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LACK OF UNIQUENESS FOR AN ELLIPTIC EQUATION WITH NONLINEAR AND NONLOCAL DRIFT POSED ON A TORUS

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ABSTRACT. We study a nonlinear and nonlocal elliptic equation posed on the flat torus. While constant solutions always exist, we show that uniqueness fails in general. Using spectral analysis and the Crandall–Rabinowitz bifurcation theorem, we prove the existence of branches of nonconstant periodic solutions bifurcating from constant states. This result is qualitative and nonconstructive. Using a conceptually different argument, we construct explicit multiple solutions for a specific one-dimensional formulation of our target problem.

1. INTRODUCTION

In this note, we investigate the following elliptic boundary-value problem with nonlinear and nonlocal drift on the torus $\mathbb{T}^d$:

$$(\star) \qquad \nabla \cdot \big(a\nabla u + u\,\Phi * \nabla u\big) = 0 \qquad \text{in } \mathbb{T}^d,$$

where $a > 0$, and Φ is a periodic interaction kernel. This type of boundary-value problem arises in the context of periodic homogenization of parabolic problems that have nonlocal drifts. Depending on the scaling of the potential Φ, problem $(\star)$ appears as a cell problem in the said context [1].

Our motivation for studying $(\star)$ stems from exploring a certain class of stochastically interacting particle systems and their mean-field limits. Concretely, in some applications one considers the evolution of N particles whose motion is biased by pairwise interactions through a potential (or kernel) Φ, which, at the hydrodynamic level, leads to a drift velocity given by a convolution field of the form $\Phi * \nabla u$, where u denotes the particle density. From this perspective, the term $u(\Phi * \nabla u)$ arising in $(\star)$ represents a macroscopic signature of possible aggregation and repulsion originally taking place at microscopic scales.

Somewhat similar nonlocal interaction mechanisms appear across fields, including collective behavior, swarming, granular media, opinion dynamics, and McKean–Vlasov-type models [**2, 3**]. In the specific setting we have in mind, the

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[1]It is worth noting that in the homogenization context, cell problems are mathematical objects typically used to incorporate microscopic information into effective coefficients.

problem ($\star$) arises in the context of multiscale models of sulphation reactions in marble (or in some other porous material), which are important for cultural heritage conservation [**7**]. The spatial distribution of reactive agents is influenced by interactions such as effective attraction/repulsion induced by the material's microstructure or by crowding effects at the pore level.

The solvability question for the boundary-value problem ($\star$) can be answered in a straightforward manner. While constant solutions are always present, we wonder about their uniqueness within a fixed-mean class. In this sense, the bifurcation result below identifies a genuinely nontrivial form of non-uniqueness. This is precisely where this note contributes: we indicate that the lack of uniqueness of solutions is somewhat inherent in this context. Our main result states that, under mild spectral assumptions on the potential Φ, nonconstant periodic solutions bifurcate from constant states while preserving the mean, naturally yielding non-uniqueness of solutions to the target problem. The Crandall–Rabinowitz bifurcation theorem will be the main tool used here to clarify the matter. For a long-time behaviour and bifurcation analysis of aggregation-diffusion systems on the torus, see also the recent related work [**4**].

2. An observation and the statement of our main result

2.1. Observation

For convenience, we let $\Omega := \mathbb{T}^d$ be the flat d-dimensional torus. We consider the equation

$$\nabla \cdot \big(a\nabla u + u\,\Phi * \nabla u\big) = 0 \qquad \text{in } \Omega, \tag{1}$$

where $a > 0$, $\Phi\colon \Omega \to \mathbb{R}$ is an even function $\Phi \in L^2(\Omega)$, and the convolution is taken in the standard sense.

From the modeling perspective, one is often primarily interested in nonnegative interaction kernels. However, the mathematical mechanism leading to non-uniqueness does not rely on the sign of Φ, but rather on the spectral condition imposed below on its Fourier coefficients. We therefore state the result for real-valued even kernels, which slightly extends the class suggested by the original modeling motivation.

It is immediate that any constant function $u \equiv c$ is a solution of (1). Thinking of the uniqueness question, it is interesting to observe that even in the corresponding linear but nonlocal situation, the uniqueness of solutions up to a constant is not guaranteed. To illustrate this point, we first consider a linearized version of (1). Namely, for any given $b \in \mathbb{R}$, we seek periodic solutions to

$$a\Delta u + b\Phi * \Delta u = 0 \qquad \text{in } \Omega. \tag{2}$$

Throughout the paper, we use the Fourier convention on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ given by

$$\widehat{f}(k) = \int_{\mathbb{T}^d} f(y)\mathrm{e}^{-2\pi \mathrm{i} k\cdot y}\,\mathrm{d}y, \qquad f(y) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k)\mathrm{e}^{2\pi \mathrm{i} k\cdot y}.$$

With this normalization, convolution satisfies

$$\widehat{\Phi * f}(k) = \widehat{\Phi}(k)\widehat{f}(k).$$

Note that, since Φ is real-valued and even, its Fourier coefficients are real and even; namely,

$$\widehat{\Phi}(k) \in \mathbb{R}, \qquad \widehat{\Phi}(-k) = \widehat{\Phi}(k), \qquad k \in \mathbb{Z}^d.$$

Rewriting (2) using Fourier series, we get

$$u(y) = \sum_{k\in\mathbb{Z}^d} \widehat{u}(k)\mathrm{e}^{2\pi \mathrm{i} k\cdot y}, \qquad \Phi(y) = \sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(k)\mathrm{e}^{2\pi \mathrm{i} k\cdot y},$$

which yields

$$-(2\pi)^2|k|^2\big(a + b\widehat{\Phi}(k)\big)\widehat{u}(k) = 0.$$

Hence, nonconstant solutions exist if and only if there exists some $k \neq 0$ such that $\widehat{\Phi}(k) = -a/b$. In this linear case, it is easy to obtain sufficient conditions on the kernel that guarantee uniqueness: either via energy estimates, requiring that $\|\Phi\|_{L^1} < a/|b|$, or by imposing that the Fourier coefficients of the kernel satisfy $\widehat{\Phi}(k) \neq -a/b$. If, for instance, $b > 0$, then all kernels with nonnegative Fourier coefficients would guarantee the uniqueness of the solution. In the nonlinear case, however, the situation becomes more complicated, and, as we will see next, for a very wide class of kernels, the uniqueness property can be actively disproved.

2.2. Statement of the main result

Let $s > d/2$. We define the spaces

$$X := \Big\{ v \in H^{s+2}(\Omega) : \int_\Omega v = 0,\ v(y) = v(-y) \Big\}, \tag{3}$$

$$Y := \Big\{ w \in H^{s}(\Omega) : \int_\Omega w = 0,\ w(y) = w(-y) \Big\}. \tag{4}$$

We next state the main result of this note:

Theorem 2.1 (Non-uniqueness of periodic solutions to (1))**.** *Assume that there exists an index $k_0 \in \mathbb{Z}^d \setminus \{0\}$ such that*

$$\widehat{\Phi}(k_0) \neq 0, \qquad \widehat{\Phi}(k) \neq \widehat{\Phi}(k_0) \quad \text{for every } k \in \mathbb{Z}^d \setminus \{\pm k_0\}. \tag{H}$$

Define $\mathcal{F}\colon \mathbb{R}\times X \to Y$ as

$$\mathcal{F}(c, v) := a\Delta v + \nabla\cdot\big((c+v)\,\Phi * \nabla(c+v)\big),$$

and set

$$c_0 := -\frac{a}{\widehat{\Phi}(k_0)}, \qquad v_0(y) := \cos(2\pi k_0\cdot y).$$

Let Z be a closed complement of $\operatorname{span}\{v_0\}$ in X, so that $X = \operatorname{span}\{v_0\} \oplus Z$. Then there exist $\varepsilon > 0$, a neighborhood U of $(c_0, 0)$ in $\mathbb{R}\times X$, and two $\mathcal{C}^1$ maps $\varphi\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$, $\psi\colon (-\varepsilon,\varepsilon) \to Z$, satisfying $\varphi(0) = c_0$ and $\psi(0) = 0$, such that

$$\mathcal{F}^{-1}(0)\cap U = \big\{(\varphi(s),\, sv_0 + s\psi(s)) :\ |s| < \varepsilon\big\} \cup \big\{(c, 0) : (c, 0) \in U\big\}.$$

In particular, a C^1 branch of nonconstant solutions to (1) bifurcates from $(c_0, 0)$.

Remark. It is worth noticing that the non-uniqueness obtained above occurs within a fixed-mean class. Indeed, for $t \neq 0$ sufficiently small,

$$u_t = \varphi(t) + tv_0 + t\psi(t)$$

is a nonconstant solution of (1). Since v_0 and $\psi(t)$ belong to X, the perturbation $tv_0 + t\psi(t)$ has zero mean. Hence u_t and the constant solution $u \equiv \varphi(t)$ have the same mean, and therefore the same mass.

Remark. If the solutions are interpreted as particle densities, then one should require the bifurcation to occur from a positive constant state. This amounts to the additional condition $\widehat{\Phi}(k_0) < 0$. Without it, the bifurcation may occur from a negative constant state, which is mathematically admissible but not consistent with a density interpretation.

Essentially, this tells us that uniqueness of the solution cannot generally be expected. In the next section, we provide a proof of this result.

3. Proof of Theorem 2.1

The main tool we use is Lemma 1.1 of [**6**]; see also [**5**, Theorem 1.7] for a more general version. For the reader's convenience and expositional clarity, we recall here a version of this theorem adapted to our notation.

Theorem 3.1 (Crandall–Rabinowitz)**.** *Let X, Y be Banach spaces, let $I \subset \mathbb{R}$ be an open interval, $V \subset X$ an open neighborhood of 0, and $\mathcal{F} \in C^2(I \times V; Y)$ in the Fréchet sense. Let $\lambda_0 \in I$ and assume:*

(i) $\mathcal{F}(\lambda, 0) = 0$ *for all* $\lambda \in I$*;*

(ii) $\ker D_u\mathcal{F}(\lambda_0, 0) = \operatorname{span}\{x_0\}$ *with* $x_0 \neq 0$*;*

(iii) $\operatorname{codim} \operatorname{Ran}(D_u\mathcal{F}(\lambda_0, 0)) = 1$*;*

(iv) $D_{\lambda u}\mathcal{F}(\lambda_0, 0)[x_0] \notin \operatorname{Ran}(D_u\mathcal{F}(\lambda_0, 0))$.

Then $(\lambda_0, 0)$ is a bifurcation point of $\mathcal{F}(\lambda, u) = 0$. More precisely, if Z is any complement of span x_0 in X, then there exists a neighborhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times X$, a small parameter $\varepsilon > 0$ and two continuous maps $\varphi\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$, $\psi\colon (-\varepsilon, \varepsilon) \to Z$, with $\varphi(0) = \lambda_0, \psi(0) = 0$, such that

$$\mathcal{F}^{-1}(0) \cap U = \{(\varphi(s),\, sx_0 + s\psi(s)) :\ |s| < \varepsilon\} \cup \{(\lambda, 0) : (\lambda, 0) \in U\}.$$

If $D_{xx}\mathcal{F}$ is also continuous, the functions φ, ψ are once continuously differentiable.

To prove the statement of Theorem 2.1, we are going to show that the conditions of the Crandall–Rabinowitz local bifurcation theorem (i.e. Theorem 3.1) are satisfied. Informally, this theorem states that for a family of equations $\mathcal{F}(\lambda, u) = 0$ with a trivial branch $u = 0$, parametrized by λ, if the linearized operator loses invertibility at some $\lambda = \lambda_0$, with a one-dimensional kernel, and the appropriate nondegeneracy condition holds, then there exists a local branch of nontrivial small amplitude solutions that bifurcates from $(\lambda_0, 0)$.

Step 1: Regularity and condition (i).

Recall the Banach spaces X, Y defined in (3). Since $s > d/2$ by hypothesis, the space $H^s(\Omega)$ is a Banach algebra (cf. [**1**, Theorem 4.39]). We recall that the operator under consideration is

$$\mathcal{F}\colon \mathbb{R} \times X \to Y, \qquad \mathcal{F}(c, v) := a\Delta v + \nabla \cdot ((c+v)\Phi * \nabla(c+v)).$$

We first verify that $\mathcal{F}$ is well defined. In fact, since $\Phi \in L^2(\mathbb{T}^d)$ and $|\mathbb{T}^d| = 1$, Hölder's inequality gives

$$\|\Phi\|_{L^1(\mathbb{T}^d)} \le \|\Phi\|_{L^2(\mathbb{T}^d)}.$$

Thus, $\Phi \in L^1(\mathbb{T}^d)$, and Young's convolution inequality may be applied in the following estimates:

(5)
$$\begin{aligned}
\|\Delta v\|_{H^s(\Omega)} &\le C\,\|v\|_{H^{s+2}(\Omega)},\\
\|\Phi * \Delta v\|_{H^s(\Omega)} &\le \|\Phi\|_{L^1(\Omega)}\,\|v\|_{H^{s+2}(\Omega)},\\
\|\nabla \cdot \big(v\,(\Phi * \nabla v)\big)\|_{H^s(\Omega)} &= \|\nabla v \cdot (\,\Phi * \nabla v) + v\,(\Phi * \Delta v)\|_{H^s(\Omega)}\\
&\le C\|\nabla v\|^2_{H^s(\Omega)}\|\Phi\|_{L^1(\Omega)} + C\|v\|_{H^s(\Omega)}\|\Phi\|_{L^1(\Omega)}\|v\|_{H^{s+2}(\Omega)}\\
&\le C\|\Phi\|_{L^1(\Omega)}(\|v\|^2_{H^{s+1}} + \|v\|_{H^s(\Omega)}\|v\|_{H^{s+2}(\Omega)})\\
&\le C\|v\|^2_{H^{s+2}(\Omega)}.
\end{aligned}$$

Consequently, $\mathcal{F}\colon \mathbb{R} \times X \to Y$ is well defined.

Note that condition (i) is trivially satisfied since constants are always solutions of the original equation.

Recall that a mapping $G\colon E \to F$ between two Banach spaces E and F is Fréchet differentiable (cf. e.g. [**9**, Chap. 4.2]) at $x \in E$ if there exists a bounded linear operator $DG(x)\colon E \to F$ such that

$$\lim_{\|h\|_E \to 0} \frac{\|G(x+h) - G(x) - DG(x)[h]\|_F}{\|h\|_E} = 0.$$

We now verify that the mapping $v \mapsto \mathcal{F}(c, v)$ is a polynomial of degree 2 whose coefficients are bounded multilinear maps, hence, it is infinitely Fréchet differentiable. In fact, we can rewrite $\mathcal{F}(c, v)$ as

$$\mathcal{F}(c, v) = a\Delta v + cAv + B(v, v) \qquad \text{for all } (c, v) \in \mathbb{R} \times X,$$

where

$$\begin{aligned}
A\colon H^{s+2}(\Omega) &\to H^s(\Omega) & Av &:= \Phi * \Delta v,\\
B\colon H^{s+2}(\Omega) \times H^{s+2}(\Omega) &\to H^s(\Omega) & B(v, w) &:= \nabla \cdot \big(v(\Phi * \nabla w)\big),
\end{aligned}$$

which are respectively linear and bilinear. Both A and B are bounded operators, as per estimates (5).

Explicitly, for any increment $(\delta c, h) \in \mathbb{R} \times X$, the first Fréchet derivative is

$$D\mathcal{F}(c, v)[\delta c, h] = a\Delta h + \delta c\, Av + c\, Ah + B(h, v) + B(v, h),$$

while the second Fréchet derivative with increments $(\delta c_1, h_1), (\delta c_2, h_2) \in \mathbb{R} \times X$ is the constant bilinear map

$$D^2\mathcal{F}(c,v)[(\delta c_1, h_1), (\delta c_2, h_2)] = \delta c_1\, Ah_2 + \delta c_2\, Ah_1 + B(h_1, h_2) + B(h_2, h_1).$$

For completeness, let $(\delta c, h) \in \mathbb{R} \times X$. Then the explicit computations give

$$\begin{aligned}\mathcal{F}(c+\delta c, v+h) - \mathcal{F}(c,v) = a\Delta h + \delta c\,(\Phi * \Delta v)\\ + \nabla\cdot\big((c+v)\,\Phi * \nabla h\big) + \nabla\cdot\big(h\,\Phi * \nabla v\big)\\ + \delta c\,(\Phi * \Delta h) + \nabla\cdot\big(h\,\Phi * \nabla h\big).\end{aligned}$$

We define the bounded linear operator $D\mathcal{F}(c,v)\colon \mathbb{R} \times H^{s+2}(\Omega) \to H^s(\Omega)$ by

$$\begin{aligned}D\mathcal{F}(c,v)[\delta c, h] := a\Delta h + \delta c\,(\Phi * \Delta v)\\ + \nabla\cdot\big((c+v)\,(\Phi * \nabla h)\big) + \nabla\cdot\big(h\,(\Phi * \nabla v)\big).\end{aligned}$$

Using $s > \frac{d}{2}$, the Banach algebra property of $H^{s+1}(\Omega)$ together with the convolution estimate $\|\Phi * f\|_{H^{s+1}(\Omega)} \le \|\Phi\|_{L^1(\Omega)}\|f\|_{H^{s+1}(\Omega)}$, the remaining terms satisfy

$$\begin{aligned}\|\delta c\,(\Phi * \Delta h)\|_{H^s(\Omega)} &\le C|\delta c|\,\|h\|_{H^{s+2}(\Omega)},\\ \|\nabla\cdot(h\,(\Phi * \nabla h))\|_{H^s(\Omega)} &\le C\|h\|^2_{H^{s+2}(\Omega)}.\end{aligned}$$

Therefore,

$$\frac{\|\mathcal{F}(c+\delta c, v+h) - \mathcal{F}(c,v) - D\mathcal{F}(c,v)[\delta c, h]\|_{H^s(\Omega)}}{\|(\delta c, h)\|_{\mathbb{R}\times H^{s+2}(\Omega)}} \xrightarrow[(\delta c,h)\to 0]{} 0,$$

which proves that $\mathcal{F}$ is Fréchet differentiable.

In particular, the derivative at $v = 0$ is

$$L_c h := D_v\mathcal{F}(c,0)h = a\Delta h + c\,(\Phi * \Delta h).$$

Similarly, the second Fréchet derivative with increments $(\delta c_1, h_1), (\delta c_2, h_2) \in \mathbb{R} \times X$ is a bilinear map, continuous from $(\mathbb{R} \times H^{s+2}(\Omega))^2$ into $H^s(\Omega)$:

$$\begin{aligned}D^2\mathcal{F}(c,v)[(\delta c_1, h_1), (\delta c_2, h_2)] = \delta c_1\,(\Phi * \Delta h_2) + \delta c_2\,(\Phi * \Delta h_1)\\ + \nabla\cdot\big(h_1\,\Phi * \nabla h_2\big) + \nabla\cdot\big(h_2\,\Phi * \nabla h_1\big).\end{aligned}$$

All subsequent derivatives vanish identically; hence $\mathcal{F}$ is infinitely differentiable in the Fréchet sense.

(ii) The kernel of L_{c_0} has dimension 1.

Since L_c is translation-invariant on $\mathbb{T}^d$, it is a Fourier multiplier. In particular, for each $k \in \mathbb{Z}^d$, we have

$$L_c\big(\mathrm{e}^{2\pi \mathrm{i} k\cdot y}\big) = -(2\pi)^2|k|^2\big(a + c\widehat{\Phi}(k)\big)\mathrm{e}^{2\pi \mathrm{i} k\cdot y} := m(k)\mathrm{e}^{2\pi \mathrm{i} k\cdot y}.$$

Hence, $\ker L_c$ is spanned by those modes for which $m(k) = 0$.

For $c_0 = -\frac{a}{\widehat{\Phi}(k_0)}$, assumption (H) implies that $m(\pm k_0) = 0$ and $m(k) \neq 0$ for every $k \neq 0, \pm k_0$.

Since we are restricting ourselves to the even subspace

$$\ker L_{c_0} = \operatorname{span}\{\cos(2\pi k_0 \cdot y)\},$$

(iii) The range of L_{c_0} has codimension 1.

We first characterize the range of L_{c_0}. By definition,

$$\mathrm{Ran}(L_{c_0}) := \{v \in Y : v = L_{c_0} g \text{ for some } g \in X\}.$$

Since L_{c_0} is translation invariant, it acts as a Fourier multiplier. Thus $v \in \mathrm{Ran}(L_{c_0})$ if and only if there exists $g \in X$ such that

$$\widehat{v}(k) = m(k)\widehat{g}(k), \qquad k \in \mathbb{Z}^d. \tag{6}$$

Since both v and g have zero mean, we must have $\widehat{v}(0) = \widehat{g}(0) = 0$.

We intend to prove

$$\mathrm{Ran}(L_{c_0}) = \Big\{v \in Y : \ \widehat{v}(k_0) = \widehat{v}(-k_0) = 0\Big\}. \tag{7}$$

Take $v \in \mathrm{Ran}(L_{c_0})$. If $k = \pm k_0$, then $m(\pm k_0) = 0$. Hence, equation (6) forces the compatibility condition

$$\widehat{v}(\pm k_0) = 0.$$

Conversely, suppose $v \in Y$ satisfies $\widehat{v}(\pm k_0) = 0$. For $k = 0, \pm k_0$, define

$$\widehat{g}(0) = 0, \qquad \widehat{g}(\pm k_0) = 0.$$

For $k \neq 0, \pm k_0$, assumption (H) implies $m(k) \neq 0$, which means that any $\widehat{v}(k)$ determines uniquely

$$\widehat{g}(k) = \frac{\widehat{v}(k)}{m(k)}.$$

We need to prove that $g \in X$. We first claim that there exists $\eta > 0$ such that

$$\left|a + c_0\widehat{\Phi}(k)\right| \geq \eta \qquad \text{for all } k \neq 0, \pm k_0.$$

Indeed, since $\Phi \in L^2(\mathbb{T}^d)$, we have $\widehat{\Phi}(k) \to 0$ as $|k| \to \infty$. Hence

$$a + c_0\widehat{\Phi}(k) \to a > 0 \qquad \text{as } |k| \to \infty.$$

Thus, the above quantity is bounded away from zero for all sufficiently large $|k|$. For the remaining modes, there are only finitely many indices, and none of them vanishes by assumption (H). This proves the claim.

Using this lower bound, together with the fact that for $k \neq 0$ one has

$$(1 + |k|^2)^2 \leq C|k|^4,$$

we obtain

$$\begin{aligned} \|g\|_{H^{s+2}}^2 &= \sum_{k \neq 0, \pm k_0} (1 + |k|^2)^{s+2} \frac{|\widehat{v}(k)|^2}{(2\pi)^4 |k|^4 |a + c_0\widehat{\Phi}(k)|^2} \\ &\leq C \sum_{k \neq 0, \pm k_0} (1 + |k|^2)^s |\widehat{v}(k)|^2 \leq C\|v\|_{H^s}^2. \end{aligned}$$

Therefore $g \in H^{s+2}$. Since v has zero mean and is even, the function g also has zero mean and is even. Hence $g \in X$ and $L_{c_0} g = v$.

This proves (7).

Since Y consists of even functions, the two conditions $\widehat{v}(k_0) = 0$ and $\widehat{v}(-k_0) = 0$ are equivalent. Thus (7) becomes a single linear constraint:

$$\mathrm{Ran}(L_{c_0}) = \Big\{ v \in Y : \ \widehat{v}(k_0) = 0 \Big\}.$$

Equivalently, there exists a nonzero continuous linear functional $\ell \colon Y \to \mathbb{R}$ given by $\ell(v) := \widehat{v}(k_0)$ such that

$$\mathrm{Ran}(L_{c_0}) = \ker \ell.$$

Hence $\mathrm{Ran}(L_{c_0})$ is a closed hyperplane in Y and $\operatorname{codim} \mathrm{Ran}(L_{c_0}) = 1$.

(iv) The transversality condition: $D_c L_c[v_0]\big|_{c=c_0} \notin \mathrm{Ran}(L_{c_0})$.

We compute

$$D_c L_c[v_0] = \Phi * \Delta v_0 = -(2\pi)^2 |k_0|^2 \widehat{\Phi}(k_0) v_0.$$

Since $\widehat{\Phi}(k_0) \neq 0$, the Fourier coefficient of $D_c L_c[v_0]\big|_{c=c_0}$ at k_0 is nonzero. On the other hand, by the range characterization proved above,

$$\mathrm{Ran}(L_{c_0}) = \{ w \in Y : \widehat{w}(k_0) = 0 \},$$

it follows that

$$D_c L_c[v_0]\big|_{c=c_0} \notin \mathrm{Ran}(L_{c_0}).$$

All the hypotheses of Theorem 3.1 are therefore satisfied, and the conclusion follows.

Note that since $\mathcal{F}(c, v) = a\Delta v + cAv + B(v, v)$ is a polynomial map with bounded multilinear coefficients, its second Fréchet derivative is continuous. Therefore, by the last assertion of Theorem 3.1, the bifurcating curves φ and ψ are continuously differentiable.

Remark. Assumption (H) may potentially be relaxed. If the kernel is higher-dimensional, then one may obtain a richer local structure of $\mathcal{F}^{-1}(0)$, possibly containing several curves or higher-dimensional sets of solutions. In this setting, one may expect nontrivial bifurcation under suitable additional nondegeneracy conditions. However, this is not proved here; such a case would require a more involved argument, for instance via a Lyapunov–Schmidt reduction or an equivariant bifurcation analysis.

4. A one-dimensional example with explicit construction of a nontrivial solution

The Crandall–Rabinowitz Theorem does not offer a construction of an explicit nonconstant solution. However, for a particular choice of kernel, one can sometimes write an explicit nontrivial solution to equation (1). This is the aim of this section: we produce such an illustrative example in dimension 1. The approach proposed here does not utilize any linearization of (1), hence it is conceptually different from the procedure motivated by Theorem 3.1.

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus and consider the particular choice of kernel

$$\Phi(x) = 2\cos(2\pi x), \qquad \Phi\colon \mathbb{T} \to \mathbb{R}.$$

Then the Fourier coefficients of Φ satisfy

$$\widehat{\Phi}(\pm 1) = 1, \qquad \widehat{\Phi}(k) = 0 \text{ for } k \neq \pm 1.$$

We seek nonconstant solutions of equation

$$0 = a u_{xx} + (u\Phi * u_x)_x, \tag{8}$$

of the form $u(y) = c + v(y)$, where $c \in \mathbb{R}$ is a constant and v is a periodic, even function with zero mean.

We expand v in its real Fourier cosine series. To distinguish the cosine-series coefficients from the complex Fourier coefficients introduced earlier, we write

$$v(x) = \sum_{n=1}^{\infty} V_n \cos(2\pi n x). \tag{9}$$

Since v is real-valued, even, and has zero mean, these coefficients satisfy

$$V_n = 2\,\widehat{v}(n) = 2\,\widehat{v}(-n), \qquad n \geq 1,$$

and $\widehat{v}(0) = 0$.

Formally differentiating the series gives

$$\begin{aligned} v_x(x) &= -2\pi \sum n\, V_n \sin(2\pi n x), \\ v_{xx}(x) &= -(2\pi)^2 \sum n^2\, V_n \cos(2\pi n x). \end{aligned}$$

Note that the convolution operation corresponds to multiplication in Fourier variables, and furthermore, $\widehat{\Phi}$ is supported on the modes ± 1. Hence, only these modes contribute. Consequently, we can write

$$\begin{aligned} \Phi * v_{xx}(x) &= -(2\pi)^2 \sum n^2\, V_n \widehat{\Phi}(n) \cos(2\pi n x) \\ &= -(2\pi)^2\, V_1 \cos(2\pi x), \\ \Phi * v_x(x) &= -2\pi\, V_1 \sin(2\pi x), \\ v\Phi * v_x(x) &= -2\pi\, V_1 \sum_{n=1}^{\infty} V_n \cos(2\pi n x)\sin(2\pi x) \\ &= -2\pi\, V_1 \sum_{n=1}^{\infty} V_n \frac{\sin((n+1)2\pi x) - \sin((n-1)2\pi x)}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned}(v\Phi * v_x)_x(x) &= -2\pi\, V_1 \sum_{n=1}^{\infty} V_n \pi\left[(n+1)\cos((n+1)2\pi x) - (n-1)\cos((n-1)2\pi x)\right] \\ &= -2\pi^2\, V_1 \left[\sum_{m=2}^{\infty} V_{m-1} m \cos(2\pi m x) - \sum_{m=0}^{\infty} V_{m+1} m \cos(2\pi m x)\right] \\ &= -2\pi^2\, V_1 \sum_{m=1}^{\infty} \left[V_{m-1} - V_{m+1}\right] m \cos(2\pi m x),\end{aligned}$$

where in the last step we used the zero-mean condition $V_0 = 0$.

Using the cosine-series coefficients V_n introduced above, finding a solution to (8) therefore amounts to determining the coefficients V_n through the recurrence relation

$$(a + c\widehat{\Phi}(m))m^2 V_m + \frac{V_1}{2} m\,(V_{m-1} - V_{m+1}) = 0, \qquad m \in \mathbb{N}^+.$$

The case $V_1 = 0$ would correspond to a constant solution. In fact, since $\widehat{\Phi}(m) = 0$ for $m \geq 2$, in such case the coefficient equation reduces to

$$am^2 V_m = 0.$$

As $a > 0$, it follows that $V_m = 0$ for every $m \geq 2$. Together with $V_0 = V_1 = 0$, this shows that $v \equiv 0$, so that $u \equiv c$ is a constant solution.

Since we are searching for a nontrivial solution, we exclude this case. The recurrence scheme becomes

$$\begin{aligned} V_2 &= 2(a+c), \qquad m = 1, \\ V_{m-1} - V_{m+1} &= -\frac{2ma}{V_1} V_m, \quad m \in \mathbb{N},\ m \geq 2. \end{aligned} \tag{10}$$

Given the structure of the recurrence relation (10), we can construct an explicit solution using the modified Bessel functions of the first kind $\{I_m\}_{m\geq 0}$; see [**8**, Chapter 9] for more details, and specifically the item 9.6.10. We recall them here:

$$I_m(z) = \sum_{j=0}^{\infty} \frac{1}{j!\,\Gamma(j+m+1)} \left(\frac{z}{2}\right)^{2j+m}, \qquad z \in \mathbb{C},\ m \in \mathbb{N} \cup \{0\}. \tag{11}$$

Interestingly, they satisfy the recurrence [**8**] (item 9.6.26):

$$I_{m-1}(z) - I_{m+1}(z) = \frac{2m}{z} I_m(z), \qquad m \in \mathbb{N}, m \geq 1,\ z \neq 0. \tag{12}$$

Motivated by this recurrence, we seek coefficients of the form

$$V_m = C I_m(z), \qquad m \in \mathbb{N}^+,$$

for some constant C to be determined later and some $z \in \mathbb{C}$, $z \neq 0$ such that $I_1(z) \neq 0$. This holds true, for example, for all $z \in \mathbb{R}^+$. With this choice, by comparing (10) and (12) we must have

$$-\frac{2ma}{I_1(z)} I_m(z) = C(I_{m-1} - I_{m+1}) = C\frac{2m}{z} I_m(z)$$

which therefore gives

$$C = -\frac{az}{I_1(z)}.$$

Moreover, the constant c in (10) (hence in the structure of the solution we are looking for) can always be chosen such that the recurrence scheme is valid for $m = 1$:

$$c = \frac{CI_2(z)}{2} - a = -a\left(\frac{zI_2(z)}{2I_1(z)} + 1\right).$$

We obtain in this way the explicit family of cosine-series coefficients

$$V_m = -\frac{az}{I_1(z)}\, I_m(z) \quad (m \geq 1), \qquad V_0 = 0, \tag{13}$$

where $z \in \mathbb{R}^+$ is a free parameter.

The only thing that remains to be proven is that the computations for the derivatives of v are justified. We therefore need to show that the series

$$\sum_{m=1}^{\infty} V_m \cos(2\pi m x) \text{ with } V_m = -\frac{az}{I_1(z)}\, I_m(z), \tag{14}$$

converges uniformly on $\mathbb{T}$, and therefore defines a smooth, even, periodic function with zero mean, which is a (classical) solution of (8). Moreover, u is obviously nonconstant.

From the series definition (11) with $z \in \mathbb{R}$ fixed, we have for every $m \geq 0$ the bound

$$|I_m(z)| \leq \sum_{j=0}^{\infty} \frac{1}{j!\,(j+m)!} \left(\frac{z}{2}\right)^{2j+m} \leq \frac{(z/2)^m}{m!} \sum_{j=0}^{\infty} \frac{(z/2)^{2j}}{(j!)^2}.$$

The last series is finite since

$$\sum_{j=0}^{\infty} \frac{(z/2)^{2j}}{(j!)^2} \leq \sum_{j=0}^{\infty} \frac{(z/2)^{2j}}{(j!)} = \mathrm{e}^{(z/2)^2}.$$

Hence, there exists $C_z > 0$ such that

$$|I_m(z)| \leq C_z \frac{(z/2)^m}{m!}.$$

Using (13), this gives

$$|V_m| \leq \frac{az}{|I_1(z)|}\, C_z \frac{(z/2)^m}{m!}.$$

Therefore

$$\sum_{m=1}^{\infty} |V_m| \leq \frac{az}{|I_1(z)|}\, C_z \sum_{m=1}^{\infty} \frac{(z/2)^m}{m!} < \infty,$$

and the cosine series (14) converges absolutely and uniformly on $\mathbb{T}$. Term-wise differentiation is also justified, since for each $\ell \geq 1$,

$$\sum_{m=1}^{\infty} (2\pi m)^{\ell} |V_m| < \infty$$

by the factorial decay of V_m. Hence $v \in C^{\infty}(\mathbb{T})$.

We now verify directly that, for each fixed $z > 0$, the function $u_z := c_z + v_z$ is a classical solution of (8), where

$$v_z(x) := -\frac{az}{I_1(z)} \sum_{m=1}^{\infty} I_m(z) \cos(2\pi m x),$$
$$c_z := -a \left(1 + \frac{z I_2(z)}{2 I_1(z)}\right).$$

Write

$$V_m := -\frac{az}{I_1(z)} I_m(z), \qquad m \geq 1, \qquad V_0 := 0,$$

so that

$$v_z(x) = \sum_{m=1}^{\infty} V_m \cos(2\pi m x).$$

The factorial decay established above justifies all the following termwise differentiations. We have

$$v_z''(x) = -4\pi^2 \sum_{m=1}^{\infty} m^2 V_m \cos(2\pi m x).$$

Moreover, since

$$\widehat{\Phi}(\pm 1) = 1, \qquad \widehat{\Phi}(m) = 0 \quad \text{for } |m| \neq 1,$$

we obtain

$$\Phi * v_z'' = -4\pi^2 \sum_{m=1}^{\infty} m^2 \widehat{\Phi}(m) V_m \cos(2\pi m x).$$

For the nonlinear term,

$$v_z(\Phi * v_z') = -2\pi V_1 \sum_{n=1}^{\infty} V_n \cos(2\pi n x) \sin(2\pi x).$$

Using

$$\cos(2\pi n x) \sin(2\pi x) = \frac{1}{2} \left[\sin\big(2\pi(n+1)x\big) - \sin\big(2\pi(n-1)x\big)\right],$$

and then differentiating and collecting equal modes, we find

$$\big(v_z(\Phi * v_z')\big)' = -2\pi^2 V_1 \sum_{m=1}^{\infty} m\big(V_{m-1} - V_{m+1}\big) \cos(2\pi m x).$$

Consequently,

$$\begin{aligned} &a v_z'' + c_z(\Phi * v_z'') + \big(v_z(\Phi * v_z')\big)' \\ &\quad = -4\pi^2 \sum_{m=1}^{\infty} \left[\big(a + c_z \widehat{\Phi}(m)\big) m^2 V_m + \frac{V_1}{2} m\big(V_{m-1} - V_{m+1}\big)\right] \cos(2\pi m x). \end{aligned}$$

It remains to verify that every coefficient in this series vanishes. The modified Bessel recurrence gives

$$I_{m-1}(z) - I_{m+1}(z) = \frac{2m}{z} I_m(z),$$

and hence

$$V_{m-1} - V_{m+1} = \frac{2m}{z} V_m.$$

For $m \geq 2$, since $V_1 = -az$ and $\widehat{\Phi}(m) = 0$, we have

$$\big(a + c_z \widehat{\Phi}(m)\big) m^2 V_m + \frac{V_1}{2} m \big(V_{m-1} - V_{m+1}\big) = a m^2 V_m - \frac{az}{2} m \frac{2m}{z} V_m = 0.$$

For $m = 1$, using $V_0 = 0$ and $\widehat{\Phi}(1) = 1$, the corresponding coefficient is

$$(a + c_z) V_1 - \frac{V_1 V_2}{2} = V_1 \left(a + c_z - \frac{V_2}{2} \right).$$

By the definitions of c_z and V_2,

$$a + c_z = -\frac{az I_2(z)}{2 I_1(z)} = \frac{V_2}{2},$$

so this coefficient also vanishes. Thus

$$a v_z'' + c_z (\Phi * v_z'') + \big(v_z (\Phi * v_z')\big)' = 0.$$

Since $u_z = c_z + v_z$ and the derivative of c_z vanishes, this is equivalent to

$$a u_z'' + \big(u_z (\Phi * u_z')\big)' = 0.$$

Therefore, u_z is a smooth nonconstant solution of (8).

5. Conclusion

We point out in this note a situation that naturally leads to *loss of uniqueness* for periodic elliptic problems that are similar to $\nabla \cdot \big(a \nabla u + u\, \Phi * \nabla u\big) = 0$ on $\mathbb{T}^d$. Essentially, we show that under a mild nondegeneracy condition on a single Fourier mode of the interaction kernel Φ, a local bifurcation argument yields branches of nonconstant periodic solutions bifurcating from constant states, thereby guaranteeing non-uniqueness near the bifurcation point. In addition, for a specific one-dimensional kernel, we construct an explicit example producing many smooth nonconstant solutions.

From the viewpoint of stochastically interacting particle systems and their corresponding mean-field models, the drift term $u\,(\Phi * \nabla u)$ represents a self-induced transport generated by pairwise interactions. This Note indicates that, even for smooth periodic interaction kernels, the associated stationary (or cell) problem may admit multiple periodic equilibria. This kind of situation is relevant in several applied settings, including multiscale descriptions of sulphation chemical reactions in porous stones or, seen more broadly, including models used in cultural heritage protection, where the dynamics of reactive agents are impacted by effective attraction or repulsion. A natural next step is to identify selection principles or scalings that pick out physically meaningful equilibria among the possibly multiple steady states.

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