COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

QUALITATIVE AND QUANTITATIVE ANALYSIS OF BLACK–SCHOLES TYPE MODELS OF PRICING DERIVATIVES ON ASSETS WITH GENERAL FUNCTION OF VOLATILITY

DISSERTATION THESIS

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2014

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Dissertation Thesis in Applied Mathematics

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DISSERTATION THESIS

Specialization:	9.1.9 Applied Mathematics
Supervising division:	Department of Applied Mathematics and Statistics
Supervisor:	prof. RNDr. Daniel Ševčovič, CSc

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UNIVERZITA KOMENSKÉHO V BRATISLAVE
FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

KVALITATÍVNA A KVANTITATÍVNA ANALÝZA MODELOV OCEŇOVANIA DERIVÁTOV AKTÍV BLACK-SCHOLESOVHO TYPU SO VŠEOBECNOU FUNKCIOU VOLATILITY

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Aim: Numerical analysis of linear and nonlinear Black-Scholes type models for pricing derivatives on assets. Generalisation of the models with variable transaction costs and the risk from an unprotected portfolio. Minimization of the total risk measure in the model by optimal choice of portfolio hedging time interval. Solving by Gamma equation and numerical scheme.

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Názov: Qualitative and Quantitative Analysis of Black-Scholes Type Models of Pricing Derivatives on Assets with General Function of Volatility

Cieľ: Matematická analýza lineárnych a nelineárnych modelov oceňovania derivátov aktív Black-Scholesovho typu. Zovšeobecnenie modelov s variabilnými transakčnými nákladmi a rizikom z nezaisteného portfólia. Minimalizácia celkového rizika v modeli voľbou optimálnej dĺžky časového kroku pri diskrétnom zaisťovaní portfólia. Riešenie pomocou Gama rovnice a numerickej schémy.

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Declaration of Authorship

I here declare that I have prepared this Dissertation Thesis without the prohibited assistance of third parties and without making use of aids other than those specified; notions taken over directly or indirectly from other sources have been identified as such.

This document has not previously been presented in identical or similar form to any other Slovak or foreign examination board.

This dissertation thesis was conducted from 2010 to 2014 under the supervision of prof. RNDr. Ševčovič, CSc at the Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Slovak Republic.

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Abstract

In this thesis we propose and analyse a generalization of the Risk Adjusted Pricing Methodology model with variable transaction costs. We introduce some useful generalizations of the original Black–Scholes equation, which have a mathematical expression through solving partial differential equations of the Black–Scholes type equation. The diffusion coefficient of the equation for the price V is a function of the derivative price. More specifically, it will be function of the expression $\Gamma = \partial_S^2 V$ the underlying asset price S and time $\tau = T - t$ to expiration. We analyse the generalized derivatives pricing models that take into account non–trivial transaction costs associated with trading the financial stock market. Furthermore, models that take into account the risk of the investor's unprotected portfolio and other useful generalization of the classical linear Black–Scholes equation will also be examined. We show that the generalizations of the classical Black–Scholes model, including the novel model, can be solved by transformation of the fully nonlinear parabolic equation into a quasilinear parabolic equation for which one can construct an effective numerical scheme for approximation of the solution. The solutions are obtained by efficient numerical discretization of the Gamma equation, based on the finite volume scheme.

Keywords: non–linear Black–Scholes equations, non–linear parabolic equation, Risk Adjusted Pricing Methodology model, variable transaction costs, risk from an unprotected portfolio, numerical solution

Abstrakt

V tejto dizartačnej práci analyzujeme a ponúkame zovšobecnenie RAPM riziko zahrňujúcej medológie s variabilnými transakčnými nákladmi. Predstavujeme niekoľko užitočných zovšeobecnení pôvodnej Black–Scholesovej rovnice, ktoré majú matematické vyjadrenie prostredníctvom riešenia parciálnej diferenciálnej rovnice Black–Scholesovho typu, pričom difúzny koeficient rovnice pre cenu V derivátu je funkciou tejto ceny. Presnejšie, ide o funkciu výrazu $\Gamma = \partial_S^2 V$, ceny podkladového aktíva S a času $\tau = T - t$ do expirácie. Analyzujeme zovšeobecnené modely oceňovania finančných derivátov zohľadňujúce netriviálne transakčné náklady spojené s obchodovaním na finančnej burze a tiež modely zohľadňujúce investorovo riziko z nezaisteného portfólia a iné užitočné zovšeobecnenia klasickej lineárnej Black–Scholesovej rovnice. Ponúkame zovšeobecnenia klasického Black–Scholesovho modelu vrátane nového modelu, ktoré možno riešiť transformáciou nelineárnej parabolickej rovnice na kvázilineárnu parabolickú rovnicu, pre ktorú možno zostrojiť efektívnu numerickú schému pre aproximáciu riešenia. Riešenia sme dosiahli prostredníctvom efektívnej numerickej diskretizácie tzv. Gama rovnice, založenej na metóde konečných objemov.

Kľúčové slová: nelineárne Black–Scholesove rovnice, nelinaárna parabolická rovnica, RAPM riziko–zahrňujúca metodológia, variabilné transakčné náklady, riziko z nezaisteného portfólia, numerické riešenie

Contents

Introduction

Pricing financial derivatives belongs to actual topics on financial markets. As markets have become more sophisticated, more complex contracts than simple buy or sell trades have been introduced. They are known as financial derivatives, derivative securities or just derivatives. There exist many kinds of financial markets, e.g. stock markets, bonds markets, currency markets or foreign exchange markets, commodity markets or futures and options markets. On option markets derivative products are traded.

A derivative is defined as a financial instrument whose value depends on (or derives from) the values of other, more basic underlying variables. Very often the variables underlying derivatives are the prices of traded assets. As an example, an asset option is a derivative whose value is dependent on the price of a asset. However, derivatives can be dependent on almost any variable.

A *European call* option is a contract with the following conditions: At a prescribed time in the future, known as the expiration date, the holder of the option may purchase a prescribed asset, known as the underlying asset for a prescribed amount, the exercise price or strike price. For the holder of the option this contract is a right, not an obligation. The other party of the contract, the writer, must sell the asset if the holder chooses to buy it. Since the option is the right with no obligation for the holder, it has some value, paid for at the time of opening the contract. The right to sell the option is called a *put option*. A put option allows its holder to sell the asset on a certain date for a prescribed amount. The writer is then obligated to buy the asset.

There are many different types of options. Along the options of the European type, which do not depend on the evolution of the price of the underlying asset, there are also the option types where this is not true or where the exercise time is not given at the time the contract is opened. To the first group belong to the so–called *Asian* options, where the pay–off is determined by the average underlying price over some time period. The second group belong to the so–called *American* options. Other types can be the barrier, binary or the other exotic options. In this thesis we will focus on European options.

Options are used for hedging but also for speculations. Hedgers use derivatives to reduce the risk that they face from potential future movements in a market variable. Speculators use them to bet on the future direction of a market variable. Arbitrageurs take offsetting positions in two or more instruments to lock in a profit.

One of the most common methods of valuing stock options is the Black–Scholes method introduced in 1973. Economists Myron Scholes and Robert Merton and theoretical physicist Fischer Black derived and analysed a pricing model by means of a solution to a certain partial differential equation.

This thesis deals with the nonlinear models of Black–Scholes type, which are becoming more and more important since they take into account many effects that are not included in the linear model.

The main goals of the thesis can be summarized as follows:

- **Review of existing nonlinear models.** We review option pricing models of the Black–Scholes type with a general function of volatility. They provide more accurate values than the classical linear model by taking into account more realistic assumptions, such as transaction costs, the risk from an unprotected portfolio, large investor's preferences or illiquid markets.

- **Novel nonlinear models.** The main goals of the thesis is to derive models with variable transaction costs. We extend the models by two more new examples of realistic variable transaction costs that are decreasing with the amount of transactions. Using the Risk adjusted pricing methodology we derive a novel option pricing model under transaction costs and risk of the unprotected portfolio.

- **Solving the model by Gamma equation.** We show that the generalizations of the classical Black–Scholes model, including the novel model, can be solved by transformation of the fully nonlinear parabolic equation into a quasilinear parabolic equation for which one can construct an effective numerical scheme for approximation of the solution.

- **Numerical scheme and experiments.** The aim of this part is to propose an efficient numerical discretization of the Gamma equation, including, in particular, the model with variable transaction costs. The numerical scheme is based on the finite volume approximation of the partial derivatives entering the equation to be solved.

The structure of the thesis is as follows:

In Chapter 1 we give a brief exposition of stochastic differential calculus. We focus on the various stochastic processes, in particular the Brownian motion, and Itō's lemma, that are central to the mathematics of the pricing of derivatives.

Chapter 2 deals with the classical Black–Scholes linear model. It contains a brief summary of the model's assumptions and the derivation of the partial differential equation governing the option price using a self–financing portfolio, delta hedging and Itō's lemma. We discuss the terminal and boundary conditions and give a explicit solution by famous Black–Scholes–Merton pricing formula [28], [23] or [36].

In Chapter 3 we recall and summarize the nonlinear Black–Scholes option pricing models and the form of models with variable transaction costs. We review for example the Jumping volatility model due to Avellaneda, Levy and Paras [4], Leland model [29], the model with investor's preferences from Barles & Soner [6], the model with linear decreasing transaction costs depending on volume of trading stocks proposed by Avellaneda, Levy and Paras [4], non–arbitrage liquidity model developed by Bakstein and Howison [5] and Risk Adjusted Pricing Methodology model proposed by Kratka [25] and its generalization by Jandačka and Ševčovič in the work [24].

The main Section 3.8 develops a general theory of models with variable transaction costs. The main idea is in defining the modified transaction cost function $\tilde{C}$ when using the transaction costs measure, defined as the expected value of a change of the transaction cost per unit time interval Δt and price S. We also give the properties of this function to confirm its generality. Subsections 3.8.1 to 3.8.4 contains special cases of transaction costs function and their modification $\tilde{C}$. We mention the constant transaction costs function used in the Leland model [29] and also the linearly decreasing one from the model studied by Amster et al. [1]. We present and analyse two more new examples of realistic variable transaction costs that are decreasing with the amount of transactions, particularly, the piecewise linear non–increasing function and the exponentially decreasing function. By considering these functions, we solved the difficulty with possibly negative transaction costs that arises in the model proposed by Amster et al. [1].

Chapter 4 brings the main contribution in the form of a novel option pricing model under the transaction costs and the risk of an unprotected portfolio. It is a model with variable transaction costs with a general modified function of transaction costs $\tilde{C}$ and at the same time there is a possibility to control the risk of an unprotected portfolio. We show that this novel model is a generalization of the Leland model [29], the model with linear decreasing transaction cost depending on the volume of transaction [1] and also of the Risk adjusted pricing methodology model [25], [24]. We give also detailed analysis behind the optimization of hedging time.

Chapter 5 we introduce the Gamma equation proposed in [24] by Jandačka and Ševčovič as the main instrument to solve the nonlinear models including the novel one. The method includes the derivation of the Gamma equation, transformation of the initial and boundary conditions and also backward transformation of the solution.

The advantage of using the transformation to the Gamma equation lies in the fact that we can use an efficient numerical scheme, introduced in Chapter 6. The construction of numerical approximation of a solution to Gamma equation is based on the derivation of a system of difference equations to be solved at every discrete time step. We give also the Mathematica code for the model with variable transaction cost. Finally we consider the modelling of a bid–ask spread and perform extensive comparisons.

Chapter 1

Concepts from Stochastic Differential Calculus

In this chapter we present some preliminary concepts. We mainly focus on random variables, stochastic processes, in particular the Brownian motion and Itō's lemma, that are central to the mathematics for the pricing of derivatives. The following definitions and theorems are mostly adapted from Ševčovič, Stehlíková and Mikula [36] and comments from Wilmott, Howison and Dewynne [42] and Hull [23].

Since the mid-1980s one can be familiar from newspapers, television or the internet the nature of financial time series. Graphs of values of major indices are quoted frequently. As an example of a financial time series, Figure 1.1 shows the S&P500 Consumer Discretionary Sector daily closing price. These kind of graphs show the variation of the value of the asset or index with time. One can not predict tomorrow's value of the asset price. From a mathematical point of view, any variable whose value changes over the time in an uncertain way is said to follow a stochastic

Figure 1.1: Daily closing price of S&P500 Consumer Discretionary Sector.
Source: `http://www.thumbcharts.com/100519/S-P-500-Consumer-Discretionary-Sector`

process. Stochastic processes can be classified as discrete time or continuous time. A discrete–time stochastic process is one where the value of the variable can change only at certain fixed points in time, whereas a continuous–time stochastic process is one where changes can take place at any time. Similarly, the stochastic processes can be also be classified as continuous variable or discrete variable.

In this chapter we present continuous-variable, continuous-time stochastic process used for modelling the movement of asset prices. Learning about this process is the first step to understanding the pricing of options and other more complicated derivatives. It should be noted that, in practise asset prices are restricted to discrete values (e.g., multiples of a cent) and changes can be observed only when the exchange is open for trading.

Almost all models of option pricing are founded on one simple model for asset price movement, involving parameters derived, for example, from historical data.

We assume that:

- the past history of the asset is fully reflected in the present price, which does not hold any further information;
- markets respond immediately to any new information about an asset.

We give an example, why this weak–form market efficiency holds.

Example 1. *Suppose that it was discovered that a particular pattern in asset prices* *always gave a 65% chance of subsequent steep price rises. Investors would attempt to buy an asset as soon as the pattern was observed, and demand for the asset would immediately rise. This would lead to an immediate rise in its price and the observed effect would be eliminated, as would any profitable trading opportunities.*

Thus the modelling of asset prices is about modelling the arrival of new information which effects the price. These unanticipated changes in the asset price with the two assumptions above are a Markov process.

A Markov process is a particular type of stochastic process where only the current value of a variable $X(s)$ is relevant for predicting the future values $X(t)$ for $t > s$. The past history of the variable, $X(u)$ for $u < s$, and the way that the present has emerged from the past are irrelevant. Markov property can be written as: $\forall\, u \le s \le t: \; P\big(X(t) < x \mid X(s)\big) = P\big(X(t) < x \mid X(s), X(u)\big)$. From a practical point of view, if the process $\{X(t), t \in I\}$ is a Markov process, then we can start generation from the given initial value s without knowing the past history of the process.

Markovian stochastic processes are the basic tool for describing such a random evolution of the asset price. Although there is a wide range of Markov processes, the most used are the Wiener process and its generalization the Brownian motion.

1.1 Wiener Process and Geometric Brownian Motion

Definition 1.1. ***Brownian motion*** $\{X(t), t \ge 0\}$ *is a t-parametric system of random variables, for which*

(i) all increments $X(t + \Delta) - X(t)$ *have normal probability distribution with the expected value* $\mu\Delta$ *and dispersion (or variance)* $\sigma^2\Delta$*,*

(ii) for any partition $t_0 = 0 < t_1 < t_2 < t_3 < ... < t_n$ *of the interval* $(0, t_n)$*, all increments* $X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})$*, are independent*

random variables with parameters according to the point (i),

(iii) $X(0) = 0$ *and trajectories* $X(t), t \geq 0$ *are continuous in almost surely.*

A Brownian motion with parameters $\mu = 0, \sigma^2 = 1$ *is called a* ***Wiener process****. The Wiener process as well as the Brownian motion are Markov process.*

Throughout the thesis, a Wiener process will be denoted by $\{w(t), t \geq 0\}$.

From the preceding definition, it immediately follows that its first two moments, the expected value and variance are

$$\mathbb{E}[w(t)] = 0, \quad Var(w(t)) = t, \tag{1.1}$$

i.e., $w(t) \sim N(0, t)$.

Moreover, the cumulative distribution function of a Wiener process for a fixed time t is given by

$$Prob(w(t) < x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\xi^2/2t} \mathrm{d}\xi. \tag{1.2}$$

A sample of five numerical realizations of a Wiener process is shown in Figure 1.2.

If $\{w(t), t \geq 0\}$ is a Wiener process, then its increments over a short time interval dt will be denoted by dw, i.e., $dw(t) = w(t + dt) - w(t)$. According to the Definition 1.1 of a Wiener process, the increments $dw(t)$ are independent in time t. Their expected value is zero, i.e., $\mathbb{E}[dw(t)] = 0$ and their dispersion $Var(dw(t)) = dt$. The increment dw can be therefore written as $dw = \Phi\sqrt{dt}$, where $\Phi \sim N(0, 1)$ is a random variable with a standardized normal distribution.

A Brownian motion $\{X(t), t \geq 0\}$ with parameters μ and σ can be also analysed by means of its increments $dX(t) := X(t + dt) - X(t)$ where dt is an infinitesimally small quantity. According to property (i) from Definition 1.1 it holds that

$$\mathbb{E}[dX(t)] = \mu dt \quad \text{and} \quad Var(dX(t)) = \sigma^2 dt = \sigma^2 Var(dw(t)).$$

It means that a Brownian motion can be characterized by its deterministic and stochastic components. Its increments $dX(t)$ can be expressed in the following form

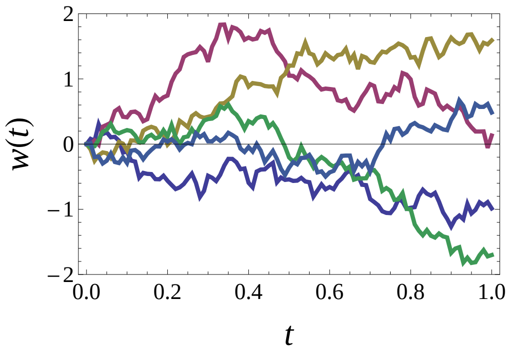


Figure 1.2: Five trajectories of a Wiener process displayed together.

of a total differential

$$dX(t) = \mu dt + \sigma dw(t). \tag{1.3}$$

Equation (1.3) is called a stochastic differential equation.

It is important to note, that the *absolute* change in the asset price is not by itself a useful quantity. For example, a change of 1€ is much more significant when the asset price is 20€ than when it is 200€. Therefore, instead of asset price, we will model the *return* of the asset, which is defined as the change in the price divided by the original value. This relative measure of the change is a better indicator of its size than any absolute measure.

Now suppose that at time t the asset price is S. Let us consider a small subsequent time interval dt, during which S changes[1] to $S + dS$. The most common model decomposes modelling the return $\frac{dS}{S}$ into two parts. One is deterministic and gives a contribution

$$\rho\, dt$$

to the return dS/S, where ρ is a measure of the average rate of growth of the asset price, also known as the drift. We assume it's a constant. The second contribution to dS/S models the random change in the asset price as a response to unexpected

[1]The notation $d\cdot$ is used for the small change in any quantity over a time interval when we consider it as an infinitesimal change.

news and effects. It is represented by

$$\sigma dw,$$

where σ is called the volatility and measures the standard deviation of the returns. It can be constant or the deterministic function of time and the underlying asset price. Putting these contributions together, we obtain the stochastic differential equation

$$\frac{dS}{S} = \rho dt + \sigma dw,$$

which is the mathematical model to generate asset prices. This stochastic process is called the geometric Brownian motion.

Under this process the return to the holder of the asset in a small period of time is normally distributed and the returns in two non–overlapping periods are independent. The value of the asset price at a future time has a lognormal distribution. The Black–Scholes–Merton model, which we cover in the next chapter, is based on the geometric Brownian motion assumption.

Definition 1.2. *If* $\{X(t), t \geq 0\}$ *is a Brownian motion with parameters* μ, σ *and* $y_0 \in \mathbb{R}^+$*, then the system of random variables* $\{Y(t), t \geq 0\}$*,*

$$Y(t) = y_0 e^{X(t)}, \quad t \geq 0,$$

is called a geometric Brownian motion.

Any geometric Brownian motion is again a Markov process. It is easy to show using Itō's lemma introduced next, that $Y(t) = y_0 e^{X(t)}$ satisfies the stochastic differential equation $\frac{dY}{Y} = \tilde{\mu} dt + \sigma dw$, where $\tilde{\mu}$ is shifted drift of ρ. Based on the explicit form of the probability distribution function of a Wiener process (1.2) we are able to compute its first two moments:

$$\mathbb{E}[Y(t)] = y_0 e^{\mu t + \frac{\sigma^2 t}{2}}, \quad Var(Y(t)) = y_0^2 e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1). \tag{1.4}$$

We say, that a random variable $\{Y(t), t \geq 0\}$ has a lognormal distribution with the

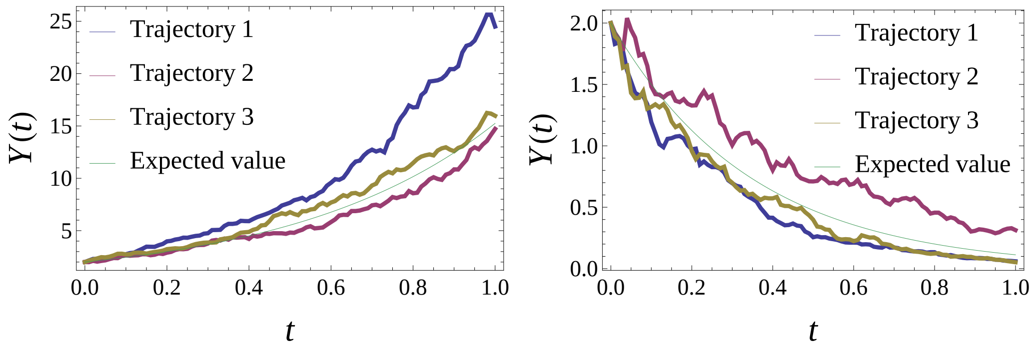


Figure 1.3: Different random realizations of a geometric Brownian motion with the positive drift $\mu > 0$ (left) and negative drift $\mu < 0$ (right).

expected value and variance given by (1.4).

1.2 Itō's Lemma

As we mentioned before, in real life asset prices are quoted at discrete intervals of time. Instead of dealing with an unmanageably large amount of data, we set up the mathematical models with a continuous time limit $dt \to 0$. It is much more efficient to solve the resulting differential equations than it is to value options by direct simulation of the random walk on a practical timescale. In order to do this, we need some technical machinery that enables us to handle the random term dw as $dt \to 0$.

The price of an option on an asset is a function of the underlying asset's price and time. More generally, we can say that the price of any derivative is a function of the stochastic variables underlying the derivative and time. For a better understanding of the behaviour of functions of stochastic variables we give an important result discovered by the mathematician K. Itō in 1951 known as Itō's lemma.

Lemma 1.1 (Itō's lemma). *Let $f(x, t)$ be a C^2 smooth function of two variables. Assume the random process $\{x(t), t \geq 0\}$ is a solution to the stochastic differential* *equation*

$$dx = \mu(x,t)dt + \sigma(x,t)dw,$$

where w is a Wiener process. Then the first differential of the function f is given by

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt, \tag{1.5}$$

and so the function f satisfies the stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dw.$$

The proof of this lemma can be found in any standard stochastic calculus book [28].

Let us give an example of a application of Itō's lemma.

Example 2. *Consider a Brownian motion $dX = \mu dt + \sigma dw$ and its function $Y(t) = f(X(t),t)$, where $f(X,t) = e^X$. By applying Itō's lemma we obtain*

$$dY = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma\frac{\partial f}{\partial X}dw = \left(\mu + \frac{\sigma^2}{2}\right)Ydt + \sigma Ydw.$$

As a consequence, we obtain for the expected value $\mathbb{E}[Y(t)]$ the following differential equation

$$d(\mathbb{E}[Y(t)]) = \left(\mu + \frac{\sigma^2}{2}\right)\mathbb{E}[Y(t)]dt,$$

from which we easily deduce that $\mathbb{E}[Y(t)] = \mathbb{E}[Y(0)]e^{(\mu+\sigma^2/2)t}$, as it was already claimed in (1.4).

Itō's lemma is a way of calculating the stochastic process followed by a function of a variable from the stochastic process followed by the variable itself. As we shall see in the next chapter, Itō's lemma plays a very important part in the pricing of derivatives.

Chapter 2

Classical Black–Scholes Linear Model

Let us move to the problem of how to obtain an accurate price of a financial derivative. The basic of pricing options on assets was given by Scholes, Black and Merton in 1973. Their theory is based on partial differential equations. It became very popular because the final pricing formula of an option is a function of a few observable variables (excluding volatility σ). Nonetheless, it is not enough to model the real markets. On the one hand we have a simple theory, very clear and interpretable. On the other hand, the simplicity is due to overly simplifying assumptions about the financial market and they do not reflect reality.

The assumptions of an idealized market in the Black–Scholes model are:

- trading is continuous in time;
- risk–free interest rate r is given and constant in time;
- stocks have zero dividend yield;
- there are no transaction costs or other fees by buying and selling the options and the assets in general;

- shares are perfectly divisible;
- the are no short position losses and risk is limited;
- the are no risk-free arbitrage opportunities, meaning that there are no opportunities instantly making a risk–free profit.

According to this Black–Scholes–Merton theory, the option price on this simplified market is obtained as a solution of the linear parabolic partial differential equation (in latter referred to as PDE).

2.1 Derivation of the Linear Black–Scholes PDE with dividend yield

We recall the derivation of the linear Black–Scholes PDE following Hull [23], Kwok [28], Ševčovič, Stehlíková and Mikula [36] and many authors dealing with this problem. Let us consider a useful generalization of the Black–Scholes equation for the case when the underlying asset is paying non–trivial continuous dividends with an annualized divided yield $q \geq 0$. Excluding the "no dividend" assumption of the simplified market yields to qualitatively the same final equation and it leads to a generalization of the problem.

Our goal is to find a mathematical model describing the price of an option $V = V(S,t)$, as a function of the underlying asset prise S and the time t. By construction of a self–financing portfolio, using the risk–free hedging principle and Itō's lemma we describe the derivation of the option price V as a solution of a PDE.

2.1.1 A Stochastic Differential Equation for the Option Price

In order to model the random evolution of the underlying asset price as a function of time $S = S(t)$ we will use the stochastic differential equation representing the geometric Brownian motion

$$dS = (\rho - q)Sdt + \sigma S dw, \tag{2.1}$$

where dS is the change of the asset value over the time interval of length dt, ρ represents a trend of underlying asset price evolution, $q \geq 0$ an annualized divided yield and σ is volatility. By dw we have denoted the differential of a standard Wiener process. The deterministic process $dS = (\rho - q)Sdt$ (i.e., $\sigma = 0$) has solution $S(t) = S(0)\exp[(\rho - q)t]$ representing exponential growth (decreasing if $\rho < 0$) of asset values observed in financial markets. Furthermore, notice that the stochastic equation (2.1) can be also written in the form

$$\frac{dS}{S} = (\rho - q)\,dt + \sigma\,dw.$$

The term $\sigma\,dw$ can be therefore understood as a random fluctuation over the trend part of the asset price. Hence the essential information is contained in the relative change dS/S and not in the absolute change in the asset price dS. Moreover, the relativized differential dS/S represents a return on the asset. Another reason is that the resulting model has to be invariant with respect to choice of units, i.e., the pricing formula should be currency unit invariant.

In the next step we derive a stochastic differential equation describing the evolution of an arbitrary smooth function (derivative) of asset price and time. Suppose that a function $V = V(S,t)$ is a smooth function of two variables, where S satisfies the stochastic differential equation (2.1). A stochastic differential equation for the function $V = V(S,t)$ can be derived by using the fundamental tool in the theory of stochastic processes - Itō's lemma 1.1. In our case, the variable S satisfies the stochastic differential equation (2.1), i.e., $dS = (\rho - q)S\,dt + \sigma\,dw$, and hence $\mu(S,t) = (\rho - q)S$, $\sigma(S,t) = \sigma S$, where ρ, σ are constants. Then a function $V(S,t)$

of the stochastic process S satisfies the following stochastic differential equation

$$dV = \left(\partial_t V + (\rho - q) S \partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V \right) dt + \sigma S \partial_S V dw. \tag{2.2}$$

To abbreviate the notation we will use:

$$\frac{\partial V}{\partial t} \equiv \partial_t V, \qquad \frac{\partial V}{\partial S} \equiv \partial_S V, \qquad \frac{\partial^2 V}{\partial S^2} \equiv \partial_S^2 V. \tag{2.3}$$

2.1.2 Self–financing Portfolio

In this step we focus on the construction of a portfolio consisting of underlying assets of the same type and one option on these assets. We will consider the self–financing portfolio. More precisely, at time t, the portfolio consists of an amount of δ units stocks with unit price S and one long position in the option (i.e. we are holders of this option) with unit price V.

The value of the portfolio is hence:

$$\Pi = V + \delta S, \tag{2.4}$$

and the change in time is as follows:

$$d\Pi = dV + \delta dS + \delta q S dt, \tag{2.5}$$

where the last term represents increase of portfolio value due to dividends in time interval dt.

The price of the option $V = V(S,t)$ is a function of the stochastic process S satisfying (2.1), therefore we can use Itō's lemma (Lemma 1.1) to obtain a stochastic differential equation for dV. Substituting the differential dV, we obtain:

$$d\Pi = \partial_S V dS + \left(\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V \right) dt + \delta dS + \delta q S dt. \tag{2.6}$$

Hence the term dS satisfies (2.1), we obtain the equation in the form:

$$d\Pi = \left(\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + \delta q S + (\rho - q) S(\partial_S V + \delta)\right) dt + (\partial_S V + \delta)\sigma S dw. \quad (2.7)$$

As we want the portfolio to be risk–free, we can eliminate all the stochastic terms introducing randomness in the equation above. The only stochastic term is represented by the differential dw of the Wiener process. This term vanishes provided that we choose the parameter δ, i.e. number of assets in the portfolio, as follows:

$$\delta = -\partial_S V. \quad (2.8)$$

By the no–arbitrage principle, the expected return of the portfolio is equal to the risk–free yield $r > 0$ of bonds, i.e.

$$d\Pi = r\Pi \, dt. \quad (2.9)$$

Using these assumptions and after some algebraic reordering, we obtain the mathematical model of the option price $V(S, t)$ given by a generalized Black–Scholes partial differential equation:

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (r - q) S \partial_S V - rV = 0, \quad (2.10)$$

where σ is the constant historical volatility of asset, $r > 0$ is the risk–free yield of bonds and $q \geq 0$ is the dividend yield. The solution $V = V(S, t)$ represents the option price as an function of underlying asset $S > 0$ and time $t \in [0, T]$.

2.1.3 Terminal and Boundary Conditions

To complete the model we have to add to the Black–Scholes partial differential equation (2.10) an additional terminal condition at expiration time T, determining the type of derivative contract. Such conditions are called terminal pay–off conditions.

By adding this condition, we can use our assumption on what type the derivative security is, for example a call option or a put option.

In the case of European call option the pay–off at expiration time is

$$V_{call}(S,T) = \max(S - E, 0), \tag{2.11}$$

where E is exercise or strike price. If the present asset price S at time T exceeds the exercise (strike) price value E then the value of the option is given as a difference $S - E$. We can also say the option is *in-the-money* [31, p. 78]. On the other hand, if the present asset price does not exceed the strike price E, then the option has no value, since it makes no sense to exercise it. In this case the option is *out-of-the-money* [31, p. 78].

An argument similar to that given above for the value of a call at expiry leads to the pay–off for a put option. Thus, the European put option value is given by the terminal pay–off condition:

$$V_{put}(S,T) = \max(E - S, 0). \tag{2.12}$$

At expiry it is worthless if $S > E$ but has the value $E - S$ for $S < E$.

The pay–off diagrams for a European call and put options are shown in Figure 2.1.

Although the two most basic structures for the pay–off are the call and the put, in principle there is no reason why an option contract cannot be written with a more general pay-off. For more detail we refer to [42].

Boundary conditions are applied for a zero stock price $S = 0$ and $S \to \infty$. In the case of European call option

$$V(0,t) = 0 \quad \text{and} \quad V(S \to \infty, t) = Se^{-q(T-t)} \quad \text{for every} \quad t \in (0,T). \tag{2.13}$$

The boundary conditions for European put option are

$$V(0,t) = Ee^{-r(T-t)} \quad \text{and} \quad V(S \to \infty, t) = 0 \quad \text{for every} \quad t \in (0,T). \tag{2.14}$$

By using the Black–Scholes partial differential equation (2.10) with specific terminal condition we obtain the value of the particular option type.

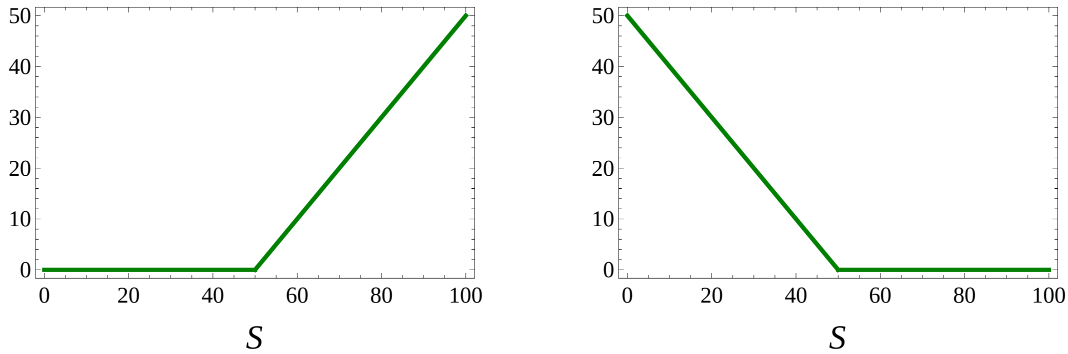


Figure 2.1: The pay–off diagrams for call (left) and put options (right) with the exercise (strike) price $E = 50$ from the perspective of the holder (**long positions**).

2.2 Alternative Positions

In the following chapters we will use some properties of pay–offs, depending on the position in the option. We note that there are four basic types of traders in options markets:

1. Buyers of calls
2. Sellers of calls
3. Buyers of puts
4. Sellers of puts.

Buyers are always referred to as having long positions and sellers as having short positions. Selling an option is also known as writing the option. Derivatives markets have been outstandingly successful helping investors diversify their risk. The main reason is that they have attracted many different types of traders and have a great deal of liquidity. When an investor wants to take one side of a contract, there is usually no problem in finding someone who is prepared to take the other side.

The short positions can be depicted by pay–off diagrams in Figure 2.2, where the pay-off functions (2.11) and (2.12) are multiplied by -1. That means, that the writer of a European call option is taking the risk of a potentially unlimited loss and must carefully design a strategy to compensate this risk.

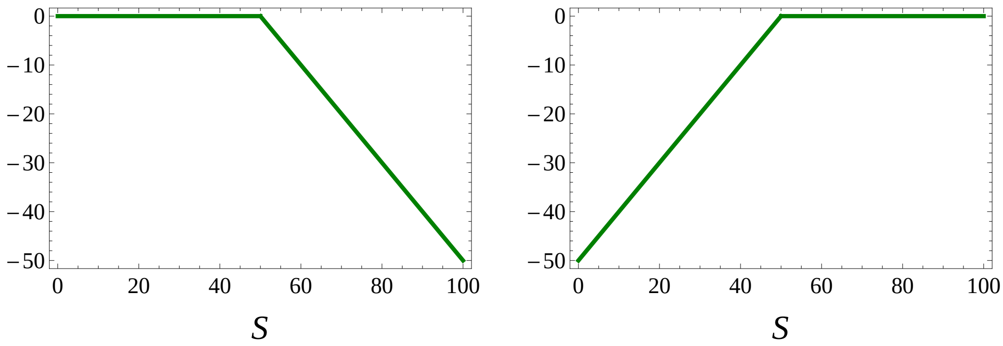


Figure 2.2: The pay–off diagrams for call (left) and put options (right) with the exercise (strike) price $E = 50$ from perspective of the writer (**short positions**).

One can ask, why would somebody hold a short position, when pay–offs have non–positive values. The answer is, there is a difference between the pay–off and profit diagrams. While a pay–off diagram simply graphs the cash value at any point in time during the lifetime of the option, a profit diagram shows us exactly what we have earned from the purchase or sell of the option and it is shifted down or up respective of the value of the current option price.

The act of selling an option is referred to as option writing. To compensate the possibly infinite risk, the option writer receives at the same time a premium (the option price).

2.3 The Explicit Solutions

Here we show the exact solutions of the Black–Scholes partial differential equation 2.10. It is obtained for example in [28], [23] or [36]. The most famous are the Black–Scholes–Merton formulas for the prices of European call and put options. This formulas read as follows:

$$\text{European call:} \qquad V(S,t) = Se^{-q(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2), \tag{2.15}$$

$$\text{European put:} \qquad V(S,t) = Ee^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1), \tag{2.16}$$

where

$$d_1 = \frac{(r - q + \frac{\sigma^2}{2})(T-t) + \ln\frac{S}{E}}{\sigma\sqrt{T-t}}, \tag{2.17}$$

$$d_2 = \frac{(r - q - \frac{\sigma^2}{2})(T-t) + \ln\frac{S}{E}}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \tag{2.18}$$

The function $N(x)$ is the cumulative probability distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\xi^2/2}d\xi$$

for the standardised normal distribution. All the remaining constants and parameters should be familiar. An example of solution is depicted in Figure 2.3. The evolution of the solution at the time, in sense of time to expiration, is illustrated in Figure 2.4 for the call option and Figure 2.5 for the put option.

2.3.1 Influence of dividends

Furthermore, it should be mentioned that the value of a call option on an underlying asset without a dividend payment, i.e. $q = 0$, is always greater than the value af a call option on an underlying asset with a dividend payment, i.e. $q > 0$. For

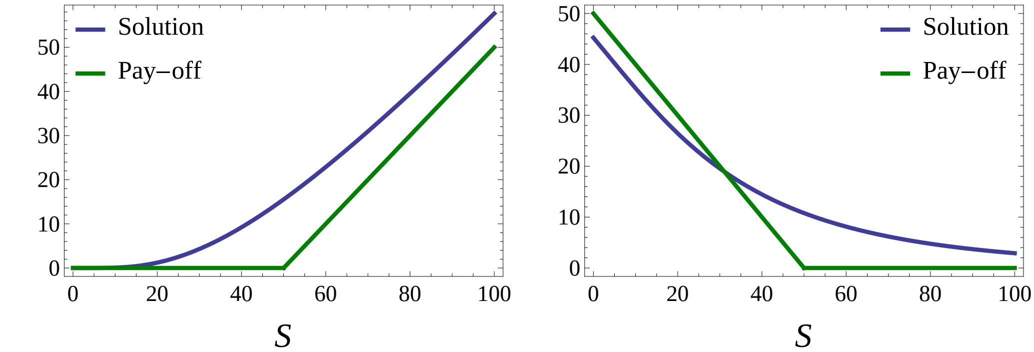


Figure 2.3: European call option value (left) and European put option value (right) with the exercise (strike) price $E = 50$ and the pay–off diagrams.

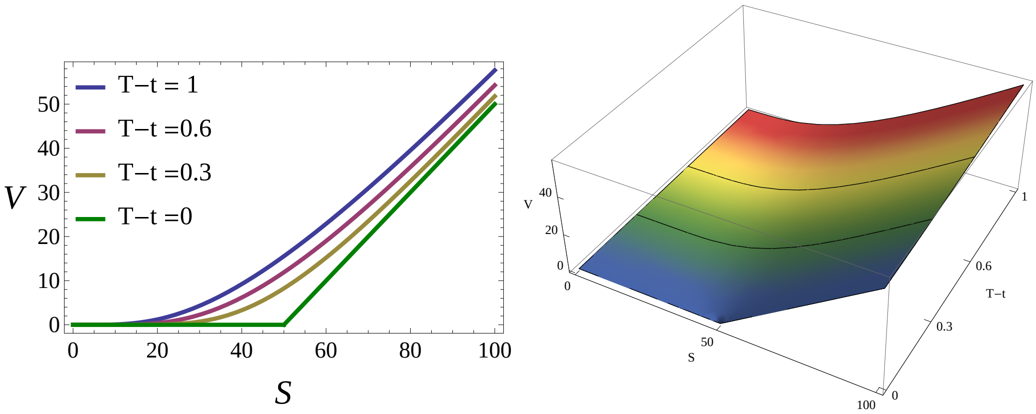


Figure 2.4: Evolution of the price of European call option in time.

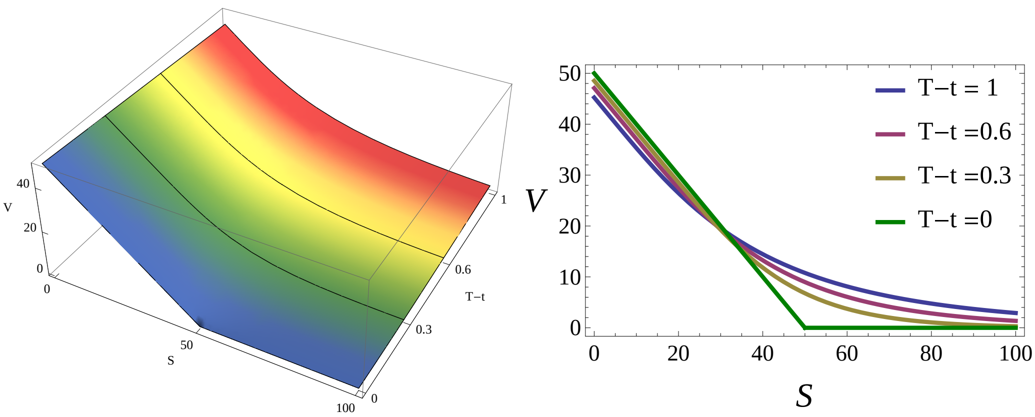


Figure 2.5: Evolution of the price of European put option in time.

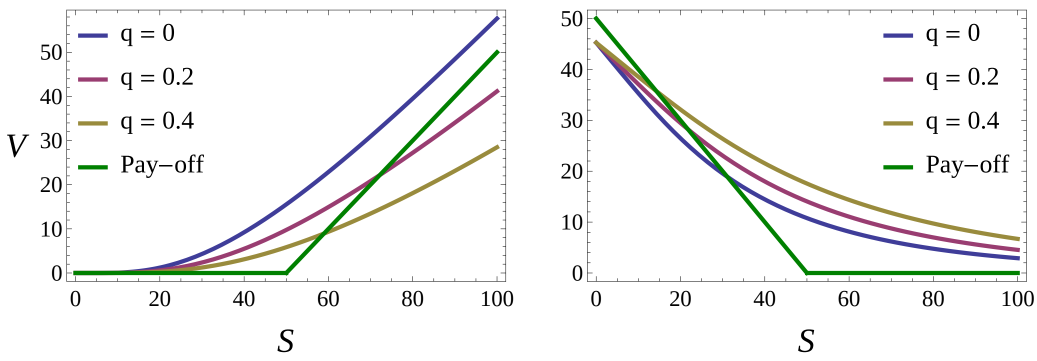


Figure 2.6: European call option with dividend yields q (left) and European put option with dividend yields q (right).

European put options on an underlying asset without a dividend payment the value is less than on the underlying asset with a dividend payment. The influence of a dividend payment is summarized in Figure 2.6.

2.4 The Role of Delta Hedging

In a financial context hedging refers to a reduction in risk. Delta hedging is the reduction of the sensitivity of a portfolio to the movement of an underlying asset. Most traders use many sophisticated hedging procedures. In general, at time t the self–financing portfolio consist of an amount of Q_S stocks with unit price S, an amount of Q_V options with the unit price V and risk–less zero–coupon bonds having total money value B. The Merton's condition of the self–financing strategy means that the purchase or sale of one of the three components has to be compensated by selling or purchasing another component of the portfolio.

A delta hedge aims to keep the value of this portfolio the same for the situation where the price of the underlying asset goes up, as for where it goes down.

As explained by equation (2.7), to eliminate the risk, we choose delta according to (2.8) as negative of the partial derivative $\partial_S V$ of the option value with respect to the price of the underlying asset. The *delta* can be also interpreted as ratio of amount of assets in short position to amount of options (for more detail see [36]):

$$\delta = -\partial_S V = \frac{Q_S}{Q_V} < 0. \tag{2.19}$$

Example 3. *If $\delta = 0.2$, then having 10 options in the long position $Q_V = 10$ should be hedged by selling 2 assets $Q_S = -2$.*

A position with a delta of zero is referred to as delta neutral.

It is important to realize that, since the delta of an option does not remain constant, the trader's position remains delta hedged (or delta neutral) for only a relatively short period of time. The hedge has to be adjusted periodically. This is known as rebalancing. It is a dynamic strategy and requires a continuous rebalancing of the portfolio and the ratio of the holdings of the asset and the derivative product.

Chapter 3

Nonlinear Volatility Black–Scholes Type Models

In this chapter we introduce some useful generalizations of the original Black–Scholes equation, which have a mathematical expression through solving partial differential equations of the Black–Scholes type in these generalized models. The diffusion coefficient $\sigma^2 S^2/2$ of equation (4.22) for the price V is a function of the derivative price. More specifically, it will be function of the expression $\Gamma = \partial_S^2 V$, the underlying asset price S and time $\tau = T - t$ to expiration.

We analyse the generalized derivatives pricing models that take into account non–trivial transaction costs associated with trading the financial stock market. Furthermore, we also analyse models that take into account the risk of the investor's unprotected portfolio and other useful generalizations of the classical linear Black–

Scholes equation.

3.1 Motivation for Studying Nonlinear Models

As we mentioned in the previous chapter, the classical linear Black–Scholes model for option pricing with constant volatility has been derived under several restrictive assumptions, complete market information[1], continuous trading, zero transaction costs, etc.

Assuming that the underlying asset S follows a geometric Brownian motion

$$dS = (\rho - q)S\,dt + \sigma S\,dw, \tag{3.1}$$

where ρ is drift, q is dividend yield, σ is volatility of underlying asset and w is a standard Wiener process, and using a construction of the self–financing portfolio we derived a partial differential equation for the option price $V(S,t)$ of the following form:

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (r-q)S\partial_S V - rV = 0, \tag{3.2}$$

where σ is the constant historical volatility of an underlying asset, $r > 0$ is the risk–free yield of bonds and $q \geq 0$ is the dividend yield.

Analysing real market data we can see there is a need of nonlinear models, where $\sigma > 0$ is now not constant, but is a function of the option price V itself. We focus on case, where volatility σ depends of second derivative $\partial_S^2 V$ of the option price (hereafter referred to a Γ), the price of an underlying asset S and the time to expiration $\tau = T - t$, as Ševčovič, Stehlíková and Mikula state in [36], i.e.

$$\hat{\sigma} = \hat{\sigma}(S\partial_S^2 V, S, \tau). \tag{3.3}$$

On the one hand, in case of the constant $\sigma > 0$ in (4.22) represents the classical Black–Scholes equation derived by Black and Scholes in [8]. On the other

[1] Any demand or supply offer is accepted or purchased.

hand, if $\sigma > 0$ is a function of a solution, equation (4.22) represents the nonlinear generalization of the Black–Scholes equation.

The motivation for studying the nonlinear Black–Scholes equation (4.22) with volatility having a general form of (3.3) arises from traditional option pricing models taking into account non–trivial transaction costs due to buying and selling assets, market feedbacks and illiquid market effects due to large traders choosing given stock–trading strategies, risk from a volatile (unprotected) portfolio or investors preferences, etc. There is an increase of interest in studying nonlinear Black–Scholes model, because it takes into account more realistic assumptions, that can impact volatility, drift and price of an asset.

One of the basic nonlinear models is the Leland model [29] which including transaction costs arising by hedging the portfolio with call or put options. This model was later extended by Hoggard, Whalley and Wilmott [22] for more general option types. Another nonlinear model is a model adjusted with jumping volatility known from Avellaneda and Paras [3]. Models including feedback and illiquid market effects due to large traders choosing given stock–trading strategies was developed by Frey and Patie [18], Frey and Stremme [19], During and et al. [14], Schönbruchen and Wilmott [37]. There is also a nonlinear generalization proposed by Barles and Sonner[6] for the description of imperfect replication and investor's preferences. Another model that takes into account risk from unprotected portfolio is proposed by Kratka [25] and Jandačka and Ševčovič in [24], [36].

One of the models dealing with transaction costs is model proposed by Grossinho and Morais [21]. The model proposed by Avellaneda, Levy and Paras [4] is aligned with the Barles and Soner model [6] where it is assumed that investor's preferences are characterized by an exponential utility function. The next is the Risk adjusted pricing methodology (RAPM) model proposed by Kratka [25] and its generalization by Jandačka and Ševčovič in the work [24]. Last but not least is the model with linear decreasing transaction costs depending on volume of trading stocks [1] by authors Amster, Averbuj, Marian and Rial with transaction costs as a function of the amount of traded assets.

In this chapter we will go into more detail through the Leland model [29] and Risk Adjusted Pricing Methodology (RAPM) model proposed by Kratka [25] and its generalization by Jandačka and Ševčovič in the work [24]. We will also use the variable transaction costs in the model following Amster, Averbuj, Mariani and Rial [1].

In sections 3.2 - 3.7 we review some of the known nonlinear models. The aim of this work is modelling in Section 3.8, with comparison to the model proposed by Amster et al. and RAPM model.

3.2 Jumping Volatility Model

Avellaneda, Levy and Paras [4] proposed a model for the description of incomplete markets and uncertain but bounded volatility. In their model we have

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \begin{cases} \sigma_+^2, & \text{if } S\partial_S^2 V > 0, \\ \sigma_-^2, & \text{if } S\partial_S^2 V < 0. \end{cases} \tag{3.4}$$

where σ_- and σ_+ represent volatility of the asset, where option is in the long position or short position respectively. When a trader buys an option contract he is opening a long position. When a trader sells an option contract that's not long, he is said to be opening a short position. The nonlinearity is when a trader switches from a long ($\partial_S^2 V > 0$) to a short position ($\partial_S^2 V < 0$). Another asset volatility is taken into account when being in a long and short position.

3.3 Leland Model

The Leland model published in paper [29] has been further generalized to more complex options strategies by Hoggard, Whalley and Wilmot in [22]. We present the derivation of a more general model in Section 3.8, of which the Leland model is

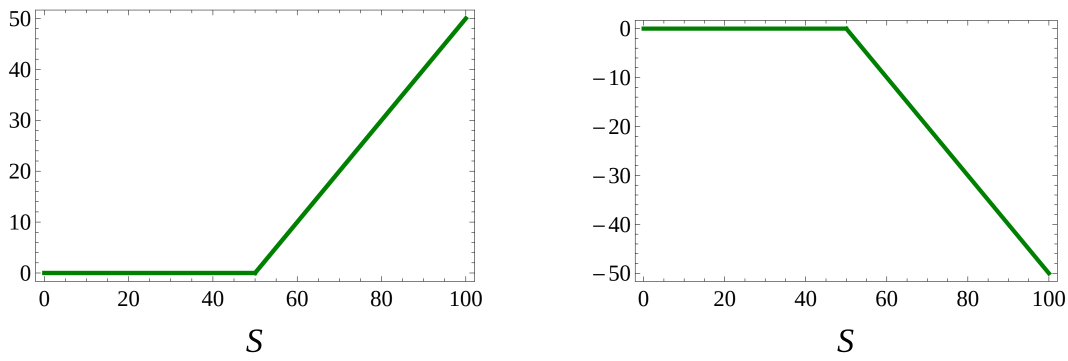


Figure 3.1: The pay–off diagrams for a call in a long position, $\partial_S^2 V > 0$ (left) and short position, $\partial_S^2 V < 0$ (right).

just a special case.

Nonlinearity in the diffusion coefficient is in the form

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2\left(1 - \mathrm{Le}\,\mathrm{sgn}\left(S\partial_S^2 V\right)\right) = \begin{cases} \sigma^2(1-\mathrm{Le}), & \text{if } S\partial_S^2 V > 0, \\ \sigma^2(1+\mathrm{Le}), & \text{if } S\partial_S^2 V < 0, \end{cases} \tag{3.5}$$

$$\text{where} \quad \mathrm{Le} = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}} \tag{3.6}$$

is the Leland number and σ is constant historical volatility, $C_0 > 0$ is a constant transaction cost per unit dollar of transaction in the assets market and Δt is the time–lag between portfolio adjustments.

Notice that for plain vanilla call or put options $\partial_S^2 V > 0$, hence $\mathrm{sgn}\,(S\partial_S^2 V) = 1$. In this case the Leland equation is again the linear Black–Scholes equation with modified constant volatility

$$\sigma_{\mathrm{Le}}^2 = \sigma^2(1-\mathrm{Le}). \tag{3.7}$$

Leland's equation is a backward parabolic PDE satisfying a terminal condition (see Subsection 2.1.3). To have a diffusion coefficient of the correct sign [35], we have to specify for the simple call and put options the upper and lower bounds of the Leland number $0 \le \mathrm{Le} < 1$. As mentioned in Ševčovič et al. [35, p. 56], the solution of the Black–Scholes equation is an increasing function of volatility parameter σ. If we fix the parameters of volatility σ, interest and dividend rate of r and q and the same expiration price of E and expiratory time T, then the price of a call or put option from the Leland model in the case of positive Leland number is less than the value obtained from the solution of Black–Scholes model. This is due to the fact that from the perspective of the holder or buyer of the option, the transaction costs are on his side. He must hedge the portfolio by selling or purchasing the shares. Therefore in this case the price obtained by the Leland equation is a price to buy the options, i.e. the *bid* price.

If the option in the self financing portfolio is in a short position, i.e., $\Pi = -V + \delta S$, by the Leland equation we obtain the price to purchase the option, the *ask* price. The final equation is exactly the same, only the diffusion coefficient has a positive sign:

$$\sigma_{\mathrm{Le}}^2 = \sigma^2(1 + \mathrm{Le}). \tag{3.8}$$

3.4 Model with Investor's Preferences

Barles & Soner derived in [6] a particular nonlinear adjusted volatility of the form

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2\Big(1 + \Psi\left(a^2 e^{r\tau} S^2 \partial_S^2 V\right)\Big), \tag{3.9}$$

where $a > 0$ includes a risk aversion of investor and also proportional transaction cost. Therefore the choice of a depends on how much risk we are willing to take. The motivation is to include preferences in order to evaluate the price of options.

The volatility correction function Ψ is the solution of the ordinary differential equation:

$$\Psi'(A) = \frac{\Psi(A) + 1}{2\sqrt{A\Psi(A)} - A}, \qquad A \neq 0, \qquad \Psi(0) = 0. \tag{3.10}$$

The function Ψ is given implicitly through relations:

$$A = \left(-\frac{\operatorname{arcsinh}\sqrt{\Psi}}{\sqrt{\Psi+1}} + \sqrt{\Psi}\right)^2, \qquad \text{if } \Psi > 0, \tag{3.11}$$

$$A = -\left(\frac{\operatorname{arcsinh}\sqrt{-\Psi}}{\sqrt{\Psi+1}} - \sqrt{-\Psi}\right)^2, \qquad \text{if } 0 > \Psi > -1 \tag{3.12}$$

(cf. Company, Navarro, Pintos and Ponsoda [12] or Dremkova, Ehrhardt [13]).

3.5 Model with Linear Decreasing Transaction Costs Depending on the Volume of Trading Stocks

Amster, Averbuj, Mariani and Rial in their work [1] assume that the costs behave as a non–increasing linear function, depending on the trading stocks needed to hedge the replicating portfolio. They proposed the model, where the transaction costs are not proportional to the amount of the transaction, but the individual cost of the transaction of each share diminishes as the number of traded shares increases. Therefore transaction cost function is given by

$$C(\xi) = C_0 - \kappa\xi, \tag{3.13}$$

where ξ is the volume of trading stocks, i.e. $\xi = |\Delta\delta|$ and $C_0, \kappa > 0$ are constants depending on the individual investor. The number of bought or sold assets depends on the one–time step change of δ, i.e. stocks hold in the portfolio. The main idea is decreasing transaction cost with increasing amount of transaction. It can be seen as a discount for a large deal attractive for large investors.

In the model studied by Amster et al. we have

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2\left(1 - \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}\operatorname{sgn}\left(S\partial_S^2 V\right) + \kappa S\partial_S^2 V\right). \tag{3.14}$$

In the framework of this model, Amster et al. obtained a nonlinear Black–

Scholes type equation and studied the stationary problem associated with appropriated boundary conditions. The authors proved the existence and uniqueness of the solution of this problem, which may be obtained as a limit of a non–interacting (non–decreasing) sequence of upper (respectively lower) solutions.

3.6 Liquidity Model

Bakstein and Howison in their paper [5] *A Non–Arbitrage Liquidity Model with Observable Parameters* in 2003 introduced the model including three of the already mentioned models namely the classical B–S, Leland and model proposed by Amster et al.. They developed a parametrised model for liquidity effects arising from the trading in an asset. Here $\hat{\sigma}^2$ is the following quadratic function of $\Gamma = \partial_S^2 V$:

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2 \Bigg(1 + \bar{\gamma}^2(1-\alpha)^2 + 2\lambda S\partial_S^2 V + \lambda^2(1-\alpha)^2 \left(S\partial_S^2 V\right)^2 + \\ + 2\sqrt{\frac{2}{\pi}}\bar{\gamma}\,\mathrm{sgn}\left(S\partial_S^2 V\right) + 2\sqrt{\frac{2}{\pi}}\lambda(1-\alpha)^2\bar{\gamma}\left|S\partial_S^2 V\right| \Bigg) \tag{3.15}$$

The parameter λ corresponds to market depth measures, i.e. scales the slope of the average transaction price. Next, γ models the relative bid–ask spreads and $\bar{\gamma}\sigma\sqrt{\Delta t} = \gamma$. Finally, α transforms the average transaction price into the next quoted price, $0 \le \alpha \le 1$.

The important point to note here is:

- with $\bar{\gamma} = 0$ and $\lambda = 0$ the equation is reduced to the classic Black–Scholes equation;

- with $\lambda = 0$ and $\alpha = 1$ the equation is reduced to the Leland equation with a positive sign for the diffusion coefficient, where

$$\mathrm{Le} = 2\bar{\gamma}\sqrt{\frac{2}{\pi}} \qquad \text{and} \qquad \bar{\gamma} = -\frac{C_0}{2\sigma\sqrt{\Delta t}};$$

- with $\alpha = 1$ the equation is reduced to the equation obtained by Amster et al.(see Subsection 3.5), where

$$\lambda = \frac{\kappa}{2} \qquad \text{and} \qquad \bar{\gamma} = -\frac{C_0}{2\sigma\sqrt{\Delta t}}.$$

3.7 Risk Adjusted Pricing Methodology Model

The next example of the Black–Scholes equation with a nonlinearly depending volatility we present is the RAPM model (Risk adjusted pricing methodology model) proposed by Kratka in [25] and revisited by Jandačka and Ševčovič in [24]. The volatility is in the following form:

$$\hat{\sigma}^2(S\partial_S^2 V, S, \tau) = \sigma^2 \left(1 - \mu \left(S\partial_S^2 V\right)^{\frac{1}{3}}\right), \tag{3.16}$$

where $\sigma > 0$ is a constant historical volatility of the asset price return and

$$\mu = 3(C_0^2 R/2\pi)^{\frac{1}{3}}, \tag{3.17}$$

where $C_0, R \geq 0$ are non–negative constants representing cost measure and the risk premium measure, respectively.

We will present detailed derivation of RAPM model in Subsection 4.4.1.

3.8 Models with Variable Transaction Costs

The aim of this section is to present a new approach taking into account variable transaction costs in a more general form of a **decreasing or non–increasing function of the amount of transactions**, $|\Delta\delta|$, per unit of time Δt, i.e. $C = C(|\Delta\delta|)$.

One of the key assumptions of the Black–Scholes analysis is the continuous rehedging of a portfolio. In connection with the transaction costs for buying and selling the underlying asset, continuing hedging would lead to an infinite number of transactions and unbounded total transaction costs. The Leland [29], and Hoggard, Whalley and Wilmott [22], models are based on a simple, but very important modification of the Black–Scholes model, which includes transaction costs and rearranging of the portfolio at discrete times. Since the portfolio is maintained at regular intervals, this means that the total transaction costs are limited.

The assumptions of our new model are in general the same as for the Black–Scholes model with the following extensions. Some of the conditions are adapted from Wilmott, Dewynne and Howison [41] and Ševčovič, Stehlíková and Mikula [35]:

[C_1] A portfolio consisting of shares and options on these shares is rearranged every Δt time units, where Δt is specified time step[2] (we do not consider the continuous limit of $\Delta t \to 0$);

[C_2] The underlying asset S follows a geometric Brownian motion at discrete time points $t_1 < t_2 < ... < t_n$, where $t_{i+1} - t_i = \Delta t$, for each $i = 1, 2, ..., n-1$. The change in share price ΔS is therefore given by the equation for a discrete geometric Brownian motion (equation), where $\Delta w = \Phi\sqrt{\Delta t}$ and Φ is a random variable with a normal probability distribution $\Phi \sim N(0,1)$

$$\Delta S = (\rho - q)S\Delta t + \sigma S \Delta w; \tag{3.18}$$

[C_3] We assume that the option price V is in the long position, which means that we keep it and the hedging of the portfolio is performed by buying and selling underlying assets;

[C_4] The transaction costs depend on the volume of transactions.

(a) Leland's approach of modelling small investors under transaction costs

We will assume that the cost C per one transaction is constant

$$C \equiv const = C_0. \tag{3.19}$$

[2] For instance, portfolio can be rearranged every day at 9:00 in the morning

We expect that market shares are purchased at a higher (so-called *ask*) price S_{ask} and sell for less (so–called *bid*) price S_{bid}. The price of S denotes the average of *ask* and *bid* price of S_{ask} and S_{bid}, i.e., $S = (S_{ask} + S_{bid})/2$. Let C_0 represent a constant percentage of the cost of the sale and purchase of a share relative to the average price of S_{ask} and S_{bid}, i.e.

$$C_0 = 2\,\frac{S_{ask} - S_{bid}}{S_{ask} + S_{bid}}, \tag{3.20}$$

then we can express the *ask* and *bid* price of the share as follows:

$$S_{ask} = S\left(1 + \frac{C_0}{2}\right), \quad S_{bid} = S\left(1 - \frac{C_0}{2}\right). \tag{3.21}$$

This means that the purchase of $\Delta\delta > 0$ or sales of $\Delta\delta < 0$ shares at a price of S, we calculate the additional cost of:

$$\frac{S}{2}C_0|\Delta\delta| = \frac{S_{ask} - S_{bid}}{2}|\Delta\delta| \tag{3.22}$$

units. This leads to the Leland equation, where

$$\mathrm{Le} = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}. \tag{3.23}$$

(b) **Modelling variable transaction costs for large investors**

Large investors can have some kind of discount, because of large transaction amounts. The more they purchase in one transaction, the less will they pay for one traded underlying asset. In general, we will assume that the cost C per one transaction is a non-increasing function of the amount of transactions, $|\Delta\delta|$, per unit of time Δt, i.e.

$$C = C(|\Delta\delta|). \tag{3.24}$$

This means that the purchase of $\Delta\delta > 0$ or sales of $\Delta\delta < 0$ shares at a price of S, we calculate the additional transaction cost per unit of time Δt:

$$\Delta TC \equiv \frac{S}{2} C(|\Delta\delta|)|\Delta\delta| \tag{3.25}$$

units;

[C_5] Expected return on the portfolio is equal to the risk-free yield of bonds.

As in the derivation of the Black–Scholes equation we synthesize portfolio Π of one option V in the so–called long position and δ amount of shares S, i.e.

$$\Pi = V + \delta S. \tag{3.26}$$

We perform a hedging of the portfolio by buying and selling shares. As an investor has a long position in a put option, he has the transaction costs, the price of such an option is less than the price of the shares, without transaction costs. In the self–financing portfolio, the change of the expected value of the portfolio at time Δt, i.e. $\Delta\Pi_t = \Pi_{t+\Delta t} - \Pi_t$ must be equal to the risk–free bond yield changes to the risk–free rate $r > 0$ for the same time interval, i.e.

$$\Delta\Pi = r\Pi\Delta t. \tag{3.27}$$

The value of the portfolio changes in reality by rearranging the portfolio (when we buy and sell shares) in rate of transaction costs. The resulting transaction costs are therefore deducted from the right side of the equation (3.26). Also during time the interval Δt the dividend value increases. The change in portfolio $\Delta\Pi$ consists precisely of the parts:

$$\Delta\Pi = \Delta(V + \delta S) + \delta q S \Delta t - \Delta TC, \tag{3.28}$$

where ΔTC represents the transaction costs for the length of the time interval Δt. From condition [C4], we know that the transaction costs are paid by equation (3.25).

Therefore

$$\Delta TC = \frac{S}{2} C(|\Delta\delta|)|\Delta\delta|. \tag{3.29}$$

For change of the portfolio $\Delta\Pi$ in the time interval Δt, the following equation is true:

$$r\Pi\Delta t = \Delta\Pi = \Delta V + \delta\Delta S + \delta q S\Delta t - \frac{S}{2} C(|\Delta\delta|)|\Delta\delta|. \tag{3.30}$$

We suppose that portfolio adjustments follow the so–called δ–hedging strategy, i.e.

$$\delta = -\partial_S V. \tag{3.31}$$

In the equation however, there still remains a stochastic term of $\Delta\delta$. From this (3.31) is the number of shares in the portfolio function of the current stock price S and time. Recall that the underlying asset fulfils

$$\Delta S = (\rho - q)S\Delta t + \sigma S \Delta w, \tag{3.32}$$

where $\Delta w = w(t + \Delta t) - w(t)$ is the increment of the Wiener process. Applying Itô's formula on $-\partial_S V$ we obtain

$$\Delta\delta \approx -\sigma S \partial_S^2 V \Delta w, \tag{3.33}$$

except for members of higher order in Δt. Using $\Phi \sim N(0, 1)$ a normally distributed random variable we obtain

$$|\Delta\delta| = \alpha|\Phi|, \quad \text{where} \tag{3.34}$$

$$\alpha := \sigma S \left|\partial_S^2 V\right| \sqrt{\Delta t}. \tag{3.35}$$

Following Leland [29], Hoggard, Whalley and Wilmott [22], Jandačka and Ševčovič [24] and Ševčovič, Stehlíková and Mikula [36] we define the transaction cost measure r_{TC} as the expected value of a change of the transaction costs per time interval.

In the case $C \equiv C_0 = const$, Leland in [29] (see also Hoggard, Whalley and Wilmott [22]) suggested the following mean value approximation of the term

$$|\Delta\delta| = \alpha|\Phi| \approx \alpha\,\mathbb{E}[|\Phi|] = \alpha\sqrt{\frac{2}{\pi}}.$$

In our generalization of the transaction costs model we assume the mean value approximation of the stochastic term $C(|\Delta\delta|)|\Delta\delta|$:

$$C(|\Delta\delta|)|\Delta\delta| \approx \mathbb{E}\big[C(|\Delta\delta|)|\Delta\delta|\big].$$

It leads to the following definition of the transaction cost measure r_{TC}.

Definition 3.1. *The transaction cost measure r_{TC} is defined as the expected value of a change of the transaction costs per unit time interval Δt and price S:*

$$r_{TC} = \frac{\mathbb{E}[\Delta TC]}{S\Delta t}.$$

Applying the formula of additional transaction costs (3.25) the transaction cost measure r_{TC} is defined as:

$$r_{TC} = \frac{1}{2}\frac{\mathbb{E}[C(|\Delta\delta|)|\Delta\delta|]}{\Delta t}, \tag{3.36}$$

where C is the transaction costs function and $\Delta\delta$ is the number of purchased $\Delta\delta > 0$ or sold $\Delta\delta < 0$ shares per unit of time Δt.

With this notation, equation (3.30) with δ–hedging (3.31) becomes transaction cost generalization of the Black–Scholes equation

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (r-q)S\partial_S V - rV - r_{TC}S = 0, \tag{3.37}$$

where r_{TC} is given as in Definition 3.1.

By increasing the time–lag Δt between portfolio adjustments, we can decrease transaction costs. Therefore, in order to minimize transaction costs, we have to take a larger time–lag Δt. On the other hand, as it will be shown in the next section, choosing larger time–lag Δt could lead to a higher investor's exposure to the risk from an unprotected portfolio.

Let us derive transaction costs measure r_{TC} taking into account the general transaction costs function. We apply the expectation operator $\mathbb{E}$ to equation (3.29) using approximation (3.33) and notation (3.34)-(3.35) and after some rearranging we obtain:

$$r_{TC} = \frac{1}{2} \frac{\mathbb{E}\left[C(\alpha|\Phi|)\alpha|\Phi|\right]}{\Delta t}, \tag{3.38}$$

where $\alpha = \sigma S \left|\partial_S^2 V\right| \sqrt{\Delta t}$ and Φ is a normally distributed random variable $\Phi \sim N(0,1)$.

To simplify notation we introduce the function $\tilde{C}(\alpha)$ defined as follows:

Definition 3.2. *Let $C = C(\xi)$; $C : \mathbb{R}_0^+ \to \mathbb{R}$ be transaction costs function. We call the function $\tilde{C} : \mathbb{R}_0^+ \to \mathbb{R}$ given by*

$$\tilde{C}(\xi) = \mathbb{E}[C(\xi|\Phi|)|\Phi|], \tag{3.39}$$

the modified function of transaction costs. Variable $\xi \geq 0$ and $\Phi \sim N(0,1)$, i.e., Φ is a random variable with a standardized normal distribution.

This definition established a relation between original $C(\alpha)$ and its modification $\tilde{C}(\alpha)$.

Applying relation (3.39) from Definition 3.2 to equation (3.38) we obtain the following expression for the transaction cost measure:

$$r_{TC} = \frac{\alpha}{2} \frac{\tilde{C}(\alpha)}{\Delta t}. \tag{3.40}$$

Its form can be derived by expanding the function C in term $C(\alpha|\Phi|)\alpha|\Phi|$ from equation (3.38) into Taylor's series:

$$
\begin{aligned}
r_{TC}\Delta t =& \frac{1}{2}\sum_{n=0}^{\infty} C_n \mathbb{E}\left[\alpha^{n+1}|\Phi|^{n+1}\right] = \frac{1}{2}\sum_{n=0}^{\infty} C_n \mathbb{E}\left[|\Phi|^{n+1}\right]\alpha^{n+1} \\
=& \frac{1}{2}\alpha\sum_{n=0}^{\infty} \tilde{C}_n \alpha^n = \frac{1}{2}\alpha\,\tilde{C}(\alpha), \quad \text{where}
\end{aligned}
$$

$$
\tilde{C}(\alpha) = \sum_{n=0}^{\infty} \tilde{C}_n \alpha^n \quad \text{and} \quad \tilde{C}_n = C_n \mathbb{E}\left[|\Phi|^{n+1}\right]. \tag{3.41}
$$

It is easy to see that the last term can be computed as follows:

$$
\mathbb{E}\left[|\Phi|^{n+1}\right] = \frac{2^{n/2+1}}{\sqrt{2\pi}}\,\Gamma\left(n/2+1\right). \tag{3.42}
$$

In more detail:

$$
\begin{aligned}
\mathbb{E}[|\Phi|^{n+1}] =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |x|^{n+1} e^{-x^2/2}\mathrm{d}x \qquad (3.43)\\
=& \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} x^{n+1} e^{-x^2/2}\mathrm{d}x \\
=& \frac{2^{n/2+1}}{\sqrt{2\pi}}\int_{0}^{\infty} t^{(n/2+1)-1} e^{-t}\mathrm{d}t \\
=& \frac{2^{n/2+1}}{\sqrt{2\pi}}\,\Gamma\left(n/2+1\right).
\end{aligned}
$$

Here the substitution $x^2/2 = t$ and properties of Gamma function

$$
\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t}\,\mathrm{dt}
$$

were used.

At this point we would like to introduce some properties of the modified function of transaction costs function $\tilde{C}$.

Proposition 3.1. *Let $\tilde{C}$ be the modified function of transaction costs defined in Definition 3.2 and C the transaction costs function from assumption $[C_4]$. Then based on C the modified transaction costs function $\tilde{C}$ has the following properties:*

(*i*) *If* $C(\xi) \geq 0$, *then* $\tilde{C}(\xi) \geq 0$.

(*ii*) *If* $C(\xi)$ *is decreasing (increasing), then* $\tilde{C}(\xi)$ *is decreasing (increasing).*

(*iii*) *If* $C(\xi)$ *is convex (concave), then* $\tilde{C}(\xi)$ *is convex (concave).*

(*iv*) *If function* $C(\xi)$ *is* C^k*–smooth, then function* $\tilde{C}(\xi)$ *is* C^k*–smooth.*

Proof. The proof is based on the properties of an expected value.

(*i*) For each $|\Phi| \geq 0$ we have $C(\xi) \geq 0$. Therefore $\tilde{C}(\xi) = \mathbb{E}\left[C(\xi|\Phi|)|\Phi|\right] \geq 0$.

(*ii*) If $\xi_1, \xi_2 \in \mathbb{R}_0^+ : \xi_1 \leq \xi_2$, then $\xi_1|\Phi| \leq \xi_2|\Phi|$. Assuming $C(\xi)$ decreasing, i.e. $C(\xi_1|\Phi|)|\Phi| \geq C(\xi_2|\Phi|)|\Phi|$, then

$$\mathbb{E}\left[C(\xi_1|\Phi|)|\Phi|\right] \geq \mathbb{E}\left[C(\xi_2|\Phi|)|\Phi|\right],$$

that means $\tilde{C}(\xi_1) \geq \tilde{C}(\xi_2)$. The proof for $\tilde{C}$ increasing is similar.

(*iii*) For any $\xi_1, \xi_2 \in \mathbb{R}_0^+$ and every $\lambda \in [0,1]$ we have:

$$\begin{aligned} C\big(\lambda\xi_1|\Phi| + (1-\lambda)\xi_2|\Phi|\big)|\Phi| \leq& \lambda C\big(\xi_1|\Phi|\big)|\Phi| + (1-\lambda)C\big(\xi_2|\Phi|\big)|\Phi| \\ \mathbb{E}\Big[C\big(\lambda\xi_1|\Phi| + (1-\lambda)\xi_2|\Phi|\big)|\Phi|\Big] \leq& \mathbb{E}\Big[\lambda C\big(\xi_1|\Phi|\big)|\Phi| + (1-\lambda)C\big(\xi_2|\Phi|\big)|\Phi|\Big] \\ \tilde{C}\big(\lambda\xi_1 + (1-\lambda)\xi_2\big) \leq& \lambda\tilde{C}\big(\xi_1\big) + (1-\lambda)\tilde{C}\big(\xi_2\big). \end{aligned} \tag{3.44}$$

The same proof works for concave $\tilde{C}$.

(*iv*) *Case* $(k=1)$ $\tilde{C}$ is also a smooth function, because it has a derivative of the first order:

$$\begin{aligned} \tilde{C}'(\xi) =& \lim_{\epsilon\to 0}\frac{\tilde{C}(\xi+\epsilon)-\tilde{C}(\xi)}{\epsilon} = \lim_{\epsilon\to 0}\mathbb{E}\left[\frac{C\big((\xi+\epsilon)|\Phi|\big)-C\big(\xi|\Phi|\big)}{\epsilon}|\Phi|\right] \\ =& \mathbb{E}\Big[C'\big(\xi|\Phi|\big)|\Phi||\Phi|\Big] = \mathbb{E}\Big(C'\big(\xi|\Phi|\big)|\Phi|^2\Big). \end{aligned}$$

Case $(k \geq 2)$ By analogy, $\tilde{C}$ has derivatives of all orders:

$$\tilde{C}^{(k)}(\xi) = \mathbb{E}\Big[C^{(k)}\big(\xi|\Phi|\big)|\Phi|^{k+1}\Big].$$

$\square$

Now we give some concrete examples of functions of variable transaction costs. At first we present a classic example of a constant function according to Leland. The aim is to derive and show their modifications, i.e. relevant functions $\tilde{C}$. The important point to note here is the introduction of two new types of functions of variable transaction costs in Subsections 3.8.3 and 3.8.4.

3.8.1 Constant Transaction Costs Function

This subsection contains a case of the constant transaction costs function and its modification $\tilde{C}$ introduced in the previous section. We refer to classical function of Leland model [29] from Section(3.3) and also assumption $[C_4]$ in Subsection 3.8.

In the Leland model the function of transaction costs C has the form:

$$C(\xi) \equiv C_0, \quad \text{for } \xi \geq 0, \tag{3.45}$$

where $C_0 > 0$ denotes constant transaction costs.

The modified transaction costs function of the Leland model according to relation (3.41) is:

$$\tilde{C}(\xi) \equiv C_0\sqrt{\frac{2}{\pi}}, \quad \text{for } \xi \geq 0, \tag{3.46}$$

where $C_0 > 0$ denotes the constant transaction costs of the original Leland model. Both of them are shown in Figure 3.2. They are depicted for the parameter value $C_0 = 0.02$.

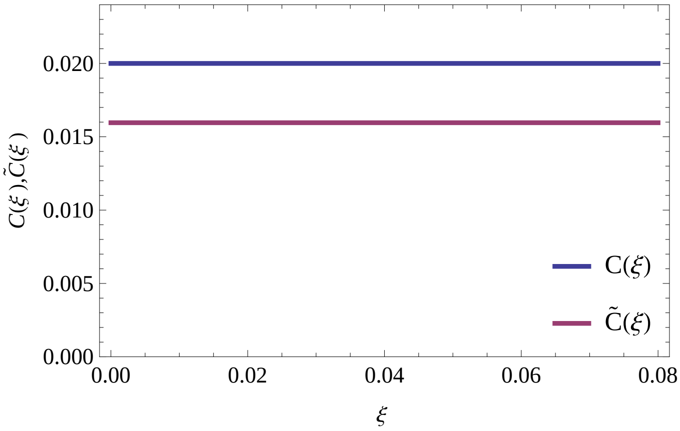


Figure 3.2: Constant transaction costs functions by Leland model.

3.8.2 Linear Decreasing Transaction Costs Function

In the model proposed by Amster et al. [1], which was introduced in Subsection 3.5, the function C is linear and decreasing:

$$C(\xi) \equiv C_0 - \kappa\xi, \quad \text{for } \xi \geq 0, \tag{3.47}$$

where $C_0 > 0$ denotes constant transaction costs and $\kappa \geq 0$ is the rate at which transaction costs decrease (measured per one transaction).

According to the relation (3.41) the modified transaction costs function of the model proposed by Amster et al. has the form:

$$\tilde{C}(\xi) \equiv C_0\sqrt{\frac{2}{\pi}} - \kappa\xi, \quad \text{for } \xi \geq 0, \tag{3.48}$$

where constants C_0 and κ are the same as in the original model.

A disadvantage of the function (3.47) lies in the fact that it may attain negative values provided the amount of transactions $|\Delta\delta|$ exceeds the critical value $\xi = |\Delta\delta| = C_0/\kappa$. For illustration see Figure 3.3. In the figure there are functions depicted for parameter values $C_0 = 0.02$ and $\kappa = 0.5$.

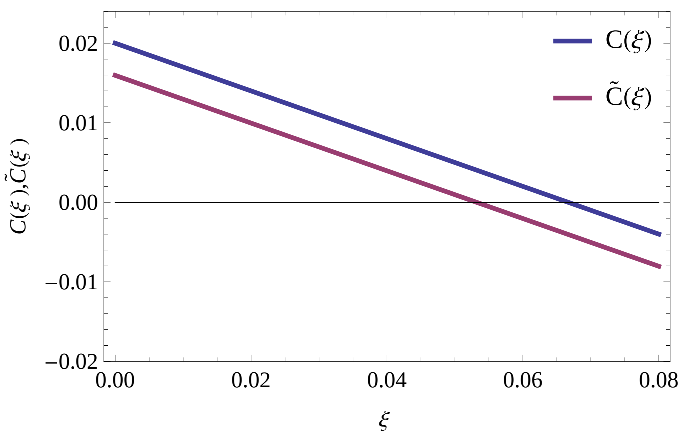


Figure 3.3: Linear decreasing transaction costs functions.

3.8.3 Piecewise Linear Non–Increasing Transaction Costs Function

In this section we present a reasonable example of realistic transaction costs that are also decreasing with the amount of transactions as in model studied by Amster. The benefit is the elimination of the problem of negative values of the linear decreasing costs function. We define the following piecewise linear function.

Definition 3.3. *We define a piecewise linear non–increasing transaction costs function as*

$$C(\xi) = \begin{cases} C_0, & \text{if} \quad 0 \leq \xi < \xi_-, \\ C_0 - \kappa(\xi - \xi_-), & \text{if} \quad \xi_- \leq \xi \leq \xi_+, \\ \underline{C}_0, & \text{if} \quad \xi \geq \xi_+. \end{cases} \tag{3.49}$$

where we assume $C_0, \kappa > 0$, *and* $0 \leq \xi_- \leq \xi_+ \leq \infty$ *to be given constants and* $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$.

This is the most realistic function, because for some small volume of traded stocks one constant amount C_0 is paid, when the volume is significant, there starts to be a discount depending on higher volume and finally some another small constant payment $\underline{C}_0$ when there are very large trades.

This function also covers classical transaction costs functions and it satisfies all

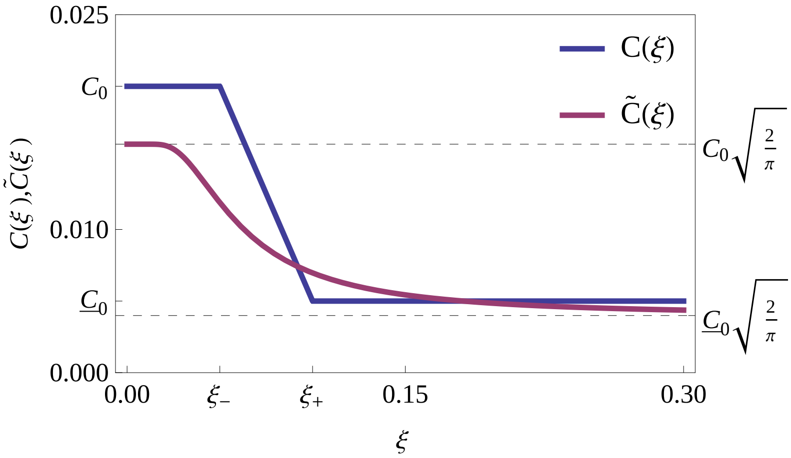


Figure 3.4: A piecewise linear transaction costs function C and its modification $\tilde{C}$.

assumptions we need when modelling and optimizing in Section 4.4. It is easy to see that this example includes all of the previous observations because in the case of:

- if $\xi_- = \xi_+ = 0$ then the function C is constant, that means it is the same as in the Leland model (Subsection 3.8.1);
- if $\xi_- = 0$ and $\xi_+ = \infty$ then the function C is linearly decreasing, i.e. the same as in the model studied by Amster (Subsection 3.8.2).

In the next part we will present the detailed derivation of the modified transaction costs function $\tilde{C}$ for this type of piecewise linear non–increasing function C. For comparison of original C and modified $\tilde{C}$ function see Figure 3.4. These functions are depicted for parameter values $C_0 = 0.02$, $\kappa = 0.3$, $\xi_- = 0.05$ and $\xi_+ = 0.1$.

Proposition 3.2. *The modified transaction costs function $\tilde{C}$ of piecewise linear function (3.49) is given by:*

$$\tilde{C}(\xi) = C_0\sqrt{\frac{2}{\pi}} - 2\kappa\xi \int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x, \quad \textit{for } \xi \geq 0. \tag{3.50}$$

Proof. Now let us derive the modified transaction cost function $\tilde{C}(\xi)$. Since the relation between $\tilde{C}(\xi)$ and $C(\xi)$ is according to Definition 3.2, the $\tilde{C}(\xi)$ can be written as:

$$
\begin{aligned}
\tilde{C}(\xi) =& \mathbb{E}\big[C(\xi|\Phi|)|\Phi|\big] = \int_{-\infty}^{\infty} C(\xi|x|)|x|\frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x = 2\int_{0}^{\infty} C(\xi x)x\frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x \\
=& -2\int_{0}^{\infty} C(\xi x)f'(x)\mathrm{d}x = -2\Big[C(\xi x)f(x)\Big]_0^{\infty} + 2\xi\int_{0}^{\infty} C'(\xi x)f(x)\mathrm{d}x \\
=& C_0\sqrt{\frac{2}{\pi}} - 2\kappa\xi\int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} f(x)\mathrm{d}x,
\end{aligned} \tag{3.51}
$$

where

$$
f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \tag{3.52}
$$

satisfies the relation

$$
f'(x) = -xf(x) \tag{3.53}
$$

and

$$
C'(\xi x) = \begin{cases} -\kappa, & \text{if } \frac{\xi_-}{\xi} \le x < \frac{\xi_+}{\xi}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.54}
$$

□

Proposition 3.3. *Let $\tilde{C}(\xi)$ be a function defined by equation (3.50). Then the $\tilde{C}(\xi)$ has the following properties:*

(i) $\tilde{C}(0) = C_0\sqrt{\frac{2}{\pi}}$;

(ii) $\tilde{C}'(\xi) = -2\kappa\int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} f(x)\mathrm{d}x + 2\kappa\left[\frac{\xi_+}{\xi}f\left(\frac{\xi_+}{\xi}\right) - \frac{\xi_-}{\xi}f\left(\frac{\xi_-}{\xi}\right)\right] < 0$ *for* $\xi > 0$;

(iii)

$$
\tilde{C}'(0) = \begin{cases} -\kappa, & \textit{if } \xi_- = 0, \\ 0, & \textit{if } \xi_- > 0; \end{cases}
$$

(iv) $\tilde{C}''(\xi) = 2\kappa\left[\frac{\xi_+^3}{\xi^4}f\left(\frac{\xi_+}{\xi}\right) - \frac{\xi_-^3}{\xi^4}f\left(\frac{\xi_-}{\xi}\right)\right] > 0$, *i.e.* $\tilde{C}$ *is a convex function if* $\xi_- = 0$;

(v) $\tilde{C}$ *need not be convex if* $\xi_- > 0$ *(see Figure 3.4);*

(vi) $\tilde{C}''(0) \equiv 0$.

Proof. (*i*) Trivial.

(*ii*) Let us derive $\tilde{C}$ from equation (3.50), then:

$$\begin{aligned}\tilde{C}'(\xi) =& -2\kappa \int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} f(x)\mathrm{d}x - 2\kappa\xi \left[f\left(\frac{\xi_+}{\xi}\right)\left(-\frac{\xi_+}{\xi^2}\right) + f\left(\frac{\xi_-}{\xi}\right)\left(-\frac{\xi_-}{\xi^2}\right)\right] \\ =& -2\kappa \int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} f(x)\mathrm{d}x + 2\kappa \left[\frac{\xi_+}{\xi} f\left(\frac{\xi_+}{\xi}\right) - \frac{\xi_-}{\xi} f\left(\frac{\xi_-}{\xi}\right)\right] < 0\end{aligned}$$

because $F(x) = -\int_0^x f(\xi)\, d\xi + xf(x)$ is decreasing function for $x > 0$, $F(0) = 0$. (By differentiation $F'(x) = xf'(x) < 0$.)

(*iii*) If $\xi_- = 0$ then $\tilde{C}'(0) = -2\kappa \int_0^\infty f(x)\mathrm{d}x = -\kappa$.

If $\xi_- > 0$ then $\tilde{C}'(0) = -2\kappa \int_\infty^\infty f(x)\mathrm{d}x = 0$.

(*iv*) Now we differentiate $\tilde{C}'(\xi)$ again and use the property $f'(x) = -xf(x)$. Then we obtain:

$$\begin{aligned}\tilde{C}''(\xi) =& 2\kappa \left[f'\left(\frac{\xi_+}{\xi}\right)\left(-\frac{\xi_+^2}{\xi^3}\right) - f'\left(\frac{\xi_-}{\xi}\right)\left(-\frac{\xi_-^2}{\xi^3}\right)\right] \\ =& 2\kappa \left[\frac{\xi_+^3}{\xi^4} f\left(\frac{\xi_+}{\xi}\right) - \frac{\xi_-^3}{\xi^4} f\left(\frac{\xi_-}{\xi}\right)\right] \\ =& 2\kappa \frac{\xi_+^3}{\xi^4} f\left(\frac{\xi_+}{\xi}\right) > 0 \quad \text{for } \xi_- = 0.\end{aligned}$$

(*v*) See the counter example shown in Figure 3.4.

(*vi*) Based on $\tilde{C}''(\xi) = 2\kappa \left[\frac{\xi_+^3}{\xi^4} f\left(\frac{\xi_+}{\xi}\right) - \frac{\xi_-^3}{\xi^4} f\left(\frac{\xi_-}{\xi}\right)\right]$ seeing that for $\xi \to 0$ the term $f\left(\frac{\xi_+}{\xi}\right)$ and also $f\left(\frac{\xi_-}{\xi}\right)$ goes to zero exponentially, whereas $\frac{\xi_+^3}{\xi^4}$ and $\frac{\xi_-^3}{\xi^4}$ go to $+\infty$ polynomially, consequently $\tilde{C}''(0) = 0$.

□

Proposition 3.4. *The function $\tilde{C}$ defined in equation (3.50) satisfies*

$$\underline{C}_0 \sqrt{\frac{2}{\pi}} \le \tilde{C}(\xi) \le C_0 \sqrt{\frac{2}{\pi}} \quad \textit{and} \tag{3.55}$$

$$\lim_{\xi\to\infty} \tilde{C}(\xi) = \underline{C}_0\sqrt{\frac{2}{\pi}} > 0, \tag{3.56}$$

where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$ *from Definition 3.3.*

Proof. We first prove (3.55). Let us start with the first inequality. We will use the upper bound of the function $f(x)$, i.e.

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \le \frac{1}{\sqrt{2\pi}}.$$

Hence equation (3.50) implies

$$\tilde{C}(\xi) > C_0\sqrt{\frac{2}{\pi}} - 2\kappa\xi\left(\frac{\xi_+}{\xi} - \frac{\xi_-}{\xi}\right)\frac{1}{\sqrt{2\pi}} \equiv [C_0 - \kappa(\xi_+ - \xi_-)]\sqrt{\frac{2}{\pi}} \equiv \underline{C}_0\sqrt{\frac{2}{\pi}}.$$

The second inequality is clear from equation (3.50), because $2\kappa\xi\int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} f(x)\mathrm{d}x$ is a positive number and hence for $\xi > 0$

$$\tilde{C}(\xi) = C_0\sqrt{\frac{2}{\pi}} - 2\kappa\xi\int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} f(x)\mathrm{d}x \le C_0\sqrt{\frac{2}{\pi}}.$$

The basic idea of the proof of equation (3.56) is to use the L'Hospital rule and the fact that $f(0) = \frac{1}{\sqrt{2\pi}}$.

$$\begin{aligned}
\lim_{\xi\to\infty} \tilde{C}(\xi) =& C_0\sqrt{\frac{2}{\pi}} - 2\kappa\lim_{\xi\to\infty}\frac{\int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} f(x)\mathrm{d}x}{\frac{1}{\xi}} \\
=& C_0\sqrt{\frac{2}{\pi}} - 2\kappa\lim_{\xi\to\infty}\frac{-\frac{\xi_+}{\xi}f\left(\frac{\xi_+}{\xi}\right) + \frac{\xi_-}{\xi}f\left(\frac{\xi_-}{\xi}\right)}{-\frac{1}{\xi^2}} \\
=& C_0\sqrt{\frac{2}{\pi}} - 2\kappa\lim_{\xi\to\infty}\left[\xi_+ f\left(\frac{\xi_+}{\xi}\right) - \xi_- f\left(\frac{\xi_-}{\xi}\right)\right] \\
=& C_0\sqrt{\frac{2}{\pi}} - \frac{2\kappa}{\sqrt{2\pi}}(\xi_+ - \xi_-) \\
=& \big[C_0 - \kappa(\xi_+ - \xi_-)\big]\sqrt{\frac{2}{\pi}} = \underline{C}_0\sqrt{\frac{2}{\pi}} > 0
\end{aligned}$$

Note that our assumption was following $C_0 - \kappa(\xi_+ - \xi_-) > 0$, see Definition 3.3. $\square$

Proposition 3.5. *Let $\tilde{C}$ be defined by equation (3.50) with properties (i)-(vi), then for all $\xi \geq 0$*

$$\tilde{C}(\xi) - \xi\tilde{C}'(\xi) + \frac{\xi^2}{2}\tilde{C}''(\xi) \geq \sqrt{\frac{2}{\pi}}\big[C_0 - \kappa(\xi_+ - \xi_-)\big] > 0. \tag{3.57}$$

Proof. The second inequality holds from Proposition 3.4

$$\lim_{\xi\to\infty} \tilde{C}(\xi) = \underline{C}_0\sqrt{\frac{2}{\pi}} > 0, \tag{3.58}$$

where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$. Let us simplify the first inequality of (3.57) and in the second step divide it by ξ:

$$\sqrt{\frac{2}{\pi}}C_0 + \left[\xi_+\left(\frac{\xi_+^2}{\xi^2} - 2\right) f\left(\frac{\xi_+}{\xi}\right) - \xi_-\left(\frac{\xi_-^2}{\xi^2} - 2\right) f\left(\frac{\xi_-}{\xi}\right)\right] \geq \sqrt{\frac{2}{\pi}}C_0 - \sqrt{\frac{2}{\pi}}\kappa(\xi_+ - \xi_-)$$
$$\frac{\xi_+}{\xi}\left(\frac{\xi_+^2}{\xi^2} - 2\right) f\left(\frac{\xi_+}{\xi}\right) - \frac{\xi_-}{\xi}\left(\frac{\xi_-^2}{\xi^2} - 2\right) f\left(\frac{\xi_-}{\xi}\right) \geq -\sqrt{\frac{2}{\pi}}(\xi_+ - \xi_-).$$

For simplicity of notation, we write x_+, x_- instead of $\frac{\xi_+}{\xi}, \frac{\xi_-}{\xi}$ respectively. Setting

$$x_\pm := \frac{\xi_\pm}{\xi},$$

where $x_- < x_+$, we can write

$$x_+(x_+^2 - 2)f(x_+) + \sqrt{\frac{2}{\pi}}x_+ \geq x_-(x_-^2 - 2)f(x_-) + \sqrt{\frac{2}{\pi}}x_- \tag{3.59}$$

This inequality holds if and only if the following function is non–decreasing:

$$h(t) \mapsto t(t^2 - 2)f(t) + \sqrt{\frac{2}{\pi}}t. \tag{3.60}$$

The function $h(t)$ is non–dependent on any parameters. An easy computation shows

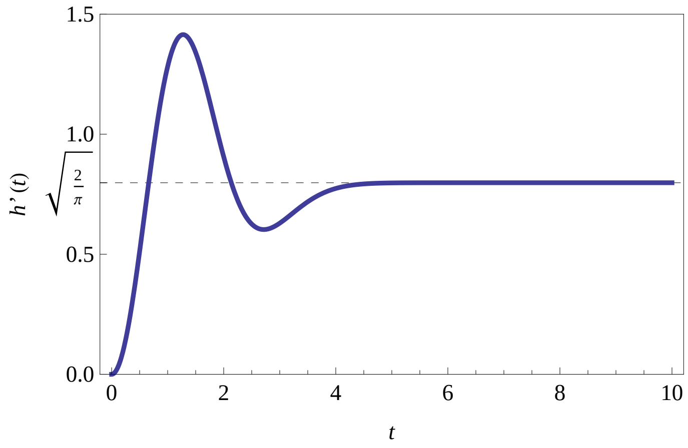


Figure 3.5: The plot of first derivative $h'(t)$ of the function $h(t) \mapsto t(t^2-2)f(t)+\sqrt{\frac{2}{\pi}}t$.

that its first derivative is non–negative, i.e. minimum of $h(t) > -\sqrt{\frac{2}{\pi}}$. We see in Figure 3.5, that $h'(0) = 0$, $h'(t)$ has only two roots and asymptotically $\lim\limits_{t\to\infty} h'(t) = \sqrt{\frac{2}{\pi}}$. Since $-\sqrt{\frac{2}{\pi}} = h(0) < h(t)$ and first derivative $h'(t) \geq 0$ is non–negative for any $t \geq 0$, this shows that $h(t)$ is non–decreasing. As $h(t)$ is non–decreasing we have (3.59) and we conclude that (3.57) holds. $\square$

We have introduced a universal and reasonable example of a realistic transaction costs function in the form of a piecewise linear function whether ξ_- is zero or not.

3.8.4 Exponentially Decreasing Transaction Costs Function

As an another example of transaction costs that are decreasing with the amount of transactions we can consider the following exponential function of the form

$$C(\xi) = C_0 \exp(-\kappa\xi), \quad \text{for } \xi \geq 0, \tag{3.61}$$

where $C_0 > 0$ and $\kappa > 0$ are given constants. Its modification:

$$\tilde{C}(\xi) = C_0\sqrt{\frac{2}{\pi}} + \sum_{n=0}^{\infty} \frac{C_0}{n!}(-\kappa\xi)^n \frac{2^{\frac{n}{2}+1}}{\sqrt{2\pi}} \Gamma\left(\frac{n}{2}+1\right), \quad \text{for } \xi \geq 0, \tag{3.62}$$

where constants are the same as in original. In Figure 3.6 these functions are depicted for parameter values $C_0 = 0.02$ and $\kappa = 100$.

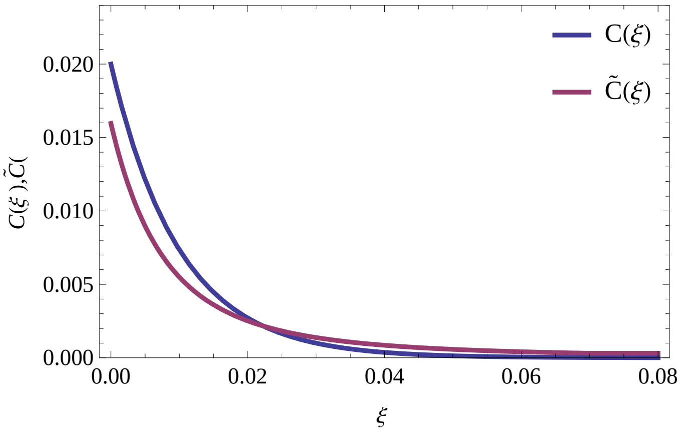


Figure 3.6: Exponential decreasing transaction costs functions.

This figure was constructed by $\tilde{C}$ of another form than (3.62). It is because in the case of Tailor's formula the number of elements should be finite and it can cause numerical problems. The value of the function for a high variable ξ goes either to $+\infty$ or to $-\infty$. For this reason we realized another expression for modified transaction costs function of the form:

$$\tilde{C} = C_0\sqrt{\frac{2}{\pi}}\,\phi(-\sqrt{2}\kappa\xi), \quad \text{for } \xi \geq 0, \quad \text{where} \tag{3.63}$$

$$\phi(x) = 1 + xe^{\frac{x^2}{4}}\left(\operatorname{erf}(x/2) + 1\right)\frac{\sqrt{\pi}}{2}. \tag{3.64}$$

Chapter 4

A Novel Option Pricing Model under Transaction Costs and Risk of the Unprotected Portfolio

The aim of this chapter is to recall how to model risk from volatile portfolio and present a novel nonlinear generalization of the classical Black–Scholes equation that incorporates both variable transaction costs and the risk arising from a volatile portfolio.

In the next section we recall how to model risk from volatile portfolio. We follow Jandačka and Ševčovič [24] modification of the original Kratka's approach [25].

4.1 Modelling Risk from an Unprotected Portfolio

Except transaction costs the nonlinearity is caused also by including risk from an unprotected portfolio. We adopt the measure r_{VP} of risk following from the unprotected portfolio on the time interval Δt from the derivation of the so–called RAMP model in [24] by Jandačka and Ševčovič, i.e. generalization of the RAPM (see Sec. 3.7) proposed by Kratka in [25].

An investor usually ask for a price compensation, in the case when the portfolio consisting of options and assets is highly volatile. Notice that exposure to risk is higher when the time–lag between portfolio adjustments is higher. Jandačka and Ševčovič propose a measure of such a risk based on the volatility of a fluctuating portfolio.

The volatility of a fluctuating portfolio can be measured by the variance of relative increments of replicating portfolio $\Pi = V + \delta S$, that is, by the term $Var(\Delta\Pi/S)$. Hence it is reasonable to introduce the measure r_{VP} as follows:

$$r_{VP} = R\frac{Var(\Delta\Pi/S)}{\Delta t}. \tag{4.1}$$

In other words, r_{VP} is proportional to the variance of the relative change of a portfolio per time interval Δt. A constant R is called the *risk premium coefficient*. It can be interpreted as the marginal value of investor's exposure to a risk.

Now applying Itō's formula to the differential $\delta\Pi = \Delta V - \delta\,\Delta S + \delta q S\Delta t$, we obtain

$$\Delta\Pi = (\partial_S V + \delta)\sigma S\Delta w + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V(\Delta w)^2 + \mathcal{G}, \tag{4.2}$$

where $\mathcal{G} = (\partial_S V + \delta)\rho S\Delta t + \partial_t V\Delta t + \delta q S\Delta t$ is a deterministic term, i.e. $\mathbb{E}[\mathcal{G}] = \mathcal{G}$ in the lowest order Δt-term approximation. Thus

$$\Delta\Pi - \mathbb{E}[\Delta\Pi] = (\partial_S V + \delta)\sigma S\Phi\sqrt{\Delta t} + \frac{1}{2}\sigma^2 S^2(\Phi^2 - 1)\partial_S^2 V\Delta t, \tag{4.3}$$

where Φ is a random variable with the standard normal distribution such that $\Delta w = \Phi\sqrt{\Delta t}$. Hence the variance of $\Delta\Pi$ can be computed as follows

$$var(\Delta\Pi) = \mathbb{E}\Big[(\Delta\Pi - \mathbb{E}[\Delta\Pi])^2\Big] = \mathbb{E}\left[\left((\partial_S V + \delta)\sigma S\Phi\sqrt{\Delta t} + \frac{1}{2}\sigma^2 S^2(\Phi^2 - 1)\partial_S^2 V\Delta t\right)^2\right].$$

Similarly, as in the previous derivation of the model we assume the δ–hedging of portfolio adjustments, i.e. we choose $\delta = -\partial_S V$. Since $\mathbb{E}[(\Phi^2 - 1)^2] = 2$ we obtain an expression for the risk premium r_{VP} in the form:

$$r_{VP} = \frac{1}{2} R\sigma^4 S^2 \left(\partial_S^2 V\right)^2 \Delta t. \tag{4.4}$$

where $R \geq 0$ is a non–negative constant representing the level of risk of the unprotected portfolio.

Notice that in this approach of Jandačka and Ševčovič the increase in the time–lag Δt between consecutive transactions leads to a linear increase of the risk from a volatile portfolio.

4.2 Option Pricing Model under Transaction Costs and Risk of the Unprotected Portfolio

The aim of this section is to present a novel nonlinear generalization of the classical Black–Scholes equation that incorporates both variable transaction costs and the risk arising from a volatile portfolio.

By adding the measures r_{TC} and r_{VP} defined in Definition 3.1 and equation (4.4) respectively, we obtain a total measure of the risk r_R given by the following relation

$$r_R = r_{TC} + r_{VP}.$$

The total risk premium r_R is a function of Δt, i.e. the time–lag between two consecutive portfolio adjustments. As both r_{TC} as well as r_{VP} depend on the time–lag Δt so does the total risk premium r_R.

In the derivation of the new nonlinear model, we take into account **the variable transaction costs and risk of the unprotected portfolio**.

We again assume that the underlying stock price pays dividends ($q \neq 0$) and follows a geometric Brownian motion (2.1) $dS = (\rho - q)Sdt + \sigma S dw$. We follow the derivation of the model from Section 3.8. Following these steps, the difference is in the change of the portfolio (equation (3.28)), here of the form:

$$\Delta\Pi = \Delta V + \delta\Delta S + \delta q S\Delta t - r_R S\Delta t, \tag{4.5}$$

where r_R is total risk $r_R = r_{TC} + r_{VP}$. This risk includes transaction costs in addition to the level of risk of the unprotected portfolio. They are being considered because a large rearranging interval Δt leads to smaller transaction costs, at the same time, however, the investor is in danger, because the portfolio is for a long time unprotected.

The transaction cost measure r_{TC} is due to a variable transaction cost $C = C(|\Delta\delta|)$ the same as we defined in equation $r_{TC}S\Delta t = \frac{S}{2}\alpha\,\tilde{C}(\alpha)$, where $\alpha = \sigma S\,|\partial_S^2 V|\,\sqrt{\Delta t}$. The measure r_{VP} of risk following from the unprotected portfolio we adopt in the form $r_{VP} = \frac{1}{2}R\sigma^4 S^2\left(\partial_S^2 V\right)^2\Delta t$. To simplify notation we use

$$\Gamma = \partial_S^2 V. \tag{4.6}$$

The final equation for the new model then is

$$\partial_t V + \frac{1}{2}\hat{\sigma}^2(S\Gamma, \Delta t)S^2\partial_S^2 V + (r-q)S\partial_S V - rV = 0, \tag{4.7}$$

with volatility having form

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2\left(1 - \tilde{C}(\sigma|S\Gamma|\sqrt{\Delta t})\frac{\operatorname{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} - R\sigma^2 S\Gamma\Delta t\right). \tag{4.8}$$

It is a generalization of the model with decreasing transaction costs studied by Amster et al., hence the model includes variable transaction costs, for example, piecewise linear non-increasing or exponentially decreasing, from section 3.8 in the form of a general function of transaction costs $\tilde{C}$. At the same time there is a possibility to control the risk of an unprotected portfolio. That means including the last term with the risk premium coefficient R, the model is in combination also with the RAPM model. In this form the nonlinear volatility (4.8) is with unprescribed time–lag interval Δt, but in Section 4.4 we will show how to find this optimal hedging time.

For the purpose of the numerical analysis (see Chapters 5 and 6) it is convenient to introduce the following function

$$\beta(H, x, \tau) \equiv \frac{1}{2}\hat{\sigma}^2(S\Gamma, S, t)S\Gamma, \tag{4.9}$$

where $H := S\Gamma$, $x = \ln S/E, \tau = T - t$.

More specifically, in our case of the RAPM based model, the function β of the novel nonlinear model reads as follows:

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \tilde{C}(\sigma|H|\sqrt{\Delta t})\frac{\text{sgn}(H)}{\sigma\sqrt{\Delta t}} - R\sigma^2 H\Delta t\right)H. \tag{4.10}$$

4.3 Special Cases of the Novel Model

In this section we give some special cases of the new model. We see that the new model is a generalization of some known nonlinear models. For different choices of $\tilde{C}$ and R we obtain the following special forms.

4.3.1 Classical Black–Scholes Model

In case of:

- no transaction costs, i.e. $C = 0$ and
- zero risk premium coefficient arising from the risk of an unprotected portfolio, i.e. $R = 0$,

the volatility function $\hat{\sigma}^2$ given by (4.8) reduces to constant volatility

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2. \tag{4.11}$$

It means the resulting equation is the same as in the classical Black–Scholes model.

4.3.2 Leland Model

As one can expect, in case of:

- constant transaction costs, i.e. $C = const$,
- zero risk premium coefficient arising from an unprotected portfolio, i.e. $R = 0$ and
- with time–lag Δt given,

the volatility function $\hat{\sigma}^2$ given by (4.8) reduces to nonlinear volatility

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2 \left(1 - \mathrm{Le}\, \mathrm{sgn}(S\Gamma)\right), \tag{4.12}$$

where $\mathrm{Le} = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}$ is the Leland number. It is exactly the same as in the Leland model setting $C = C_0$, i.e. $\tilde{C}(\alpha) \equiv C_0\sqrt{2/\pi}$ (compare with Section 3.3).

4.3.3 Model with Linear Decreasing Transaction Costs Depending on the Volume of Trading Stocks

Similarly, by setting:

- the transaction costs as a non–constant $C \neq const$, for example is linearly decreasing, i.e.
$$C\left(|\Delta\delta|\right) = C_0 - \kappa|\Delta\delta|,$$
- the risk premium coefficient arising from unprotected portfolio equal to zero, i.e. $R = 0$ and
- by given time–lag Δt,

the volatility function $\hat{\sigma}^2$ given by (4.8) reduces to nonlinear volatility

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2 \left(1 - \mathrm{Le}\, \mathrm{sgn}(S\Gamma) + \kappa S\Gamma\right), \tag{4.13}$$

where $\mathrm{Le} = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}$ is the Leland number (compare with the model proposed by Amster et al. in Section 3.5).

4.3.4 RAPM Model with Fixed Time–Lag Interval

An interesting example arises by setting

- the transaction costs as constant, i.e. $C = const$,
- the risk premium coefficient arising from unprotected portfolio not equal to zero, i.e. $R \neq 0$ and
- the time–lag Δt given and fixed.

Then the volatility function $\hat{\sigma}^2$ given by (4.8) reduces to nonlinear volatility of the form

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2 \left(1 - \left(\frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}\mathrm{sgn}(S\Gamma) + R\sigma^2 S\Gamma\Delta t\right)\right) \tag{4.14}$$

It is an interesting example, because it is similar to the Leland model, but it also contains the term due to an unprotected portfolio and the total volatility could be smaller. If we chose optimal hedging time Δt, we would obtain the volatility of original RAPM model. We refer to Subsection 4.4.1.

4.3.5 RAPM Model with Variable Transaction Costs with Fixed Time–Lag Interval

We obtain an another example by setting

- the transaction costs as a non–constant $C \neq const$, for example, a linearly decreasing function from model proposed by Amster et al., i.e.

$$C\left(|\Delta\delta|\right) = C_0 - \kappa|\Delta\delta|,$$

- the risk premium coefficient arising from an unprotected portfolio not equal to zero, i.e. $R \neq 0$ and
- the time–lag Δt given.

Then the volatility function $\hat{\sigma}^2$ given by (4.8) reduces to a nonlinear volatility of the form:

$$\hat{\sigma}^2(S\Gamma, \Delta t) = \sigma^2 \left(1 - \left(\frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}\operatorname{sgn}(S\Gamma) + R\sigma^2 S\Gamma\Delta t\right) + \kappa S\Gamma\right). \tag{4.15}$$

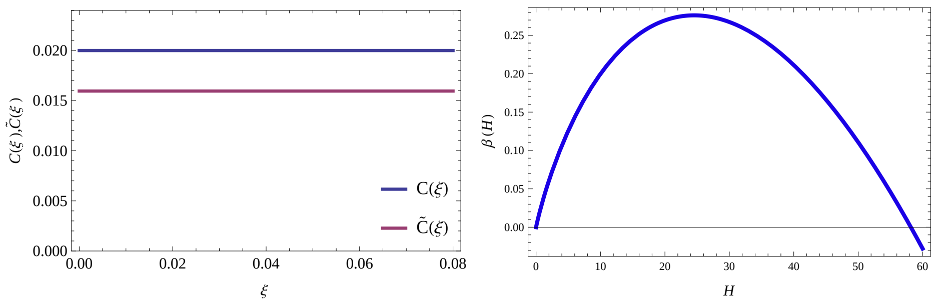
It is a combination of volatility from the model proposed by Amster et al. and the RAPM Model with an unprescribed time–lag interval.

Next we give an examples of variable transaction costs functions and corresponding β functions in Figure 4.1. In cases a) and b) the function β need not to be strictly increasing in a H variable making thus the resulting equation (see Chapter 5) backward parabolic for some values of H. On the other hand, for linearly decreasing or piecewise linear transaction costs function the function β is strictly increasing, it follows Gamma equation is parabolic.

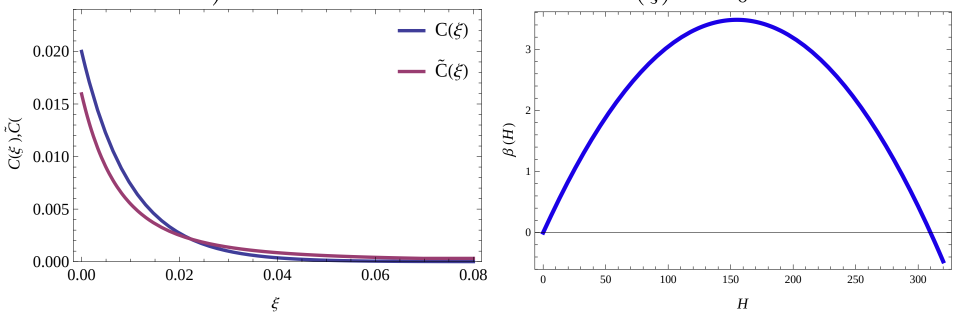
4.4 RAPM Based Models with the Optimal Choice of Hedging Time Δt

Our task is now to minimize the total risk of the portfolio to find the optimal time Δt when rehedging the portfolio. Clearly, in order to minimize transaction costs, we have to take a larger time–lag Δt. On the other hand, a larger time interval Δt means higher risk exposure for the investor, because an increase in the time–lag interval Δt between two consecutive transactions leads to a linear increase of the risk from a volatile portfolio.

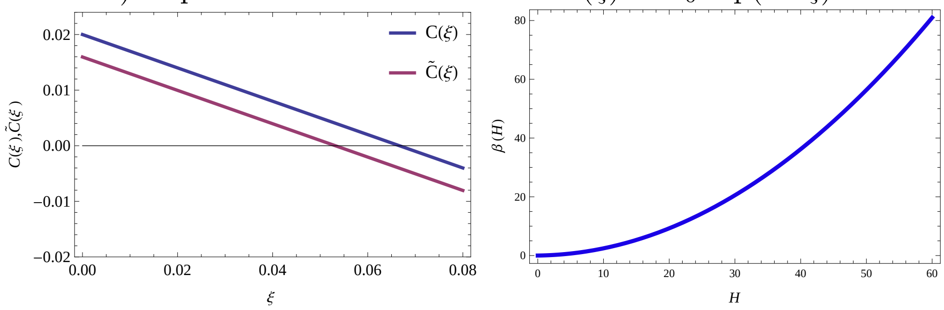
In the first part of this section we will review the basic idea proposed by Jandačka and Ševčovič in the RAPM model [24] for constant transaction costs and



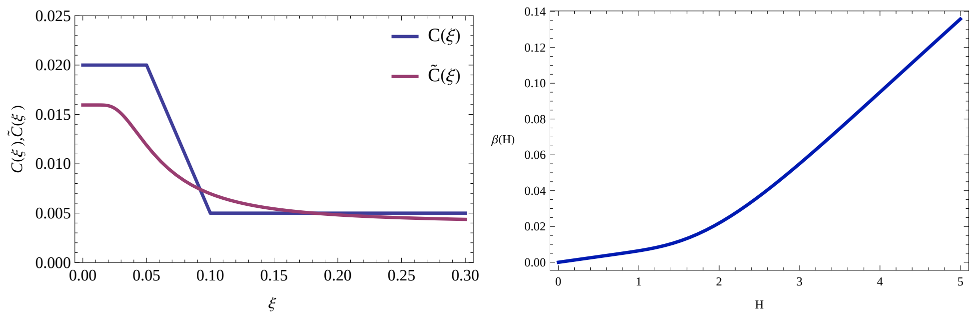
a) Constant transaction costs $C(\xi) = C_0$.



b) Exponential transaction costs $C(\xi) = C_0 \exp(-\kappa\xi)$.



c) Linearly decreasing costs $C(\xi) = C_0 - \kappa\xi$.



d) Piecewise linear non–increasing function.

Figure 4.1: The graph of different types of transaction costs function C with its modification $\tilde{C}$ (left) and corresponding function $\beta(H)$ (right) for the parameter values $R = 10$ and $\sigma = 0.3$.

in the second part we will give a general approach when the variable transaction costs function will be taken into account. We postulate the basic assumptions on admissible transformed functions of transaction costs $\tilde{C}$.

4.4.1 Classical RAPM Model

In this subsection, we will discuss the choice of an optimal time interval between two consecutive portfolio adjustments according to Jandačka and Ševčovič in the paper [24]. The name of the model is the Risk adjusted pricing methodology (RAPM) model. They analysed a model for pricing derivative securities in the presence of both transaction costs as well as the risk from a volatile portfolio. Note, that the transaction costs are constant, i.e. same as in the Hogard, Whalley and Wilmott extension of the Leland model. It is a modification of the Kratka model [25] to be mathematically well proposed and scale invariant. These two important features were missing in the original model of Kratka.

The model is based on the Black–Scholes parabolic PDE in which the transaction costs are constant and the risk from a volatile portfolio is described by the variance of the synthesized portfolio. Transaction costs as well as the volatile portfolio risk depend on the time lag between two consecutive transactions.

We recall that the key idea of the Black–Scholes theory is to examine the differential of the portfolio $\Pi = V + \delta S$ consisting of one option and an amount of δ stocks with unit price S. We recall the change of the portfolio (4.5) consist of these parts

$$\Delta\Pi = \Delta V + \delta\Delta S + \delta q S\Delta t - r_R S\Delta t,$$

where r_R is total risk $r_R = r_{TC} + r_{VP}$. Minimizing this sum yields to the optimal length of the hedge interval.

At first will derive the coefficient of transaction costs r_{TC}. We will assume, for the moment, that there is no risk from the volatile portfolio, i.e., $r_{VP} = 0$. As we mentioned before, the transaction costs are constant. The coefficient $C_0 =$

$(S_{ask} - S_{bid})/2$ denotes the round trip transaction cost per unit dollar of transaction, i.e. $S = (S_{ask} + S_{bid})/2$ is the mid value. This means that the purchase of $\Delta\delta > 0$ or sales of $\Delta\delta < 0$ shares at a price of S makes the transaction cost by the value:

$$\frac{S}{2} C_0 |\Delta\delta| = \frac{S_{ask} - S_{bid}}{2} |\Delta\delta| \tag{4.16}$$

units. Following Leland's approach [22], using Itō's formula and assuming δ–hedging, that is, $\delta = -\partial_S V$ (see Subsection 2.4) of a portfolio Π, the coefficient r_{TC} is derived and is given by the formula

$$r_{TC} = \frac{C_0 |\Gamma| \sigma S}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}} \tag{4.17}$$

(cf. [22, equation 3]).

Next we recall the expression for the risk premium r_{VP} derived already in Section 4.1. The risk from the volatile portfolio is of the form

$$r_{VP} = \frac{1}{2} R \sigma^4 S^2 \Gamma^2 \Delta t.$$

where $R \geq 0$ is non–negative constant representing the level of risk of the unprotected portfolio.

By increasing the time–lag interval Δt between portfolio adjustments, we can decrease transaction costs. Therefore, in order to minimize transaction costs, we have to take larger time–lag Δt. On the other hand, a larger time interval Δt means higher risk exposure for the investor, because an increase in the time–lag interval Δt between two consecutive transactions leads to a linear increase of the risk from a volatile portfolio.

Now move to solution of this problem of minimizing the value of the total risk premium $r_R = r_{TC} + r_{VP}$.In order to find the optimal value of Δt, we have to minimize the following function:

$$\Delta t \mapsto r_R = r_{TC} + r_{VP} = \frac{C_0|\Gamma|\sigma S}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}} + \frac{1}{2} R\sigma^4 S^2 \Gamma^2 \Delta t. \tag{4.18}$$

A graph of the total risk premium as a function of the time–lag Δt is depicted in the Figure 4.2. The unique minimum of the function is attained at the time–lag

$$\Delta t_{opt} = \frac{K^2}{\sigma^2 |S\Gamma|^{2/3}}, \quad \text{where } K = \left(\frac{C_0}{R} \frac{1}{\sqrt{2\pi}} \right)^{1/3}. \tag{4.19}$$

Therefore the minimal value of the function $\Delta t \mapsto r_R(\Delta t)$ we have

$$r_R(\Delta t_{opt}) = \frac{3}{2} \left(\frac{C_0^2 R}{2\pi} \right)^{1/3} \sigma^2 |S\Gamma|^{4/3}. \tag{4.20}$$

Finally by taking the optimal value of the total risk coefficient r_R, we get the

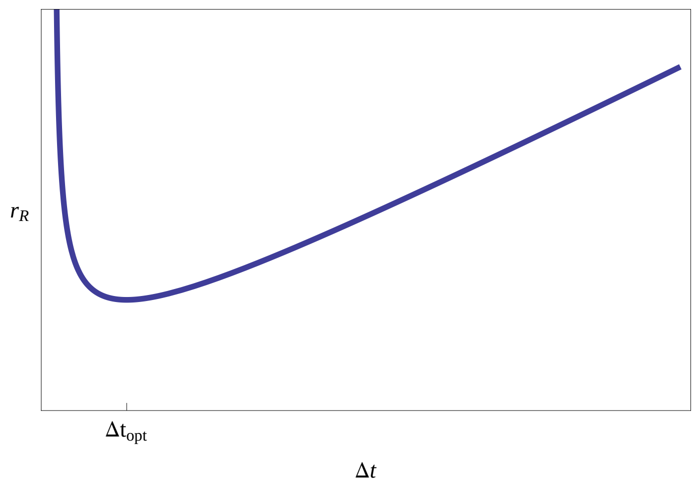


Figure 4.2: The function of total risk premium $\Delta t \mapsto r_R(\Delta t) = r_{TC} + r_{VP}$ attains its unique minimum at the point Δt_{opt}, i.e. optimal time–lag between two consecutive portfolio adjustments.

following generalization of the Black–Scholes equation

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (r-q)S\partial_S V - rV - r_R S = 0, \tag{4.21}$$

can be written as the following nonlinear parabolic equation:

$$\partial_t V + \frac{\sigma^2}{2} S^2 \left(1 + \mu \left(S\partial_S^2 V\right)^{1/3}\right) \partial_S^2 V + (r-q) S \partial_S V - rV = 0, \tag{4.22}$$

where $\mu = 3(\bar{C}^2 R/(2\pi))^{\frac{1}{3}}$ and Γ^p with $\Gamma = \partial_S^2 V$ and $p = 1/3$ stands for the signed power function, i.e., $\Gamma^p = |\Gamma|^{p-1}\Gamma$. We note that the equation is a backward parabolic PDE if and only if the function

$$\beta(H) = \frac{\sigma^2}{2}(1 + \mu H^{1/3}) H \tag{4.23}$$

is an increasing function in the variable $H := S\Gamma = S\partial_S^2 V$. It is satisfied if $\mu \geq 0$ and $H \geq 0$.

4.4.2 Optimal Choice of Hedging Time Δt in the Novel Model

Our task now is minimization of the total measure of risk. We will choose Δt as the arg min of $r_R = r_R(t)$, i.e.

$$\min_{\Delta t > 0} r_R = \min_{\Delta t > 0} (r_{TC} + r_{VP}).$$

It can be also viewed as the argument of maximum of the variance function (4.10) $\hat{\sigma} = \hat{\sigma}^2(S\Gamma, \Delta t)$, this means

$$\max_{\Delta t > 0} \hat{\sigma}^2(S\Gamma, \Delta t) = \max_{\Delta t > 0} \sigma^2 \left(1 - \tilde{C}\left(\sigma|S\Gamma|\sqrt{\Delta t}\right) \frac{\mathrm{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} - R\sigma^2 S\Gamma\Delta t\right),$$

i.e. finding the time interval where $\tilde{C}\left(\sigma|S\Gamma|\sqrt{\Delta t}\right) \frac{\mathrm{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} + R\sigma^2 S\Gamma\Delta t$ attains its minimum value:

$$\Delta t_* = \arg\min_{\Delta t > 0} \left(\tilde{C}\left(\sigma|S\Gamma|\sqrt{\Delta t}\right) \frac{\mathrm{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} + R\sigma^2 S\Gamma\Delta t\right). \tag{4.24}$$

In the following definition, we will postulate the basic assumptions on admissible transformed functions of transaction costs $\tilde{C}$. These assumptions will enable us to use such functions for the generalization of a risk adjusted model for pricing the derivatives of the underlying assets.

Definition 4.1. *Let $C : \mathbb{R}_0^+ \to \mathbb{R}$ be a transaction costs function. We say C is an admissible transaction costs measure if the following conditions are satisfied for the modified transaction costs function $\tilde{C} = \mathbb{E}\big[C(\xi|\Phi|)|\Phi|\big]$:*

(H_1) $\tilde{C}(0) > 0$, $\tilde{C}'(\xi) \le 0$ *for all* $\xi \ge 0$ *and*

(H_2) $\tilde{C}(\xi) - \xi\tilde{C}'(\xi) + \frac{\xi^2}{2}\tilde{C}''(\xi) \ge 0$ *for all* $\xi \ge 0$.

As an example we can consider a piecewise linear transaction costs function from Subsection 3.8.3.

Let us proceed by analysing the minimisation problem. Our aim is to find an optimal hedging time Δt_*. We first rewrite the minimized function from (4.24) as follows:

$$\tilde{C}\left(\sigma|S\Gamma|\sqrt{\Delta t}\right)\frac{\operatorname{sgn}(S\Gamma)}{\sigma\sqrt{\Delta t}} + R\sigma^2 S\Gamma\Delta t = \tilde{C}(\alpha)\frac{H}{\alpha} + \frac{R}{H}\alpha^2,$$

where $H = S\Gamma = S\partial_S^2 V$ and $\alpha = \sigma|S\Gamma|\sqrt{\Delta t} = \sigma|H|\sqrt{\Delta t}$.

Define an auxiliary variable $\tau = \sqrt{\Delta t}$ and an auxiliary function $\varphi = \varphi(\Delta t)$ as follows:

$$\varphi(\tau) = \tilde{C}(b\tau)\frac{H}{b\tau} + \frac{Rb^2}{H}\tau^2,$$

where $b = \sigma|H|$. It is easy to see that

$$\tilde{C}(\alpha)\frac{H}{\alpha} + \frac{R}{H}\alpha^2 = \varphi(\tau). \tag{4.25}$$

Proposition 4.1. *The function $\varphi(\tau)$ attains its unique positive minimum $\Delta t_* > 0$ provided that the function C is admissible transaction costs function.*

Proof. The first and second derivatives of $\varphi(\tau)$ are as follows:

$$\begin{aligned}\varphi'(\tau) &= \tilde{C}'(b\tau)\frac{H}{\tau} - \tilde{C}(b\tau)\frac{H}{b\tau^2} + \frac{2Rb^2}{H}\tau,\\ \varphi''(\tau) &= \tilde{C}''(b\tau)\frac{bH}{\tau} - \tilde{C}'(b\tau)\frac{2H}{\tau^2} + \tilde{C}(b\tau)\frac{2H}{b\tau^3} + 2R\sigma^2 H.\end{aligned}$$

Next, we denote $\xi := b\tau$. Using the assumption (H_2) from the Definition 4.1 we obtain

$$\varphi''(\tau) = 2R\sigma^2 H + \frac{2\sigma^2 H^3}{\xi^3}\left(\tilde{C}(\xi) - \xi\tilde{C}'(\xi) + \frac{\xi^2}{2}\tilde{C}''(\xi)\right) > 0.$$

Therefore, the function $\tau \mapsto \varphi(\tau)$ is strictly convex by (H_2).

Analysing the value of the first derivative at zero, we can demonstrate that (H_2) from Definition 4.1 is a sufficient condition to show that $\varphi'(\tau_*) = 0$ has a root. We only need to show that the value of the first derivative at zero is equal to $-\infty$. It is easy to compute

$$\varphi'(0) = -bH\lim_{\xi\to 0}\frac{\tilde{C}(\xi) - \xi\tilde{C}'(\xi)}{\xi^2} = -bH\lim_{\xi\to 0}\frac{\tilde{C}(0)}{\xi^2} = -\infty$$

because $\tilde{C}'$ is bounded and $\lim_{\xi\to 0}\xi\tilde{C}'(\xi) = 0$ and $\tilde{C}(0) = C_0\sqrt{\frac{2}{\pi}} = const$, see Proposition 3.3. Finally, as the function φ is defined for non–negative variable only and $\varphi'(0) = -\infty$ and the function is strictly convex for $\tau > 0$ for this reason it attains its unique minimum at some point τ_*, which can be found by solving the equation $\varphi'(\tau_*) = 0$ (see Figure 4.3). □

Proposition 4.2. *The optimum value $\Delta t = \tau_*^2$ is attained where $\xi_* = b\tau_*$ solves the equation*

$$\tilde{C}(\xi_*) - \tilde{C}'(\xi_*)\xi_* = \nu\xi_*^3, \tag{4.26}$$

where

$$\nu := \frac{2R}{H^2} = \frac{2R}{S^2\Gamma^2}.$$

Proof. This result arises from the proof of Proposition 4.1. From there we know that $\varphi'(\tau_*) = 0$ has a solution, i.e., equation (4.26) has a solution. □

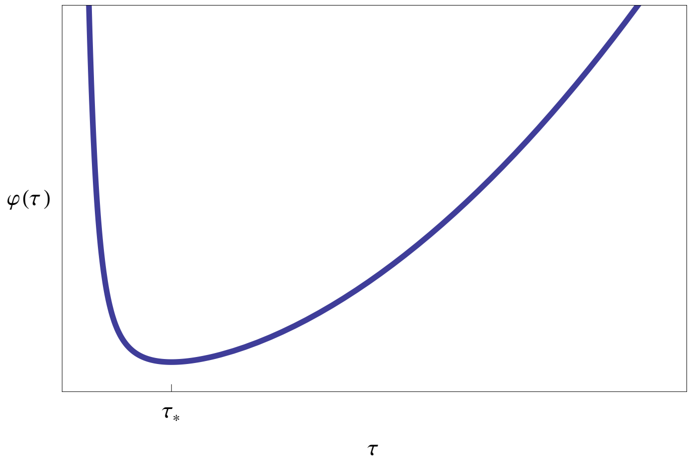


Figure 4.3: The function $\varphi(\tau)$ attains its unique minimum at the point τ_*.

For the maximum value of variance we obtain the following relation

$$\hat{\sigma}^2(S\Gamma, \Delta t_*) = \sigma^2(1 - \varphi(\tau_*)) = \sigma^2 \left(1 - \tilde{C}(\xi_*)\frac{H}{\xi_*} - \frac{R}{H}\xi_*^2\right).$$

which can be inserted into the modified Black–Scholes equation

$$\partial_t V + \frac{1}{2}\hat{\sigma}^2(S\Gamma, \Delta t_*)S^2\partial_S^2 V + (r-q)S\partial_S V - rV - r_R S = 0. \tag{4.27}$$

The expression $\hat{\sigma}^2(S\Gamma, \Delta t_*)$ emerging in (4.27) has the form

$$\hat{\sigma}^2(S\Gamma, \Delta t_*) \quad = \quad \sigma^2\left(1 - \psi(S\Gamma)\right),$$

where the function $\psi = \psi(H)$ is defined in an implicit way

$$\psi(S\Gamma) = \tilde{C}(\xi_*)\frac{H}{\xi_*} + \frac{R}{H}\xi_*^2. \tag{4.28}$$

We already know, that for given $H = S\Gamma$, we have the unique solution ξ_* of the implicit equation (4.26). This equation can be cast into an equivalent form

$$H^2\left(\tilde{C}(\xi_*) - \tilde{C}'(\xi_*)\xi_*\right) = 2R\xi_*^3. \tag{4.29}$$

Finally, by inserting the expression for $r_R S$ into the modified Black–Scholes equation (4.27), we obtain the following RAMP equation, which takes into account the variable transaction costs

$$\partial_t V + \frac{1}{2}\sigma^2 S(S\Gamma - S\Gamma\psi(S\Gamma)) + (r-q)S\partial_S V - rV = 0.$$

If we define an auxiliary function

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \psi(H)\right) H, \tag{4.30}$$

then the modified Black–Scholes equation becomes

$$\partial_t V + S\beta\left(H\right) + (r-q)S\partial_S V - rV = 0. \tag{4.31}$$

The advantage of this novel model is that many of the known models are included, for example the Leland model, and the model studied by Amster et al.. We can extend analysis by using a more realistic piecewise linear non–increasing function (see Subsection 3.8.3).

In Figure 4.1 some examples of functions $H \mapsto \beta(H)$ are shown for different types of $\tilde{C}(\xi)$; the value of the risk parameter was set to $R = 10$ and the historical volatility $\sigma = 0.3$.

Example 4. *For the linear decreasing transaction costs function given by the model studied by Amster, i.e. $C(\xi) = C_0 - \kappa\xi$, the function can be expressed analytically. The equation (4.29) has the following form*

$$H^2\left(C_0 - \kappa\xi_* - (-\kappa)\xi_*\right) = \frac{\pi^2 R}{2}\xi_*^3,$$

and therefore, similarly as in the classical RAMP model

$$\xi_* = \left(\frac{C_0}{2R}\sqrt{\frac{2}{\pi}}H^2\right)^{\frac{1}{3}}.$$

By inserting ξ_* into (4.28), we obtain for the function $\psi(H)$ the following relation

$$\psi(H) = \left(\sqrt{\frac{2}{\pi}}\frac{C_0}{\xi_*} - \kappa\right) H + \frac{R}{H}\xi_*^2 = \mu H^{\frac{1}{3}} - \kappa H,$$

where $\mu = 3(C_0^2 R/(2\pi))^{\frac{1}{3}}$ and and H^p with $H = S\partial_S^2 V$ and $p = 1/3$ stands for the signed power function, i.e., $H^p = |H|^{p-1}H$. Thus the function β has the form

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \mu H^{1/3} + \kappa H\right) H$$

and is depicted in Figure 4.4. Note, that function β is increasing for $\frac{\mu^3}{\kappa} < (\frac{27}{8\sqrt{2}})^2 \approx 5.6953125$.

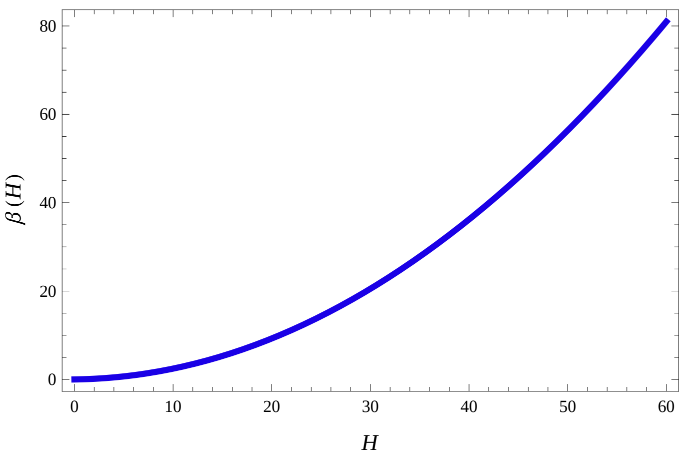


Figure 4.4: The function $\beta(H)$ for the linear decreasing transaction costs function.

4.5 Summary of the $\beta(H)$-Functions for the Non-linear Models

In this section we review the possible types of function $\beta(H)$ in the modified nonlinear Black–Scholes equation of the form:

$$\partial_t V + S\beta\,(H) + (r-q)S\partial_S V - rV = 0.$$

As has been mentioned, the function $\beta(H)$ varies depending on chosen model. We can write the form of function $\beta(H)$ from the volatility of a general type (3.3) by the following relation:

$$\beta(H, S, \tau) = \frac{1}{2}\hat{\sigma}^2(S\Gamma, S, t)S\Gamma. \tag{4.32}$$

- **Classical Black–Scholes model**

  In the linear B–S model the volatility as a function of H has the form $\hat{\sigma}^2(H) = \sigma^2$, and therefore the β function can be written as:

$$\beta(H) = \frac{\sigma^2}{2}H. \tag{4.33}$$

  This is the most simple example, let us move to the β function of models including transaction costs.

- **Leland model**

  The first model with transaction costs is the Leland model. Here we distinguish between *bid* and *ask* price of the option we want to obtain by solving the equation.

$$\hat{\sigma}^2(H) = \begin{cases} \sigma^2(1-\mathrm{Le}), & \text{if} \quad H > 0, \\ \sigma^2(1+\mathrm{Le}), & \text{if} \quad H < 0, \end{cases} \tag{4.34}$$

  where $\mathrm{Le} = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}$ is the Leland number.

  Therefore for pricing the option we can use one of these β functions:

$$\beta_{bid}(H) = \frac{\sigma^2}{2}\Big(1 - \mathrm{Le}\Big)H, \tag{4.35}$$

$$\beta_{ask}(H) = \frac{\sigma^2}{2}\Big(1 + \mathrm{Le}\Big)H. \tag{4.36}$$

- **Model with investor's preferences**

  In the model of Barles & Soner [6] we note that here the function β really depends on all the variables x, τ and H. We did already use the further transformation (5.2) of the space variable, i.e., $x = \ln(S/E)$.

$$\beta(H, x, \tau) = \frac{\sigma^2}{2}\Big(1 + \Psi\left(Ea^2 \mathrm{e}^{r\tau + x} H\right)\Big)H. \tag{4.37}$$

- **Liquidity model**

  In the case of the model proposed by Bakstein and Howison the function β depends again only on H:

$$\begin{aligned}\beta(H) =& \frac{\sigma^2}{2}\left(1 + \bar{\gamma}^2(1-\alpha)^2 + 2\lambda H + \lambda^2(1-\alpha)^2 H^2 + \sqrt{\frac{2}{\pi}}2\bar{\gamma}\mathrm{sgn}(H)\right. \\ &\left. + \sqrt{\frac{2}{\pi}}2\lambda(1-\alpha)^2\bar{\gamma}H\mathrm{sgn}(H)\right)H. \end{aligned} \tag{4.38}$$

- **Model with linear decreasing transaction costs depending on the volume of trading stocks**

  We refer to this model also as the model with decreasing transaction costs in scale. The first formula can be used in the model to obtain the *bid* option price and the second one for the *ask* option price:

$$\beta_{bid}(H) = \frac{\sigma^2}{2}\left(1 - \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}} + \kappa H\right)H, \tag{4.39}$$

$$\beta_{ask}(H) = \frac{\sigma^2}{2}\left(1 + \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}} - \kappa H\right)H. \tag{4.40}$$

- **RAPM model**

  This model takes into account both the risk arising from non–trivial transaction costs as well as the risk from an unprotected portfolio. Their sum is subject to minimization. By minimizing the total risk premium we obtain the optimal length of the hedging interval. The governing equation is backward parabolic PDE if and only if the function $\beta(H)$ is an increasing function in variable H:

$$\beta_{bid}(H) = \frac{\sigma^2}{2}\left(1 - \mu H^{1/3}\right) H, \tag{4.41}$$

$$\beta_{ask}(H) = \frac{\sigma^2}{2}\left(1 + \mu H^{1/3}\right) H, \tag{4.42}$$

where $\mu = 3\left(\frac{C_0^2 R}{2\pi}\right)^{1/3}$.

- **Novel Option Pricing Model under Transaction Costs and Risk of the Unprotected Portfolio**

  In this chapter we introduced the novel model, where the function $\beta(H)$ has the form:

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \tilde{C}(\sigma|H|\sqrt{\Delta t})\frac{\operatorname{sgn}(H)}{\sigma\sqrt{\Delta t}} - R\sigma^2 H\Delta t\right) H \tag{4.43}$$

  - **Model with variable transaction costs**

    If we consider in the last formula the coefficient of risk premium equal to zero, i.e., $R = 0$, we obtain the model with variable transaction costs derived in the previous chapter. We can choose any variable transaction costs function, for example piecewise linear non–increasing function or exponentially decreasing and its corresponding modification $\tilde{C}$. Such a model can be solved by means of the Gamma equation with the $\beta(H)$ function:

$$(\textit{bid}\text{ price}) \qquad \beta(H) = \frac{\sigma^2}{2}\left(1 - \frac{\tilde{C}(\sigma|H|\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}}\right) H. \tag{4.44}$$

$$(\textit{ask}\text{ price}) \qquad \beta(H) = \frac{\sigma^2}{2}\left(1 + \frac{\tilde{C}(\sigma|H|\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}}\right) H. \tag{4.45}$$

– **RAPM model with variable transaction costs**

If we consider the risk premium coefficient $R > 0$, we obtain the RAPM model with variable transaction costs analysed in this chapter. We include it into this summary as one of the examples of function $\beta(H)$ in particular form:

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \psi(H)\right) H, \tag{4.46}$$

where the function $\psi(H)$ is defined in an implicit way by equation (4.28). For the special case of linear decreasing transaction costs in model proposed by Amster et al., we have

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \mu H^{1/3} + \kappa H\right) H, \quad \text{where} \quad \mu = 3\left(\frac{C_0^2 R}{2\pi}\right)^{1/3}. \tag{4.47}$$

Chapter 5

Gamma Equation

In this section, we introduce the Gamma equation proposed in the article [24] by Jandačka and Ševčovič (see also Ševčovič, Stehlíková and Mikula [36, p. 174]). The goal is to present the transformation of the the nonlinear Black–Scholes equation into a quasilinear parabolic equation.

Let us consider the previously mentioned modified nonlinear Black–Scholes equation with the nonlinear volatility of a general type included in the β function

$$\partial_t V + S\beta(H) + (r-q)S\partial_S V - rV = 0, \quad S > 0, t \in (0,T), \tag{5.1}$$

where the form of the function $\beta(H), H = S\Gamma$ depends on the model we use (see Section 4.5).

The idea how to analyse and solve this equation is based on the transformation method. We consider the standard change of independent variables, as usual in the classical Black–Scholes theory [8]:

$$x := \ln(S/E),\ x \in (-\infty, \infty), \quad \text{and} \quad \tau := T - t, \tau \in (0, T). \qquad (5.2)$$

The transformation of the space, $x = \ln(S/E)$, stretches the domain to the whole set of real numbers. Substituting $\tau = T - t$ transforms the backward parabolic differential equation to a forward one. Since the equation (5.1) contains the term $S\Gamma = S\partial_S^2 V$ it is convenient to use the following transformation:

$$H(x, \tau) := S\Gamma = S\partial_S^2 V(S, t). \qquad (5.3)$$

After this transformation β can be a function of H, x and τ, i.e. $\beta = \beta(H, x, \tau)$.

5.1 Derivation of the Gamma Equation

The so–called Γ *equation* can be obtained if we compute the second derivative of the equation (5.1) with respect to x according to Jandačka and Ševčovič [24] (see also Ševčovič, Stehlíková and Mikula in [36], Mikula and Kútik in [26] and [27]).

Theorem 5.1. *Function $V = V(S, t)$ is a solution to (5.1) if and only if $H = H(x, \tau)$ solves*

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + (r - q)\partial_x H - qH, \qquad (5.4)$$

where β is a composed function

$$\beta = \beta(H(x, \tau), x, \tau).$$

For the readers convenience we present a detailed derivation of the Gamma equation. Let us compute the second derivative of the equation (5.1) with respect to x. Note that $\frac{\mathrm{d}S}{\mathrm{d}x} = S$. The derivatives of the terms in the equation (5.1) are:

$$
\begin{aligned}
\partial_x^2(\partial_t V) =& \partial_x \left(\partial_{tx}^2 V \partial_x S\right) = \partial_{tSS}^3 S^2 + \partial_x(\partial_t V),\\
\partial_x^2(S(\beta(S\Gamma))) =& \partial_x(\partial_x(S\beta(H))) = \partial_x(S\beta(H) + S\partial_x\beta(H))\\
=& S\partial_x\beta(H) + S\partial_x^2\beta(H) + \partial_x(S\beta(H)),\\
\partial_x^2\big((r-q)S\partial_S V\big) =& \partial_x\big((r-q)S\partial_S V + (r-q)S^2\partial_S^2 V\big)\\
=& (r-q)2S^2\partial_S^2 V + (r-q)S^3\partial_S^3 V + \partial_x\big((r-q)S\partial_S V\big),\\
\partial_x(-rV) =& \partial_x(-rS\partial_S V) = -rS^2\partial_S^2 V + \partial_x(-rV).
\end{aligned}
$$

Summing up all of the terms back into equation gives

$$
\begin{aligned}
\partial_{tSS}^3 S^2 + S\partial_x^2\beta(H)S\partial_x\beta(H) + (r-q)S^2\partial_S^2 V + (r-q)S^3\partial_S^3 V - qS^2\partial_S^2 V&\\
+\partial_x\big(\partial_t V + S\beta(H) + (r-q)S\partial_S - rV\big) = 0&,
\end{aligned}
$$

where the last term is equal to zero, because equation (5.4) holds. After the vanishing of the last term and dividing by S the equation has the form

$$
-\partial_{tSS}^3 S = \partial_x^2\beta(H) + \partial_x\beta(H) + (r-q)S\partial_S^2 V + (r-q)S^2\partial_S^3 V - qS\partial_S^2 V \tag{5.5}
$$

and can be rewritten into the terms of H

$$
\partial_\tau H = \partial_x^2\beta(H) + \partial_x\beta(H) + (r-q)\partial_x H - qH,
$$

(cf. [36, Chapter 11]) where

$$
\begin{aligned}
\partial_\tau H =& \partial_x(S\partial_S^2 V) = S\partial_{tSS}^3 V\partial_\tau t = -S\partial_{tSS}^3 V,\\
(r-q)\partial_x H =& (r-q)\partial_x(S\partial_S^2 V) = (r-q)S\partial_S^2 V + (r-q)S^2\partial_S^3 V \quad \text{and}\\
-qH =& -qS\partial_S^2 V.
\end{aligned}
$$

5.1.1 Transformation of the Initial and Boundary Conditions

In order to solve the equation (5.4) we have to transform in terms of $H(x,\tau)$ initial and boundary conditions for a European call (2.11) and European put (2.12) options as well. The solution H has at $\tau = 0$ the form

$$H(x,0) = \bar{H}(x), \quad x \in \mathbb{R}, \tag{5.6}$$

where $\bar{H}(x) := \delta(x)$ is the Dirac delta function [36, p. 174]. Recall that the Dirac function is a function in a distributional sense such that

$$\int_{-\infty}^{\infty} \delta(x - x_0)\Phi(x)\mathrm{d}x = \Phi(x_0), \qquad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1, \tag{5.7}$$

for any smooth function Φ.

In terms of $H(x,\tau)$ the boundary conditions for a European call and European put options at $x = \pm\infty$ are

$$H(-\infty,\tau) = H(\infty,\tau) = 0, \quad \tau \in (0,T). \tag{5.8}$$

In the case of call and put options we modify the boundary conditions (5.8) at $x = \pm L$ to Dirichlet boundary conditions, i.e.

$$H(-L,\tau) = H(L,\tau) = 0\,, \quad \tau \in (0,T)\,.$$

The initial condition given by Dirac delta function is approximated as follows:

$$\bar{H}(x) = N'(d)/(\hat{\sigma}\sqrt{\tau^*}),$$

where $\tau^* > 0$ is sufficiently small, $N(d)$ is the cumulative distribution function of the normal distribution, and $d = \left(x + (r - q - \hat{\sigma}^2/2)\tau^*\right)/\hat{\sigma}\sqrt{\tau^*}$. It corresponds to the value $H = S\partial_S^2 V$ of a call (put) option valued by a linear Black–Scholes equation with a constant volatility $\hat{\sigma} > 0$ at the time $T - \tau^*$ close to expiry T when the time

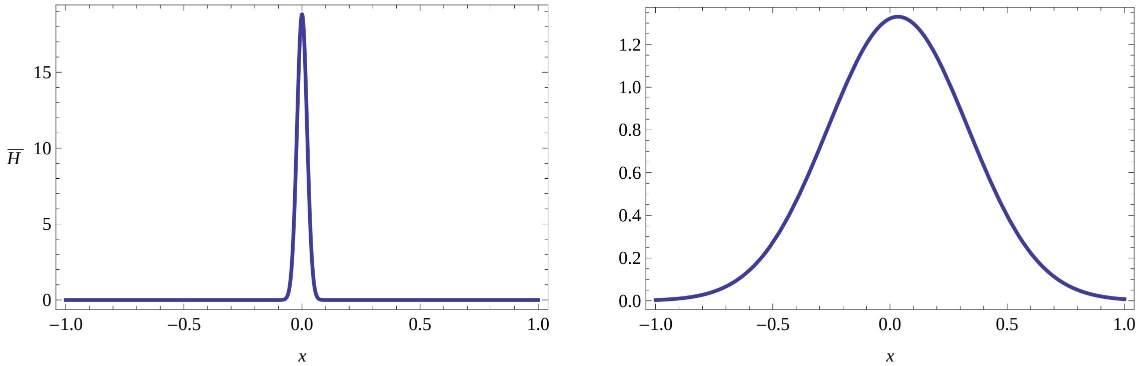


Figure 5.1: Plots of the initial approximation of the function $\bar{H}(x)$ for τ^* sufficiently small (left) and the solution profile $H(x,T)$ at $\tau = T$ (right).

parameter $0 < \tau^* \ll 1$ is sufficiently small. See Figure 5.1.

5.1.2 Backward Transformation of the Solution

The following proposition and proof are adopted from the paper written by Jandačka and Ševčovič [24] (see also Ševčovič, Stehlíková and Mikula in [36], Mikula and Kútik in [26] and [27]).

Proposition 5.1. *The option price $V = V(S,t)$ can be directly computed from the solution $H(x,\tau)$ of the Γ equation*

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + (r-q)\partial_x H - qH,$$

i.e., the transformation of the modified Black–Scholes equation (5.1), where $H(x,\tau) = S\Gamma = S\partial_S^2 V(S,t)$ and $x = \ln(S/E)$, $x \in (-\infty,\infty)$, and $\tau = T - t, \tau \in (0,T)$.

Proof. We see the relation from function $H(x,\tau)$ defined in (5.3). In the case of a call option we have

$$\partial_S V(S,t) = \partial_S V(0,t) + \int_0^S \frac{1}{s} H(\ln(s/E), T-t)\mathrm{d}s = \int_{-\infty}^{\ln(S/E)} H(x, T-t)\mathrm{d}x,$$

from which we deduce, by integration,

$$V(S,t) = \int_{-\infty}^{\infty} (S - Ee^x)^+ H(x, T-t)\mathrm{d}x, \tag{5.9}$$

because $\partial_S V(0,t) = V(0,t) = 0$ for the **call option**.

Similarly, for the **put option** computed from known $H(x,\tau)$, we obtain the following formula:

$$V(S,t) = \int_{-\infty}^{\infty} (Ee^x - S)^+ H(x, T-t)\mathrm{d}x. \tag{5.10}$$

□

Chapter 6

Computational Results

The purpose of this chapter is to derive a robust numerical scheme for solving the Γ equation. The construction of numerical approximation of a solution H to (5.4) is based on a derivation of a system of difference equations corresponding to (5.4) to be solved at every discrete time step. We give also the Mathematica source using the model with variable transaction cost. Next we show the modelling of the bid–ask spread and perform extensive comparisons of the solutions of the models.

6.1 Numerical Scheme for the Full Space–Time Discretization and for Solving the Γ-Equation

In this section we present the numerical scheme adopted from the paper by Jandačka and Ševčovič [24] in order to solve the Γ equation (5.4) for a general function $\beta = \beta(H, x, \tau)$ including, in particular, the case of the model with variable transaction costs. The efficient numerical discretization is based on the finite volume approximation of the partial derivatives entering (5.4). The resulting scheme is semi–implicit in a finite–time difference approximation scheme.

For numerical reasons we restrict the spacial interval to $x \in (-L, L)$ where $L > 0$ is sufficiently large. Since $S = Ee^x$ it is now a restricted interval of underlying stock values, $S \in (Ee^{-L}, Ee^L)$. From a practical point of view, it is sufficient to take $L \approx 1.5$ in order to include the important range of values of S.

For the purpose of construction of a numerical scheme, the time interval $[0, T]$ is uniformly divided with a time step $k = T/m$ into discreet points τ_j, where $j = 0, 1, ..., m$, $\tau_j = jk$. We also take the spacial interval $[-L, L]$ with uniform division with a step $h = L/n$, into discreet points $x_i = ih$, where $i = -n, \ldots, n$.

Now the homogeneous Dirichlet boundary conditions on new discrete values representing the initial condition are $H_i^0 = \bar{H}(x_i)$ where $x_i = ih$.

The numerical algorithm is semi–implicit in time. Notice that the term $\partial_x^2 \beta$, where $\beta = \beta(H(x, \tau), x, \tau)$ can be expressed in the form

$$\partial_x^2 \beta = \partial_x \left(\beta'_H(H, x, \tau) \partial_x H + \beta'_x(H, x, \tau) \right),$$

where β'_H and β'_x are partial derivatives of the function $\beta(H, x, \tau)$ with respect to H and x, respectively:

$$\partial_x \beta = \beta'_H \partial_x H + \beta'_x, \tag{6.1}$$

$$\partial_x^2 \beta = \beta'_H \partial_x^2 H + \beta''_{HH} (\partial_x H)^2 + 2\beta''_{xH} \partial_x H + \beta''_{xx}. \tag{6.2}$$

In the discretization scheme, the nonlinear terms $\beta'_H(H, x, \tau)$ and $\beta'_x(H, x, \tau)$ are evaluated from the previous time step τ_{j-1} whereas linear terms are solved at the current time level.

Such a discretization leads to a solution of linear systems of equations at every discrete time level.

The next steps are as follows, at first, we replace the time derivative by the time difference, approximate H in nodal points by the average value of neighbouring segments, then we collect all linear terms at the new time level j and by taking all the remaining terms from the previous time level $j-1$ we obtain a *tridiagonal system* for the solution vector $H^j = (H^j_{-n+1}, \dots, H^j_{n-1}) \in \mathbb{R}^{2n-1}$:

$$a_i^j H_{i-1}^j + b_i^j H_i^j + c_i^j H_{i+1}^j = d_i^j, \quad H_{-n}^j = 0, \quad H_n^j = 0\,, \tag{6.3}$$

where $i = -n+1, \dots, n-1$ and $j = 1, \dots, m$.

The coefficients of the tridiagonal matrix are given by

$$\begin{aligned}
a_i^j &= -\frac{k}{h^2}\beta'_H(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1}) + \frac{k}{2h} r\,,\\
c_i^j &= -\frac{k}{h^2}\beta'_H(H_i^{j-1}, x_i, \tau_{j-1}) - \frac{k}{2h} r\,,\\
b_i^j &= 1 - (a_i^j + c_i^j)\,,\\
d_i^j &= H_i^{j-1} + \frac{k}{h}\Big(\beta(H_i^{j-1}, x_i, \tau_{j-1}) - \beta(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1})\\
&\quad +\beta'_x(H_i^{j-1}, x_i, \tau_{j-1}) - \beta'_x(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1})\Big)\,.
\end{aligned}$$

It means that the vector H^j at the time level τ_j is a solution to the system of linear equations $\mathbf{A}^j\, H^j = d^j$, where the $(2n-1)\times(2n-1)$ matrix $\mathbf{A}^j$ is defined as

$$\mathbf{A}^j = \begin{pmatrix}
b^j_{-n+1} & c^j_{-n+1} & 0 & \cdots & 0\\
a^j_{-n+2} & b^j_{-n+2} & c^j_{-n+2} & & \vdots\\
0 & \cdot & \cdot & \cdot & 0\\
\vdots & \cdots & a^j_{n-2} & b^j_{n-2} & c^j_{n-2}\\
0 & \cdots & 0 & a^j_{n-1} & b^j_{n-1}
\end{pmatrix}. \tag{6.4}$$

To solve the tridiagonal system in every time step in a fast and effective way, we can use the simple LU – matrix decomposition. The key idea is in decomposition of a matrix $\mathbf{A}$ into a product of two matrices, i.e., $\mathbf{A} = \mathbf{L}.\mathbf{U}$, where L is lower and U is an upper triangular matrix respectively (for more details see example [36, Chapter 10]).

According to (5.9) and (5.10) the option price $V(S, T-\tau_j)$ can be constructed

Table 6.1: The Mathematica source code for implicit method of the solution to the Gamma equation.

```
Needs["LinearAlgebra`Tridiagonal`"];

(* Model parameters *)
sigma = 0.3; r = 0.011; q = 0.; T = 1; X = 25;

(* Numerical parameters *)
n = 500;   m = 200; h = 0.005;   k = T/m;

(* Initial function - approximation of the Dirac function *)
taustar = 0.001;
Hinit[x_] :=
Exp[- ((x + (r-q- sigma^2/2)taustar)/(sigma Sqrt[taustar]))^2/2]/
(sigma Sqrt[taustar]Sqrt[2 Pi]);

Hfn = Table[Hinit[i h], {i, -n + 1, n - 1}];
Hsol[0] = Hfn;

(* The main time loop *)
For[j = 1, j <= m, 1,
    {
(* Definition of the triadiagonal matrix *)
      a = Table[
          If[i == -n + 1,
            -(k/h^2)bfunH[ 0.  , (i - 1) h, j k]  + 0.5(k/h) r,
            -(k/h^2)bfunH[ Hfn[[ i + n - 1]] , (i - 1) h, j k] + 0.5(k/h) r  ],
          { i, -n + 1, n - 1} ];

      c = Table[
          -(k/h^2)bfunH[ Hfn[[ i + n]] , i h, j k ]- 0.5(k/h) r,
          { i, -n + 1, n - 1} ];

      b = Table[1 - a[[i + n]] - c[[i + n]],
          {i, -n + 1, n - 1}];

      a = Table[a[[i]], {i, 2, 2 n - 1}];
      c = Table[c[[i]], {i, 1, 2n - 2}];

      d = Hfn + (k/h) Table[
              If[
                i == -n + 1, (bfun[ Hfn[[1]], i h, j k ] -
                    bfun[0, i h, j k]),
                 (bfun[ Hfn[[i + n ]], i h, j k] -
                    bfun[Hfn[[i - 1 + n]], i h, j k ])
                ],  {i, -n + 1, n - 1}]
          + (k/h)*Table[
              If[ i == -n + 1, (bfunx[ Hfn[[1]], i h, j k ] -
                    bfunx[0, i h, j k]),
                 (bfunx[ Hfn[[i + n ]], i h, j k] -
                    bfunx[Hfn[[i - 1 + n]], i h, j k ])
                ],  {i, -n + 1, n - 1}];

       Hfn = TridiagonalSolve[a, b, c, d]; Hsol[j] = Hfn;  j++;
      }];

ListPlot[Hfn];

(* Reconstruction of the option price from the solution H *)
V[S_]:=h Sum[Max[S-X Exp[i h], 0] Hsol[m][[i+n]], {i,-n+1,n-1}];
```

from the discrete solution H_i^j as follows:

$$\begin{aligned}
&\text{(call option)} \qquad & V(S, T-\tau_j) &= h\sum_{i=-n}^{n}(S - Ee^{x_i})^+ H_i^j, \\
&\text{(put option)} \qquad & V(S, T-\tau_j) &= h\sum_{i=-n}^{n}(Ee^{x_i} - S)^+ H_i^j,
\end{aligned}$$

for $j = 1, \ldots, m$.

In Table 6.1 we present a simple source code in the Mathematica language for the finite difference approximation of the Gamma equation. The function β corresponds to the model with variable transaction costs (with piecewise linear non–increasing function) and its implementation in Mathematica environment is shown in Table 6.2.

For other nonlinear Black–Scholes models one has to modify the function $\beta = \beta(H, x, \tau)$ accordingly.

6.2 Numerical results for the nonlinear model with variable transaction costs

In this section we present the numerical results for the approximation of the option price. Recall that we solve nonlinear models of the Black–Scholes type, particularly, the novel option pricing model under transaction costs and risk of the unprotected portfolio.

Into the numerical scheme enters the $\beta(H)$ function derived in Chapter 4 given in (4.10) as:

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \tilde{C}(\sigma|H|\sqrt{\Delta t})\frac{\operatorname{sgn}(H)}{\sigma\sqrt{\Delta t}} - R\sigma^2 H\Delta t\right)H,$$

where $\tilde{C}$ is the modified transaction cost function. For numerical experiments we take the coefficient of risk premium equal to zero, i.e., $R = 0$. Hence we notice that the nonlinearity arises from the transaction costs. Hence we take the optimal

Table 6.2: The Mathematica source code for definition of the variable transaction cost function $C(\xi)$ and $\tilde{C}(\xi)$ as well as the function $\beta(H)$.

```
(* Definition of variable TC function *)
C0 = 0.02;
kappa = 0.3;
ximinus = 0.05;
xiplus = 0.1;
Cfun[xi_] :=  If[xi < ximinus, C0,
   If[xi < xiplus, C0 - kappa*(xi - ximinus),
    C0 - kappa*(xiplus - ximinus)]];

* The function tilde C *)
Ctilde[xi_] :=  If[ximinus < 10*xi,
   Sqrt[2/Pi]*C0 -
    2*kappa*xi*
     Integrate[Exp[-s^2/2]/Sqrt[2*Pi], {s, ximinus/xi, xiplus/xi}],
   Sqrt[2/Pi]*C0] ;

(* Time interval between two rehedgements *)
Deltat = 1/261.;

(* Definition of beta function *)
bfun[H_, x_, tau_] := (sigma^2/2) (1 -
     Ctilde[sigma*Abs[H]*Sqrt[Deltat]]/(sigma*Sqrt[Deltat])) H ;


(* Numerical differentiation of beta function *)
sm = 0.001;
bfunH[H_, x_, tau_] = (bfun[H + sm, x, tau] - bfun[H, x, tau])/sm;
bfunx[H_, x_, tau_] = (bfun[H, x + sm, tau] - bfun[H, x, tau])/sm;
```

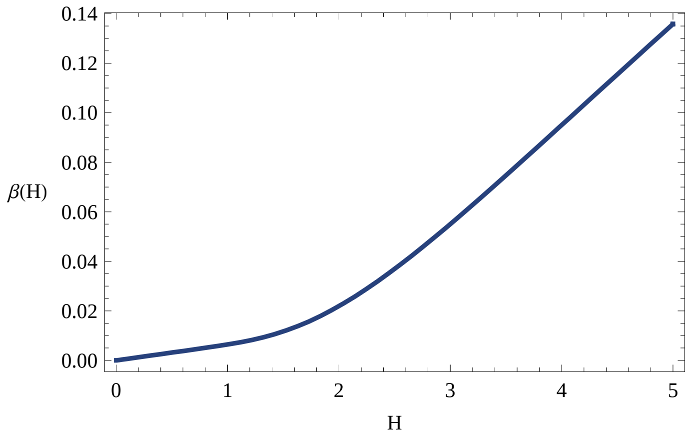


Figure 6.1: The shape of the function $\beta(H)$ corresponding to the nonlinear model with variable transaction costs.

hedging time, Δt, as fixed. Though, it is possible to do the numerical experiments for the case $R > 0$ and Δt is optimal, however we will not do the optimization for the hedging time Δt.

Refer to the shape of the $\beta(H)$ function for this particular model given in the Figure 6.1.

From the variable transaction costs functions listed in Subsections 3.8.1-3.8.4 we choose the piecewise linear non–increasing function, depicted in Figure 6.2. In practise it means that for some small volume of traded stocks one constant amount C_0 is paid; when the volume is significant, there starts to be a discount depending on a higher volume and finally there is another small constant payment $\underline{C}_0$ when the trades are very large.

The piecewise linear non–increasing transaction costs function is defined as:

$$C(\xi) = \begin{cases} C_0, & \text{if} \quad 0 \leq \xi < \xi_-, \\ C_0 - \kappa(\xi - \xi_-), & \text{if} \quad \xi_- \leq \xi \leq \xi_+, \\ \underline{C}_0, & \text{if} \quad \xi \geq \xi_+. \end{cases} \tag{6.5}$$

where we assume C_0, $\kappa > 0$, and $0 \leq \xi_- \leq \xi_+ \leq \infty$ to be given constants and $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$.

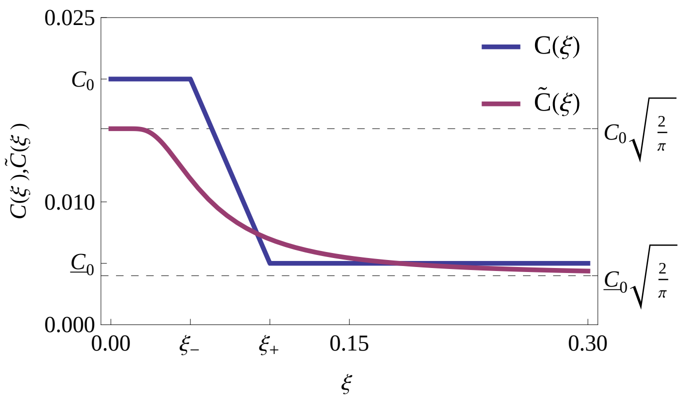


Figure 6.2: The shape of the function $C(\xi)$ for parameter values $C_0 = 0.02, \kappa = 0.3, \xi_- = 0.05, \xi_+ = 0.1$ (blue line). The corresponding function $\tilde{C}(\xi)$ is given by the red line.

Table 6.3: Parameter values used for computation of the numerical solution.

Parameter and Value	
$C_0 = 0.02$	$T = 1.$
$\kappa = 0.3$	$E = 25$
$\xi_- = 0.05$	$r = 0.011$
$\xi_+ = 0.1$	$m = 200$
$\Delta_t = 0.00383142$	$n = 250$
$\sigma = 0.3$	$h = 0.01$
$\sigma_{min} = 0.112511$	$\tau^* = 0.005$
$\sigma_{max} = 0.265828$	$R = 0$

The parameter values used in our computations are given in the Table 6.3.

According to Proposition 3.4 the function $\tilde{C}$ satisfies the following inequality (3.55):

$$\underline{C}_0\sqrt{\frac{2}{\pi}} \le \tilde{C}(\xi) \le C_0\sqrt{\frac{2}{\pi}}.$$

In what follows, we show that this restriction holds also for the numerical solution. That means, the solution of the nonlinear equation with variable transaction costs $\tilde{C}$ will be always between the solution of the Black–Scholes equation with constant transaction costs (i.e. the Leland model) with higher C_0 and lower $\underline{C}_0$ respectively. Values $\underline{C}_0\sqrt{2/\pi}$ and $C_0\sqrt{2/\pi}$ correspond to the modified transaction costs function

Table 6.4: Bid Option values of the numerical solution of nonlinear model in comparison to B–S with constant volatility.

S	$V_{\sigma^2_{max}}$	V_{vtc}	$V_{\sigma^2_{min}}$
20	0.709	0.127	0.029
23	1.752	0.844	0.421
25	2.768	1.748	1.258
28	4.723	3.695	3.474
30	6.256	5.321	5.327

$\tilde{C}$ in the case when $\tilde{C}$ is constant.

For Δt sufficiently small, we have from Proposition 3.4 that the equation to be solved is parabolic. For any value of ξ_+ and ξ_-, the $\tilde{C}(\xi)$ will lie between the values $\underline{C}_0\sqrt{2/\pi}$ and $C_0\sqrt{2/\pi}$ and the solutions will be ordered in this manner:

$$V_{\sigma^2_{min}}(S,t) \leq V_{vtc}(S,t) \leq V_{\sigma^2_{max}}(S,t) \quad \forall S, t.$$

In the Table 6.4 we present the option values for different prices of the underlying asset achieved by a numerical solution.

In Figure 6.3 we present the graphs of solution $V_{vtc} := V(S,t)$, as well as that of $\Delta(S,t) = \partial_S V(S,t)$, for various times $t \in \{0, T/3, 2T/3\}$. The upper dashed line corresponds to the solution of the linear Black–Scholes equation with volatility $\hat{\sigma}^2_{max} = \sigma^2\left(1 - \underline{C}_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right)$, where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$, and the lower dashed line corresponds to the solution with volatility $\hat{\sigma}^2_{min} = \sigma^2\left(1 - C_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right)$.

Note that at the beginning the solution of nonlinear model is closer to the lower bound and later moves closer to the upper one. It can be interpreted as follows: at the beginning of the contract the holder of the portfolio is not required to perform many operations to hedge. Therefore he does not have high volumes of transactions and pays the cost of C_0. With the impending expiry time it is necessary to hedge the portfolio and so trade in high volume, and so the investor pays lower transaction costs, i.e. $\underline{C}_0$.

The dependence of the option price on time $t \in (0,T)$ for $S \in \{20, 23, 25\}$ is

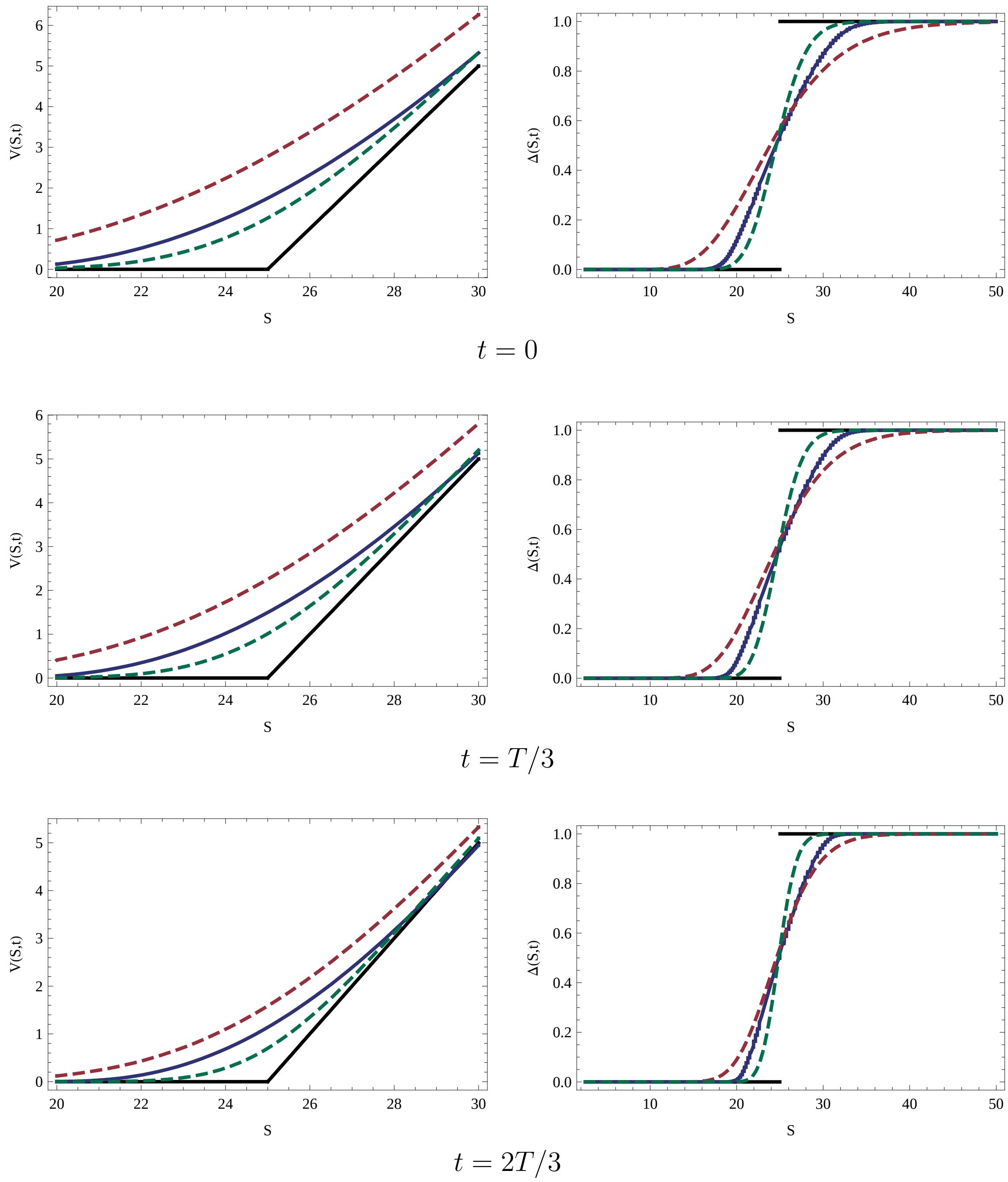


Figure 6.3: Solution $V(S,t)$ for $t=0, t=T/3, t=2T/3$ (left) and corresponding $\Delta(S,t)=\partial_S V(S,t)$ of the call option.

shown on Figure 6.4. In Figure 6.4 we also see how the price converges to zero value at the expiration time. One can observe a very rapid change in convergence to zero for an increasing underlying asset value.

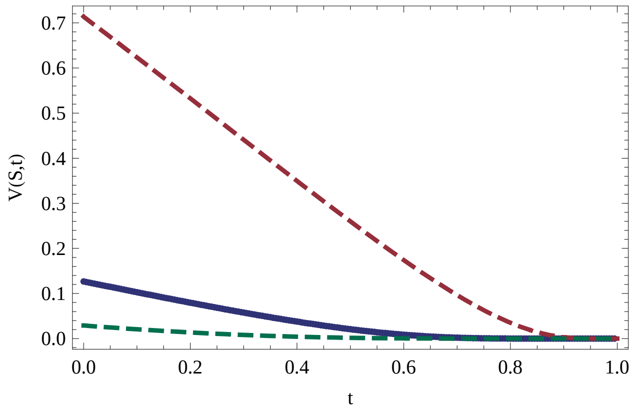
6.2.1 Modelling Option Bid–Ask Spreads by Using the Novel Model With Variable Transaction Costs

In the data sets of real markets quotes, there are listed two different option prices $V_{bid} < V_{ask}$ called bid and ask price representing offers for buying and selling options, respectively.

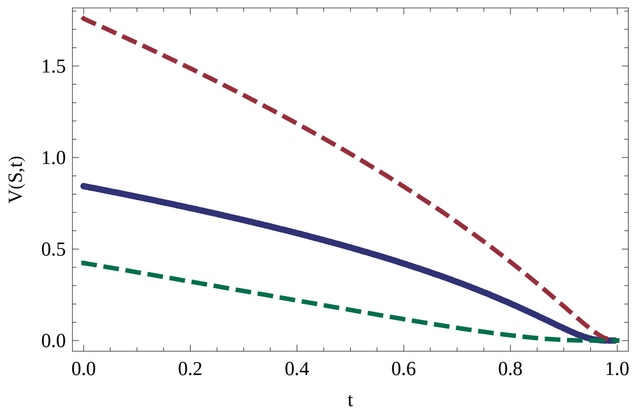
The bid is the price that someone is willing to pay for a security at a specific point in time, whereas the ask is the price at which someone is willing to sell. The difference between the two prices is called the bid–ask spread. In other words, if you're a seller, you receive the lower price (the bid), and if you're a buyer, you pay the higher price (the ask).

In the presented novel nonlinear model with variable transaction costs, the holder of a long positioned option bears transaction costs for maintaining the delta hedged portfolio by buying and selling assets. It turned out that the price of an option under the presence of transaction costs is always less that the option price on asset not paying transaction costs, i.e. $V_{vtc} < V_{bs}$. From the point of view of a perspective holder who wants to buy a long positioned option, the price obtained by novel model V_{vtc} can be therefore identified with the *bid* price of the option.

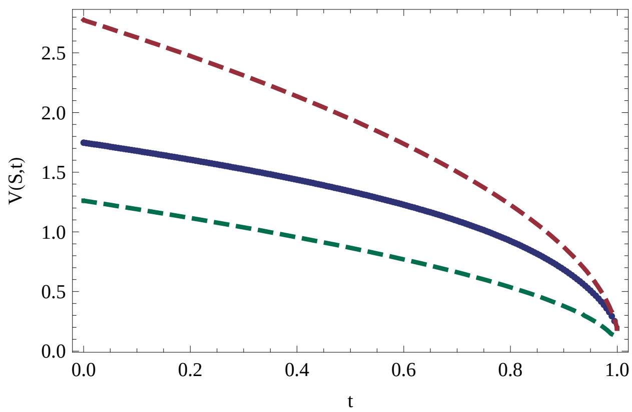
In order to derive a pricing equation for a short positioned call option then we have to take into account that the pay–off diagram $V^{sp}(S,T) = -(S-E)^+$ is a concave function and so is the solution $V^{sp}(S,t)$. Hence, sign $\left(\frac{\partial^2 V^{sp}}{\partial S^2}\right) = -1$. The same conclusion is true for the put option. The explanation is adopted from Ševčovič, Stehlíková and Mikula [36]. In this case the governing equation changes slightly. Namely, in front of the last coefficient of the nonlinear sigma function there



$S = 20$



$S = 23$



$S = 25 \equiv E$

Figure 6.4: Solution $V(S,t)$ for $S = 20, S = 23, S = 25 \equiv E$ and $t \in [0,T]$.

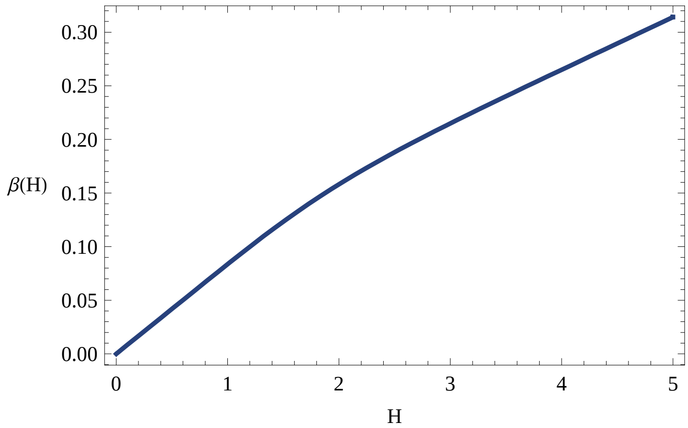


Figure 6.5: The shape of the function $\beta(H)$ corresponding to the nonlinear model with variable transaction costs.

is a reversed sign:

$$\hat{\sigma}^2(S\partial_S^2 V, \Delta t) = \sigma^2 \left(1 + \frac{\tilde{C}(\sigma|S\partial_S^2 V|\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}}\right). \tag{6.6}$$

Therefore the equation modelling the higher ask option price is

$$\partial_t V + \frac{1}{2}\hat{\sigma}^2(S\partial_S^2 V, \Delta t)S^2\partial_S^2 V + (r-q)S\partial_S V - rV = 0 \tag{6.7}$$

with volatility given by (6.6).

6.2.2 Ask-Option Prices by Using the Novel Model

After the transformation of the last equation, with the volatility given (6.6) to the Gamma equation, we obtain the function $\beta(H)$ in the following form:

$$\beta(H) = \frac{\sigma^2}{2}\left(1 + \frac{\tilde{C}(\sigma|H|\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}}\right) H.$$

This can be easily replaced in the Mathematica code in the Table 6.2 for definition of the $\beta(H)$ function. The shape of $\beta(H)$ is depicted in Figure 6.5.

Using the same numerical scheme, we obtained the following results for the higher

Table 6.5: Ask Option values of the numerical solution of nonlinear model in comparison to B–S with constant volatility.

S	$V_{\sigma^2_{min}}$	V_{vtc}	$V_{\sigma^2_{max}}$
20	1.150	1.686	1.729
23	2.344	3.015	3.067
25	3.403	4.119	4.168
28	5.338	6.060	6.102
30	6.820	7.514	7.549

ask option price. The lower bound is given by option value from the linear Black–Scholes equation with volatility $\hat{\sigma}^2_{min} = \sigma^2 \left(1 + \underline{C}_0 \sqrt{\frac{2}{\pi}} \frac{1}{\sigma\sqrt{\Delta t}}\right)$, where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$, and the upper bound corresponding to solution with volatility $\hat{\sigma}^2_{max} = \sigma^2 \left(1 + C_0 \sqrt{\frac{2}{\pi}} \frac{1}{\sigma\sqrt{\Delta t}}\right)$.

The solutions is ordered in this manner:

$$V_{\sigma^2_{min}}(S,t) \leq V_{vtc}(S,t) \leq V_{\sigma^2_{max}}(S,t) \quad \forall S, t.$$

The interpretation is similar to the previous one. Note that with the time approaching expiration the value of the option is going from the upper bound to the lower bound, i.e., first the amount of transactions is small for higher price per one unit of traded assets and later the amount of transactions is high for lower price per one unit of traded assets.

In Figure 6.6 we present the graphs of solution $V_{vtc} := V(S,t)$ as well as $\Delta(S,t) = \partial_S V(S,t)$ for various times $t \in \{0, T/3, 2T/3\}$. The lower dashed line corresponds to the solution of the linear Black–Scholes equation with volatility

$$\hat{\sigma}^2_{min} = \sigma^2 \left(1 + \underline{C}_0 \sqrt{\frac{2}{\pi}} \frac{1}{\sigma\sqrt{\Delta t}}\right),$$

where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$, and upper dashed line corresponds to the solution with volatility

$$\hat{\sigma}^2_{max} = \sigma^2 \left(1 + C_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right).$$

The dependence of the option price at the time $t \in (0, T)$ for $S \in \{20, 23, 25\}$ is shown on Figure 6.7. In the Figure 6.7 we see how the sequence converges to zero value at the time of the expiration. One can observe the very rapid change in the convergence to zero for an increasing underlying asset value. To conclude the numerical experiments we illustrate the Bid–Ask spread in Figure 6.8.

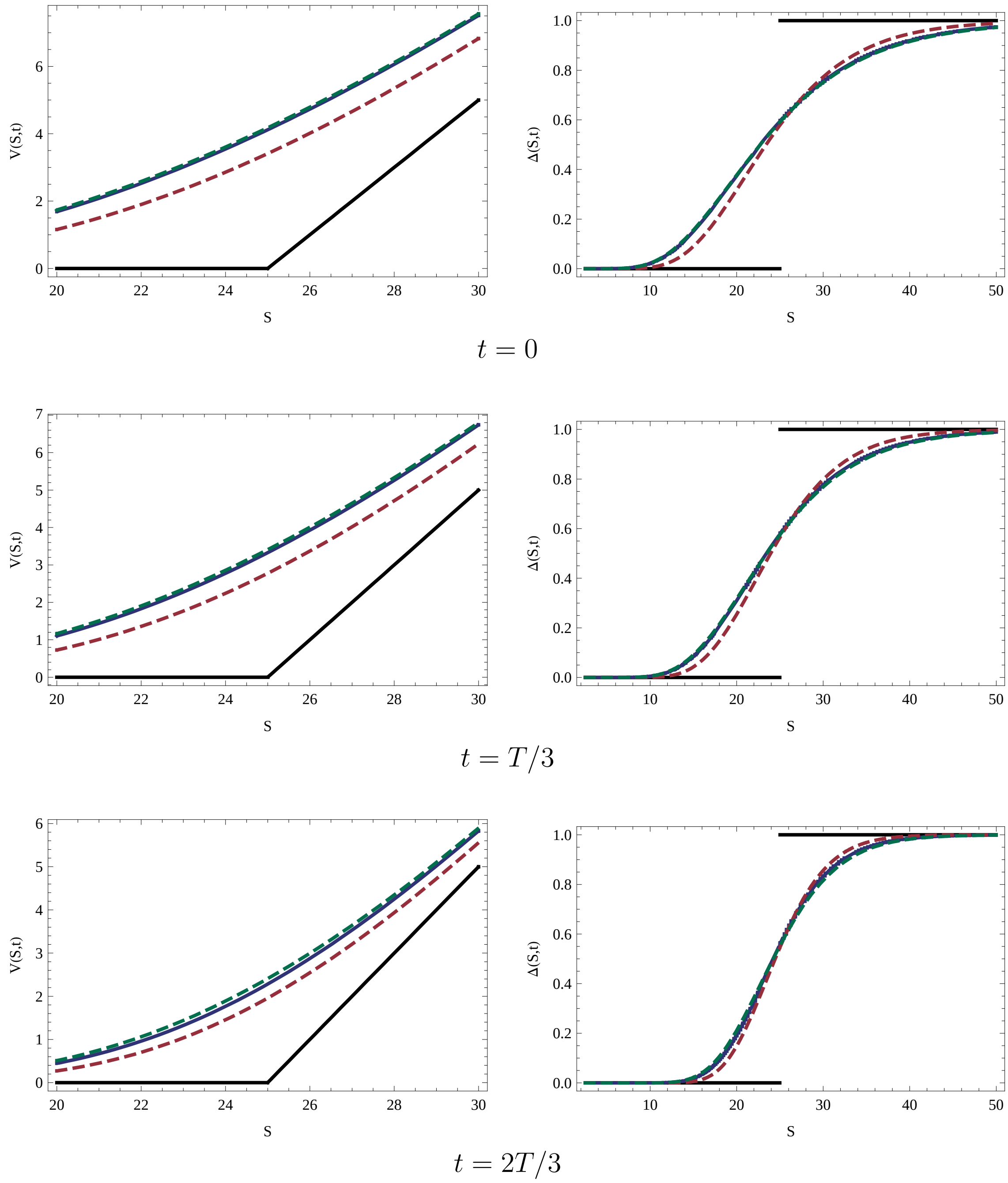
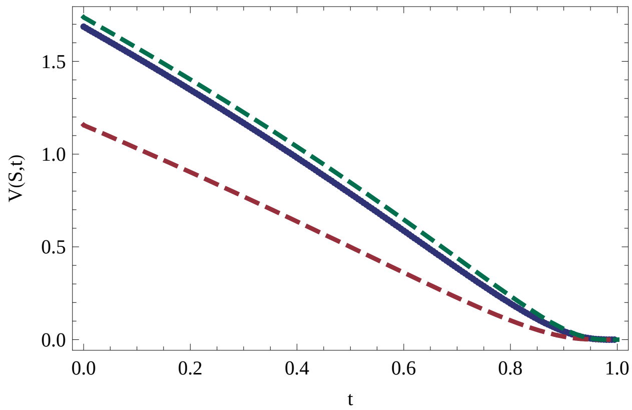
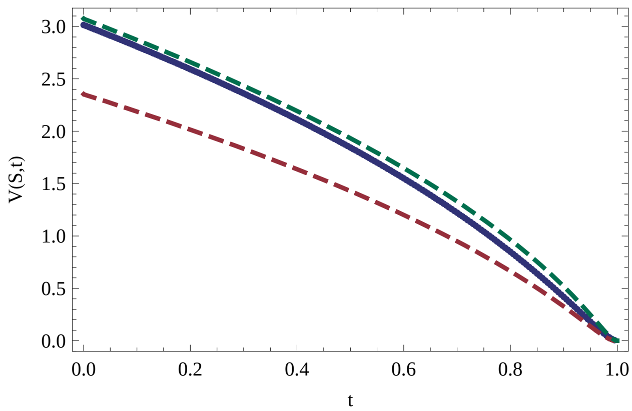


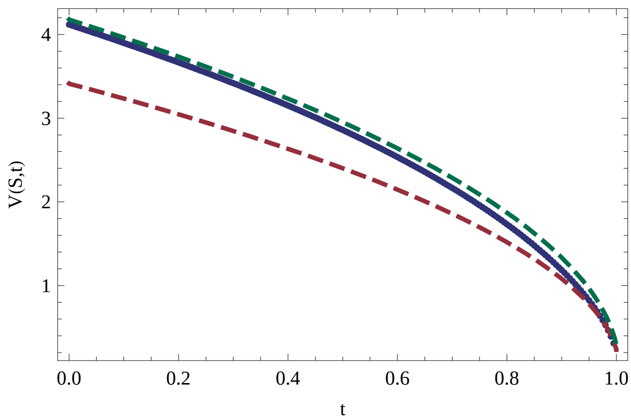
Figure 6.6: Solution $V(S,t)$ for $t = 0, t = T/3, t = 2T/3$ (left) and corresponding $\Delta(S,t) = \partial_S V(S,t)$ of the call option.



$S = 20$

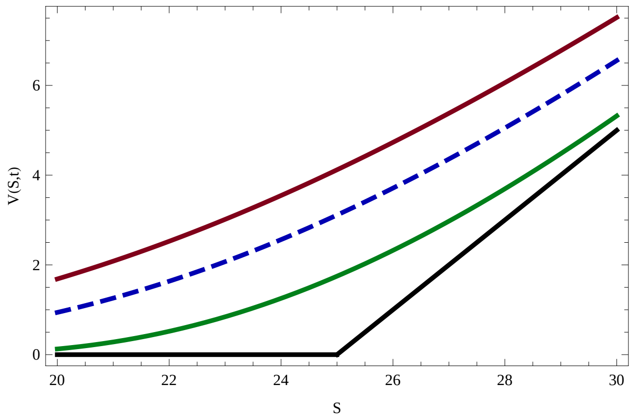


$S = 23$

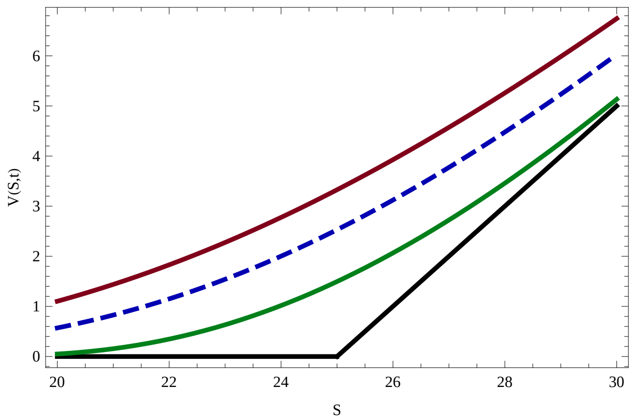


$S = 25 \equiv E$

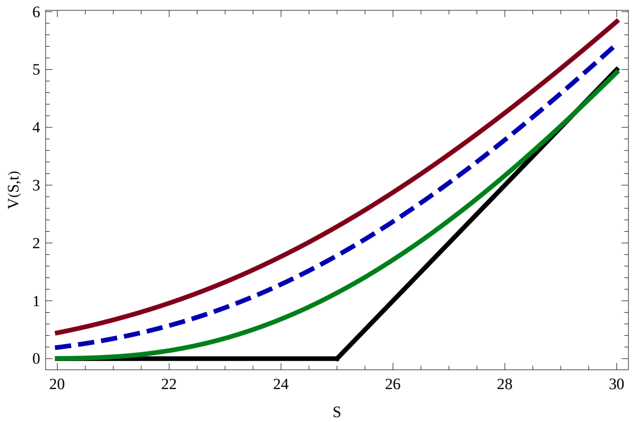
Figure 6.7: Solution $V(S, t)$ for $S = 20, S = 23, S = 25 \equiv E$ and $t \in 0, T]$.



$t = 0$



$t = T/3$



$t = 2T/3$

Figure 6.8: A comparison of the bid (green line) and ask (red line) option prices computed by means of the model with variable transaction costs for $t = 0, t = T/3, t = 2T/3$. The middle dashed line is the option price computed from the Black–Scholes equation.

Conclusions

In this thesis we analysed recent topics on pricing derivatives by means of the solutions to nonlinear Black–Scholes equations. We presented various nonlinear generalizations of the classical Black–Scholes theory arising when modelling illiquid and incomplete markets, in the presence of a dominant investor in the market, etc. We did show that, in presence of variable transaction costs and risk from an unprotected portfolio, the resulting novel pricing model is a nonlinear extension of the Black–Scholes equation in which the diffusion coefficient is no longer constant and it depends on the option price itself.

In Chapter 2 we developed the theory of models with variable transaction costs. The main idea was in defining the modified transaction cost function $\tilde{C}$ when using the transaction costs measure. We also studied the properties of this function to confirm its generality. We presented and analysed two more new examples of realistic variable transaction costs that are decreasing with the amount of transactions, particularly, the piecewise linear nonincreasing function and the exponentially decreasing function. By considering these functions, we solved the difficulty with possibly negative transaction costs that arises in the model proposed by Amster et al. [1]. We developed the Risk adjusted pricing methodology using variable transaction cost instead of constant. In Chapter 3.8 we analysed the optimal choice of hedging time as a problem of maximizing the variance to cover the most negative scenario.

We have also shown how to solve the presented nonlinear Black–Scholes models numerically. In particular, we solved the model with piecewise linear non–increasing function of transaction costs. The main idea was in the transformation of the governing equation into the Gamma equation. Into this equation enters $\beta(H)$ function corresponding to the chosen model.

In order to solve the Gamma equation we used an efficient numerical discretization. The numerical scheme was based on the finite volume scheme. By numerical solution we obtained the values of the options and showed that when the modified transaction costs function is bounded, then the solution of the novel nonlinear model lies between the solutions of the Black–Scholes equation with constant transaction costs of upper and lower bound.

In general it is difficult to find an explicit solution of general nonlinear models of the Black–Schholes type. An extension of this thesis can be in application of other numerical schemes to deal with the problem of derivative pricing. To solve Gamma equation it is possible to use the scheme of Casabán, Company, Jódar and Pintos [11], the modern schemes by Niu Cheng–hu, Zhou Sheng–Wu [32] and also the scheme designed by Kútik and Mikula [27]. There exist also some explicit solutions for special type of nonlinear models that are known from Bordag and Frey in [9] and [10] to compare the results. Another extension could be the consideration of the other types of financial derivatives, for example American options.

List of Symbols

$\beta(.)$	Beta function
$C(.)$	transaction costs function
$\tilde{C}(.)$	modified transaction costs function
$\mathbb{E}$	expected value
$C_0, \underline{C}_0, \kappa > 0$	parameters of a transaction costs function
$N(.,.)$	normal distribution with specified parameters
$Prob(.)$	probability of an event
$q > 0$	divided yield
$r > 0$	risk–free interest rate
r_{TC}	transaction cost measure
r_{VP}	risk from the unprotected portfolio
r_R	total measure of the risk
$R > 0$	risk premium coefficient
$\rho > 0$	drift of the asset price
S	asset price
$\sigma > 0$	volatility of the asset price
$\sigma(.,.)$	volatility function of a process
$\hat{\sigma}(.,.,.)$	general volatility function
$\mu(.,.)$	drift function of a process
T	expiration time
t	actual time
τ	time to maturity
V	option price
V_{vtc}	approximation of option price in model with variable transaction costs
$Var(.)$	variance of a random variable
$w, w(t)$	Wiener process
$\Gamma(.)$	Gamma function

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