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# EVALUATIONS OF DETERMINANTS OF SEVERAL MATRICES WHOSE ENTRIES SATISFY INHOMOGENEOUS RECURRENCE RELATIONS

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ABSTRACT. Let $P = [P_{i,j}]_{i,j\geq 0}$ be an infinite matrix whose entries satisfy the recurrence relation $P_{i,j} = xP_{i,j-1} + yP_{i-1,j-1} + zP_{i-1,j} + f(i,j)$ for $i,j \geq 1$, where $x$, $y$, and $z$ are arbitrary constants, and $f$ is a function mapping $\mathbb{N}\times\mathbb{N}$ to $\mathbb{C}$. The purpose of this article is twofold: First, it provides a factorization of $P$ for specific values of $x$, $y$, and $z$, as well as the function $f$. Second, it presents several evaluations of the determinants of such matrices.

## 1. Introduction

For an infinite matrix $A = [A_{i,j}]_{i,j\geq 0}$ and any integer $n \geq 1$, as usual, by $A(n)$ we denote the principal $n \times n$ submatrix of $A$, that is $A(n) = [A_{i,j}]_{0\leq i,j<n}$. Note that we index matrices starting at $(0,0)$. We let $I$ denote the identity matrix.

Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two sequences that start with a common first term, denoted by $\alpha_0 = \beta_0 (:= \gamma_0)$. Let $x$, $y$, and $z$ be constants, and let $f$ be a function mapping $\mathbb{N}\times\mathbb{N}$ to $\mathbb{C}$. An infinite matrix $P_{\alpha,\beta,f}^{[x,y,z]} = [P_{i,j}]_{i,j\geq 0}$ is called a *Pascal-type matrix* (associated with sequences $\alpha$, $\beta$, constants $x$, $y$, and $z$, and the function $f$) if its entries satisfy the following recurrence relation:

$$P_{i,j} = xP_{i,j-1} + yP_{i-1,j-1} + zP_{i-1,j} + f(i,j), \quad \text{for } i,j \geq 1,$$

with initial conditions given by $P_{i,0} = \alpha_i$ and $P_{0,j} = \beta_j$ for $i,j \geq 0$.

Let us first assume that $f(i,j) = 0$. In this case, we are dealing with a *homogeneous* recurrence relation. For different choices of constants $x$, $y$, and $z$, as well as the sequences $\alpha$ and $\beta$, these matrices have been investigated in the literature

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(see, for instance, [**1, 7, 9, 12, 14**]). Additionally, it is straightforward to verify (see also [**1, 3**]) that if $\alpha = \beta = (1, 1, 1, \ldots)$, then we have

$$P^{[1,0,1]}_{\alpha,\beta,0}(n) = L(n) \cdot I(n) \cdot U(n), \tag{1}$$

where $L(n)$ is the lower-triangular Pascal matrix:

$$L(n) = [L_{i,j}]_{0\le i,j<n} = \left(\binom{i}{j}\right)_{0\le i,j<n} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n-1}{0} & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-1} \end{pmatrix},$$

and $U(n) = L(n)^t$, the superscript $t$ denotes transpose. In particular, we obtain

$$\det(P^{[1,0,1]}_{\alpha,\beta,0}(n)) = 1.$$

Given a sequence $\omega = (\omega_i)_{i\ge 0}$, we define the *binomial transform* $\hat{\omega} = (\hat{\omega}_i)_{i\ge 0}$ and the *inverse binomial transform* $\check{\omega} = (\check{\omega}_i)_{i\ge 0}$ of $\omega$ as follows:

$$\hat{\omega}_i = \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \omega_k \quad \text{and} \quad \check{\omega}_i = \sum_{k=0}^{i} \binom{i}{k} \omega_k.$$

It is routine to check that (see also [**7**, Lemma 2.1]):

$$\hat{\check{\omega}} = \check{\hat{\omega}} = \omega. \tag{2}$$

Let $k$ be a positive integer. A matrix of the form

$$T^{[k]}(\infty) = \begin{pmatrix} \gamma_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\ \alpha_1 & \gamma & k\beta_1 & k\beta_2 & k\beta_3 & \ddots \\ \alpha_2 & k\alpha_1 & k\gamma & k^2\beta_1 & k^2\beta_2 & \ddots \\ \alpha_3 & k\alpha_2 & k^2\alpha_1 & k^2\gamma & k^3\beta_1 & \ddots \\ \alpha_4 & k\alpha_3 & k^2\alpha_2 & k^3\alpha_1 & k^3\gamma & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is called a *k-quasi-Töeplitz* matrix. A $k$-quasi-Töeplitz matrix is determined by its first column $\alpha = (\alpha_i)_{i\ge 0} = (\alpha_0, \alpha_1, \alpha_2, \ldots)$, first row $\beta = (\beta_i)_{i\ge 0} = (\beta_0, \beta_1, \beta_2, \ldots)$, with $\alpha_0 = \beta_0 = \gamma_0$, and the $(1,1)$-th entry $\gamma$, henceforth, we will use $T^{[k]}_{\alpha,\beta,\gamma}(\infty)$ to describe the infinite matrix $T^{[k]}(\infty)$. In particular, this reduces to the usual definition of a Töeplitz matrix when $k = 1$ and $\gamma = \gamma_0$, and in order to have a convenient notation, we write $T_{\alpha,\beta}(\infty)$ in place of 1-quasi-Töeplitz matrix $T^{[1]}_{\alpha,\beta,\gamma_0}(\infty)$.

It is worthwhile pointing out that in [**7**, Theorem 3.1] (see also [**12**]), we obtained a *factorization* of the Pascal-type matrices $P^{[1,0,1]}_{\alpha,\beta,0}(n)$, as a product of the lower-triangular Pascal matrix $L(n)$, the Töeplitz matrix $T_{\hat{\alpha},\hat{\beta}}(n)$ and $U(n) = L(n)^t$, that is

$$P^{[1,0,1]}_{\alpha,\beta,0}(n) = L(n) \cdot T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n), \tag{3}$$

(compare with equation (1)). In fact, we obtained a *connection* between Pascal-type matrices $P^{[1,0,1]}_{\alpha,\beta,0}(n)$ and Töeplitz matrices $T_{\hat{\alpha},\hat{\beta}}(n)$, and in particular, we concluded that

$$\det(P^{[1,0,1]}_{\alpha,\beta,0}(n)) = \det(T_{\hat{\alpha},\hat{\beta}}(n)).$$

One of the aims of this article is to derive a matrix factorization similar to (3) for the Pascal-type matrices $P^{[1,y,1]}_{\alpha,\beta,f}(n)$, where $f(i,j) = c$. It is important to note that when $c \neq 0$, we are dealing with an *inhomogeneous* recurrence relation. We can articulate this more precisely with the following theorem.

**Theorem 1.1.** *Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two sequences starting with a common first term $\alpha_0 = \beta_0 (= \gamma_0)$, and let $f(i,j) = c$ be an arbitrary constant. Then, for any positive integer $n$, we have*

$$P^{[1,y,1]}_{\alpha,\beta,f}(n) = L(n) \cdot T^{[1+y]}_{\hat{\alpha},\hat{\beta},\gamma}(n) \cdot U(n),$$

*where $\gamma = (1+y)\gamma_0 + c$. In particular, we have*

$$\det(P^{[1,y,1]}_{\alpha,\beta,f}(n)) = \det(T^{[1+y]}_{\hat{\alpha},\hat{\beta},\gamma}(n)).$$

**Remark.** In Theorem 1.1, if we put $c = y = 0$, then we obtain Theorem 3.1 in [**7**].

If $\alpha = (\alpha_i)_{i\geq 0}$ is a constant sequence, then

$$\hat{\alpha}_i = \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \alpha_k = (-1)^i \alpha_0 \sum_{k=0}^{i} (-1)^k \binom{i}{k} = \begin{cases} \alpha_0 & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

and so, $\hat{\alpha} = (\hat{\alpha}_i)_{i\geq 0} = (\alpha_0, 0, 0, 0, \ldots)$. Conversely, if $\hat{\alpha} = (\hat{\alpha}_i)_{i\geq 0} = (\alpha_0, 0, 0, 0, \ldots)$, then it is easy to see that $\alpha$ is a constant sequence. Therefore, if $\alpha$ (resp. $\beta$) is a constant sequence, then the $k$-quasi-Töeplitz matrix $T^{[k]}_{\hat{\alpha},\hat{\beta},\gamma}(n)$ is lower triangular (resp. upper triangular), and so we obtain

$$\det(T^{[k]}_{\hat{\alpha},\hat{\beta},\gamma}(n)) = k^{\binom{n-1}{2}} \gamma_0 \gamma^{n-1}.$$

The following is now an immediate consequence of Theorem 1.1.

**Corollary 1.2.** *Let $n$ be a positive integer. In the notation of Theorem* 1.1*, if $\alpha$ or $\beta$ is a constant sequence, then we have*

$$\det(P^{[1,y,1]}_{\alpha,\beta,f}(n)) = \det(T^{[1+y]}_{\hat{\alpha},\hat{\beta},\gamma}(n)) = (1+y)^{\binom{n-1}{2}} \gamma_0 [(1+y)\gamma_0 + c]^{n-1}.$$

**Remark.** In Corollary 1.2, if we put $y = c = 0$, then we obtain

$$\det(P^{[1,0,1]}_{\alpha,\beta,0}(n)) = \gamma_0^n,$$

which is Corollary 4.11 in [**7**]. Moreover, in Corollary 1.2, if we put $y = \gamma_0 = 1$ and $c = 0$, then we obtain

$$\det(P^{[1,1,1]}_{\alpha,\beta,0}(n)) = 2^{\binom{n}{2}},$$

which is Theorem 1 in [**5**] for $x = \rho = \sigma = 1$.

It is necessary to point out that several evaluations of the determinants of such matrices have been done in [**1, 2, 5, 6, 7, 8, 9, 10, 11**]. In the following, by choosing the appropriate sequences $\alpha$ and $\beta$, we obtain an explicit formula for the general term of the sequence of principal minors for the infinite matrix $P_{\alpha,\beta,y,c}$ as stated in Theorem 1.1.

**Corollary 1.3.** *Let $n$ be a positive integer. In the notation of Theorem* 1.1*, if $\alpha_i = \beta_i = 2^i$ and $f(i,j) = c$, then $\det(P^{[1,0,1]}_{\alpha,\beta,f}(n)) = c^{n-1}$.*

*Proof.* Since

$$\hat{\alpha}_i = \hat{\beta}_i = \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \beta_k = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} 2^k = 1,$$

and $\gamma = 1 + c$, it follows by Theorem 1.1 that

$$\det(P^{[1,0,1]}_{\alpha,\beta,f}(n)) = \det(T^{[1]}_{\hat{\alpha},\hat{\beta},1+c}(n)),$$

and it is routine to check that

$$\det(T^{[1]}_{\hat{\alpha},\hat{\beta},1+c}(n)) = \det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1+c & 1 & \dots & 1 \\ 1 & 1 & 1+c & \ddots & 1 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & 1 & 1 & 1+c \end{pmatrix} = c^{n-1},$$

as required. □

Our next theorem presents a variation of the previous theorem.

**Theorem 1.4.** *Let $\alpha = \beta = (1, 1, 1, \dots)$ and let $f(i,j) = ai + bj$, where $a, b \in \mathbb{C}$. Then, for any positive integer $n$, we have*

$$P^{[1,1,1]}_{\alpha,\beta,f}(n) = L(n) \cdot \Big( [1] \oplus T^{[2]}_{\delta,\epsilon,\gamma}(n-1) \Big) \cdot U(n),$$

*where*

$$\begin{aligned} \delta &= (a + b + 2, a, 0, 0, \dots), \\ \epsilon &= (a + b + 2, b, 0, 0, \dots), \\ \gamma &= 2(a + b + 2). \end{aligned}$$

*In particular, we have*

$$\det(P^{[1,1,1]}_{\alpha,\beta,f}(n)) = \det(T^{[2]}_{\delta,\epsilon,\gamma}(n-1)).$$

*Also, if $D_n = \det(P^{[1,1,1]}_{\alpha,\beta,f}(n))$, then $D_n$ satisfies the following recursion:*

$$D_n = \begin{cases} 1 & \text{if } n = 1, \\ a + b + 2 & \text{if } n = 2, \\ 2^{n-2}(a + b + 2)D_{n-1} - 4^{n-3}abD_{n-2} & \text{if } n \geq 3. \end{cases} \tag{4}$$

**Remark.** (a) Using the method for solving linear recurrence relations with *non-constant coefficients* presented in [**4**], we obtain a closed-form expression for the sequence $(D_n)_{n\geq 3}$. Suppose $a$ and $b$ are *non-zero* real numbers, and let $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ be the sequences both satisfying the same recurrence relation

$$\alpha_n = 2^{n-2}(a+b+2)\alpha_{n-1} - 4^{n-3}ab\alpha_{n-2},$$
$$\beta_n = 2^{n-2}(a+b+2)\beta_{n-1} - 4^{n-3}ab\beta_{n-2},$$

for $n \geq 3$, and the initial conditions

$$\alpha_1 = 0, \quad \alpha_2 = 1,$$
$$\beta_1 = 1, \quad \beta_2 = 0.$$

Choose $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\alpha_n c + \beta_n \neq 0$. Now, we define two functions $\lambda\colon \mathbb{N}_{\geq 1} \to \mathbb{R}$ and $\mu\colon \mathbb{N}_{\geq 2} \to \mathbb{R}$ as

$$\lambda(1) = c, \ \lambda(n-1) = \frac{\alpha_n c + \beta_n}{\alpha_{n-1}c + \beta_{n-1}},$$
$$\mu(n) = 2^{n-1}(a+b+2) - \lambda(n).$$

Then, we obtain the closed form of $(D_n)_{n\geq 3}$ as

$$D_n = (a+b+2-c)\prod_{i=2}^{n-1}\mu(i)\left(1 + \sum_{k=3}^{n-1}\prod_{i=k}^{n-1}\frac{\lambda(i)}{\mu(i)} + \frac{a+b+2}{a+b+2-c}\prod_{i=2}^{n-1}\frac{\lambda(i)}{\mu(i)}\right).$$

For example, for $n = 3$, we have

$$\begin{aligned} D_3 &= (a+b+2-c)\mu(2)\left(1 + 0 + \frac{a+b+2}{a+b+2-c}\cdot\frac{\lambda(2)}{\mu(2)}\right) \\ &= (a+b+2-c)\mu(2) + (a+b+2)\lambda(2) \\ &= (a+b+2-c)(ab/c) + (a+b+2)\left[2(a+b+2) - (ab)/c\right] \\ &= 2(a+b+2)^2 - ab = 2a^2 + 8a + 2b^2 + 3ab + 8b + 8. \end{aligned}$$

(b) In Theorem 1.4, if $ab = 0$, then we obtain

$$D_n = 2^{\binom{n-1}{2}}(a+b+2)^{n-1}.$$

(c) In Theorem 1.4, if $a+b+2 = 0$, then we obtain

$$D_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}}2^{\frac{(n-3)(n-1)}{2}}(ab)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

We conclude the introduction with some notation and terminology that will be used throughout the article. Given a matrix $A$, we denote by $\mathrm{R}_i(A)$ and $\mathrm{C}_j(A)$ the row $i$ and the column $j$ of $A$, respectively. The notation $A^t$ represents the transpose of $A$.

## 2. Proof of Theorem 1.1

*Proof of Theorem* 1.1. For convenience, we will denote $P_{\alpha,\beta,y,c}(n)$ and $T^{[1+y]}_{\alpha,\beta,\gamma}(n)$ respectively, as $P = P(n) = [P_{i,j}]_{0\le i,j<n}$ and $T = T(n) = [T_{i,j}]_{0\le i,j<n}$. We want to prove that

$$P(n) = L(n) \cdot T(n) \cdot U(n).$$

First, we claim that

$$P(n) = L(n) \cdot Q(n),$$

where $L = L(n) = [L_{i,j}]_{0\le i,j<n}$ is the lower-triangular Pascal matrix with

$$L_{i,j} = \begin{cases} 0 & \text{if } i < j, \\ \binom{i}{j} & \text{if } i \ge j, \end{cases}$$

and $Q = Q(n) = [Q_{i,j}]_{0\le i,j<n}$, with $Q_{i,0} = \hat{\alpha}_i$, $Q_{0,j} = \beta_j$, and

$$Q_{i,j} = \begin{cases} Q_{i,j-1} + (1+y)Q_{i-1,j-1} + c & \text{if } i = 1, j \ge 1, \\ Q_{i,j-1} + (1+y)Q_{i-1,j-1} & \text{if } i > 1, j \ge 1. \end{cases} \tag{5}$$

For instance, when $n = 3$, the matrix $Q(3)$ is given by

$$Q(3) = \begin{pmatrix} \gamma & \beta_1 & \beta_2 \\ -\gamma + \alpha_1 & y\gamma + \alpha_1 + c & y\gamma + \alpha_1 + (1+y)\beta_1 + 2c \\ \gamma - 2\alpha_1 + \alpha_2 & -y\gamma + (y-1)\alpha_1 + \alpha_2 & y^2\gamma + 2y\alpha_1 + \alpha_2 + yc \end{pmatrix}.$$

Note that the entries of $L(n)$ satisfy the following recurrence relation:

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad i,j \ge 1. \tag{6}$$

For the proof of the claimed factorization, we will compute the $(i,j)$-th entry of $LQ$, that is

$$(LQ)_{i,j} = \sum_{k=0}^{n-1} L_{i,k} Q_{k,j}.$$

In fact, it suffices to show that $\mathrm{R}_0(LQ) = \mathrm{R}_0(P)$, $\mathrm{C}_0(LQ) = \mathrm{C}_0(P)$ and

$$(LQ)_{i,j} = (LQ)_{i,j-1} + y(LQ)_{i-1,j-1} + (LQ)_{i-1,j} + c \quad \text{for } 1 \le i,j < n. \tag{7}$$

First, suppose that $i = 0$. Then, we obtain

$$(LQ)_{0,j} = \sum_{k=0}^{n-1} L_{0,k} Q_{k,j} = L_{0,0} Q_{0,j} = 1\beta_j = \beta_j,$$

and so, $\mathrm{R}_0(LQ) = \mathrm{R}_0(P) = (\beta_0,\ \beta_1,\ \ldots,\ \beta_{n-1})$.

Next, suppose that $i \ge 1$ and $j = 0$. In this case, we have

$$(LQ)_{i,0} = \sum_{k=0}^{n-1} L_{i,k} Q_{k,0} = \sum_{k=0}^{n-1} \binom{i}{k} \hat{\alpha}_k = \check{\hat{\alpha}}_i = \alpha_i, \quad \text{(by (2))},$$

and so, $\mathrm{C}_0(LQ) = \mathrm{C}_0(P) = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})^t$.

Finally, we must establish equation (7). Therefore, suppose that $1 \le i, j < n$, and write $y_1 = 1 + y$. In this situation, using equations (5) and (6), we obtain

$$\begin{aligned}
(LQ)_{i,j} &= \sum_{k=0}^{n-1} L_{i,k}Q_{k,j} = L_{i,0}Q_{0,j} + L_{i,1}Q_{1,j} + \sum_{k=2}^{n-1} L_{i,k}Q_{k,j} \\
&= \beta_j + L_{i,1}(Q_{1,j-1} + y_1 Q_{0,j-1} + c) + \sum_{k=2}^{n-1} L_{i,k}(Q_{k,j-1} + y_1 Q_{k-1,j-1}) \\
&= \beta_j + ic + \sum_{k=1}^{n-1} L_{i,k}Q_{k,j-1} + y_1 \sum_{k=1}^{n-1} L_{i,k}Q_{k-1,j-1} \\
&= \beta_j + ic + (LQ)_{i,j-1} - L_{i,0}Q_{0,j-1} + y_1 \sum_{k=1}^{n-1} (L_{i-1,k-1} + L_{i-1,k})Q_{k-1,j-1} \\
&= \beta_j + ic + (LQ)_{i,j-1} - \beta_{j-1} + y_1 \sum_{k=0}^{n-2} L_{i-1,k}Q_{k,j-1} + \sum_{k=1}^{n-1} L_{i-1,k} y_1 Q_{k-1,j-1} \\
&= \beta_j + ic + (LQ)_{i,j-1} - \beta_{j-1} + y_1 \sum_{k=0}^{n-1} L_{i-1,k}Q_{k,j-1} \\
&\quad + L_{i-1,1}(Q_{1,j} - Q_{1,j-1} - c) + \sum_{k=2}^{n-1} L_{i-1,k}(Q_{k,j} - Q_{k,j-1}) \\
&= \beta_j + ic + (LQ)_{i,j-1} - \beta_{j-1} + y_1 (LQ)_{i-1,j-1} \\
&\quad - (i-1)c + \sum_{k=1}^{n-1} L_{i-1,k}Q_{k,j} - \sum_{k=1}^{n-1} L_{i-1,k}Q_{k,j-1} \\
&= \beta_j + c + (LQ)_{i,j-1} - \beta_{j-1} + y_1 (LQ)_{i-1,j-1} \\
&\quad + (LQ)_{i-1,j} - L_{i-1,0}Q_{0,j} - (LQ)_{i-1,j-1} + L_{i-1,0}Q_{0,j-1} \\
&= \beta_j + c + (LQ)_{i,j-1} - \beta_{j-1} + y_1 (LQ)_{i-1,j-1} \\
&\quad + (LQ)_{i-1,j} - \beta_j - (LQ)_{i-1,j-1} + \beta_{j-1} \\
&= (LQ)_{i,j-1} + y(LQ)_{i-1,j-1} + (LQ)_{i-1,j} + c,
\end{aligned}$$

which is equation (7), as required.

Next, we claim that $Q(n) = T(n) \cdot U(n)$. Let $U = U(n) = [U_{i,j}]_{0 \le i,j < n}$. Note that we have

$$U_{i,j} = \begin{cases} 0 & \text{if } j < i, \\ \binom{j}{i} & \text{if } j \ge i, \end{cases}$$

and also it is easy to see that

$$U_{i,j} = U_{i,j-1} + U_{i-1,j-1}, \quad 1 \le i, j < n. \tag{8}$$

Using similar arguments as the previous part, it suffices to show that

$$\mathrm{R}_0(TU) = \mathrm{R}_0(Q), \qquad \mathrm{C}_0(TU) = \mathrm{C}_0(Q),$$

and

$$(TU)_{i,j} = \begin{cases} (TU)_{i,j-1} + (1+y)(TU)_{i-1,j-1} + c & \text{if } i = 1,\ j \geq 1 \\ (TU)_{i,j-1} + (1+y)(TU)_{i-1,j-1} & \text{if } i > 1,\ j \geq 1. \end{cases} \tag{9}$$

As before, the proof of our claim requires some calculations. On the one hand, we must evaluate the following sum:

$$(TU)_{0,j} = \sum_{k=0}^{n-1} T_{0,k}U_{k,j} = \sum_{k=0}^{n-1} \hat{\beta}_k \binom{j}{k} = \check{\hat{\beta}}_j = \beta_j = Q_{0,j}, \quad \text{(by (2))},$$

which implies that $\mathrm{R}_0(TU) = \mathrm{R}_0(Q)$, as desired. On the other hand, we have

$$(TU)_{i,0} = \sum_{k=0}^{n-1} T_{i,k}U_{k,0} = T_{i,0}U_{0,0} = \hat{\alpha}_i \cdot 1 = \hat{\alpha}_i = Q_{i,0},$$

and so, $\mathrm{C}_0(TU) = \mathrm{C}_0(Q)$.

Finally, we assume that $1 \leq i, j < n$, write $y_1 = 1+y$, and establish equation (9). We distinguish two cases: $i = 1$, $j \geq 1$ and $i > 1$, $j \geq 1$.

If $i = 1$, $j \geq 1$, then using equation (8), we obtain

$$\begin{aligned}
(TU)_{1,j} &= \sum_{k=0}^{n-1} T_{1,k}U_{k,j} = T_{1,0}U_{0,j} + T_{1,1}U_{1,j} + \sum_{k=2}^{n-1} T_{1,k}U_{k,j} \\
&= -\gamma + \alpha_1 + (y_1\gamma + c)j + y_1 \sum_{k=2}^{n-1} T_{0,k-1}(U_{k,j-1} + U_{k-1,j-1}) \\
&= -\gamma + \alpha_1 + (y_1\gamma + c)j + y_1 \sum_{k=2}^{n-1} T_{0,k-1}U_{k,j-1} + y_1 \sum_{k=2}^{n-1} T_{0,k-1}U_{k-1,j-1} \\
&= -\gamma + \alpha_1 + (y_1\gamma + c)j + \sum_{k=2}^{n-1} T_{1,k}U_{k,j-1} + y_1 \sum_{k=1}^{n-2} T_{0,k}U_{k,j-1} \\
&= -\gamma + \alpha_1 + (y_1\gamma + c)j + (TU)_{1,j-1} - T_{1,0}U_{0,j-1} - T_{1,1}U_{1,j-1} \\
&\quad + y_1(TU)_{0,j-1} - (1+y)T_{0,0}U_{0,j-1} - y_1T_{0,n-1}U_{n-1,j-1} \\
&= -\gamma + \alpha_1 + (y_1\gamma + c)j + (TU)_{1,j-1} + \gamma - \alpha_1 - (y_1\gamma + c)(j-1) \\
&\quad + y_1(TU)_{0,j-1} - y_1\gamma - 0 \quad \text{(note that } U_{n-1,j-1} = 0) \\
&= (TU)_{1,j-1} + y_1(TU)_{0,j-1} + c,
\end{aligned}$$

which is the first part of equation (9).

We may therefore assume that $i > 1$, $j \geq 1$. Again, using equation (8), we observe that

$$\begin{aligned}
(TU)_{i,j} &= \sum_{k=0}^{n-1} T_{i,k}U_{k,j} = T_{i,0}U_{0,j} + \sum_{k=1}^{n-1} T_{i,k}U_{k,j} \\
&= T_{i,0} + \sum_{k=1}^{n-1} T_{i,k}(U_{k,j-1} + U_{k-1,j-1}) \quad \text{(note that } U_{0,j} = 1\text{)} \\
&= T_{i,0} + \sum_{k=1}^{n-1} T_{i,k}U_{k,j-1} + \sum_{k=1}^{n-1} T_{i,k}U_{k-1,j-1} \\
&= T_{i,0} + \sum_{k=0}^{n-1} T_{i,k}U_{k,j-1} - T_{i,0} + y_1 \sum_{k=1}^{n-1} T_{i-1,k-1}U_{k-1,j-1} \\
&\quad \text{(note that } U_{0,j-1} = 1 \text{ and } T_{i,k} = (1+y)T_{i-1,k-1}\text{)} \\
&= (TU)_{i,j-1} + (1+y)\sum_{k=0}^{n-2} T_{i-1,k}U_{k,j-1} \\
&= (TU)_{i,j-1} + y_1 \sum_{k=0}^{n-1} T_{i-1,k}U_{k,j-1} - (1+y)T_{i-1,n-1}U_{n-1,j-1} \\
&= (TU)_{i,j-1} + y_1(TU)_{i-1,j-1} \quad \text{(note that } U_{n-1,j-1} = 0\text{)},
\end{aligned}$$

which is the second part of equation (9). The proof of $P(n) = L(n) \cdot T(n) \cdot U(n)$ is now complete. The last assertion is immediate from $\det(L(n)) = \det(U(n)) = 1$. □

## 3. Proof of Theorem 1.4

*Proof of Theorem* 1.4. For convenience, we will let $P = P^{[1,1,1]}_{\alpha,\beta,f}(n)$, $L = L(n)$, $T = [1] \oplus T^{[2]}_{\delta,\epsilon,\gamma}(n-1)$, and $U = U(n)$. To prove the claimed factorization, we compute the $(i,j)$-entry of $LTU$, that is

$$(LTU)_{i,j} = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}T_{r,s}U_{s,j}.$$

A general proof scheme involves showing that

$$\begin{aligned}
\mathrm{R}_0(LTU) &= \mathrm{R}_0(P) = (1,1,\ldots,1), \\
\mathrm{C}_0(LTU) &= \mathrm{C}_0(P) = (1,1,\ldots,1),
\end{aligned}$$

followed by verifying the other entries in the recursion relation:

$$(LTU)_{i,j} = (LTU)_{i,j-1} + (LTU)_{i-1,j-1} + (LTU)_{i-1,j} + ai + bj \tag{10}$$

for $1 \leq i, j \leq n-1$.

Let us do the necessary calculations. First, if $i = 0$, we find that

$$(LTU)_{0,j} = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{0,r}T_{r,s}U_{s,j} = \sum_{s=0}^{n-1} T_{0,s}U_{s,j} = 1 \cdot U_{0,j} = 1,$$

which implies that $\mathrm{R}_0(LTU) = \mathrm{R}_0(P) = (1, 1, \ldots, 1)$. Next, if $j = 0$, we conclude

$$(LTU)_{i,0} = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}T_{r,s}U_{s,0} = \sum_{r=0}^{n-1} L_{i,r}T_{r,0} = L_{i,0} \cdot 1 = 1,$$

showing that $\mathrm{C}_0(LTU) = \mathrm{C}_0(P) = (1, 1, \ldots, 1)$.

Finally, we need to establish equation (10). Assuming now that $1 \le i, j \le n-1$, we have

$$\begin{aligned}
(LTU)_{i,j} &= \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}T_{r,s}U_{s,j} = \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}T_{r,s}U_{s,j} \\
&= 1 + \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}T_{r,s}\left(U_{s-1,j-1} + U_{s,j-1}\right) \ \text{(by (8))} \\
&= 1 + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i,r}T_{r,s}U_{s-1,j-1} + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} \\
&\quad + \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}T_{r,s}U_{s,j-1} - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1} \\
&= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} \left(L_{i-1,r-1} + L_{i-1,r}\right)T_{r,s}U_{s-1,j-1} + (LTU)_{i,j-1} \\
&= \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}T_{r,s}U_{s-1,j-1} + \sum_{s=1}^{n-1} L_{i-1,0}T_{1,s}U_{s-1,j-1} \\
&\quad + \sum_{r=2}^{n-1} L_{i-1,r-1}T_{r,1}U_{0,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s-1,j-1} + (LTU)_{i,j-1} \\
&= 2\sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}T_{r-1,s-1}U_{s-1,j-1} + 2 + ai + bj \\
&\quad + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}\left(U_{s,j} - U_{s,j-1}\right) + (LTU)_{i,j-1} \\
&\quad \text{(by (8) and the structure of } T\text{)} \\
&= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s,j-1} + 2 + ai + bj + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s,j} + (LTU)_{i,j-1} \\
&= (LTU)_{i-1,j-1} - 1 + 2 + ai + bj + (LTU)_{i-1,j} - 1 + (LTU)_{i,j-1} \\
&= (LTU)_{i,j-1} + (LTU)_{i-1,j-1} + (LTU)_{i-1,j} + ai + bj,
\end{aligned}$$

as required. This completes the first part of the theorem.

The determinant formula follows immediately from the established factorization. □

## References


**1.** Bacher R., *Determinants of matrices related to the Pascal triangle*, J. Théor. Nombres Bordeaux **14**(1)(2002), 19–41.

**2.** Cheon G. S., Hwang S. G., Rim S. H. and Song S. Z., *Matrices determined by a linear recurrence relation among entries*, Linear Algebra Appl. **373** (2003), 89–99.

**3.** Edelman A. and Strong G., *Pascal matrices*, Amer. Math. Monthly **111**(3) (2004), 189–197.

**4.** Farris M., Luntzlara N., Miller S. J., Shao L. and Wang M., *Recurrence relations and Benford's law*, Stat. Methods Appl. **30**(3) (2021), 797–817.

**5.** Krattenthaler C., *Evaluations of some determinants of matrices related to the Pascal triangle*, Sém. Lothar. Combin. **47** (2001/02), Art. B47g.

**6.** Moghaddamfar A. R., *Determinants of several matrices associated with Pascal's triangle*, Asian-Eur. J. Math. **3**(1) (2010), 119–131.

**7.** Moghaddamfar A. R. and Pooya S. M. H., *Generalized Pascal triangles and Toeplitz matrices*, Electron. J. Linear Algebra **18** (2009), 564–588.

**8.** Moghaddamfar A. R., Pooya S. M. H., Salehy S. Navid and Salehy S. Nima, *More calculations on determinant evaluations*, Electron. J. Linear Algebra **16** (2007), 19–29.

**9.** Moghaddamfar A. R., Pooya S. M. H., Salehy S. Navid, and Salehy S. Nima, *On the matrices related to the m-arithmetic triangle*, Linear Algebra Appl. **432** (2010), 53–69.

**10.** Moghaddamfar A. R., Salehy S. Navid, and Salehy S. Nima, *The determinants of matrices with recursive entries*, Linear Algebra Appl. **428**(11–12) (2008), 2468–2481.

**11.** Moghaddamfar A. R., Salehy S. Navid, and Salehy S. Nima, *Evaluating some determinants of matrices with recursive entries*, Bull. Korean Math. Soc. **46**(2) (2009), 331–346.

**12.** Tan M., *Matrices associated to biindexed linear recurrence relations*, Ars Combin. **86** (2008), 305–319.

**13.** Sloane N. J. A. et al., The On-Line Encyclopedia of Integer Sequences, 2023. Available at `https://oeis.org`.

**14.** Zakrajšek H. and Petkovšek M., *Pascal-like determinants are recursive*, Adv. Appl. Math. **33**(3) (2004), 431–450.



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