RM - 16.10.2026

piatok 16. októbra 2026 14:43

späť na GLM:

1.) rozdelenie $y \rightarrow$ z exp. triedy

- n meraní $\Rightarrow$ $f(y_i \mid \theta_i, \varphi) = e^{\frac{y_i\theta_i - b(\theta_i)}{a_i(\varphi)} + c(y_i, \varphi)}$, $i = 1, \dots, n$ (1)

$y_1, \dots, y_n$ - nez.

2.) vzťah medzi x a y

↳ link: $g(\mu_i) = x_i^T\beta$, kde $\mu_i = E[y_i]$ (2)

$\rightarrow$ GLM: $y_1, \dots, y_n$, ktoré spĺňajú (1) a (2)

- g - link function, napr. $g(\pi_i) = \ln\left(\frac{\pi_i}{1-\pi_i}\right) \Rightarrow$ link: $\ln\left(\frac{\pi_i}{1-\pi_i}\right) = x_i^T\beta$

$g(\mu_i) = \mu_i \Rightarrow$ (2): $\mu_i = x_i^T\beta$

- ako voliť g?
  - historicky (pre konkrétne modely)

kanonická link funkcia: taká $g(\mu)$, že $g(\mu) = \theta(\mu)$ ↳ kanonický parameter

$\Rightarrow g(\mu_i) = x_i^T\beta$

$\Rightarrow \theta_i = x_i^T\beta$

$\rightarrow$ vyjadriť θ ako fcia μ a to položiť ako $g()$

Pr.: • $y \sim N(\mu, \sigma^2)$: $\theta = \mu$

$\Rightarrow g(\mu) = \theta = \mu$

g - identita $\Rightarrow$ klasická lin. reg.: (1) $y_i \sim N(\mu_i, \sigma^2)$ $\forall i$, nez.

(2) $\mu_i = x_i^T\beta$

• $y \sim Alt(\pi)$: $\theta = \ln\left(\frac{\pi}{1-\pi}\right)$

$\pi = E(y)$ $\Rightarrow g(\pi) = \theta = \ln\left(\frac{\pi}{1-\pi}\right)$ - kan. link

$\Rightarrow$ logistická reg.: (1) $y_i \sim Alt(\pi_i)$ $\forall i$, nez.

(2) $\ln\left(\frac{\pi_i}{1-\pi_i}\right) = x_i^T\beta$

Odhadovanie:

GLM: (1) $y_i \sim f(y_i \mid \theta_i, \varphi) = e^{\frac{y_i\theta_i - b(\theta_i)}{a_i(\varphi)} + c(y_i, \varphi)}$ $\forall i$, nez.

(2) $g(\mu_i) = x_i^T\beta$ $\forall i$ $\Rightarrow \hat\beta = ?$

$[\mu_i = E(y_i) = b'(\theta_i)]$

MMV (MLE): $L(\beta) = \prod_{i=1}^n f(y_i \mid \theta_i, \varphi) = \prod_{i=1}^n e^{(\quad)} \rightarrow \max_\beta$

$\Leftrightarrow \ell(\beta) = \sum_{i=1}^n \frac{y_i\theta_i - b(\theta_i)}{a_i(\varphi)} + c(y_i, \varphi) \rightarrow \max_\beta$ ... μ: fcia β, θ: fcia μ

$\frac{\partial \ell(\beta)}{\partial \beta} = \sum_{i=1}^n \frac{y_i \cdot \frac{\partial \theta_i}{\partial \beta} - b'(\theta_i) \cdot \frac{\partial \theta_i}{\partial \beta}}{a_i(\varphi)} + 0_k = 0_k$

$\theta_i = \theta_i(\mu_i(\beta))$ $\mu_i = b'(\theta_i)$ / $\cdot \frac{\partial}{\partial \theta_i}$

$\theta_i = \theta_i(\mu_i(\beta))$

$\Rightarrow \frac{\partial \theta_i}{\partial \beta} = \frac{\partial \theta_i}{\partial \mu_i} \cdot \frac{\partial \mu_i}{\partial \beta}$

$q \times 1 \quad 1 \times 1 \quad q \times 1$

$\mu_i = b'(\theta_i) \quad / \cdot \frac{\partial}{\partial \theta_i}$

$\frac{\partial \mu_i}{\partial \theta_i} = b''(\theta_i)$

$\Downarrow$ ← ako s číslami, lebo $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$

$\frac{\partial \theta_i}{\partial \mu_i} = \frac{1}{b''(\theta_i)}$

$\Rightarrow \frac{\partial l(\beta)}{\partial \beta} = \sum_{i=1}^{n} \underbrace{y_i \cdot \frac{1}{b''(\theta_i)} \cdot \frac{\partial \mu_i}{\partial \beta} - b'(\theta_i) \cdot \frac{1}{b''(\theta_i)} \cdot \frac{\partial \mu_i}{\partial \beta}}_{a_i(\varphi)} = 0_q$

$\sum_{i=1}^{n} \left(y_i \cdot \frac{1}{a_i(\varphi) \cdot b''(\theta_i)} - b'(\theta_i) \cdot \frac{1}{a_i(\varphi) \cdot b''(\theta_i)} \right) \cdot \frac{\partial \mu_i}{\partial \beta} = 0_q$

$E(y_i) = b'(\theta_i)$

$D(y_i) = a_i(\varphi) \cdot b''(\theta_i)$

$(*) \quad \sum_{i=1}^{n} \frac{y_i - E(y_i)}{D(y_i)} \cdot \frac{\partial \mu_i}{\partial \beta} = 0_q$

→ numerické riešenie

Pr.: pre lin. reg.: $y_i \sim N(\mu_i, \sigma^2)$, nez. $\Rightarrow E(y_i) = \mu_i = x_i^T \beta$

$x_i^T \beta = \mu_i \quad D(y_i) = \sigma^2$

$\Rightarrow \frac{\partial \mu_i}{\partial \beta} = x_i$

$\Rightarrow \quad (*) \quad \sum_{i=1}^{n} \frac{y_i - x_i^T \beta}{\sigma^2} \cdot x_i = 0_q$

$\sum_{i=1}^{n} y_i x_i = \sum_{i=1}^{n} \underbrace{x_i^T \beta}_{1 \times 1} x_i$

$\underbrace{\sum_{i=1}^{n} y_i x_i}_{X^T y} = \underbrace{\sum_{i=1}^{n} x_i x_i^T}_{X^T X} \beta$

$X^T y = X^T X \beta$ – Norm. rovnice ✓

Testovanie hypotéz:

- pomocou log-likelihoodu $l(\beta)$

→ konkrétne: $2\left(l(\hat{\beta}_{Full}) - l(\hat{\beta}_{máš}) \right)$

↑ model, kt. uvažujeme

↳ plný model: má toľko parametrov, koľko je pozorovaní

: premenšie: toľko parametrov, koľko je rôznych x_i $(i = 1, \dots, n)$

prišlo to z: $\frac{L(\hat{\beta}_{Full})}{L(\hat{\beta}_{máš})} \quad / \ln(\ )$

$l(\hat{\beta}_{Full}) - l(\hat{\beta}_{máš})$

$l(\beta) = \sum_{i=1}^{n} \frac{y_i \theta_i - b(\theta_i)}{a_i(\varphi)} + c(y_i, \varphi)$

$\hat{\theta} = \hat{\theta}_{máš}$

$\tilde{\theta} = \hat{\theta}_{Full}$

$\Rightarrow 2\left(l(\hat{\beta}_{Full}) - l(\hat{\beta}_{máš}) \right) = 2\left[\sum_{i=1}^{n} \left(\frac{y_i \tilde{\theta}_i - b(\tilde{\theta}_i)}{a_i(\varphi)} + c(y_i, \varphi) \right) - \sum_{i=1}^{n} \left(\frac{y_i \hat{\theta}_i - b(\hat{\theta}_i)}{a_i(\varphi)} + c(y_i, \varphi) \right) \right] = (\square)$

$\ell(x|\beta_{full}) - \ell(x|\beta_{red}) = 2\left[\sum_{i=1}^{n}\left(\frac{y_i\theta_i - b(\theta_i)}{a_i(\varphi)} + c(y_i,\varphi)\right) - \sum_{i=1}^{n}\left(\frac{y_i\tilde{\theta}_i - b(\tilde{\theta}_i)}{a_i(\varphi)} + c(y_i,\varphi)\right)\right] = (\square)$

predpokladajme: $a_i(\varphi) = \varphi / w_i$ pre známe w_i

$\Rightarrow (\square) = \sum_{i=1}^{n} \frac{2w_i\left(y_i(\tilde{\theta}_i - \hat{\theta}_i) - b(\tilde{\theta}_i) + b(\hat{\theta}_i)\right)}{\varphi} = \text{deviancia, označ. } Dev \text{ (deviance)} = \frac{Dev}{\varphi} = \text{škálovaná (scaled) deviancia}$

- čím vyššia deviancia, tým horší model

$H_0: R\beta = r \quad$ vs. $\quad H_1: R\beta \neq r$

↳ $R: q \times k$ matica, $rank(R) = q$

- máme model: mod
- vytvoríme submodel, kde H_0 platí: sub $\Rightarrow$ $H_0 \Leftrightarrow$ sub stačí

$\Rightarrow$ deviancie: Dev_{mod}, Dev_{sub}

- ak H_0 platí: $Dev_{sub} \approx Dev_{mod}$
- ak H_0 neplatí: $Dev_{sub} >> Dev_{mod}$: ovela horší

platí: $\frac{Dev_{sub} - Dev_{mod}}{\varphi} \overset{\cdot}{\underset{H_0}{\sim}} \chi^2_q$ ← # obmedzení, resp. rozdiel medzi # št. voľných modelu a sub modelu

$\Rightarrow H_0$ zam. ak $> \chi^2_q(\alpha)$ ← krit. h.

$\varphi = ?$ → niekt. modely nemajú φ $\Rightarrow$ žiaden problém ($\varphi \equiv 1$)

↳ inak ho typicky neporúčame $\Rightarrow$ odhad: $\hat{\varphi}$ $\Rightarrow$ $\frac{[Dev_{sub} - Dev_{mod}]/q}{\hat{\varphi}} \overset{\cdot}{\underset{H_0}{\sim}} F_{q,\infty}$ (závisí od rozdelenia $\hat{\varphi}$)

- pre $N(\mu, \sigma^2)$: klasicky $\hat{\varphi} = \frac{RSS}{n-k}$ | máme presné $F_{q,n-k}$ — STAT
- H_0 zam. ak $STAT > F_{q,\infty}(\alpha)$

- rozdelenia sú iba približné
- ∃ iné testy

reziduá: $\hat{\varepsilon}_i = y_i - \hat{y}_i$ → stačia pre lin. reg., lebo $D(y_i) = \sigma^2\ \forall i$

↳ vo všeob. GLM: $D(y_i) = a_i(\varphi) \cdot b''(\theta_i)$

$\Rightarrow$ Pearsonove reziduá (Pearson residuals) $r_{P,i} = \frac{y_i - \hat{\mu}_i}{\sqrt{b''(\hat{\theta}_i)}}$

deviančné reziduá (deviance r.) $r_{D,i} = sgn(y_i - \hat{\mu}_i)\cdot\sqrt{d_i}$ ↑ jeden sčítanec v deviancii: $d_i = 2w_i\left(y_i(\tilde{\theta}_i - \hat{\theta}_i) - b(\tilde{\theta}_i) + b(\hat{\theta}_i)\right)$

Rho: $y_i \sim Alt(\pi_i)$, nez.

$\underbrace{\ln\left(\frac{\pi_i}{1-\pi_i}\right)}_{logit(\pi_i)} = x_i^T\beta \iff \pi_i = \frac{e^{x_i^T\beta}}{1 + e^{x_i^T\beta}}$

kontrétne: π_i: pp. srdcovej choroby

$\ln\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_1 + \beta_2 \text{vyška}_i + \beta_3 \text{cig}_i \quad / \exp()$

$\underbrace{\ln\left(\frac{u_i}{1-\pi_i}\right)}_{\text{logit}(\pi_i)} = x_i^T\beta \quad \Leftrightarrow \pi_i = \frac{e^{x_i^T\beta}}{1+e^{x_i^T\beta}}$

$\underbrace{\ln\left(\frac{\pi_i}{1-\pi_i}\right)}_{\text{logit}(\pi_i)} = \beta_1 + \beta_2 \text{výška}_i + \beta_3 \text{cig}_i \quad / \exp()$

$\hat\beta_1 = -4{,}5$ ← nastiica fcia

$\hat\beta_2 = 0{,}025 \Rightarrow$ pri väčšej výške odhadujeme väčšiu pp. srdcovej chorroby

$\hat\beta_3 = 0{,}023 \Rightarrow$ −11− pri cigarety

$\underbrace{\frac{\hat\pi_i}{1-\hat\pi_i}}_{\text{šance (odds)}} = e^{\hat\beta_1 + \hat\beta_2 \text{výška}_i + \hat\beta_3 \cdot \text{cig}_i}$

šance (odds) ... šance, že na šachte padne 3 sú 1:5 ($\frac{1}{5}$)

výška$_i$ ↦ výška$_i$ + 1

$\Rightarrow$ šance$_{nové} = e^{\hat\beta_1 + \hat\beta_2 \text{výška}_i + \hat\beta_2 + \hat\beta_3 \text{cig}_i} = \text{šance}_{staré} \cdot e^{\hat\beta_2}$: $e^{\hat\beta_2}$ – násobný nárast odhadnutých šancí na infarkt pri náraste výšky o 1

$e^{\hat\beta_2} = 1{,}026 \longrightarrow$ nárast na 1,026-násobok (o 2,6 %)

$e^{\hat\beta_3} = 1{,}023$

$\pi_i = \frac{e^{x_i^T\beta}}{1+e^{x_i^T\beta}}$