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SEMI-SYMMETRIC CONDITIONS ON GENERALIZED COMPLEX SPACE FORMS

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ABSTRACT. The aim of this paper is to study H-projective curvature and pseudo-projective curvature tensors P and $\widetilde{P}$ by using semi-symmetry and two-semi-symmetry of these tensors on generalized complex space forms. Also generalized complex space forms are studied under the conditions $P \cdot S = 0$ and $\widetilde{P} \cdot S = 0$.

1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvature of a manifold determines the curvature tensor completely. A Riemannian manifold with the constant sectional curvature c is called a *real-space-form*, and its curvature tensor satisfies the equation

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

Models for these spaces are the Euclidean spaces ($c = 0$), the spheres ($c > 0$) and the hyperbolic spaces ($c < 0$).

A similar situation can be found in the study of complex manifold from a Riemannian point of view. For each plane P in the tangent space $T_x(M)$, the sectional curvature $K(P)$ is defined to be $K(P) = R(X,Y,X,Y) = g(R(X,Y)Y,X)$, where $\{X,Y\}$ is an orthonormal basis for P. If P is invariant by J, then $K(P)$ is called the holomorphic sectional curvature by P. The holomorphic sectional curvature $K(P)$ is given by

$$K(P) = R(X,JY,X,JY) = g(R(X,JY)JY,X).$$

If $K(P)$ is constant for all J-invariant planes P in $T_x(M)$ and for all points $x \in M$, then M is called a space of a constant holomorphic sectional curvature or a complex space form. A complex space form is defined to be a simply connected complete Kaehlerian manifold of a constant holomorphic sectional curvature [**6**].

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The curvature tensor of a complex space form is given by

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ],$$

the models now are C^n, CP^n and CH^n, depending on $c = 0$, $c > 0$ or $c < 0$.

More generally, an almost Hermition manifold M is called a *generalized complex space form* $M(f_1, f_2)$ if its Riemannian curvature tensor R satisfies

$$\begin{aligned} R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY \\ - g(Y,JZ)JX + 2g(X,JY)JZ\} \end{aligned} \tag{1.1}$$

for all $X, Y, Z \in TM$, where f_1 and f_2 are smooth functions on M [**12, 13**]. In [**12**], an important obstruction for such a space was presented by F. Tricerri and L. Vanhecke: If M is connected, dim≥ 6, and f_2 is not identically zero, then M is a complex space form(in particular, f_1 and f_2 must be constant). In 1989 the author Z. Olszak [**9**] worked on existence of a generalized complex space form. The authors Pablo Alegre and Alfonso Carriazo studied structures on generalized Sasakian space forms [**1**]. The authors U. C. De and A. Sarkar studied nature of a generalized Sasakian space form under some conditions regarding a projective curvature tensor [**4**]. The authors Venkatesha and B. Sumangala [**14**], Mehmet Atceken [**7**], S. Yadav and A. K. Srivastava [**15**] studied on a generalized space form satisfying certain conditions on an M-projective curvature tensor, concircular curvature and pseudo projective curvature tensor $\widetilde{P}$ satisfying $R \cdot \widetilde{P} = 0$ and many authors studied on generalized Sasakian space form [**8**]. The authors M. C. Bharathi and C. S. Bagewadi [**3**] extended the study to W_2 curvature, conharmonic and concircular curvature tensors on generalized complex space forms.

Motivated by these ideas, in this paper, we made an attempt to study a H-projective curvature tensor and a pseudo-projective curvature tensor in a generalized complex space form. The paper is organized as follows. In Section 2, we give definitions, notations and basic results of generalized complex space forms. The Section 3 deals with H-projective semi-symmetric, H-projective two-semi-symmetric tensors and $P \cdot S = 0$ on generalized complex space forms. Lastly, the in Section 4, deals with pseudo-projective semi-symmetric, pseudo-projective two-semi-symmetric tensors and $\widetilde{P} \cdot S = 0$ on generalized complex space forms.

2. Preliminaries

A Kaehler manifold is an even-dimensional manifold M^n, where $n = 2k$ with a complex structure J and a positive-definite metric g which satisfy the following conditions [**11**]

$$J^2 = -I, \qquad g(JX, JY) = g(X,Y) \qquad \text{and} \qquad \nabla J = 0,$$

where ∇ means covariant derivation according to the Levi-Civita connection.

The scalar curvature $r = \Sigma S(e_i, e_i)$ implies

$$(\nabla_X S)(e_i, e_i) = \nabla_X r = \mathrm{dr}(X).$$

Let Q be the Ricci operator defined by $g(QX,Y)=S(X,Y)$,

$$(\nabla_Z S)(X,Y)=g((\nabla_Z Q)(X),Y).$$

Taking $Y=Z=e_i$ and summation over i in the above equation, we get

$$(\nabla_{e_i} S)(X,e_i)=g((\nabla_{e_i} Q)(X),e_i).$$

Then

$$\begin{aligned}(\operatorname{div} Q)(X)&=\operatorname{tr}(Z\to(\nabla_Z Q)(X))\\&=\sum g((\nabla_{e_i} Q)(X),e_i).\end{aligned}$$

But it is known [**2**] that $(\operatorname{div} Q)(X)=\frac{1}{2}\operatorname{dr}(X)$. Hence $(\nabla_{e_i} S)(X,e_i)=\frac{1}{2}\operatorname{dr}(X)$.

For a generalized complex space form $M(f_1,f_2)$, we have

$$S(X,Y)=\{(n-1)f_1+3f_2\}g(X,Y), \tag{2.1}$$

$$QX=[(n-1)f_1+3f_2]X, \tag{2.2}$$

$$r=n[(n-1)f_1+3f_2], \tag{2.3}$$

where $n=2k$. S is the Ricci tensor, Q is the Ricci operator and r is a scalar curvature of the space form $M(f_1,f_2)$.

Given an n dimensional (where $n=2k$) Kaehler manifold M, an H-projective curvature tensor P and a pseudo-projective curvature tensor $\widetilde{P}$ are given by

$$\begin{aligned}P(X,Y)Z=&R(X,Y)Z-\frac{2}{n+2}[S(Y,Z)X-S(X,Z)Y-S(JY,Z)JX\\&+S(JX,Z)JY+S(JX,Y)JZ-S(JY,X)JZ].\end{aligned} \tag{2.4}$$

$$\begin{aligned}\widetilde{P}(X,Y)Z=&aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y]\\&-\frac{r}{n}\Big[\frac{a}{n-1}+b\Big][g(Y,Z)X-g(X,Z)Y].\end{aligned} \tag{2.5}$$

Symmetry of the manifold is the most important property among all its geometrical properties. Symmetry of the manifold basically depends on the curvature tensor and the Ricci tensor of the manifold.

Definition 2.1 ([**5, 10**])**.** An n-dimensional generalized complex space form is said to be

- Semi-symmetric if it satisfies

$$(R(X,Y)\cdot R)(U,V,W)=0 \qquad \text{for all } X,Y\in\chi(M),$$

- H-projective-semi-symmetric if it satisfies

$$(R(X,Y)\cdot P)(U,V,W)=0 \qquad \text{for all } X,Y\in\chi(M),$$

- Pseudo-projective-semi-symmetric if it satisfies

$$(R(X,Y)\cdot \widetilde{P})(U,V,W)=0 \qquad \text{for all } X,Y\in\chi(M).$$

Definition 2.2. A n-dimensional generalized complex space form is said to be

- Two-semi-symmetric if it satisfies
$$(R(X,Y)\cdot\nabla R)(U,V,W,Z)=0 \qquad \text{for all } X,Y\in\chi(M),$$
- H-projective-two-Semi-symmetric if it satisfies
$$(R(X,Y)\cdot\nabla P)(U,V,W,Z)=0 \qquad \text{for all } X,Y\in\chi(M),$$
- Pseudo-projective-two-Semi-symmetric if it satisfies
$$(R(X,Y)\cdot\nabla \widetilde{P})(U,V,W,Z)=0 \qquad \text{for all } X,Y\in\chi(M).$$

3. H-projective curvature tensor on a generalized complex space form

Let R and P satisfy the equation $R\cdot P=0$. Then for any tangent vectors X,Y,U,V and W, the above implies

$$(R(X,Y)\cdot P)(U,V,W)=0,$$

i.e.,

$$\begin{aligned}&R(X,Y)P(U,V)W-P(R(X,Y)U,V)W\\&-P(U,R(X,Y)V)W-P(U,V)R(X,Y)W=0.\end{aligned}$$

Taking inner product with Z, we have

$$\begin{aligned}&g(R(X,Y)P(U,V)W,Z)-g(P(R(X,Y)U,V)W,Z)\\&-g(P(U,R(X,Y)V)W,Z)-g(P(U,V)R(X,Y)W,Z)=0.\end{aligned}\tag{3.1}$$

Using equations (1.1) and (2.4) in (3.1) and putting $X=V=e_i$ after simplification, and again putting $Y=Z=e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and then taking summation over i $(1\le i\le n)$ after simplification, we get

$$S(U,W)=\frac{2f_1(n-2)-4f_2}{f_1(n^2-5n+2)-2nf_2}r\cdot g(U,W).$$

So that $M(f_1,f_2)$ is an Einstein manifold.
Hence we have the following result.

Theorem 3.1. *If an n-dimensional generalized complex space form $M(f_1,f_2)$ satisfies $R\cdot P=0$, then it is an Einstein manifold.*

Taking covariant derivative of (2.4), we get

$$\begin{aligned}(\nabla_W P)(X,Y)Z=&(\nabla_W R)(X,Y)Z-\frac{2}{n+2}[(\nabla_W S)(Y,Z)X\\&-(\nabla_W S)(X,Z)Y-(\nabla_W S)(JY,Z)JX\\&+(\nabla_W S)(JX,Z)JY+(\nabla_W S)(JX,Y)JZ\\&-(\nabla_W S)(JY,X)JZ].\end{aligned}\tag{3.2}$$

$$(\operatorname{div}R)(X,Y)Z=(\nabla_Z S)(X,Y)-(\nabla_Y S)(X,Z).\tag{3.3}$$

Consider the equation $R \cdot \nabla P = 0$ satisfied by R and P. Then for any tangent vectors X, Y, U, V, W and Z, it can be written as

$$(R(X,Y) \cdot \nabla P)(U,V,W,Z) = 0,$$

i.e.,

$$\begin{aligned} R(X,Y)(\nabla_U P)(V,W)Z - (\nabla_{R(X,Y)U} P)(V,W)Z - (\nabla_U P)(R(X,Y)V,W)Z \\ - (\nabla_U P)(V,R(X,Y)W)Z - (\nabla_U P)(V,W)R(X,Y)Z = 0. \end{aligned}$$

Taking inner product with the vector field T, we have

$$\begin{aligned} & g(R(X,Y)(\nabla_U P)(V,W)Z,T) - g((\nabla_{R(X,Y)U} P)(V,W)Z,T) \\ & - g((\nabla_U P)(R(X,Y)V,W)Z,T) - g((\nabla_U P)(V,R(X,Y)W)Z,T) \\ & - g((\nabla_U P)(V,W)R(X,Y)Z,T) = 0. \end{aligned} \tag{3.4}$$

Using equations (1.1), (3.3) and (3.2) in (3.4) and putting $X = V = e_i$ after simplification, and again putting $Y = Z = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and then taking summation over i, $(1 \le i \le n)$ after simplification we get

$$\left\{ \frac{f_1(-f_1(5+n) - 2f_2 n(n+1))}{n+3} \right\} \mathrm{dr}(U) = 0$$

$$\mathrm{dr}(U) = 0 \qquad \text{for all } U.$$

This implies r is constant. Hence we have the following result.

Theorem 3.2. *If an n-dimensional generalized complex space form $M(f_1, f_2)$ satisfies $R \cdot \nabla P = 0$, then its scalar curvature is constant.*

Let S and P satisfy the equation $P \cdot S = 0$. Subsequently for any tangent vectors X, Y, U and V, this equation readily reads as

$$(P(X,Y) \cdot S)(U,V) = 0,$$

i.e.,

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0. \tag{3.5}$$

Using equation (1.1) and (2.4) in equation (3.5) and putting $X = V = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i $(1 \le i \le n)$ after simplification again using equation (2.3) after simplification, we get

$$S(Y,U) = \frac{(n+2)}{(3n^2 - 4n + 4)f_1 + (6n-12)f_2 + 16} r g(Y,U).$$

That is $M(f_1, f_2)$ is an Einstein manifold. Hence we have the following result.

Theorem 3.3. *An n-dimensional generalized complex space form $M(f_1, f_2)$ satisfying a H-projective curvature tensor $P \cdot S = 0$ is an Einstein manifold.*

4. Pseudo-projective curvature tensor on generalized complex space form

Consider the Semi-symmetric condition $R \cdot \widetilde{P} = 0$ satisfied by R and $\widetilde{P}$. Then for any tangent vectors X, Y, U, V and W, this equation can be expressed as

$$(R(X,Y) \cdot \widetilde{P})(U,V,W) = 0,$$

i.e.,

$$\begin{aligned} &R(X,Y)\widetilde{P}(U,V)W - \widetilde{P}(R(X,Y)U,V)W \\ &- \widetilde{P}(U,R(X,Y)V)W - \widetilde{P}(U,V)R(X,Y)W = 0. \end{aligned}$$

Taking inner product with Z, we have

$$\begin{aligned} &g(R(X,Y)\widetilde{P}(U,V)W,Z) - g(\widetilde{P}(R(X,Y)U,V)W,Z) \\ &- g(\widetilde{P}(U,R(X,Y)V)W,Z) - g(\widetilde{P}(U,V)R(X,Y)W,Z) = 0. \end{aligned} \tag{4.1}$$

Using equations (1.1) and (2.5) in (4.1), putting $X = V = e_i$ after simplification, and again putting $Y = Z = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and then taking summation over i $(1 \le i \le n)$ after simplification, we get

$$S(U,W) = \frac{f_1(bn(n-1)(n-2)) + f_2(-a + 3a(n-1) + 3b(n-1)^2 - b(n-1))}{n(n-1)[f_1(n(n-2)) + f_2(2a - 4b + 3bn)]} rg(U,W).$$

That is $M(f_1, f_2)$ is an Einstein manifold. Hence we have the following result.

Theorem 4.1. *If an n-dimensional generalized complex space form $M(f_1, f_2)$ satisfies $R \cdot \widetilde{P} = 0$, then it is an Einstein manifold.*

Taking covariant derivative of (2.5), we get

$$\begin{aligned} (\nabla_W \widetilde{P})(X,Y)Z &= a(\nabla_W R)(X,Y)Z + b[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y] \\ &- \frac{\mathrm{dr}(W)}{n}\big[\frac{a}{n-1} + b\big][g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.2}$$

Let R and $\widetilde{P}$ satisfy $R \cdot \nabla \widetilde{P} = 0$. Subsequently, for any tangent vectors X, Y, U, V, W and Z, this equation takes the following form

$$(R(X,Y) \cdot \nabla \widetilde{P})(U,V,W,Z) = 0,$$

i.e.

$$\begin{aligned} R(X,Y)(\nabla_U \widetilde{P})(V,W)Z - (\nabla_{R(X,Y)U}\widetilde{P})(V,W)Z - (\nabla_U \widetilde{P})(R(X,Y)V,W)Z \\ -(\nabla_U \widetilde{P})(V,R(X,Y)W)Z - (\nabla_U \widetilde{P})(V,W)R(X,Y)Z = 0. \end{aligned}$$

Taking inner product with the vector field T, we have

$$\begin{aligned} &g(R(X,Y)(\nabla_U \widetilde{P})(V,W)Z,T) - g((\nabla_{R(X,Y)U}\widetilde{P})(V,W)Z,T) \\ &- g((\nabla_U \widetilde{P})(R(X,Y)V,W)Z,T) - g((\nabla_U \widetilde{P})(V,R(X,Y)W)Z,T) \\ &- g((\nabla_U \widetilde{P})(V,W)R(X,Y)Z,T) = 0. \end{aligned} \tag{4.3}$$

Using equations (1.1), (3.3) and (4.2) in (4.3) and putting $X = V = e_i$ after simplification, and again putting $Y = Z = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and then taking summation over i $(1 \leq i \leq n)$ after simplification, we get

$$\left\{b((n-1)f_1^2 - 3nf_2) + a\big(f_1 f_2 - \frac{3}{2}\big)\right\} \mathrm{dr}(U) = 0$$

$$\mathrm{dr}(U) = 0 \qquad \text{for all } U.$$

This implies r is constant. Hence we have the following result.

Theorem 4.2. *If an n-dimensional generalized complex space form $M(f_1, f_2)$ satisfies $R \cdot \nabla \widetilde{P} = 0$, then its scalar curvature is constant.*

Consider the equation $\widetilde{P} \cdot S = 0$ satisfied by S and $\widetilde{P}$. Then for any tangent vectors X, Y, U and V, it can be written as follows

$$(\widetilde{P}(X,Y) \cdot S)(U,V) = 0,$$

i.e.,

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0. \tag{4.4}$$

Using equation (1.1) and (2.5) in equation (4.4) and putting $X = V = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, $(1 \leq i \leq n)$ and again using equation (2.3) after simplification we get

$$S(U,Y) = \frac{[-b(n-1)^2 f_1 - 3af_2 - 3b(n-1)f_2]n}{[bn^2 f_1(n-1) + af_1(n-1) + 3bn^2 f_2 - an^2 f_1](n-1)} rg(U,Y).$$

Thus $M(f_1, f_2)$ is an Einstein manifold. Hence we have the following result.

Theorem 4.3. *If an n-dimensional generalized complex space form $M(f_1, f_2)$ satisfying a pseudo-projective curvature tensor $\widetilde{P} \cdot S = 0$ is an Einstein manifold.*

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