COMENIUS UNIVERSITY, BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS



PORTFOLIO IMPLICATIONS OF PERSISTENT SHIFTS IN THE REAL INTEREST RATE

MASTER'S THESIS

2021 Bc. Martin Vojtek

COMENIUS UNIVERSITY, BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

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MASTER'S THESIS

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Department:	Department of Applied Mathematics and Statistics
Supervisor:	Pavol Povala, PhD.

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Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

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Názov: Portfolio implications of persistent shifts in the real interest rate
Portfólio implikácie perzistentných zmien v reálnej úrokovej sadzbe

Anotácia: Kvantifikácia úžitkovej straty v dôsledku ignorovania perzistentných zmien reálnej úrokovej miery (riziko reinvestovania). Jedným z kľúčových prínosov práce je vyvinúť a implementovať simuláciu, ktorá nám pomôže kvantifikovať úžitkovú stratu, ktorú by investor utrpel, ak by ignoroval perzistentné zmeny reálnej úrokovej miery. Táto simulácia nám umožňuje zvážiť rôzne konfigurácie procesu reálnych úrokových mier.

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Annotation: Quantification of the utility loss from ignoring persistent variation in the real interest rate (reinvestment risk). One of the key contributions of the thesis is to develop and implement a simulation setup with the help of which we quantify the utility loss an investor would incur when ignoring the persistent variation in real interest rates. The simulation setup allows us to consider various configurations of the real rate process.

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Abstrakt

VOJTEK, Martin: Portfólio implikácie perzistentných zmien v reálnej úrokovej sadzbe [Diplomová práca], Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky; školiteľ: Pavol Povala, PhD., Bratislava, 2021, 56s

Táto diplomová práca skúma reálne úrokové sadzby, konkrétne jej komponent, prirodzenú (rovnovážnu) úrokovú sadzbu. Ako v práci popisujeme, prirodzená úroková sadzba v posledných desaťročiach podliehala perzistentným zmenám. Avšak, mnoho modelov alokácie aktív funguje za predpokladu stacionarity reálnych úrokových mier. Preto, vytvárame a implementujeme simuláciu pomocou Double Auto-Regressive procesu, ktorá nám umožňuje zohľadniť perzistenciu aj stacionaritu v reálnych úrokových sadzbách. Na základe tejto simulácie skonštruujeme výnosovú krivku, ktorá odráža nielen jej súčasný stav ale aj jej predpovede do budúcnosti. Takýto odhad vývoja výnosovej krivky nám umožňuje formovať očakávania o návratnosti dlhopisov, čo vedie ku kľúčovému prínosu tejto práce: Kvantifikácií úžitkovej straty, ktorú by investor utrpel, ak by ignoroval perzistentné zmeny v reálnych úrokových sadzbách (riziko reinvestovania).

Kľúčové slová: reálne úrokové miery, prirodzená úroková miera, dlhopisy, výnosová krivka, autoregresné procesy, riziko reinvestovania, perzistentné zmeny

Abstract

VOJTEK, Martin: Portfolio implications of persistent shifts in the real interest rate [Master's Thesis], Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics; Supervisor: Pavol Povala, PhD., Bratislava, 2021, 56p

This thesis explores real interest rates, particularly its component, the natural (equilibrium) rate of interest. As we show, the natural rate of interest has been a subject of persistent shifts throughout the last decades. However, many asset allocation models work under the assumption of stationarity of real interest rates. Therefore, we develop and implement a simulation using the Double Auto-Regressive process, which allows us to account for both persistence and stationarity in real interest rates. Based on this simulation, we construct a yield curve that does reflect not only its current state but also its predictions into the future. Having such yield curve estimation allows us to form bond returns expectations, which leads to the key contribution of this thesis: Quantifying a utility loss an investor would occur if ignoring persistent variation in real interest rates (reinvestment risk).

Keywords: real interest rates, the natural rate of interest, bonds, yield curve, auto-regressive processes, reinvestment risk, persistent shifts

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Introduction

One of the most important decisions every investor must make is asset allocation. To make a well-informed decision on the appropriate proportion of risky assets (stocks) and risk-free assets (bonds) in their portfolio, one has to form expectations about properties of returns on these assets. These expectations are ultimately driven by macroeconomic variables such as inflation, real output growth or fluctuations in labour share. The real interest rate is an important component of the return driver on both stocks and bonds. This thesis explores the implications of the variation in real interest rates, particularly its key determinant - natural rate of interest, for bonds in the portfolio of a long-term investor.

Traditional strategic asset allocation works under the assumption that real interest rates are stationary [6]. Recent empirical evidence calls this assumption into question, however. Authors of [17] show that the natural rate of interest in the US has been on its decline in past decades. Following this finding, [12] show that the steady decline in the natural rate of interest was accelerated by the Global Financial Crisis (GFC) between 2007 and 2009 and that this decline is not specific to the US, but it is common to all major markets. Once this has been established, [2] have shown that it is crucial to account for the time-varying natural rate of interest when estimating the term structure of interest rates. The goal of this thesis is to continue the research based on those findings and form the bond returns expectations under these assumptions. By creating a simulation setup of real interest rates as well as bond returns we will emphasize the importance of accounting for persistent shifts in real interest rates and quantify the utility loss an investor would incur if ignoring these premises.

This thesis is split into 6 sections. In the first and key section, we will implement time series model introduced by [11] to simulate the natural rate of interest following the assumption that it is subject to persistent changes. In the second section, we will follow the lead of [15] to quantify the impact of those persistent changes on bond yields. This will allow us to construct a yield curve that does not only reflect its current state but also its predictions into the future. In the third section, we will show how estimated bonds yields are transformed into bonds returns and we will discuss the impact of various bond choices on investors potential return using various metrics, including utility function.

In section 4, we will come back to the long-recognised theory that the natural rate of interest can be considered a constant and we will evaluate the differences in potential returns when following this assumption compared to allowing the natural rate of interest to be a subject of persistent shifts. Since the year 2020 was dominated by the COVID-19 pandemic, our modelling approach was also affected for this period, as we will show in section 5. Therefore some adjustments will have to be done and we will also show the differences in expectations an investor had before the pandemic has hit the world and afterwards. In the final section of the thesis, section 6, we will compare the findings of our simulation to the historical evidence.

1 The natural rate of interest

The idea of the natural rate of interest was introduced in the 19th century by Swedish economist Knut Wicksell. It is defined as a real interest rate that is consistent with output equaling its natural rate of growth and constant inflation, see e.g. [12]. Rephrasing this definition less formally, the natural rate of interest is the real interest rate that is neither expansionary nor contractionary when the economy is at full employment. Throughout the thesis, we will refer to it as an r^*.

While r^* as a concept is relatively old, it has been largely neglected by academic research until after the Global Financial Crisis (GFC) in 2007-2009. The key factor that has likely contributed to renewed interest in r^* is a relatively large and abrupt downward move in nominal yields on government bonds during the GFC – especially in the US. This downward move was not driven by a corresponding move in inflation expectations, and it turned out to be persistent, as we can see in figure (1).

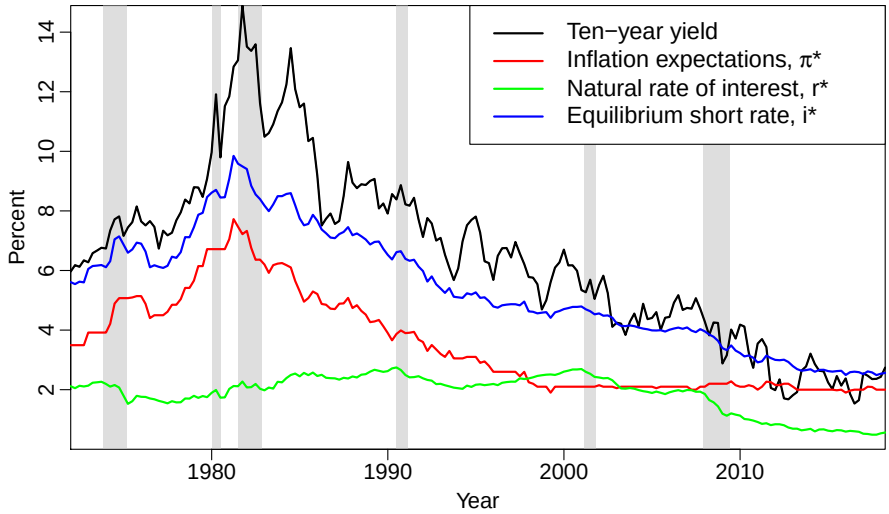


Figure 1: Ten-year US nominal yield and macroeconomic trends. Source: [2]

Before describing the empirical properties of r^*, it is useful to explore its drivers. The definition suggests that the most important driver of r^* is the growth rate of economy output. This relationship is considered to be a direct proportionality, i.e. r^* moves one-for-one with real output growth, see e.g. [17] for more details.

Because central banks generally target yields on short-term safe and liquid assets, authors of [8] and [16] emphasize the importance of safety and liquidity properties of

r^* using Treasury and corporate bond yields of various maturities, inflation and survey expectations. They show that as the premium for safety and liquidity increases, interest rates, including the natural rate of interest, decrease. In this thesis, we will model the following three drivers of r^*: GDP growth, liquidity and safety.

Authors of [12] show that r^* has declined since the turn of the century, with a sharp drop occurring during the GFC, as can be seen in Figure (1). When relating this decline to the three drivers described above, we could ascribe the steady decline of r^* to a steady decline in real output growth in advanced economies.

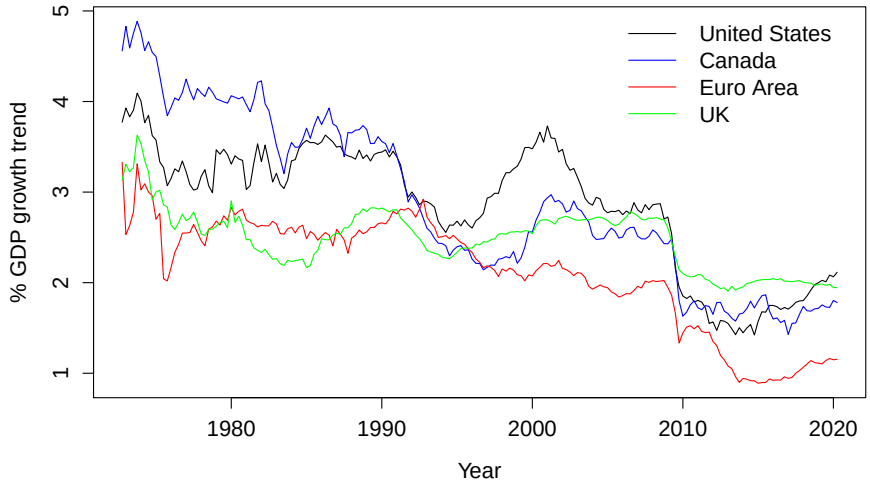


Figure 2: Quarterly estimates of GDP trend growth in major economies. Sample period is Q1:1972 through Q3:2020. Source: [12]

There are multiple potential and not mutually exclusive explanations of the decline, such as slower technological progress, demographic transition associated with ageing baby boom generation, increased longevity and others. For further details on this topic, see [5], [18].

The sharp drop we mentioned earlier can be explained by the GFC that occurred between 2007 and 2009. It appears that since then, investors value the safety and liquidity properties of government bond securities more than prior to the GFC – and are willing to pay a premium for these properties. It is important to note that the sharp decline in r^* is not specific to the United States but is common to all major markets, as shown in Figure 3.

Now that we have described drivers of r^*, reported its decline and explained the reasons

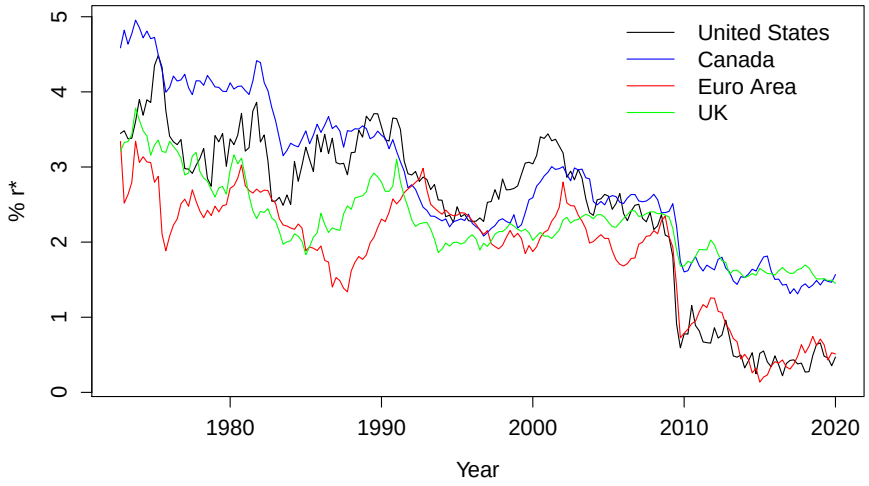


Figure 3: Quarterly estimates of r_t^* in major economies. Sample period is Q1:1972 through Q3:2020. Source: [12]

behind it, we explore the implications of the decline for the economy and investors.

Authors of [14] have precisely described the effects of low interest rates on the economy and monetary policy, including the effects such as poor economic performance when using traditional policy rules, inability to meet the inflation target and other factors influencing economic and price stability, all of which are exacerbated during crises. In this thesis, we will focus on investors and portfolio implications of low interest rates environment. It is intuitive that a lower r_t^* persistently depresses yields on bonds. We will explore this more formally by decomposing government bond yields in the next section. This assumption might cause investors to increase their allocation in stocks. On the other hand, authors of [19] show by the decomposition of stocks returns that bonds are not the only ones being affected by the decrease of the natural rate of interest. Quite the opposite, they show that long-run expected return on stocks has declined significantly through its risk-less component r^*.

Based on reasons described above and many more, r^* has established itself as a key concept for most of the investment decisions one has to encounter. That is also why it is an inseparable part of this thesis and we shall dive into it.

1.1 Modelling natural rate of interest

It is essential to mention that r^* is only a theoretical concept and therefore, it can not be observed and it is a subject of modelling and estimation.

Following the previous subsection, we are modelling r^* as a sum of 3 components: link to long term expectations about real output growth, safety and liquidity demand effects.

Based on the work of [20] we start by representing realized real output growth g_t which has a permanent component τ_t^g, that is a highly persistent component accounting for the long-run growth path and a transitory component g_t^c describing recurrent fluctuations around this growth path:

$$g_t = \tau_t^g + g_t^c \tag{1}$$

From econometric standpoint the difference in between the two is that while shocks in τ_t^g are long-lasting, shocks in g_t^c are only temporary and tend to revert back to the level given by the permanent component of g_t.

The permanent component of real output growth is unobservable and investors apply a constant-gain learning rule to form expectations about long-term output growth. There is a lot of literature based on learning rules, e.g. [9]. The key advantage of such rules, is that investors do not need a complex model containing all macroeconomic relationships, instead they learn about them based on past experiences and historical data.

Constant-gain learning rule is given by:

$$\overline{g_t} = (1 - v) \sum_{i=0}^{t-1} (v^i g_{t-1}) \tag{2}$$

where v is the forgetting factor assigning weight to individual observations and we refer to t as to look-back parameter.

It is believed that $\overline{g_t}$ is the best real-time estimate of τ_t^g in the presence of structural breaks.

We continue by describing the natural rate of interest r^* as follows:

$$r^* = \overline{g_t} - s_t \tag{3}$$

where s_t represents effects from demand for safety and liquidity and therefore can be split as $s_t = s_t^{safe} + s_t^{liq}$.

Now that we have described how we are approaching r^*, we move onto the next step: its modelling and forecasting.

As shown in [21], interest rates are extremely persistent, meaning the shocks happen to be very long-lived. This persistence in real interest rates is usually high enough for tests of stationarity to fail rejecting the null hypothesis of data containing a unit root. When modelling data with ARMA methodology, unit root is known to induce non-stationarity. Yet interest rates are believed to be ultimately mean-reverting and thus, stationary. The reason is most economic models predict that real interest rates possess a long-run equilibrium value, determined by the long-run rate of potential output growth and population growth, risk aversion and more.

Therefore we can not use Vector Auto-Regressive (VAR) models to ensure a good fit of r^*. Whilst its I(0) variation satisfies our stationarity condition but fails to match persistence in data, I(1) does quite the opposite. To overcome this intricate situation, we follow the approach presented in [11] and we introduce the Double Auto-Regressive model, henceforth referred to as DAR.

1.2 Double Auto-Regressive Model

This process can be expressed by the following set of equations:

$$\begin{aligned} X_{t+1} &= \mu + \Phi X_t + \Omega_{t+1}^{\frac{1}{2}} \epsilon_{t+1}, \\ \Omega_{t+1} &= \Sigma_0 \Sigma_0' + \Sigma_1 X_t X_t' \Sigma_1', \\ \epsilon_{t+1} &\sim i.i.d.\, \mathcal{N}(0, I_p) \end{aligned} \tag{4}$$

Let us mention few remarks. First, VAR is a special case of DAR, where $\Sigma_1 = 0$. In VAR, if we set Φ as an identity matrix, we would get a non-stationary process. On the other hand, when considering DAR, stationarity is not induced only by conditional mean Φ, but also by conditional volatility, which is described by Σ_1, determining the sensitivity of the conditional volatility to the level of the process realized in the previous period. Therefore setting Φ as an identity matrix does not eliminate stationarity and we are able to obtain the combination of desired properties of r^*. Second, the DAR model can also generate the jump behaviour of interest rates by a sequence of shocks with the same sign.

These shocks accumulate in the conditional variance so that when a shock of the opposite sign arrives, it is weighted by a large conditional variance that pushes the process rapidly downwards. Thirdly, stationarity induced by conditional volatility is not equivalent to mean-reversion in the traditional sense, where the process reverts back to the level given by unconditional mean. Instead, the process will tend to spend the most time at the level at which the conditional variance is low, i.e. at zero. When series of innovations occur, our second remark kicks in and pushes the process back to its near-zero level [11].

Now that we have described how we want to induce stationarity, it comes only natural to find a way to prove to ourselves that this assumption is correct, meaning we need a stationarity test for our data. The only problem with standard stationarity tests, such as the Augmented Dickey-Fuller test, Phillips–Perron test is that they are unit root tests, and we have defined our process as a process containing the unit root. Therefore we have to look for an alternative way to test stationarity.

The authors of [22] have shown that process given by (4) is stationary if it satisfies the following equation:

$$\gamma(\phi, \Sigma_1) = \lim_{\xi \to \infty} \left[\mathbb{E} \left(\log \| \prod_{t=1}^{\xi} (\phi + e_t) \| \right) \right] < 0 \tag{5}$$

where e_t is a $p \times p$ matrix that is i.i.d. normally distributed with mean zero and covariance matrix defined as $\mathbb{E}(e_t \otimes e_t) = \Sigma_1 \otimes \Sigma_1$. Such an equation is referred to as a top Lyapunov exponent being strictly negative and it tells us that stationarity can be induced by both conditional mean and conditional variance.

1.3 Calibration of the model

In this subsection, we will bring the model given by (4) to the data and calibrate its parameters to model r^*.

The object of modelling is X_t which can be expressed in the following form:

$$X_t = \begin{bmatrix} \tau_t^g \\ g_t^c \\ s_t^{safe} \\ s_t^{liq} \end{bmatrix} \tag{6}$$

and ultimately $r^* = (\overrightarrow{1, 1, -1, -1})X_t$, where the constant-gain rule to the first two was applied beforehand the summation.

The following table shows parameters after calibration:

	μ	Φ	Σ_0	Σ_1
τ_t^g	$0.575 \cdot 10^{-2}$	1	$0.191 \cdot 10^{-2}$	0.04
g_t^c	0	0	$0.382 \cdot 10^{-2}$	0.03
s_t^{safe}	$0.115 \cdot 10^{-2}$	0.995	$0.625 \cdot 10^{-3}$	0.01
s_t^{liq}	$0.675 \cdot 10^{-3}$	0.995	$0.625 \cdot 10^{-3}$	0.01

Table 1: Calibrated parameters of DAR

We calibrated the parameters using the following information, e.g. μ parameter for τ_t^g has been estimated from US GDP data, g_t^c was taken from [20], s_t^{safe} and s_t^{liq} were inspired by [16]. Parameters for Φ were empirically chosen to induce persistence in data, Σ_0 and Σ_1 were also chosen empirically, whilst the procedure of setting Σ_0 followed results of [20], Σ_1 was calibrated to induce stationarity. More specified procedure, as well as setting the starting parameters, can be seen in the replication code available in Appendix A on page 54. As will be seen later on, we chose to model r^* in the horizon of 80 quarters as this translates into 20 years, which is long enough to observe the significant role of persistent factors, but it also is a relevant time horizon for long-term investors. The simulation allows to alter the modelling time-horizon.

After obtaining time series for individual elements of X_t, we connected simulation results of g_t to historical data acquired from [25] and applied constant gain learning algorithm (2) to receive the best estimate possible as constant gain learning is argued to provide the sunspot equilibrium of rational expectations according to [9].

Finally, we put all components together, obtaining time series for r^*.

Taking into account that this time series is stochastic, we are not satisfied with just one outcome of our model. Therefore we ran this simulation 1000 times to ensure that the median of individual time series will give us a reliable estimate of r^* and more importantly, we will be able to capture the variance resulting from the fundamentals of r^*.

Figure (4) presents the result of our simulation with yellow line accounting for sim-

ulation median, whereas blue colour represents prediction intervals accounting for 50%, 68%, 90% and 95%, meaning, e.g. that with a probability of 95%, our estimate of r^* will be inside of the lightest blue interval. The 68% interval might seem like a rather random selection, but this interval has established itself in most of r^* literature such as [17] and [8], so we decided to include it as well from the comparability point of view.

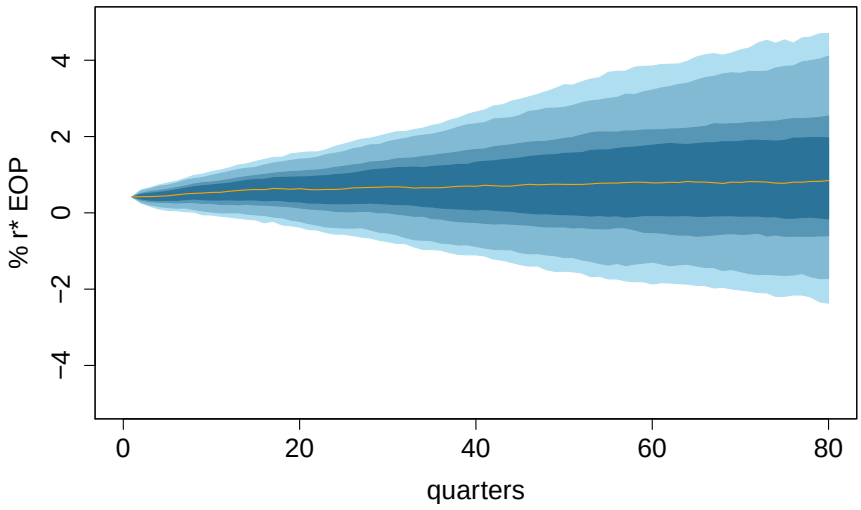


Figure 4: 1000 simulations of base case r^* using DAR. Yellow line is median.

The stationarity check, using (5) based on Φ and Σ_1, confirms that this condition is fulfilled as the top Lyapunov exponent is negative (-0.0585). This is in line with [22] as they show that as long as $0 < \Sigma_1^2 < 2.42$ is fulfilled, stationarity from conditional volatility of the DAR process is induced.

As we can see in Figure (4), median r^* is projected to stay fairly low with a slight push upwards, only seen when looking at the raw numbers from simulation results. It is no surprise that the median line is fairly horizontal when considering this many simulations. What is more interesting is the variance of r^*, totally contradictory to the assumption of r^* being a constant. The importance of this observation will be explored later on. In the next section, we will put r^* into the context of bond yields.

2 Term structure of interest rates

As we have described in the previous section, r^* is subject to persistent shifts, as it was the case during GFC. It has been more than 10 years since the sharp drop of r^* and it does not seem to revert to its mean. As described in [2], there have been numerous attempts to connect macroeconomic variables to the term structure of interest rates, but the natural rate of interest has been very much overlooked. Most term structure analyses assume that this variable is best described by a constant. In this section, we present term structure based on r^* as we would like to build foundation for quantifying the difference when and when not accounting for persistent changes in r^* later in this thesis.

2.1 Theoretical framework

Following [15], the yield of an n-period Treasury bond can be expressed as shown in equation (7):

$$r_t^n = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}_t(r_{t+i}) + b_t^n, \tag{7}$$

where the term on the left is referred to as Expectation Hypothesis (EH) and b_t^n is term premium. EH term can be understood as the average future short rate r_t expected over the life of the bond, see e.g. [7].

Short rate is obtained via a stochastic process which is very sensitive to minor variations resulting in major implications in long-horizon forecasts associated with bond yields predictions, as authors of [15] show. Therefore the approach of isolating long-horizon implications is welcome. Thus we define the endpoint of the short rate process $r_\infty^{(t)}$ as the limiting conditional forecast implied by that process:

$$r_\infty^{(t)} \equiv \lim_{s \to \infty} \mathbb{E}_t(r_s) \tag{8}$$

where t denotes the time subscript of the information set on which the expectations are conditioned.

Although, short rates might be subject to occasional shifts, which we refer to as a cyclical part of the real rate c_t. Those shifts reflect cyclical variation in the real rate and its weight in bond yields is decreasing with increasing maturity as (8) holds stronger. This thesis abstracts from shifts in inflation, which are discussed e.g. in [2]. Therefore we

can express real bond yields as a sum of endpoint, cyclical part of the real rate and term premium:

$$r_t^n = r_\infty^{(t)} + c_t^n + b_t^n, \tag{9}$$

where c_t^n depends on the maturity and takes form of $\left(\frac{1}{n}\sum_{i=0}^{n-1} h^i\right) c_t$. This form is in line with equation (8), as the longer the maturity is, the closer we approach $r_\infty^{(t)}$ and c_t has a smaller weight in the equation. We will explain what h stands for in the next subsection.

The authors of [15] discuss various variations of an endpoint, such as a constant endpoint, moving-average endpoint or a shifting endpoint. Shifting endpoint provided the best results as it allows for explicit shifts in the endpoint resulting from the dependence of the endpoint on the conditioning information set. Those authors assume that the endpoint is fixed over the horizon of forecasts originating in any given period. In other words, $\mathbb{E}_t(r_\infty^{(t+k)}) = r_\infty^{(t)}$, for any information set that includes data not in t ($k \geq 0$) [15]. Such an estimate for the endpoint can be understood as an equilibrium short rate adjusted for inflation. At this point, this is a well known definition from the first section of the thesis. What those authors call shifting endpoint is equivalent to what we refer to as r^*. Therefore we can rewrite equation (9) as follows:

$$r_t^n = r_t^* + c_t^n + b_t^n, \tag{10}$$

or more precisely accounting for maturity variation in c_t:

$$r_t^n = r_t^* + \left(\frac{1}{n}\sum_{i=0}^{n-1} h^i\right) c_t + b_t^n, \tag{11}$$

2.2 Modelling term structure of interest rates

Now that we have described what shifting endpoint is and we have already estimated it in the previous section, let us explore the second term of the equation, i.e. cyclical part of real rate, which translates the current stance of monetary policy into the real interest rates.

At first, to calibrate the process for c_t, we used data collected by the Federal Reserve Board of United States on Treasury inflation-protected securities (TIPS) available here: [13]. TIPS are securities issued by the US government and they are designed to protect

investors from a decline in the purchasing power of their money. They do so by increasing in their principal value as inflation rises, while the interest rate varies with the adjusted principal value of the security.

Unfortunately, TIPS are only issued at the lowest maturity of 2 years. To obtain a reliable estimation of c_t we would have to assume that $r_t^8 \doteq r_t^1$ (n in quarters) and also that $b_t^8 \doteq 0$. The reason behind those assumptions is that if we want to calibrate the process for c_t^n we want n to be equal to 1 and b_t to be equal to zero, see equation (12). Using equation (11):

$$c_t = \frac{r_t - r_t^* - b_t^n}{\sum_{i=0}^{n-1} h^i} \overset{\text{n=1}}{=} r_t - r_t^* - b_t^1 \overset{\text{b=0}}{=} r_t - r_t^* \tag{12}$$

This way, we are able to estimate c_t, which is assumed to follow an AR(1) process, as it is argued in [7]. Since we concentrated persistent variation in r^*, we are left with transitory variation c_t. Asset pricing models are consistent with the Markovian structure, therefore it is unreasonable to assume that higher order AR processes should be considered. In other words, a rational investor translates all information into current asset price and past prices should not have any influence. From an empirical standpoint, c_t is well represented by AR(1) process consistent with the Markovian structure.

After making these simplifications and estimating c_t we were not entirely satisfied with the results. We decided to make a robustness check by creating an artificial TIPS that would be of a 3-month maturity, which actually translates into n = 1 from equation (12) and also, the assumption of zero term premium would be much more realistic.

To create a 3-month artificial TIPS (r_t^{real}) we only need 2 time series. One describing nominal yield on a widely spread 3-month Treasury bill (r_t^{nom}) and series for inflation expectations 3-months ahead (i_t^1). The formula then goes as follows:

$$r_t^{real} = r_t^{nom} - i_t^1 \tag{13}$$

Data for Treasury bills are available at [4], while inflation expectations data and estimation procedures are here [1].

The difference between 2-year TIPS and created 3-month artificial TIPS can be seen in figure (5). Prior to GFC, the yields seemed to resemble each other, however, the differences escalated after the crisis. Overall, taking 2-year TIPS seems to be too much of

an offset, which is supported even by the Student's t-test rejecting the hypothesis of equal means of those time series. Therefore we decided to stick with created 3-month artificial TIPS and proceed with the estimation.

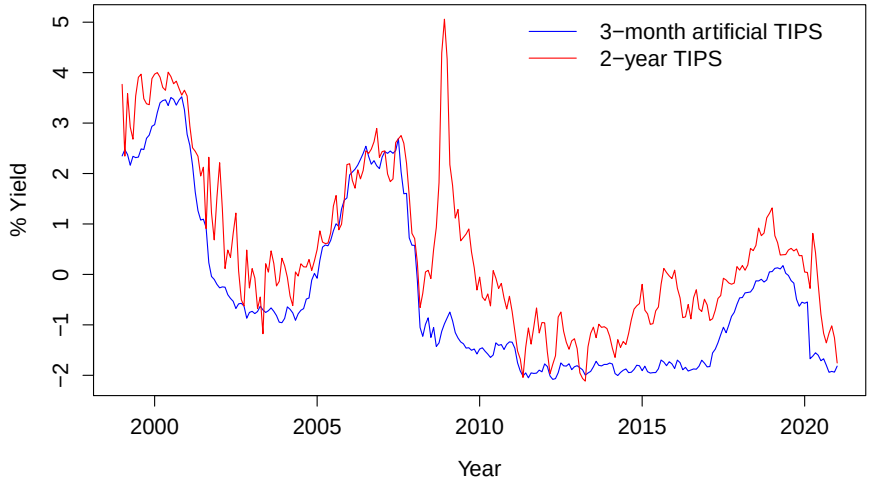


Figure 5: Monthly Treasury yields. Sample period is M1:1999 through M1:2021.

Calibrated AR(1) process for c_t can be expressed as follows:

$$c_t = -0.0015 + 0.9131 c_{t-1} + u_t \tag{14}$$

where $u_t \sim \mathcal{N}(0, 1.818 \cdot 10^{-5})$. Such process is again stochastic and therefore, just one simulation will not do a satisfactory job, instead it will also be a subject of numerous simulations. To execute those simulations, we will revisit the DAR model described by formula (4) and remember that setting Σ_1 equal to zero reduces our model to ordinary AR(1). Therefore we will expand our model by adding c_t and add the following row to table (1):

	μ	Φ	Σ_0	Σ_1
c_t	$-0.146 \cdot 10^{-2}$	0.913	$1.818 \cdot 10^{-5}$	0

Table 2: Parameters of c_t in DAR model

while our object of modelling (6) will be augmented as follows:

$$X_t = \begin{bmatrix} \tau_t^g & g_t^c & s_t^{safe} & s_t^{liq} & c_t \end{bmatrix}' \tag{15}$$

Before plotting the results of this estimation, let us add the last component of bond yields, which is the term premium b_t^n. Term premium of interest rates reflects the compensation to investors for holding a bond for a specific period of time. Coming back to (7), we defined what an Expectation Hypothesis is. To put it into context, EH tells us that the expected return from holding a long bond to maturity should be the same as the expected return from rolling over a series of short bonds with a total maturity equal to that of a long-term bond. Though this is reasonable to assume, it ignores interest rate risk, resulting in uncertainty of return on a long bond. To sum this up, term premium b_t^n represents deviations from EH and provides compensation for interest rate risk.

To model b_t^n, we once again follow the approach of [15], who show the full derivation of term premium resulting in equation (16). We assume a constant risk premium that only depends on the bond's time to maturity n obtaining the following equation:

$$\begin{aligned} b^n = &- \frac{1}{n}\sigma^2 \sum_{k=2}^{n} \left[\beta h^{k-2} + \sum_{j=2}^{k-1} [\beta h^{k-j-1}] \right] \\ &- \frac{1}{n}\sigma^2 \sum_{k=2}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} (h^{j-i-1} h^{k-i-1}) \end{aligned} \tag{16}$$

where n is the time to maturity of a bond, β and σ are risk parameters, such as $\beta\sigma^2$ can be interpreted as the reward for bearing a unit of risk, meaning $-\beta\sigma$ is the price of risk and lastly, h accounts for the degree of persistence of the short rate, assuming a simplification, that short rate follows an AR(1) model [15]. It is the same h as can be seen in equation (11) and it is equal to an AR(1) coefficient from equation (14). Therefore term premium will not be modelled as an auto-regressive process, it will be expressed as a function of external factors.

While β and σ are chosen empirically, h was calculated quantitatively. Authors of [15] designed equation (16) to work well on a monthly h while our h was obtained using quarterly data, which implies a lower degree of persistence. We, therefore, decided to transform the quarterly AR coefficient, let us call it h_q to monthly one h_m in the following fashion:

$$h_q^{\frac{1}{3}} = h_m \tag{17}$$

This approach relies on an assumption of zero mean of white noise process, but it also has

a strong econometric foundation called the half-life of interest rates (derived in Appendix B on page 56), that takes the following form:

$$HL_q = -\frac{\ln(2)}{\ln(h_q)} \tag{18}$$

which gives us the half-life of interest rates in quarters. It is important to note that half-life represents the speed of mean reversion and by transforming our coefficient to a monthly one, we want our process to have the same half-life, only expressed in a different unit. Therefore our HL_m (in months) is equal to 3-times HL_q (in quarters) and h_m can be calculated as follows:

$$h_m = e^{-\frac{\ln(2)}{HL_m}} = e^{-\frac{\ln(2)}{3HL_q}} = e^{-\frac{\ln(2)}{\frac{-3\ln(2)}{\ln(h_q)}}} = e^{\ln(h_q^{\frac{1}{3}})} = h_q^{\frac{1}{3}} \tag{19}$$

At the very moment, the US yield curve is at so called upward sloping shape, which means that investors expect the economy to grow, resulting in increased inflation and therefore increased interest rates. To hedge themselves against this kind of risk, they expect to be paid a positive term premium on their bond. We calibrated parameters β and σ from equation (16) using root-mean-square error (RMSE) according to the term premium that could be observed at the start of 2021. RMSE was minimised via the Nelder-Mead simplex method. The obtained function for b_t^n is in figure (6).

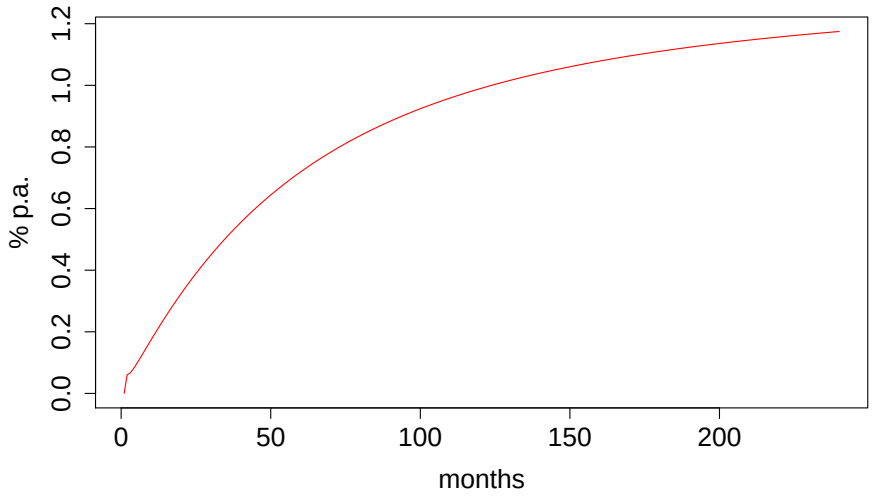


Figure 6: Term premium of interest rates for maturity from 0 to 20 years

2.3 Estimation results

The last step to forecast bond yields is to revisit equation (10) and put all of its components together. Figure (7) shows the yield estimation of a 3-month Treasury Bill and 20-year Treasury Bond. As one can see, the volatility is much more significant in the first case as this is where the cyclical part of real rate plays a more significant role in the overall estimation.

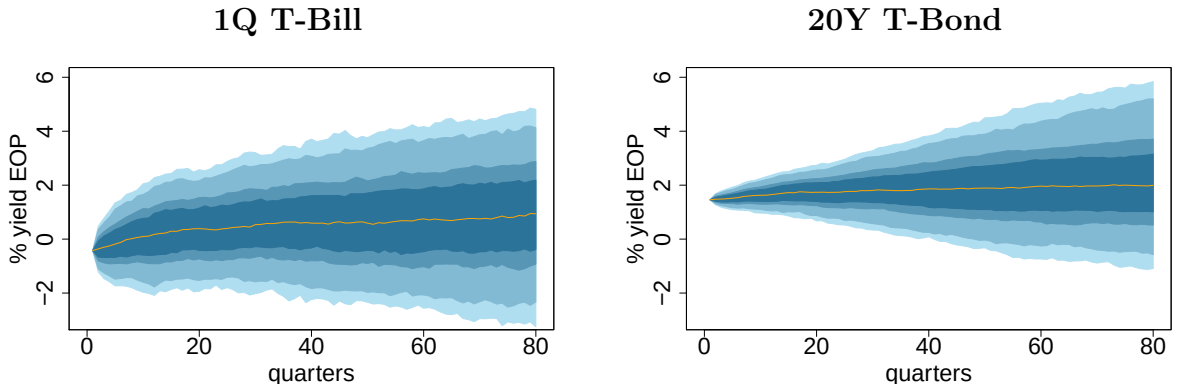


Figure 7: Estimated yield on a 3-month Treasury Bill (left) and 20-year Treasury Bond (right) based on 1000 simulations. Yellow line is median.

Having yields for every possible maturity allows us to construct a yield curve. Because we do not only observe yields that are present at the moment, but we also try to predict them into the future, our yield curve takes on a 3-dimensional shape. Every simulation we run gives a slightly different yield curve, therefore we decided to show its median in figure (8). Since such a choice does not show any persistent shifts based on our model, we also show one random yield curve from the simulation. This simulation, in a way, resembles the situation after GFC when yields suddenly dropped and stayed low ever since. Random yield curve choice can be seen in figure (9).

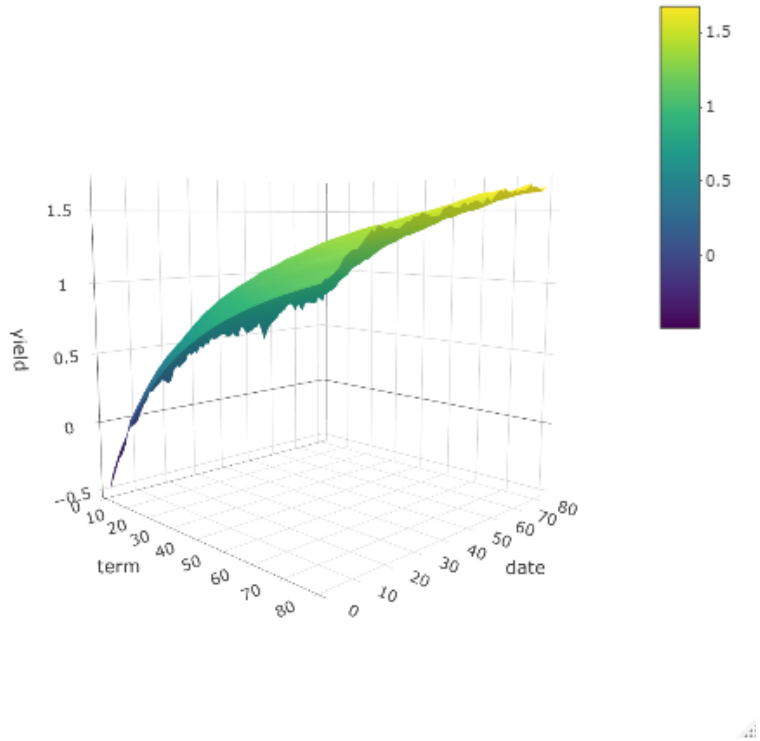


Figure 8: Median yield curve based on 1000 simulations

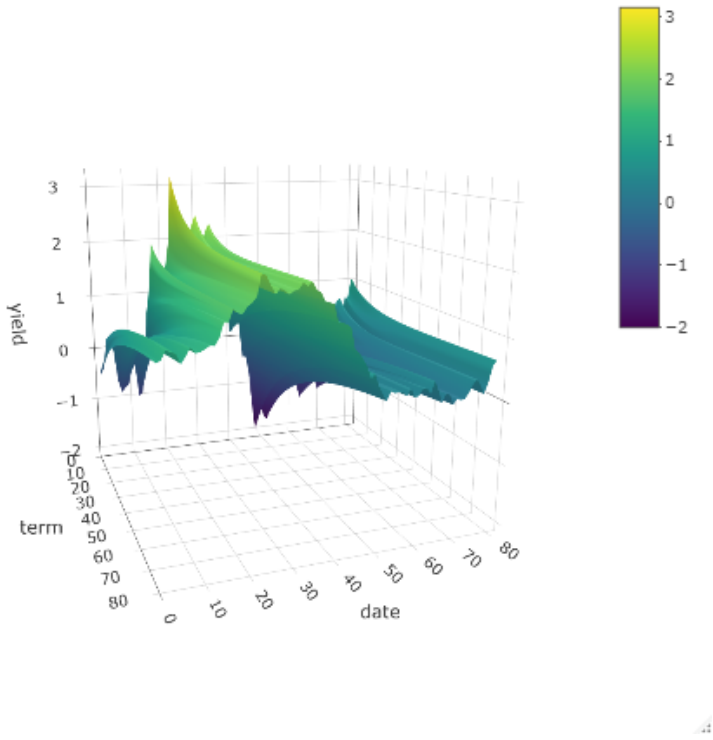


Figure 9: Randomly chosen yield curve simulation

3 Bond returns

Having the yield curve expectations is crucial to the investor, but still, it is not enough to make a well-informed decision on the investment strategy. To make such a decision, the investor must form expectations of investment returns. This chapter transforms the results from the previous section to calculations of bond returns such that the investor is able to anticipate the mean and volatility of his returns throughout the whole investment period.

3.1 Theoretical framework

An investor interested in bonds has many options. The most simple one is to buy a bond matching his investment period and hold it until maturity. In this case, the investor receives the yield promised at the time of issuance. Let us again consider the 20-year investment period. The most natural option is to buy a 20-year bond. In this subsection following theory from [23], if this bond was issued at a yield r, then the return investor receives after the holding period is $e^{nr} - 1$, where n is the bond's maturity. In our case the investor has a return equal to $e^{20r} - 1$.

Another option for the investor is to buy shorter-term bonds and reinvest the proceeds from the maturing bond. If the investor chooses to do so, he bets on bond yields to rise so that he could reinvest his money at a higher yield. In this case, his yields become compounded and we can calculate his return ϱ using the following formula:

$$\varrho = \prod_i (1 + r_i) - 1 \tag{20}$$

The most extreme approach following this strategy would be buying 3-month Treasury Bills and reinvesting the yield every quarter. In a 20-year investment horizon, this approach would result in the investor compounding 80 short term yields. We will show the implications of such an approach later in this section.

Another approach to investing in bonds is not to hold them until maturity but to benefit from the change in the bond price. First, let us describe the relationship between yields and bond prices. Assuming continuously compounded bond yield, or in other words yield-to-maturity, we can define the relationship between the price of a bond and its yield

by the following equation:

$$\frac{1}{P_t^n} = e^{nr} \tag{21}$$

As we can see, if yields rise, bond prices fall and vice versa. This is reasonable, as any investor would prefer a higher-yielding bond for the same price and therefore investor looking to sell his lower-yielding bond must decrease its price to attract a potential buyer.

When calculating returns in between maturities, it is useful to convert the price of a bond to its logarithmic form. By reordering elements of equation (21) and transforming it to logs, we obtain:

$$p_t^n = -nr_t^n \tag{22}$$

The log return from holding a zero-coupon bond from time t to time $t+s$ is calculated as the relative change of the bond price.

$$\rho_{t+s}^n = p_{t+s}^{n-s} - p_t^n \tag{23}$$

It is important to notice that the bond's maturity decreases with time. By merging equation (22) and (23), we arrive at the formula for calculating log returns on a bond with maturity n.

$$\rho_{t+s}^n = -n(r_{t+s}^{n-s} - r_t^n) + sr_{t+s}^{n-s} \tag{24}$$

Clearly, the return on a bond is calculated as follows:

$$\varrho_{t+s}^n = e^{\rho_{t+s}^n} - 1 \tag{25}$$

To make sure the formula at the start of this subsection is correct, let us return to the topic of holding a bond until maturity. Using equation (24) in case of holding until maturity so that $n = s$ the elements with $n - s$ cancel out, so we are left with log return $\rho = nr$ and return $\varrho = e^{nr} - 1$.

3.2 Empirical results

The figure below presents a difference between 2 of the most extreme approaches a bond investor can choose. The one on the left in figure (10) presents rolling a 3-month Treasury

Bill. As we can see, there is a decent amount of uncertainty regarding investor's return at the end of an investment period. The return standard deviation (s.d.) increases with time as the investor undergoes a reinvestment risk every quarter. At the end of an investment horizon, the s.d. is equal to 22.3%, with a mean return of only 12.5%. On the other hand, looking at the figure on the right, the investor is certain about his investment return if we exclude the possibility of the government going bankrupt. What the investor can not be certain about is his return if he decides to sell the bond prior to maturity. The largest s.d. in his return can be seen somewhere in the middle of the investment horizon as nearing maturity, the investor is effectively holding a short-term bond, so the fluctuations are not that significant. If he decides not to sell early, the return is locked in from the time of buying the bond and, in this case, equal to 33.8%. These differences are mostly driven by the reinvestment risk the investor undergoes when choosing the strategy on the left as the risk is being accumulated every quarter. The other source of increased risk of this strategy is from 3-month Treasury Bill yields as they are more volatile than long-term bonds. More on this topic and detailed statistics for bond returns will be presented in the next section.

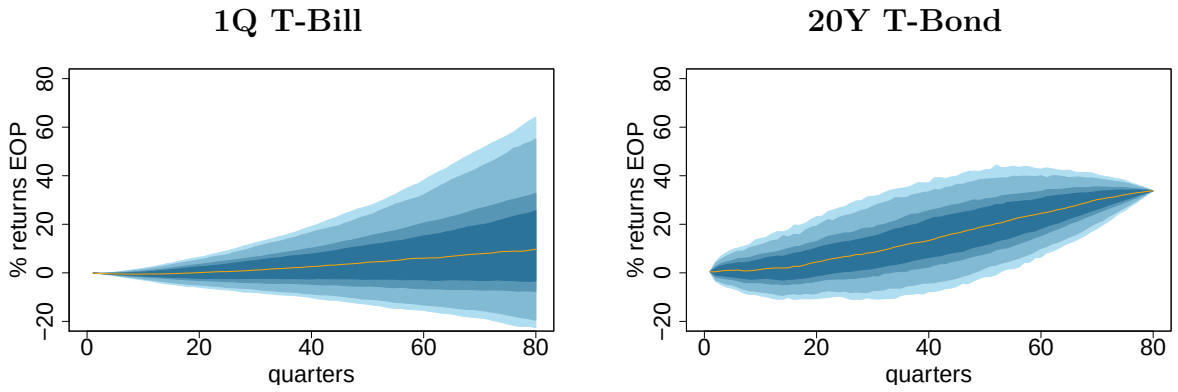


Figure 10: Anticipated returns of rolling 3-month Treasury Bill throughout whole investment period (left) compared to buy and hold strategy of a 20-year Treasury bond (right) based on 1000 simulations. Yellow line is median.

The simulation we developed does not only provide those two extreme examples. One can choose any Treasury security and examine its yields and returns. To show an example, figure (11) provides a look at 2 of possible strategies. The one on the left presents rolling a 2-year Treasury Note and the one on the right presents buying a 10-year Treasury Bond

and buying the second one once the first matures. As we can see in the figure, returns from holding the bond to maturity resemble the strategy of buying a 20-year bond. Only in this case, the period is shortened to 2 and 10 years, respectively. After that, the investor buys the next bond at an unknown yield.

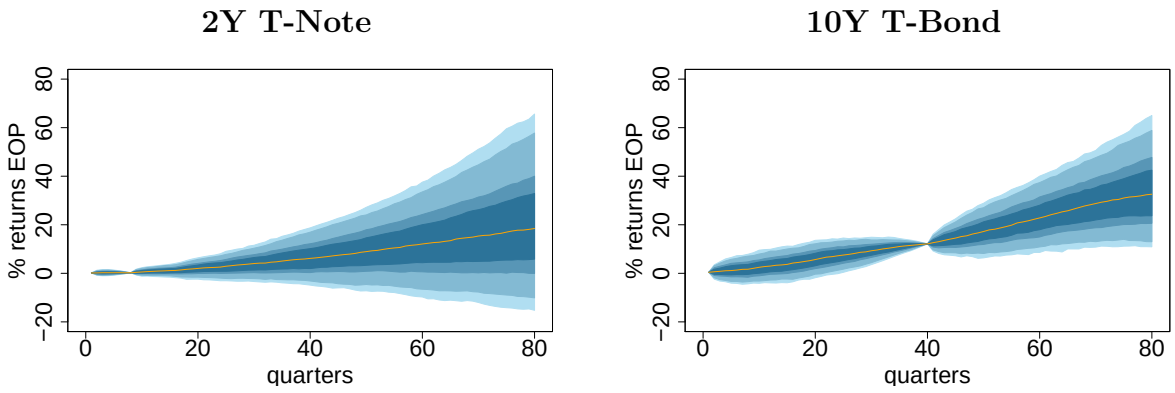


Figure 11: Anticipated returns of rolling 2-year Treasury Note throughout whole investment period (left) compared to buy and hold strategy of a 10-year Treasury bond (right) twice based on 1000 simulations. Yellow line is median.

Whilst investor buying 2-year Notes walks away with a mean return of 20.4% with s.d of 21.1%, the investor buying 10-year bonds received 33.8% mean return with 14% s.d.. Generally speaking, the shorter the maturity of Treasury securities in the rolling strategy, the lower are the expectations in terms of mean return. On the other hand, the standard deviation is sharply increasing the more reinvestment risk investor undergoes. In the next section, a table regarding percentiles of possible investors return will be presented.

3.3 Utility function

As we could see, there is a risk-free strategy of buying a bond matching investment horizon and there are also other strategies with potentially higher returns, but also with a higher risk. Therefore, the concept of utility comes in handy as it provides us with a measure of the trade-off between risk and reward or, in the mathematical sense of variance σ^2 and expected value, or mean, μ defined as the first two moments of the data. Such a group of utility functions are considered to be consistent with mean-variance preferences. There are many utility functions with such preferences, mostly dependent on the distribution of the data, in our case, of returns. We narrowed down the selection to functions represented

in [6] as to: exponential utility, power utility and quadratic utility. Even though there is disagreement in academic literature, many economists believe that exponential utility assumes a normal distribution of returns. In our case, accumulated returns at the end of the investment period. We have performed various normality tests on our data. First, we constructed a density plot as well as a quantile-quantile plot to get a visual grasp of the data, which can be seen in figure (12).

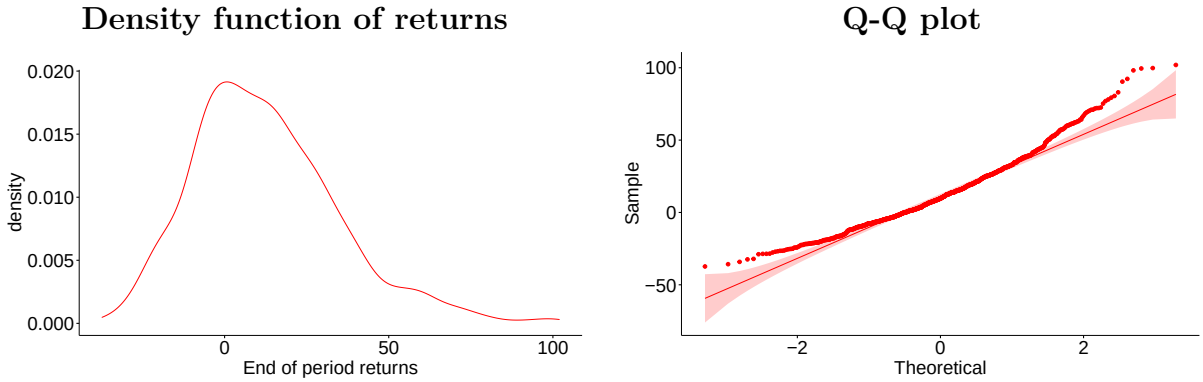


Figure 12: Density function of returns (left) and quantile-quantile plot (right) based on 1000 simulations.

The density function resembles the density function of normal distribution, however, a slight skewness is visible. The Q-Q plot showing a correlation between our data and the data from normal distribution also shows that some of our data are outside the quantile of normal distribution. Since we were not able to determine whether the differences between our data and data from normal distribution were or were not significantly different from each other purely on a visual basis, we performed a Shapiro-Wilk test of normality. Because the p-value was less than 0.05, we rejected the hypothesis of the distributions being not significantly different.

Seeing that the skewness from the figure (12) resembled the density function of log-normal distribution, our second attempt was to use the power utility function. By taking logs of our data and using the Shapiro-Wilk test of normality again, we took advantage of the fact that if the log-data are normal, then the original data are log-normal. Unfortunately, the p-value was once again less than 0.05, implying our data were not log-normal too. To not make a statement whether such distributions are or are not necessary to use exponential or power utility functions, for simplicity we decided to use quadratic utility

Strategy	Utility Score
1-quarter Treasury Bill	5.12
2-year Treasury Note	13.78
10-year Treasury Bond	30.92
20-year Treasury Bond	33.81

Table 3: Utility scores of various strategies based on quadratic utility function with $\lambda = 3$.

function as it does not require any distributional assumptions and also, for the sake of comparing few bond strategies, such choice is more than sufficient.

Following [6], the quadratic utility function is defined as follows:

$$\mathbb{E}(U) = \mu - \frac{1}{2}\lambda\sigma^2 \tag{26}$$

where μ equals expected return, σ standard variation and λ denotes risk aversion. Coefficient λ is believed to be in the range 1-5, where $\lambda = 1$ signals the lowest risk aversion from the range. In the thesis, we decided to choose $\lambda = 3$, implying median risk aversion preferences. Table (3) presents utility scores from the strategies presented in figures (10) and (11).

As we can see from table (3), none of the mentioned strategies has better performance than a risk-free option of buying and holding a bond matching the investment horizon in terms of the utility function defined as (26). Why this finding comes as no surprise will be explored in the next section when comparing our modelling approach to a counterfactual one.

4 Constant r-star

As we mentioned in section (2), most term structure models (almost all of them prior to GFC) have assumed a constant r^*. Therefore, this chapter's subject matter is to create a counterfactual approach consistent with the assumption of constant r^* and to compare it to the approach from previous sections.

To create the counterfactual, we revisit equation (11), where the calibration for c_t^n as well as for b_t^n remains unchanged to emphasize the effects r^* has on bond yields. To make it comparable to the DAR approach, we set the natural rate of interest equal to the starting point of simulated r^* to ensure that r_0^n is the same for both approaches and we will leave it unchanged throughout the whole investment period.

Figure (13) describes the difference in predicted yield of a 3-month Treasury Bill in the upper part of a figure and 20-year Treasury Bond in the bottom part, meanwhile, figures on the left reflect r^* simulated using DAR and figures on the right have r^* set to a constant.

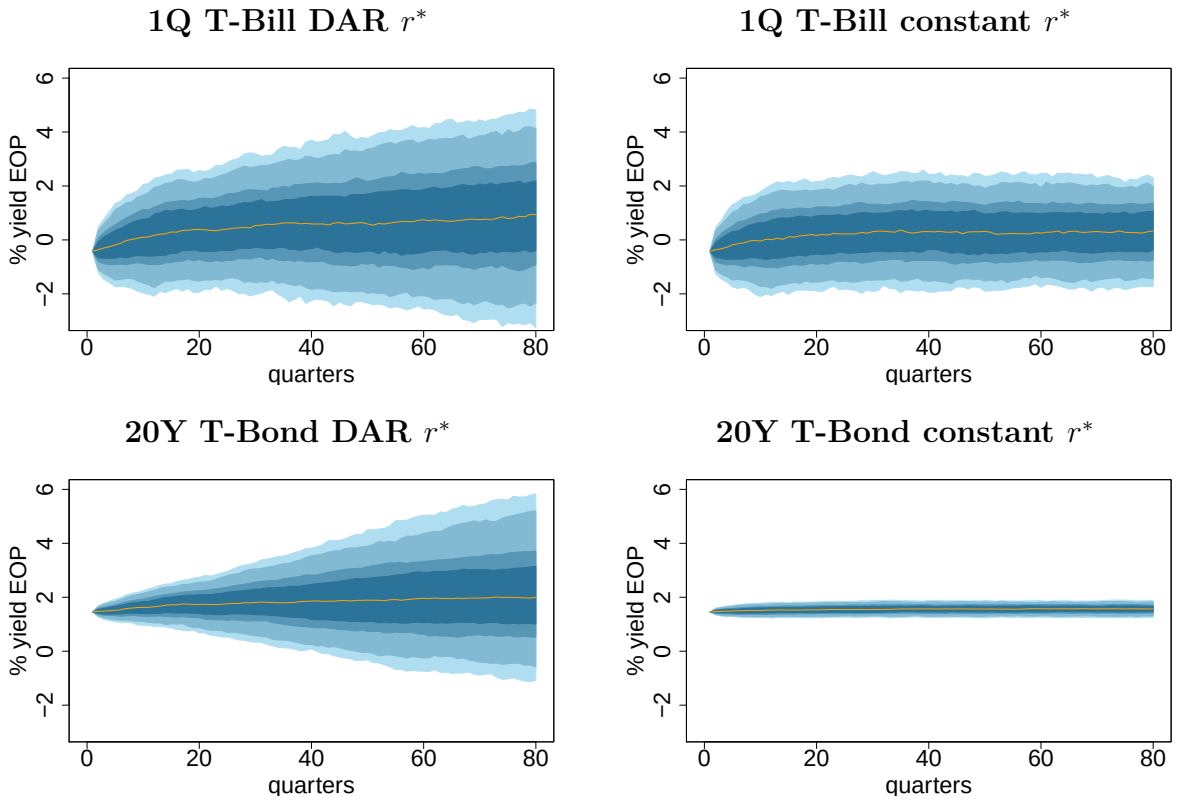


Figure 13: Estimated yields comparing r^* approaches based on 1000 simulations.

As we can see, the difference is much more significant in the case of a 20-year bond. If we go back to equation (11), we can see that the only variation in yields is now due to cyclical part of real rate, which is highly dependent on bond's maturity according to [15]. For reminder $c_t^n = \left(\frac{1}{n}\sum_{i=0}^{n-1} h^i\right) c_t$, therefore the role cyclical part of real rate plays decreases with increasing maturity.

What is more important to the investor based on his model, what returns can he expect to have and most importantly, what losses he might incur. Figure (14) presents such results in the same manner it was regarding yields in figure (13).

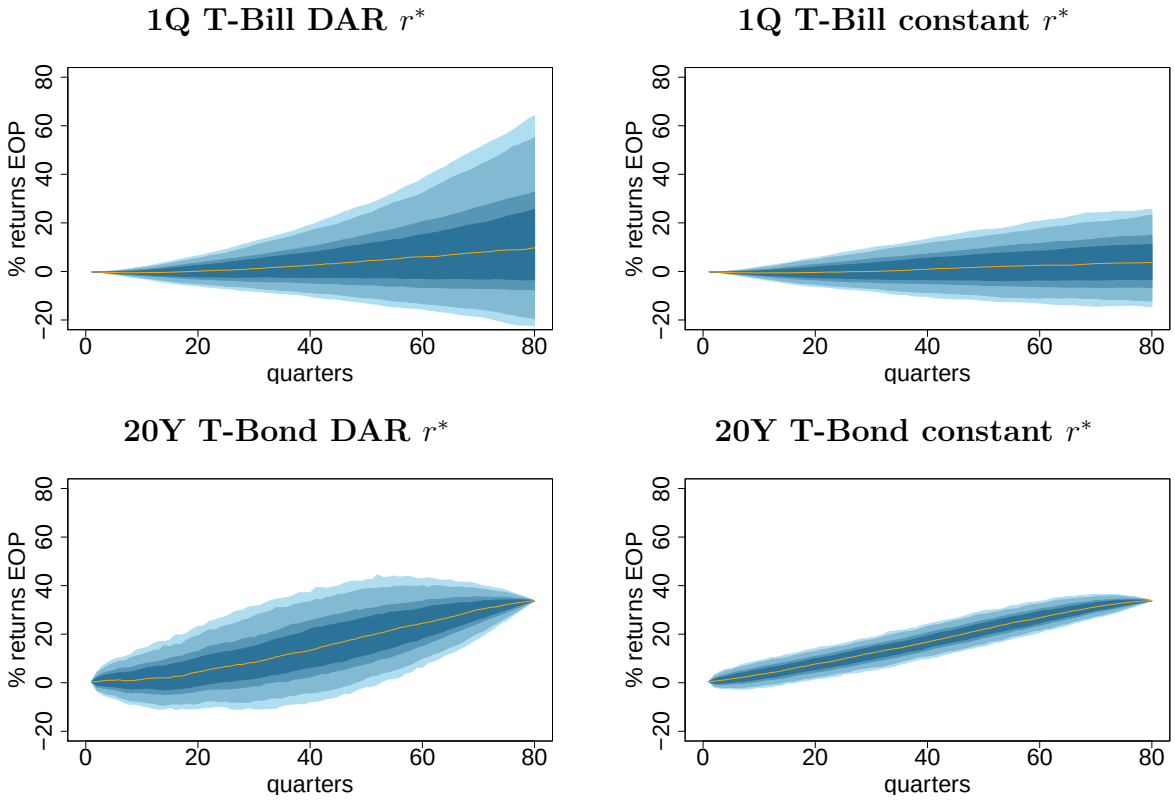


Figure 14: Estimated returns comparing r^* approaches based on 1000 simulations.

It is now apparent that based on our simulations, there is not a negligible amount of additional reinvestment risk caused by fluctuations of r^*. To get a better grasp of those differences, we also present a table (4) describing return percentiles of possible outcomes throughout mentioned investment strategies when accounting for and when ignoring changes in the natural rate of interest.

As we can see, not dependent on the model, the only safe choice an investor can make is to buy a bond matching his investment period and hold it to maturity. By doing

so, he protects himself from potential losses but also misses out on potential gains. If his investment style is more aggressive or he wants to bet on rising yields, he chooses a bond of shorter maturity. This is where our contribution becomes significant as we show that accounting for persistent changes in r^* resulting in persistent changes in yields accounts for a large variation in potential returns compared to the model that perceives r^* as constant. Based on our simulations, when rolling a 3-month Treasury Bill, the 90^{th} percentile accounts for a 41% gain when simulating r^* using DAR and only 19% gain when considering r^* a constant. On the other hand, the 10^{th} percentile accounts for a 12% loss compared to only a 9% loss DAR versus constant r^* approach. For more in-depth results, please see table (4) or consider running the simulation yourself using the replication code provided in Appendix A.

Percentile	1%	5%	10%	25%	50%	75%	90%	95%	99%
1-quartal Treasury Bill									
DAR r^*	-27.13	-19.49	-12.27	-3.32	9.73	25.58	40.61	55.15	77.02
Constant r^*	-16.87	-12.25	-9.01	-3.45	3.71	11.07	18.85	23.19	33.03
2-year Treasusy Note									
DAR r^*	-19.56	-10.11	-5.28	5.84	18.46	32.89	48.41	57.79	78.46
Constant r^*	-6.63	-1.67	1.64	6.58	12.02	18.22	23.34	26.23	31.47
10-year Treasury Bond									
DAR r^*	7.11	13.00	16.85	23.69	32.59	42.48	52.71	58.87	73.90
Constant r^*	19.31	22.18	23.24	25.73	28.48	31.07	33.82	35.26	38.03
20-year Treasury Bond									
DAR r^*	33.81	33.81	33.81	33.81	33.81	33.81	33.81	33.81	33.81
Constant r^*	33.81	33.81	33.81	33.81	33.81	33.81	33.81	33.81	33.81

Table 4: Estimated returns comparing r^* approaches based on 1000 simulations. Values in percentages.

Besides percentile results, we also present table (5), which shows mean, median and standard deviation of simulation results in the half of investment period (HOP) and at the end of the investment period (EOP). What is not surprising after seeing the previous table,

	HOP[1] mean	HOP median	HOP s.d	EOP[2] mean	EOP median	EOP s.d
1-quartal Treasury Bill						
DAR r^*	3.04	2.57	7.64	12.58	9.73	22.30
Constant r^*	1.11	0.90	6.14	4.36	3.71	10.74
2-year Treasury Note						
DAR r^*	6.29	6.15	6.03	20.44	18.46	21.08
Constant r^*	4.67	4.47	4.52	12.32	12.02	8.52
10-year Treasury Bond						
DAR r^*	12.12	12.12	0.00	33.87	32.59	14.02
Constant r^*	12.12	12.12	0.00	28.48	28.48	4.03
20-year Treasury Bond						
DAR r^*	13.96	13.34	11.58	33.81	33.81	0.00
Constant r^*	16.85	16.70	3.62	33.81	33.81	0.00

Table 5: Estimated returns comparing r^* approaches based on 1000 simulations. Values in percentages.

standard deviation when accounting for changes in r^* is much higher in all of the cases. Interestingly enough, the table shows that none of the listed strategies provides better results both in mean and median compared to simply buying and holding a bond with maturity matching ones investment period. Such finding confirms the results obtained with the help of the utility function presented in table (3), as the mean is largest for the 20-year Treasury Bond (except for the 10-Year Treasury Bond, which is only slightly larger) so that even if the investor was completely risk-tolerant, it would not make any sense for him to choose any other strategy based on the utility score. The higher the risk

[1]HOP = Half of period

[2]EOP = End of period

aversion is, the less sense it makes, as the utility score would only lower. Therefore table (4) might give more hope to the risk-seeking investor in terms of potential returns.

The table below concludes this section as it transforms mean and variance of end of period returns from table (5) into utility scores according to ones risk aversion based on equation (26) to emphasize potential utility loss from considering r^* a constant.

Risk aversion coefficient	1	2	3	4	5
1-quarter Treasury Bill					
DAR r^*	10.09	7.61	5.12	2.64	0.15
Constant r^*	3.79	3.21	2.63	2.06	1.48
2-year Treasury Note					
DAR r^*	18.22	16.00	13.77	11.55	9.33
Constant r^*	11.96	11.59	11.23	10.87	10.51
10-year Treasury Bond					
DAR r^*	32.89	31.91	30.92	29.94	28.96
Constant r^*	28.39	28.31	28.23	28.15	28.07
20-year Treasury Bond					
DAR r^*	33.81	33.81	33.81	33.81	33.81
Constant r^*	33.81	33.81	33.81	33.81	33.81

Table 6: Utility scores comparing r^* approaches based on 1000 simulations.

The table (6) suggests that the investor accounting for persistent changes in r^* should be much more aware of the reinvestment risk in case of short-term Treasury securities as with increasing risk aversion his utility score sharply decreases compared to an investor who's model considers r^* a constant. Such choice, as empirically shown, might result in a significant amount of utility loss. Intuitively, if the investor intents to sell his long-term bond prior to the maturity, significant utility loss would also be present according to figure (14).

5 Robustness

In all of our previous work, we have been using data until 2019. The reason for this is, the year 2020 was marked by the pandemic COVID-19. This pandemic had numerous social, health but also economic consequences. The GDP of almost every country was deeply influenced by lockdowns and widespread economy closures. Considering the USA, the quarter-over-quarter GDP growth in Q2 2020 was -36% as all of the economy closed, the next quarter on the other hand, as all the restrictions were relaxed QoQ GDP in Q3 2020 jumped to 30%. In a year-over-year measure, all of the quarters in 2020 suffered a GDP decline compared to the previous year. Figure (15) shows how much of the outliers these GDP data are compared to the historical data.

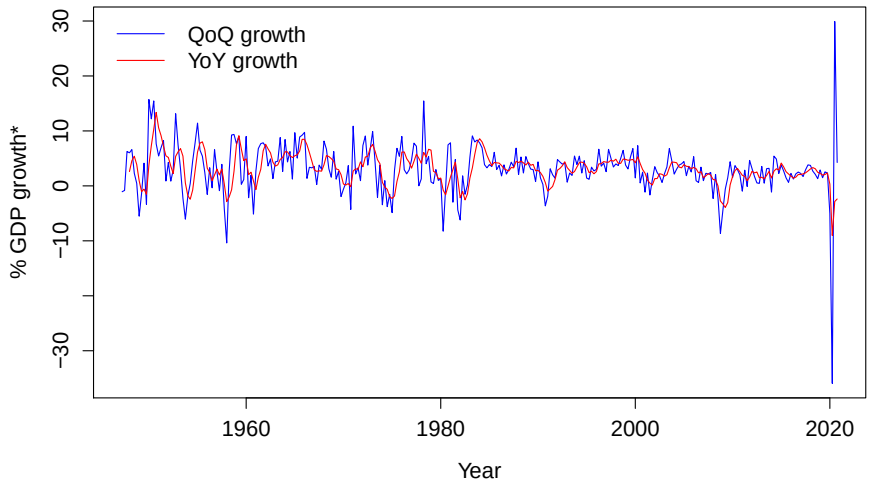


Figure 15: Quarterly Gross Domestic Product growth in the USA. Sample period is Q2:1947 through Q4:2020. Source: [25]

As it is evident from equation (3), our model relies heavily on GDP growth. Even applying the constant gain learning algorithm described by equation (2) did not manage to mitigate the pandemic's effects. Our model is designed to represent well the effects of persistent changes in the economy, meanwhile, we can not consider economic shutdowns and re-openings nowhere near to be persistent. Figure (16) represents the r^* estimation if using our model.

The jump behaviour at the start of a simulation is solely caused by exceptional volatility of GDP growth at the end of our learning historical data, as constant gain learning

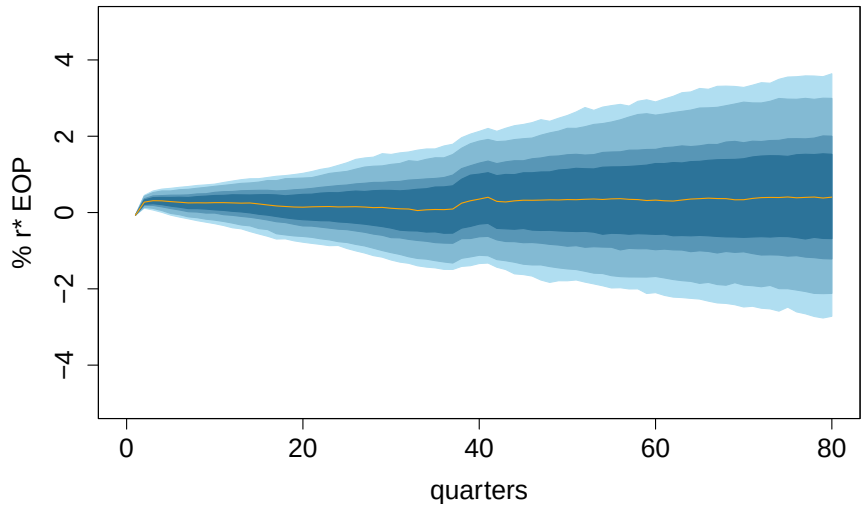


Figure 16: 1000 simulations of non-adjusted r^* influenced by COVID-19.

assigns the biggest weight to the most recent observations. The issues, unfortunately, do not end at the start of the period. Somewhere around 40. quarter one could see another deterministic jump in the forecast. This is because in constant gain learning (2), we set the look-back parameter to 40 quarters, so once the simulation reaches this number, the effects of pandemic outliers are no longer visible. After these observations, we can conclude that the model does not do a satisfactory job when countering significant outliers at the end of its learning period. This comes as no surprise as the GDP growth based model was inspired by [17] and [12] from New York Federal Reserve, who stopped sharing their real-time r^* estimates due to extraordinary volatility in GDP growth related to the COVID-19 pandemic and they released a note on adjusting their model to account for the pandemic situation.

Therefore we decided to estimate r^* in 2020 in the following fashion. Federal Reserve Bank of Philadelphia regularly conducts a Survey of Professional Forecasters (SPF) on various macroeconomic variables [10]. Revisiting equation (11), we can compare r^* to 3-month Treasury Bill assuming that $b_t^1 \doteq 0$ and simultaneously, if we pick the long term forecast of r_t^1 and subtract the long term forecast of CPI inflation, we might come close to the estimate of r^* as the cyclical part of real rate is also suppressed by the median log-term estimate of inflation. This way, we should end up somewhere close to the equilibrium or natural rate of interest. To maintain the structure of the model, we calculated *pro forma* GDP growth that would account for r^* calculated using the SPF approach using the

following equation:

$$g_t = r^* - s_t \tag{27}$$

where s_t account for liquidity and safety demand effects. Based on this approach, we are able to use our simulation model and, at the same time, deal with extraordinary volatility in GDP caused by the COVID-19 pandemic.

The pandemic's aftermath can be mostly seen in declined r^* as the first time in history it reached negative values. The simulation forecast of r^* based on such starting value is represented by figure (17).

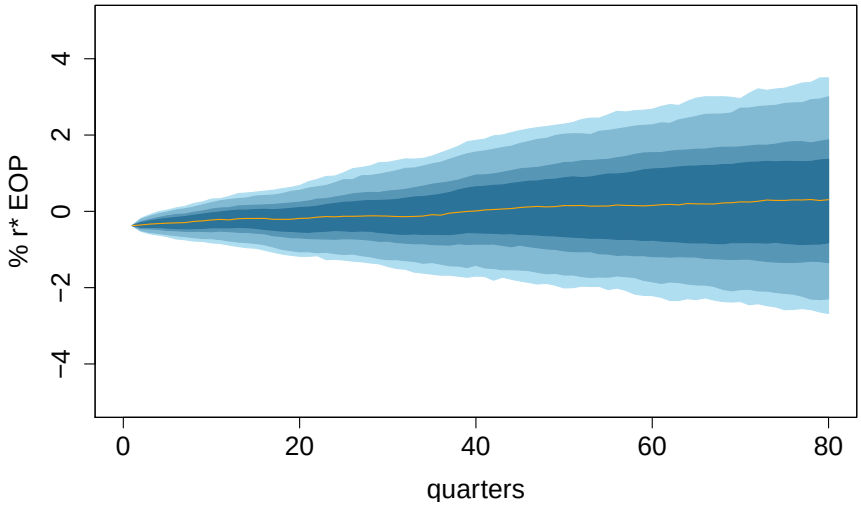


Figure 17: 1000 simulations of adjusted r^* influenced by COVID-19.

Such a downward shift of the natural rate of interest has caused almost the whole real yield curve to end up in negative values. This state of the yield curve is not very beneficial for the economy, which is well understood by FED, who predict its rise in the next decades. Such behaviour is well represented by our model, as yields slowly raise towards the fundamental determinants of our calibration. The median yield curve based on the post-2020 r^* estimate can be seen in figure (18).

The negative yields on short to mid-term treasuries and only slightly positive values on long-term treasuries significantly impact returns the investor might expect in his investment horizon. Figure (19) describes again the most extreme strategies of rolling a 3-month Treasury Bill and buy and hold strategy of a bond matching investment horizon.

Since the differences in possible anticipated returns based on whether we allow for shifts in r^* or not were very similar to those in tables (4) and (5), we decided to skip

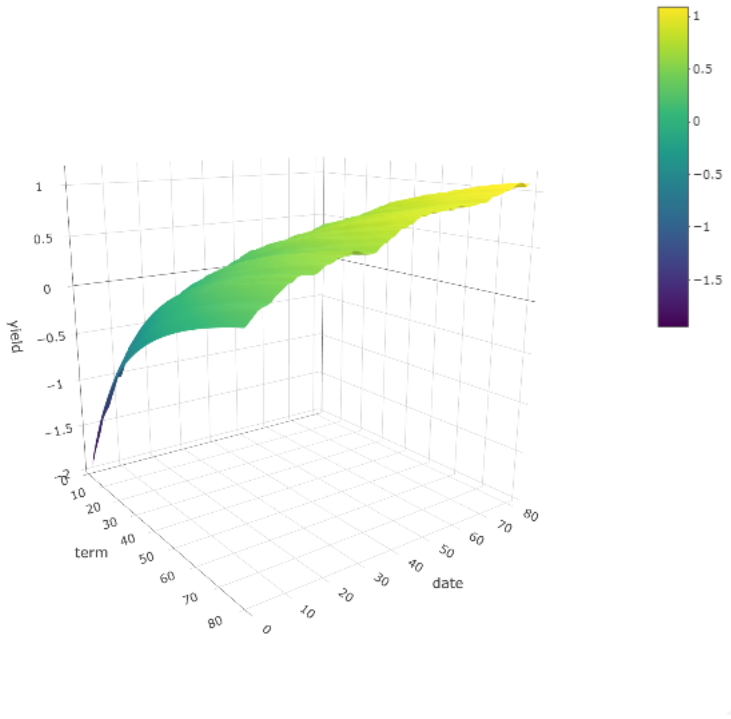


Figure 18: Median yield curve based on 1000 simulations accounting for 2020 data

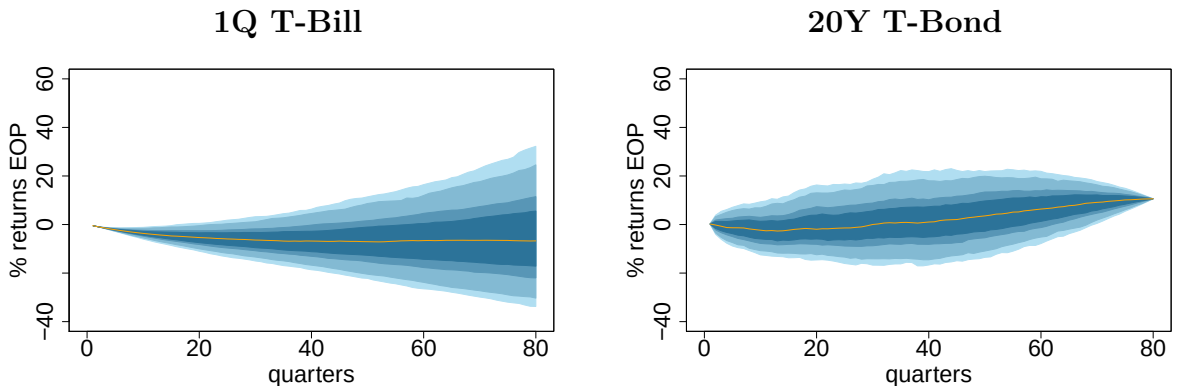


Figure 19: Anticipated returns of rolling 3-month Treasury Bill throughout whole investment period (left) compared to buy and hold strategy of a 20-year Treasury bond (right) based on 1000 simulations. Yellow line is median. Data adjusted for the pandemic.

on providing more additional information emphasizing the importance of accounting for those shifts, instead we provide a similar table, only this time comparing the returns an investor could expect before the COVID-19 pandemic and after the year 2020.

Percentile	1%	5%	10%	25%	50%	75%	90%	95%	99%
1-quartal Treasury Bill									
Pre C-19[3]	-27.13	-19.49	-12.27	-3.32	9.73	25.58	40.61	55.15	77.02
Post C-19[4]	-36.75	-30.32	-25.74	-17.14	-6.73	5.54	15.60	24.58	42.13
2-year Treasusy Note									
Pre C-19	-19.56	-10.11	-5.28	5.84	18.46	32.89	48.41	57.79	78.46
Post C-19	-30.11	-23.29	-19.01	-10.04	-0.23	11.14	22.77	34.59	53.79
10-year Treasury Bond									
Pre C-19	7.11	13.00	16.85	23.69	32.59	42.48	52.71	58.87	73.90
Post C-19	-10.97	-5.90	-2.88	3.06	10.35	18.56	26.77	31.91	44.66
20-year Treasury Bond									
Pre C-19	33.81	33.81	33.81	33.81	33.81	33.81	33.81	33.81	33.81
Post C-19	10.59	10.59	10.59	10.59	10.59	10.59	10.59	10.59	10.59

Table 7: Estimated returns comparing r^* approaches based on 1000 simulations. Values in percentages.

As the yields were depressed after the pandemic, the expected returns are also somewhat lower. Throughout all strategies, the expected returns at the end of an investment period are about 20% lower. Buying a 20-year Treasury Bond might be a good solution for protecting the investor against inflation, but it is nowhere near being a tool for building wealth. Despite the figure (19), the investor is also not likely to gain from selling the bond prior to maturity as from an empiric standpoint, yields will not soar any lower. Another lowering of long term yields would make the investors choose to hold on to cash instead. Quite the opposite, yields are expected to rise, which would result in the depreciation of investors bond. A much more intuitive solution, if only looking at the basket of Treasury securities, would be buying a short term bond to mitigate at least some portion of inflation and wait for long term bonds to reach at least pre-pandemic levels, as 33.81% return

[3]Pre C-19 = Simulation based on pre-pandemic setup - data for 2020 not used

[4]Post C-19 = Simulation based on post-pandemic setup - including 2020 data

	HOP mean	HOP median	HOP s.d	EOP mean	EOP median	EOP s.d
1-quartal Treasury Bill						
Pre C-19	3.04	2.57	7.64	12.58	9.73	22.30
Post C-19	-6.76	-6.81	6.39	-5.14	-6.73	17.11
2-year Treasury Note						
Pre C-19	6.29	6.15	6.03	20.44	18.46	21.08
Post C-19	-3.52	-3.55	5.22	1.64	-0.23	17.26
10-year Treasury Bond						
Pre C-19	12.12	12.12	0.00	33.87	32.59	14.02
Post C-19	1.42	1.42	0.00	11.34	10.35	11.75
20-year Treasury Bond						
Pre C-19	13.96	13.34	11.58	33.81	33.81	0.00
Post C-19	1.26	1.03	9.62	10.59	10.59	0.00

Table 8: Estimated returns comparing r^* approaches based on 1000 simulations. Values in percentages.

throughout the investment horizon is much more appealing than 10.59% return available at the time of writing this thesis.

6 Historical evidence

To conclude this thesis, we decided to show how US Treasury securities have performed in the past compared to our future simulation-based results. One has to realize that the Treasury market situation at the start of 2001, 20 years ago, offered completely different interest rates to investors for the purchase of their securities. According to `https://www.treasury.gov/` at the start of Q2 in 2001, the yield curve in the US was relatively flat, offering a 4.2% nominal yield p.a. for a 3-month Treasury Bill and about 5% and 5.5% nominal yield p.a. for 10 and 20-year Treasury Bonds respectively. Thus, investors holding their Treasuries until maturity were rewarded much higher than are investors nowadays, but also, as we have seen from equation (21), the price of those long term bonds have risen significantly over time as the yields were lowering. Suppose an investor has been holding a 20-year bond yielding 5% for 19 years, now he has a bond that matures in a year with the same expected yield. On the other hand, all the other investors are offered to buy newly issued bonds with the same maturity, yielding not even 0.1%. The price of his bond is large compared to the new ones. To showcase this phenomenon, we use data on 3-month Treasury Bills from [4] and data on Treasury Bond returns of various maturities obtained from [3]. Those data are from indices divided into 3 maturity brackets: 1-3 year Treasury Notes (short), 7-10 year Treasury Bonds (middle) and 20+ year Treasury Bonds (long). These indices are constructed so that the average remaining maturity of the bonds inside is moving somewhere inside the designated range. That implies that bonds inside the index are not held until maturity, rather they are traded, so the maturity condition is fulfilled. Whilst it is questionable whether this is the optimal way to invest in bonds, for retail investors, it often is the only way, as they do not always have access to buying a bond itself.

We have calculated the cumulative returns of both T-Bill and indices using equation (20). Since both data sets were in nominal terms, we also adjusted the returns for inflation, using CPI inflation available here: [26]. As it is generally known, the transformation from nominal returns to real returns can be executed using the following formula:

$$\varrho^{real} = \left(\frac{1+\varrho^{nom}}{1+i}\right) - 1 \tag{28}$$

where i denotes CPI inflation rate.

Figure (20) describes the cumulative returns from the above-described investment options when investing 20 years ago and holding the index until now or repurchasing T-Bills.

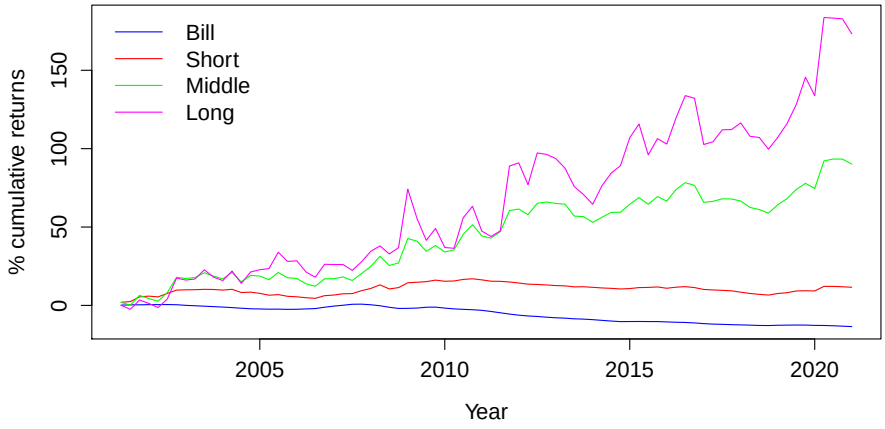


Figure 20: Cumulative returns from investments into treasuries. Sample period is Q2:2001 through Q1:2021.

The downward shift in real interest rates can be mostly seen from the blue line, as after the GFC, real rates of 3-month Treasury Bills have been most of the time negative, which suppressed the returns at the end of the period to about -13.5%. As for returns from Barclays Bloomberg Indices, the longer the maturity bracket, the more returns the investor was rewarded. Starting with the short index, the return after 20-years has only been 11.5%, for middle ranged 90%, and for the highest maturity bracket index, the return was about 173.3%, which in annualized terms accounts for about 5.2% real return, due to falling interest rates. Such return is almost as good of a performance as by S&P 500 represented stock market.

One could ask why are we comparing bond indices to singular bonds modelled in our simulation. The reason is quite technical. If we remember equation (24), if wanting to calculate historical bond returns on a 20-year Treasury Bond, we would need bond yields of every maturity ranging from 1 quarter to 80 quarters. Whilst we were able to do that from simulated data, there are not such historical data as bonds are not denominated for every possible maturity.

Although there is a way around it, which is to use a Nelson-Siegel-Svensson (NSS)

historical yield curve estimation which infers what the yields would be across the maturity spectrum from the prices of existing securities. With the help of NSS we were able to obtain historical real rates and replicate the procedure from previous sections to construct a yield curve and calculate historical bond returns. To obtain historical real rates, we followed the approach of [13]. The 6-parameter yield curve estimation introduced by Svensson as a refinement of traditional Nelson-Siegel functional form can be expressed using the following equation:

$$r_t^n = \beta_0 + \beta_1 \frac{1 - e^{-\frac{n}{\tau_1}}}{\frac{n}{\tau_1}} + \beta_2 [\frac{1 - e^{-\frac{n}{\tau_1}}}{\frac{n}{\tau_1}} - e^{-\frac{n}{\tau_1}}] + \beta_3 [\frac{1 - e^{-\frac{n}{\tau_2}}}{\frac{n}{\tau_2}} - e^{-\frac{n}{\tau_2}}] \tag{29}$$

where n is maturity in years and the other parameters are well explained in [13], which is also the source of the historical parameters used in this thesis. If interested, consider reading the original paper [24], where equation (29) was introduced.

Since we wanted to make the estimation comparable to the simulation results, we readjusted equation (29) so that the maturity is expressed in quarters. Accounting for the continuity and monotonicity of the functional form in between the annual maturities, we readjusted the function in the following way:

$$r_t^n = \beta_0 + \beta_1 \frac{1 - e^{-\frac{n}{4\tau_1}}}{\frac{n}{4\tau_1}} + \beta_2 [\frac{1 - e^{-\frac{n}{4\tau_1}}}{\frac{n}{4\tau_1}} - e^{-\frac{n}{4\tau_1}}] + \beta_3 [\frac{1 - e^{-\frac{n}{4\tau_2}}}{\frac{n}{4\tau_2}} - e^{-\frac{n}{4\tau_2}}] \tag{30}$$

As authors of [13] indicate, the functional form is not stable for low maturities (less than 2 years), and therefore, we follow their approach of linearly down-weighting the yield curve from maturity of 8 quarters to 1 quarter for smooth behaviour of the short-end of the curve over time. To do so, we used the spread between 3-month artificial TIPS and a 2-year TIPS from figure the (5). The resulted yield curve is presented by figure (21).

Once we had the entire yield curve, we followed bond maths from section (3) to calculate historical returns on the same treasuries we discussed throughout the thesis. We exclude the 3-month Treasury Bill, as we already presented it in figure (20). In contrast to this figure, in figure (22) investor does not buy an index, he follows the same strategy as it was in the case of the simulations, i.e. he buys the security, holds it to maturity, but we also see his potential balance if he chooses to sell in between the maturities.

As we can see, 20 years ago, the investor was best off buying a bond matching his investment horizon. This is simply because of the falling interest rates. For instance, an

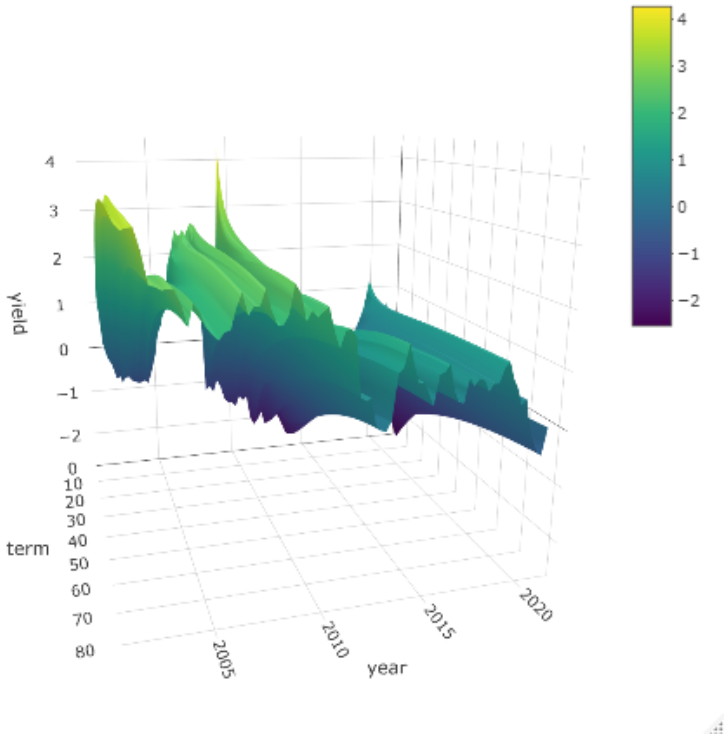


Figure 21: Historical evolution of the real yield curve. Sample period is Q2:2001 through Q1:2021.

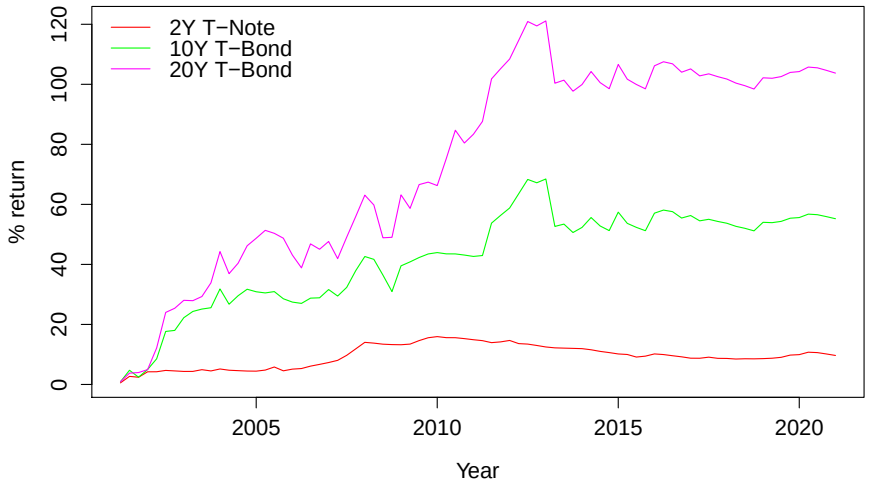


Figure 22: Cumulative returns from investments into treasuries. Sample period is Q2:2001 through Q1:2021.

investor buying a 10-year Treasury Bond received a 3.5% p.a. real yield on his investment, although, 10 years later, he reinvested at only 0.8% p.a. His total return at the end of the investment period accounted for about 55.2%. Investor rolling 2-year Treasury Note just

suffered the lowering yields, walking away after 20 years with only a 9.6% real return. He managed to outperform the investor rolling Treasury Bills, as he stayed a little longer in the environment of higher yields and he also collected some term premium. The investor buying a 20-year Treasury Bond received a nice compensation of 103.7%.

Conclusion

In this thesis, we explored the natural rate of interest, its implications on bonds yields and, ultimately, bonds returns. We described what drives r^*, what was its state throughout the last decades and followed by forecasting it into the future. When forecasting r^*, we put emphasis on its links to macroeconomic variables, as well as properties of securities that are valued by investors the most, i.e. its liquidity and safety. This forecast was made using the Double Auto-Regressive model, which, as we believe, provides the best tools to capture the most significant properties of r^* and therefore gives us the most reliable source of future expectations. Results of this estimation can be seen in figure (4). It is important to notice that r^* is modelled to be a subject of persistent shifts, which results in significant volatility contradictory to the historical research considering it a constant. We followed by linking the natural rate of interest to bond yields. Using historical data, we estimated the cyclical part of real rate with AR(1) process. Subsequently, we estimated the term premium as a function of external factors described in equation (16) to provide the most relevant reflection of the present shape of the term structure of interest rates. Our estimation of term premium is in figure (6). Once we had all components of bond yields estimated, we put them all together and provided an estimate of the yields on Treasury securities of all possible maturities. Since we were able to forecast yields for all maturities, we constructed a 3D yield curve that represents its current state and possible future development. The median yield curve based on 1000 simulations can be seen in figure (8), yet to grasp a better understanding of how our simulation is designed, see figure (9) as it is a specimen of a persistent change in real interest rates.

Since bond investors do not always hold their bonds to maturity, they also tend to cash in from the bonds price fluctuations we followed by calculating bond returns. We quantified the reinvestment risk of rolling short term Treasury securities and we also showed the volatility of returns if selling a bond prior to its maturity. These findings are well represented by figures (10) and (11).

One of this thesis's key contributions is the quantification of additional uncertainty of returns when accounting for persistent shifts in r^* compared to the counterfactual of assuming r^* to be constant. We consider figure (14) the highlight of the thesis as it shows that accounting for variation in the natural rate of interest plays a not negligible role in

forming expectations about future returns from bond investments and that ignoring this behaviour of r^* might result in a significant amount of utility loss (6).

Throughout writing this thesis, it was a very turbulent social, health-related but also economic period. Such a period marked by unprecedented volatility of GDP growth proposed a challenge to our model. Therefore, if accounting for the year 2020, we readjusted our model, provided an estimate of current r^* and estimated bond yields and returns accordingly. We dedicated tables (7) and (8) to outline the differences in expectations an investor had prior to the pandemic and afterwards according to our model.

Finally, we described the historical returns from investments into Treasury securities using bond indices as well as bonds themselves based on Nelson-Siegel-Svensson yield curve estimation to allow the reader to compare the results of our simulation with historical evidence. Following the differences, we discussed the change in the bond market that occurred in the last 2 decades.

This thesis's key contribution is to help investors form expectations about the potential returns of their bond investments. Our work gives convincing evidence that accounting for variation in r^* is necessary to make well-informed decisions. Our simulation, available in Appendix A (6), allows for various settings any individual feels are accurate at the current state of the economy as well as it is easy to fill up the model with up to date data that are available to everyone.

We admit that the simulation might have its imperfections, as our goal was to emphasize the effects of r^*. Possible enhancements of the simulation might include allowing for time variation in term premium consistent with shifts of r^* or adding inflation expectations as an explanatory variable of nominal bond yields. If one wished to continue the research helping the investor in his asset allocation, accounting for r^* shifts in stock return expectations would be more than welcome. Only then we would have all we need to perform a natural rate of interest based portfolio choice.

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Appendix A

Replication code for the simulation setup, plots in this thesis as well as underlying data to successfully run the simulation can be found at `https://github.com/MartinVojtek/masters_thesis`. The rest of this appendix serves as a simple guidance to the files that go with this thesis.

A.1 Data

- US_GDP.xlsx - data on US GDP sourced from [25], adjusted to various scenarios, containing calculations of QoQ and YoY GDP growth
- cyclical_part_of_real_rate.xlsx - data on 3-month Treasury Bill [4] as well as inflation expectations [1], containing calculations of c_t
- TermPremiumCalibration.xlsx - file describing estimation of Term Premium based on outputs of the simulation
- HLW_GDP_in_world.xlsx - data created by [12] on the state of the GDP growth trend throughout the last decades
- HLW_rstar_in_world.xlsx - data created by [12] on the state of the r^* throughout the last decades
- 2yTIPSvs3mATIPS.xlsx - data comparing 2-year TIPS [13] and artificial TIPS to size up the differences
- US_CPI.xlsx - data on historical CPI inflation from [26] used for calculation of real historical returns
- nss_par_data.xlsx - data with historical parameters of NSS from [13], as well as data used in cyclical_part_of_real_rate.xlsx

A.2 Codes

All codes listed below can be run separately.

- simulation.r - main code of the thesis, runs simulations of r^*, estimates bond yields and bond returns, constructs a 3D yield curve. All of the research results are based on this file.
- RMSE_TermPremium.r - code used to estimate σ and β in calibration of term premium of interest rates
- TIPS_comparison.r - code used to plot figure (5)
- plot_HLW_world_GDP.r - code used to plot figure (2)
- plot_HLW_world_rstar.r - code used to plot figure (3)
- plot_US_GDP.r - code used to plot figure (15)
- historical_returns.r - code used to calculate historical nominal and real returns on treasury securities, gives figure (20), underlying data are not included as that would violate copyrights
- nss.r - code used to estimate historical real yield curve using NSS functional form and calculate returns based off of it, gives figures (21) and (22)

Appendix B

B.1 Half-life of interest rates

Let c_t denote a weakly stationary AR(1) process describing cyclical part of real rate defined as:

$$c_t = \delta + \phi c_{t-1} + \epsilon_t \tag{1}$$

Weak stationarity implies that

$$\mathbb{E}(c_t) = \mu \tag{2}$$

for all t, where μ is a constant defined as follows:

$$\mu = \frac{\delta}{1-\phi} \tag{3}$$

By subtracting μ from both sides of (1), we obtain:

$$c_t - \mu = \phi(c_{t-1} - \mu) + \epsilon_t \tag{4}$$

or

$$Y_t = \phi Y_{t-1} + \epsilon_t \tag{5}$$

where $Y_t = c_t - \mathbb{E}(c_t)$ is defined as the distance to the mean.

Considering the current distance to the stationary mean is Y_t, we expect the half-life to halve the distance to the stationary mean in time $t + HL$:

$$\mathbb{E}(Y_{t+HL}) = \frac{1}{2} Y_t \tag{6}$$

By revisiting equation (5) we get that $\mathbb{E}(Y_{t+HL}) = \phi^{HL} Y_t$ which leads to:

$$\phi^{HL} Y_t = \frac{1}{2} Y_t \tag{7}$$

Using few algebraic simplifications we obtain the formula for half-life:

$$\phi^{HL} Y_t = \frac{1}{2} Y_t \Leftrightarrow \phi^{HL} = \frac{1}{2} \Leftrightarrow \ln(\phi^{HL}) = \ln\left(\frac{1}{2}\right) \Leftrightarrow HL \ln(\phi) = \ln\left(\frac{1}{2}\right) \Leftrightarrow HL = -\frac{\ln(2)}{\ln(\phi)} \tag{8}$$