) model proposed by Kratka [22] and its generalization by Jandačka and Ševčovič in the work [21]. Last but not least is the model with linear decreasing transaction costs depending on volume of trading stocks [1] by authors Amster, Averbuj, Marian and Rial with transaction costs as a function of the amount of traded assets.

In this section we will go into more detail through the Leland model [26] and Risk Adjusted Pricing Methodology (RAPM) model proposed by Kratka [22] and its generalization by Jandačka and Ševčovič in the work [21]. We will also use the variable transaction costs in the model following Amster, Averbuj, Mariani and Rial [1].

In section 1.1 - 1.6 we review some of the known nonlinear models. The aim of this work is modelling in Section 1.7, with comparison to the model proposed by Amster et al. and RAPM model.

1.1 Jumping Volatility Model

Avellaneda, Levy and Paras [4] proposed a model for the description of incomplete markets and uncertain but bounded volatility. In their model we have

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \begin{cases} \sigma_+^2, & \text{if } S\partial_S^2 V > 0, \\ \sigma_-^2, & \text{if } S\partial_S^2 V < 0. \end{cases} \tag{2}$$

where σ_- and σ_+ represent volatility of the asset, where option is in the long position or short position respectively.

1.2 Leland Model

The Leland model published in paper [26] has been further generalized to more complex options strategies by Hoggard, Whalley and Wilmot in [19]. We present the derivation of a more general model in Section 1.7, of which the Leland model is just a special case.

Nonlinearity in the diffusion coefficient is in the form

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2\left(1 - \mathrm{Le}\,\mathrm{sgn}\left(S\partial_S^2 V\right)\right) = \begin{cases} \sigma^2(1-\mathrm{Le}), & \text{if } S\partial_S^2 V > 0, \\ \sigma^2(1+\mathrm{Le}), & \text{if } S\partial_S^2 V < 0, \end{cases} \tag{3}$$

where $\text{Le} = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}$ is the Leland number and σ is constant historical volatility, $C_0 > 0$ is a constant transaction cost per unit dollar of transaction in the assets market and Δt is the time–lag between portfolio adjustments.

1.3 Model with Investor's Preferences

Barles & Soner derived in [6] a particular nonlinear adjusted volatility of the form

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2\Big(1 + \Psi\left(a^2 e^{r\tau} S^2 \partial_S^2 V\right)\Big), \tag{4}$$

where $a > 0$ includes a risk aversion of investor and also proportional transaction cost.

1.4 Model with Linear Decreasing Transaction Costs Depending on the Volume of Trading Stocks

Amster, Averbuj, Mariani and Rial in their work [1] assume that the costs behave as a non–increasing linear function, depending on the trading stocks needed to hedge the replicating portfolio. They proposed the model, where the transaction costs are not proportional to the amount of the transaction, but the individual cost of the transaction of each share diminishes as the number of traded shares increases. Therefore transaction cost function is given by

$$C(\xi) = C_0 - \kappa\xi, \tag{5}$$

where ξ is the volume of trading stocks, i.e. $\xi = |\Delta\delta|$ and $C_0, \kappa > 0$ are constants depending on the individual investor. The number of bought or sold assets depends on the one–time step change of δ, i.e. stocks hold in the portfolio. The main idea is decreasing transaction cost with increasing amount of transaction. It can be seen as a discount for a large deal attractive for large investors.

1.5 Liquidity Model

Bakstein and Howison in their paper [5] *A Non–Arbitrage Liquidity Model with Observable Parameters* in 2003 introduced the model including three of the already mentioned models namely the classical B–S, Leland and model proposed by Amster et al.. They developed a parametrised model for liquidity effects arising from the trading in an asset. Here $\hat{\sigma}^2$ is the following quadratic function of $\Gamma = \partial_S^2 V$:

$$\begin{aligned}\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2\Bigg(& 1 + \bar{\gamma}^2(1-\alpha)^2 + 2\lambda S\partial_S^2 V + \lambda^2(1-\alpha)^2\left(S\partial_S^2 V\right)^2 + \\ & + 2\sqrt{\frac{2}{\pi}}\bar{\gamma}\,\mathrm{sgn}\left(S\partial_S^2 V\right) + 2\sqrt{\frac{2}{\pi}}\lambda(1-\alpha)^2\bar{\gamma}\left|S\partial_S^2 V\right|\Bigg)\end{aligned} \tag{6}$$

1.6 Risk Adjusted Pricing Methodology Model

The next example of the Black–Scholes equation with a nonlinearly depending volatility we present is the RAPM model (Risk adjusted pricing methodology model) proposed by Kratka in [22] and revisited by Jandačka and Ševčovič in [21]. The volatility is in the following form:

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2 \left(1 - \mu \left(S\partial_S^2 V\right)^{\frac{1}{3}}\right), \tag{7}$$

where $\sigma > 0$ is a constant historical volatility of the asset price return and

$$\mu = 3(C_0^2 R/2\pi)^{\frac{1}{3}}, \tag{8}$$

where $C_0, R \geq 0$ are non–negative constants representing cost measure and the risk premium measure, respectively.

1.7 Models with Variable Transaction Costs

The aim of this section is to present a new approach taking into account variable transaction costs in a more general form of a **decreasing or non–increasing function of the amount of transactions**, $|\Delta\delta|$, per unit of time Δt, i.e. $C = C(|\Delta\delta|)$.

One of the key assumptions of the Black–Scholes analysis is the continuous rehedging of a portfolio. In connection with the transaction costs for buying and selling the underlying asset, continuing hedging would lead to an infinite number of transactions and unbounded total transaction costs. The Leland [26], and Hoggard, Whalley and Wilmott [19], models are based on a simple, but very important modification of the Black–Scholes model, which includes transaction costs and rearranging of the portfolio at discrete times. Since the portfolio is maintained at regular intervals, this means that the total transaction costs are limited.

The assumptions of our new model are in general the same as for the Black–Scholes model with the following extensions. Some of the conditions are adapted from Wilmott, Dewynne and Howison [32] and Ševčovič, Stehlíková and Mikula [29]:

Modelling variable transaction costs for large investors

Large investors can have some kind of discount, because of large transaction amounts. The more they purchase in one transaction, the less will they pay for one traded underlying asset. In general, we will assume that the cost C per one transaction is a non-increasing function of the amount of transactions, $|\Delta\delta|$, per unit of time Δt, i.e.

$$C = C(|\Delta\delta|). \tag{9}$$

This means that the purchase of $\Delta\delta > 0$ or sales of $\Delta\delta < 0$ shares at a price of S, we calculate the additional transaction cost per unit of time Δt:

$$\Delta TC \equiv \frac{S}{2} C(|\Delta\delta|)|\Delta\delta| \tag{10}$$

units;

1.7.1 Constant Transaction Costs Function

This subsection contains a case of the constant transaction costs function and its modification $\tilde{C}$ introduced in the previous section. We refer to classical function of Leland model [26] from Section(1.2) and also assumption $[C_4]$ in Subsection 1.7.

In the Leland model the function of transaction costs C has the form:

$$C(\xi) \equiv C_0, \quad \text{for } \xi \geq 0, \tag{11}$$

where $C_0 > 0$ denotes constant transaction costs.

The modified transaction costs function of the Leland model is:

$$\tilde{C}(\xi) \equiv C_0\sqrt{\frac{2}{\pi}}, \quad \text{for } \xi \geq 0, \tag{12}$$

where $C_0 > 0$ denotes the constant transaction costs of the original Leland model. Both of them are shown in Figure 1. They are depicted for the parameter value $C_0 = 0.02$.

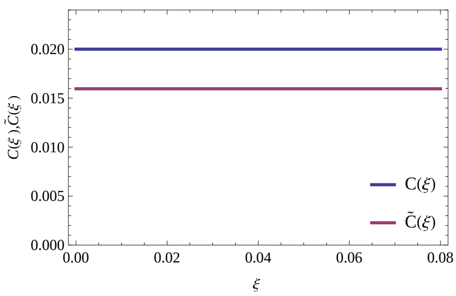


Figure 1: Constant transaction costs functions by Leland model.

1.7.2 Linear Decreasing Transaction Costs Function

In the model proposed by Amster et al. [1], which was introduced in Subsection 1.4, the function C is linear and decreasing:

$$C(\xi) \equiv C_0 - \kappa\xi, \quad \text{for } \xi \geq 0, \tag{13}$$

where $C_0 > 0$ denotes constant transaction costs and $\kappa \geq 0$ is the rate at which transaction costs decrease (measured per one transaction).

The modified transaction costs function of the model proposed by Amster et al. has the form:

$$\tilde{C}(\xi) \equiv C_0\sqrt{\frac{2}{\pi}} - \kappa\xi, \quad \text{for } \xi \geq 0, \tag{14}$$

where constants C_0 and κ are the same as in the original model.

A disadvantage of the function (13) lies in the fact that it may attain negative values provided the amount of transactions $|\Delta\delta|$ exceeds the critical value $\xi = |\Delta\delta| = C_0/\kappa$. For illustration see Figure 2. In the figure there are functions depicted for parameter values $C_0 = 0.02$ and $\kappa = 0.5$.

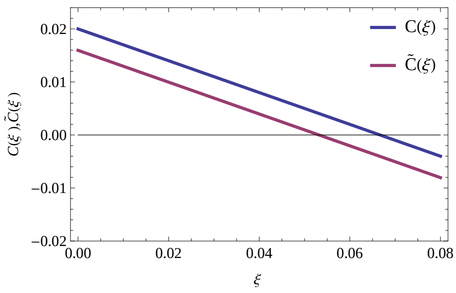


Figure 2: Linear decreasing transaction costs functions.

1.7.3 Piecewise Linear Non–Increasing Transaction Costs Function

In this section we present a reasonable example of realistic transaction costs that are also decreasing with the amount of transactions as in model studied by Amster. The benefit is the elimination of the problem of negative values of the linear decreasing costs function. We define the following piecewise linear function.

Definition 1.1. *We define a piecewise linear non–increasing transaction costs function as*

$$C(\xi) = \begin{cases} C_0, & \textit{if} \quad 0 \le \xi < \xi_-, \\ C_0 - \kappa(\xi - \xi_-), & \textit{if} \quad \xi_- \le \xi \le \xi_+, \\ \underline{C}_0, & \textit{if} \quad \xi \ge \xi_+. \end{cases} \tag{15}$$

where we assume $C_0, \kappa > 0$, *and* $0 \le \xi_- \le \xi_+ \le \infty$ *to be given constants and* $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$.

This is the most realistic function, because for some small volume of traded stocks one constant amount C_0 is paid, when the volume is significant, there starts to be a discount depending on higher volume and finally some another small constant payment $\underline{C}_0$ when there are very large trades.

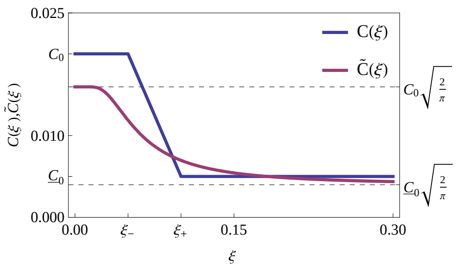


Figure 3: A piecewise linear transaction costs function C and its modification $\tilde{C}$.

This function also covers classical transaction costs functions and it satisfies all assumptions we need when modelling and optimizing in Section 2.2. It is easy to see that this example includes all of the previous observations because in the case of:

- if $\xi_- = \xi_+ = 0$ then the function C is constant, that means it is the same as in the Leland model ;
- if $\xi_- = 0$ and $\xi_+ = \infty$ then the function C is linearly decreasing, i.e. the same as in the model studied by Amster.

In the next part we will present the detailed derivation of the modified transaction costs function $\tilde{C}$ for this type of piecewise linear non–increasing function C. For comparison of original C and modified $\tilde{C}$ function see Figure 3. These functions are depicted for parameter values $C_0 = 0.02$, $\kappa = 0.3$, $\xi_- = 0.05$ and $\xi_+ = 0.1$.

Proposition 1.1. *The modified transaction costs function $\tilde{C}$ of piecewise linear function (15) is given by:*

$$\tilde{C}(\xi) = C_0\sqrt{\frac{2}{\pi}} - 2\kappa\xi \int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x, \quad \textit{for } \xi \geq 0. \tag{16}$$

Proposition 1.2. *Let $\tilde{C}(\xi)$ be a function defined by equation (16). Then the $\tilde{C}(\xi)$ has the following properties:*

(*i*) $\tilde{C}(0) = C_0\sqrt{\frac{2}{\pi}}$;

(*ii*) $\tilde{C}'(\xi) = -2\kappa\int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} f(x)\mathrm{d}x + 2\kappa\left[\frac{\xi_+}{\xi}f\left(\frac{\xi_+}{\xi}\right) - \frac{\xi_-}{\xi}f\left(\frac{\xi_-}{\xi}\right)\right] < 0 \quad \textit{for } \xi > 0$;

(*iii*)

$$\tilde{C}'(0) = \begin{cases} -\kappa, & \textit{if } \xi_- = 0, \\ 0, & \textit{if } \xi_- > 0; \end{cases}$$

(*iv*) $\tilde{C}''(\xi) = 2\kappa\left[\frac{\xi_+^3}{\xi^4}f\left(\frac{\xi_+}{\xi}\right) - \frac{\xi_-^3}{\xi^4}f\left(\frac{\xi_-}{\xi}\right)\right] > 0$, *i.e. $\tilde{C}$ is a convex function if $\xi_- = 0$;*

(*v*) *$\tilde{C}$ need not be convex if $\xi_- > 0$ (see Figure 3);*

(*vi*) $\tilde{C}''(0) \equiv 0$.

Proposition 1.3. *The function $\tilde{C}$ defined in equation (16) satisfies*

$$\underline{C}_0\sqrt{\frac{2}{\pi}} \leq \tilde{C}(\xi) \leq C_0\sqrt{\frac{2}{\pi}} \quad \textit{and} \tag{17}$$

$$\lim_{\xi\to\infty} \tilde{C}(\xi) = \underline{C}_0\sqrt{\frac{2}{\pi}} > 0, \tag{18}$$

where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$ from Definition 1.1.

Proposition 1.4. *Let $\tilde{C}$ be defined by equation (16) with properties (i)-(vi), then for all $\xi \geq 0$*

$$\tilde{C}(\xi) - \xi\tilde{C}'(\xi) + \frac{\xi^2}{2}\tilde{C}''(\xi) \geq \sqrt{\frac{2}{\pi}}\left[C_0 - \kappa(\xi_+ - \xi_-)\right] > 0. \tag{19}$$

We have introduced a universal and reasonable example of a realistic transaction costs function in the form of a piecewise linear function whether ξ_- is zero or not.

1.7.4 Exponentially Decreasing Transaction Costs Function

As an another example of transaction costs that are decreasing with the amount of transactions we can consider the following exponential function of the form

$$C(\xi) = C_0 \exp(-\kappa\xi), \quad \text{for } \xi \geq 0, \tag{20}$$

where $C_0 > 0$ and $\kappa > 0$ are given constants. Its modification:

$$\tilde{C}(\xi) = C_0\sqrt{\frac{2}{\pi}} + \sum_{n=0}^{\infty} \frac{C_0}{n!}(-\kappa\xi)^n \frac{2^{\frac{n}{2}+1}}{\sqrt{2\pi}} \Gamma\left(\frac{n}{2}+1\right), \quad \text{for } \xi \geq 0, \tag{21}$$

where constants are the same as in original. In Figure 4 these functions are depicted for parameter values $C_0 = 0.02$ and $\kappa = 100$.

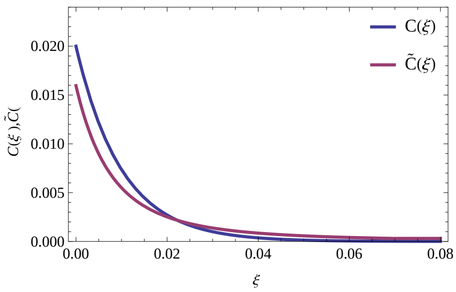


Figure 4: Exponential decreasing transaction costs functions.

This figure was constructed by $\tilde{C}$ of another form than (21). It is because in the case of Tailor's formula the number of elements should be finite and it can cause numerical problems. The value of the function for a high variable ξ goes either to $+\infty$ or to $-\infty$. For this reason we realized another expression for modified transaction costs function of the form:

$$\tilde{C} = C_0\sqrt{\frac{2}{\pi}}\,\phi(-\sqrt{2}\kappa\xi), \quad \text{for } \xi \geq 0, \quad \text{where} \tag{22}$$

$$\phi(x) = 1 + xe^{\frac{x^2}{4}}\left(\operatorname{erf}(x/2)+1\right)\frac{\sqrt{\pi}}{2}. \tag{23}$$

2 A Novel Option Pricing Model under Transaction Costs and Risk of the Unprotected Portfolio

The aim of this section is to present a novel nonlinear generalization of the classical Black–Scholes equation that incorporates both variable transaction costs and the risk arising from a volatile portfolio.

By adding the measures r_{TC} and r_{VP} we obtain a total measure of the risk r_R given by the following relation

$$r_R = r_{TC} + r_{VP}.$$

The total risk premium r_R is a function of Δt, i.e. the time–lag between two consecutive portfolio adjustments. As both r_{TC} as well as r_{VP} depend on the time–lag Δt so does the total risk premium r_R.

In the derivation of the new nonlinear model, we take into account **the variable transaction costs and risk of the unprotected portfolio**.

We again assume that the underlying stock price pays dividends ($q \neq 0$) and follows a geometric Brownian motion $dS = (\rho - q)Sdt + \sigma S dw$. The difference is in the change of the portfolio, here of the form:

$$\Delta\Pi = \Delta V + \delta\Delta S + \delta q S\Delta t - r_R S\Delta t, \tag{24}$$

where r_R is total risk $r_R = r_{TC} + r_{VP}$. This risk includes transaction costs in addition to the level of risk of the unprotected portfolio. They are being considered because a large rearranging interval Δt leads to smaller transaction costs, at the same time, however, the investor is in danger, because the portfolio is for a long time unprotected.

The transaction cost measure r_{TC} is due to a variable transaction cost $C = C(|\Delta\delta|)$ the same as we defined in equation $r_{TC}S\Delta t = \frac{S}{2}\alpha\,\tilde{C}(\alpha)$, where $\alpha = \sigma S\,|\partial_S^2 V|\,\sqrt{\Delta t}$. The measure r_{VP} of risk following from the unprotected portfolio we adopt in the form $r_{VP} = \frac{1}{2}R\sigma^4 S^2\left(\partial_S^2 V\right)^2\Delta t$. To simplify notation we use

$$\Gamma = \partial_S^2 V. \tag{25}$$

The final equation for the new model then is

$$\partial_t V + \frac{1}{2}\hat{\sigma}^2(S\Gamma, \Delta t)S^2\partial_S^2 V + (r-q)S\partial_S V - rV = 0, \tag{26}$$

with volatility having form

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2\left(1 - \tilde{C}(\sigma|S\Gamma|\sqrt{\Delta t})\frac{\mathrm{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} - R\sigma^2 S\Gamma\Delta t\right). \tag{27}$$

It is a generalization of the model with decreasing transaction costs studied by Amster et al., hence the model includes variable transaction costs, for example, piecewise linear non-increasing or exponentially decreasing, from section 1.7 in the form of a general function of transaction costs $\tilde{C}$. At the same time there is a possibility to control the risk of an unprotected portfolio. That means including the last term with the risk premium coefficient R, the model is in combination also with the RAPM model. In this form the nonlinear volatility (27) is with unprescribed time–lag interval Δt, but in Subsection 2.2 we will show how to find this optimal hedging time.

For the purpose of the numerical analysis it is convenient to introduce the following function

$$\beta(H, x, \tau) \equiv \frac{1}{2}\hat{\sigma}^2(S\Gamma, S, t)S\Gamma, \tag{28}$$

where $H := S\Gamma$, $x = \ln S/E, \tau = T - t$.

More specifically, in our case of the RAPM based model, the function β of the novel nonlinear model reads as follows:

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \tilde{C}(\sigma|H|\sqrt{\Delta t})\frac{\mathrm{sgn}(H)}{\sigma\sqrt{\Delta t}} - R\sigma^2 H\Delta t\right)H. \tag{29}$$

2.1 Special Cases of the Novel Model

In this section we give some special cases of the new model. We see that the new model is a generalization of some known nonlinear models. For different choices of $\tilde{C}$ and R we obtain the following special forms.

2.1.1 Model with Linear Decreasing Transaction Costs Depending on the Volume of Trading Stocks

Similarly, by setting:

- the transaction costs as a non–constant $C \neq const$, for example is linearly decreasing, i.e.
$$C\left(|\Delta\delta|\right) = C_0 - \kappa|\Delta\delta|,$$
- the risk premium coefficient arising from unprotected portfolio equal to zero, i.e. $R = 0$ and
- by given time–lag Δt,

the volatility function $\hat{\sigma}^2$ given by (27) reduces to nonlinear volatility

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2\left(1 - \mathrm{Le}\,\mathrm{sgn}(S\Gamma) + \kappa S\Gamma\right), \tag{30}$$

where $\mathrm{Le} = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}$ is the Leland number (compare with the model proposed by Amster et al. in Section 1.4).

2.1.2 RAPM Model with Variable Transaction Costs with Fixed Time–Lag Interval

We obtain an another example by setting

- the transaction costs as a non–constant $C \neq const$, for example, a linearly decreasing function from model proposed by Amster et al., i.e.
$$C\left(|\Delta\delta|\right) = C_0 - \kappa|\Delta\delta|,$$

- the risk premium coefficient arising from an unprotected portfolio not equal to zero, i.e. $R \neq 0$ and
- the time–lag Δt given.

Then the volatility function $\hat{\sigma}^2$ given by (27) reduces to a nonlinear volatility of the form:

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2 \left(1 - \left(\frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}\mathrm{sgn}(S\Gamma) + R\sigma^2 S\Gamma\Delta t\right) + \kappa S\Gamma\right). \tag{31}$$

It is a combination of volatility from the model proposed by Amster et al. and the RAPM Model with an unprescribed time–lag interval.

2.2 RAPM Based Models with the Optimal Choice of Hedging Time Δt

Our task is now to minimize the total risk of the portfolio to find the optimal time Δt when rehedging the portfolio. Clearly, in order to minimize transaction costs, we have to take a larger time–lag Δt. On the other hand, a larger time interval Δt means higher risk exposure for the investor, because an increase in the time–lag interval Δt between two consecutive transactions leads to a linear increase of the risk from a volatile portfolio.

In the first part of this section we will review the basic idea proposed by Jandačka and Ševčovič in the RAPM model [21] for constant transaction costs and in the second part we will give a general approach when the variable transaction costs function will be taken into account. We postulate the basic assumptions on admissible transformed functions of transaction costs $\tilde{C}$.

2.2.1 Classical RAPM Model

In this subsection, we will discuss the choice of an optimal time interval between two consecutive portfolio adjustments according to Jandačka and Ševčovič in the paper [21]. The name of the model is the Risk adjusted pricing methodology (RAPM) model.

The coefficient r_{TC} is given by the formula

$$r_{TC} = \frac{C_0|\Gamma|\sigma S}{\sqrt{2\pi}}\frac{1}{\sqrt{\Delta t}} \tag{32}$$

(cf. [19, equation 3]).

Next we recall the expression for the risk premium r_{VP}. The risk from the volatile portfolio is of the form

$$r_{VP} = \frac{1}{2}R\sigma^4 S^2\Gamma^2\Delta t.$$

where $R \geq 0$ is non–negative constant representing the level of risk of the unprotected portfolio.

By increasing the time–lag interval Δt between portfolio adjustments, we can decrease transaction costs. Therefore, in order to minimize transaction costs, we have to take larger time–lag Δt. On the other hand, a larger time interval Δt means higher risk exposure for the investor, because an increase in the time–lag interval Δt between two consecutive transactions leads to a linear increase of the risk from a volatile portfolio.

Now move to solution of this problem of minimizing the value of the total risk premium $r_R = r_{TC} + r_{VP}$.In order to find the optimal value of Δt, we have to minimize the following function:

$$\Delta t \mapsto r_R = r_{TC} + r_{VP} = \frac{C_0|\Gamma|\sigma S}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}} + \frac{1}{2} R\sigma^4 S^2 \Gamma^2 \Delta t. \tag{33}$$

A graph of the total risk premium as a function of the time–lag Δt is depicted in the Figure 5. The unique minimum of the function is attained at the time–lag

$$\Delta t_{opt} = \frac{K^2}{\sigma^2 |S\Gamma|^{2/3}}, \quad \text{where } K = \left(\frac{C_0}{R} \frac{1}{\sqrt{2\pi}} \right)^{1/3}. \tag{34}$$

Therefore the minimal value of the function $\Delta t \mapsto r_R(\Delta t)$ we have

$$r_R(\Delta t_{opt}) = \frac{3}{2} \left(\frac{C_0^2 R}{2\pi} \right)^{1/3} \sigma^2 |S\Gamma|^{4/3}. \tag{35}$$

Finally by taking the optimal value of the total risk coefficient r_R, we get the

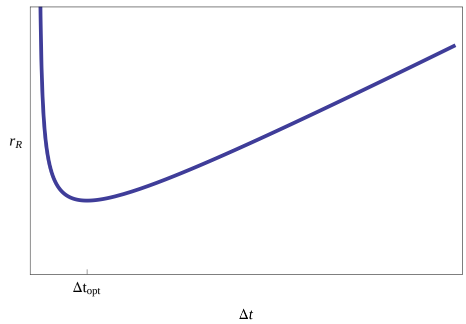


Figure 5: The function of total risk premium $\Delta t \mapsto r_R(\Delta t) = r_{TC} + r_{VP}$ attains its unique minimum at the point Δt_{opt}, i.e. optimal time–lag between two consecutive portfolio adjustments.

following generalization of the Black–Scholes equation

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (r - q) S \partial_S V - rV - r_R S = 0, \tag{36}$$

can be written as the following nonlinear parabolic equation:

$$\partial_t V + \frac{\sigma^2}{2} S^2 \left(1 + \mu \left(S \partial_S^2 V\right)^{1/3}\right) \partial_S^2 V + (r-q) S \partial_S V - rV = 0, \tag{37}$$

where $\mu = 3(\bar{C}^2 R/(2\pi))^{\frac{1}{3}}$ and Γ^p with $\Gamma = \partial_S^2 V$ and $p = 1/3$ stands for the signed power function, i.e., $\Gamma^p = |\Gamma|^{p-1}\Gamma$. We note that the equation is a backward parabolic PDE if and only if the function

$$\beta(H) = \frac{\sigma^2}{2}(1 + \mu H^{1/3}) H \tag{38}$$

is an increasing function in the variable $H := S\Gamma = S\partial_S^2 V$. It is satisfied if $\mu \geq 0$ and $H \geq 0$.

2.2.2 Optimal Choice of Hedging Time Δt in the Novel Model

Our task now is minimization of the total measure of risk. We will choose Δt as the arg min of $r_R = r_R(t)$, i.e.

$$\min_{\Delta t > 0} r_R = \min_{\Delta t > 0} (r_{TC} + r_{VP}).$$

It can be also viewed as the argument of maximum of the variance function (29) $\hat{\sigma} = \hat{\sigma}^2(S\Gamma, \Delta t)$, this means

$$\max_{\Delta t > 0} \hat{\sigma}^2(S\Gamma, \Delta t) = \max_{\Delta t > 0} \sigma^2 \left(1 - \tilde{C}\left(\sigma |S\Gamma| \sqrt{\Delta t}\right) \frac{\operatorname{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} - R\sigma^2 S\Gamma \Delta t\right),$$

i.e. finding the time interval where $\tilde{C}\left(\sigma |S\Gamma| \sqrt{\Delta t}\right) \frac{\operatorname{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} + R\sigma^2 S\Gamma \Delta t$ attains its minimum value:

$$\Delta t_* = \arg\min_{\Delta t > 0} \left(\tilde{C}\left(\sigma |S\Gamma| \sqrt{\Delta t}\right) \frac{\operatorname{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} + R\sigma^2 S\Gamma \Delta t\right). \tag{39}$$

In the following definition, we will postulate the basic assumptions on admissible transformed functions of transaction costs $\tilde{C}$. These assumptions will enable us to use such functions for the generalization of a risk adjusted model for pricing the derivatives of the underlying assets.

Definition 2.1. *Let $C : \mathbb{R}_0^+ \to \mathbb{R}$ be a transaction costs function. We say C is an admissible transaction costs measure if the following conditions are satisfied for the modified transaction costs function $\tilde{C} = \mathbb{E}\big[C(\xi|\Phi|)|\Phi|\big]$:*

(H$_1$) $\tilde{C}(0) > 0$, $\tilde{C}'(\xi) \leq 0$ *for all* $\xi \geq 0$ *and*

(H$_2$) $\tilde{C}(\xi) - \xi\tilde{C}'(\xi) + \frac{\xi^2}{2}\tilde{C}''(\xi) \geq 0$ *for all* $\xi \geq 0$.

As an example we can consider a piecewise linear transaction costs function from Subsection 1.7.3.

Proposition 2.1. *The function $\varphi(\tau)$ attains its unique positive minimum $\Delta t_* > 0$ provided that the function C is admissible transaction costs function.*

Proposition 2.2. *The optimum value $\Delta t = \tau_*^2$ is attained where $\xi_* = b\tau_*$ solves the equation*

$$\tilde{C}(\xi_*) - \tilde{C}'(\xi_*)\xi_* = \nu\xi_*^3, \tag{40}$$

where

$$\nu := \frac{2R}{H^2} = \frac{2R}{S^2\Gamma^2}.$$

For the maximum value of variance we obtain the following relation

$$\hat{\sigma}^2(S\Gamma, \Delta t_*) = \sigma^2(1 - \varphi(\tau_*)) = \sigma^2\left(1 - \tilde{C}(\xi_*)\frac{H}{\xi_*} - \frac{R}{H}\xi_*^2\right).$$

which can be inserted into the modified Black–Scholes equation

$$\partial_t V + \frac{1}{2}\hat{\sigma}^2(S\Gamma, \Delta t_*)S^2\partial_S^2 V + (r-q)S\partial_S V - rV - r_R S = 0. \tag{41}$$

The expression $\hat{\sigma}^2(S\Gamma, \Delta t_*)$ emerging in (41) has the form

$$\hat{\sigma}^2(S\Gamma, \Delta t_*) \quad = \quad \sigma^2\left(1 - \psi(S\Gamma)\right),$$

where the function $\psi = \psi(H)$ is defined in an implicit way

$$\psi(S\Gamma) = \tilde{C}(\xi_*)\frac{H}{\xi_*} + \frac{R}{H}\xi_*^2. \tag{42}$$

We already know, that for given $H = S\Gamma$, we have the unique solution ξ_* of the implicit equation (40). This equation can be cast into an equivalent form

$$H^2\left(\tilde{C}(\xi_*) - \tilde{C}'(\xi_*)\xi_*\right) = 2R\xi_*^3. \tag{43}$$

Finally, by inserting the expression for $r_R S$ into the modified Black–Scholes equation (41), we obtain the following RAMP equation, which takes into account the variable transaction costs

$$\partial_t V + \frac{1}{2}\sigma^2 S(S\Gamma - S\Gamma\psi(S\Gamma)) + (r-q)S\partial_S V - rV = 0.$$

If we define an auxiliary function

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \psi(H)\right)H, \tag{44}$$

then the modified Black–Scholes equation becomes

$$\partial_t V + S\beta\left(H\right) + (r-q)S\partial_S V - rV = 0. \tag{45}$$

The advantage of this novel model is that many of the known models are included, for example the Leland model, and the model studied by Amster et al.. We can extend analysis by using a more realistic piecewise linear non–increasing function.

Example 1. *For the linear decreasing transaction costs function given by the model studied by Amster, i.e.* $C(\xi) = C_0 - \kappa\xi$*, the function can be expressed analytically. The equation (43) has the following form*

$$H^2 \left(C_0 - \kappa\xi_* - (-\kappa)\xi_*\right) = \frac{\pi^2 R}{2}\xi_*^3,$$

and therefore, similarly as in the classical RAMP model

$$\xi_* = \left(\frac{C_0}{2R}\sqrt{\frac{2}{\pi}}H^2\right)^{\frac{1}{3}}.$$

By inserting ξ_* *into (42), we obtain for the function* $\psi(H)$ *the following relation*

$$\psi(H) = \left(\sqrt{\frac{2}{\pi}}\frac{C_0}{\xi_*} - \kappa\right)H + \frac{R}{H}\xi_*^2 = \mu H^{\frac{1}{3}} - \kappa H,$$

where $\mu = 3(C_0^2 R/(2\pi))^{\frac{1}{3}}$ *and and* H^p *with* $H = S\partial_S^2 V$ *and* $p = 1/3$ *stands for the signed power function, i.e.,* $H^p = |H|^{p-1}H$*. Thus the function* β *has the form*

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \mu H^{1/3} + \kappa H\right)H.$$

Note, that function β *is increasing for* $\frac{\mu^3}{\kappa} < (\frac{27}{8\sqrt{2}})^2 \approx 5.6953125$.

3 Gamma Equation

In this section, we introduce the Gamma equation proposed in the article [21] by Jandačka and Ševčovič (see also Ševčovič, Stehlíková and Mikula [30, p. 174]). The goal is to present the transformation of the the nonlinear Black–Scholes equation into a quasilinear parabolic equation.

Let us consider the previously mentioned modified nonlinear Black–Scholes equation with the nonlinear volatility of a general type included in the β function

$$\partial_t V + S\beta(H) + (r - q)S\partial_S V - rV = 0, \quad S > 0, t \in (0, T), \tag{46}$$

where the form of the function $\beta(H), H = S\Gamma$ depends on the model we use.

The idea how to analyse and solve this equation is based on the transformation method. We consider the standard change of independent variables, as usual in the classical Black–Scholes theory [7]:

$$x := \ln(S/E),\ x \in (-\infty, \infty), \quad \text{and} \quad \tau := T - t, \tau \in (0, T). \tag{47}$$

The transformation of the space, $x = \ln(S/E)$, stretches the domain to the whole set of real numbers. Substituting $\tau = T - t$ transforms the backward parabolic differential equation to a forward one. Since the equation (46) contains the term $S\Gamma = S\partial_S^2 V$ it is convenient to use the following transformation:

$$H(x,\tau) := S\Gamma = S\partial_S^2 V(S,t). \tag{48}$$

After this transformation β can be a function of H, x and τ, i.e. $\beta = \beta(H, x, \tau)$.

The so–called Γ *equation* can be obtained if we compute the second derivative of the equation (46) with respect to x according to Jandačka and Ševčovič [21] (see also Ševčovič, Stehlíková and Mikula in [30], Mikula and Kútik in [23] and [24]).

Theorem 3.1. *Function $V = V(S,t)$ is a solution to (46) if and only if $H = H(x,\tau)$ solves*

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + (r-q)\partial_x H - qH, \tag{49}$$

where β is a composed function

$$\beta = \beta(H(x,\tau), x, \tau).$$

4 Computational Results

The purpose of this section is to derive a robust numerical scheme for solving the Γ equation. The construction of numerical approximation of a solution H to (49) is based on a derivation of a system of difference equations corresponding to (49) to be solved at every discrete time step. We give also the Mathematica source using the model with variable transaction cost. Next we show the modelling of the bid–ask spread and perform extensive comparisons of the solutions of the models.

4.1 Numerical Scheme for the Full Space–Time Discretization and for Solving the Γ-Equation

In this section we present the numerical scheme adopted from the paper by Jandačka and Ševčovič [21] in order to solve the Γ equation (49) for a general function $\beta = \beta(H, x, \tau)$ including, in particular, the case of the model with variable transaction costs. The efficient numerical discretization is based on the finite volume approximation of the partial derivatives entering (49). The resulting scheme is semi–implicit in a finite–time difference approximation scheme.

For numerical reasons we restrict the spacial interval to $x \in (-L, L)$ where $L > 0$ is sufficiently large. Since $S = Ee^x$ it is now a restricted interval of underlying stock values, $S \in (Ee^{-L}, Ee^{L})$. From a practical point of view, it is sufficient to take $L \approx 1.5$ in order to include the important range of values of S.

For the purpose of construction of a numerical scheme, the time interval $[0, T]$ is uniformly divided with a time step $k = T/m$ into discreet points τ_j, where $j = 0, 1, ..., m$, $\tau_j = jk$. We also take the spacial interval $[-L, L]$ with uniform division with a step $h = L/n$, into discreet points $x_i = ih$, where $i = -n, \ldots, n$.

Now the homogeneous Dirichlet boundary conditions on new discrete values representing the initial condition are $H_i^0 = \bar{H}(x_i)$ where $x_i = ih$.

The numerical algorithm is semi–implicit in time. Notice that the term $\partial_x^2 \beta$, where $\beta = \beta(H(x,\tau), x, \tau)$ can be expressed in the form

$$\partial_x^2 \beta = \partial_x \left(\beta'_H(H,x,\tau)\partial_x H + \beta'_x(H,x,\tau) \right),$$

where β'_H and β'_x are partial derivatives of the function $\beta(H,x,\tau)$ with respect to H and x, respectively:

$$\partial_x \beta = \beta'_H \partial_x H + \beta'_x, \tag{50}$$

$$\partial_x^2 \beta = \beta'_H \partial_x^2 H + \beta''_{HH}(\partial_x H)^2 + 2\beta''_{xH}\partial_x H + \beta''_{xx}. \tag{51}$$

In the discretization scheme, the nonlinear terms $\beta'_H(H,x,\tau)$ and $\beta'_x(H,x,\tau)$ are evaluated from the previous time step τ_{j-1} whereas linear terms are solved at the current time level.

Such a discretization leads to a solution of linear systems of equations at every discrete time level.

The next steps are as follows, at first, we replace the time derivative by the time difference, approximate H in nodal points by the average value of neighbouring segments, then we collect all linear terms at the new time level j and by taking all the remaining terms from the previous time level $j-1$ we obtain a *tridiagonal system* for the solution vector $H^j = (H^j_{-n+1}, \ldots, H^j_{n-1}) \in \mathbb{R}^{2n-1}$:

$$a_i^j H_{i-1}^j + b_i^j H_i^j + c_i^j H_{i+1}^j = d_i^j, \quad H_{-n}^j = 0, \quad H_n^j = 0\,, \tag{52}$$

where $i = -n+1, \ldots, n-1$ and $j = 1, \ldots, m$.

The coefficients of the tridiagonal matrix are given by

$$\begin{aligned}
a_i^j &= -\frac{k}{h^2}\beta'_H(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1}) + \frac{k}{2h} r\,, \\
c_i^j &= -\frac{k}{h^2}\beta'_H(H_i^{j-1}, x_i, \tau_{j-1}) - \frac{k}{2h} r\,, \\
b_i^j &= 1 - (a_i^j + c_i^j)\,, \\
d_i^j &= H_i^{j-1} + \frac{k}{h}\Big(\beta(H_i^{j-1}, x_i, \tau_{j-1}) - \beta(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1}) \\
&\quad + \beta'_x(H_i^{j-1}, x_i, \tau_{j-1}) - \beta'_x(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1}) \Big)\,.
\end{aligned}$$

It means that the vector H^j at the time level τ_j is a solution to the system of linear equations $\mathbf{A}^j\, H^j = d^j$, where the $(2n-1)\times(2n-1)$ matrix $\mathbf{A}^j$ is defined as

$$\mathbf{A}^j = \begin{pmatrix}
b_{-n+1}^j & c_{-n+1}^j & 0 & \cdots & 0 \\
a_{-n+2}^j & b_{-n+2}^j & c_{-n+2}^j & & \vdots \\
0 & \cdot & \cdot & \cdot & 0 \\
\vdots & \cdots & a_{n-2}^j & b_{n-2}^j & c_{n-2}^j \\
0 & \cdots & 0 & a_{n-1}^j & b_{n-1}^j
\end{pmatrix}. \tag{53}$$

To solve the tridiagonal system in every time step in a fast and effective way, we can use the simple LU – matrix decomposition. The key idea is in decomposition of a matrix $\mathbf{A}$ into a product of two matrices, i.e., $\mathbf{A} = \mathbf{L}.\mathbf{U}$, where L is lower and U is an upper triangular matrix respectively (for more details see example [30, Chapter 10]).

The option price $V(S, T - \tau_j)$ can be constructed from the discrete solution H_i^j as follows:

$$\begin{aligned} &\text{(call option)} \qquad & V(S, T - \tau_j) &= h \sum_{i=-n}^{n} (S - Ee^{x_i})^+ H_i^j, \\ &\text{(put option)} \qquad & V(S, T - \tau_j) &= h \sum_{i=-n}^{n} (Ee^{x_i} - S)^+ H_i^j, \end{aligned}$$

for $j = 1, \ldots, m$.

4.2 Numerical results for the nonlinear model with variable transaction costs

In this section we present the numerical results for the approximation of the option price. Recall that we solve nonlinear models of the Black–Scholes type, particularly, the novel option pricing model under transaction costs and risk of the unprotected portfolio.

Into the numerical scheme enters the $\beta(H)$ function derived given in (29) as:

$$\beta(H) = \frac{\sigma^2}{2} \left(1 - \tilde{C}(\sigma |H| \sqrt{\Delta t}) \frac{\operatorname{sgn}(H)}{\sigma \sqrt{\Delta t}} - R\sigma^2 H \Delta t \right) H,$$

where $\tilde{C}$ is the modified transaction cost function. For numerical experiments we take the coefficient of risk premium equal to zero, i.e., $R = 0$. Hence we notice that the nonlinearity arises from the transaction costs. Hence we take the optimal hedging time, Δt, as fixed. Though, it is possible to do the numerical experiments for the case $R > 0$ and Δt is optimal, however we will not do the optimization for the hedging time Δt.

From the variable transaction costs functions we choose the piecewise linear non–increasing function. In practise it means that for some small volume of traded stocks one constant amount C_0 is paid; when the volume is significant, there starts to be a discount depending on a higher volume and finally there is another small constant payment $\underline{C}_0$ when the trades are very large.

The piecewise linear non–increasing transaction costs function is defined as:

$$C(\xi) = \begin{cases} C_0, & \text{if} \quad 0 \le \xi < \xi_-, \\ C_0 - \kappa(\xi - \xi_-), & \text{if} \quad \xi_- \le \xi \le \xi_+, \\ \underline{C}_0, & \text{if} \quad \xi \ge \xi_+. \end{cases} \tag{54}$$

Table 1: Parameter values used for computation of the numerical solution.

Parameter and Value	
$C_0 = 0.02$	$T = 1.$
$\kappa = 0.3$	$E = 25$
$\xi_- = 0.05$	$r = 0.011$
$\xi_+ = 0.1$	$m = 200$
$\Delta_t = 0.00383142$	$n = 250$
$\sigma = 0.3$	$h = 0.01$
$\sigma_{min} = 0.112511$	$\tau^* = 0.005$
$\sigma_{max} = 0.265828$	$R = 0$

where we assume C_0, $\kappa > 0$, and $0 \leq \xi_- \leq \xi_+ \leq \infty$ to be given constants and $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$.

The parameter values used in our computations are given in the Table 1.

According to Proposition 1.3 the function $\tilde{C}$ satisfies the following inequality (17):

$$\underline{C}_0\sqrt{\frac{2}{\pi}} \leq \tilde{C}(\xi) \leq C_0\sqrt{\frac{2}{\pi}}.$$

In what follows, we show that this restriction holds also for the numerical solution. That means, the solution of the nonlinear equation with variable transaction costs $\tilde{C}$ will be always between the solution of the Black–Scholes equation with constant transaction costs (i.e. the Leland model) with higher C_0 and lower $\underline{C}_0$ respectively. Values $\underline{C}_0\sqrt{2/\pi}$ and $C_0\sqrt{2/\pi}$ correspond to the modified transaction costs function $\tilde{C}$ in the case when $\tilde{C}$ is constant.

For Δt sufficiently small, we have from Proposition 1.3 that the equation to be solved is parabolic. For any value of ξ_+ and ξ_-, the $\tilde{C}(\xi)$ will lie between the values $\underline{C}_0\sqrt{2/\pi}$ and $C_0\sqrt{2/\pi}$ and the solutions will be ordered in this manner:

$$V_{\sigma^2_{min}}(S,t) \leq V_{vtc}(S,t) \leq V_{\sigma^2_{max}}(S,t) \quad \forall S,t.$$

In the Table 2 we present the option values for different prices of the underlying asset achieved by a numerical solution.

In Figure 6 we present the graphs of solution $V_{vtc} := V(S,t)$, as well as that of $\Delta(S,t) = \partial_S V(S,t)$, for various times $t \in \{0, T/3, 2T/3\}$. The upper dashed line corresponds to the solution of the linear Black–Scholes equation with volatility $\hat{\sigma}^2_{max} = \sigma^2\left(1 - \underline{C}_0\sqrt{\frac{2}{\pi}\frac{1}{\sigma\sqrt{\Delta t}}}\right)$, where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$, and the lower dashed line corresponds to the solution with volatility $\hat{\sigma}^2_{min} = \sigma^2\left(1 - C_0\sqrt{\frac{2}{\pi}\frac{1}{\sigma\sqrt{\Delta t}}}\right)$.

Note that at the beginning the solution of nonlinear model is closer to the lower bound and later moves closer to the upper one. It can be interpreted as follows: at the beginning of the contract the holder of the portfolio is not required to perform

Table 2: Bid Option values of the numerical solution of nonlinear model in comparison to B–S with constant volatility.

S	$V_{\sigma^2_{max}}$	V_{vtc}	$V_{\sigma^2_{min}}$
20	0.709	0.127	0.029
23	1.752	0.844	0.421
25	2.768	1.748	1.258
28	4.723	3.695	3.474
30	6.256	5.321	5.327

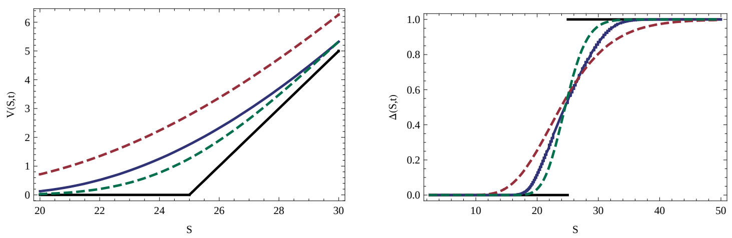
many operations to hedge. Therefore he does not have high volumes of transactions and pays the cost of C_0. With the impending expiry time it is necessary to hedge the portfolio and so trade in high volume, and so the investor pays lower transaction costs, i.e. $\underline{C}_0$.

Conclusions

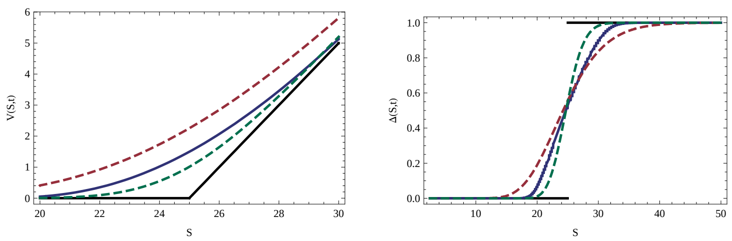
In this thesis we analysed recent topics on pricing derivatives by means of the solutions to nonlinear Black–Scholes equations. We presented various nonlinear generalizations of the classical Black–Scholes theory arising when modelling illiquid and incomplete markets, in the presence of a dominant investor in the market, etc. We did show that, in presence of variable transaction costs and risk from an unprotected portfolio, the resulting novel pricing model is a nonlinear extension of the Black–Scholes equation in which the diffusion coefficient is no longer constant and it depends on the option price itself.

In Section 2 we developed the theory of models with variable transaction costs. The main idea was in defining the modified transaction cost function $\tilde{C}$ when using the transaction costs measure. We also studied the properties of this function to confirm its generality. We presented and analysed two more new examples of realistic variable transaction costs that are decreasing with the amount of transactions, particularly, the piecewise linear nonincreasing function and the exponentially decreasing function. By considering these functions, we solved the difficulty with possibly negative transaction costs that arises in the model proposed by Amster et al. [1]. We developed the Risk adjusted pricing methodology using variable transaction cost instead of constant. We analysed the optimal choice of hedging time as a problem of maximizing the variance to cover the most negative scenario.

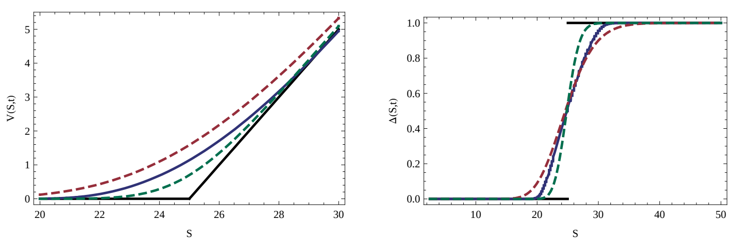
We have also shown how to solve the presented nonlinear Black–Scholes models numerically. In particular, we solved the model with piecewise linear non–increasing function of transaction costs. The main idea was in the transformation of the governing equation into the Gamma equation. Into this equation enters $\beta(H)$ function corresponding to the chosen model.



$t = 0$



$t = T/3$



$t = 2T/3$

Figure 6: Solution $V(S,t)$ for $t = 0, t = T/3, t = 2T/3$ (left) and corresponding $\Delta(S,t) = \partial_S V(S,t)$ of the call option.

In order to solve the Gamma equation we used an efficient numerical discretization. The numerical scheme was based on the finite volume scheme. By numerical solution we obtained the values of the options and showed that when the modified transaction costs function is bounded, then the solution of the novel nonlinear model lies between the solutions of the Black–Scholes equation with constant transaction costs of upper and lower bound.

In general it is difficult to find an explicit solution of general nonlinear models of the Black–Schholes type. An extension of this thesis can be in application of other numerical schemes to deal with the problem of derivative pricing. To solve Gamma equation it is possible to use the scheme of Casabán, Company, Jódar and Pintos [10], the modern schemes by Niu Cheng–hu, Zhou Sheng–Wu [27] and also the scheme designed by Kútik and Mikula [24]. There exist also some explicit solutions for special type of nonlinear models that are known from Bordag and Frey in [8] and [9] to compare the results. Another extension could be the consideration of the other types of financial derivatives, for example American options.

References

[1] Amster, P., Averbuj, C. G., Mariani, M. C., Rial, D. (2005): A Black–Scholes option pricing model with transaction costs. *J. Math. Anal. Appl.* **303**, 688–695.

[2] Ankudinova J., Ehrhardt, M. (2008): On the numerical solution of nonlinear Black–Scholes equations. *Computers and Mathematics with Applications* **56**, 799–812.

[3] Avellaneda, M., Paras, A. (1994): Dynamic Hedging Portfolios for Derivative Securities in the Presence of Large Transaction Costs. *Applied Mathematical Finance* **1**, 165–193.

[4] Avellaneda, M., Levy, A., Paras, A. (1995): Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance* **2**, 73–88.

[5] Bakstein, D., Howison, S. (2003): A Non–Arbitrage Liquidity Model with Observable Parameters. *Mathematical Finance.* (Submitted).

[6] Barles, G., Soner, H. M. (1998): Option Pricing with transaction costs and a nonlinear Black–Scholes equation. *Finance Stochast.* **2**, 369-397.

[7] Black, F., Scholes, M. (1973): The pricing of options and corporate liabilities. *J. Political Economy* **81**, 637–654.

[8] Bordag, L. A., Frey, R. (2008): Pricing Options in Illiquid Markets: Symmetry Reductions and Exact Solutions. *In: Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, Editor: Mathias Ehrhardt, 83–109.

[9] Bordag, L. A. (2009): Study of the risk–adjusted pricing methodology model with methods of Geometrical Analysis. *Stochastics: An International Journal of Probability and Stochastic Process*, **Vol. 00,** No. 00, 1-11.

[10] Casabán, M. C., Company, R., Jódar, L., Pintos, J. R. (2011): Numerical analysis and computing of a non–arbitrage liquidity model with observable parameters for derivatives. *Computers an Mathematics with Applications* **61**, 1951–1956.

[11] Company R., Navarro E., Pintos J.R. and Ponsoda E. (2008): Numerical solution of linear and nonlinear Black-Scholes option pricing equations. *Computers an Mathematics with Applications* **56**, 813—821.

[12] Dremkova, E., Ehrhardt, M. (2010): High-order compact method for nonlinear Black–Scholes option pricing equations of American options. *International Journal of Computer Mathematics* **Vol. 00,** No. 00, 1–18.

[13] During, B., Fournier, M., Jungel, A. (2003): High order compact finite difference schemes for a nonlinear Black–Scholes equation. *Int. J. Appl. Theor. Finance* **7**, 767–789.

[14] Ehrhardt, M.(2008): Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing. Nova Science Publishers, Inc., New York, ISBN 978-1-60456-931-5.

[15] Fabião F., Grossinho, M.R. (2009): Positive solutions of a Dirichlet problem for a stationary nonlinear Black—Scholes equation. *Nonlinear Analysis: Theory, Methods & Applications* **71**(10), 4624–4631.

[16] Frey, R., Patie, P. (2002): Risk Management for Derivatives in Illiquid Markets: A Simulation Study. *In: Advances in Finance and Stochastics*, Springer, Berlin, 137–159.

[17] Frey, R., Stremme, A. (1997): Market Volatility and Feedback Effects from Dynamic Hedging. *Mathematical Finance* **4**, 351–374.

[18] Grossinho, M.R., Morais, E. (2009): A note on a stationary problem for a Black-Scholes equation with transaction costs. *International Journal of Pure and Applied Mathematics* **51**(4), 557–565.

[19] Hoggard, T., Whalley, A. E., Wilmott, P. (1994): Hedging option portfolios in the presence of transaction costs. *Advances in Futures and Options Research* **7**, 21–35.

[20] Hull, J. C. (1989): Options, Futures and Other Derivative Securities. Prentice Hall.

[21] Jandačka, M., Ševčovič, D. (2005): On the risk adjusted pricing methodology based valuation of vanilla options and explanation of the volatility smile. *Journal of Applied Mathematics* **3**, 235–258.

[22] Kratka, M. (1998): No Mystery Behind the Smile. *Risk* **9**, 67–71.

[23] Kútik P., Mikula K. (2011): Finite Volume Schemes for Solving Nonlinear Partial Differential Equations in Financial Mathematics, in Proceedings of the Sixth International Conference on Finite Volumes in Complex Applications, Prague, June 6-10, 2011, Springer Proceedings in Mathematics, Volume 4, 2011, Eds: J.Fořt, J. Fürst, J.Halama, R. Herbin and F.Hubert, 643-561, DOI: 10.1007/978-3-642-20671-9

[24] Kútik P. (2013): Numerical Solution of Partial Differential Equations in Financial Mathematics, Dissertation Thesis, STU, Bratislava.

[25] Kwok, Y. K. (1998): Mathematical Models of Financial Derivatives. Springer-Verlag.

[26] Leland, H. E. (1985): Option pricing and replication with transaction costs. *Journal of Finance* **40** 1283–1301.

[27] Niu Cheng–hu, Zhou Sheng–wu. (2012): Numerical solution of non–arbitrage liquidity model based on uncertain volatility. *Journal of East China Normal University (Natural Science)* **1**.

[28] Ševčovič, D. (2008): Parciálne diferenciálne rovnice a ich aplikácie. IRIS, vydavateľstvo a tlač, s.r.o.

[29] Ševčovič, D., Stehlíková, B., Mikula, K. (2009): Analytické a numerické metódy oceňovania finančných derivátov. Nakladateľstvo STU, Bratislava, 200 strán, ISBN 978-80-227-3014-3.

[30] Ševčovič, D., Stehlíková, B., Mikula, K. (2011): Analytical and numerical methods for pricing financial derivatives. Nova Science Publishers, Inc., Hauppauge, ISBN 978-1-61728-780-0 (Hardcover), ISBN 978-1-61761-350-0 (ebook).

[31] Schönbucher, P., Wilmott, P. (2000): The feedback-effect of hedging in illiquid markets. *SIAM Journal of Applied Mathematics* **61** 232–272.

[32] Wilmott, P., Dewynne, J., Howison, S.D. (1995): Option Pricing: Mathematical Models and Computation. UK: Oxford Financial Press.

Author's publication

- Option pricing, In: ISCAMI 2012. - Ostrava: Universitas Ostraviensis, 2012. - S. 83

Conferences

- ISCAMI 2012: International Student Conference on Applied Mathematics and Informatics. Malenovice, 10.-13.5.2012
- ALGORITMY 2012: Conference on Scientific Computing. Vysoké Tatry, Podbanské, 9.-14. 9. 2012

Co-solver in grants

- APVV–0104–12: Modern methods for solving nonlinear partial differential equations in financial mathematics
- DAAD 2012: NL-BS-AO: Numerical Solution of nonlinear Black-Scholes equations for American Options

Grants

- UK/615/2012: G-12-314-00: Nonlinear generalizations of Black-Scholes equation
- UK/637/2013: G-13-149-00: RAMP model with variable transaction costs