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EXISTENCE, UNIQUENESS, AND CONTINUOUS DEPENDENCE ON PARAMETERS OF PERIODIC SOLUTIONS FOR A SECOND-ORDER NEUTRAL ITERATIVE DIFFERENTIAL EQUATION

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ABSTRACT. This paper investigates a class of second-order neutral iterative differential equations. Using a hybrid analytical approach that combines the Krasnoselskii fixed point theorem, the Banach fixed point theorem, and the Green's function technique, we establish new sufficient conditions that guarantee the existence, uniqueness, and continuous dependence on parameters of periodic solutions. To demonstrate the applicability and effectiveness of the theoretical results, two illustrative examples are provided. In conclusion, the findings presented in this work extend and complement several existing contributions in the literature.

1. INTRODUCTION

Iterative functional differential equations, which involve both derivatives and iterates of the unknown function, have attracted considerable attention in recent years. This interest is largely motivated by their promising applications and their ability to accurately describe a wide range of phenomena. Despite this growing interest, the number of publications devoted to this topic remains relatively limited. This is mainly due to the fact that the study of iterative differential equations is still a challenging task, requiring more delicate and sophisticated analysis than that needed for classical functional differential equations. Indeed, many researchers have encountered, and continue to encounter, significant difficulties in this area. One of the principal obstacles lies in the lack of sufficient theoretical frameworks and effective analytical methods. Moreover, the presence of iterative terms introduces additional complexities that further complicate the analysis. For these reasons, only a limited number of authors have addressed iterative differential equations. Nevertheless, over the past two decades, substantial progress has been made in the study of such equations, particularly for first- and second-order cases (see [**1**]–[**4**], [**6**], and [**8**]–[**19**], [**21**], [**23**]). However, a careful examination

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of the existing literature reveals a significant research gap: the study of periodic solutions for second-order neutral differential equations involving iterative terms remains almost completely unexplored. Most existing works focus on boundary value problems (non-periodic) or non-neutral equations.

In [**11**], Kaufmann investigated the existence and uniqueness of solutions for the following second-order iterative differential equation with nested states ($x(x(t))$:

$$x''(t) = f\left(t, x(t), x^{[2]}(t)\right), \quad a \leq t \leq b,$$

where $x^{[2]}(t) = x(x(t))$, and the solutions satisfy one of the boundary conditions $x(a) = a$, $x(b) = b$ or $x(a) = b$, $x(b) = a$.

In [**9**], the authors established the existence of positive solutions for the following second-order iterative differential equation subject to integral boundary conditions:

$$\begin{aligned} &x''(t) + f\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right) = 0, \quad 0 < t < b, \\ &x(0) = 0, \quad \alpha \int_0^{\eta} x(s)\,\mathrm{d}s = x(b) \text{ with } \eta \in (0,b), \quad \alpha \in \mathbb{R}^*, \end{aligned}$$

where the iterates are defined by

$$x^{[0]}(t) = t,\ x^{[1]}(t) = x(t),\ x^{[2]}(t) = x(x(t)),\ \ldots,\ x^{[n]}(t) = x^{[n-1]}(x(t)).$$

Furthermore, in [**8**], the following second-order iterative differential equation with two-point boundary conditions:

$$\begin{aligned} &x''(t) = h\left(t, x(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right), \quad -b \leq t \leq b, \\ &x(-b) = \eta_1, \quad x(b) = \eta_2, \qquad \eta_1, \eta_2 \in [-b, b], \end{aligned}$$

was investigated using the Banach and Schauder fixed point theorems. For more on iterative differential equations, we refer the reader to [**4**], [**5**], [**7**], [**10**], [**12**], and the references therein.

To the best of our knowledge, there exist very few results concerning periodic solutions of neutral differential equations involving iterative terms. In fact, it appears that only a limited number of works have addressed such equations; in particular, we may cite [**12**].

Motivated by the above discussion and to fill this identified research gap, the main objective of the present paper is to investigate the following class of second-order iterative neutral differential equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + p(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t) + q(t)x(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[h\left(t, x(t), x^{[2]}(t)\, x(t-\tau(t))\right)\right], \tag{1}$$

where $p, q \in \mathcal{C}(\mathbb{R}, (0,\infty))$, $\tau \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ are T-periodic functions and satisfied $\int_0^T p(s)\,\mathrm{d}s > 0$, $\int_0^T q(s)\,\mathrm{d}s > 0$, $\tau(t) \geq \tau^* > 0$, $x^{[2]}(t) = x(x(t))$ and $h \in \mathcal{C}^2\left([0,T] \times \mathbb{R}^2, (0,\infty)\right)$ is a T-periodic function with respect to the first argument and satisfied the following Lipschitz condition:

$$\exists \ell_1, \ell_2 > 0 : |h(t, x_1, y_1) - h(t, x_2, y_2)| \leq \ell_1 |x_1 - x_2| + \ell_2 |y_1 - y_2|. \tag{2}$$

The main objective of this paper is to establish verifiable criteria ensuring the existence, uniqueness, and stability of periodic solutions to equation (1). The approach adopted herein is based on transforming the studied equation into an equivalent integral equation, followed by the application of Krasnoselskii's and Banach's fixed point theorems, together with the use of certain properties of the associated Green's function. To overcome the difficulties arising from the presence of iterative terms, our strategy consists in selecting an appropriate Banach space along with a suitable subset. Indeed, the proper choice of these functional settings constitutes the cornerstone of our method. Several considerations guide this choice, including simplicity, consistency with known biological or physical phenomena, the ability to control the iterative terms, and the fulfillment of the required mathematical conditions. Moreover, this framework facilitates the effective application of the selected fixed point theorems.

The novelty and specific contributions of this work can be summarized as follows:

(i) New sufficient conditions are established for the existence, uniqueness, and continuous dependence on parameters of periodic solutions, the latter being obtained for this class of equations for the first time.

(ii) Compared with closely related works [**8**], [**9**] and [**11**], which deal with boundary value problems without neutral structure or periodic conditions, our results address periodic solutions, neutral derivatives, and continuous dependence on parameters simultaneously.

(iii) The hybrid approach combining Krasnoselskii's fixed point theorem, Banach's fixed point theorem, and Green's function technique is tailored to overcome the difficulties arising from the iterative terms.

Recent developments in iterative boundary value problems [**1**], eigenvalues for iterative systems of higher order fractional boundary value problems [**2**], new contraction methods for fractional inclusions without using the selection property [**20**], as well as fractional neutral functional q-differential equations with applications to particle dynamics [**22**], further motivate our study and confirm the timeliness of this research direction.

The remainder of this paper is organized as follows. In Section 2, we present the necessary notations, assumptions, and preliminary results that will be used throughout the paper. Section 3 is devoted to the study of the existence, uniqueness, and continuous dependence of periodic solutions to equation (1). In Section 4, two illustrative examples are provided to demonstrate the applicability of the obtained results. Finally, a brief conclusion is given in the last section.

2. Preliminaries

In this section, we introduce some notations, assumptions, and preliminary results that are essential for the proofs of our main results.

Let

$$P_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}),\ x(t+T) = x(t),\ T > 0\}.$$

Denote the space of continuous T-periodic functions, equipped with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

Then $(P_T, \|.\|)$ forms a Banach space.

Moreover, for given constants ℓ_3, $\ell_4 > 0$, define

$$\mathbb{M} = \{x \in P_T : \|x\| \le \ell_3,\ |x(t_2) - x(t_1)| \le \ell_4 |t_2 - t_1|,\ \forall t, t_1, t_2 \in [0,T]\},$$

The set $\mathbb{M}$ is a convex and compact subset of P_T.

It is worth noting that, for al x_1, x_2 in $\mathbb{M}$, the following estimate holds:

$$\begin{aligned}
&\left|x_1(s-\tau(t))\, x_1^{[2]}(t) - x_2(s-\tau(t))\, x_2^{[2]}(t)\right| \\
&\le |x_1(s-\tau(t))| \left|x_1^{[2]}(t) - x_2^{[2]}(t)\right| + \left|x_2^{[2]}(t)\right| |x_1(s-\tau(t)) - x_2(s-\tau(t))| \\
&\le |x_1(s-\tau(t))|\, |x_1(x_1(t)) - x_1(x_2(t))| + |x_1(s-\tau)|\, |x_1(x_2(t)) - x_2(x_2(t))| \\
&\quad + \left|x_2^{[2]}(t)\right| |x_1(s-\tau(t)) - x_2(s-\tau(t))|.
\end{aligned}$$

Consequently, we obtain the following estimate, which will be used in the sequel:

$$\left|x_1(s-\tau(t))\, x_1^{[2]}(t) - x_2(s-\tau(t))\, x_2^{[2]}(t)\right| \le \ell_3(\ell_4+2)|x_1(t) - x_2(t)|. \tag{3}$$

For convenience, we introduce the following notations:

$$\begin{aligned}
&\rho_1 = \max_{t\in[0,T]} |a(t)|, && \rho_2 = \max_{t\in[0,T]} |a'(t)|, \\
&\rho_3 = \max_{t\in[0,T]} |b(t)|, && \rho_4 = \max_{t\in[0,T]} |p(t)|, \\
&\delta_0 = \frac{1}{\exp\left(\int_0^T b(v)\,\mathrm{d}v\right) - 1}, && \delta_1 = \frac{1}{\exp\left(\int_0^T a(v)\,\mathrm{d}v\right) - 1}, \\
&A = \int_0^T p(u)\,\mathrm{d}u, && B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(q(u))\,\mathrm{d}u\right) \\
&M_1 = \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right), && M_2 = \frac{1}{2}\left(A + \sqrt{A^2 - 4B}\right), \\
&\underline{g} = \frac{T}{\left(\mathrm{e}^{M_2} - 1\right)^2}, && \overline{g} = \frac{T\exp A}{\left(\mathrm{e}^{M_1} - 1\right)^2}, \\
&E_1(t,s) = \frac{\exp\left(\int_t^s b(v)\,\mathrm{d}v\right)}{\delta_0}, && \kappa_1 = \frac{\exp\left(\int_0^T b(v)\,\mathrm{d}v\right)}{\delta_0} \\
&h_0 = \max_{t\in[0,T]} |h(t,0,0)|, && \mu_1 = \rho_1^2 + \rho_2, \\
&\mu_2 = h_0 + (\ell_1 + \ell_2)\ell_3, && \mu_3 = \ell_1\ell_4 + \ell_2\ell_3\ell_4(2\ell_4 + 1)
\end{aligned}$$

In addition, we impose the following hypotheses:

$$\Lambda_1 = \mu_2(1 + T(\mu_1\overline{g} + \rho_4\kappa_1)) \le \ell_3, \tag{4}$$

$$\Lambda_2 = \mu_1\mu_2(2\overline{g} + \rho) + \kappa_1\mu_2\rho_4(T\rho_3 + 2) + \mu_3 \le \ell_4, \tag{5}$$

and

$$\Lambda_3 = \kappa_1\rho_4\left(\ell_1 + \ell_2\ell_3\left(\ell_4 + 2\right)\right) < 1. \tag{6}$$

Lemma 2.1 ([**19**]). *Let*

$$R_1 = \max_{t\in[0,T]}\left|\frac{1}{\exp\left(A\right)-1}\int_t^{t+T}\exp\left(\int_t^s p\left(u\right)\mathrm{d}u\right)q(s)\mathrm{d}s\right|,$$

and

$$Q_1 = \left(1+\exp\left(A\right)\right)^2 R_1^2.$$

If

$$\frac{R_1\left[\exp\left(A\right)-1\right]}{Q_1 T} \geq 1, \tag{7}$$

then there are two T-periodic functions $a\in\mathcal{C}^1\left(\left[0,T\right],\mathbb{R}\right)$ and $b\in\mathcal{C}\left(\left[0,T\right],\mathbb{R}\right)$ such that

$$b\left(t\right)>0, \quad \int_0^T a\left(u\right)\mathrm{d}u>0, \quad \frac{\mathrm{d}}{\mathrm{d}t}b(t)+a(t)b(t)=q(t),$$

and

$$a\left(t\right)+b(t)=p(t),$$

for all $t\in\mathbb{R}$.

Moreover, if $k\in P_T$, then the equation

$$x''(t)+p(t)x'(t)+q(t)x(t)=k\left(t\right),$$

admits a T-periodic solution, which can be expressed as

$$x(t)=\int_t^{t+T}G(t,s)k(s)\mathrm{d}s,$$

where the associated Green's function $G(t,s)$ is given by

$$\begin{aligned}G(t,s) = {} & \frac{1}{\delta_0\delta_1}\int_t^s \exp\left\{\int_t^u b\left(v\right)\mathrm{d}v+\int_u^s a\left(v\right)\mathrm{d}v\right\}\mathrm{d}u\\ & +\frac{1}{\delta_0\delta_1}\int_s^{t+T}\exp\left\{\int_t^u b\left(v\right)\mathrm{d}v+\int_u^{s+T}a\left(v\right)\mathrm{d}v\right\}\mathrm{d}u.\end{aligned}$$

Corollary 2.2 ([**23**]). *Let $t,s\in[0,T]$. Then the Green's function $G(t,s)$ introduced in Lemma* 2.1 *satisfies the following properties:*

$$\begin{aligned}&G(t,t+T)=G(t,t), \quad G(t+T,s+T)=G(t,s),\\ &\frac{\partial}{\partial s}G(t,s)=a(s)G(t,s)-E_1\left(t,s\right),\\ &\frac{\partial}{\partial t}G(t,s)=-b(t)G(t,s)+E_2\left(t,s\right).\end{aligned}$$

Moreover, If $A^2\geq 4B$, then

$$0<\underline{g}\leq G(t,s)\leq\overline{g}, \ \left|E_1\left(t,s\right)\right|\leq\kappa_1. \tag{8}$$

Corollary 2.3 ([**3**])**.** *Let $t, s \in [0,T]$. Then the Green's function $G(t,s)$ introduced in Lemma* 2.1 *satisfies*

$$\frac{\partial^2}{\partial s^2}G(t,s) = \left(a^2(s) + a'(s)\right) G(t,s) - p(s)E_1(t,s). \tag{9}$$

Lemma 2.4 ([**4**])**.** *Let $t_1, t_2 \in [0,T]$. Then the following estimate holds for the Green function $G(t,s)$:*

$$\int_{t_1}^{t_1+T} |G_2(t_2,s) - G_2(t_1,s)|\,\mathrm{d}s \leq \rho\,|t_2 - t_1|, \tag{10}$$

where

$$\rho = T\mathrm{e}^{2M_2}\delta_0\delta_1\left[T\left(2\mathrm{e}^{2M_2}+1\right)\frac{\rho_3}{\delta_0} + \mathrm{e}^{M_2} + 1\right].$$

3. Existence of periodic solutions

In this section, we present results concerning the existence, uniqueness, and stability of periodic solutions to equation (1).

To begin, we establish a useful equivalence between equation (1) and a corresponding integral equation.

Lemma 3.1. *Assume that condition* (7) *is satisfied. Let $x \in P_T \cap \mathcal{C}^2(\mathbb{R},\mathbb{R})$. Then the following assertions are equivalent:*

i) *x is a solution of equation* (1)

ii) *x is a solution of the integral equation*

$$\begin{aligned} x(t) &= h\left(t, x(t), x^{[2]}(t)\,x(t-\tau(t))\right) \\ &\quad + \int_t^{t+T} \left(a^2(s) + a'(s)\right) h\left(s, x(s), x^{[2]}(s)\,x(s-\tau(s))\right) G(t,s)\,\mathrm{d}s \\ &\quad - \int_t^{t+T} p(s)\,h\left(s, x(s), x^{[2]}(s)\,x(s-\tau(s))\right) E_1(t,s)\,\mathrm{d}s. \end{aligned} \tag{11}$$

Proof. Let $x \in P_T \cap \mathcal{C}^2(\mathbb{R},\mathbb{R})$. Following the approach used in the proof of Lemma 2.1, we can write

$$\begin{aligned} x(t) &= \int_t^{t+T} \left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2} h\left(s, x(s), x^{[2]}(s)\,x(s-\tau(s))\right)\right\} G(t,s)\,\mathrm{d}s \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}s} h\left(s, x(s), x^{[2]}(s)\,x(s-\tau(s))\right)\right]_t^{t+T} [G(t,s)]_t^{t+T} \\ &\quad - \int_t^{t+T} \left\{\frac{\mathrm{d}}{\mathrm{d}s} h\left(s, x(s), x^{[2]}(s)\,x(s-\tau(s))\right)\right\} \left\{\frac{\mathrm{d}}{\mathrm{d}s} G(t,s)\right\} \mathrm{d}s. \end{aligned}$$

From Corollary 2.2, it follows that

$$\left[\frac{\mathrm{d}}{\mathrm{d}s} h\left(s, x(s), x^{[2]}(s)\,x(s-\tau(s))\right)\right]_t^{t+T} [G(t,s)]_t^{t+T} = 0,$$

and

$$\left[-h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right)\left\{\frac{d}{\mathrm{d}s}G(t,s)\right\}\right]_t^{t+T}$$
$$= h\left(t, x(t), x^{[2]}(t)\, x(t-\tau(s))\right).$$

Using equation (9), we then have

$$-\int_t^{t+T}\left\{\frac{\mathrm{d}}{\mathrm{d}s}h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right)\right\}\left\{\frac{\mathrm{d}}{\mathrm{d}s}G(t,s)\right\}\mathrm{d}s$$
$$= \left[-h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right)\left\{\frac{\mathrm{d}}{\mathrm{d}s}G(t,s)\right\}\right]_t^{t+T}$$
$$+\int_t^{t+T} h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right)\left\{\frac{\mathrm{d}^2}{\mathrm{d}s}G(t,s)\right\}$$
$$= h\left(t, x(t), x^{[2]}(t)\, x(t-\tau(t))\right)$$
$$+\int_t^{t+T}\left(a^2(s)+a'(s)\right)h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right)G(t,s)$$
$$-\int_t^{t+T} p(s)h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right)E_1(t,s).$$

Conversely, if x satisfies the integral equation (11), differentiation with respect to t immediately recovers equation (11), completing the proof. □

In order to apply the Krasnoselskii fixed point theorem for a sum of two operators, we first construct an operator L that can be expressed as the sum of two operators L_1 and L_2. Let $L_1, L_2 \colon \mathbb{M} \longrightarrow P_T$ be defined by

(12)
$$(L_1x)(t) = \int_t^{t+T}\left(a^2(s)+a'(s)\right)h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right)G(t,s)\,\mathrm{d}s,$$

and

(13)
$$(L_2x)(t) = h\left(t, x(t), x^{[2]}(t)\, x(t-\tau(t))\right) - \int_t^{t+T} p(s)\, h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right)E_1(t,s)\,\mathrm{d}s.$$

Owing to the T-periodicity of the functions involved, we have $(L_1x)(t+T) = (L_1\varphi)(t)$ and $(L_2x)(t+T) = (L_2\varphi)(t)$ for all $x \in \mathbb{M}$. Moreover, by Lemma 3.1, the existence of solutions to equation (1) is equivalent to the existence of fixed points of the operator $L = L_1 + L_2$.

Theorem 3.2. *Let $\tau \in \mathbb{M}$, and assume that condition (4)–(7) hold. Then equation (1) admits at least one T-periodic solution $x \in \mathbb{M}$.*

Proof. The proof proceeds in three steps.

Step 1: We show that L_1 is a compact operator.

Let $x_1,\, x_2 \in \mathbb{M}$. Using conditions (2) and (8), we have

$$\begin{aligned}
&|(L_1x_1)(t) - (L_1x_2)(t)| \\
&\leq \overline{g}\left(\rho_1^2 + \rho_2\right) \int_t^{t+T} \ell_1 |x_1(s) - x_2(s)| \, \mathrm{d}s \\
&\quad + \overline{g}\left(\rho_1^2 + \rho_2\right) \int_t^{t+T} \ell_2 \left| x_1^{[2]}(s)\, x_1(t - \tau(s)) - x_2^{[2]}(s)\, x_2(t - \tau(s)) \right| \mathrm{d}s.
\end{aligned}$$

Applying estimate (3), we obtain

$$\begin{aligned}
\sup_{t\in[0,T]} |(L_1x_1)(t) - (L_1x_2)(t)| &= \|L_1x_1 - L_1x_2\| \\
&\leq T\mu_1\overline{g}\left(\ell_1 + \ell_2\ell_3\left(\ell_4 + 2\right)\right)\|x_1 - x_2\|,
\end{aligned}$$

which shows that L_1 is Lipschitz continuous, and hence continuous. By the Arzelà–Ascoli theorem, the set $\mathbb{M}$ is compact in P_T, implying that L_1 is a compact operator.

Step 2: We show that $L_1x_1 + L_2x_2 \in \mathbb{M}$.
For $t \in [0,T]$ and $x_1,\, x_2 \in \mathbb{M}$. We have

$$\begin{aligned}
&|(L_1x_1)(t) + (L_2x_2)(t)| \\
&\leq \int_t^{t+T} \left(a^2(s) + |a'(s)|\right) h\left(s, x_1(s), x_1^{[2]}(s)\, x_1(t - \tau(s))\right) G(t,s)\, \mathrm{d}s \\
&\quad + \int_t^{t+T} |p(s)| \left| h\left(s, x_2(s), x_2^{[2]}(s)\, x_2(t-\tau)\right)\right| |E_1(t,s)|\, \mathrm{d}s \\
&\quad + \left| h\left(t, x_2(t), x_2^{[2]}(t)\, x_2(t - \tau(t))\right)\right|.
\end{aligned}$$

Using (2), we set

$$h\left(t, x_2(t), x_2^{[2]}(t)\, x_2(t - \tau(t))\right) \leq \mu_2, \tag{14}$$

and, together with (4), (8) and (14), we get

$$\begin{aligned}
\sup_{t\in[0,T]} |(L_1x_1)(t) + (L_2x_2)(t)| &= \|L_1x_1 + L_2x_2\| \\
&\leq \mu_2\left(1 + T\left(\mu_1\overline{g} + \rho_4\kappa_1\right)\right) \\
&\leq \ell_3.
\end{aligned} \tag{15}$$

Next, for $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$, we have

$$\int_{t_1}^{t_1+T} |E(t_2,s) - E(t_1,s)|\, \mathrm{d}s \leq T\rho_3\kappa_1 |t_2 - t_1|, \tag{16}$$

and

$$
\begin{aligned}
&|(L_1x_1)(t_2)+(L_1x_2)(t_2)-(L_1x_1)(t_1)+(L_1x_2)(t_1)|\\
&\leq \int_{t_2}^{t_1}\left(a^2(s)+|a'(s)|\right)\left|h\left(s,x_1(s),x_1^{[2]}(s)\,x_1(t-\tau(s))\right)\right|G(t_2,s)\,\mathrm{d}s\\
&\quad+\int_{t_1}^{t_1+T}\left(a^2(s)+|a'(s)|\right)\left|h\left(s,x_1(s),x_1^{[2]}(s)\,x_1(t-\tau(s))\right)\right|\\
&\qquad\times|G(t_2,s)-G(t_1,s)|\,\mathrm{d}s\\
&\quad+\int_{t_1+T}^{t_2+T}\left(a^2(s)+|a'(s)|\right)\left|h\left(s,x_1(s),x_1^{[2]}(s)\,x_1(t-\tau(s))\right)\right|G(t_2,s)\,\mathrm{d}s\\
&\quad+\int_{t_2}^{t_1}|p(s)|\left|h\left(s,x_2(s),x_2^{[2]}(s)\,x_2(t-\tau(s))\right)\right||E_1(t_2,s)|\,\mathrm{d}s\\
&\quad+\int_{t_1}^{t_1+T}|p(s)|\left|h\left(s,x_2(s),x_2^{[2]}(s)\,x_2(t-\tau(s))\right)\right||E_1(t_2,s)-E_1(t_1,s)|\,\mathrm{d}s\\
&\quad+\int_{t_1+T}^{t_2+T}|p(s)|\left|h\left(s,x_2(s),x_2^{[2]}(s)\,x_2(t-\tau(s))\right)\right||E_1(t_2,s)|\,\mathrm{d}s\\
&\quad+\left|h\left(t_2,x_2(t_2),x_2^{[2]}(t_2)\,x_2(t_2-\tau(t))\right)-h\left(t_1,x_2(t_1),x_2^{[2]}(t_1)\,x_2(t_1-\tau(t))\right)\right|.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\left|h\left(t_2,x_2(t_2),x_2^{[2]}(t_2)\,x_2(t_2-\tau(t_2))\right)-h\left(t_1,x_2(t_1),x_2^{[2]}(t_1)\,x_2(t_1-\tau(t_1))\right)\right|\\
&\leq \ell_1|x_2(t_2)-x_2(t_1)|+\ell_2\left|x_2^{[2]}(t_2)\,x_2(t_2-\tau(t_2))-x_2^{[2]}(t_1)\,x_2(t_1-\tau(t_1))\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|x_2^{[2]}(t_2)\,x_2(t_2-\tau(t_2))-x_2^{[2]}(t_1)\,x_2(t_1-\tau(t_1))\right|\\
&\leq x_2(t_2-\tau(t_2))\left|x_2^{[2]}(t_2)-x_2^{[2]}(t_1)\right|\\
&\quad+x_2^{[2]}(t_1)\,|x_2(t_2-\tau(t_2))-x_2(t_1-\tau(t_1))|\\
&\leq \ell_3\ell_4|x_2(t_2)-x_2(t_1)|+\ell_3\ell_4|t_2-t_1+\tau(t_1)-\tau(t_2)|\\
&\leq \ell_3\ell_4^2|t_2-t_1|+\ell_3\ell_4|t_2-t_1|+\ell_3\ell_4^2|t_2-t_1|\\
&\leq \ell_3\ell_4(2\ell_4+1)|t_2-t_1|.
\end{aligned}
$$

From which we infer that

$$
\begin{aligned}
&\left|h\left(t_2,x_2(t_2),x_2^{[2]}(t_2)\,x_2(t_2-\tau)\right)-h\left(t_1,x_2(t_1),x_2^{[2]}(t_1)\,x_2(t_1-\tau)\right)\right|\\
&\leq \ell_1|x_2(t_2)-x_2(t_1)|+\ell_2\left|x_2^{[2]}(t_2)\,x_2(t_2-\tau)-x_2^{[2]}(t_1)\,x_2(t_1-\tau)\right|\\
&\leq \ell_1\ell_4+\ell_2\ell_3\ell_4(2\ell_4+1)|t_2-t_1|.
\end{aligned}
\tag{17}
$$

According to (5), (8), (14), (16) and (17), we have

$$\begin{aligned} &|(L_1x_1)(t_2)+(L_1x_2)(t_2)-(L_1x_1)(t_1)+(L_1x_2)(t_1)| \\ &\leq (\mu_1\mu_2(2\overline{g}+\rho)+\kappa_1\mu_2\rho_4(T\rho_3+2)+\mu_3)|t_2-t_1| \\ &\leq \ell_4|t_2-t_1|. \end{aligned} \tag{18}$$

Step 3: We show that L_2 is a contraction.

Let x_1, $x_2 \in \mathbb{M}$. In view of (3), (8) and (14), we obtain

$$\begin{aligned} &|(L_2x_1)(t)-(L_2x_2)(t)| \\ &\leq \kappa_1\rho_4\int_t^{t+T}\left|h\left(s,x_1(s),x_1^{[2]}(s)\,x_1(s-\tau(s))\right)-h\left(s,x_2(s),x_2^{[2]}(s-\tau(s))\right)\right|\mathrm{d}s \\ &\leq \kappa_1\rho_4T(\ell_1+\ell_2\ell_3(\ell_4+2))|x_1-x_2|. \end{aligned}$$

We conclude by (6) that L_2 is a contraction.

Finally, as a consequence of the previous three steps, all conditions of the Krasnoselskii fixed point theorem are satisfied, so operator L_1+L_2 has at least one fixed point in $\mathbb{M}$ which is a periodic solution of equation (1). □

Now, we will give the sufficient conditions that allow us to establish the existence of a unique periodic solution of equation (1).

Theorem 3.3. *Assume that $\tau \in \mathbb{M}$ and conditions (4)–(6) hold. If*

$$\Lambda_4 = T(\ell_1+\ell_2\ell_3(\ell_4+2))(\overline{g}\mu_1+\kappa_1\rho_4) < 1, \tag{19}$$

then equation (1) admits a unique periodic solution $x \in \mathbb{M}$.

Proof. By using the same technique as that in the proof of the first and third steps of the Theorem 3.2, we can establish the following estimate:

$$\begin{aligned} &\|(L_1+L_2)(x_1)-(L_1+L_2)(x_2)\| \\ &\leq T(\ell_1+\ell_2\ell_3(\ell_4+2))(\overline{g}\mu_1+\kappa_1\rho_4)\|x_1-x_2\| \end{aligned}$$

for all x_1, $x_2 \in \mathbb{M}$.

According to condition (19) we deduce that L_1+L_2 is a contraction. By virtue of the Banach fixed point theorem L_1+L_2 admits a unique fixed point in $\mathbb{M}$. Therefore, equation (1) owns a unique solution in $\mathbb{M}$. □

The next important question is whether the unique periodic solution of equation (1) depends continuously on the parameters. This property is particularly significant in applications, since small perturbations in model parameters or measured data should produce only small changes in the predicted periodic solution. We establish this continuous dependence in the following theorem.

Theorem 3.4. *The unique solution obtained in Theorem* 3.3 *depends continuously on the function h.*

Proof. Let x be the unique solution of equation (1) and $\widetilde{x}$ be a solution of equation (1) with a small perturbation in the function h. So x and $\widetilde{x}$ satisfying the following integral equations:

$$\begin{aligned} x(t) &= h\left(t, x(t), x^{[2]}(t)\, x(t-\tau(t))\right) \\ &\quad + \int_t^{t+T} \left(a^2(s) + a'(s)\right) h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right) G(t,s)\, \mathrm{d}s \\ &\quad - \int_t^{t+T} p(s)\, h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right) E_1(t,s)\, \mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} \widetilde{x}(t) &= \widetilde{h}\left(t, \widetilde{x}(t), \widetilde{x}^{[2]}(t)\, \widetilde{x}(t-\tau(t))\right) \\ &\quad + \int_t^{t+T} \left(a^2(s) + a'(s)\right) \widetilde{h}\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s)\, \widetilde{x}(s-\tau(s))\right) G(t,s)\, \mathrm{d}s \\ &\quad - \int_t^{t+T} p(s)\, \widetilde{h}\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s)\, \widetilde{x}(s-\tau(s))\right) E_1(t,s)\, \mathrm{d}s, \end{aligned}$$

where $\widetilde{h}$ is the perturbed function.

Estimating the difference between $x(t)$ and $\widetilde{x}(t)$, we have

$$\begin{aligned} &|x(t) - \widetilde{x}(t)| \\ &\le \left|h\left(t, x(t), x^{[2]}(t)\, x(t-\tau(t))\right) - \widetilde{h}\left(t, \widetilde{x}(t), \widetilde{x}^{[2]}(t)\, \widetilde{x}(t-\tau(t))\right)\right| \\ &\quad + (\mu_1 \overline{g} + \kappa_1 \rho_4) \int_t^{t+T} \Big|h\left(s, x(s), x^{[2]}(s)\, x(s-\tau(s))\right) \\ &\qquad\qquad - \widetilde{h}\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s)\, \widetilde{x}(s-\tau(s))\right)\Big| \mathrm{d}s. \end{aligned}$$

Since

$$\begin{aligned} &\left|h\left(t, x(t), x^{[2]}(t)\, x(t-\tau(t))\right) - \widetilde{h}\left(t, \widetilde{x}(t), \widetilde{x}^{[2]}(t)\, \widetilde{x}(t-\tau(t))\right)\right| \\ &\le \left|h\left(t, x(t), x^{[2]}(t)\, x(t-\tau(t))\right) - h\left(t, \widetilde{x}(t), \widetilde{x}^{[2]}(t)\, \widetilde{x}(t-\tau(t))\right)\right| \\ &\quad + \left|h\left(t, \widetilde{x}(t), \widetilde{x}^{[2]}(t)\, \widetilde{x}(t-\tau(t))\right) - \widetilde{h}\left(t, \widetilde{x}(t), \widetilde{x}^{[2]}(t)\, \widetilde{x}(t-\tau(t))\right)\right| \\ &\le \left\|h - \widetilde{h}\right\| + \left(\ell_1 + \ell_2 \ell_3 (\ell_4 + 2)\right) |x(t) - \widetilde{x}(t)|, \end{aligned}$$

then

$$|x(t) - \widetilde{x}(t)| \le \left\|h - \widetilde{h}\right\| + M\, |x(t) - \widetilde{x}(t)|.$$

Making use of (19), one derives

$$\|x - \widetilde{x}\| \le \frac{1}{1 - \Lambda_4} \left\|h - \widetilde{h}\right\|.$$

The proof finishes. □

4. Examples

In this section, we will present two examples to show the feasibility and effectiveness of our main findings.

4.1. Example 1

Consider the following equation:

$$\begin{aligned}(20)\quad &\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)+(0.23)\frac{\mathrm{d}}{\mathrm{d}t}x(t)+(0.01)x(t)\\ &=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(0.2+0.2\sin^2 2\pi t+(0.034152)x(t)+(0.023144)x^{[2]}(t)x\left(t-\cos^2 2\pi t\right)\right).\end{aligned}$$

We take

$$\mathbb{M}=\left\{x\in P_T:\|x\|\le\frac{\pi}{3},\ |x(t_2)-x(t_1)|\le 3\pi|t_2-t_1|,\ \forall t,t_1,t_2\in[0,T]\right\}.$$

In this case

$$a(t)=0.058211,\quad b(t)=0.17179,$$

and

$$\begin{aligned}&A=0.23,\quad R_1\approx 0.043478,\quad Q_1\approx 0.0096431,\quad \frac{R_1[A-1]}{Q_1T}\approx 1.166>1,\\ &A^2=0.0529,\quad 4B=0.04,\quad M_1=0.058211,\quad M_2=0.171\,79,\quad \delta_0=5.335\,4,\\ &\delta_1=16.684,\quad \rho_1=0.058211,\quad \rho_2=0.058211=0,\quad \rho_3=0.17179,\quad \rho_4=0.23\\ &\rho=289.98,\quad \overline{g}\approx 350.33,\quad \kappa_1\approx 0.22256,\quad \mu_1=0.0033885,\quad \ell_1=0.034152,\\ &\ell_2=0.023144,\quad \mu_2=0.46,\quad \mu_3=4.\,856\,0,\quad h_0=0.4.\end{aligned}$$

So, we have

$$\begin{aligned}&\Lambda_1\approx 1.029\,6<\ell_3=\frac{\pi}{3}\approx 1.0472,\\ &\Lambda_2\approx 6.451\,2<\ell_4=3\pi\approx 9.4248,\\ &\Lambda_3\approx 0.015922<1\\ &\Lambda_4\approx 0.38516<1.\end{aligned}$$

Thus all the requirements of Theorem 3.3 and Theorem 3.4 are fulfilled, thus we infer that equation (20) owns a unique periodic solution which depends continuously on parameters.

4.2. Example 2

Consider the following equation:

$$\begin{aligned}(21)\quad &\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)+(0.23)\frac{\mathrm{d}}{\mathrm{d}t}x(t)+(0.01)x(t)\\ &=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(0.3+0.03\sin^2 2\pi t+(0.09)x(t)+(0.09)x^{[2]}(t)x\left(t-\cos^2 2\pi t\right)\right).\end{aligned}$$

We take

$$\mathbb{M} = \{x \in P_T : \|x\| \leq 1.5, \quad |x(t_2) - x(t_1)| \leq 15\,|t_2 - t_1|, \forall t, t_1, t_2 \in [0, T]\}.$$

We have

$$a(t) = 0.058211, \quad b(t) = 0.17179,$$

and

$$A = 0.23, \quad R_1 \approx 0.043478, \quad Q_1 \approx 0.0096431, \quad \frac{R_1[A-1]}{Q_1 T} \approx 1.166 > 1,$$
$$A^2 = 0.0529, \quad 4B = 0.04, \quad M_1 = 0.058211, \quad M_2 = 0.171\,79, \quad \delta_0 = 5.335\,4,$$
$$\delta_1 = 16.684, \quad \rho_1 = 0.058211, \quad \rho_2 = 0.058211 = 0, \quad \rho_3 = 0.17179, \quad \rho_4 = 0.23$$
$$\rho = 289.98, \quad \overline{g} \approx 350.33, \quad \kappa_1 \approx 0.22256, \quad \mu_1 = 0.0033885, \quad \ell_1 = 0.09,$$
$$\ell_2 = 0.09, \quad \mu_2 = 0.6, \quad \mu_3 \approx 10.688, \quad h_0 = 0.33.$$

Then

$$\Lambda_1 \approx 1.343 < \ell_3 = 1.5,$$
$$\Lambda_2 \approx 12.768 < \ell_4 = 15,$$
$$\Lambda_3 \approx 0.122\,09 < 1,$$
$$\Lambda_4 \approx 2.953\,3 > 1.$$

Condition (19) in Theorem 3.3 does not hold while all the hypotheses of Theorem 3.2 are satisfied. Then equation (21) admits a periodic solution which is not necessarily unique.

Thus, the theoretical conditions are sharp, since Example 1 satisfies the uniqueness conditions of Theorem 3.2, whereas Example 2 satisfies only the existence conditions of Theorem 3.3.

5. Conclusion

During the past few decades, a lot of authors have paid close attention to iterative differential equations which offer highly accurate descriptions of many real phenomena. In this article, we have discussed a second order neutral differential equation with iterative terms. We mainly focus on establishing some new sufficient conditions that ensure the existence, uniqueness, and continuous dependence of periodic solutions for equation (1). The proofs of the main results have relied on the Banach and Krasnoselskii fixed point theorems and some useful properties of a Green's function. Firstly, the first step our study is to convert the equation into an integral one whose kernel is Green's function. Then, we derived a set of sufficient criteria, whose successful verification leaded to the application of the Krasnoselskii fixed-point theorem. Finally, under an additional condition, the uniqueness and continuous dependence on parameters of the periodic solutions are established by using the Banach fixed point theorem. Two examples are given to illustrate the correctness and effectiveness of the main outcomes.

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